H-spaces, semiperfect rings and self-homotopy equivalences

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We use the theory of semiperfect rings to derive decomposition theorems for H- and coH-spaces, which generalize the results of C. Wilkerson [23]. The results are then used to prove reducibility of self-homotopy equivalences for arbitrary p-local H- and coH-spaces.

Introduction

The group of self-homotopy equivalences $\mathrm{Aut}(X)$ of a space X consists of homotopy classes of maps $X \to X$ that are homotopy equivalences (see [22] for a detailed survey of the subject). In [15] the second author proved a decomposition theorem for groups of self-homotopy equivalences of product spaces, under the assumption that the self-equivalences in question are in certain sense reducible, and then in [17] he derived a series of computable homological criteria that imply reducibility. These criteria are based on the detection of nilpotent self-maps, and so are best suited for the study of atomic spaces (i.e. spaces, for which every self-map is either nilpotent or an equivalence, cf. [1]). It turns out that indecomposable p-complete H-spaces are always atomic while in general indecomposable p-local H-spaces are not. Consequently, the classical theorem of C. Wilkerson [23] that finite p-local H-spaces (and simply-connected finite p-local coH-spaces) admit essentially unique decompositions into indecomposable factors did not provide sufficient information to imply reducibility for p-local spaces. Motivated by the results obtained in [18] we use the theory of semiperfect rings to show that the endomorphism sets of indecomposable factors of p-local H-spaces are in a certain sense local. Furthermore, semiperfect rings turned out to be a powerful organizational principle that allow a unified and simplified treatment of decomposition theorems for both p-complete and p-local, H- and coH-spaces. The improved versions of the Wilkerson's theorem are strong enough to imply reducibility of self-equivalences and hence a canonical decomposition of the group of self-homotopy equivalences of p-local H- and coH-spaces.

Following is the outline of the contents. The first section contains basic definitions and facts about semiperfect rings and near-rings. In the second section we relate the properties of indecomposable factors of H- and coH-spaces to the properties of idempotents in the homotopy and homology representation of their self-maps, and then use the theory of semiperfect rings to prove that decompositions of H- and coH-spaces whose factors are strongly indecomposable are essentially unique. We conclude the section with a result on the triviality of the Mislin genus. In the third section we prove that for p-complete and p-local H- and coH-spaces indecomposable factors are indeed strongly indecomposable and deduce the strong form of Wilkerson's theorem that all such spaces admit unique decompositions into strongly indecomposable factors. In the last section we study the reducibility of self-homotopy equivalences for arbitrary p-complete and p-local H- and coH-spaces.

We will assume throughout that spaces under consideration are pointed (although the base-point is systematically omitted from the notation) and have the homotopy type of a connected CW complex. All maps and homotopy classes are base-point preserving, and we do not distinguish notationally between a map and its homotopy class.

1. Algebraic preliminaries

One of the main goals of this paper is to emphasize the role of semiperfect rings in homotopy theory. Recall that the *Jacobson radical* $J(R)$ of a ring R is defined as the intersection of all maximal ideals of R. If R is an artinian ring then its radical is a nilpotent ideal, and the famous Wederburn–Artin theorem states that $R/J(R)$ is a finite product of full matrix rings over division algebras. Moreover, by another fundamental result, the Krull–Schmidt theorem, every finitely generated module over an artinian ring admits a unique decomposition into a direct sum of strongly indecomposable modules (which are, by defintion, modules whose endomorphism ring is a local ring).

Semiperfect rings are rings which possess two properties of artinian rings that are crucial for the Krull–Schmidt theory: a ring R is *semiperfect* (cf. [9, Ch. 8]) if

(a) $R/J(R)$ is semisimple artinian (i.e. a product of full matrix rings over division algebras), and

(b) Every idempotent in $R/J(R)$ can be lifted to an idempotent in R.

An idempotent $e \in R$ is *primitive* if it cannot be decomposed as a sum of nontrivial orthogonal idempotents, and it is *local* if eRe is a local ring. Since local rings have no nontrivial idempotents, a local idempotent is a fortiori primitive. The characteristic property of semiperfect rings is that they admit an essentially unique complete set of orthogonal local idempotents. More precisely, R is semiperfect if and only if there are pairwise orthogonal local idempotents $e_1, \ldots, e_n \in R$ such that $e_1+\cdots+e_n = 1$, (cf. [9, Theorem 23.6]). Moreover, if $e'_1, \ldots, e'_m$ is another complete set of orthogonal primitive idempotents in R, then $m = n$ and there is an invertible element $u \in R$ such that, after a suitable permutation of indices, $e'_i = u^{-1}e_i u$ [9, Proposition 23.5 and Remark 23.7]. This implies that a module is a finite direct sum of strongly indecomposable modules if and only if its endomorphism ring is semiperfect [9, Theorem 23.8].

Semiperfect rings turn out to be a common generalization of local rings and artinian rings. Let us list a few examples that are relevant for homotopy theory:

Example 1.1. Let $\widehat{\mathbb{Z}}_P$ be the ring of integers completed at a finite set of primes P. Every finite algebra over $\widehat{\mathbb{Z}}_P$ is semiperfect [9, Example 23.3].

Example 1.2. Let $\mathbb{Z}_{(p)}$ denote the ring of integers localized at the prime p, and let M be a finitely generated $\mathbb{Z}_{(p)}$-module. Every subring of finite index in the endomorphism ring $\mathrm{End}(M)$ is semiperfect [19, Corollary 4.3].

Example 1.3. A ring whose additive group is a finitely generated abelian group is semiperfect if and only if it is finite. (This follows from [4, Theorems 2.5 and 2.6].)

Another algebraic structure that naturally arises in homotopy theory is that of a near-ring. A *near-ring* $(N, +, \cdot)$ is a set endowed with two operations such that $(N, +)$ is a group (not necessarily abelian), $(N, \cdot)$ is a monoid and only one of the distributivity laws is satisfied, either $(x + y)z = xz + yz$ (for right near-rings) or $z(x + y) = zx + zy$ (for left near-rings). A general reference on near-rings is [20].

Example 1.4. If G is an H-group, i.e. an associative H-space with inverses, then $\mathrm{End}(G) = [G, G]$ is a right near-ring with respect to H-addition and composition. If C is a coH-group, then $\mathrm{End}(C) = [C, C]$ is a left near-ring with respect to coH-addition and composition. In particular, $\mathrm{End}(\Omega X)$ and $\mathrm{End}(\Sigma X)$ are near-rings for any space X (see [3, Chapter 3]).

