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On the classification of $\mathcal{F}$-fibrations

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Abstract

We begin this paper by studying the construction of principal fibrations associated to $\mathcal{F}$-fibrations (that is to say, fibrations whose fibres are objects of a fixed category $\mathcal{F}$). We prove that under certain conditions we can even define an inverse construction (in the sense of May–Stasheff). Finally, with the aid of the Dold–Lashof–Fuchs classification theorem, we give a classification theorem for numerable $\mathcal{F}$-fibrations. The paper is developed entirely within the framework of the category of k-spaces. © 1998 Elsevier Science B.V.

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Introduction

It is well known that in the theory of fibre bundles, transition functions can be used very effectively to produce new bundles; examples include the construction of associated principal bundles, and the construction of fibre bundles associated with a principal bundle. The lack of transition functions in the general theory of fibrations thwarts any direct attempt to translate such constructions of bundles into the theory of fibrations. Nevertheless, it is known that in the theory of fibrations one can associate to each fibration a

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"principal" fibration in which we have an action of a topological monoid determined by a set of self-homotopy equivalences of the typical fibre.

We begin the paper by showing that the transition from a fibration to a principal fibration can be carried out already at the level of $\mathcal{F}$-arrows. An $\mathcal{F}$-arrow is a surjective map $p: E \to B$ whose fibers $p^{-1}(b)$, $b \in B$, are objects of a category $\mathcal{F}$ of structured topological spaces which contains a given structured space F as a distinguished element. The category $\mathcal{F}$-called a *category of fibers*—is required to satisfy a certain number of conditions which reflect the properties of the fibers (cf. [9]). As in May's work (see [9]), $\mathcal{F}$ produces a principal category of fibers Prin $\mathcal{F}$ (we prefer Stasheff's notation as in [15]) whose distinguished object is the topological monoid of self-equivalences of F in $\mathcal{F}$. We then obtain a Prin $\mathcal{F}$-arrow, which is principal in the sense that the topological monoid we just defined acts fibrewise on the right of the total space. We then discuss the general notion of *principal category of fibers* $\mathcal{H}$ having for distinguished object a topological monoid H and describe some conditions which allows us to transform a principal $\mathcal{H}$-arrow into an $\mathcal{F}$-arrow (the first of these conditions being that H must act on the left of F). To this end we extend the familiar amalgamated product construction to the level of $\mathcal{H}$-arrows and $\mathcal{F}$-arrows and we study some of their properties.

We next study the theory of numerable $\mathcal{F}$-fibrations, namely $\mathcal{F}$-arrows which are locally homotopy trivial over a numerable open covering of their base space; notice that according to P. Booth [2] a numerable $\mathcal{F}$-fibration is a Dold fibration under mild conditions on the base space B (weak-Hausdorffness and paracontractibility, e.g., a CW-complex). In this setting we are much closer to bundle theory, as we can prove that under suitable conditions there is a bijection between the set of all equivalence classes of numerable $\mathcal{F}$-fibrations and the set of all equivalence classes of numerable principal Prin $\mathcal{F}$-fibrations.

At that point, it seems natural to look for a classification theorem for numerable $\mathcal{F}$-fibrations. A theorem of this kind was obtained by M. Fuchs for principal $\mathcal{H}$-fibrations (see [6]); Fuchs' method consists in a clever modification of a well-known construction due to A. Dold and R. Lashof [5].[3] His method can be extended to cover the numerable case and allows us to prove a general classification theorem for numerable $\mathcal{F}$-fibrations. We end the paper by applying this theorem to some standard classes of $\mathcal{F}$-fibrations, producing in this way a classification theorem for each of them.

0. Topological preliminaries

All spaces considered in this paper are objects of the category kTOP of *k-spaces* and continuous functions; all categorical constructions are therefore performed in this category.

We recall that a k-space is a space endowed with the final topology with respect to all incoming maps from compact[4] Hausdorff spaces. Equivalently, kTOP is the coreflective

[3] It should be noted that Fuchs' construction requires the topological monoid H to have a homotopy inverse.
[4] We do *not* use the French convention and therefore our compact spaces are not necessarily Hausdorff.

hull of the category of compact Hausdorff spaces in the category TOP of all topological spaces and thus it comes equipped with a functor $k : \mathrm{TOP} \to \mathrm{kTOP}$, the *k-ification functor*, which re-topologizes every topological space as a k-space, giving it this final topology and includes each hom-set of continuous functions into the corresponding hom-set of kTOP. We recall that kTOP is complete and cocomplete: colimits, such as quotients, sums and direct limits, are homeomorphic to the usual colimits in TOP; however, limits, such as subspaces and product spaces, must be k-ified.

The main advantage in dealing with k-spaces is that kTOP is *cartesian closed*. For any k-spaces X and Y the exponential space Y^X is the space of maps from X to Y, topologized with the k-ified compact-open topology. We recall for future use some well-known properties of kTOP deriving from cartesian closure (see, e.g., [1, Chapter 27]):

- *Exponential law*: for any k-spaces X, Y and Z there is a homeomorphism
$$Z^{X\times Y} \cong (Z^Y)^X,$$
given by the rule $g \mapsto (x \mapsto [y \mapsto g(x,y)])$.
- *Preservation of colimits*: the functor $X \times -$ (or $- \times X$) applied to a colimit diagram gives rise to a colimit diagram, for any k-space X.
- *Continuity of compositions*: for any k-spaces X, Y and Z and for every map $f : X \to Y$ the functions $X^Z \times Z^Y \to X^Y$, $(g,h) \mapsto gh$ (composition), $Z^Y \to Z^X$, $h \mapsto hf$ (precomposition by f) and $X^Z \to Y^Z$, $g \mapsto fg$ (post-composition by f) are continuous.
- *Exponentiation of homotopies*: for any k-spaces X, Y and Z and for any homotopy $G : X \times I \to Y$ there is a homotopy $X^Z \times I \to Y^Z$ given by the rule $(g,t) \mapsto [z \mapsto G(g(z),t)]$.
- *Self-maps monoid*: for any k-space X the space of self-maps X^X has a natural structure of topological monoid,[5] where the multiplication is the composition of self-maps.

For more information about k-spaces see [8], [10] and [16, Section 5, Example (ii)].

