ON TAMURA'S IDENTITY $yx = f(x, y)$ IN GROUPS

PRIMOŽ MORAVEC

Abstract. The purpose of this paper is to study the groups satisfying the law $yx = x^{a_1}y^{b_1}x^{a_2}y^{b_2}$, where a_i and b_i are positive integers. Tamura posed a problem as to when such a law implies that a group is abelian. We show that, apart from obvious reasons why such a variety should contain non-abelian groups, every elementary amenable or residually finite group satisfying such a law has to be abelian.

1. Introduction

Tully showed in his unpublished work [8] that if a semigroup S satisfies an identity $yx = x^m y^n$, where m and n are fixed positive integers, then S is commutative. This led Tamura [7] to study semigroup identities of the form $yx = f(x, y)$ where $f(x, y)$ is a fixed semigroup word of the form $f(x, y) = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_s}y^{a_s}$, where a_i and b_i are positive integers and $s \geq 1$. Tamura showed that the law $yx = f(x, y)$ implies commutativity in semigroups if and only it implies commutativity for groups. This rises a natural question:

Question 1.1 ([7], Problem 1, item 4)**.** *Which laws $yx = f(x, y)$ of the above form imply commutativity in groups?*

Apart from the above mentioned Tully's result, it appears that only Stein [6] recently dealt with this question and partially solved this for $s = 2$, and some other special types of laws. Generalizations of Tamura's identity in semigroups have been considered by Putcha and Weissglass [3, 4], and Kowol [1].

The purpose of this paper is address Tamura's problem for the variety $\mathcal{T}_{a_1,b_1,a_2,b_2}$ of groups satisfying the law

$$yx = x^{a_1}y^{b_1}x^{a_2}y^{b_2}. \tag{1.1.1}$$

Our approach will be different from the one taken by Stein [6], who mainly considered the cases when some of the parameters a_1, b_1, a_2, b_2 are equal to 1. We prove the following result:

Theorem 1.2. *Denote*

$$\begin{aligned} d &= \gcd\{a_1 + a_2 - 1, b_1 + b_2 - 1\}, \\ g &= \gcd\{a_1, b_2, d\}, \\ c &= a_2 b_1 - 1, \\ f &= \gcd\left\{d, \binom{d}{2}, c\right\}. \end{aligned}$$

Then the Tamura variety $\mathcal{T}_{a_1,b_1,a_2,b_2}$ contains non-abelian groups in the following cases:

(a) *When there exists a positive integer $e > 2$ such that $a_1 \equiv 0 \mod e$, $b_1 \equiv 1 \mod e$, $a_2 \equiv 1 \mod e$ and $b_2 \equiv 0 \mod e$.*

Date: August 27, 2018.

The author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0222, and projects No. J1-8132, J1-7256 and N1-0061).

(b) *When at least one of a_2 and b_1 is equal to 1, and $g \neq 1$.*
(c) *$a_2 \neq 1$, $b_1 \neq 1$, and $f \neq 1$.*

When we exclude the above cases and assume $g \neq 1$ and $f = 1$, then

(d) *Every elementary amenable or residually finite group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian.*
(e) *If $g \in \{2,3,4,6\}$ and d/g is a prime, then every group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian.*

The proof and more detailed information about the above cases are given in Section 2. Close inspection of the proof reveals that Theorem 1.2 roughly says that there are the following possible scenarios: either all groups in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ are abelian for trivial reasons, or there are trivial reasons why this variety contains non-abelian groups, or every reasonably nice group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian.

2. The law $yx = x^{a_1}y^{b_1}x^{a_2}y^{b_2}$

We first introduce some notations. If e is a positive integer, then $\mathcal{B}_e$ is the variety of groups of exponent dividing e, that is, groups satisfying the law $x^e = 1$. By $\mathcal{CB}_e$ we denote the variety determined by the law $[x^e, y] = 1$. The variety of locally finite groups satisfying the identity $x^e = 1$ will be denoted by $\mathcal{R}_e$; this is a variety by the positive solution of the Restricted Burnside Problem [9].