Homomorphisms of near-rings are defined in the same way as homomorphisms of rings. In particular, if $\phi\colon N \to R$ is a homomorphism from a near-ring N to a ring R, its image $\mathrm{im}\,\phi$ is a subring of R. A homomorphism ϕ between near-rings is *unit-reflecting* if an element is invertible if and only if its image by ϕ is invertible.

Following Maxson [11] we define a near-ring N to be *local* if the set $\mathfrak{m} = N \smallsetminus U(N)$ of non-units in N is an N-subgroup. In this case $\mathfrak{m}$ is the unique maximal N-subgroup and $\mathfrak{m} = J_2(N)$, where $J_2(N)$ is one of the several possible generalizations of the Jacobson radical to near-rings [20, Chapter 5]. This definition coincides with the usual notion of localness when N is a ring.

Lemma 1.5. *Let $\phi\colon N \to N'$ be a unit-reflecting homomorphism of near-rings. If the near-ring N' is local, then N is also local.*

Proof. Let $\mathfrak{m}' \leq N'$ be the unique maximal N'-subgroup. The homomorphism ϕ is unit-reflecting, so every ϕ-preimage of a unit is a unit, i.e. $\phi^{-1}(U(N')) = U(N)$. Hence $\phi^{-1}(\mathfrak{m}') = \phi^{-1}(N' \smallsetminus U(N')) = N \smallsetminus U(N) \subset N$ is an N-subgroup and N is local. □

We conclude the section with an auxiliary result which will allow us to relate homomorphisms between modules over a local ring to homomorphisms between modules over the corresponding residue field.

Lemma 1.6. *Let R be a commutative local ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$, and let $f\colon M \to M'$ be a homomorphism of finitely generated R-modules.*

(a) If $f \otimes \mathbb{1}_k\colon M \otimes k \to M' \otimes k$ is an isomorphism, then f is surjective.

(b) If $f \otimes \mathbb{1}_k$ is an isomorphism, and M and M' are isomorphic noetherian R-modules, then f is an isomorphism. In other words, the ring homomorphism

$$\mathrm{End}(M) \to \mathrm{End}(M \otimes k),\ f \mapsto f \otimes \mathbb{1}_k$$

is unit-reflecting whenever M is noetherian.

Proof. (a): Let d be the dimension of $M \otimes k \cong M' \otimes k$ as a vector space over k. There are surjections $q\colon R^d \to M$, $q'\colon R^d \to M'$ and, by freeness of R^d, there is also an endomorphism $\phi\colon R^d \to R^d$, which is a lift for f, i.e. $q'\phi = fq$.

By the assumptions $f \otimes \mathbb{1}_k$ is an isomorphism. Since $q \otimes \mathbb{1}_k$ and $q' \otimes \mathbb{1}_k$ are isomorphisms by construction, it follows that $\phi \otimes \mathbb{1}_k$ is an isomorphism as well. As R^d is free and R is commutative, $\det(\phi)$ makes sense, and we have

$$\det(\phi)(\mathrm{mod}\ \mathfrak{m}) \equiv \det(\phi \otimes \mathbb{1}_k).$$

The determinant on the right is invertible in k, which means that $\det(\phi)$ is invertible in R, since R is local. Therefore ϕ is an isomorphism. But then $q'\phi = fq$ implies f is epic.

(b): When $M \cong M'$, it is enough to consider $f\colon M \to M$. As M is noetherian, the sequence $\ker f \le \ker f^2 \le \ker f^3 \le \cdots$ stabilizes, i.e. $\ker f^n = \ker f^{n+1}$ for some n. By (a), f is surjective, so every $x \in \ker f$ is of the form $x = f^n(y)$, $y \in M$. Then: $0 = f(x) = f^{n+1}(y) = f^n(y) = x$, so $\ker f = 0$ and f is monic. □

2. Uniqueness of decompositions of H- and coH-spaces

We are going to use the theory of semiperfect rings to study homotopy decompositions of H- and coH-spaces. In particular, we will show that under certain conditions analogous to those of the Krull–Schmidt–Azumaya theorem for modules (cf. [9], [21]), any two decompositions of an H- or coH-space are equivalent. Because of the duality between H- and coH-spaces it will be sufficient to consider only one case.

Given a space X let $H_*(X)$ and $\pi_*(X)$ be the graded groups

$$H_*(X) = \bigoplus_{i=1}^{\dim X} H_i(X) \qquad \text{and} \qquad \pi_*(X) = \bigoplus_{i=1}^{\dim X} \pi_i(X),$$

and let α_X and β_X be the homomorphisms

$$\alpha = \alpha_X\colon [X, X] \to \mathrm{End}(H_*(X)), \quad \alpha_X\colon f \mapsto f_*$$

and

$$\beta = \beta_X\colon [X, X] \to \mathrm{End}(\pi_*(X)), \quad \beta_X\colon f \mapsto f_\sharp.$$

Observe that our π_* is the direct sum of π_i's is only up to the dimension of X, even though higher homotopy groups of X may be nontrivial. With this convention $\pi_*(X)$ is finitely generated whenever $\pi_i(X)$ is finitely generated for each i and X is either finite-dimensional, or its higher homotopy groups vanish.

In general $[X, X]$ is a monoid with respect to the composition of maps, and α and β are homomorphisms of monoids. However, if C is a coH-space, then $\alpha(f+g) = \alpha(f)+\alpha(g)$, and, if G is an H-space, then $\beta(f+g) = \beta(f)+\beta(g)$. Also, connected H-spaces always have inverses [25, Theorem 1.3.1], as do simply-connected coH-spaces [2, Proposition 1.13]. We will denote by $-f$ either the left or the right additive inverse of f, although these inverses may be distinct for general H- and coH-spaces. Then we also have $\alpha(-f) = -\alpha(f)$ for coH-inverses and $\beta(-f) = -\beta(f)$ for H-inverses. The following lemma is a restatement of well-known facts.

Lemma 2.1.

(a) *If G is a connected H-space, then β_G is a unit-reflecting homomorphism, and* $\operatorname{im}\beta_G$ *is a subring of* $\operatorname{End}(\pi_*(G))$.

(b) *If C is a simply-connected coH-space, then α_C is a unit-reflecting homomorphism, and* $\operatorname{im}\alpha_C$ *is a subring of* $\operatorname{End}(H_*(C))$.