In this paper the symbols $\cong$ and $\simeq$ are used to indicate homeomorphisms and homotopy equivalences, respectively. The identity of the space X is denoted by id_X while pr_i represents the projection onto the ith factor of a product. The capital Latin letter I stands for the real interval $[0,1]$.

1. $\mathcal{F}$-arrows

Let $\mathcal{F}$ be a nonempty category endowed with a faithful functor $U : \mathcal{F} \to \mathrm{kTOP}$, i.e., $\mathcal{F}$ is a concrete category over kTOP via the forgetful functor U. We think of $\mathcal{F}$-objects as spaces with an additional structure and in the sequel we will not distinguish between the space UX and the $\mathcal{F}$-object X.

An *$\mathcal{F}$-arrow* (also called $\mathcal{F}$-space by May in [9]) is a continuous function $p : E \to B$ between the spaces E and B, where the fibers $E_b = p^{-1}(b)$ are structured spaces in $\mathcal{F}$,

[5] I.e., a Hopf space with a strictly associative multiplication possessing a strict unit.

for every $b \in B$; to avoid possible pathologies we impose the additional conditions that p is surjective and B is a T_1-space. We use either notation (E, p, B) or p to indicate an $\mathcal{F}$-arrow. An *$\mathcal{F}$-map* from the $\mathcal{F}$-arrow (D, q, A) to the $\mathcal{F}$-arrow (E, p, B) is a pair of maps $g : D \to E$, $\bar{g} : A \to B$ such that $pg = \bar{g}q$ and

$$g|_{D_a} : D_a \to E_{\bar{g}(a)} \in \mathcal{F}(D_a, E_{\bar{g}(a)})$$

for every $a \in A$. Two $\mathcal{F}$-maps are *$\mathcal{F}$-homotopic* if there exists a homotopy $(G, \overline{G}) : (D \times I, q \times \mathrm{id}_I, A \times I) \to (E, p, B)$ between them, which is an $\mathcal{F}$-map.[6] Two $\mathcal{F}$-arrows are *$\mathcal{F}$-homotopy equivalent* if there is an $\mathcal{F}$-map between them that has an $\mathcal{F}$-map as $\mathcal{F}$-homotopy inverse. In particular, we have the concept of *$\mathcal{F}$-homotopy equivalence over B* for $\mathcal{F}$-arrows with same base space B: an $\mathcal{F}$-map $(g, \mathrm{id}_B) : q \to p$ is an *$\mathcal{F}$-homotopy equivalence over B* if there exists an $\mathcal{F}$-map $(g', \mathrm{id}_B) : p \to q$ such that the compositions $(g, \mathrm{id}_B)(g', \mathrm{id}_B)$ and $(g', \mathrm{id}_B)(g, \mathrm{id}_B)$ are $\mathcal{F}$-homotopic to the appropriate identity $\mathcal{F}$-maps, via $\mathcal{F}$-homotopies $(G, \mathrm{pr}_1) : p \times \mathrm{id}_I \to p$ and $(G', \mathrm{pr}_1) : q \times \mathrm{id}_I \to q$; in this case we say that the $\mathcal{F}$-arrows (D, q, B) and (E, p, B) are *$\mathcal{F}$-homotopy equivalent over B*.

As the members of $\mathcal{F}$ will ultimately be taken to be fibers of fibrations, we specialize the category $\mathcal{F}$ by requiring that in $\mathcal{F}$ all morphisms are $\mathcal{F}$-homotopy equivalences over a point and that it contains a distinguished object F such that $\mathcal{F}(F, X) \neq \emptyset$ for every space X of $\mathcal{F}$. We call such an $\mathcal{F}$ a *category of fibers* (compare with [9, 4.1]). In what follows we will denote categories of fibers with capital calligraphic letters and the corresponding distinguished objects with the corresponding capital latin letter (e.g., F is the distinguished object of $\mathcal{F}$, H of $\mathcal{H}$, ...).

In [9,11] one can find several examples of categories of fibers. Here we give some of them for further reference.

Example 1.1. $\mathcal{F}$ is the category consisting of all spaces of the homotopy type of a fixed finite CW-complex F, with morphisms all homotopy equivalences between such spaces. The forgetful functor is the inclusion functor of $\mathcal{F}$ in kTOP.

Example 1.2. Let G be a topological group and let F be a fixed effective left G-space. We take $\mathcal{F}$ to be the category whose objects are all left G-spaces X together with a G-equivariant homeomorphism $\varphi : F \to X$; the distinguished object of $\mathcal{F}$ is F together with the identity map 1_F and $\mathcal{F}(X, X')$ consists of all maps of the form $\varphi' g \varphi^{-1}$, $g \in G$. The functor U forgets the G-structure and the homeomorphism φ.

Example 1.3. Given a based fixed finite CW-complex F, denote by $\mathcal{F}$ the category consisting of all based spaces homeomorphic to F, with morphisms all based homeomorphisms between such spaces. The functor U forgets the base-point. This category is suitable for a theory of *ex-fibrations*.

Let H be a fixed topological monoid. A category of fibers with distinguished object H is called a *principal category of fibers* $\mathcal{H}$ if every object is a space with a right H-

[6] We implicitly assume that $\mathcal{F}$ is closed under products with the singleton space $*$ and we always identify the products $X \times *$ and $* \times X$ with X.

action and every morphism is H-equivariant. The functor U forgets the H-structure. We require, in addition, that for every object X of $\mathcal{H}$ and for every $x \in X$ the *orbit map* $\bar{x} : H \to X$, $\bar{x}(h) := xh$ is a morphism of $\mathcal{H}$ (compare with [9, 4.2]).

Example 1.4. Let H be a topological monoid with multiplication $\mu : H \times H \to H$. Let $\mathcal{H}$ be the category defined as follows. The objects of $\mathcal{H}$ are *all* spaces X with a right H-action $\mu_X : X \times H \to X$ and the same H-equivariant homotopy type as H. The morphisms of $\mathcal{H}$ are *all* H-equivariant homotopy equivalences. Finally, we require that for all $X \in \mathcal{H}$ and every $x \in X$,

$$\mu_X(x, -) : H \to X, \quad h \mapsto \mu(x, h)$$

is a morphism of $\mathcal{H}$.

