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Spaces with high topological complexity

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By a formula of Farber, the topological complexity $\mathrm{TC}(X)$ of a $(p-1)$-connected m-dimensional CW-complex X is bounded above by $(2m+1)/p+1$. We show that the same result holds for the monoidal topological complexity $\mathrm{TC}^{\mathrm{M}}(X)$. In a previous paper we introduced various lower bounds for $\mathrm{TC}^{\mathrm{M}}(X)$, such as the nilpotency of the ring $H^*(X \times X, \Delta(X))$, and the weak and stable (monoidal) topological complexity $\mathrm{wTC}^{\mathrm{M}}(X)$ and $\sigma\,\mathrm{TC}^{\mathrm{M}}(X)$. In general, the difference between these upper and lower bounds can be arbitrarily large. In this paper we investigate spaces with topological complexity close to the maximal value given by Farber's formula. We show that in these cases the gap between the lower and upper bounds is narrow and $\mathrm{TC}(X)$ often coincides with the lower bounds.

1. Introduction

Topological complexity was introduced by Farber in [5] as a measure of the discontinuity of robot motion planning algorithms. A *motion planning algorithm* in a space X is a rule that takes as input a pair of points $x, y \in X$ and returns a path in X starting at x and ending at y. We are interested in the minimal number of rules that are continuously dependent on the input, and that are sufficient to connect any two points of X. The formal definition is as follows. Let X^I be the space of paths in X (endowed with the compact-open topology), and let $p\colon X^I \to X \times X$ be the fibration given by $p(\alpha) = (\alpha(0), \alpha(1))$. A continuous choice of paths between given end points corresponds to a continuous section of p. However, a global section exists if and only if X is contractible (see [5]), so for a general space we may ask how many local sections are needed to cover all possible pairs of end points.

Definition 1.1. The *topological complexity* $\mathrm{TC}(X)$ of a space X is the least integer n for which there exist an open cover $\{U_1, U_2, \ldots, U_n\}$ of $X \times X$ and sections $s_i\colon U_i \to X^I$ of the fibration $p\colon X^I \to X \times X$.

Observe that this definition is just a special case of the *Schwarz genus* [15] or the *sectional category* of James [12]. In an attempt to extend certain standard techniques of homotopy theory, in particular of the Lyusternik–Schnirelmann (LS) category to the topological complexity, Iwase and Sakai [11] introduced the following concept.

DEFINITION 1.2. The *monoidal topological complexity* $\mathrm{TC}^{\mathrm{M}}(X)$ of a space X is the least integer n for which there exist an open cover $\{U_1, U_2, \dots, U_n\}$ of $X \times X$ such that $\Delta(X) \subset U_i$, and sections $s_i\colon U_i \to X^I$ of the fibration $p\colon X^I \to X \times X$ such that $s_i(x, x) = c_x$, the constant path in x.

In other words, for the monoidal topological complexity we consider only the motion planning algorithms that satisfy the natural requirement that the robot motion should be constant whenever the starting and ending points coincide. In a sense, the relation between the ordinary and the monoidal topological complexity is analogous to the relation between the standard definition of the LS-category and the alternative definition introduced by Whitehead (see [1, § 1.6]). It is well known that for locally nice spaces Whitehead's definition of the LS-category coincides with the original one. As for the topological complexity, Iwase and Sakai [11] claimed that $\mathrm{TC}^{\mathrm{M}}(X) = \mathrm{TC}(X)$ for every locally finite simplicial complex X, but in [11, erratum] they retracted the claim and proved that for X as above the difference between the two invariants is at most 1, i.e. $\mathrm{TC}(X) \leqslant \mathrm{TC}^{\mathrm{M}}(X) \leqslant \mathrm{TC}(X) + 1$. They also proved that the two versions of topological complexity have the same value when X admits a cover with some special properties (see [11, erratum]). Up to now the most general result regarding the equality between TC and TC^{M} was proposed by Dranishnikov [4], who used obstruction theory to show that they coincide when the topological complexity of X exceeds a certain estimate depending on the dimension and the connectivity of X.

The importance of the monoidal topological complexity comes both from the previously mentioned practical considerations and from its strong relation with the LS-category. In fact, Iwase and Sakai [11] found a useful characterization of the monoidal topological complexity as the fibrewise pointed LS-category (see § 2 for details), which makes the above analogy even clearer. In [8] we exploited this new approach and introduced several lower bounds for $\mathrm{TC}^{\mathrm{M}}(X)$ that refine previously known estimates. Nevertheless, these bounds need not be precise, and in fact one can always construct spaces for which the difference between the estimate and the actual value of $\mathrm{TC}^{\mathrm{M}}(X)$ is arbitrarily large. In this paper we investigate an interesting phenomenon, already observed in the case of the LS-category: when the topological complexity of X is close to a certain upper bound that can be computed from the dimension and connectivity of X, the lower bounds are also good approximations. Crucial here is the theorem of Dranishnikov (see [4] and § 2 for details), which implies that TC and TC^{M} are equal when TC is close to this upper bound.

The paper has the following structure. In the next section we describe a diagram of fibrewise pointed spaces that relates the two principal approaches to the monoidal topological complexity, and recall the definitions of the main lower bounds for $\mathrm{TC}^{\mathrm{M}}(X)$, namely, the nilpotency of the ring $H^*(X \times X, \Delta(X))$, the weak monoidal topological complexity $\mathrm{wTC}^{\mathrm{M}}(X)$ and the stable monoidal topological complexity $\sigma\,\mathrm{TC}^{\mathrm{M}}(X)$. The remaining sections are dedicated to various upper and lower estimates of the topological complexity.