Proof. The first claim is just the Whitehead theorem and its generalizations, while the second part follows from the fact that β and α respect the additive structure on $[G, G]$ and $[C, C]$ as noted above. □

Given a self-map $f\colon X \to X$, we denote by $\operatorname{Tel}(f)$ *the telescope of* f, i.e.

$$\operatorname{Tel}(f) = \varinjlim(X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots),$$

and $X \hookrightarrow \operatorname{Tel}(f)$ denotes the inclusion of X into the base level of the telescope. For an idempotent $f_\sharp$, the results of [7] imply that $\pi_*(\operatorname{Tel}(f)) \cong \operatorname{im} f_\sharp$.

Let us call a space X $\times$-*decomposable* if $X \simeq Y \times Z$ for nontrivial spaces Y and Z. Dually, a space X is $\vee$-*decomposable* if $X \simeq Y \vee Z$ for nontrivial Y and Z. Decomposability of H- and coH-spaces can be detected by looking at the idempotents in the image of β and α.

Proposition 2.2.

(a) *A connected H-space G is $\times$-decomposable if and only if* $\operatorname{im}\beta_G$ *contains a nontrivial idempotent.*

(b) *A simply-connected coH-space C is $\vee$-decomposable if and only if* $\operatorname{im}\alpha_C$ *contains a nontrivial idempotent.*

Proof. A nontrivial decomposition $G \simeq G_1 \times G_2$ determines nontrivial idempotents

$$e_1\colon G \xrightarrow{p_1} G_1 \xrightarrow{j_1} G \quad \text{and} \quad e_2\colon G \xrightarrow{p_2} G_2 \xrightarrow{j_2} G.$$

Clearly $e_{1\sharp}, e_{2\sharp} \in \operatorname{im}\beta_G$ are also idempotents, and $e_{1\sharp} + e_{2\sharp} = \mathbb{1}_{G\sharp}$. They are both nontrivial: otherwise, if one of them, say $e_{1\sharp}$ equals $\mathbb{1}_{G\sharp}$, then e_1 is invertible by

Lemma 2.1 and idempotent, hence $e_1 = \mathbb{1}_G$, which contradicts the nontriviality of G_2.

For the reverse implication, assume that $\epsilon_\sharp$ is a nontrivial idempotent in $\operatorname{im}\beta_G$. Then $\mathbb{1}_{G\sharp} - \epsilon_\sharp = (\mathbb{1}_G - \epsilon)_\sharp$ is also idempotent and the composition

$$G \xrightarrow{\Delta} G \times G \hookrightarrow \operatorname{Tel}(\epsilon) \times \operatorname{Tel}(\mathbb{1}_G - \epsilon)$$

induces an isomorphism

$$\pi_*(G) \to \operatorname{im}\epsilon_\sharp \oplus \operatorname{im}(\mathbb{1}_G - \epsilon)_\sharp$$

of homotopy groups. Both telescopes are nontrivial, since $\epsilon_\sharp$ and $(\mathbb{1}_G - \epsilon)_\sharp$ are nontrivial idempotents.

In the dual case, the diagonal $\Delta\colon G \to G \times G$ has to be replaced by the actual coH-structure map $\nu\colon C \to C \vee C$. □

One important feature of homotopy theory is that decompositions into indecomposable factors are in general not unique. The first examples of spaces which have different decompositions as wedges of indecomposable coH-spaces appeared in stable homotopy theory (see [5]). Unstably, there is the famous *Hilton–Roitberg criminal* $E_{5\omega}$, which is an H-manifold such that $S^3 \times \operatorname{Sp}(2)$ and $S^3 \times E_{5\omega}$ are homotopy equivalent (even diffeomorphic), but $\operatorname{Sp}(2) \not\simeq E_{5\omega}$ [12, Example B].

An analogous situation arises in algebra where direct sum decompositions into indecomposable modules are in general not unique. However, the uniqueness is restored if the summands are strongly indecomposable, i.e. if their endomorphism rings are local (cf. [9, Definition 19.12], [21, Definition 2.9.11]), which motivates the following definition for spaces:

Definition 2.3. A connected H-space G is *strongly $\times$-indecomposable* if $\operatorname{im}\beta_G$ is a local ring. A simply-connected coH-space C is *strongly $\vee$-indecomposable* if $\operatorname{im}\alpha_C$ is a local ring.

By Example 1.4 we know that $[G, G]$ is a near-ring if G is an H-group. Thus, Lemmas 2.1 and 1.5 imply that an H-group G is strongly $\times$-indecomposable if and only if $[G, G]$ is a local near-ring. Similarly, a coH-group C is strongly $\vee$-indecomposable if and only if $[C, C]$ is a local near-ring.

Since local rings have only trivial idempotents, every strongly indecomposable H- or coH-space is indecomposable by Proposition 2.2. The converse is false however; for S^3 we have $\operatorname{im}\beta \cong \mathbb{Z}$ and $\operatorname{im}\alpha \cong \mathbb{Z}$, hence S^3 is $\times$- and $\vee$-indecomposable, but is neither strongly $\times$- nor $\vee$-indecomposable. The distinction between indecomposability and strong indecomposability disappears for finite p-complete and p-local spaces, as we will see in Theorems 3.1 and 3.3 below.

Notice that every decomposition $\mathbb{1}_{G\sharp} = \epsilon_{1\sharp} + \cdots + \epsilon_{n\sharp} \in \operatorname{im}\beta$, where $\epsilon_{i\sharp}$'s are pairwise orthogonal idempotents, yields a decomposition $G \simeq G_1 \times \cdots \times G_n$ by setting $G_i = \operatorname{Tel}(\epsilon_i)$, as in the proof of Proposition 2.2. This decomposition determines idempotents $e_i = j_i p_i \in [G, G]$ and it is clear that $\beta(e_i) = \beta(\epsilon_i)$, i.e. $e_{i\sharp} = \epsilon_{i\sharp}$. Let $\beta_i = \beta_{G_i}$: then the ring homomorphisms

$$\operatorname{im}\beta_i \to e_{i\sharp}(\operatorname{im}\beta)e_{i\sharp},\ f_{i\sharp} \mapsto j_{i\sharp} f_{i\sharp} p_{i\sharp}$$

and

$$e_{i\sharp}(\operatorname{im}\beta)e_{i\sharp} \to \operatorname{im}\beta_i,\ e_{i\sharp}f_\sharp e_{i\sharp} \mapsto p_{i\sharp}f_\sharp j_{i\sharp}$$

are (well defined and) inverse to each other, hence $\operatorname{im}\beta_i \cong e_{i\sharp}(\operatorname{im}\beta)e_{i\sharp}$. In particular *$G_i$ is $\times$-indecomposable if and only if $e_{i\sharp}$ is a primitive idempotent*, and *G_i is strongly $\times$-indecomposable if and only if $e_{i\sharp}$ is a local idempotent.* Hence we obtain:

Proposition 2.4.

