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Fibrations between mapping spaces

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ABSTRACT

We study the behaviour of fibre maps under exponentiation, i.e. given a fibration $p: E \to B$ we ask for which spaces X is the induced map between mapping spaces $p_*: E^X \to B^X$ also a fibration. If X is a locally compact space, the positive answer follows easily by the exponential law so in this paper we consider more general spaces and show that the preservation of fibrations is related to the local homotopy properties of the space X. For example, if p is a Dold fibration and X admits a deformation retraction on a compactly generated space, then the induced map p_* is also a Dold fibration. Similar results hold for Hurewicz fibrations with unique path-lifting and for covering spaces, and can be furthermore extended to spaces that admit some numerable cover, whose elements preserve fibration property.

1. Introduction

Given a map $p: E \to B$ and a space X we will denote by E^X and B^X the spaces of continuous maps from X to E and B endowed with the compact-open topology and by $p_*: E^X \to B^X$ the induced map $p_*: f \mapsto p \circ f$. It is often important to know which properties of p are shared by p_*. In this paper we study spaces that *preserve* fibrations, in the sense that if p belongs to some class of fibrations (e.g. Hurewicz fibrations, Dold fibrations or covering spaces) then the induced map p_* belongs to the same class.

1.1. Prior work

The basic case arises when X is locally compact and $p: E \to B$ is a Hurewicz fibration (i.e. it has the homotopy lifting property for arbitrary spaces – cf. [5]). Then we may use the exponential law to transform the homotopy lifting diagram

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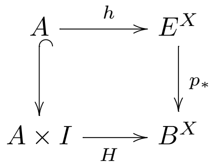
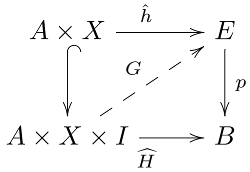


into the diagram



The map G exists by the homotopy lifting property of p, and its adjoint $\widehat{G}: A \times I \to E^X$ yields the lifting of H, which proves that p_* is also a Hurewicz fibration. Smrekar [7,8] was able to extend the above argument to the case where X is any compactly generated space. However, the direct approach breaks down when X is not compactly generated as in that case the exponential law is not available. Of course, a possible way out would be to work in the context of k-spaces (see [5]), but that requires a modification of topologies on products and mapping spaces, and actually answers a different question. In a related work Apery [1] considered covering spaces and showed that they are preserved by contractible spaces, compact CW-complexes and (possibly infinite) graphs.

1.2. Our contribution

All results of this work are derived in one way or the other from Lemma 2.3 which can be roughly stated as follows: if X has a deformation retract that preserves a class of fibrations, then X preserves the same class of fibrations, up to a homotopy. The precise statement and the proof are quite technical, as they involve a roundabout definition of a lifting function, and a delicate proof of its continuity. Lemma 2.3 has various consequences that are thoroughly studied in the second part of the paper, where we apply the lemma to several important classes of fibrations.

The first comprehensive class of fibrations that we consider are Hurewicz fibrations with unique path lifting (see [9]) and we prove Theorem 3.2: if $p: E \to B$ is a Hurewicz fibration with unique path-lifting, and if X admits a numerable open or a locally finite closed cover such that each of its elements can be deformed to a compactly generated subset, then $p_*: E^X \to B^X$ is also a Hurewicz fibration with unique path lifting. In particular, these fibrations are preserved by all locally contractible spaces or more generally, by all spaces that are locally homotopy equivalent to a compactly generated space.

Then we study classical covering spaces, which can be viewed as locally trivial Hurewicz fibrations with discrete fibres. Local triviality and the topology of the fibre are both affected by basic operations on fibrations like compositions, products and construction of mapping spaces (cf. [9]). Nevertheless, in Theorem 3.8 we prove that every (strongly) locally contractible space X with finitely many path-components preserves all covering spaces.

Since our construction of the lifting function is based on a homotopy deformation, it is not surprising that for general fibrations we obtain a lifting only up to a homotopy. That is why a more natural context for our approach is given by Dold fibrations (see [5, Section 1.2]): our main result is Theorem 3.13 stating that Dold fibrations are preserved by every X which admits a finite numerable covering, whose elements can be deformed to subspaces that preserve Dold-fibrations.

1.3. Outline

The main body of the paper is divided in two parts. We begin the first part by recalling standard characterizations in terms of lifting functions of various classes of fibrations. We then construct a function that will serve as a lifting function for fibrations between mapping spaces and prove that it is well-defined and continuous. In the second part we discuss three main types of fibrations, namely Hurewicz fibrations with unique path lifting, covering spaces and Dold fibrations, and prove corresponding preservation theorems for each of these classes.

2. Construction of a lifting function

Every map $p: E \to B$ induces a map $\pi: E^I \to E \times B^I$, $\pi: \alpha \mapsto (\alpha(0), p \circ \alpha)$. The map π is not surjective, in fact its image is the subspace

$$E \sqcap B^I := \{(e, \alpha) \in E \times B^I \mid p(e) = \alpha(0)\} \subset E \times B^I.$$

A *lifting function* for p is a section of π, that is, a map $\Gamma: E \sqcap B^I \to E^I$ such that $\pi \circ \Gamma$ is the identity map on $E \sqcap B^I$. Several classes of fibrations can be characterized in terms of lifting functions: in particular p is a Hurewicz fibration if and only if it admits a continuous lifting function Γ, and p is a fibration with unique path-lifting property if and only if Γ is a homeomorphism. We postpone a slightly more involved description of lifting functions that define Dold fibrations to Section 3.3.