Given a category of fibers $\mathcal{F}$, there is a standard way to construct a principal category of fibers with distinguished object the topological monoid $\mathcal{F}(F, F)$ (compare with [9, 4.3]). Let $\operatorname{Prin}\mathcal{F}$[7] be the image of $\mathcal{F}$ via the functor

$$\mathcal{F}(F, -) : \mathcal{F} \to \text{kTOP}.$$

We give to $\operatorname{Prin}\mathcal{F}$ an additional algebraic structure as follows. The distinguished object is $\mathcal{F}(F, F)$ with the structure of a topological monoid induced by the self-map monoid $F^F \supset \mathcal{F}(F, F)$. Each object of $\operatorname{Prin}\mathcal{F}$ has a natural right $\mathcal{F}(F, F)$-space structure with action given by composition, that is to say, if X is an object of $\mathcal{F}$, the map $\mathcal{F}(F, X) \times \mathcal{F}(F, F) \to \mathcal{F}(F, X)$ given by the rule $(\alpha, h) \mapsto \alpha h$ is the required (continuous) action; the morphisms of $\operatorname{Prin}\mathcal{F}$ are equivariant with respect to this action. A straightforward computation which uses the exponentiation of homotopies shows that $\operatorname{Prin}\mathcal{F}$ is a principal category of fibers called *the principal category of fibers associated to* $\mathcal{F}$. The functor $\mathcal{F}(F, -)$ can therefore be interpreted as a functor $\mathcal{F}(F, -) : \mathcal{F} \to \operatorname{Prin}\mathcal{F}$ from a category of fibers to its associated principal category of fibers.

Notice that if $\mathcal{H}$ is a principal category of fibers, then $\operatorname{Prin}\mathcal{H}$ can be identified with $\mathcal{H}$ in the following sense: for every object X of $\mathcal{H}$ define

$$\Lambda_X : \mathcal{H}(H, X) \to X, \qquad \Lambda_X(\varphi) := \varphi(u_H),$$

where u_H is the unit of H, and

$$\Gamma_X : X \to \mathcal{H}(H, X), \qquad \Gamma_X(x) := \bar{x}, \quad \bar{x}(h) := xh.$$

A straightforward verification shows that Λ_H is an isomorphism of topological monoids, with inverse Γ_H; furthermore, for every X, Λ_X is a Λ_H-equivariant homeomorphism, with inverse Γ_X. Thus a principal category of fibers can be identified in a coherent way with its associated principal category of fibers.

We now introduce a functor (modeled after the well-known method of changing fibers in the theory of fibre bundles) that enables us to move from a principal category of fibers to a different one. Let $\mathcal{H}$ be a principal category of fibers. If X and Y are spaces with

[7] We find it convenient to extend Stasheff's standard notation to the level of categories of fibers.

a right and left H-action, respectively, the *amalgamated product* $X \times_H Y$ of X and Y is defined as the coequalizer of the maps

$$\theta_1, \theta_2 : X \times H \times Y \to X \times Y,$$

given by $\theta_1(x, h, y) := (xh, y)$ and $\theta_2(x, h, y) := (x, hy)$, for every $x \in X$, $h \in H$ and $y \in Y$. Alternatively, we can describe $X \times_H Y$ as the quotient of $X \times Y$ with respect to the equivalence relation generated by the requirement that (xh, y) and (x, hy) are equivalent for all $x \in X$, $h \in H$ and $y \in Y$.

For a fixed space F with left H-action, the amalgamated product construction gives rise to a functor

$$- \times_H F : \mathcal{H} \to \text{kTOP},$$

whose value on spaces is given by the rule $X \mapsto X \times_H F$ and a morphism $f : X \to X'$ in $\mathcal{H}$ yields the map $f \times_H \mathrm{id}_F$ defined by $[x, y] \mapsto [f(x), y]$. Since f is equivariant, $f \times_H \mathrm{id}_F$ is well-defined, continuous and is uniquely determined by the universal coequalizer property of $X \times_H F$. Let $\mathcal{H} \times_H F$ be the image of $\mathcal{H}$ by this functor. A simple computation shows that $\mathcal{H} \times_H F$ is a category of fibers with distinguished object $H \times_H F$, which is homeomorphic to F via the homeomorphism $[h, x] \mapsto hx$. Again, we can interpret $- \times_H F$ as a functor $- \times_H F : \mathcal{H} \to \mathcal{H} \times_H F$.

If $\mathcal{F}$ is a category of fibers, its distinguished object F has a natural left $\mathcal{F}(F, F)$-action and gives rise to $\operatorname{Prin} \mathcal{F} \times_{\mathcal{F}(F,F)} F$. This category is intimately related to $\mathcal{F}$, because every space of $\operatorname{Prin} \mathcal{F} \times_{\mathcal{F}(F,F)} F$ is of the form $\mathcal{F}(F, X) \times_{\mathcal{F}(F,F)} F$ for some space X of $\mathcal{F}$, and is of the same homotopy type as X (and thus of F). In fact, for every object X of $\mathcal{F}$, the function

$$\mathrm{ev}_X : \mathcal{F}(F, X) \times_{\mathcal{F}(F,F)} F \to X, \qquad \mathrm{ev}_X[\alpha, x] := \alpha(x),$$

which we call the *evaluation map for* X, is well-defined and continuous; moreover, for every choice of $\varphi \in \mathcal{F}(F, X) \neq \emptyset$ and of an $\mathcal{F}$-homotopy inverse $\overline{\varphi}$ for φ, the function

$$\overline{\mathrm{ev}}_X : X \to \mathcal{F}(F, X) \times_{\mathcal{F}(F,F)} F, \qquad \overline{\mathrm{ev}}_X(x) := [\varphi, \overline{\varphi}(x)]$$

is well-defined and continuous. The composition

$$\overline{\mathrm{ev}}_X \mathrm{ev}_X[\alpha, x] = [\varphi, \overline{\varphi}\alpha(x)] = [\varphi\overline{\varphi}\alpha, x]$$

is homotopic to the identity, while

$$\mathrm{ev}_X \overline{\mathrm{ev}}_X(x) = \varphi\overline{\varphi}(x)$$

is even $\mathcal{F}$-homotopic to the identity. Notice that the definition of $\overline{\mathrm{ev}}_X$ is independent, up to homotopy, of the choices of φ and $\overline{\varphi}$.