(a) *A connected H-space G has a decomposition $G \simeq G_1 \times \cdots \times G_n$, with all G_i's $\times$-indecomposable, if and only if $\operatorname{im}\beta_G$ has a complete set of pairwise orthogonal primitive idempotents. Moreover, the factors of this decomposition are strongly $\times$-indecomposable, if and only if $\operatorname{im}\beta_G$ is a semiperfect ring.*

(b) *A simply-connected coH-space C has a decomposition $C \simeq C_1 \vee \cdots \vee C_n$, with all C_i's $\vee$-indecomposable, if and only if $\operatorname{im}\alpha_C$ has a complete set of pairwise orthogonal primitive idempotents. Moreover, the summands of this decomposition are strongly $\vee$-indecomposable, if and only if $\operatorname{im}\alpha_C$ is a semiperfect ring.*

It is a well-known algebraic fact that every right noetherian ring has a complete set of orthogonal primitive idempotents. In particular, if $\pi_*(G)$ is finitely generated as an abelian group (i.e. as a $\mathbb{Z}$-module), then $\operatorname{im}\beta \leq \operatorname{End}(\pi_*(G))$ is noetherian. It follows that every connected H-space G with f.g. $\pi_*(G)$ admits a decomposition $G \simeq G_1 \times \cdots \times G_n$, with all G_i's $\times$-indecomposable. Dually, every simply-connected coH-space C with f.g. $H_*(C)$ admits a decomposition $C \simeq C_1 \vee \cdots \vee C_n$, with all C_i's $\vee$-indecomposable.

We are now ready to formulate unique decomposition theorems for H- and coH-spaces. Both formulations closely resemble the Krull–Schmidt–Azumaya theorem as in [9, Theorem 19.21].

Theorem 2.5.

(a) *Assume $G \simeq G_1 \times \cdots \times G_n \simeq \overline{G}_1 \times \cdots \times \overline{G}_m$ are two decompositions of a connected H-space G, with $\overline{G}_i$'s $\times$-indecomposable and G_i's strongly $\times$-indecomposable. Then $m = n$ and, after reindexing, $G_i \simeq \overline{G}_i$ for all i.*

(b) *Assume $C \simeq C_1 \vee \cdots \vee C_n \simeq \overline{C}_1 \vee \cdots \vee \overline{C}_m$ are two decompositions of a simply-connected coH-space C, with $\overline{C}_i$'s $\vee$-indecomposable and C_i's strongly $\vee$-indecomposable. Then $m = n$ and, after reindexing, $C_i \simeq \overline{C}_i$ for all i.*

Proof. There are idempotents $e_i = j_i p_i$ and $\bar{e}_i = \bar{j}_i \bar{p}_i \in [G, G]$ corresponding to each decomposition, and

$$\mathbb{1}_{G\sharp} = e_{1\sharp} + \cdots + e_{n\sharp} = \bar{e}_{1\sharp} + \cdots + \bar{e}_{m\sharp}.$$

As G_i's are strongly $\times$-indecomposable, $\operatorname{im}\beta$ is semiperfect by Proposition 2.4. By the same proposition, idempotents $\bar{e}_{i\sharp}$ are primitive.

The decomposition of $\mathbb{1}_{G\sharp}$ in a semiperfect ring $\operatorname{im}\beta$ is then unique in the sense that $m = n$ and there is a unit $u_\sharp \in U(\operatorname{im}\beta)$, such that $\bar{e}_{i\sharp} = u_\sharp^{-1} e_{i\sharp} u_\sharp$ after reindexing. Since $u_\sharp$ is an isomorphism, $u\colon G \to G$ is a homotopy equivalence by the Whitehead theorem. The compositions

$$G_i \xrightarrow{j_i} G \xrightarrow{u^{-1}} G \xrightarrow{\bar{p}_i} \overline{G}_i \qquad \text{and} \qquad \overline{G}_i \xrightarrow{\bar{j}_i} G \xrightarrow{u} G \xrightarrow{p_i} G_i$$

become inverse to each other after applying π_*, so each of them is a homotopy equivalence. □

Observe that in Theorem 2.5 we do not claim that the factors are equivalent as H- or coH-spaces. Moreover, as shown by products of $K(\mathbb{Z}, n)$'s and wedges of spheres, the uniqueness of the factorization does not imply that $\operatorname{im}\beta$ or $\operatorname{im}\alpha$ are semiperfect rings (i.e. the factors of the unique decomposition need not be strongly indecomposable).

Corollary 2.6.

(a) Let G be a finite connected H-space. Then G is a finite product of strongly $\times$-indecomposable spaces if and only if $\pi_*(G)$ is finite.

(b) Let C be a finite simply-connected coH-space. Then C is a finite wedge of strongly $\vee$-indecomposable spaces if and only if $H_*(C)$ is finite.

Proof. Since G is a finite H-space, $\pi_*(G)$ is finitely generated, and $\operatorname{im}\beta$ is also finitely generated as an abelian group.

If $G \simeq G_1 \times \cdots \times G_n$, with G_i's strongly $\times$-indecomposable, then $\operatorname{im}\beta$ is semiperfect by Proposition 2.4. Therefore $\operatorname{im}\beta$ is finite by Example 1.3. For each $\gamma \in \pi_*(G)$, any multiple of γ can be realized by composing γ with the self-map $k \cdot \mathbb{1}_G = \mathbb{1}_G + \cdots + \mathbb{1}_G\colon G \to G$ (where $+$ denotes the H-addition on G and the bracketing may be chosen arbitrarily), and since $\mathbb{1}_{G\sharp} \in \operatorname{im}\beta$ has finite additive order, then γ also has finite order, so $\pi_*(G)$ is finite.