Unless otherwise stated, the spaces under consideration are assumed to have the homotopy type of a finite CW-complex. We do not distinguish notationally between a map and its homotopy class. Standard notation for maps is 1 for the identity map, $\Delta_n\colon X \to X^n$ for the diagonal map $x \mapsto (x, \ldots, x)$, pr_i for the projection from a product to the ith factor, and $\mathrm{ev}_{0,1}$ for the evaluation of a path in X^I to the end points. When considering the LS-category of a space, we always use the non-normalized version (so that the category of a contractible space is equal to 1).

2. Preliminaries

Recall that a *fibrewise pointed space* over a *base* B is a topological space E, together with a *projection* $p\colon E \to B$ and a *section* $s\colon B \to E$. Fibrewise pointed spaces over a base B form a category, and the notions of fibrewise pointed maps and fibrewise pointed homotopies are defined in an obvious way. We refer the reader to [13, 14] for more details on fibrewise constructions. In [11], Iwase and Sakai considered the product $X \times X$ as a fibrewise pointed space over X by taking the projection to the first component and the diagonal section Δ as in the diagram $X \xrightarrow{\Delta} X \times X \xrightarrow{\mathrm{pr}_1} X$. Their description of the topological complexity is based on the following result.

THEOREM 2.1 (Iwase and Sakai [11]). *The topological complexity* $\mathrm{TC}(X)$ *of* X *is equal to the least integer* n *for which there exists an open cover* $\{U_1, U_2, \ldots, U_n\}$ *of* $X \times X$ *such that each* U_i *is compressible to the diagonal via a fibrewise homotopy.*

The monoidal topological complexity $\mathrm{TC}^{\mathrm{M}}(X)$ *of* X *is equal to the least integer* n *for which there exists an open cover* $\{U_1, U_2, \ldots, U_n\}$ *of* $X \times X$ *such that each* U_i *contains the diagonal* $\Delta(X)$ *and is compressible to the diagonal via a fibrewise pointed homotopy.*

Iwase and Sakai [11] proved that $\mathrm{TC}(X) \leqslant \mathrm{TC}^{\mathrm{M}}(X) \leqslant \mathrm{TC}(X) + 1$ and that $\mathrm{TC}(X) = \mathrm{TC}^{\mathrm{M}}(X)$ when the minimal cover $\{U_1, U_2, \ldots, U_n\}$ meets certain technical assumptions. In a somewhat different vein, Dranishnikov proved the following result.

THEOREM 2.2 (Dranishnikov [4, theorem 2.5]). *If* X *is a* $(p-1)$*-connected simplicial complex such that* $\mathrm{TC}(X) > (\dim(X) + 1)/p$*, then* $\mathrm{TC}(X) = \mathrm{TC}^{\mathrm{M}}(X)$.

In the spirit of [14] we say that an open set $U \subseteq X \times X$ is *fibrewise categorical* if it is compressible to the diagonal by a fibrewise homotopy, and U is *fibrewise pointed categorical* if it contains the diagonal $\Delta(X)$ and is compressible onto it by a fibrewise pointed homotopy. In this sense, $\mathrm{TC}^{\mathrm{M}}(X)$ is the minimal n such that $X \times X$ can be covered by n fibrewise pointed categorical sets, i.e. $\mathrm{TC}^{\mathrm{M}}(X)$ is precisely the fibrewise pointed Lyusternik–Schnirelmann category of the fibrewise pointed space $X \xrightarrow{\Delta} X \times X \xrightarrow{\mathrm{pr}_1} X$. The main advantage of this alternative formulation is that it is more geometrical, since it only involves the space X and its square $X \times X$, and does not refer to function spaces.

The standard machinery of the LS-category can be extended to the fibrewise setting. In particular, we can take the standard Whitehead and Ganea characterizations of the LS-category (see [1, ch. 2]) and transpose them to the fibrewise pointed setting to obtain alternative characterizations of the monoidal topological complexity. As often happens in the fibrewise context, however, the standard notation for the various fibrewise constructions becomes excessively complicated and difficult to read. In an attempt to avoid this inconvenience, we use more intuitive notation (introduced in [8]) based on the analogy between fibrewise constructions and semi-direct products. Indeed, whenever we perform a pointed construction (e.g. a wedge or a smash product) on some fibrewise space, the fibres of the resulting space depend on the choice of base points, and we view this effect as an action of the base on the fibres. In this way we obtain the following diagram (analogous to the diagram from [1, p. 49]), in which all spaces are fibrewise pointed over X, and all maps preserve fibres and sections.

$$\begin{array}{ccc}
X \ltimes G_n X & \xrightarrow{1\ltimes\widehat{\Delta}_n} & X \ltimes W^n X \\
\big\downarrow{\scriptstyle 1\ltimes p_n} & & \big\downarrow{\scriptstyle 1\ltimes i_n} \\
X \ltimes X & \xrightarrow{1\ltimes\Delta_n} & X \ltimes \Pi^n X \\
\big\downarrow{\scriptstyle 1\ltimes q'_n} & & \big\downarrow{\scriptstyle 1\ltimes q_n} \\
X \ltimes G_{[n]} X & \xrightarrow[1\ltimes\tilde{\Delta}_n]{} & X \ltimes \wedge^n X
\end{array} \tag{2.1}$$