So far we studied the composite functor $(- \times_{\mathcal{F}(F,F)} F) \circ \operatorname{Prin}$; we can take a step further and compose it with the functor Prin, obtaining the functor $\operatorname{Prin} \circ (- \times_{\mathcal{F}(F,F)} F) \circ \operatorname{Prin}$. However, since the range of $- \times_{\mathcal{F}(F,F)} F$ is in kTOP and not in $\mathcal{F}$, the latter composition is in general not well defined without some additional assumptions. Take an object Y of $\operatorname{Prin} \mathcal{F}$, of the form $Y = \mathcal{F}(F, X)$ for some object X of $\mathcal{F}$; choose φ and $\overline{\varphi}$ as before and assume that:

(a) $Y \times_{\mathcal{F}(F,F)} F$ is an object of $\mathcal{F}$;

(b) ev_X is an $\mathcal{F}$-homotopy equivalence with $\mathcal{F}$-homotopy inverse $\overline{\mathrm{ev}}_X$.

Under these assumptions the spaces Y and $\mathcal{F}(F, Y \times_{\mathcal{F}(F,F)} F)$ are Prin $\mathcal{F}$-homotopy equivalent. In fact the *coevaluation map*

$$\mathrm{coev}_Y : Y \to \mathcal{F}(F, Y \times_{\mathcal{F}(F,F)} F), \qquad \mathrm{coev}_Y(\alpha) := \overline{\mathrm{ev}}_X \alpha$$

and the map

$$\overline{\mathrm{coev}}_Y : \mathcal{F}(F, Y \times_{\mathcal{F}(F,F)} F) \to Y, \qquad \overline{\mathrm{coev}}_Y(\beta) := \mathrm{ev}_X \beta$$

are well-defined and continuous, as they are postcomposition maps by the $\mathcal{F}$-morphisms $\overline{\mathrm{ev}}_X$ and ev_X.[8] Furthermore, the exponentiation of the $\mathcal{F}$-homotopy $\mathrm{ev}_X \overline{\mathrm{ev}}_X \simeq \mathrm{id}_X$ produces, by restriction to $Y \times I \to Y$, a Prin $\mathcal{F}$-homotopy $\overline{\mathrm{coev}}_Y \mathrm{coev}_Y \simeq \mathrm{id}_Y$. Similarly, the $\mathcal{F}$-homotopy $\overline{\mathrm{ev}}_X \mathrm{ev}_X \simeq \mathrm{id}_{Y \times_{\mathcal{F}(F,F)} F}$ yields a Prin $\mathcal{F}$-homotopy $\mathrm{coev}_Y \overline{\mathrm{coev}}_Y \simeq \mathrm{id}_{\mathcal{F}(F, Y \times_{\mathcal{F}(F,F)} F)}$.

Finally, let us recall that $\overline{\mathrm{ev}}_X$ was independent up to homotopy of the choices of φ and $\overline{\varphi}$ in their $\mathcal{F}$-homotopy classes. Now assumption (b) implies that this independence is up to $\mathcal{F}$-homotopy equivalence and that the definition of coev_Y is independent of the same choice up to Prin $\mathcal{F}$-homotopy equivalence.

The considerations made so far on the homotopy equivalences

$$\mathcal{F}(F, X) \times_{\mathcal{F}(F,F)} F \simeq X \quad \text{and} \quad Y \simeq \mathcal{F}(F, Y \times_{\mathcal{F}(F,F)} F)$$

can be extended to a more general case. Let $\mathcal{F}$ be a category of fibers, let $\mathcal{H}$ be a principal category of fibers with $H = \mathcal{F}(F, F)$ and suppose that the following conditions hold:

(i) The equality $H = \mathcal{F}(F, F)$ extends to a lifting of the functor $\mathcal{F}(F, -) : \mathcal{F} \to$ kTOP to $\mathcal{H}$ along the forgetful functor $U : \mathcal{H} \to$ kTOP, which preserves the right $\mathcal{F}(F, F)$-structure of $\mathcal{F}(F, X)$ for every object X of $\mathcal{F}$.

(ii) The homeomorphism $F \cong H \times_H F$ extends to a lifting of the functor $- \times_H F$: $\mathcal{H} \to$ kTOP to $\mathcal{F}$ along the forgetful functor $U : \mathcal{F} \to$ kTOP, and the previous homeomorphism becomes an $\mathcal{F}$-isomorphism.

(iii) For every space X of $\mathcal{F}$ and Y of $\mathcal{H}$, the evaluation and coevaluation maps are $\mathcal{F}$-morphisms and $\mathcal{H}$-morphisms, respectively.

In this case[9] we say that the pair $(\mathcal{F}, \mathcal{H})$ is *complete*. In particular, if $(\mathcal{F}, \mathrm{Prin}\,\mathcal{F})$ is a complete pair, we say that $\mathcal{F}$ is a *complete category of fibers*. Notice that condition (iii) implies that evaluation is a natural transformation $\mathrm{ev} : - \times_H F \circ \mathcal{F}(F, -) \to \mathrm{id}_{\mathcal{F}}$ and the coevaluation is a natural transformation $\mathrm{coev} : \mathrm{id}_{\mathcal{H}} \to \mathcal{F}(F, -) \circ - \times_H F$; hence there is an equivalence between the homotopy category of $\mathcal{F}$ (morphisms are $\mathcal{F}$-homotopy classes of $\mathcal{F}$-morphisms) and the homotopy category of $\mathcal{H}$ (morphisms are $\mathcal{H}$-homotopy classes of $\mathcal{H}$-morphisms).

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[8] Notice that, for every $\alpha \in Y$ and for every $x \in F$, $\mathrm{coev}_Y(\alpha)(x) = [\varphi, \overline{\varphi}\alpha(x)] = [\varphi\overline{\varphi}\alpha, x]$ and hence $\mathrm{coev}_Y(\alpha)$ is homotopic to the map $(x \to [\alpha, x])$. This fact is the reason for the name we have chosen.

[9] Notice that the three axioms (i)–(iii) are not independent; for example, we have already shown that the fact that the coevaluation is an $\mathcal{H}$-morphisms follows from (i), (ii) and from the fact that the evaluation is an $\mathcal{F}$-morphisms. However, one can easily verify that the three axioms are noncontradictory, and we prefer to state them in this form to completely explain the situation.

