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Recognizing the second derived subgroup of free groups

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ABSTRACT

The commutator subgroup of a free group is the subgroup consisting of elements contained in the kernel of every homomorphism from the free group to the integers. We give a similar characterization of the second derived subgroup of a free group. Specifically, we show that the second derived subgroup of a free group is equal to the intersection of the kernels of all homomorphisms into a solvable deficiency 1 group.

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1. Introduction and main result

In a free group, the commutator subgroup is the set of all words such that the exponent sum of every element of the free generating set is zero. Expressed differently, the commutator subgroup is the intersection of all kernels of homomorphisms from the free group to the integers. Here we prove a similar characterization for the second derived subgroup by considering the intersection of kernels of homomorphisms of a free group into a fixed solvable deficiency 1 group.

Theorem 1. *Let F be a free group and B any deficiency 1, solvable group which is not virtually abelian. Then*

$$\bigcap_{\phi \in \hom(F,B)} \ker(\phi) = F'',$$

where F'' is the second derived subgroup of F. Thus the evaluation map e gives rise to an exact sequence

$$1 \to F'' \to F \xrightarrow{e} B^{\hom(F,B)}.$$

Recall that the deficiency of a group H is the largest number n such that H has a presentation in which the number of generators exceeds the number of relations by n. Since the second commutator subgroup of a free group has infinite index in the commutator subgroup, it is necessary to consider homomorphisms into a fixed solvable deficiency 1 group that is not virtually abelian.

We shall derive Theorem 1 from the following:

Theorem 2. *Let G be a free metabelian group and B any solvable deficiency 1 group which is not virtually abelian. Then*

$$\bigcap_{\phi \in \hom(G,B)} \ker(\phi) = 1.$$

Expressed differently, the evaluation $e : G \to B^{\hom(G,B)}$ is a monomorphism.

2. Proofs

Notation. For sets A and B, we will denote the set of functions from A to B by B^A and, when A and B are groups, the set of homomorphisms from A to B by $\hom(A, B)$. When A and B are groups, B^A carries a natural group structure as follows. For ϕ and ψ in B^A, the product is the map $\phi\psi$ given by $\phi\psi(a) = \phi(a)\psi(a)$ for $a \in A$.

For elements x, y in a group G, let $x^y = yxy^{-1}$ denote the conjugation of x by y and $[x, y] = xyx^{-1}y^{-1}$ denote the commutator of x and y. We will denote the commutator subgroup of G by G' and the second commutator subgroup by G''.

The first step to proving Theorem 2 is to characterize solvable deficiency 1 groups by applying results of Eckmann and Gildenhuys. Eckmann in [2, Corollary 3.3] showed that a finitely presented amenable group G of deficiency 1 is either isomorphic to the integers $\mathbb{Z}$ or has cohomological dimension 2. When G has cohomological dimension 2, it follows from [3, Theorem 5] that G is a Baumslag Solitar group $B(1,m) = \langle t, a \mid tat^{-1} = a^m \rangle$. One thus obtains the following result, which was first proved by Hillman in [5, Corollary to Theorem 6] for torsion free groups. We will use the following, as stated in [8]:

Theorem 3. *If H is a solvable group of deficiency 1, then H is isomorphic to a Baumslag–Solitar group $B(1,m) = \langle t, a \mid tat^{-1} = a^m \rangle$.*

The Baumslag–Solitar group $B(1,m)$ has a normal subgroup, which we will denote by A, that as a $\mathbb{Z}$-module is generated by $\{a^{t^i} : i \in \mathbb{Z}\}$. Then $B(1,m)$ can be written as a semidirect product $A \rtimes \langle t \rangle$. There is an isomorphism from $B(1,m)$ onto the semidirect product

$$B = \mathbb{Z}\left[\frac{1}{m}\right] \rtimes \mathbb{Z},$$

where

$$(a,\lambda)(a',\lambda') = (a + m^{\lambda} a', \lambda + \lambda'), \quad a \in \mathbb{Z}\left[\frac{1}{m}\right], \ \lambda, \lambda' \in \mathbb{Z}.$$

The isomorphism is induced by sending $a^{t^i} \mapsto (m^i, 0)$ for $i \in \mathbb{Z}$ and $t \mapsto (0,1)$.