We now give a precise description of the spaces involved: $X \ltimes X$ denotes the fibrewise pointed space $X \xrightarrow{\Delta} X \times X \xrightarrow{\mathrm{pr}_1} X$; $X \ltimes \Pi^n X$ is the fibrewise pointed space

$$X \xrightarrow{(1,\Delta_n)} \{(x, y_1, \ldots, y_n) \in X \times X^n\} \xrightarrow{\mathrm{pr}_1} X,$$

which can be easily recognized as the n-fold fibrewise pointed product of $X \ltimes X$; $X \ltimes W^n(X)$ is the fibrewise pointed space

$$X \xrightarrow{(1,\Delta_n)} \{(x, y_1, \ldots, y_n) \in X \ltimes \Pi^n X \mid \exists j\colon y_j = x\} \xrightarrow{\mathrm{pr}_1} X,$$

the n-fold fibrewise pointed fat wedge of $X \ltimes X$. The Whitehead-type characterization of the monoidal topological complexity (see [8, theorem 3] and [11, § 6]) is that $\mathrm{TC}^{\mathrm{M}}(X)$ is the least integer n such that the map $1 \ltimes \Delta_n \colon X \ltimes X \to X \ltimes \Pi^n X$ can be compressed into $X \ltimes W^n X$ by a fibrewise pointed homotopy:

$$\begin{array}{ccc}
 & & X \ltimes W^n X \\
 & \overset{g}{\nearrow} & \big\downarrow{\scriptstyle 1\ltimes i_n} \\
X \ltimes X & \xrightarrow[1\ltimes\Delta_n]{} & X \ltimes \Pi^n X
\end{array}$$

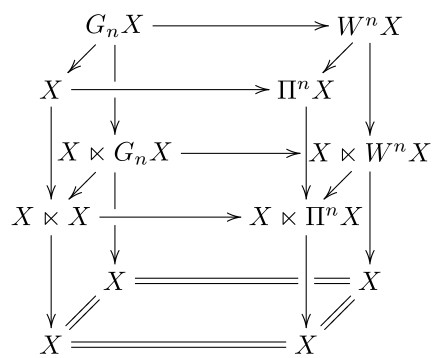
For the description of $X \ltimes G_n X$, we first need the fibrewise path space $X \ltimes PX$, defined as the fibrewise pointed space

$$X \xrightarrow{x \mapsto c_x} X^I \xrightarrow{\mathrm{ev}_0} X,$$

where $c_x \colon I \to X$ is the constant path in x. Observe that the evaluation at the end points determines a fibrewise pointed map $\mathrm{ev}_{0,1} \colon X \ltimes PX \to X \ltimes X$. The nth fibrewise Ganea space $X \ltimes G_n X$ is defined as the n-fold fibrewise reduced join of the path fibration $\mathrm{ev}_{0,1}\colon X^I \to X \times X$ (viewed as a subspace of the n-fold join $X^I * \cdots * X^I$):

$$X \ltimes G_n X := *^n_{X\times X} X^I = *^n_{X\ltimes X} X \ltimes PX.$$

The Ganea-type characterization of the monoidal topological complexity (see [8, corollary 4]) is $\mathrm{TC}^{\mathrm{M}}(X)$, the least integer n such that the fibrewise pointed map $1 \ltimes p_n\colon X \ltimes G_n X \to X \ltimes X$ admits a section. Note that the fibres of these constructions are, respectively, the spaces X, $\Pi^n X$, $W^n X$, PX and $G_n X$ (the nth Ganea space). The base point, however, is different on each fibre, and this is expressed by the semi-direct product notation. This notation also applies to maps. We can summarize the relations between these spaces in a diagram of fibrewise pointed spaces over X:



Note that all the horizontal squares are fibrewise pointed homotopy pullbacks.

The diagram (2.1) is obtained by extending the middle square with the fibrewise cofibres of the maps $1 \ltimes p_n\colon X \ltimes G_n X \to X \ltimes X$ and $1 \ltimes i_n\colon X \ltimes W_n X \to X \ltimes \Pi^n X$, which we denote, respectively, by $1 \ltimes q'_n\colon X \ltimes X \to X \ltimes G_{[n]} X$ and $1 \ltimes q_n\colon X \ltimes X \to X \ltimes \wedge^n X$. Note that, with some extra effort, we can fit all these constructions of fibrewise pointed spaces into a unified framework. This was done in [8, appendix].

We conclude this section with a brief overview of lower bounds for the monoidal topological complexity (see [8] for more details). For any ring of coefficients R we denote by $\mathrm{nil}_R(X) := \mathrm{nil}(H^*(X \times X, \Delta(X); R))$ the nilpotency of the ideal $H^*(X \times X, \Delta(X); R) \lhd H^*(X \times X; R)$. Furthermore, let $\mathrm{wTC}^{\mathrm{M}}(X)$, the *weak monoidal topological complexity* of X, be the least integer m such that the composition

$$X \ltimes X \xrightarrow{1\ltimes\Delta^m} X \ltimes \Pi^n X \xrightarrow{1\ltimes q^m} X \ltimes \wedge^m X$$

is fibrewise pointed homotopic to the section. Finally, let $\sigma\,\mathrm{TC}^{\mathrm{M}}(X)$, the *stable monoidal topological complexity* of X, be the minimal n such that some suspension

$$1 \ltimes \Sigma^i p_n\colon X \ltimes \Sigma^i G_n(X) \to X \ltimes \Sigma^i X$$

admits a section. By [8, theorem 12] we have, for any ring R, that the relations

$$\mathrm{nil}_R(X) \leqslant \mathrm{wTC}^{\mathrm{M}}(X) \leqslant \mathrm{TC}^{\mathrm{M}}(X) \quad \text{and} \quad \mathrm{nil}_R(X) \leqslant \sigma\,\mathrm{TC}^{\mathrm{M}}(X) \leqslant \mathrm{TC}^{\mathrm{M}}(X)$$

hold, while $\mathrm{wTC}^{\mathrm{M}}(X)$ and $\sigma\,\mathrm{TC}^{\mathrm{M}}(X)$ are in general unrelated.

