$*$-ORDERABLE SEMIGROUPS

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ABSTRACT. Fix a $*$-orderable field k. We introduce the class of *$*$-orderable semigroups* as those semigroups with involution S for which the semigroup algebra kS endowed with the canonical involution admits a $*$-ordering. It is shown that this class is a quasivariety that is locally and residually closed. A cancellative *nilpotent* semigroup with involution is proved to be $*$-orderable if and only if it has unique extraction of roots. In general this equivalence fails, although every $*$-orderable semigroup has unique extraction of roots.

1. INTRODUCTION

It is well known and easy to see that a semigroup algebra kS admits a total ordering if and only if the field k is formally real (i.e., admits a total ordering) and S is a cancellative orderable semigroup. The case of $*$-orderability of kS is much harder. The notion of a $*$-ordering has been extended from division rings to general noncommutative rings in a series of papers by Marshall [10, 11] and Craven [3]. Recently $*$-orderability of group algebras has been investigated in [2, 7]; see also [6, Section 4.2]. The aim of this paper is to extend some of the results from groups to semigroups. For this we introduce $*$-orderable semigroups as those semigroups with involution S for which the semigroup algebra kS with the induced involution admits a $*$-ordering. We show that being a $*$-orderable semigroup is a local and residual property and deduce that the class of all $*$-orderable semigroups is a quasivariety. All this is done in Section 2.

The third section contains some results on nilpotent semigroups, which are used in Section 4 to give examples of $*$-orderable semigroups. Nilpotent semigroups were first introduced by Mal'cev [8], and later independently by Neumann and Taylor [14]. This class of semigroups has been studied widely, including in [13]. Our results give rise to another proof of the fact that free algebras admit $*$-orderings. This was first proved non-constructively in [3] and a nice, constructive proof was later given by Cimprič [1]. In Section 4 we also prove that every $*$-orderable semigroup has unique extraction of roots. The last section contains a discussion of the dependence on the base field. We show that the

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$*$-orderability of kS essentially depends at most on the type of involution on the $*$-orderable field k, not on the field itself.

2. $*$-orderable semigroups form a quasivariety

Definition. Let A be a domain with involution and $\operatorname{Sym} A$ the set of its symmetric elements. A subset $P \subseteq \operatorname{Sym} A$ is called a *$*$-ordering* if $P + P \subseteq P$, P is closed under the *Jordan multiplication* $\{a, b\} := ab + ba$, $P \cup -P = \operatorname{Sym} A$, $P \cap -P = \{0\}$ and for every $r \in A$ we have $rPr^* \subseteq P$. In other words, P is a total ordering of the Jordan algebra $(\operatorname{Sym} A, +, \{\ ,\ \})$ closed under $*$-conjugation for elements from A. (Note that our $*$-orderings are usually called "support $\{0\}$ $*$-orderings".)

The theory of $*$-ordered rings was started by Marshall [10, 11], connections with $*$-ordered division rings were studied in [3] and a more detailed study of the corresponding valuation theory and examples is given in [2, 1, 5, 6]. We refer the reader to these papers and the references therein for more details on $*$-orderings.

Throughout the paper k will denote a $*$-orderable field with involution $*$. As we will be dealing with unital semigroup algebras which are domains, unless mentioned otherwise we assume that *semigroups are cancellative* and *contain an identity element* denoted by 1. This element is preserved under subsemigroups and homomorphic images.

Definition. A semigroup S with involution σ (i.e., an antiautomorphism of order 2) is *$*$-orderable* if the semigroup algebra kS with the induced involution

$$\left(\sum_{s \in S} \lambda_s s\right)^* = \sum_{s \in S} \lambda_s^* \sigma(s)$$

admits a $*$-ordering.

The aim of this section is to study the class of *all* $*$-orderable semigroups. We show that this class is 1^{st}-order axiomatizable and even a quasivariety (sometimes called a universal Horn class) in the language of semigroups with involution. Moreover, we show that being a $*$-orderable semigroup is a local and residual property. All this generalizes our results for groups given in [7].

Let us recall two notions usually used in group theory. If $\mathfrak{P}$ is a property of $*$-semigroups, we say that a semigroup S is *locally-$\mathfrak{P}$* if every finitely generated $*$-subsemigroup of S is in $\mathfrak{P}$. S is called *residually-$\mathfrak{P}$* if for every $a, b \in S$ with $a \neq b$ there exists a $*$-invariant congruence $\sim$ such that $a \not\sim b$ and $S/\!\sim\, \in \mathfrak{P}$. Equivalently, there is a surjective $*$-homomorphism $\varphi : S \to T \in \mathfrak{P}$ such that $\varphi(a) \neq \varphi(b)$.