Let $x_1, x_2, \ldots, x_n$ be elements of a group G. We write $x_1^{x_2} = x_2^{-1}x_1x_2$. The commutator of x_1 and x_2 is defined by $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, and we inductively define $[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$ for $n \geq 3$. For other unexplained notions we refer to [5].

We begin by exhibiting a case when $\mathcal{T}_{a_1,b_1,a_2,b_2}$ contains non-abelian groups for trivial reasons.

Proposition 2.1. *Suppose there exists a positive integer $e > 2$ such that $a_1 \equiv 0 \mod e$, $b_1 \equiv 1 \mod e$, $a_2 \equiv 1 \mod e$ and $b_2 \equiv 0 \mod e$. Then the variety $\mathcal{T}_{a_1,b_1,a_2,b_2}$ contains $\mathcal{B}_e$. In particular, if $e > 2$, then $\mathcal{T}_{a_1,b_1,a_2,b_2}$ contains non-abelian groups.*

The proof is straightforward and thus omitted. In the rest of this section we assume that the parameters a_1, b_1, a_2, b_2 do not satisfy the condition stated in Proposition 2.1 for any $e > 2$.

Lemma 2.2. *Let $G \in \mathcal{T}_{a_1,b_1,a_2,b_2}$ and $x, y \in G$.*
(a) $x^{a_1+a_2-1} = x^{b_1+b_2-1} = 1$,
(b) $[y, x] = [y^{a_2}, x^{b_1}]$,
(c) $[y, x, x^{b_2}] = [y, x, x^{a_1}] = 1$,
(d) *$[y, x]^{a_1}$ and $[y, x]^{b_2}$ belong to $Z(\langle x, y\rangle)$.*

Proof. Let G satisfy the identity (1.1.1). If we plug in $x = 1$ or $y = 1$, we get (a). Now (1.1.1) can be rewritten as $yx = x^{1-a_2}y^{b_1}x^{a_2}y^{1-b_1}$ which is equivalent to $[x, y^{-1}] = [x^{a_2}, y^{-b_1}]$. From here we easily get (b). The equation $[x, y^{-1}] = [x^{a_2}, y^{-b_1}]$ can also be written as $[y, x]^{y^{-1}} = [y^{b_1}, x^{a_2}]^{y^{-b_1}}$, which, together with (b), implies $[y, x] = [y, x]^{y^{1-b_1}} = [y, x]^{y^{b_2}}$. Similarly, $[y, x] = [y, x]^{x^{a_1}}$, hence we obtain (c).

By (a), (b) and (c) we have that $[y, x, x] = [[y, x]^{a_2}, x^{b_1}] = [[y, x]^{a_2}, x \cdot x^{-b_2}] = [[y, x]^{a_2}, x^{-b_2}][[y, x]^{a_2}, x]^{x^{-b_2}} = [[y, x]^{a_2}, x]^{x^{-b_2}} = [([y, x]^{x^{-b_2}})^{a_2}, x^{x^{-b_2}}] = [[y, x]^{a_2}, x]$. This implies that $[y, x]^{-1}[y, x]^{a_2} = [y, x]^{a_1}$ commutes with x, and hence also with y. Similarly we prove the remaining part of (d). □

Remark 2.3. It is straightforward to see that every group satisfying the laws $x^{a_1+a_2-1} = x^{b_1+b_2-1} = 1$ and $[y, x] = [y^{a_2}, x^{b_1}]$ belongs to $\mathcal{T}_{a_1,b_1,a_2,b_2}$.