The commutator subgroup G' of a metabelian group G is a right $\mathbb{Z}[G/G']$-module for the conjugation induced action of G/G'. When $G = F[x_1, \ldots, x_k]$ is the free metabelian group by a result of S. Bachmuth in [1], see also [6, p. 185] and a generalization to free groups by W. Tomaszewski in [7] (compare also [4, Theorem 5.2]), one can give a precise description of that module as follows:

Theorem 4 *(Bachmuth 1965). For the free metabelian group G with free metabelian generating set $X := \{x_1, \ldots, x_k\}$, the second commutator quotient G'/G'' is a free $\mathbb{Z}$-module with basis*

$$[x_i, x_j]^{x_i^{d_i} \cdots x_k^{d_k}}, \quad 1 \le i < j \le k.$$

This theorem shows that every element w in G'/G'' can be written in a unique fashion

$$\begin{aligned} w &= \prod_{1 \le i < j \le n} [x_i, x_j]^{u_{ij}}, \\ u_{ij} &= \sum_{d_i, \ldots, d_k \in \mathbb{Z}} a_{ij d_i \cdots d_k} x_i^{d_i} \cdots x_k^{d_k} \in \mathbb{Z}\left[x_i, \ldots, x_k, \frac{1}{x_1}, \ldots, \frac{1}{x_k}\right], \end{aligned} \tag{1}$$

where $a_{ijd_i\cdots d_k} \in \mathbb{Z}$ and the sum is finite. The terms w_{ij} are then finite *Laurent* polynomials in the variables $x_1, \ldots, x_k$.

Proof of Theorem 2. Since $\mathbb{Z}$, $\mathbb{Z}^2$, and the Klein bottle group are virtually abelian, a solvable group of deficiency 1 that is not virtually abelian is isomorphic to $B(1,m)$ where $m \neq 0, \pm 1$ by Theorem 3. Thus to prove Theorem 2, we need only consider homomorphisms into a fixed Baumslag–Solitar group $B(1,m)$ with $|m| > 1$.

Let G be a free metabelian group and $K := \bigcap_{\phi \in \hom(G,B)} \ker(\phi)$. By way of contradiction, suppose that $K \neq \{1\}$ and fix $w \in K \backslash \{1\}$.

Let X be a set of free generators of G and assume for a moment that it is infinite. Then w is a word in elements from a finite subset S of X and thus, first factoring the normal closure of $X \setminus S$ reduces the situation to the finitely generated case.

Hence, we may assume that G is finitely and freely generated by some set $X := \{x_1, \ldots, x_k\}$.

Observe that w must belong to G'; otherwise there would exist a homomorphism $\phi : G \to \langle (0,1) \rangle \cong \mathbb{Z}$ with $\phi(w) \neq 1$. The same argument implies $k \geq 2$.

By Bachmuth's result (Theorem 4), we can assume that w is written in normal form as in Eq. (1). We will induct on the number k of generators of G. We already concluded that $k \neq 1$ and we will now show that $k \geq 3$.

Suppose that $k = 2$. Then $w = [x_1, x_2]^u$ where $u = \sum_{i,j \in \mathbb{Z}} a_{ij} x_1^i x_2^j$ and a_{ij} is nonzero for only finitely many pairs (i,j). Thus we can find distinct $\lambda, \mu \in \mathbb{N}$ such that if $a_{ij}, a_{i'j'} \neq 0$ and $\lambda i + \mu j = \lambda i' + \mu j'$, then $(i,j) = (i', j')$. Thus, for any $\nu \in \mathbb{N}$, if $a_{ij}, a_{i'j'} \neq 0$ and $\nu\lambda i + \nu\mu j = \nu\lambda i' + \nu\mu j'$, then $(i,j) = (i',j')$.