Lemma 1 (Lemma 3.1 in [7])**.** *The class of $*$-orderable rings is 1^{st}-order axiomatizable in the language of $*$-rings.*

This lemma says that the class of all $*$-orderable rings can be described by a set of 1^{st}-order axioms in the language of rings with involution. It can be proved using a suitable generalization of the Artin-Schreier theory for $*$-orderable rings: Marshall [10] proved that a ring with involution A admits a $*$-ordering if and only if $T_0 \cap -T_0 = \{0\}$, where T_0 denotes the smallest $*$-preordering of A. The condition $T_0 \cap -T_0 = \{0\}$ can be written as a countable set of 1^{st}-order sentences in the language of $*$-rings.

Theorem 2. *The class of all $*$-orderable semigroups is 1^{st}-order axiomatizable in the language $\mathscr{L}_* = (\,\cdot\,,\,*\,)$ of semigroups with involution.*

Proof. Recall the following result from model theory: a class of structures is 1^{st}-order axiomatizable if and only if it is closed under ultraproducts, isomorphic copies and substructures (cf. [9, IV.8.3 Corollary 5]). It is clear that the class of all $*$-orderable semigroups is closed under isomorphic copies and substructures. All we have to show is that the class is closed under ultraproducts.

Let S_i $(i \in I)$ be $*$-orderable semigroups and set $S := \prod_{i\in I} S_i/\mathscr{U}$ for some ultrafilter $\mathscr{U}$ on I. This ultraproduct is taken in the language of $*$-semigroups. We claim that S is $*$-orderable. Write $A_i := kS_i$. As usual, we endow A_i with the canonical involution. Define $A := \prod_{i\in I} A_i/\mathscr{U}$ (in the language of rings with involution). By Lemma 1, A is $*$-orderable. Clearly, $S \hookrightarrow A$. We have to prove more: kS embeds into A so as to extend the canonical embedding $S \hookrightarrow A$. As the possibility is only one, we denote this mapping $kS \to A$ by Φ. Assume $\Phi(\lambda_1 s_1 + \cdots + \lambda_r s_r) = 0$ for some $0 \neq \lambda_j \in k$ and $s_j \in S$. We may assume that $s_j \neq s_k$ for $j \neq k$. Write $s_k = \big[(s_i^{(k)})_{i\in I}\big]$ for $k = 1, \ldots, r$. Then the given equality implies $\big[(\lambda_1 s_i^{(1)})_{i\in I} + \cdots + (\lambda_r s_i^{(r)})_{i\in I}\big] = 0$. Hence $J := \{i \in I \mid \sum_{k=1}^r \lambda_k s_i^{(k)} = 0\} \in \mathscr{U}$. As all λ_k are nonzero, for all $i \in J$ there must be an index $\ell \in \{1, \ldots, r\}$ such that $s_1^{(i)} = s_\ell^{(i)}$. Hence

$$J \subseteq \{i \in I \mid s_1^{(i)} = s_2^{(i)}\} \cup \cdots \cup \{i \in I \mid s_1^{(i)} = s_r^{(i)}\}.$$

Since $\mathscr{U}$ is an ultrafilter, this implies $\{i \in I \mid s_1^{(i)} = s_2^{(i)}\} \cup \cdots \cup \{i \in I \mid s_1^{(i)} = s_r^{(i)}\} \in \mathscr{U}$. Thus one of these sets must be in $\mathscr{U}$, say $\{i \in I \mid s_1^{(i)} = s_2^{(i)}\} \in \mathscr{U}$. But this implies $s_1 = s_2$, a contradiction. ∎

Any axiomatization will necessarily consist of an infinite number of axioms as the class of *not* $*$-orderable semigroups is not elementary (cf. [6, Proposition 13]).

In order to prove that the class of $*$-orderable semigroups is a quasivariety, it suffices to show that this class is closed under direct products [9, V.11.1 Theorem 2]. Unable to give a direct proof, we instead proceed to show that this class is residually closed and then deduce the closedness under direct products. In the course of the proof we reduce the problem to the finitely generated case, so we start by proving that the class of $*$-orderable semigroups is locally closed.

Proposition 3. *A semigroup with involution S is $*$-orderable if and only if its every finitely generated $*$-subsemigroup is.*

Proof. It is clear that *all* $*$-subsemigroups of a $*$-orderable semigroup are $*$-orderable. In particular, this holds for finitely generated $*$-subsemigroups of S.