Denote
$$d = \gcd\{a_1 + a_2 - 1, b_1 + b_2 - 1\}.$$
Note that $\mathcal{T}_{a_1,b_1,a_2,b_2} \subseteq \mathcal{B}_d$. Let $G \in \mathcal{T}_{a_1,b_1,a_2,b_2}$. If $d = 1$, then G is a trivial group by Lemma 2.2 (a). If $d = 2$, then G has exponent dividing 2, and is thus abelian. Thus we assume from here on that $d > 2$. We also denote
$$g = \gcd\{a_1, b_2, d\}.$$
Then Lemma 2.2 gives
$$[x, y]^g \in Z(\langle x, y\rangle) \tag{2.3.1}$$
and
$$[x, y, x^g] = [x, y, y^g] = 1 \tag{2.3.2}$$
for all $x, y \in G$, where $G \in \mathcal{T}_{a_1,b_1,a_2,b_2}$. Furthermore, we have the following:

Lemma 2.4. *Every $G \in \mathcal{T}_{a_1,b_1,a_2,b_2}$ satisfies the identity $[y^g, x^{d/g}] = 1$.*

Proof. Write $d = g\delta$. Then the Tamura identity implies $yx^\delta = y^{b_1}x^{a_2\delta}y^{b_2}$. This can be rewritten as $y^{b_2}x^\delta = x^{(1-a_1)\delta}y^{b_2} = x^\delta y^{b_2}$. Therefore G satisfies the identity $[y^{b_2}, x^\delta] = 1$. Similarly, G also satisfies $[y^{a_1}, x^\delta] = 1$. This gives the result. □

In some cases this can be strengthened as follows:

Proposition 2.5. *If $a_2 = 1$ or $b_1 = 1$, then $\mathcal{T}_{a_1,b_1,a_2,b_2} = \mathcal{B}_d \cap \mathcal{CB}_g$.*

Proof. We only prove the case when $a_2 = 1$, the other one is similar. If $G \in \mathcal{T}_{a_1,b_1,1,b_2}$, then Lemma 2.2 implies that G also satisfies the laws $x^d = x^{a_1} = 1$ and $yx = y^{b_1}xy^{b_2}$. The latter implies the law $y^{b_2}x = xy^{b_2}$, thus $G \in \mathcal{B}_d \cap \mathcal{CB}_g$.

Conversely, let $G \in \mathcal{B}_d \cap \mathcal{CB}_g$. Then $x^{a_1}y^{b_1}xy^{b_2} = y^{b_1+b_2}x^{a_1+1} = yx$, hence $G \in \mathcal{T}_{a_1,b_1,1,b_2}$. □

From now on suppose that $a_2 \neq 1$ and $b_1 \neq 1$. We separate two main cases:

Case 1. Assume $g = 1$. Then Lemma 2.2 (d) implies that every two-generator subgroup of G is nilpotent of class ≤ 2 and thus, in particular, G is nilpotent of class ≤ 3 [5, p. 373, 12.3.6]. Take $x, y \in G$, and denote
$$c = a_2b_1 - 1.$$
Then Lemma 2.2 (a) gives that $G \in \mathcal{B}_d$, whereas (b) yields $[y, x]^c = 1$. Conversely, note that the laws $[x, y, y] = [x, y]^c = x^d = 1$ imply the Tamura law (1.1.1). For,
$$\begin{aligned}
x^{a_1}y^{b_1}x^{a_2}y^{b_2} &= x \cdot x^{-a_2}y^{b_1}x^{a_2}y \cdot y^{-b_1}\\
&= xy[y, x^{-a_2}y^{b_1}x^{a_2}][x^{a_2}, y^{-b_1}]\\
&= xy[x, y^{-1}]^{a_2b_1}\\
&= xy[y, x]^{a_2b_1}\\
&= xy[y, x]\\
&= yx.
\end{aligned}$$
Now, the expansion $1 = (xy)^d = x^dy^d[y, x]^{\binom{d}{2}}$, see [5, p. 141, 5.3.5], implies $[y, x]^{\binom{d}{2}} = 1$. Thus the order of $[y, x]$ divides $\gcd\{d, \binom{d}{2}, c\} = \gcd\{\tilde{d}, c\}$, where
$$\tilde{d} = \begin{cases} d & : \ d \text{ odd} \\ d/2 & : \ d \text{ even} \end{cases}.$$
It is straightforward to check that
$$G_{c,d} = \mathrm{pc}\langle g_1, g_2, g_3 \mid g_1^d = g_2^d = g_3^{\gcd\{\tilde{d},c\}} = 1, [g_2, g_1] = g_3\rangle$$