3. Dimension and category estimates

In this section we determine the highest possible values for $\mathrm{TC}(X)$ and $\mathrm{TC}^{\mathrm{M}}(X)$ based on the connectivity, the dimension and the LS-category of X. Note that we use the non-normalized definitions of TC, TC^{M} and LS-category (i.e. $\mathrm{cat}(X) \leqslant n$ if there exists a cover $\{U_1, \ldots, U_n\}$ of X such that each U_i is contractible to a point in X).

Farber [6, theorem 5.2] used general results on the Schwarz genus to obtain the following basic estimate: if X is a $(p-1)$-connected CW-complex, then

$$\mathrm{TC}(X) < \frac{2\dim(X)+1}{p} + 1,$$

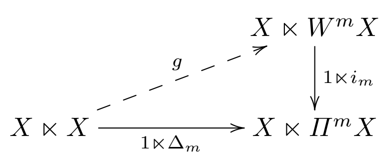
so, in particular, if $\dim(X) = np + r$ for $0 \leqslant r < p$, then

$$\mathrm{TC}(X) \leqslant \begin{cases} 2n+1 & \text{if } 2r \leqslant p, \\ 2n+2 & \text{if } 2r > p. \end{cases}$$

The Whitehead-type characterization of the monoidal topological complexity described in § 2 yields an analogous upper bound for $\mathrm{TC}^{\mathrm{M}}(X)$. In fact, the inclusion $i_m \colon W^m X \hookrightarrow \Pi^m X$ of the fat wedge into the product is an mp-equivalence (i.e. $(i_m)_* \colon [P, W^m X] \to [P, \Pi^m X]$ is bijective for every polyhedron P of $\dim(P) < mp$, and surjective for $\dim(P) \leqslant mp$). It now follows from the fibrewise obstruction theory (see [2, proposition 2.15]) that the induced function between fibrewise homotopy classes of maps over X,

$$(1 \ltimes i_m)_* \colon [X \ltimes X, X \ltimes W^m X]_X \to [X \ltimes X, X \ltimes \Pi^m X]_X,$$

is surjective for $2(np + r) \leqslant mp$, which is to say that there exists a lifting in the diagram



By plugging in $m = 2n + 1$ or $m = 2n + 2$ we get the desired estimates.

It is not surprising that we get the same upper estimates for $\mathrm{TC}(X)$ and $\mathrm{TC}^{\mathrm{M}}(X)$, as they fall in the region where Dranishnikov's theorem guarantees that they are equal. In fact, we have the following result.

Proposition 3.1. *Let X be a $(p-1)$-connected $(np + r)$-dimensional complex, $0 \leqslant r < p$. Then, $\mathrm{TC}(X) = \mathrm{TC}^{\mathrm{M}}(X)$ in the following cases:*

(a) $\mathrm{TC}(X) > n + 1$ *or*

(b) $\mathrm{TC}(X) \geqslant 2n$ *or*

(c) $\mathrm{TC}(X) = 2n - 1$ *and* $r + 1 < p$.

Proof. According to theorem 2.2 we need only show that $\mathrm{TC}(X) > (\dim(X)+1)/p$.

(a) Assume that $\mathrm{TC}(X) > n+1$. Then,

$$\mathrm{TC}(X) > \frac{np+p}{p} = \frac{np+r+1}{p} - \frac{r+1-p}{p} = \frac{\dim(X)+1}{p} + 1 - \frac{r+1}{p}.$$

Since $r < p$, we have $1-(r+1)/p \geqslant 0$. Hence, $\mathrm{TC}(X) > (\dim(X)+1)/p$.

(b) If $n \geqslant 2$, then $\mathrm{TC}(X) \geqslant 2n > n+1$, and the result follows from (a). The same is true for $n = 1$ and $\mathrm{TC}(X) > 2$. If $\mathrm{TC}(X) = 2$, then, by [10, theorem 1], X is homotopy equivalent to an odd-dimensional sphere. It is easy to check that $\mathrm{TC}(S^k) = \mathrm{TC}^{\mathrm{M}}(S^k)$ for all k.

(c) If $n \geqslant 3$, then $\mathrm{TC}(X) = 2n-1 > n+1$, and the result follows from (a). If $n = 1$, then $\mathrm{TC}(X) = 1$ implies that X is contractible and $\mathrm{TC}^{\mathrm{M}}(X) = 1$. Finally, if $n = 2$, then

$$\mathrm{TC}(X) = 3 = n+1 = \frac{\dim(X)+1}{p} + 1 - \frac{r+1}{p}.$$

Since $r+1 < p$, we have $1-(r+1)/p > 0$, and so $\mathrm{TC}(X) > (\dim(X)+1)/p$.