For the converse implication let $\mathcal{S}$ denote the set of all finitely generated $*$-subsemigroups of S. For every $r \in \mathbb{N}$ and $x_1, \dots, x_r \in S$ we define

$$I_{x_1,\dots,x_r} := \{T \in \mathcal{S} \mid x_1, \dots, x_r \in T\}.$$

Obviously, $I_{x_1,\dots,x_r} \neq \varnothing$ and $I_{x_1,\dots,x_r} \cap I_{y_1,\dots,y_s} = I_{x_1,\dots,x_r,y_1,\dots,y_s}$. This shows that the sets $I_{x_1,\dots,x_r}$ form a filter base. Let $\mathscr{U}$ denote an ultrafilter of $\mathcal{S}$ containing $I_{x_1,\dots,x_r}$ for every $r \in \mathbb{N}$ and $x_1, \dots, x_r \in S$.

We construct a mapping $\varphi : S \to \prod_{T\in\mathcal{S}} T/\mathscr{U} =: S^\star$ as follows. For $s \in S$ let s^T denote s if $s \in T$ and 1 otherwise (note that by our assumptions, $1 \in T$). Define $\varphi(s) := [s^T]_{T\in\mathcal{S}}$. It is easy to see that φ is a semigroup $*$-homomorphism. Moreover,

$$\varphi(s) = \varphi(t) \;\Leftrightarrow\; \{T \in \mathcal{S} \mid s^T = t^T\} \in \mathscr{U} \;\Leftrightarrow\; \{T \in \mathcal{S} \mid s^T \neq t^T\} \notin \mathscr{U}.$$

We have used the fact that $\mathscr{U}$ is an ultrafilter and Łoś's fundamental theorem on ultraproducts [16, Theorem 4.3]. If $s \neq t$, then $\{T \in \mathcal{S} \mid s^T \neq t^T\}$ contains $I_{s,t} \in \mathscr{U}$ which contradicts the fact that $\mathscr{U}$ is a filter. This shows that φ is injective.

All this proves that S is a $*$-subsemigroup of $S^\star$. By Theorem 2 and our assumption, $S^\star$ is $*$-orderable. Hence so is S, as desired. ∎

Again, this proposition can be proved using Marshall's extension of the classical Artin-Schreier theory to $*$-rings by observing that each of the axioms for the $*$-orderability of kS involves only *finitely* many elements from S.

Corollary 4. *A residually $*$-orderable semigroup is $*$-orderable.*

Proof. Let S be residually $*$-orderable. If S is not $*$-orderable, then by Proposition 3 a finitely generated $*$-subsemigroup T of S is not $*$-orderable although it is residually $*$-orderable. In other words, we may assume that S is finitely generated.

Since S is residually $*$-orderable and finitely generated, it is countable. Thus we may choose countably many $*$-congruences $\sim_i$ $(i \in \mathbb{N})$ such that $S/{\sim_i}$ is $*$-orderable for every $i \in \mathbb{N}$ and $\bigcap_{i\in\mathbb{N}} {\sim_i} = \operatorname{diag}(S)$. Without loss of generality, ${\sim_i} \neq \operatorname{diag}(S)$ for every i. We have a canonical homomorphism $\varphi : S \to \prod_{i\in\mathbb{N}} S/{\sim_i}$, mapping $s \mapsto \big([s]_{\sim_i}\big)_{i\in\mathbb{N}}$ for $s \in S$. It is easy to see that φ is a $*$-embedding.

Let $\mathscr{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and $S^\star := \prod_{i\in\mathbb{N}} \big(S/{\sim_i}\big)/\mathscr{U}$. Let Ψ denote the canonical homomorphism $\prod_{i\in\mathbb{N}} S/{\sim_i} \to S^\star$ and $\Phi := \Psi \circ \varphi$. We claim that Φ is a $*$-embedding. Obviously, $\Phi(x) = \Phi(y)$ if and only if $J := \{i \in \mathbb{N} \mid x \sim_i y\} \in \mathscr{U}$. Since $\mathscr{U}$ is nonprincipal, J is infinite. So $\bigcap_{i\in J} {\sim_i} = \operatorname{diag}(S)$ and thus $x = y$. As before this implies that S is $*$-orderable. ∎

Corollary 5. *The class of all $*$-orderable semigroups is closed under direct products.*

Proof. Let S_λ for $\lambda \in \Lambda$ be a set of $*$-orderable semigroups. By the preceding corollary, it suffices to prove that $\prod_{\lambda\in\Lambda} S_\lambda$ is residually $*$-orderable. For this let $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} S_\lambda$ be two different elements. There is some $\mu \in \Lambda$ with $x_\mu \neq y_\mu$. But then the canonical projection $\prod_{\lambda\in\Lambda} S_\lambda \to S_\mu$ maps $(x_\lambda)_{\lambda\in\Lambda}$ and $(y_\lambda)_{\lambda\in\Lambda}$ into different elements, finishing the proof. ■

By combining Theorem 2 and Corollary 5, we obtain:

Theorem 6. *The class of all $*$-orderable groups is a quasivariety.*

3. Nilpotent semigroups

In this section semigroups are not necessarily cancellative and need not contain 1. Let x, y be elements of a semigroup S and let $z_0, z_1, z_2, \dots$ be elements of $S^1 := S \cup \{1\}$. Put $\boldsymbol{z} = (z_0, z_1, z_2, \dots)$ and let $\boldsymbol{S^1}$ be the direct product of countably many copies of S^1. Consider the sequence of elements of S defined inductively as $q_0(x, y, \boldsymbol{z}) = x$ and

$$q_{n+1}(x, y, \boldsymbol{z}) = q_n(x, y, \boldsymbol{z}) z_n q_n(y, x, \boldsymbol{z}).$$

A semigroup S is said to be *nilpotent of class n* if it satisfies the identity $q_n(x, y, \boldsymbol{z}) = q_n(y, x, \boldsymbol{z})$ for all $x, y \in S$, $\boldsymbol{z} \in \boldsymbol{S^1}$ and n is the least positive integer with this property. S is *nilpotent* if it is nilpotent of class n for some $n \in \mathbb{N}$. The notion of a nilpotent semigroup was introduced by Neumann and Taylor [14]. They showed that a group is nilpotent of class n in the classical sense if and only if it satisfies the above identity. Additionally, it was proved that n-nilpotency of a cancellative semigroup S implies that S has a group of fractions SS^{-1} which is also nilpotent of class n.

In the cancellative case, there is an alternative way to define nilpotent semigroups. Let S be a cancellative semigroup and define a relation ζ on S by the rule

$$(a, b) \in \zeta \Leftrightarrow asb = bsa \text{ for all } s \in S^1.$$

Lemma 7. *Let S be a cancellative semigroup and let ζ be as above. Then ζ is a congruence on S and S/ζ is a cancellative semigroup.*

Proof. It is clear that ζ is reflexive and symmetric. To prove that ζ is transitive, let $(a, b) \in \zeta$, $(b, c) \in \zeta$ and $s \in S$. Since b and c commute, we obtain $b(as)c = c(asb) = cbsa = bcsa$, hence $(a, c) \in \zeta$. To prove that ζ is invariant under multiplication, let $(a, b) \in \zeta$ and $c, s \in S$. Then $a(casc)b = (bca)sca = acbsca$, hence $(ca, cb) \in \zeta$. Similarly we obtain $(ac, bc) \in \zeta$, so ζ is a congruence on S.

Now, let $a, b, c \in S$ and suppose $(ac, bc) \in \zeta$. First of all, this implies that ac and bc commute, hence $acb = bca$. Besides that, we have $acsb = bcsa$ for every $s \in S$. Replacing s by b, we obtain $acb^2 = bcba$, hence $bcab = bcba$. This implies that a and b commute. We get $ac(as)b = (bca)sa = acbsa$, hence $(a, b) \in \zeta$. Similarly we prove that $(ca, cb) \in \zeta$ implies $(a, b) \in \zeta$. This concludes the proof. ■

We therefore define a sequence of cancellative semigroups $S_0, S_1, \dots$ inductively by $S_0 = S$ and $S_{i+1} = S_i/\zeta$. Since every S_i is a homomorphic image of S, there exists a cancellative congruence ζ_i on S such that $S_i = S/\zeta_i$. By induction on i we see that for $a, b \in S$ we have $(a,b) \in \zeta_i$ if and only if $q_i(a,b,\boldsymbol{z}) = q_i(b,a,\boldsymbol{z})$ for all $\boldsymbol{z} \in \boldsymbol{S^1}$. It is now clear that S is nilpotent of class n if and only if n is the least positive integer with the property $\zeta_n = S \times S$. It is also clear that in the case when S embeds into its group of fractions G, then $(a,b) \in \zeta_n$ if and only if $ab^{-1} \in Z_n(G)$, where $Z_n(G)$ is the n-th term of the upper central series of G defined inductively by $Z_0(G) = 1$ and $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$ for $n \geqslant 0$.

We say that a semigroup S has *unique extraction of roots* if $a^n = b^n$ implies $a = b$ for all $a, b \in S$ and $n \in \mathbb{N}$. If this conclusion only holds if $ab = ba$, then S is said to have *unique extraction of roots for commuting elements.* Furthermore, we say that a semigroup S is *torsion-free* if all monogenic subsemigroups of S generated by non-identity elements are infinite. That is, if $a^n = a^m$ for $a \in S$ and $n, m \in \mathbb{N}$, then $n = m$. It is not difficult to see that a cancellative semigroup with unique extraction of roots is torsion-free. It is well known that at least in the case of nilpotent groups the converse is true as well (this can be proved by an easy induction argument). On the other hand, there exist cancellative nilpotent torsion-free semigroups that do not have unique extraction of roots, as the following example shows.