If $(\mathcal{F}, \mathcal{H})$ is a complete pair, we can think to $\mathcal{H}$ as an enlargement of Prin $\mathcal{F}$. This enlargement is needed in the sequel, since Prin $\mathcal{F}$ can be to small for our proposes (see the hypothesis on $\mathcal{H}$ in Theorem 3.1).

In view of previous remarks the condition that requires a category of fibers to be a complete category of fibers does not seem to be unnatural. Indeed the categories of fibers described in the specific examples given before are all complete categories of fibers.

Observe that the theory of $\mathcal{F}$-arrows is a *fibrewise* theory and thus can be seen in the light of the fibrewise topology, as described, for example, in [7]. We are going to extend the previous constructions from fibers to fibrewise constructions on arrows. Let $\mathcal{H}$ be a principal category of fibers. By definition, an $\mathcal{H}$-arrow is endowed with a right H-action on each fibre; however, a global H-action on the total space which induces the given actions on the fibers might not exist. If such a global action exists, we will call the arrow a *principal* $\mathcal{H}$-arrow. More precisely, a *principal* $\mathcal{H}$-arrow is an $\mathcal{H}$-arrow (D, q, B) with a fibrewise right H-action $\mu : D \times H \to D$ such that $(\mu, q) : (D \times H, pr_1, D) \to (D, q, B)$ is an $\mathcal{H}$-map.

Now assume that $(\mathcal{F}, \mathcal{H})$ is a complete pair. For every $\mathcal{F}$-arrow (E, p, B) define the principal $\mathcal{H}$-arrow

$$\coprod_{b \in B} \mathcal{F}(F, E_b) \to B, \qquad (F \xrightarrow{\alpha} E_b) \mapsto b,$$

where the total space is considered as a subspace of E^F and the global right H-action on it is given by composition. We call this arrow *the principal $\mathcal{H}$-arrow associated to p* and we will denote it by $(\operatorname{Prin} E, \operatorname{Prin} p, B)$. Let $\mathbb{A}_{\mathcal{F}}$ (respectively $\mathbb{PA}_{\mathcal{H}}$) be the category with objects all $\mathcal{F}$-arrows (respectively all principal $\mathcal{H}$-arrows) and with morphisms all $\mathcal{F}$-maps (respectively all $\mathcal{H}$-maps). Now, the above construction defines a functor $\operatorname{Prin} : \mathbb{A}_{\mathcal{F}} \to \mathbb{PA}_{\mathcal{H}}$, whose value on morphisms is given by post-composition:

$$\big[(f, \bar{f}) : q \to p\big] \mapsto \big[(f \circ -, \bar{f}) : \operatorname{Prin} q \to \operatorname{Prin} p\big].$$

Indeed, we can go one step further. Let $\operatorname{Ho} \mathbb{A}_{\mathcal{F}}$ (respectively $\operatorname{Ho} \mathbb{PA}_{\mathcal{H}}$) be the homotopy category of $\mathbb{A}_{\mathcal{F}}$ (respectively of $\mathbb{PA}_{\mathcal{H}}$): objects are all $\mathcal{F}$-arrows (respectively principal $\mathcal{H}$-arrows) and morphisms are $\mathcal{F}$-homotopy classes of $\mathcal{F}$-maps (respectively $\mathcal{H}$-homotopy classes of $\mathcal{H}$-maps). The previous functor goes to the homotopy level, that is to say:

Proposition 1.1. *If $(\mathcal{F}, \mathcal{H})$ is a complete pair, then the principal $\mathcal{H}$-arrow construction defines a functor*

$$\operatorname{Prin} : \operatorname{Ho} \mathbb{A}_{\mathcal{F}} \to \operatorname{Ho} \mathbb{PA}_{\mathcal{H}}.$$

Proof. We have only to show that Prin turns $\mathcal{F}$-homotopies into $\mathcal{H}$-homotopies. If $(G, \overline{G}) : (D \times I, q \times \mathrm{id}_I, A \times I) \to (E, p, B)$ is an $\mathcal{F}$-homotopy between the $\mathcal{F}$-maps $(f, \bar{f}), (g, \bar{g}) : (D, q, A) \to (E, p, B)$, let $G^F : D^F \times I \to E^F$ be the exponentiation of G and let $\widetilde{G}$ be the restriction of G^F to $\operatorname{Prin} D$. Simple calculations show that $(\widetilde{G}, G)$ is an $\mathcal{H}$-homotopy between the $\mathcal{H}$-maps $(f \circ -, \bar{f})$ and $(g \circ -, \bar{g})$. □

Conversely, for every principal $\mathcal{H}$-arrow (D, q, B), the amalgamated product gives rise to the $\mathcal{F}$-arrow $(D \times_H F, q \times_H \mathrm{id}_F, B)$, where $D \times_H F$ is the quotient space of $D \times F$ with respect to the equivalence relation induced by the requirement that (dh, x) and (d, hx) are equivalent for $d \in D$, $x \in F$ and $h \in H$. The projection $q \times_H \mathrm{id}_F$ is given by $[d, x] \mapsto q(d)$ and is well-defined because the action of H on D preserves fibers. This construction is functorial on $\mathbb{PA}_{\mathcal{H}}$, its value on morphisms being given in the obvious way.

Proposition 1.2. *If $(\mathcal{F}, \mathcal{H})$ is a complete pair, then the amalgamated product construction defines a functor*

$$- \times_H F : \mathrm{Ho}\,\mathbb{PA}_{\mathcal{H}} \to \mathrm{Ho}\,\mathbb{A}_{\mathcal{F}}.$$

Proof. A straightforward verification shows that the amalgamated product by F transforms $\mathcal{H}$-homotopies into $\mathcal{F}$-homotopies. □

Remark 1.1. The condition that $(\mathcal{F}, \mathcal{H})$ is complete is not fully needed in both Propositions 1.1 and 1.2; in the former proposition we only need conditions (i) and (iii), while in the latter we need only (ii) and (iii). A similar remark applies elsewhere.