For the reverse implication observe than $\operatorname{im}\beta$ is finite and therefore semiperfect. Now use Proposition 2.4 to obtain the required decomposition of G. □

For a nilpotent, finite type space X, the *Mislin genus* of X is the set $\mathcal{G}(X)$ of all homotopy classes $[Y]$, such that Y is of finite type and $X_{(p)} \simeq Y_{(p)}$ for all primes p, see [12]. Finite H-spaces with $\operatorname{im}\beta$ a semiperfect ring have trivial Mislin genus:

Corollary 2.7.

(a) If G is a finite connected H-space with $\pi_*(G)$ finite then the Mislin genus of G is trivial, i.e. $\mathcal{G}(G) = \{[G]\}$.

(b) If C is a finite simply-connected coH-space with $H_*(C)$ finite then the Mislin genus of C is trivial, i.e. $\mathcal{G}(C) = \{[C]\}$.

Proof. According to [12, Theorem 2], or [26, Theorem 2.12], there is a positive integer k, such that $G^k \simeq Y^k$ for each $[Y] \in \mathcal{G}(G)$. Then Y is an H-space, $\pi_*(Y)$ is finite, and by Corollary 2.6 we have $G \simeq G_1 \times \cdots \times G_n$ and $Y \simeq Y_1 \times \cdots \times Y_m$ with G_i's and Y_i's strongly $\times$-indecomposable. Hence $G_1^k \times \cdots \times G_n^k \simeq Y_1^k \times \cdots \times Y_m^k$, and this decomposition is unique by theorem 2.5. So we must have $km = kn$ and $Y_i \simeq G_i$ after a suitable reordering of Y_i's, which means $Y \simeq G$. □

Corollary 2.7 is a special case of [13, Theorem 2].

3. Decompositions of p-complete and p-local H- and coH-spaces

From the previous section we know that every finite H- or coH-space admits decompositions into indecomposable factors. These decompositions are not necessarily unique, unless we know that the factors are strongly indecomposable. In this section we are going to prove that the difference between 'indecomposable' and 'strongly indecomposable' disappears for finite p-complete and p-local H- and coH-spaces. From this fact we can immediately deduce unique decomposition theorems for such spaces (cf. Freyd [5], Gray [6], Margolis [10], Wilkerson [23] and Xu [24]).

Let us first consider the case of p-complete spaces:

Theorem 3.1. *Let P be a finite set of primes.*

- *(a) Every connected finite P-complete H-space G admits a unique decomposition into strongly $\times$-indecomposable factors.*
- *(b) Every simply-connected finite P-complete coH-space C admits a unique decomposition into strongly $\vee$-indecomposable summands. (The wedge must be interpreted in P-complete sense, i.e. $C \simeq (C_1 \vee \cdots \vee C_n)\widehat{}_P$.)*

Proof. Since $\pi_*(G)$ is a f.g. $\widehat{\mathbb{Z}}_P$-module, $\operatorname{im}\beta_G$ is a finite $\widehat{\mathbb{Z}}_P$-algebra. Hence $\operatorname{im}\beta_G$ is semiperfect by Example 1.1. Now use Proposition 2.4 and Theorem 2.5. □

The comparison with the standard proofs of this theorem as given by K. Xu in [24] and B. Gray in [6] clearly shows the power of semiperfect rings.

To prove the p-local version of the decomposition theorem, we need the following result, which appeared in various forms in the literature, so we give only a brief proof of it.

Lemma 3.2.

- *(a) Let G be a connected p-local H-space, such that $\pi_*(G)$ is a f.g. $\mathbb{Z}_{(p)}$-module. Then the ring $\operatorname{im}\beta_G$ has finite additive index in $\operatorname{End}(\pi_*(G))$.*
- *(b) Let C be a simply-connected p-local coH-space, such that $H_*(C)$ is a f.g. $\mathbb{Z}_{(p)}$-module. Then the ring $\operatorname{im}\alpha_C$ has finite additive index in $\operatorname{End}(H_*(C))$.*

Proof. Let P be a Postnikov section of G such that $\pi_*(P) = \pi_*(G)$. There exists a rational homotopy equivalence $u\colon P \to K$, where K is a finite product of Eilenberg–MacLane spaces $K(\mathbb{Z}_{(p)}, n_j)$, such that $\pi_1(u)$ is an epimorphism. The (graded) homotopy group of the homotopy fibre of u is a finite p-group, let d denote its order. By [8, Proposition 2.13 and Theorem 2.14] the Moore–Postnikov decomposition of u admits a principal refinement

$$P = Y_k \to Y_{k-1} \to \cdots \to Y_1 \to Y_0 = K,$$

such that each $Y_{i+1} \to Y_i$ is a principal fibration induced by a map $Y_i \to K(A_i, m_i)$, where A_i's are finite p-groups. Since d is the product of orders of groups A_i, the map $d \cdot \mathbb{1}_K\colon K \to K$ can be lifted along the tower of principal fibrations yielding a map $\tilde{u}\colon K \to P$, such that $u\tilde{u} = d \cdot \mathbb{1}_K$. As chosen, d will kill the torsion in

$\pi_*(P) = \pi_*(G)$, and letting $v := d \cdot \tilde{u}$, the endomorphism $(vu)_\sharp\colon \pi_*(P) \to \pi_*(P)$ is the multiplication by d^2.

Let $\phi\colon \pi_*(G) \to \pi_*(G)$ be an endomorphism of $\pi_*(G) = \pi_*(P)$. Every endomorphism of $\pi_*(K)$ can be realized by a self-map of K, so there is a $g\colon K \to K$, such that $g_\sharp = u_\sharp \phi v_\sharp$. Let $f\colon G \to G$ be the lifting of the map $vgu\colon P \to P$. Then $f_\sharp = (vu)_\sharp \phi (vu)_\sharp = d^4 \cdot \phi$ on $\pi_*(G)$, and, since $\pi_*(G)$ is finitely generated, the additive index of $\operatorname{im}\beta_G$ in $\operatorname{End}(\pi_*(G))$ is finite. □

Wilkerson's decomposition theorem now follows from Example 1.2 in the same way as Theorem 3.1 followed from Example 1.1.

Theorem 3.3.

(a) Let G be a connected p-local H-space, such that $\pi_*(G)$ is a f.g. $\mathbb{Z}_{(p)}$-module. Then G has a unique decomposition as a product $G \simeq G_1 \times \cdots \times G_n$ of strongly $\times$-indecomposable H-spaces.

(b) Let C be a simply-connected p-local coH-space, such that $H_*(C)$ is a f.g. $\mathbb{Z}_{(p)}$-module. Then C has a unique decomposition as a wedge $C \simeq C_1 \vee \cdots \vee C_n$ of strongly $\vee$-indecomposable coH-spaces.