Example. Let G be the group generated by a and b, subject to the following relations:

$$(ab^{-1})^3 = [[a,b],a] = [[a,b],b] = 1.$$

Then G is nilpotent of class 2, and the relations imply that $1 = (ab^{-1})^3 = a^3b^{-3}[b^{-1},a]^3$, therefore $[b,a]^3 = a^3b^{-3}$. As $[a,b]$ is central in G, the latter relation gives that $b^3[b^3,a] = a^3$, which is equivalent to $a^3 = b^3$. Hence we conclude that $[a,b]^3 = 1$. This shows that a, b, $[b,a]$ form a polycyclic generating sequence (cf. [17, p. 394]) in G satisfying the following power relations:

$$a^3 = b^3 \quad \text{and} \quad [b,a]^3 = 1.$$

Thus every element of G can be uniquely written in the form

$$a^k b^m [b,a]^n,$$

where $k \in \mathbb{Z}$, $m, n \in \{0,1,2\}$. If $t \in \mathbb{N}$, then

$$(a^k b^m [b,a]^n)^t = a^{kt} b^{mt} [b,a]^{mk\binom{t}{2}+nt}.$$

This shows that the torsion subgroup of G is given by

$$\tau(G) = \langle ab^{-1} \rangle \times \langle [a,b] \rangle \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

Now let S be the submonoid of G generated by a and b. Then S is nilpotent of class 2, torsion-free, and G is its group of fractions. It is clear that S does not have unique extraction of roots, since $a^3 = b^3$.

A semigroup S is said to be orderable if there is a linear ordering on S which is compatible with the multiplication on S:

$$\forall a, b, c \in S : \; a < b \Rightarrow ac < bc \wedge ca < cb.$$

It is clear that every orderable semigroup is cancellative. For cancellative *nilpotent* semigroups we have the following characterizations of orderability:

Proposition 8 (cf. Proposition 9, Chapter 10 in [15]). *Let S be a cancellative nilpotent semigroup and let G be its group of fractions. The following assertions are equivalent:*

(i) *S is an orderable semigroup.*
(ii) *G is an orderable group.*
(iii) *G is torsion-free.*
(iv) *S has unique extraction of roots.*
(v) *S has unique extraction of roots for commuting elements.*
(vi) *There exists an integer i such that the group $Z_{i+1}(G)/Z_i(G)$ is torsion-free and the semigroup S has unique extraction of roots for elements belonging to the same ζ_i-classes.*

Proof. We start by proving that the statements (i)-(v) are equivalent. By a well known result of Mal'cev, (ii) and (iii) are equivalent. It is also straightforward to see that (i) and (ii) are equivalent. Let us show that (ii) implies (iv). Let $<$ be an ordering on G and suppose there exist $s, t \in S$, $s \neq t$ such that $s^n = t^n$ for some $n > 1$. We may assume that $s < t$. But then $s^n < t^n$, which is a contradiction. As (iv) clearly implies (v), we are left with the proof of the fact that (v) implies (iii). Suppose G is not torsion-free. Then $Z(G)$ contains a nontrivial element $u = st^{-1}$ $(s, t \in S)$ of finite order n. Hence $s^n = (ut)^n = u^n t^n = t^n$ and clearly $st = ts$, which leads to a contradiction. Thus G is torsion free.

Now, it is clear that (iv) implies (vi). Suppose there is an integer i such that S has unique extraction of roots for elements which are ζ_i-equivalent. We want to show that S also has unique extraction of roots for ζ_j-equivalent elements for all $j \geq i$. We prove this by induction on j. For $j = i$ this is our assumption, so we may assume $j > i$. Let $a, b \in S$ be such that $(a, b) \in \zeta_j$ and $a^n = b^n$ for some positive integer n. Then we have $ab^{-1} \in Z_j(G)$, so $a = bc$ for some $c \in Z_j(G)$. Since we have $[c, x] \in Z_{j-1}(G)$ for all $x \in G$, we obtain $a^n = (bc)^n \equiv b^n c^n \mod Z_{j-1}(G)$, hence $(ab^{-1})^n \in Z_{j-1}(G)$. But since $Z_{i+1}(G)/Z_i(G)$ is torsion-free, the same is true for $Z_j(G)/Z_{j-1}(G)$, hence $ab^{-1} \in Z_{j-1}(G)$, or equivalently, $(a, b) \in \zeta_{j-1}$. By the induction hypothesis, $a = b$. Finally, taking $j = c$, where c is the nilpotency class of S, we conclude that S has unique extraction of roots, which proves our assertion. ∎

Recall that a nilpotent group is orderable if and only if it is torsion-free. Proposition 8 and the example preceding it show that there exist cancellative torsion-free nilpotent semigroups that are not orderable.