Remark 2.1. The image of a fibration map is always a union of path components of the base, so in the theory of fibrations one often assumes that the base space B is path-connected and that the projection p is surjective. However, the mapping spaces are usually path disconnected so it is more adequate for our discussion to allow the possibility that B has several path-components and that some of them are not contained in the image of p. Observe that characterization by means of lifting functions is not affected, because only paths in B that are contained in the image of p appear in $E \sqcap B^I$.

If $p: E \to B$ is a Hurewicz fibration then its lifting function $\Gamma: E \sqcap B^I \to E^I$ induces a continuous map $\Gamma_*: (E \sqcap B^I)^X \to (E^I)^X$. The covariant mapping space functors commute with pullbacks (see [7, Lemma A.3]), hence $(E \sqcap B^I)^X$ is homeomorphic to $E^X \sqcap (B^I)^X$. On the other hand, if X is locally compact (or even compactly generated) then $(B^I)^X$ and $(E^I)^X$ are respectively homeomorphic to $(B^X)^I$ and $(E^X)^I$, and so Γ_* may be viewed as a lifting function for $p_*: E^X \to B^X$ (this is essentially the approach followed by [7]). However, if X is not compactly generated we run into trouble, as it is not clear at all why $\Gamma_*: E^X \sqcap (B^X)^I \to (E^X)^I$ should be continuous. Thus for more general spaces we need a different approach.

In this paper we will not require the exponent space X to have any particular topological property, the only restrictions on X will be of homotopy theoretical nature. Some mild topological assumptions are nevertheless needed to guarantee that the composition of maps is continuous as a function of two variables. Therefore we will assume throughout that the spaces E and B are regular (topology separates points from closed sets) but not necessarily Hausdorff. Then we may freely use the fact [4, Proposition 2.98] (adapted as in [8, Proposition 3], see also [6, Lemma 2.6]) that the composition map

$$\circ: B^L \times L^Y \to B^Y \tag{1}$$

is continuous, whenever Y is (locally) compact (observe that the continuity of composition is usually stated under more stringent assumptions).

Let $p: E \to B$ be a Hurewicz fibration with lifting function $\Gamma: E \sqcap B^I \to E^I$. We will say that X is a p *preserving extension* if there exists a subspace $A \subset X$ such that the following assumptions are satisfied:

(E1) A is a deformation retract of X, and
(E2) A *preserves* p, i.e. $p_*: E^A \to B^A$ is a Hurewicz fibration.

Every space that admits a compactly generated deformation retract (in particular, every contractible space) is a p preserving extension for all fibrations p.

By assumption (E1) there is a retraction $r: X \to A$ which is a left inverse to the inclusion $i: A \hookrightarrow X$ and there is a homotopy $K: X \times I \to X$ between $i \circ r$ and the identity on X. We will denote by $\widehat{K}: X \to X^I$ the (continuous) adjoint function of K. Furthermore, by assumption (E2), there is a continuous lifting function $\Gamma^A: E^A \sqcap (B^A)^I \to (E^A)^I$.

We are going to define a function

$$\Gamma^X: E^X \sqcap \left(B^X\right)^I \to \left(E^X\right)^I$$

by an explicit formula: given $f: X \to E$ and $H: I \to B^X$ that satisfy the compatibility relation $p \circ f = H(0)$ let $\Gamma^X(f, H) \in (E^X)^I$ be given as

$$\left[\Gamma^X(f,H)(t)\right](x) := \Gamma\left(\left[\Gamma^A\left(f \circ i, H(-) \circ i\right)(t)\right]\left(r(x)\right), H(t) \circ \widehat{K}(x)\right)(1). \tag{2}$$

For the benefit of the reader we attempt an explanation of this complicated expression. First step is the compression of X to A by means of the retraction r. Then the paths in B that are described by points $ir(x) \in A$ along H are lifted by Γ^A to paths in E originating at $f(ir(x))$. Thus we get a function that correctly lifts the restrictions of f and H over A. In the final step we construct the extension over entire X by lifting the paths given by the deformation K. The basic motivation for this construction is that in the original approach the main problem was to prove that Γ^X is well-defined and continuous. Our alternative approach avoids this difficulty because in the formula (2) the lifting function is applied only to arguments in A, where it is continuous by assumption (E2).

Lemma 2.2. *Γ^X is correctly defined, i.e. the expression (2) defines a function from $E^X \sqcap (B^X)^I$ to $(E^X)^I$.*

Proof. We first check that expression (2) makes sense for every $(f, H) \in E^X \sqcap (B^X)^I$: in fact $p \circ f \circ i = H(0) \circ i$ so Γ^A is defined. Furthermore

$$p\left(\left[\Gamma^A\left(f \circ i, H(-) \circ i\right)(t)\right]\left(r(x)\right)\right) = H(t)\left(ir(x)\right) = H(t)\left(\widehat{K}(x)(0)\right),$$

so the 'external' Γ is also defined.