Proof. Lemma 3.2 and Example 1.2 imply that $\operatorname{im}\beta_G$ is semiperfect. Then, by Proposition 2.4, G has a decomposition into strongly $\times$-indecomposable factors, and by Theorem 2.5 this decomposition is unique. □

We conclude this section with a useful result which essentially says that strong indecomposability of finite p-complete or p-local H- and coH-spaces can be detected by looking only at the mod p data. For a p-complete or p-local space X we consider the homomorphisms

$$\alpha' = \alpha'_X\colon [X,X] \to \operatorname{End}(H_*(X) \otimes \mathbb{Z}/p), \quad \alpha'_X\colon f \mapsto f'_* := f_* \otimes \mathbb{1}_{\mathbb{Z}/p}$$

and

$$\beta' = \beta'_X\colon [X,X] \to \operatorname{End}(\pi_*(X) \otimes \mathbb{Z}/p), \quad \beta'_X\colon f \mapsto f'_\sharp := f_\sharp \otimes \mathbb{1}_{\mathbb{Z}/p}.$$

Proposition 3.4.

(a) Let G be a finite connected p-complete or p-local H-space. Then G is strongly $\times$-indecomposable if and only if $\operatorname{im}\beta'_G$ contains no nontrivial idempotents.

(b) Let C be a finite simply-connected p-complete or p-local coH-space. Then C is strongly $\vee$-indecomposable if and only if $\operatorname{im}\alpha'_C$ contains no nontrivial idempotents.

Proof. Since $\operatorname{im}\beta'$ is finite and has no idempotents other than $\mathbb{0}'_{G\sharp}$ and $\mathbb{1}'_{G\sharp}$, it is local. The homomorphism

$$\operatorname{im}\beta \to \operatorname{im}\beta',\ f_\sharp \mapsto f_\sharp \otimes \mathbb{1}_{\mathbb{Z}/p}$$

is unit-reflecting by Lemma 1.6, so $\operatorname{im}\beta$ is also local by Lemma 1.5. This proves the 'if' part.

For the 'only if' part observe that every quotient of a local ring is a local ring. □

4. Reducibility of self-homotopy equivalences

Given a self-map f of $X_1 \times \cdots \times X_n$ and $I, K \in \{X_1, \ldots, X_n\}$, let $f_I := p_I f$ and $f_{IK} := p_I f j_K$, where p_I is the projection of the product onto I, and j_K is the the inclusion of K into the product as a slice. We will say that a self-homotopy equivalence $f \in \mathrm{Aut}(X_1 \times \cdots \times X_n)$ is *reducible* if $f_{II} \in \mathrm{Aut}(I)$ for all $I \in \{X_1, \ldots, X_n\}$. Denote by $\langle g_1, \ldots, g_n\rangle \colon Z \to X_1 \times \cdots \times X_n$ the unique map determined by maps $g_i \colon Z \to X_i$. By [16, Lemma 2.1], if a self-equivalence $f = \langle f_{X_1}, \ldots, f_{X_n}\rangle$ of $X_1 \times \cdots \times X_n$ is reducible, then $\langle p_{X_1}, f_{X_2}, \ldots, f_{X_n}\rangle$ (and by induction $\langle f_{X_1}, \ldots, f_{X_n}\rangle$ with any number of components substituted by respective projections) is in $\mathrm{Aut}(X_1 \times \cdots \times X_n)$. The reducibility of self-maps of a wedge $X_1 \vee \cdots \vee X_n$ is defined analogously.

The concept of reducibility plays a central role in the study of self-homotopy equivalences of products and wedges. For a thorough discussion of the related results see [17], in particular the Introduction and Section 5. Moreover, in Section 6 of the same paper we studied the reducibility of self-homotopy equivalences of products of atomic spaces. The results are well suited to the study of complete H- and coH-spaces but not for local spaces, because in general the indecomposable p-local H- and coH-spaces are not atomic. We are going to show that results on reducibility can be extended to products and wedges of strongly indecomposable (or even more generally, strongly irreducible) factors. As we have just proved, these results will be in particular valid for all finite p-local H- and simply-connected coH-spaces.

As the reducibility does not depend on the H- or coH-structure, we first introduce a notion, originally due to Mislin [14], which is a generalization of strong indecomposability to arbitrary spaces.

Definition 4.1. Given a space X the set $\mathrm{End}(X) := [X, X]$ is said to be *π_*-local* if, for arbitrary maps $f, g \colon X \to X$, $\pi_*(f) + \pi_*(g) = \mathbb{1}_{\pi_*(X)}$ implies that f or g is a homotopy equivalence. Similarly, the set $\mathrm{End}(X)$ is *H_*-local* if $H_*(f) + H_*(g) = \mathbb{1}_{H_*(X)}$ implies that f or g is a homotopy equivalence.

A space X is *irreducible* if it has no nontrivial retracts. We call X *strongly π_*-irreducible* if it is irreducible and $\mathrm{End}(X)$ is π_*-local, and *strongly H_*-irreducible* if it is irreducible and $\mathrm{End}(X)$ is H_*-local.

The following proposition is an immediate consequence of results proved in Section 2.

Proposition 4.2.

(a) *For a connected H-space G the following properties are equivalent:*
 – *G is strongly $\times$-indecomposable;*
 – *G is strongly π_*-irreducible;*
 – $\mathrm{End}(G)$ *is π_*-local.*

(b) *For a simply-connected coH-space C the following properties are equivalent:*
 – *C is strongly $\vee$-indecomposable;*
 – *C is strongly H_*-irreducible;*
 – $\mathrm{End}(C)$ *is H_*-local.*

Local spheres $S^n_{(p)}$ provide examples of strongly π_*-irreducible, which are in general not H-spaces, while the local Eilenberg–MacLane spaces $K(\mathbb{Z}_{(p)}, n)$ are strongly H_*-irreducible spaces, which are not coH-spaces.

The role of strong irreducibility is clear from the following results.

Lemma 4.3.