For later use we record the following:

Lemma 9. *A free semigroup endowed with an involution is residually "cancellative nilpotent $*$-semigroup with unique extraction of roots".*

Proof. Let S be a free semigroup with an involution. Note that the involution preserves lengths of words, so is essentially given by a permutation of a free generating set. If G denotes the corresponding free group, then the involution of S extends uniquely to G. Let $a, b \in S$ be two different elements. Then $ab^{-1} \in G \setminus \{1\}$ and since G is residually torsion-free nilpotent, there is some $k \in \mathbb{N}$ such that $ab^{-1} \notin \sqrt{\gamma_k(G)}$. Here γ_k denotes the k-th term of the lower central series and $\sqrt{H} := \{g \in G \mid \exists n \in \mathbb{N} : g^n \in H\}$ is the isolator. It is easy to see that $G/\sqrt{\gamma_k(G)}$ is torsion-free nilpotent and the image of ab^{-1} in $G/\sqrt{\gamma_k(G)}$ is not identity. Moreover, as the given involution of S preserves lengths of words, $\sqrt{\gamma_k(G)}$ is $*$-invariant. Hence $G/\sqrt{\gamma_k(G)}$ can be given the natural involution and the canonical projection $G \to G/\sqrt{\gamma_k(G)}$ is a $*$-homomorphism. Composing this map with the natural embedding $S \to G$ gives the desired map. ■

4. Examples and non-examples

For this section we assume that $k = \mathbb{C}$, so we deal with semigroup algebras over $\mathbb{C}$ endowed with the complex conjugation involution. The non-dependence on the field chosen is discussed in the next section.

As mentioned in the introduction, every $*$-orderable semigroup is cancellative. It is also easy to see that every $*$-orderable semigroup is torsion-free (we call a semigroup S *torsion-free* if $a^n = 1$ implies $a = 1$ for every $a \in S$ and $n \in \mathbb{N}$). It is known that every $*$-orderable *group* is orderable. It is not known whether the same holds true for $*$-orderable semigroups. Therefore the following result is of interest.

Theorem 10. *Every $*$-orderable semigroup has unique extraction of roots.*

In order to prove this, we need to recall the connection between $*$-orderings and valuations.

Definition. If A is a domain and Γ is an ordered cancellative abelian semigroup, then an onto mapping $v : A \to \Gamma \cup \{\infty\}$ is a *valuation* if:

- (V_1) $v(x) = \infty$ if and only if $x = 0$,
- (V_2) $v(xy) = v(x) + v(y)$ for all $x, y \in A^\times$,
- (V_3) $v(x + y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in A^\times$.

Here $A^\times := A \setminus \{0\}$. The valuation v is called *quasi-commutative* if

$$v(ab - ba) > v(ab) \quad \text{for all } a, b \in A^\times.$$

In this case $v(\{a, b\}) = v(ab + ba) = v(ab)$ for all $a, b \in A$.

To each $*$-ordering $P \subseteq A$ a *natural valuation* v_P can be associated as follows. The $*$-ordering P gives an order relation $\leqslant$ on $\operatorname{Sym} A$, which induces the archimedean equivalence $\approx$ on $\operatorname{Sym} A$. We extend the latter to the whole

A by declaring, for all $0 \neq a, b \in A$, that $a \prec b$ if $aa^* \leqslant nbb^*$ for some integer n, and $a \approx b$ if $a \prec b$ and $b \prec a$. Denote by $v_P(a)$ the equivalence class of $0 \neq a \in A$ and $v_P(0) := \infty$. Then the relation $\prec$ induces an ordering of the set $\Gamma_P = v_P(R \setminus \{0\})$. By [10, Theorem 3.3], the binary operation $v_P(a)+v_P(b) := v_P(ab)$ is well-defined on Γ_P, so Γ_P becomes an ordered abelian semigroup. Marshall [10, Theorem 3.3] proved that v_P is a valuation that is $*$-invariant, i.e., $v_P(a^*) = v_P(a)$ for every $a \in A$. Furthermore, if A contains a central skew element i satisfying $i^2 = -1$, then by a result of the first author [5, Theorem 2], v_P is quasi-commutative. For more on classical valuation theory we refer the reader to [4].