□

We conclude that when the topological complexity is close to the dimension-connectivity estimate it coincides with the monoidal topological complexity. In addition, that estimate can in some cases be further improved using the LS-category. In fact, [1, theorem 1.50] asserts that the LS-category of a $(p-1)$-connected CW-complex X is bounded by

$$\mathrm{cat}(X) \leqslant \frac{\dim(X)}{p} + 1,$$

while by [5, theorem 5] we have that

$$\mathrm{TC}(X) \leqslant 2 \cdot \mathrm{cat}(X) - 1.$$

Therefore, if X is $(p-1)$ connected and $(n \cdot p + r)$ dimensional, then $\mathrm{cat}(X) \leqslant n+1$ and, hence, $\mathrm{TC}(X) \leqslant 2n+1$. As we see, in roughly half of the cases the category estimate gives us a more accurate upper bound than the dimension-connectivity estimate. This fact combined with proposition 3.1 yields the following theorem.

THEOREM 3.2. *If X is a $(p-1)$-connected complex of dimension $np+r$, $n \in \mathbb{Z}$, $0 \leqslant r < p$, then* $\mathrm{TC}^{\mathrm{M}}(X) \leqslant 2n+1$.

We also obtain the following useful corollary, which essentially says that if the topological complexity of a space is high with respect to its dimension and connectivity, then its LS-category must be maximal.

COROLLARY 3.3. *Let the space X be $(p-1)$ connected and $(np+r)$ dimensional, $0 \leqslant r < p$. If* $\mathrm{TC}^{\mathrm{M}}(X) \geqslant 2n$, *then* $\mathrm{cat}(X) = n+1$.

Proof. By proposition 3.1 and [5, theorem 5] we have

$$2n \leqslant \mathrm{TC}^{\mathrm{M}}(X) = \mathrm{TC}(X) \leqslant 2\,\mathrm{cat}(X) - 1;$$

thus, $\mathrm{cat}(X) \geqslant n+1$. On the other side, by [1, theorem 1.50], $\mathrm{cat}(X) \leqslant n+1$. □

4. Cohomological estimates

In § 2 we mentioned the classical lower bound for the topological complexity of a space X, namely, $\mathrm{nil}_R(X)$, the nilpotency of the ideal $H^*(X \times X, \Delta(X); R)$. There is an analogous lower bound for the LS-category, given by the nilpotency of the reduced cohomology ring $H^*(X, *; R)$, viewed as an ideal in $H^*(X; R)$. Note that in the literature these results are more often expressed in terms of the relation between the normalized LS-category and the cup length of X, and between the normalized topological complexity and the zero-divisors cup length of X (see [1, 5, 7]).

In general, both estimates give relatively crude bounds for $\mathrm{cat}(X)$ and $\mathrm{TC}(X)$, respectively. Nevertheless, in certain cases, when the category of X is maximal with respect to the dimension and connectivity of X, one can show that the nilpotency of the reduced cohomology ring with suitable coefficients gives the precise value of the LS-category of X. A similar phenomenon arises in the case of the topological complexity, as we now show.

Let X be a $(p-1)$-connected $(p \geqslant 2)$ and np-dimensional complex, and let us assume for simplicity that $H_p(X)$ is cyclic. Then, $\mathrm{cat}(X) \leqslant n+1$ by [1, theorem 1.50], and $\mathrm{TC}(X) \leqslant 2n+1$ by theorem 3.2. We assume that $\mathrm{TC}(X) = 2n+1$. Corollary 3.3 then implies that $\mathrm{cat}(X) = n+1$, so, by a theorem of James [12] (see also [1, proposition 5.3]), there exists a cohomology class $\alpha \in H^p(X; H_p(X))$ such that $0 \neq \alpha^n \in H^{np}(X; H_p(X))$ (in fact, α is the class that corresponds to the identity under the identification $H^p(X; H_p(X)) = \mathrm{Hom}(H_p(X), H_p(X))$). The element $\alpha \times 1 - 1 \times \alpha \in H^p(X \times X; H_p(X))$ then clearly satisfies $\Delta^*(\alpha \times 1 - 1 \times \alpha) = 0$ (where $\Delta\colon X \to X \times X$ is the diagonal map). Therefore, we may consider $\alpha \times 1 - 1 \times \alpha$ as an element in $H^p(X \times X, \Delta(X); H_p(X))$.

We compute the cup-product power $(\alpha \times 1 - 1 \times \alpha)^{2n}$. To this end we recall the commutation formula (see [3, ch. 7]) for the cup product in $H^*(X \times X)$:

$$(\alpha \times \beta) \smile (\gamma \times \delta) = (-1)^{|\beta| \cdot |\gamma|} (\alpha \smile \gamma) \times (\beta \smile \delta).$$

If p is even, then $1 \times \alpha_X$ and $\alpha_X \times 1$ commute, and we obtain

$$\begin{aligned}
(\alpha \times 1 - 1 \times \alpha)^{2n} &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (1 \times \alpha)^{2n-k} \smile (\alpha \times 1)^k \\
&= (-1)^n \binom{2n}{n} (1 \times \alpha)^n \smile (\alpha \times 1)^n \\
&= (-1)^n \binom{2n}{n} \alpha^n \times \alpha^n
\end{aligned}$$

as an element of $H^*(X \times X, \Delta(X); H_p(X))$ (note how most summands above are 0, because one of the factors is in cohomology above the dimension). If p is odd and

n is even, then we get a similar result, because then $(\alpha\times 1-1\times\alpha)^2=\alpha^2\times 1+1\times\alpha^2$, and so

$$(\alpha\times 1-1\times\alpha)^{2n}=(-1)^{n/2}\binom{n}{n/2}\alpha^n\times\alpha^n.$$

We may summarize the above computations in the following proposition.