(a) If X is a retract of the product $Y \times Z$ and $\mathrm{End}(X)$ *is π_*-local, then X is a retract of Y or Z.*

(b) If X is a retract of the wedge $Y \vee Z$ and $\mathrm{End}(X)$ *is H_*-local, then X is a retract of Y or Z.*

Proof. Let $i\colon X \to Y \times Z$ and $r\colon Y \times Z \to X$ be such that $ri = \mathbb{1}_X$. Then

$$\mathbb{1}_{X\sharp} = r_\sharp i_\sharp = r_\sharp (j_{Y\sharp} p_{Y\sharp} + j_{Z\sharp} p_{Z\sharp}) i_\sharp = (r j_Y p_Y i)_\sharp + (r j_Z p_Z i)_\sharp.$$

Since $[X,X]$ is π_*-local, at least one of the compositions

$$r j_Y p_Y i\colon X \to Y \to X \quad \text{or} \quad r j_Z p_Z i\colon X \to Z \to X$$

is a homotopy equivalence, therefore X is a retract of Y or Z. □

Corollary 4.4.

(a) Let X and $Y_1, \ldots, Y_k$ be strongly π_*-irreducible spaces. If X is a retract of $Y_1 \times \cdots \times Y_k$, then $X \simeq Y_i$ for some i.

(b) Let X and $Y_1, \ldots, Y_k$ be strongly H_*-irreducible spaces. If X is a retract of $Y_1 \vee \cdots \vee Y_k$, then $X \simeq Y_i$ for some i.

In the next lemma we deal with self-maps of X^n and $X^{\vee n}$. Let $p_i\colon X^n \to X$ be the projection onto the i^{th} factor and $j_i\colon X \to X^n$ be the inclusion of the i^{th} factor. For an $f\colon X^n \to X^n$, let $f_{ik} = p_i f j_k\colon X \to X$. The same notation will be used for $\vee$-analogues.

Lemma 4.5.

(a) Let X be a connected space with $\mathrm{End}(X)$ *π_*-local and let f be a self-map of X^n. If f_{ii} is an equivalence for each i and f_{ik} is not an equivalence whenever $i \neq k$, then $f \in$* $\mathrm{Aut}(X^n)$.

(b) Let X be a simply-connected space with $\mathrm{End}(X)$ *H_*-local and let f be a self-map of $X^{\vee n}$. If f_{ii} is an equivalence for each i and f_{ik} is not an equivalence whenever $i \neq k$, then $f \in$* $\mathrm{Aut}(X^{\vee n})$.

Proof. (a): The proof is by induction on n. For $n = 1$ the lemma is obvious. We will write $X' = X^{n-1}$, $X^n = X \times X'$ and identify $\pi_*(X^n) = \pi_*(X \times X')$ with $\pi_*(X) \oplus \pi_*(X')$. The homomorphism induced by $f\colon X \times X' \to X \times X'$ can then be written in matrix form

$$\begin{pmatrix} f_{XX\sharp} & f_{XX'\sharp} \\ f_{X'X\sharp} & f_{X'X'\sharp} \end{pmatrix} = \begin{pmatrix} f_{XX\sharp} & 0 \\ f_{X'X\sharp} & \phi \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1}_{X\sharp} & f_{XX\sharp}^{-1} f_{XX'\sharp} \\ 0 & \mathbb{1}_{X'\sharp} \end{pmatrix},$$

where $\phi = f_{X'X'\sharp} - f_{X'X\sharp} f^{-1}_{XX\sharp} f_{XX'\sharp}$. We will write this matrix product as $F = L \cdot U$.

There are several things to notice. The matrix U is invertible;

$$U^{-1} = \begin{pmatrix} \mathbb{1}_{X\sharp} & -f^{-1}_{XX\sharp} f_{XX'\sharp} \\ 0 & \mathbb{1}_{X'\sharp} \end{pmatrix},$$

and induced by the map $u := \langle f^{-1}_{XX} p_X f, p_{X'} \rangle \colon X \times X' \to X \times X'$, hence u is a homotopy equivalence by the Whitehead theorem. The matrix L is then induced by the map $\ell := fu^{-1} \colon X \times X' \to X \times X'$, and we conclude that ϕ is induced by $\ell_{X'X'}$, i.e. $\phi = \ell_{X'X'\sharp}$. Hence

$$f_{X'X'\sharp} = (f_{X'X} f^{-1}_{XX} f_{XX'})_\sharp + \ell_{X'X'\sharp}.$$

Also, since $u^{-1}_\sharp = U^{-1}$, we have that $-f^{-1}_{XX\sharp} f_{XX'\sharp} = (u^{-1})_{XX'\sharp}$, and we can rearrange above equation to read

$$\ell_{X'X'\sharp} = (f_{X'X}(u^{-1})_{XX'})_\sharp + f_{X'X'\sharp}.$$

Note that $f_{XX} = f_{11}$, and composing the last two equations with $p_{i\sharp}$ from the left and $j_{k\sharp}$ from the right for $i, k \geq 2$ yields

$$f_{ik\sharp} = (f_{i1} f_{11}^{-1} f_{1k})_\sharp + \ell_{ik\sharp} \quad \text{and} \quad \ell_{ik\sharp} = (f_{i1}(u^{-1})_{1k})_\sharp + f_{ik\sharp}.$$

For $i \geq 2$, the map $f_{i1} \colon X \to X$ is not an equivalence, therefore $f_{i1} f_{11}^{-1} f_{1k}$ and $f_{i1}(u^{-1})_{1k}$ are not self-equivalences of X. Since $\mathrm{End}(X)$ is π_*-local, we thus conclude that either f_{ik} and ℓ_{ik} are both equivalences, or are both *not* equivalences. Hence $\ell_{X'X'} \colon X^{n-1} \to X^{n-1}$ also satisfies the assumptions of the lemma, and is therefore an equivalence by our inductive hypothesis. But then the matrix L in the product $F = L \cdot U$ is also invertible, so f is a self-equivalence of X^n by the Whitehead theorem.

(b): This time we also sketch the proof of the dual case, which, although strictly dual, is not done by simply replacing π_* with H_*.

Put $X' = X^{\vee(n-1)}$, $X^{\vee n} = X \vee X'$ and identify $H_*(X^{\vee n}) = H_*(X \vee X')$ with $H_*(X) \oplus H_*(X')$. We then write $f_* \colon H_*(X \vee X') \to H_*(X \vee X')$ in matrix form

$$\begin{pmatrix} f_{XX*} & f_{XX'*} \\ f_{X'X*} & f_{X'X'*} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{X*} & 0 \\ f_{X'X*} f^{-1}_{XX*} & \mathbb{1}_{X'*} \end{pmatrix} \cdot \begin{pmatrix} f_{XX*} & f_{XX'*} \\ 0 & \phi \end{pmatrix},$$

where $\phi = f_{X'X'*} - f_{X'X*} f^{-1}_{XX*} f_{XX'*}$. Again, we write this product as $F = L \cdot U$.