Proof of Theorem 10. Let S be a $*$-orderable semigroup and let $s, t \in S$ satisfy $s^n = t^n$ for some $n \geqslant 2$. Take a $*$-ordering of $\mathbb{C}S$ and let v denote the natural valuation associated to it. By the above, v is quasi-commutative. We make use of the following identity:

$$2a^{k+1} - 2b^{k+1} = \{a^k - b^k, a + b\} + (b^k a - ab^k) + (ba^k - a^k b), \qquad (\star)$$

which holds for all $a, b \in \mathbb{C}S$ and $k \in \mathbb{N}$.

As $s^n = t^n$, $nv(s) = v(s^n) = v(t^n) = nv(t)$ and so $v(s) = v(t)$. Here we distinguish two cases. If $n = 2$, then from $s^2 = t^2$ it follows that s and t^2 commute. Thus

$$0 = st^2 - t^2 s = (st - ts)t + tst - t^2 s = (st - ts)t + t(st - ts)$$

and hence

$$\infty = v(0) = v(st^2 - t^2 s) = v\big((st - ts)t + t(st - ts)\big) = v(st - ts) + v(t).$$

This gives $st = ts$ and $0 = s^2 - t^2 = (s - t)(s + t)$, so $s = t$.

Now let $n \geqslant 3$. By replacing t with $\xi_n t$ or $\xi_n^2 t$ (where ξ_n is a primitive n-th root of unity), we may assume that $v(s) = v(t) = v(s - t) = v(s + t)$. By induction on k we prove that $v(s^k - t^k) = kv(t)$. This obviously holds for $k = 1$. Assume that it holds up to k and use $(\star)$ for $a = s$ and $b = t$. Since $v(t^k s - st^k) > (k + 1)v(t)$ and $v(ts^k - s^k t) > (k + 1)v(t)$, we obtain $v(s^{k+1} - t^{k+1}) = v\big(\{s^k - t^k, s + t\}\big) = v(s^k - t^k) + v(t) = (t + 1)v(k)$, as desired. But now $\infty = v(s^n - t^n) = nv(t)$, hence $t = 0$ and also $s = 0$ finishing the proof. ∎

To give examples of $*$-orderable semigroups, we use results from Section 3.

Example. A cancellative nilpotent semigroup S with involution σ is $*$-orderable if and only if it has unique extraction of roots (for commuting elements). By Proposition 8 this is the case if and only if S is orderable.

Proof. Assume S has unique extraction of roots for commuting elements. Since S is nilpotent, it has a group of right fractions G. If the involution σ extends to G, then $\sigma(st^{-1}) = \sigma(t)^{-1}\sigma(s)$. So we must check that this is well defined. Suppose $st^{-1} = uv^{-1}$. Then there exist $a, b \in S$ satisfying $sa = ub$ and $ta = vb$. Applying σ we obtain $\sigma(a)\sigma(s) = \sigma(b)\sigma(u)$ and $\sigma(a)\sigma(t) = \sigma(b)\sigma(v)$. Since G is also a group of left fractions of S, we have $\sigma(t)^{-1}\sigma(s) = \sigma^{-1}(v)\sigma(u)$. Thus σ is an involution of G. By Proposition 8, G is a torsion-free nilpotent group and for such groups it is shown in [2] that the corresponding group algebra $\mathbb{C}G$ admits $*$-orderings independent of the involution chosen. This proves one directions, and the reverse implication follows from Theorem 10. ∎

From Proposition 3 it follows that a cancellative locally nilpotent $*$-semigroup with unique extraction of roots is $*$-orderable. Similarly, Corollary 4 shows that every residually "cancellative nilpotent $*$-semigroup with unique extraction of roots" is $*$-orderable. In particular, by Lemma 9 this applies to free semigroups endowed with an involution and gives a new proof of the fact that free algebras over $\mathbb{C}$ admit $*$-orderings as observed first in [3].

In general, a cancellative $*$-semigroup with unique extraction of roots need not be $*$-orderable. This is shown even for the group case in [2, 7], where examples of metabelian orderable groups that are not $*$-orderable are given.

5. (Non-)Dependence on the base field

We defined a semigroup S to be $*$-orderable if the semigroup algebra kS allowed a $*$-ordering for a *fixed* $*$-orderable field k. In fact, this notion is under mild additional assumptions *independent* of the field k chosen and depends only on the *type of involution* on k. Recall that an involution on a field is of the 2^{nd}-*kind* if it is nontrivial and of the 1^{st}-*kind* otherwise.

Theorem 11. *Let k be a formally real field with an involution of the 1^{st}-kind. Then for a semigroup S the following are equivalent:*

(i) *kS admits a $*$-ordering.*
(ii) *$\mathbb{R}S$ admits a $*$-ordering.*

Proof. (i) $\Rightarrow$ (ii): Fix a $*$-ordering of kS. Since the involution on k is of the 1^{st}-kind, the $*$-ordering on kS induces a total ordering of k.