PROPOSITION 4.1. *Let X be a $(p-1)$-connected np-dimensional finite complex, where np is even. Assume, furthermore, that $H_p(X)$ is cyclic without $\binom{2n}{n}$- or $\binom{n}{n/2}$-torsion. The following are then equivalent:*

(1) $\mathrm{TC}(X)=2n+1$,

(2) $\mathrm{nil}_{H_p(X)}(X)=2n+1$,

(3) $\mathrm{cat}(X)=n+1$,

(4) $\mathrm{nil}\,\hat{H}^*(X;H_p(X))=n+1$.

Farber and Grant [7] proved that the above relation between the topological complexity and the nilpotency of the cohomology ring $H^*(X\times X,\Delta(X);H_p(X))$ holds without the assumption that $H_p(X)$ is cyclic. In fact, [7, theorem 2.2] states that, for a $(p-1)$-connected np-dimensional finite complex X, $\mathrm{TC}(X)=2n+1$ if and only if $\mathrm{nil}\,H^*(X\times X,\Delta(X);H_p(X))=2n+1$. To this end they extended the definition of nilpotency to cup products with coefficients in an abelian group, and applied obstruction theory results from [15]. In that case, however, we lose the strong relation between the topological complexity and category in the sense that maximal category (relative to the dimension and connectivity) does not imply maximal topological complexity, as the example of odd-dimensional spheres shows.

If np, the dimension of X, is odd, we obtain a different relation between the topological complexity and the category. Assume again that $H_p(X)$ is cyclic, and denote by α the element of $H^p(X;H_p(X))$ corresponding to the identity map $H_p(X)\to H_p(X)$. Then,

$$(\alpha\times 1-1\times\alpha)^{2n}=(\alpha^2\times 1+1\times\alpha^2)^n=\sum_{k=0}^{n}\binom{n}{k}\alpha^{2k}\times\alpha^{2n-2k}=0,$$

because n is odd and $\alpha^{n+1}=0$, so in every summand at least one of the powers of α is 0. Since by (the proof of) [7, theorem 2.2] (see also [15]) $(\alpha\times 1-1\times\alpha)^{2n}$ is the only obstruction to the existence of a section for the Schwarz fibration, we conclude that $\mathrm{TC}(X)\leqslant 2n$. If $\mathrm{TC}(X)=2n$, then, by corollary 3.3, $\mathrm{cat}(X)=n+1$; therefore, we get $\alpha^n\neq 0$ as above. A straightforward computation then yields that

$$(\alpha\times 1-1\times\alpha)^{2n-1}=\binom{n-1}{(n-1)/2}(\alpha^n\times\alpha^{n-1}-\alpha^{n-1}\times\alpha^n).$$

Thus, we get the following result, which complements proposition 4.1.

PROPOSITION 4.2. *Let X be a $(p-1)$-connected np-dimensional finite complex, where np is odd. Assume, furthermore, that $H_p(X)$ is cyclic and does not have $\binom{n-1}{(n-1)/2}$-torsion. The $\mathrm{TC}(X)\leqslant 2n$ and the following are then equivalent:*

(1) $\mathrm{TC}(X) = 2n$,

(2) $\mathrm{nil}_{H_p(X)}(X) = 2n$,

(3) $\mathrm{cat}(X) = n + 1$,

(4) $\mathrm{nil}\, \hat{H}^*(X; H_p(X)) = n + 1$.

5. Weak complexity estimates

As we already know, the topological complexity of a $(p-1)$-connected $(np+r)$-dimensional space is at most $2n+1$. In this section we use the fibrewise Blakers–Massey theorem to relate the topological complexity to the more accessible weak monoidal topological complexity. Recall that the weak monoidal topological complexity of X, denoted by $\mathrm{wTC}^{\mathrm{M}}(X)$, is the minimal m such that the composition

$$X \ltimes X \xrightarrow{1 \ltimes \Delta^m} X \ltimes \Pi^n X \xrightarrow{1 \ltimes q^m} X \ltimes \wedge^m X$$

is fibrewise trivial (i.e. fibrewise homotopic to the section). By [8, theorem 12] we have that $\mathrm{nil}_R(X) \leqslant \mathrm{wTC}^{\mathrm{M}}(X) \leqslant \mathrm{TC}(X)$, so in general the weak topological complexity is a better approximation for the topological complexity than the cohomological estimate. In our discussion we need the following consequence of the fibrewise Blakers–Massey theorem.

THEOREM 5.1. *Let X be a finite complex of dimension at most m, and let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a fibrewise pointed cofibration sequence of fibrewise pointed bundles over X. Assume that the fibres of A and C are, respectively, a-connected and c-connected. The sequence

$$[Z, A]_X \xrightarrow{f_*} [Z, B]_X \xrightarrow{g_*} [Z, C]_X$$

of fibrewise pointed homotopy classes is then exact for every fibrewise pointed bundle Z over X, whose fibres are of dimension at most $a + c - m$.