2. Numerable $\mathcal{F}$-fibrations

It is possible to obtain sharper results if we restrict our discussion to the category of numerable $\mathcal{F}$-fibrations. An $\mathcal{F}$-arrow (E, p, B) is said to be a *numerable $\mathcal{F}$-fibration* if there exists a numerable covering $\mathcal{U}$ of B (that is to say, an open covering admitting a refinement given by the co-zero sets of a partition of unity—see [4]) such that, for every $U \in \mathcal{U}$, the $\mathcal{F}$-arrow (E_U, p_U, U), obtained by pulling back (E, p, B) over the inclusion $U \hookrightarrow B$, is $\mathcal{F}$-homotopy equivalent to the trivial $\mathcal{F}$-arrow $(U \times F, \mathrm{pr}_1, U)$.

Proposition 2.1. *Let $(\mathcal{F}, \mathcal{H})$ be a complete pair. If (E, p, B) is a numerable $\mathcal{F}$-fibration then* $(\mathrm{Prin}\,E, \mathrm{Prin}\,p, B)$ *is a principal numerable $\mathcal{H}$-fibration.*

Proof. The numerability of p implies that for every $U \in \mathcal{U}$, there is an $\mathcal{F}$-homotopy equivalence $\psi : E_U \simeq U \times F$ over U. The functor Prin gives rise to an $\mathcal{H}$-homotopy equivalence over U

$$\overline{\psi} : \mathrm{Prin}(E_U) \to \mathrm{Prin}(U \times F)$$

which, when composed with obvious $\mathcal{H}$-homeomorphisms, yields the $\mathcal{H}$-homotopy equivalence

$$(\mathrm{Prin}\,E)_U \xrightarrow{\cong} \mathrm{Prin}(E_U) \xrightarrow{\overline{\psi}} \mathrm{Prin}(U \times F) \xrightarrow{\cong} U \times H$$

over U. □

Proposition 2.2. *Let $(\mathcal{F}, \mathcal{H})$ be a complete pair. If (D, q, B) is a principal numerable $\mathcal{H}$-fibration, then $(D \times_H F, q \times_H \mathrm{id}_F, B)$ is a numerable $\mathcal{F}$-fibration.*

Proof. The numerability of q implies that for every $U \in \mathcal{U}$, there is an $\mathcal{H}$-homotopy equivalence ϕ: $D_U \simeq U \times H$ over U. The functor $- \times_H F$ gives rise to an $\mathcal{F}$-homotopy equivalence over U

$$\overline{\phi}\colon\ D_U \times_H F \simeq (U \times H) \times_H F.$$

We complete the proof by composing $\overline{\phi}$ with the $\mathcal{F}$-homeomorphisms over U

$$(D \times_H F)_U \cong D_U \times_H F$$

and

$$(U \times H) \times_H F \cong U \times F, \qquad [(b, h), x] \mapsto (b, h(x)),$$

where the inverse of this last map is given by

$$U \times F \to (U \times H) \times_H F, \qquad (b, x) \mapsto [(b, \mathrm{id}_F), x]. \qquad \square$$

The following result, which allows the gluing together of the various local homotopy equivalences, is the main tool for handling numerable $\mathcal{F}$-fibrations.

Theorem 2.1 (Dold–May, [13, Theorem 7.2.4]). *If an $\mathcal{F}$-map over B is an $\mathcal{F}$-homotopy equivalence when restricted over every open set of some numerable covering of the base, then this map is an $\mathcal{F}$-homotopy equivalence over B.*

Let us denote by $\mathbb{N}_{\mathcal{F}}$ (respectively $\mathbb{PN}_{\mathcal{H}}$) the category of numerable $\mathcal{F}$-fibrations (respectively principal numerable $\mathcal{H}$-fibrations). Also, denote by $\mathrm{Ho}\,\mathbb{N}_{\mathcal{F}}$ and $\mathrm{Ho}\,\mathbb{PN}_{\mathcal{H}}$ the homotopy categories of $\mathbb{N}_{\mathcal{F}}$ and $\mathbb{PN}_{\mathcal{H}}$, respectively.

Theorem 2.2. *If $(\mathcal{F}, \mathcal{H})$ is a complete pair, then the functors* Prin *and $- \times_H F$ induce an equivalence of categories between* $\mathrm{Ho}\,\mathbb{N}_{\mathcal{F}}$ *and* $\mathrm{Ho}\,\mathbb{PN}_{\mathcal{H}}$.

Proof. Because $(\mathcal{F}, \mathcal{H})$ is a complete pair, the image of the numerable $\mathcal{F}$-fibration (E, p, B) by $(- \times_H F) \circ \mathrm{Prin}$ is the numerable $\mathcal{F}$-fibration $(\mathrm{Prin}\, E \times_H F, \mathrm{Prin}\, p \times_H \mathrm{id}_F, B)$. The maps

$$\mu_E : \mathrm{Prin}\, E \times_H F \to E, \qquad \mu_E[\alpha, x] := \alpha(x)$$

determine a natural transformation $\mu : (- \times_H F) \circ \mathrm{Prin} \to \mathrm{id}_{\mathrm{Ho}\,\mathbb{N}_{\mathcal{F}}}$. According to the previous theorem it suffices to show that the restriction of μ_E over every U in the numerable covering of B is an $\mathcal{F}$-homotopy equivalence. To this end, pick a local trivialization $\varphi = (\varphi_1, \varphi_2) : E_U \to U \times F$ and consider the following diagram:

$$\begin{array}{ccccc}
(\mathrm{Prin} E \times_H F)_U & \xrightarrow{\cong} & \mathrm{Prin}(E_U) \times_H F & \xrightarrow{f \times_H \mathrm{id}_F} & (U \times H) \times_H F \\
\downarrow{\scriptstyle \mu_E} & & & & \downarrow{\scriptstyle g} \\
E_U & & \xrightarrow{\quad\varphi\quad} & & U \times F
\end{array}$$

where the first horizontal map is the obvious $\mathcal{F}$-homeomorphism over U, the map

$$f:\mathrm{Prin}(E_U)\to U\times H,\qquad f:\alpha\mapsto(\varphi_1\alpha(F),\varphi_2\alpha)$$

is an $\mathcal{H}$-homotopy equivalence over U, while $g:[(b,h),x]\mapsto(b,h(x))$ is an $\mathcal{F}$-homeomorphism over U. It is easy to check that the diagram commutes, hence μ_E is an $\mathcal{F}$-homotopy equivalence. Notice that μ_E restricted to the fibre over every $b\in B$ is the evaluation map ev_{E_b}.