It remains to verify that the function $\Gamma^X(f, H)$ maps I into continuous functions from X to E, and that $\Gamma^X(f, H): I \to E^X$ is continuous. The first claim is obvious as $\Gamma^X(f, H)(t)$ is clearly a composition of continuous functions of the variable x. In order to prove the second claim we consider the adjoint function $G: I \times X \to E$, given by $G(t, x) := [\Gamma^X(f, H)(t)](x)$. In general G may not be continuous. In fact, by the definition of the compact-open topology, $\Gamma^X(f, H)$ is continuous if and only if the restrictions $G|_{\{t\} \times X}$ for every $t \in I$ and $G|_{I \times C}$ for every compact subset $C \subset X$ are continuous.

We first consider the continuity of $G|_{\{t\} \times X}$: for fixed t the functions

$$\Gamma^A\left(f \circ i, H(-) \circ i\right)(t): A \to E \quad \text{and} \quad H(t): X \to B$$

are continuous, and thus so are the maps $x \mapsto [\Gamma^A(f \circ i, H(-) \circ i)(t)](r(x))$ and $x \mapsto H(t) \circ \widehat{K}(x)$, being compositions with fixed continuous maps. We conclude that $x \mapsto G(t, x)$ is continuous for every $t \in I$.

To prove the continuity of $G|_{I\times C}$ we fundamentally use the compactness of C. First observe that the map

$$(t,x) \mapsto \big[\Gamma^A\big(f\circ i, H(-)\circ i\big)(t)\big]\big(r(x)\big)$$

is continuous, because it can be decomposed as

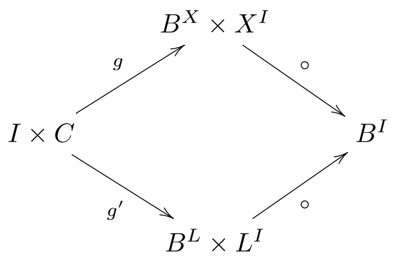
$$I\times C \xrightarrow{\Gamma^A(f\circ i,H(-)\circ i)\times r} E^A\times r(C) \longrightarrow E^{r(C)}\times r(C) \longrightarrow E,$$

where $E^A \to E^{r(C)}$ is the continuous restriction map, and the compactness of $r(C)$ implies the continuity of the evaluation $E^{r(C)}\times r(C)\to E$.

On the other hand, the map

$$I\times C \to B^I, \qquad (t,x)\mapsto H(t)\circ \widehat{K}(x)$$

is continuous because it may be factored as follows. Since C is compact, then so is $L := K(C\times I)$, which proves that $\widehat{K}(x)\in L^I$. Thus we have a commutative diagram



where $g(t,x) = (H(t),\widehat{K}(x))$ and $g'(t,x) = (H(t)|_L,\widehat{K}(x))$ are both continuous, and the composition map $\circ\colon B^L\times L^I \to B^I$ is continuous because L is compact. We conclude that $G|_{I\times C}$ is continuous because all maps that appear in its definition are continuous. □

Lemma 2.3. *The function Γ^X is continuous.*

Proof. The idea of the proof is similar as in the previous lemma. To prove the continuity of Γ^X it is sufficient to prove the continuity of its adjoint map $E^X \sqcap (B^X)^I\times I\to E^X$, and for this it is sufficient to show that the function

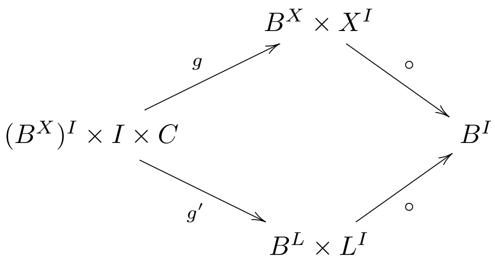
$$G\colon E^X\sqcap \big(B^X\big)^I\times I\times X\to E, \qquad G\colon (f,H,t,x)\mapsto \big[\Gamma^X(f,H)(t)\big](x)$$

becomes continuous when it is restricted over sets of the form $\{(f,H,t)\}\times X$ and $E^X\sqcap (B^X)^I\times I\times C$ for every compact subset $C\subset X$.

The continuity of G for fixed values of f,H and t was already verified in the proof of the previous lemma. For the second claim it is sufficient to prove the continuity of the function

$$\big(B^X\big)^I\times I\times C\to B^I, \qquad (H,t,x)\mapsto H(t)\circ\widehat{K}(x).$$

Similarly as before we observe that $L := K(C\times I)$ is compact, so the above function may be factorized as



where $g(H,t,x) = (H(t), \widehat{K}(x))$ and $g'(H,t,x) = (H(t)|_L, \widehat{K}(x))$ are both continuous since they involve evaluation over the compact I, and the composition $\circ: B^L \times L^I \to B^I$ is continuous because L is compact. □

3. Fibrations

The results of the previous section can be used to construct mappings Γ^X whenever X has a cover whose elements can be deformed into spaces that preserve fibrations, for instance into compactly generated ones. However, there is a price to pay for the use of a deformation homotopy to attain the continuity of Γ^X. In fact, it turns out that for general fibrations the map Γ^X may not be a lifting function in the strict sense, but only up to a vertical homotopy. In this section we first consider two important cases where Γ^X is a strict lifting function, and then we turn to the general case which has a natural interpretation in terms of Dold fibrations.