The matrix L is invertible;

$$L^{-1} = \begin{pmatrix} \mathbb{1}_{X*} & 0 \\ -f_{X'X*} f^{-1}_{XX*} & \mathbb{1}_{X'*} \end{pmatrix},$$

and induced by the map $\ell := \langle f j_X f^{-1}_{XX}, j_{X'} \rangle \colon X \vee X' \to X \vee X'$, hence ℓ is an equivalence. The matrix U is then induced by $u := \ell^{-1} f \colon X \vee X' \to X \vee X'$, so ϕ is induced by $u_{X'X'}$. Hence

$$f_{X'X'*} = (f_{X'X} f^{-1}_{XX} f_{XX'})_* + u_{X'X'*}.$$

Since $\ell_*^{-1} = L^{-1}$, we have $-f_{X'X*}f_{XX*}^{-1} = (\ell^{-1})_{X'X*}$, and we also obtain

$$u_{X'X'*} = ((\ell^{-1})_{X'X}f_{XX'})_* + f_{X'X'*}.$$

From this point, we follow the same argument as in the proof of (a), swapping the roles of ℓ and u, and replacing π_*-local with H_*-local. □

Proposition 4.6.

(a) *Let X be a connected space with* $\mathrm{End}(X)$ *π_*-local. If X is not a retract of Y, then the self-homotopy equivalences of $X^n \times Y$ are reducible, i.e.* $f_{X^nX^n} \in \mathrm{Aut}(X^n)$ *and* $f_{YY} \in \mathrm{Aut}(Y)$ *for each* $f \in \mathrm{Aut}(X^n \times Y)$.

(b) *Let X be a simply-connected space with* $\mathrm{End}(X)$ *H_*-local. If X is not a retract of Y, then the self-homotopy equivalences of $X^{\vee n} \vee Y$ are reducible, i.e.* $f_{X^{\vee n}X^{\vee n}} \in \mathrm{Aut}(X^{\vee n})$ *and* $f_{YY} \in \mathrm{Aut}(Y)$ *for each* $f \in \mathrm{Aut}(X^{\vee n} \vee Y)$.

Proof. Besides $p_{X^n}\colon X^n \times Y \to X^n$ and $p_Y\colon X^n \times Y \to Y$, we also denote by $p_i\colon X^n \times Y \to X$ the projection onto the i^{th} X-factor. Denote the corresponding inclusions by j_{X^n}, j_Y and j_i. For each i, k, and $f \in \mathrm{Aut}(X^n \times Y)$ we then have

$$\begin{aligned} p_{i\sharp}j_{k\sharp} &= p_{i\sharp}f_\sharp f_\sharp^{-1}j_{k\sharp} = p_{i\sharp}f_\sharp(j_{X^n\sharp}p_{X^n\sharp} + j_{Y\sharp}p_{Y\sharp})f_\sharp^{-1}j_{k\sharp} \\ &= f_{iX^n\sharp}(f^{-1})_{X^nk\sharp} + f_{iY\sharp}(f^{-1})_{Yk\sharp}. \end{aligned}$$

Note that $p_{i\sharp}j_{k\sharp} = \mathbb{0}_{X\sharp}$ if $i \neq k$, and $p_{i\sharp}j_{i\sharp} = \mathbb{1}_{X\sharp}$. Since X is not a retract of Y, $f_{iY}(f^{-1})_{Yk}\colon X \to Y \to X$ is not an equivalence. Hence $f_{iX^n}(f^{-1})_{X^nk}$ is not an equivalence if $i \neq k$, and $f_{iX^n}(f^{-1})_{X^ni}$ is an equivalence for each i, because $\mathrm{End}(X)$ is π_*-local. It follows from Lemma 4.5 that $f_{X^nX^n}(f^{-1})_{X^nX^n}$ is a self-equivalence of X^n, so $f_{X^nX^n} \in \mathrm{Aut}(X^n)$. Now, [17, Proposition 2.1] states that self-homotopy equivalences of $W \times Y$ are reducible if and only if $f \in \mathrm{Aut}(W \times Y)$ implies $f_{WW} \in \mathrm{Aut}(W)$, so we are done. □

The following is the main result of this section.

Theorem 4.7.

(a) *If $X_1, \ldots, X_k$ are mutually inequivalent, strongly π_*-irreducible spaces, then the self-homotopy equivalences of $X_1^{n_1} \times \cdots \times X_k^{n_k}$ are reducible.*

(b) *If $X_1, \ldots, X_k$ are mutually inequivalent, simply-connected, strongly H_*-irreducible spaces, then the self-homotopy equivalences of $X_1^{\vee n_1} \vee \cdots \vee X_k^{\vee n_k}$ are reducible.*

Proof. We can assume by induction that the self-equivalences of $X_2^{n_2} \times \cdots \times X_k^{n_k}$ are reducible. By Corollary 4.4 X_1 is not a retract of $X_2^{n_2} \times \cdots \times X_k^{n_k}$, so by Proposition 4.6 the self-homotopy equivalences of $X_1^{n_1} \times X_2^{n_2} \times \cdots \times X_k^{n_k}$ are reducible. □

By Theorems 3.1 and 3.3, every finite p-complete or p-local connected H-space G admits a unique decomposition as a product of strongly $\times$-indecomposable factors. By collecting together equivalent factors we can write $G \simeq G_1^{n_1} \times \cdots \times G_k^{n_k}$, where each G_i is strongly π_*-irreducible. Similarly, every finite p-complete or p-local simply-connected coH-space C can be decomposed as $C \simeq C_1^{\vee n_1} \vee \cdots \vee C_k^{\vee n_k}$, where the summands C_i are strongly H_*-irreducible and mutually inequivalent.

Corollary 4.8.

(a) If G is a finite p-complete or p-local connected H-space, then the self-homotopy equivalences of G are reducible with respect to the decomposition $G \simeq G_1^{n_1} \times \cdots \times G_k^{n_k}$.

(b) If C is a finite p-complete or p-local simply-connected coH-space, then the self-homotopy equivalences of C are reducible with respect to the decomposition $C \simeq C_1^{\vee n_1} \vee \cdots \vee C_k^{\vee n_k}$.

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