Applying the same method we obtain that for every principal numerable $\mathcal{H}$-fibration (D,p,B) the maps

$$\nu_D:D\to\mathrm{Prin}(D\times_H F),\qquad [\nu_D(d)](x):=[d,x]$$

determine a natural equivalence between the identity on $\mathrm{Ho}\,\mathbb{PN}_{\mathcal{H}}$ and the functor $\mathrm{Prin}\circ(-\times_H F)$. Notice again that ν_D restricted to the fibre over every $b\in B$ is the coevaluation map coev_{D_b}. □

This theorem implies that the classification of numerable $\mathcal{F}$-fibrations is the same thing as the classification of principal numerable $\mathcal{H}$-fibrations, provided $(\mathcal{F},\mathcal{H})$ is a complete pair. A consequence of this assertion will be made clear in the next corollary; to this end, we need some definitions. Let $\mathcal{E}_{\mathcal{F}}:\mathrm{kTOP}\to\mathrm{SET}$ be the contravariant functor that assigns to every $B\in\mathrm{kTOP}$ the set $\mathcal{E}_{\mathcal{F}}(B)$ of $\mathcal{F}$-homotopy equivalence classes of all numerable $\mathcal{F}$-fibrations over B and to every map $f:B\to A$, the function $\mathcal{E}_{\mathcal{F}}(A)\to\mathcal{E}_{\mathcal{F}}(B)$ which associates to each class $\xi\in\mathcal{E}_{\mathcal{F}}(A)$ the class in $\mathcal{E}_{\mathcal{F}}(B)$ determined by the pullback along f of any numerable $\mathcal{F}$-fibration representing ξ. Similarly, we define a functor $\mathcal{E}_{\mathcal{H}}:\mathrm{kTOP}\to\mathrm{SET}$ which takes spaces into sets of all $\mathcal{H}$-homotopy classes of principal numerable $\mathcal{H}$-fibrations and maps $f:B\to A$ into functions $\mathcal{E}_{\mathcal{F}}(A)\to\mathcal{E}_{\mathcal{F}}(B)$ which associate to each class $\xi\in\mathcal{E}_{\mathcal{H}}(A)$ the class in $\mathcal{E}_{\mathcal{H}}(B)$ determined by the pullback along f of any principal numerable $\mathcal{H}$-fibration representing ξ.

Corollary 2.1. *If $(\mathcal{F},\mathcal{H})$ is a complete pair, then the functors $\mathcal{E}_{\mathcal{F}}$ and $\mathcal{E}_{\mathcal{H}}$ are naturally equivalent. In particular, if $\mathcal{F}$ is a complete category of fibers, then $\mathcal{E}_{\mathcal{F}}$ and $\mathcal{E}_{\mathrm{Prin}\,\mathcal{F}}$ are naturally equivalent.*

Proof. Define a natural transformation $\varphi:\mathcal{E}_{\mathcal{F}}\to\mathcal{E}_{\mathcal{H}}$ as follows: for every k-space B and every class $[(E,p,B)]$ in $\mathcal{E}_{\mathcal{F}}(B)$ set

$$\varphi_B:\big[(E,p,B)\big]\mapsto\big[(\mathrm{Prin}\,E,\mathrm{Prin}\,p,B)\big].$$

The previous theorem implies that this is indeed a well-defined transformation. Similarly, using the functor $-\times_{\mathcal{H}}F$, construct a natural transformation $\psi:\mathcal{E}_{\mathcal{H}}\to\mathcal{E}_{\mathcal{F}}$; it is easy to check that ψ is the inverse of φ. □

3. Classification of numerable $\mathcal{F}$-fibrations

The classical Dold–Lashof construction (cf. [5]) associates to every topological monoid H a classifying space B_∞ (often denoted by BH to emphasize the dependence on H).

Unfortunately, that construction gives rise to a quasifibration instead of a fibration; hence, the Dold–Lashof classification applies only to principal quasifibrations. In [15] Stasheff circumvented this difficulty by showing that B_∞ in fact classifies principal fibrations. On the other side, relying on the additional requirement that the monoid H is group-like, i.e., that H has a homotopy inverse, Fuchs (see [6]) was able to improve the original Dold–Lashof method in order to obtain a classifying fibration. More precisely, his construction associates to every principal H-fibration (E, p, B) (notation of [6]) a universal principal H-fibration $(E_\infty, p_\infty, B_\infty)$. It is not difficult to verify that Fuchs' approach preserves numerability. Moreover, a careful analysis of Fuchs' construction reveals that some fibers are replaced by the mapping cylinders of the orbit maps

$$\bar{x}: H \to X, \qquad \bar{x}: h \mapsto xh,$$

where X is an object in $\mathcal{H}$ and $x \in X$ (we refer the reader to the original paper [6] for this fact). Hence we obtain the following result:

Theorem 3.1. *Let $\mathcal{H}$ be a principal category of fibers which contains the mapping cylinders of all orbit maps and such that the distinguished topological monoid H has a homotopy inverse. If (E, p, B) is a numerable principal $\mathcal{H}$-fibration, then $(E_\infty, p_\infty, B_\infty)$, as given by the Dold–Lashof–Fuchs construction, is also a numerable principal $\mathcal{H}$-fibration and E_∞ is contractible.*

For a more detailed account on the Dold–Lashof–Fuchs construction see [12].

Remark 3.1. We wish to observe that it is impossible to avoid the assumption that H has a homotopy inverse: indeed, a topological monoid H has an inverse if, and only if, it is the fibre of a principal fibration (E, p, B) with E contractible. To show that the condition is necessary, take the principal category of fibers $\mathcal{H}$ of Example 1.4 and apply the Dold–Lashof–Fuchs construction to the trivial principal $\mathcal{H}$-fibration $H \to *$. The condition is sufficient by Lemma 3.2 of [3]. A useful condition that guarantees the existence of a homotopy inverse for H can be found in [14]: if H is a *paracontractible* topological monoid then it has a homotopy inverse, unique up to homotopy (a space is paracontractible if it admits a locally finite partition of unity such that the inclusion map of each open support is nullhomotopic). For example, if F is a finite CW-complex then the monoid of all homotopy self-equivalences of F is paracontractible and Sibson's result [14] applies. For an arbitrary category of fibers $\mathcal{F}$, the topological monoid $\mathcal{F}(F, F)$ does not necessarily have a homotopy inverse, nonetheless the first three examples described in Section 1 give rise to topological monoids with inverse.