provides a consistent polycyclic presentation of the group $G_{c,d}$; the notation pc means that other commutator relations between the generators are trivial. Note that $G_{c,d}$ satisfies the laws $[x,y,y]=[x,y]^c=x^d=1$, therefore $G_{c,d}\in\mathcal{T}_{a_1,b_1,a_2,b_2}$. We conclude that, in this case, all groups satisfying (1.1.1) are abelian if and only if $\gcd\{\tilde{d},c\}=1$.

Case 2. From here on we assume that

$$g\neq 1. \tag{2.5.1}$$

If $\gcd\{\tilde{d},c\}\neq 1$, then the above argument shows that there exist groups in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ which are nilpotent of class precisely 2, hence non-abelian. Thus we may additionally assume that

$$\gcd\{\tilde{d},c\}=1. \tag{2.5.2}$$

In this case, every group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ that is nilpotent of class ≤ 2 is actually abelian. It is easy to see that this implies that every residually nilpotent group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian.

Proposition 2.6. *Suppose (2.5.1) and (2.5.2) hold. Then every finite group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian.*

Proof. Let G be a counterexample of smallest possible order. Then all proper sections of G are abelian; in particular, every proper subgroup of G is abelian. By the above, G is not nilpotent and thus $Z(G)$ is trivial. It now follows from a result of Miller and Moreno [2] that $G=\langle t\rangle\ltimes Q$, where t is an element of order p^n, and $Q=G'=\langle x_1,x_2,\ldots,x_m\rangle$ is an elementary abelian q-group of order q^m; here p and q are different primes. The action of t on Q can be chosen in the following way: $x_i^t=x_{i+1}$ for $i=1,2,\ldots,m_1$, and $x_m^t=x_1^{-c_0}x_2^{-c_1}\cdots x_m^{-c_{m-1}}$, where $p(u)=c_0+c_1u+\ldots+c_{m-1}u^{m-1}+u^m$ is an irreducible divisor of $(u^p-1)/(u-1)$ in $\mathbb{F}_q[u]$. We have that $\exp G=p^nq$. This implies that p^nq divides d. As G is not nilpotent, we conclude that q divides $\gcd\{a_1,b_2\}$. Therefore we may write $a_1=q\alpha_1$, $b_1=q\beta_1+1$, $a_2=q\alpha_2+1$, $b_2=q\beta_2$ for some positive integers $\alpha_1,\beta_1,\alpha_2,\beta_2$. We get

$$tx_i=x_i^{q\alpha_1}t^{q\beta_1+1}x_i^{q\alpha_2+1}t^{q\beta_2}=t^{-q\beta_2}\cdot tx_i\cdot t^{q\beta_2},$$

therefore $t^{q\beta_2}\in Z(G)=1$. This shows that p^n divides β_2 and, as p^n divides $\beta_1+\beta_2$, it follows that p^n divides β_1 as well. Similarly, both α_1 and α_2 are divisible by p^n. This shows that $a_1\equiv 0 \mod p^nq$, $b_1\equiv 1 \mod p^nq$, $a_2\equiv 1 \mod p^nq$ and $b_2\equiv 0 \mod p^nq$. As $p^nq>2$, this contradicts the assumption made at the beginning of this section. □

Corollary 2.7. *Suppose (2.5.1) and (2.5.2) hold. Then every residually finite or locally finite group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian.*

In particular, the above corollary shows that if (2.5.1) and (2.5.2) hold, and if $\mathfrak{X}$ is any class of groups with the property that $\mathfrak{X}\cap\mathcal{B}_d\subseteq\mathcal{R}_d$, then every group in $\mathcal{T}_{a_1,b_1,a_2,b_2}\cap\mathfrak{X}$ is abelian. Thus, if $d\in\{2,3,4,6\}$, then every group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian, see [9]. An example of such a variety is $\mathcal{T}_{2,3,3,4}$.