Define $\phi_\nu : G \to B$, by sending $x_1 \mapsto (0, \nu\lambda)$ and $x_2 \mapsto (1, \nu\mu)$ and then extending ϕ to an element in $\hom(G,B)$ via the universal property of the free metabelian group $G = F[x_1, x_2]$. Since $w = [x_1, x_2]^u$ where u is a finite sum $u = \sum_{i,j \in \mathbb{Z}} a_{ij} x_1^i x_2^j$, an elementary computation yields

$$(0,0) = \phi_\nu(w) = \left((m^{\nu\lambda} - m^{\nu\mu}) \sum_{i,j \in \mathbb{Z}} a_{ij} m^{\nu\lambda i + \nu\mu j}, 0 \right).$$

The first coordinate being an element of the subring $\mathbb{Z}[1/m]$ of $\mathbb{Q}$ and $\lambda \neq \mu$ imply

$$\sum_{i,j} a_{ij} m^{\nu\lambda i + \nu\mu j} = 0.$$

Since this sum is 0 for all $\nu \in \mathbb{N}$ and the exponents of m are distinct whenever a_{ij} is nonzero, we can deduce that $a_{ij} = 0$ for all $i, j \in \mathbb{Z}$. Thus $u = 0$ which implies that $w = 1$, a contradiction.

We will now assume $k \geq 3$. If ψ is any homomorphism from G to a free metabelian subgroup L with rank strictly less than k, then $\psi(w) \in \bigcap_{\phi \in \hom(L,B)} \ker(\phi)$. By induction, $\bigcap_{\phi \in \hom(L,B)} \ker(\phi) = 1$ and thus $w \in \ker(\psi)$.

Let $\phi_1 : G \to L := F[x_2, \ldots, x_k]$ be the endomorphism that sends $x_1 \mapsto 1$ and $x_i \mapsto x_i$ for $2 \le i \le k$. Then, for $2 \le i < j \le k$, we can deduce that $u_{ij} = 0$ in Eq. (1). Thus $w = \prod_{1<i\le n} [x_1, x_i]^{u_{1i}}$ where $u_{1i} = \sum_{d_1,\ldots,d_k\in\mathbb{Z}} a_{1id_1\cdots d_k} x_1^{d_1} \cdots x_k^{d_k} \in \mathbb{Z}\left[x_1, \ldots, x_k, \frac{1}{x_1}, \ldots, \frac{1}{x_k}\right]$.

For $2 \le l \le k$ and $\lambda > 0$, let $\phi_{\lambda,l} : G \to F[x_1, \ldots, x_k] \le G$ be the endomorphism that sends the generator $x_l \mapsto x_1^\lambda$ and sends $x_i \mapsto x_i$ for $i \ne l$.

Then in Eq. (1) the term $[x_1, x_l]^{u_{1l}}$ is sent to 1 by $\phi_{\lambda,l}$ and for $i \notin \{1, l\}$ the term $[x_1, x_i]^{u_{1i}}$ is sent to $[x_1, x_i]^{v_i}$ where v_i results from u_{1i} by plugging in x_1^λ for x_l.

Pick $i \ne l$. We can write u_{1i} as a finite sum of the form

$$u_{1i} = \sum_{m,n\in\mathbb{Z}} a_{imn} x_1^m x_l^n,$$

where a_{imn} is a Laurent-series in the variables x_j for $j \notin \{1, l\}$. Thus under the endomorphism $\phi_{\lambda,1}$ the exponent of u_i becomes $v_i = \sum_{m,n\in\mathbb{Z}} a_{imn} x_1^{m+\lambda n}$. By choosing λ sufficiently large such that the absolute value all exponents of x_1 are pairwise distinct, we can see that the coefficients a_{imn} must all vanish and thus the term u_{1i} must also vanish.

Letting l vary from 2 to k implies $u_{1i} = 0$ for all $1 \le i \le k$ and hence $w = 1$, a final contradiction. □

Proof of Theorem 1. As we already noted, Theorem 3 implies that B is a Baumslag–Solitar group $B = B(1, m)$ with $m \ne 0, \pm 1$. Let $N = \bigcap_{\phi\in\hom(F,B)} \ker(\phi)$.

Since B is metabelian, it follows that $F'' \subset N$. Suppose there exists $g \in N \setminus F''$. Since F/F'' is free metabelian, Theorem 2 implies that there is a homomorphism $\phi : F/F'' \to B$ with $\phi(gF''/F'') \ne 1$. Letting $\pi : F \to F/F''$ denote the canonical quotient homomorphism, we see that $\phi \circ \pi(g) \ne 1$, contradicting $g \in N$. Hence $N \le F''$ holds as well. □

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