THE NON-ABELIAN TENSOR PRODUCT OF POLYCYCLIC GROUPS IS POLYCYCLIC

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1. Introduction

Let M and N be groups acting on each other on the left. The action of M on N is written ${}^m n$, and the action of N on M is written ${}^n m$, where $m \in M$, $n \in N$. The groups M and N are assumed to act upon themselves by conjugation ${}^y x = yxy^{-1}$. These actions are said to be *compatible* if

$$
\begin{aligned}
{}^{{}^m n}m' &= {}^m({}^n({}^{m^{-1}}m')),\\
{}^{{}^n m}n' &= {}^n({}^m({}^{n^{-1}}n')),
\end{aligned}
$$

for $m, m' \in M$ and $n, n' \in N$. The *non-abelian tensor product* $M \otimes N$ is the group generated by the symbols $m \otimes n$ with defining relations

$$
\begin{aligned}
mm' \otimes n &= ({}^m m' \otimes {}^m n)(m \otimes n),\\
m \otimes nn' &= (m \otimes n)({}^n m \otimes {}^n n'),
\end{aligned}
$$

where $m, m' \in M$ and $n, n' \in N$. When $M = N$ and all actions are conjugations, the group $M \otimes M$ is called the *non-abelian tensor square* of M. Note also that whenever the groups M and N act trivially on each other, then their tensor product $M \otimes N$ is isomorphic to the usual tensor product $M^{\mathrm{ab}} \otimes N^{\mathrm{ab}}$ of the abelianisations. The concept of the non-abelian tensor product of groups was introduced by Brown and Loday in [3], following the ideas of Dennis [5]. This construction has its origins in algebraic K-theory as well as in homotopy theory, and it has become interesting from a purely group-theoretical point of view since the paper of Brown, Johnson and Robertson [4].

Non-abelian tensor products of groups subject to various finiteness conditions have been studied by several authors. Ellis [6] proved that if M and N are finite, then $M \otimes N$ is also finite. Nakaoka [9] showed that if the group $[M, N] = \langle m\, {}^n m^{-1} : m \in M, n \in N \rangle$ is solvable, then so is $M \otimes N$. In [2], Blyth, Morse and Redden proved that tensor squares of polycyclic groups are also polycyclic. The purpose of this note is to extend this result to arbitrary non-abelian tensor products of groups. Our main result goes as follows.

Theorem. *Let M and N be polycyclic groups acting on each other in a compatible way. Then the group $M \otimes N$ is also polycyclic.*

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This theorem is proved in Section 2. In Section 3 we obtain a generating set for $M \otimes N$ in the case when M and N are normal polycyclic subgroups of a common overgroup, in terms of polycyclic generating sequences of M and N.

2. Proof of the main theorem

Let G and H be groups. A *crossed module* is a group homomorphism $\mu : G \to H$ together with an action of H on G satisfying $\mu({}^h g) = h\mu(g)h^{-1}$ and ${}^{\mu(g)}g' = gg'g^{-1}$ for all $g, g' \in G$ and $h \in H$. An immediate consequence of the above definition is that if $\mu : G \to H$ is a crossed module, then $\operatorname{im}\mu$ is a normal subgroup of H and $\ker\mu$ is central in G.

Now we proceed to the proof of Theorem. Let M and N be polycyclic groups acting compatibly on each other. Let G be the Peiffer product [10] of M and N. To be more precise, let $G = M * N/IJ$, where I an J are the normal closures in $M * N$ of $\langle {}^n mnm^{-1}n^{-1} : m \in M, n \in N\rangle$ and $\langle {}^m nmn^{-1}m^{-1} : m \in M, n \in N\rangle$, respectively. Note that G is an image of the semidirect product $M \rtimes N$, hence G is polycyclic. Let $\mu : M \to G$ and $\nu : N \to G$ be the natural maps and denote $\bar{M} = \mu(M)$ and $\bar{N} = \nu(N)$. Then $\bar{M}$ and $\bar{N}$ are normal subgroups of G and $G = \bar{M}\bar{N}$. As $\mu : M \to G$ and $\nu : N \to G$ are crossed modules, it follows that $\ker\mu$ is central in M and $\ker\nu$ is central in N. A similar argument as in the proof of [4, Proposition 9] shows that we have an exact sequence

$$(M \otimes \ker\nu) \times (\ker\mu \otimes N) \xrightarrow{\iota} M \otimes N \longrightarrow \bar{M} \otimes \bar{N} \longrightarrow 1,$$