Pick a nontrivial ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and form the ultrapower

$$R_1 := (kS)^{\mathbb{N}}/\mathscr{U} \supseteq (k^{\mathbb{N}}/\mathscr{U})S.$$

By definition, R_1 comes equipped with a $*$-ordering. Let v_1 denote the natural $*$-valuation of R_1. Clearly, $\mathbb{Q}^{\mathbb{N}}/\mathscr{U} \subseteq k^{\mathbb{N}}/\mathscr{U}$. Thus the residue division ring k_{v_1} contains $\mathbb{R}$ [12, Theorem II]. (This is even true for the restriction of v_1 to $k^{\mathbb{N}}/\mathscr{U}$.) k_{v_1} admits an archimedean $*$-ordering, so is either $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ [5, Proposition 3].

We continue by forming the completion $(R_2, v_2) := (\widetilde{R_1}, \widetilde{v_2})$. By the Krull-Baer theorem [5, Theorem 7] (or a lengthy straightforward computation), the $*$-ordering extends from R_1 to R_2 (essentially via density) and v_2 is the corresponding natural valuation. Clearly, $(\widetilde{k^{\mathbb{N}}/\mathscr{U}})S \subseteq R_2$. By construction, the residue division ring of v_2 is k_{v_1}. Moreover, $(\widetilde{k^{\mathbb{N}}/\mathscr{U}}, v_2)$ is immediate over $(k^{\mathbb{N}}/\mathscr{U}, v_1)$, so the corresponding residue fields coincide.

We claim that the real algebraic numbers $\mathbb{Q}_{\text{alg}}$ are contained in $\widetilde{k^{\mathbb{N}}/\mathscr{U}}$. Choose an arbitrary real algebraic number r and let $p \in \mathbb{Q}[X]$ be its minimal polynomial. Then $\bar{p} \in k_{v_2}[X]$ has a root $r = \bar{x} \in k_{v_2}$ for some $x \in \widetilde{k^{\mathbb{N}}/\mathscr{U}}$. Let us form the field $\mathbb{Q}(x) \subseteq \widetilde{k^{\mathbb{N}}/\mathscr{U}}$ and denote $u := v_2|_{\mathbb{Q}(x)}$. By the dimension inequality, u is of rank 1. Moreover, the completion $(\widetilde{\mathbb{Q}(x)}, \widetilde{u})$ of $(\mathbb{Q}(x), u)$ is contained in $(\widetilde{k^{\mathbb{N}}/\mathscr{U}}, v_2)$. $\widetilde{u}$ is complete of rank 1, hence henselian by Hensel's lemma [4, Theorem 1.3.1]. So p has a root $\rho \in \widetilde{\mathbb{Q}(x)} \subseteq \widetilde{k^{\mathbb{N}}/\mathscr{U}}$ satisfying $\bar{\rho} = r$. This shows that $\mathbb{Q}_{\text{alg}} \subseteq \widetilde{k^{\mathbb{N}}/\mathscr{U}}$.

The field $\mathbb{Q}_{\text{alg}}$ is real closed and hence elementarily equivalent to $\mathbb{R}$ by Tarski's transfer principle [16, Section 6.6]. Hence by Frayne's lemma [16, Lemma 4.12] there exists a nonempty set I, an ultrafilter $\mathscr{W}$ on I and an embedding $\mathbb{R} \hookrightarrow \mathbb{Q}_{\text{alg}}^I/\mathscr{W}$. So $\mathbb{R}$ can be embedded in the ultrapower $R_2^I/\mathscr{W}$. This shows that $\mathbb{R}S$ embeds into $(R_2^I/\mathscr{W})S$ and thus admits a $*$-ordering.

(ii) $\Rightarrow$ (i): If a semigroup algebra kS admits a $*$-ordering and K is a $*$-subfield of k, then KS also admits a $*$-ordering. Hence we may assume k is real closed.

Suppose $\mathbb{R}G$ admits a $*$-ordering. We can embed k into an ultrapower of $\mathbb{R}$ by Frayne's lemma, and this in turn yields an embedding of kS into an ultrapower of the semigroup algebra $\mathbb{R}S$. Hence kS embeds into a $*$-orderable domain, so is $*$-orderable itself. ■

As a corollary (of the proof) we obtain:

Corollary 12. *Let K be a formally real field with the trivial involution and let $k = K(\sqrt{-1})$ be endowed with the involution $\sqrt{-1}^* = -\sqrt{-1}$. Then for a semigroup S the following are equivalent:*

(i) *kS admits a $*$-ordering.*
(ii) *$\mathbb{C}S$ admits a $*$-ordering.*

We do not know whether $*$-orderability of $\mathbb{C}S$ is equivalent to that of $\mathbb{R}S$.

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