3.1. Fibrations with unique path lifting

Fibrations with unique path lifting property are a natural extension of the concept of covering spaces, so for example, most of the results in the chapter dedicated to covering spaces in Spanier's classical textbook [9] are stated in these more general terms. We have already mentioned that a map $p: E \to B$ is a (Hurewicz) fibration if and only if the projection

$$\pi: E^I \to E \sqcap B^I, \qquad \pi: \alpha \mapsto \big(p \circ \alpha, \alpha(0)\big)$$

admits a continuous section $\Gamma: E \sqcap B^I \to E^I$. A fibration has unique path lifting if for each $(e, \alpha) \in E \sqcap B^I$ there is a unique path $\widetilde{\alpha}: I \to E$ starting at e and lifting α. Clearly, unique path lifting is equivalent to π being injective, and hence a homeomorphism. It is well-known that every covering space is a fibration with unique path-lifting (cf. [9, p. 69]), and if the base space B is 'nice' (locally path-connected and semi-locally simply connected) the coverings and fibrations with unique path lifting and locally path-connected total space E coincide – see [9, Theorem 2.4.10]. However, over more general spaces the theory of fibrations with unique path lifting is richer and has more pleasant properties than covering spaces. In particular, fibrations with unique path lifting are preserved by arbitrary products, compositions and inverse limits, which is not the case for covering spaces (cf. [9, Section 2.2]). One of the main motivations for this paper was to determine when fibrations with unique path lifting are preserved by the mapping space construction.

Proposition 3.1. *Let $p: E \to B$ be a fibration with unique path lifting, and let X be a p preserving extension. Then $p_*: E^X \to B^X$ is a fibration with unique path lifting.*

Proof. By the results of the previous section it is sufficient to check that $\Gamma^X: E^X \sqcap (B^X)^I \to (E^X)^I$ is the inverse of $\pi: (E^X)^I \to E^X \sqcap (B^X)^I$. We first compute

$$\pi\big(\Gamma^X(f,H)\big) = \big(\Gamma^X(f,H)(0), p_* \circ \Gamma^X(f,H)\big)$$

Since $H(0) = p \circ f$ we have

$$\big[\Gamma^X(f,H)(0)\big](x) = \Gamma\big(fir(x), p \circ f \circ \widehat{K}(x)\big)(1),$$

which is the end point of the path that begins at $fir(x)$ and whose projection is a path in B from $p(fir(x))$ to $p(f(x))$. As p is a fibration with unique path lifting, we conclude that $\Gamma^X(f,H)(0) = f$. Furthermore

$$\big[p_* \circ \Gamma^X(f,H)(t)\big](x) = H(t) \circ \widehat{K}(x)(1) = H(t)(x),$$

so $p_* \circ \Gamma^X(f,H) = H$, and therefore $\pi(\Gamma^X(f,H)) = (f,H)$.

On the other hand, for $F \in (E^X)^I$

$$\begin{aligned}\big[\Gamma^X\big(\pi(F)\big)(t)\big](x) &= \big[\Gamma^X\big(F(0), p_* \circ F\big)(t)\big](x)\\ &= \Gamma\big(\big[\Gamma^A\big(F(0)\circ i, p_* \circ F(-)\circ i\big)(t)\big]\big(r(x)\big), p_* \circ F(t) \circ \widehat{K}(x)\big)(1)\\ &= \Gamma\big(F(t)\big(ir(x)\big), p_* \circ F(t) \circ \widehat{K}(x)\big)(1) = F(t)(x),\end{aligned}$$

again because p has unique path liftings. It follows that $\Gamma^X(\pi(F)) = F$. □

Let us now assume that X has an open cover $\mathcal{U}$ whose elements are p preserving extensions. For each $(f,H) \in E^X \sqcap (B^X)^I$ and $U \in \mathcal{U}$ we have $(f|_U, H(-)|_U) \in E^U \sqcap (B^U)^I$, so we may compute $\Gamma^U(f|_U, H(-)|_U) \in (E^U)^I$. Moreover, whenever $U, V \in \mathcal{U}$ have non-empty intersection, then the uniqueness of liftings implies that $\Gamma^U(f|_U, H(-)|_U)(-)|_{U\cap V} = \Gamma^V(f|_V, H(-)|_V)(-)|_{U\cap V}$. It follows that the partial liftings Γ^U uniquely define a function $\Gamma^X: E^X \sqcap (B^X)^I \to (E^X)^I$. A similar argument can be used if instead of an open cover we consider a closed, locally finite cover $\mathcal{U}$. It remains to prove the continuity of Γ^X. We consider the case of locally finite closed covers first.