where ι is induced by $(m \otimes n', m' \otimes n) \mapsto (m \otimes n')(m' \otimes n)$. Furthermore, it can be seen that $\operatorname{im}\iota$ is a central subgroup of $M \otimes N$, and that $\ker\mu$ and $\ker\nu$ act trivially on N and M, respectively. We thus have [6] that $M \otimes \ker\nu \cong I(M) \otimes_{\mathbb{Z}M} \ker\nu$ and $ker\mu \otimes N \cong \ker\mu \otimes_{\mathbb{Z}N} I(N)$, where $I(M)$ and $I(N)$ are the augmentation ideals in $\mathbb{Z}M$ and $\mathbb{Z}N$, respectively. Consider the homomorphism $\kappa : I(M) \otimes_{\mathbb{Z}M} \ker\nu \to \ker\nu$ induced by $(m-1) \otimes a \mapsto {}^m aa^{-1}$, where $m \in M$, $a \in \ker\nu$. Note that $\ker\kappa \cong H_1(M, \ker\nu)$ [8], hence $\ker\kappa$ is finitely generated by a result of Baumslag, Cannonito and Miller [1]. We thus have that the group $I(M) \otimes_{\mathbb{Z}M} \ker\nu$ is finitely generated, and the same conclusion holds true for $\ker\mu \otimes_{\mathbb{Z}N} I(N)$. It follows from here that the group $(M \otimes \ker\nu) \times (\ker\mu \otimes N)$ is finitely generated, whence $\operatorname{im}\iota$ is also finitely generated. As $\bar{M} \otimes \bar{N} \cong (M \otimes N)/\operatorname{im}\iota$, it suffices to show that $\bar{M} \otimes \bar{N}$ is polycyclic. Thus from now on we may assume that M and N are normal subgroups of G and $G = MN$. Define

$$M \wedge N = (M \otimes N)/D,$$

where $D = \langle x \otimes x : x \in M \cap N\rangle$, and let K be the kernel of the commutator map $M \wedge N \to [M, N]$. In order to finish the proof of Theorem it suffices to show that K and D are finitely generated. By [3, Theorem 4.5] we have an exact sequence

$$\longrightarrow H_3(G/M) \oplus H_3(G/N) \longrightarrow K \longrightarrow H_2(G) \longrightarrow .$$

Since G, G/M and G/N are polycyclic, $H_2(G)$, $H_3(G/M)$ and $H_3(G/N)$ are finitely generated by [1]. From here we conclude that K is polycyclic. As for the group D, consider the map $\phi : (M \cap N) \times (M \cap N) \to M \otimes N$ defined by $\phi(g, h) = (g \otimes h)(h \otimes g)$. It is straightforward to verify that ϕ is a bilinear map.

Thus, if $(M \cap N)/[M,N] = \langle x_1[M,N], \dots, x_r[M,N]\rangle$, then D is generated by the set $\{(x_i \otimes x_j)(x_j \otimes x_i), x_i \otimes x_i : i,j = 1,\dots,r\}$. This conludes the proof.

3. Generating sets

Let M and N be polycyclic groups acting compatibly on each other. For computational reasons it would be convenient to obtain a generating set for $M \otimes N$ in terms of polycyclic generating sequences of M and N. For the non-abelian tensor squares of polycyclic groups this has been done in [2]. Here we use a similar approach, following Ellis and Leonard [7]. Let J denote the normal subgroup of $M * N$ normally generated by the elements $x[m,n]x^{-1}[\bar{n},\bar{m}]$ for $m \in M$, $n \in N$, $x \in M \cup N$, where $\bar{m} = xmx^{-1}$ and $\bar{n} = xnx^{-1}$. Then there is an isomorphism [7]

$$((M \otimes N) \rtimes N) \rtimes M \cong (M * N)/J.$$

This isomorphism restricts to an isomorphism $M \otimes N \cong [\bar{M}, \bar{N}]$, where $\bar{M}$ and $\bar{N}$ are the normal closures in $(M*N)/J$ of M and N. Thus the algorithm is the following. First note that $\bar{M}$ and $\bar{N}$ are polycyclic, since they can be embedded into the group $((M \otimes N) \rtimes N) \rtimes M$ which is polycyclic by our main theorem. Thus one can obtain polycyclic generating sequences $\bar{m}_1, \dots, \bar{m}_k$ and $\bar{n}_1, \dots, \bar{n}_l$ of $\bar{M}$ and $\bar{N}$, respectively. By [2, Lemma 22], the group $[\bar{M}, \bar{N}]$, which is isomorphic to $M \otimes N$, can be generated by the set

$$\{[\bar{m}_i^{\epsilon_i}, \bar{n}_j^{\delta_j}] : 1 \le i \le k, 1 \le j \le l\}$$

where

$$\epsilon_i = \begin{cases} 1 & : \ |\bar{m}_i| < \infty \\ \pm 1 & : \ |\bar{m}_i| = \infty \end{cases} \qquad \text{and} \qquad \delta_j = \begin{cases} 1 & : \ |\bar{n}_j| < \infty \\ \pm 1 & : \ |\bar{n}_j| = \infty \end{cases}.$$

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