According to [6, Section 5] and to [13, Theorem 7.2.13], p_∞ is a *classifying* fibration. More precisely, let $\mathcal{E}_\mathcal{H}$ be the contravariant functor defined in Section 2 (just before Corollary 2.1) and let $[-, B_\infty]$ be the usual contravariant homotopy class functor.

Theorem 3.2. *Let $\mathcal{H}$ be as in the previous theorem. There is a natural equivalence*

$$\phi: [-, B_\infty] \cong \mathcal{E}_\mathcal{H}, \qquad \phi_B([h]) := [p],$$

where $[p]$ is the $\mathcal{H}$-equivalence class of the numerable principal $\mathcal{H}$-fibration p obtained as the pullback of p_∞ via any representative $h \in [h]$.

The category $\mathcal{H}$ of Example 1.4 is closed under the construction of mapping cylinders of orbit maps. In fact, there is a natural strong deformation retraction of the mapping cylinder $C_{\bar{x}}$ of an orbit map $\bar{x} : H \to X$ on X which can be used to induce an H-structure on $C_{\bar{x}}$. Therefore we have the following:

Corollary 3.1. *Assume that $\mathcal{H}$ is a principal category of fibers as in Example 1.4 and that its distinguished object H has a homotopy inverse. Then the functor $[-, B_\infty]$ is naturally equivalent to $\mathcal{E}_\mathcal{H}$.*

Combining Theorem 3.2 with Theorem 2.2, we obtain the Main Theorem of this section:

Theorem 3.3. *Let $(\mathcal{F}, \mathcal{H})$ be a complete pair, with $\mathcal{H}$ as before. The numerable $\mathcal{F}$-fibration $(E_\infty \times_H F, p_\infty \times_H \mathrm{id}_F, B_\infty)$ classifies numerable $\mathcal{F}$-fibrations over k-spaces.*

Let us now consider the examples given in Section 1.

If we take the category of fibers $\mathcal{F}$ of Example 1.1 and the principal category of fibers $\mathcal{H}$ of Example 1.4, with $H = \mathcal{F}(F, F)$ then, clearly $(\mathcal{F}, \mathcal{H})$ is a complete pair and $\mathcal{H}$ contains all the needed mapping cylinders.

Similarly, if $\mathcal{F}$ is the category of Example 1.3 we can take as principal category of fibers the category $\mathcal{H} = \mathrm{Prin}\,\mathcal{F}$. Notice that the amalgamated products $\mathcal{F}(F, X) \times_H F$ are naturally pointed: if $\star$ is the basepoint of F, and if φ and φ' are elements of $\mathcal{F}(F, X)$, then

$$[\varphi, \star] = [\varphi, \varphi^{-1}\varphi'(\star)] = [\varphi', \star],$$

so we can take $[\varphi, \star]$ as the base point of $\mathcal{F}(F, X) \times_H F$ and this choice is independent on the choice of $\varphi \in \mathcal{F}(F, X)$. This natural way to assign basepoints to the spaces of the form $\mathcal{F}(F, X) \times_H F$ gives rise to a lifting to $\mathcal{F}$ of the functor $- \times_H F$. A straightforward computation shows that the remaining requirements are satisfied: $(\mathcal{F}, \mathcal{H})$ is a complete pair and $\mathcal{H}$ contains the mapping cylinders of the orbit maps.

Corollary 3.2. *Let $\mathcal{F}$ be the category of fibers of either Example 1.1 or Example 1.3 and let H be defined as above. Then the numerable $\mathcal{F}$-fibration $(E_\infty \times_H F, p_\infty \times_H \mathrm{id}_F, B_\infty)$ classifies numerable $\mathcal{F}$-fibrations over k-spaces.*

Example 1.2 is more difficult to treat. In fact, the objects of $\mathrm{Prin}\,\mathcal{F}$ are homeomorphic to G and the morphisms are homeomorphisms.[10] Consequently, the mapping cylinders

[10] In fact, they are homeomorphic to $\mathcal{F}(F, F)$, which in turn can have a coarser topology than the one of G. If this were the case, one could adopt the standard solution of retopologizing G, giving it this finer topology from the beginning; compare with [9, Example 6.11].

of orbit maps are homeomorphic to $G \times I$, thus a category of fibers closed with respect to the construction of mapping cylinders of orbit maps should contain at least all spaces homeomorphic to $G \times I^n$. This gives a suggestion on how to modify the category $\mathcal{F}$ so to make the theory work. Given an effective left G-space F let $\mathcal{F}$ be the category whose objects are all pairs of the form $(X \times I^n, \varphi)$ where X is a left G-space, n is a nonnegative integer and $\varphi: F \to G$ is a G-equivariant homeomorphism. In addition, $\mathcal{F}((X \times I^n, \varphi), (X' \times I^m, \varphi'))$ consists of all maps of the form

$$X \times I^n \xrightarrow{\mathrm{pr}_1} X \xrightarrow{\varphi^{-1}} F \xrightarrow{g\cdot -} F \xrightarrow{\varphi'} X' \hookrightarrow X' \times I^m$$

for some $g \in G$. The distinguished object of $\mathcal{F}$ is $(F, 1_F)$. Similarly, the objects of the category $\mathcal{H}$ are all spaces of the form $Y \times I^n$ with Y homeomorphic to G. The distinguished object of $\mathcal{H}$ is the group G itself. At this point one can verify that $(\mathcal{F}, \mathcal{H})$ is a complete pair and that $\mathcal{H}$ contains the mapping cylinders of orbit maps. Again, Theorem 3.3 applies and the resulting classification is up to equivariant equivalence.

Remark 3.2. Principal numerable $\mathcal{H}$-fibrations are locally homotopy trivial, rather than locally trivial: they are an intermediate concept between principal G-bundles and principal fibrations. Our result gives a classification for the class of principal G-bundles embedded in this wider class of principal numerable $\mathcal{H}$-fibrations. To induce from this a classification of G-bundles, we enlarged the categories $\mathcal{F}$ and $\mathcal{H}$ to satisfy the requirements of Theorem 3.3. This produces a classification for the class of G-bundles embedded in a wider class of locally homotopy trivial fibrations.

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