Theorem 3.2 *(Preservation of fibrationswith unique path lifting). If $p: E \to B$ is a fibration with unique path lifting, and if $\mathcal{U}$ is a locally finite closed cover of X whose elements are p preserving extensions, then $p_*: E^X \to B^X$ is also a fibration with unique path lifting.*

Proof. In order to prove that $\Gamma^X: E^X \sqcap (B^X)^I \to (E^X)^I$ is continuous, let (f,H) be a point in $E^X \sqcap (B^X)^I$, such that $\Gamma^X(f,H)$ is contained in the subbasic open set $\langle K, \langle C, W\rangle\rangle$, where $K \subseteq I$ and $C \subseteq X$ are compact and $W \subseteq E$ is open. By the local finiteness of the cover $\mathcal{U}$ there exists a finite subcover $U_1, \ldots, U_n$ for the compact set C. For $x \in U_i \cap C$ the value of $[\Gamma^X(f,H)(t)](x)$ is defined by Γ^{U_i} so for $i = 1, \ldots, n$ we have $[\Gamma^X(f,H)(K)](U_i \cap C) = [\Gamma^{U_i}(f|_{U_i}, H|_{U_i})(K)](U_i \cap C) \subseteq W$. As Γ^{U_i} are continuous, there are neighbourhoods V_i for (f,H) such that $\Gamma^X(V_i) \subseteq \langle K, \langle U_i \cap C, W\rangle\rangle$. Then clearly $V := V_1 \cap \ldots \cap V_n$ is a neighbourhood of (f,H) such that $\Gamma^X(V) \subseteq \langle K, \langle C, W\rangle\rangle$, which concludes the proof of the continuity of Γ^X. □

As we see from the above proof, to show the continuity of Γ^X one must be able to decompose every compact subset of X as a union of compact sets subordinated to $\mathcal{U}$, which may not be the case for a general open cover of X. Nevertheless, if $\mathcal{U}$ is numerable, then by a theorem of Derwent [2] there exists a locally finite partition of unity whose supports refine $\mathcal{U}$ so that Theorem 3.2 applies. Another sufficient condition is that every compact subset of X is normal (topology separates closed subsets, which is automatically satisfied if X is Hausdorff). Thus we obtain the following result.

Corollary 3.3. *Let $p: E \to B$ be a fibration with unique path lifting, and let $\mathcal{U}$ be an open cover of X whose elements are p preserving extensions. If $\mathcal{U}$ is numerable or if X is Hausdorff, then $p_*: E^X \to B^X$ is also a fibration with unique path lifting.*

Although the formulation of the above results is relative to the fibration p, our main interest is in the preservation of all fibrations, and that is covered by the following result.

Corollary 3.4. *Assume that X has a cover $\mathcal{U}$ such that each element of $\mathcal{U}$ can be deformed to some compactly generated subspace, and assume furthermore that $\mathcal{U}$ is numerable open or locally finite closed, or that X is Hausdorff. Then X preserves all fibrations with unique path lifting. In particular, every locally contractible Hausdorff space preserves fibrations with unique path lifting.*

3.2. Covering spaces

A fibration with unique path lifting $p: E \to B$ is a covering space if it is locally trivial and has discrete fibres. More precisely, for every $b \in B$ the fibre $p^{-1}(b)$ is a discrete subspace of E, and b has an open neighbourhood $U \subset B$ such that $p^{-1}(U)$ is homeomorphic to $p^{-1}(b) \times U$ by a homeomorphism that commutes with the projection to U. Both the local triviality and the discreteness of the fibre are often destroyed by various operations like compositions, (infinite) products and inverse limits (cf. [9, Section 2.2]). The following example shows that the same problem arises for mapping spaces.

Example 3.5. Let $p: \mathbb{R} \to S^1$ be the standard covering map, given by $p(t) := e^{it}$, and let $\mathbb{N}$ denote the set of natural numbers endowed with the discrete topology. Then $\mathbb{N}$ is locally compact, so the induced map $p_*: \mathbb{R}^{\mathbb{N}} \to (S^1)^{\mathbb{N}}$ is obviously a fibration with unique path lifting. However, p_* is not a covering space, because its fibres are clearly non-discrete, being infinite products of non-trivial spaces. Neither it is locally trivial, as every metric ball around a point in $(S^1)^{\mathbb{N}}$ has non-trivial fundamental group, therefore it cannot be homeomorphic to a component of its preimage in $\mathbb{R}^{\mathbb{N}}$.

Let x be a point in X, U an open subset of B and $\widetilde{U}$ and open subset of E, such that the restriction of the covering projection $p: \widetilde{U} \to U$ is a homeomorphism. Then we may consider the subbasic open sets $\langle x, U\rangle = \{f: X \to B \mid f(x) \in U\}$ and $\langle x, \widetilde{U}\rangle = \{\tilde{f}: X \to E \mid \tilde{f}(x) \in \widetilde{U}\}$, and observe that $p_*: E^X \to B^X$ restricts to a continuous map $\bar{p}_*: \langle x, \widetilde{U}\rangle \to \langle x, U\rangle$. Moreover, if X is connected, then the unique lifting property for covering spaces (see [9, Theorem 2.2.2]) implies that $\bar{p}_*$ is injective.

Lemma 3.6. *If x is a strong deformation retract of X, then $\bar{p}_*: \langle x, \widetilde{U}\rangle \to \langle x, U\rangle$ is a homeomorphism.*

Proof. The surjectivity of $\bar{p}_*$ follows from the lifting criterion [9, Lemma 4.2.2]. To prove the continuity of its inverse, we define maps

$$u: \langle x, U\rangle \to E^X, \qquad u: f \mapsto \mathrm{const}_{\tilde{x}},$$

where $\tilde{x}$ is the unique point in $\widetilde{U}$ such that $p(\tilde{x}) = f(x)$, and

$$v: \langle x, U\rangle \to \left(B^X\right)^I, \qquad \left[v(f)(t)\right](x) := f\left(K(x,t)\right),$$

where $K: X \times I \to X$ is a strong deformation retraction of X to x with $K(-,0) = \mathrm{const}_x$, $K(-,1) = \mathrm{Id}_X$. Clearly, $p_*(u(f)) = v(f)(0)$ so we have a continuous map $(u,v): \langle x, U\rangle \to E^X \sqcap (B^X)^I$. Now it is easy to verify that the composition

$$\langle x, U\rangle \xrightarrow{(u,v)} E^X \sqcap (B^X)^I \xrightarrow{\Gamma^X} (E^X)^I \xrightarrow{\mathrm{ev}_1} \langle x, \widetilde{U}\rangle \subset E^X$$

is the continuous inverse of $\bar{p}_*$. □

In view of the above lemma we will say that a space is *strongly contractible* if it admits a strong deformation to a point.