Proof. We denote by $i_g \colon F(g) \to B$ and $i_f \colon F(f) \to A$ the fibrewise pointed homotopy fibres of the maps g and f, respectively. Moreover, the homotopy fibre of i_g may be identified as $j \colon \Omega_X(C) \to F(g)$, where $\Omega_X(C)$ is the fibrewise pointed loop space of C (see [2, § I.13]). By the lifting property of homotopy fibres, there exist fibrewise pointed maps u, v such that the following diagram commutes:

$$\begin{array}{ccccccc}
F(f) & \xrightarrow{i_f} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \| & & \| \\
\Omega_X(C) & \xrightarrow[j]{} & F(g) & \xrightarrow[i_g]{} & B & \xrightarrow[g]{} & C
\end{array}$$

By the fibrewise version of the Blakers–Massey theorem as formulated in [2, proposition 2.18], the map $v \colon F(f) \to \Omega_X(C)$ is an $(a + c - m)$-equivalence. The maps u

and v induce a commutative ladder between the exact homotopy sequences of the fibre sequences $F(f) \to A \to B$ and $\Omega_X(C) \to F(g) \to B$, from which we conclude that u is also an $(a+c-m)$-equivalence. Therefore, for every fibrewise pointed bundle Z over X we obtain the following commutative diagram:

$$\begin{array}{ccccc} [Z,A]_X & \xrightarrow{f_*} & [Z,B]_X & \xrightarrow{g_*} & [Z,C]_X \\ {\scriptstyle u_*}\downarrow & & \| & & \| \\ [Z,F(g)]_X & \xrightarrow[(i_g)_*]{} & [Z,B]_X & \xrightarrow[g_*]{} & [Z,C]_X \end{array}$$

whose bottom line is exact, being a part of the Puppe exact sequence. Assuming that the dimension of the fibres of Z is at most $a+c-m$, u_* is surjective, which implies that the top line of the diagram is also exact. □

We now consider a space X that is $(p-1)$ connected and $(np+r)$ dimensional. If $2r \geqslant p$, then by obstruction theory every fibrewise map $X \ltimes X \to X \ltimes \wedge^{2n+2} X$ is fibrewise trivial, so $\mathrm{wTC}^{\mathrm{M}}(X) \leqslant 2n+2$. However, we have already proved that $\mathrm{TC}(X) \leqslant 2n+1$, so if $\mathrm{wTC}^{\mathrm{M}}(X)$ is one less than the bound given by the obstruction theory, then we have *a fortiori* that $\mathrm{wTC}^{\mathrm{M}}(X) = \mathrm{TC}(X)$. It remains to consider the case $2r < p$. We need the following lemma.

Lemma 5.2. *Let X be a $(p-1)$-connected $(np+r)$-dimensional space with $2r+1 < p$. If $\mathrm{wTC}^{\mathrm{M}}(X) \leqslant 2n$, then $\mathrm{TC}(X) \leqslant 2n$.*

Proof. Under these assumptions the fat wedge $W^{2n}X$ is $(p-1)$ connected, while the smash product $\wedge^{2n}X$ is $(2np-1)$ connected. Therefore, by theorem 5.1 the sequence of fibrewise homotopy classes

$$[X \ltimes X, X \ltimes W^{2n}X]_X \xrightarrow{(1\ltimes i_{2n})_*} [X \ltimes X, X \ltimes \Pi^{2n}X]_X \xrightarrow{(1\ltimes q_{2n})_*} [X \ltimes X, X \ltimes \wedge^{2n}X]_X$$

is exact whenever $np+r \leqslant (p-1)+(2np-1)-(np+r)$, that is, if $2r+1<p$. If $\mathrm{wTC}^{\mathrm{M}}(X) = 2n$, then $(1 \ltimes q_{2n})_*(1 \ltimes \Delta_{2n})$ is trivial, which, by exactness, implies that $1 \ltimes \Delta_{2n}$ is in the image of $(1 \ltimes i_{2n})_*$. Therefore, there exists a fibrewise lift of $1 \ltimes \Delta_{2n}$ to $X \ltimes W^{2n}X$, so $\mathrm{TC}^{\mathrm{M}}(X) \leqslant 2n$ and, finally, $\mathrm{TC}(X) \leqslant 2n$. □

We can now summarize the relations between the topological complexity and the weak monoidal topological complexity when both are close to the maximal values given by the dimension estimate. In [9, theorem 25], García Calcines and Vandembroucq prove a similar result using the weak topological complexity wTC, which they also define via the Whitehead characterization, but employing the unpointed version of sectional category instead of the fibrewise pointed LS-category.

Theorem 5.3. *Let X be a $(p-1)$-connected $(np+r)$-dimensional space, $0 \leqslant r < p$. Each of the following conditions then implies that $\mathrm{TC}(X) = \mathrm{wTC}^{\mathrm{M}}(X)$:*

(a) $\mathrm{wTC}^{\mathrm{M}}(X) = 2n+1$,

(b) $\mathrm{wTC}^{\mathrm{M}}(X) = 2n$ *and* $2r+1<p$,

(c) $\mathrm{wTC}^{\mathrm{M}}(X) = 2n-1$, $\mathrm{wcat}(X) = n$ *and* $r+1<p$.