We also have that, under the above conditions, every solvable group in $\mathcal{T}_{a_1,b_1,a_2,b_2}$ is abelian. From here it is easy to see that the same holds for the elementary amenable groups.

Proposition 2.8. *Let $G\in\mathcal{T}_{a_1,b_1,a_2,b_2}$ and suppose that (2.5.1) and (2.5.2) hold. Assume that $g\in\{2,3,4,6\}$ and that d/g is a prime. Then G is abelian.*

Proof. Without loss of generality we may assume that G is a two-generator group. Denote $\delta=d/g$, and let $H=G/G^{\delta}$. We claim that H is nilpotent of class ≤ 2. We may assume that $\delta > 2$. For a non-negative integer k, define $\bar{k}$ to be k reduced mod δ. Then $H \in \mathcal{T}(\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2)$, and $\bar{a}_1+\bar{a}_2-1, \bar{b}_1+\bar{b}_2-1 \in \{0, \delta\}$. If one of these values is equal to δ, and if one of $\bar{a}_1, \bar{b}_2$ is non-zero, then $\gcd\{\bar{a}_1+\bar{a}_2-1, \bar{b}_1+\bar{b}_2-1, \bar{a}_1, \bar{b}_2\} = 1$, and thus H is nilpotent of class ≤ 2 by Case 1. The remaining case, $\bar{a}_1 = \bar{b}_2 = 0$, $\bar{a}_2 = \bar{b}_1 = 1$ shows that $G \in \mathcal{T}_{\alpha_1\delta, \beta_1\delta+1, \alpha_2\delta+1, \beta_2\delta}$, but this case was excluded from the consideration. This proves our statement, that is, $\gamma_3(G) \leq G^\delta$.

By [9], G/G^g is solvable, thus there exists a positive integer i such that $G^{(i)} \leq G^g$. We may assume that $i \geq 2$. Then $G^{(i+1)} = [G^{(i)}, G^{(i)}] \leq [G^g, G^\delta] = 1$. Thus G is solvable and therefore abelian. □

References

[1] G. Kowol, *Conditions for the commutativity of semigroups*, Proc. Amer. Math. Soc. 56 (1976), 85–88.

[2] G. A. Miller, and H. C. Moreno, *Non-abelian groups in which every subgroup is abelian*, Trans. Amer. Math. Soc. **4** (1903), no. 4, 398–404.

[3] M. S. Putcha, and J. Weissglass, *Semigroups satisfying variable identities*, Semigroup Forum **3** (1971/1972), no. 1, 64–67.

[4] M. S. Putcha, and J. Weissglass, *Semigroups satisfying variable identities. II*, Trans. Amer. Math. Soc. 168 (1972), 113–119.

[5] D. J. S. Robinson, *A course in the theory of groups. Second edition.* Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1996.

[6] S. Stein, *Semigroup identities and proofs*, Algebra Universalis **71** (2014), no. 4, 359–373.

[7] T. Tamura, *Semigroups satisfying identity $xy = f(x, y)$*, Pacific J. Math. **31** (1969) 513–521.

[8] E. J. Tully, *Semigroups satisfying an identity of the form $xy = y^m x^n$*, unpublished manuscript.

[9] M. Vaughan-Lee, *The restricted Burnside problem. Second edition.* The Clarendon Press, Oxford University Press, New York, 1993.

Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Slovenia

E-mail address: `primoz.moravec@fmf.uni-lj.si`