Corollary 3.7. *If $p: E \to B$ is a covering space and X is strongly contractible then $p_*: E^X \to B^X$ is also a covering space.*

The study of the general case is considerably complicated by the fact that the map $p_*: E^X \to B^X$ is usually not surjective, and that the existence of a lifting of some $f: X \to B$ depends on the homotopy class of f and on the choice of a point on the fibre. Therefore it only makes sense to consider the restrictions of p_* over each homotopy class in B^X separately.

Theorem 3.8 *(Preservation of coverings). Let $p: E \to B$ be a covering space. If X is path connected and admits a cover by strongly contractible open subspaces, then the restriction of $p_*: E^X \to B^X$ over each homotopy class in B^X is a covering space.*

Proof. Since X is path connected we may assume that all sets in a strongly contractible cover $\{V_\lambda\}$ of X admit a contraction to the same point $x_0 \in X$. Let C be a homotopy class in B^X and assume that $\widetilde{C} := p_*^{-1}(C) \subset E^X$ is non-empty (as otherwise there is nothing to prove). Furthermore let $U \subset X$ be an elementary open set for p and let $\widetilde{U}$ be an open subset of E, such that $p: \widetilde{U} \to U$ is a homeomorphism, and such that $\widetilde{C} \cap \langle x_0, \widetilde{U}\rangle$ is non-empty (we are not assuming that the covering p is regular so there may be components of $p^{-1}(U)$ that do not intersect images of liftings of elements from C). Consider the following diagram

$$\begin{array}{ccc}
\widetilde{C} \cap \langle x_0, \widetilde{U}\rangle = \widetilde{C} \cap \{\tilde{f} \in E^X \mid \tilde{f}(x_0) \in \widetilde{U}\} & \xrightarrow{\tilde{f} \mapsto (\tilde{f}|_{V_\lambda})} & \prod_\lambda \{\tilde{f} \in E^{V_\lambda} \mid \tilde{f}(x_0) \in \widetilde{U}\} \\
\downarrow{\scriptstyle \bar{p}_*} & & \downarrow{\scriptstyle \approx} \\
C \cap \langle x_0, U\rangle = C \cap \{f \in B^X \mid f(x_0) \in U\} & \xrightarrow[f \mapsto (f|_{V_\lambda})]{} & \prod_\lambda \{f \in B^{V_\lambda} \mid f(x_0) \in U\}
\end{array}$$

The map $\bar{p}_*: \widetilde{C} \cap \langle x_0, \widetilde{U}\rangle \to C \cap \langle x_0, U\rangle$ is a continuous bijection: injectivity follows from the uniqueness of liftings and the connectivity of X, while the surjectivity is implied by the choice of C and $\widetilde{U}$. The two horizontal maps are given by restrictions and are clearly injective, while the right-hand vertical map is a homeomorphism by Lemma 3.6. One can easily verify that these facts imply that $\bar{p}_*$ is open and hence a homeomorphism. It follows that each $C \cap \langle x_0, U\rangle$ is evenly covered by sets of the form $\widetilde{C} \cap \langle x_0, \widetilde{U}\rangle$, therefore the restriction $p_*: \widetilde{C} \to C$ is a covering space. □

The last result can be easily extended to the case when X has finitely many components.

Corollary 3.9. *Let $p: E \to B$ be a covering space and assume that X has finitely many clopen path-components that admit strongly contractible open covers. Then the restriction of $p_*: E^X \to B^X$ over each homotopy class in B^X is a covering space.*

Proof. If $X_1, \ldots, X_n$ are the clopen path-components of $X = X_1 \sqcup \ldots \sqcup X_n$ then $E^X \to B^X$ is the topological product of maps $E^{X_i} \to B^{X_i}$, and it is well-known that the property of being a covering projections is preserved by finite products. □

Remark 3.10. If a covering projection $p: E \to B$ is not regular then the global structure of the fibre space $p_*: E^X \to B^X$ can be quite complicated. In particular, the fibres of p_* over different points usually have different cardinality even when p_* is surjective. This means that Theorem 25 of [1] stating that $p_*: E^X \to B^X$ is a covering space if X is a finite CW complex and p_* is surjective, is false without the assumption that p is regular, as can be seen from the following example.