Proof. Theorem 3.2 states that $\mathrm{TC}(X) \leqslant 2n+1$, so the first claim is obvious. If $\mathrm{wTC}^{\mathrm{M}}(X) = 2n$, then by lemma 5.2 we have $\mathrm{TC}(X) \leqslant 2n$; hence, $\mathrm{wTC}^{\mathrm{M}}(X) = \mathrm{TC}(X)$. Finally, if $\mathrm{wcat}(X) = n$ and $r+1 < p$, then [16, theorem 2.2] implies that $\mathrm{cat}(X) = n$; hence, $\mathrm{TC}(X) \leqslant 2n-1$. □

6. Stable complexity estimates

The stable monoidal topological complexity is another lower bound for the topological complexity that is in general better than the cohomological estimate. Its properties are in a certain sense dual to the properties of the weak monoidal topological complexity, although the two estimates are in general incommensurable. Recall that the monoidal topological complexity $\mathrm{TC}^{\mathrm{M}}(X)$ can be defined as the minimal n for which the fibrewise Ganea construction $1 \ltimes p_n \colon X \ltimes G_n(X) \to X \ltimes X$ admits a section. The stable topological complexity $\sigma\,\mathrm{TC}^{\mathrm{M}}(X)$ is the minimal n such that some suspension $1 \ltimes \Sigma^i p_n \colon X \ltimes \Sigma^i G_n(X) \to X \ltimes \Sigma^i X$ admits a section. Clearly, $\sigma\,\mathrm{TC}^{\mathrm{M}}(X) \leqslant \mathrm{TC}^{\mathrm{M}}(X)$, while $\mathrm{nil}_R(X) \leqslant \sigma\,\mathrm{TC}^{\mathrm{M}}(X)$ by [8, theorem 12].

The following lemma is the fibrewise version of the classical result that a suspension map $\Sigma f \colon \Sigma Y \to \Sigma Z$ admits a section if and only if the quotient map $q \colon Z \to C_f$ is nulhomotopic (see, for example, [1, proposition B.12]).

Lemma 6.1. *Let $1 \ltimes f \colon X \ltimes Y \to X \ltimes Z$ be a fibrewise pointed map. The fibrewise suspension map $1 \ltimes \Sigma f \colon X \ltimes \Sigma Y \to X \ltimes \Sigma Z$ then admits a section if and only if the projection to the homotopy fibre $1 \ltimes q \colon X \ltimes Z \to X \ltimes C_f$ is fibrewise homotopy trivial.*

We use this lemma as the inductive step in the following.

Lemma 6.2. *Let X be a $(p-1)$-connected $(np+r)$-dimensional space with $2r+1 < p$. If $\sigma\,\mathrm{TC}^{\mathrm{M}}(X) \leqslant 2n$, then $\mathrm{TC}(X) \leqslant 2n$.*

Proof. By the definition of $\sigma\,\mathrm{TC}^{\mathrm{M}}(X)$, there exists an integer i such that the map

$$1 \ltimes \Sigma^i p_{2n} \colon X \ltimes \Sigma^i G_{2n}(X) \to X \ltimes \Sigma^i X$$

admits a section, so, by lemma 6.1, the map

$$1 \ltimes \Sigma^{i-1} q_{2n} \colon X \ltimes \Sigma^{i-1} X \to X \ltimes \Sigma^{i-1} G_{[2n]}$$

is fibrewise homotopy trivial. Since $\Sigma^{i-1} G_n$ is $(p+i-2)$ connected and $\Sigma^{i-1} G_{[2n]}$ is $(2np+i-2)$ connected, theorem 5.1 implies that the induced function

$$[X \ltimes \Sigma^{i-1} X, X \ltimes \Sigma^{i-1} G_{2n}]_X \xrightarrow{(1 \ltimes \Sigma^{i-1} p_{2n})_*} [X \ltimes \Sigma^{i-1} X, X \ltimes \Sigma^{i-1} X]_X$$

is surjective whenever $2(np+r) + (i-1) \leqslant (2n+1)p + 2i - 4$, and the preimage of the identity map on $X \ltimes \Sigma^{i-1} X$ is clearly a section of $1 \ltimes \Sigma^{i-1} p_{2n}$. In particular, if $2(np+r) \leqslant (2np-1)$, then we can inductively conclude that the maps $1 \ltimes \Sigma^{i-1} p_{2n}, 1 \ltimes \Sigma^{i-2} p_{2n}, \dots, 1 \ltimes p_{2n}$ admit a section, so $\mathrm{TC}^{\mathrm{M}}(X) \leqslant 2n$ and $\mathrm{TC}(X) \leqslant 2n$. □

We may now formulate a result that is analogous to theorem 5.3, and that summarizes the relations between the topological complexity and the stable monoidal topological complexity when both are close to the maximal values given by the dimension estimate.

THEOREM 6.3. *Let X be a $(p-1)$-connected $(np+r)$-dimensional space, $0 \leqslant r < p$. Each of the following conditions then implies that* $\mathrm{TC}(X) = \sigma\,\mathrm{TC}^{\mathrm{M}}(X)$:

(a) $\sigma\,\mathrm{TC}^{\mathrm{M}}(X) = 2n+1$,

(b) $\sigma\,\mathrm{TC}^{\mathrm{M}}(X) = 2n$ *and* $2r+1 < p$,

(c) $\sigma\,\mathrm{TC}^{\mathrm{M}}(X) = 2n-1$, $\sigma\,\mathrm{cat}(X) = n$ *and* $r+1 < p$.

Proof. Only the last case requires some comment. Clearly, $\mathrm{TC}(X) \geqslant 2n-1$. If, on the other hand, $\sigma\,\mathrm{cat}(X) = n$, then, by [1, proposition 2.56], $\mathrm{cat}(X) = n$, so $\mathrm{TC}(X) \leqslant 2n-1$. □

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