Let B be the Eilenberg–MacLane space for the symmetric group S_3, and let $p: E \to B$ be the covering projection corresponding to a subgroup of order two. Then p is a non-regular three-sheeted covering projection. Every map $f: \mathbb{R}P^2 \to B$ can be lifted to E: in fact, either the image of $\pi_1(f)$ is trivial, and then f has three distinct liftings, or the image of $\pi_1(f)$ has two elements and equals one of the conjugates of $\pi_1(p)$, so that there is exactly one lifting for f. It follows that $p_*: E^{\mathbb{R}P^2} \to B^{\mathbb{R}P^2}$ is surjective but it is not a covering space in the strict sense (used in [1]), because the fibres have different cardinality.

3.3. Dold fibrations

Let $p: E \to B$ be a Hurewicz fibration with lifting function $\Gamma: E \sqcap B^I \to E^I$, and let $\Gamma^X: E^X \sqcap (B^X)^I \to (E^X)^I$ be defined as in formula (2). If X is a p preserving extension then we have already verified (in the proof of Proposition 3.1) that $p_*(\Gamma^X(f,H)) = H$. On the other side, the relation $\Gamma^X(f,H)(0) = f$ was only proved under the assumption that p is a fibration with unique path lifting. In general,

$$\begin{aligned}\big[\Gamma^X(f,H)(0)\big](x) &= \Gamma\big(\big[\Gamma^A\big(f\circ i, H(-)\circ i\big)(0)\big]\big(r(x), H(0)\circ \widehat{K}(x)\big)\big)(1)\\ &= \Gamma\big(f\big(ir(x)\big), p\circ f\circ \widehat{K}(x)\big)(1)\end{aligned}$$

so that $p([\Gamma^X(f,H)(0)](x)) = p(f(K(x,1))) = p(f(x))$, therefore $p_*(\Gamma^X(f,H)(0)) = p_*(f)$. We conclude that $\Gamma^X(f,H)(0)$ and f may not coincide, but they are always contained in the same fibre of the projection $p_*: E^X \to B^X$. This last observation puts us squarely in the context of Dold fibrations.

Dold fibrations (see [5, Section 1.2]) are a class of maps $p: E \to B$ similar to Hurewicz fibrations in the following sense: for any map $f: Y \to E$ and a homotopy $H: Y \times I \to B$, starting at $H(-,0) = p \circ f$, there is a homotopy $\widetilde{H}: Y \times I \to E$ that covers H (i.e. $p: \widetilde{H} = H$). However, for a Hurewicz fibration p we also require that $\widetilde{H}(-,0) = f$, while for p to be a Dold fibration we only require that $\widetilde{H}(-,0)$ and f are vertically homotopic. Dold fibrations arise very naturally in the homotopy theory of fibrations because the *fibre homotopy equivalence* (which is the basic equivalence relation between fibrations) does not preserve Hurewicz fibrations. However, it preserves Dold fibrations and in fact, every map that is fibre homotopy equivalent to a Hurewicz fibration is automatically a Dold fibration (cf. [5, Proposition 1.10]).

Dold fibrations have the following characterization in term of lifting functions (cf. [5, Theorem 1.11]). Let B^I_{st} denote the set of 'half stationary' paths in B, defined by $B^I_{\mathrm{st}} := \{\alpha \in B^I \mid \alpha(t) = \alpha(0) \text{ for } t \le \frac{1}{2}\}$. Then $p: E \to B$ is a Dold fibration if there exists a lifting function $\Gamma: E \sqcap B^I_{\mathrm{st}} \to E^I$, such that $\pi \circ \Gamma$ is the inclusion of $E \sqcap B^I_{\mathrm{st}}$ in $E \sqcap B^I$. Somewhat less technically, p is a Dold fibration if it admits a lifting function that splits in two parts: first is a vertical homotopy between two maps with image in the same fibre of p, followed by the usual lifting of a homotopy in the base. If the vertical part may be taken to be constant then p is in fact a Hurewicz fibration. As usual, we say that a Dold fibration is *regular* if the lifting function can be chosen so that constant paths in B lift to constant paths in E. By the results of [3] every fibration over a metric base space and every locally trivial fibration over a paracompact base space is regular.

Let $p: E \to B$ be a Dold fibration with a lifting function $\Gamma: E \sqcap B^I_{\mathrm{st}} \to E^I$: then formula (2) clearly gives a well-defined function $\Gamma^X: E^X \sqcap (B^X)^I_{\mathrm{st}} \to (E^X)^I$. Moreover, if X admits a deformation retraction to some subspace A that preserves Dold fibrations, then Γ^X is continuous by Lemma 2.3.

Proposition 3.11. *Let $p: E \to B$ be a regular Dold fibration, and assume that X admits a deformation retraction to some subspace A that preserves Dold fibrations. Then $p_*: E^X \to B^X$ is a Dold fibration. Moreover, if A preserves regular Dold fibrations then p_* is also regular.*

Proof. Assume that Γ is a regular lifting function. We already know that $\Gamma^X\colon E^X \sqcap (B^X)^I_{\mathrm{st}} \to (E^X)^I$ is well-defined and continuous, and that $p_*(\Gamma^X(f,H)) = H$. Moreover, we know that for every $f \in E^X$ the maps $\Gamma^X(f,H)(0)$ and f are in the same fibre of p_*, so in order to construct a Dold lifting function for p_* it is sufficient to combine Γ^X with a vertical homotopy of E^X which is defined as follows.

Recall that $\widehat{K}\colon X \to X^I$ denotes the adjoint of the homotopy $K\colon X \times I \to X$ between $i \circ r$ and the identity. For $s \in I$ let $[\widehat{K}_s(x)](t) := \widehat{K}(x)(1 - s(1-t))$, so that $\widehat{K}_0(x) = \mathrm{const}_x$, and $\widehat{K}_1(x) = \widehat{K}(x)$. Then we may define a vertical homotopy $D\colon E^X \times I \to E^X$ by the formula

$$D(f,s)(x) := \Gamma\big(f\big(K(x,1-s)\big), p \circ f \circ \widehat{K}_s(x)\big)(1),$$

and check that by regularity of Γ

$$D(f,0)(x) = \Gamma\big(f(x), p \circ f \circ \mathrm{const}_x\big)(1) = f(x),$$

while

$$D(f,1)(x) = \Gamma\big(f\big(ir(x)\big), p \circ f \circ \widehat{K}(x)\big)(1) = \big[\Gamma^X(f,H)(0)\big](x).$$

Finally, we join D and Γ^X to obtain the function $\overline{\Gamma}^X\colon E^X \sqcap (B^X)^I_{\mathrm{st}} \to (E^X)^I$

$$\big(\overline{\Gamma}^X(f,H)\big)(t) := \begin{cases} D(f,4t); & t \in [0,\frac{1}{4}] \\ \Gamma^X(f,H)(2t-\frac{1}{2}); & t \in [\frac{1}{4},\frac{1}{2}] \\ \Gamma^X(f,H)(t); & t \in [\frac{1}{2},1], \end{cases}$$

which is clearly a lifting function for p_*.

If Γ^A is a regular lifting function then so is Γ^X, which proves the second claim. □

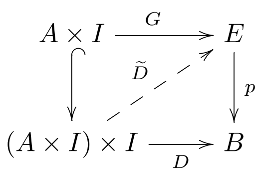
As usual, in order to extend the above result to general spaces, we will require the existence of a suitable cover. We will also use the following standard property of a (Dold) fibration $p\colon E \to B$. If two homotopies $H, K\colon A \times I \to E$ have the same projection $L = p \circ H = p \circ K\colon A \times I \to B$ and the same initial stage $H|_{A\times 0} = K|_{A\times 0}$ then their final stages $H|_{A\times 1}$ and $K|_{A\times 1}$ are vertically homotopic. In fact, one can combine the maps

$$G(a,t) := \begin{cases} H(a,1-2t); & t \le \frac{1}{2} \\ K(a,2t-1); & t \ge \frac{1}{2} \end{cases}$$

and

$$D(a,t,s) := \begin{cases} L(a,(1-s)(1-2t)); & t \le \frac{1}{2} \\ L(a,(1-s)(2t-1)); & t \ge \frac{1}{2} \end{cases}$$

to obtain the lifting diagram



and verify that the final stage of $\widetilde{P}$ is the required vertical homotopy between $H|_{A\times 1}$ and $K|_{A\times 1}$.

Lemma 3.12. *Assume that $\{U, V\}$ is a numerable open cover of X, such that U, V and $U \cap V$ preserve a Dold fibration $p: E \to B$. Then X preserves p as well.*

Proof. For every $(f, H) \in E^X \sqcap (B^X)^I_{\mathrm{st}}$ the restrictions $f|_U$ and $H|_U$ are clearly compatible, so we may compute $\Gamma^U(f|_U, H|_U) \in (E^U)^I$, and similarly $\Gamma^V(f|_V, H|_V) \in (E^V)^I$. The so obtained maps in general do not coincide over $U \cap V$, but they share the same initial stage and cover the same homotopy, so by the above remark, there exists a vertical homotopy $D: (E^X \sqcap (B^X)^I_{\mathrm{st}}) \times I \to (E^{U \cap V})^I$ relating their restrictions over $U \cap V$.

Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinated to the cover $\{U, V\}$. Then the formula

$$\big[\Gamma^X(f,H)(t)\big](x) := \begin{cases} [\Gamma^U(f|_U, H|_U)(t)](x); & x \in U - V \\ [D(f, H, \rho_V(x))(t)](x); & x \in U \cap V \\ [\Gamma^V(f|_V, H|_V)(t)](x); & x \in V - U \end{cases}$$

defines a lifting function $\Gamma^X: E^X \sqcap (B^X)^I_{\mathrm{st}} \to (E^X)^I$. In particular, continuity of Γ^X follows from the numerability of the cover as in the proof of Corollary 3.3 □

The inductive application of the last Lemma yields our main result about Dold fibrations between mapping spaces:

Theorem 3.13 *(Preservation of Dold fibrations). Let $p: E \to B$ be a Dold fibration, and assume that X has a numerable cover $\{A_1, \ldots, A_n\}$, such that each A_i and each intersection $(A_1 \cup \ldots \cup A_{i-1}) \cap A_i$ admit a deformation retraction to some subspace that preserves p (as a Dold fibration). Then $p_*: E^X \to B^X$ is also a Dold fibration.*

The assumption in the theorem regarding the intersection of the elements of the cover is analogous to the usual definition of a 'good' cover and regrettably we have not find a way to avoid it. The most useful instance of the theorem is the following:

Corollary 3.14. *Let $p: E \to B$ be a Dold fibration and let X admit a finite numerable cover such that the intersection of any subfamily of its elements is contractible. Then $p_*: E^X \to B^X$ is also a Dold fibration.*

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