GENERAL THEORY OF LIFTING SPACES

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Abstract. In his classical textbook on algebraic topology Edwin Spanier developed the theory of covering spaces within a more general context of lifting spaces (i.e. Hurewicz fibrations that admit unique liftings of paths). Among other results Spanier proved that for every space X there exists a universal lifting space, which however need not be simply connected, unless the base space X is semi-locally simply connected. The question on what exactly is the fundamental group of the universal space was left unanswered. In this paper we develop a general theory of lifting spaces and show that for connected and locally path connected X the fundamental group of the universal space is precisely the intersection of all Spanier groups associated to open covers of X, and that the later coincides with the shape kernel of X. Then we examine in more detail the class of lifting spaces that arise as inverse limits of coverings and derive relations between their group of deck transformations and the fundamental group. Among inverse limit of coverings a special role is played by inverse limits of universal coverings over polyhedral expansions of X that are in many aspects resembling the universal coverings, in particular when X is a locally path-connected space. In the final section we consider lifting spaces over non-locally path connected base and relate them to the fibration properties of the so called hat space construction.

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1. Introduction

The theory of covering spaces is best suited for spaces with nice local behaviour when it gives a complete correspondence between coverings of X and the subgroups of its fundamental group $\pi_1(X)$. Attempts to extend the covering space theory to more general spaces include Fox overlays [5], and more recently, universal path-spaces by Fisher and Zastrow [3], see also [4], and Peano covering maps by Brodskiy, Dydak, Labuzs and Mitra [1]. Presently there is an animated debate concerning the correct way to define generalized covering spaces, fundamental groups and other related concepts in order to retain as much of the original theory as possible.

In the chapter of the textbook [10] dedicated to covering spaces Spanier considered coverings as a special case of a more general concept of fibrations

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with unique path-lifting property. These fibrations turn out to be much more flexible than coverings, in particular, one can always construct the universal fibration over a given base space X, where universality is interpreted as being the initial object in the corresponding category. This universal object however need not be simply connected, and the basic question of what is its fundamental group is left open in Spanier's book.

As a part of his approach Spanier characterized subgroups of $\pi_1(X)$ that give rise to covering spaces by showing that a covering subgroup of $\pi_1(X)$ must contain all $\mathcal{U}$-small loops with respect to some cover $\mathcal{U}$ of X. Consequently elements of $\pi_1(X)$ that are small with respect to all covers of X cannot be 'unravelled' in any covering space and even in any fibration with unique path-lifting property. This result was one of the motivations for our work.

In this paper we give a systematic treatment of lifting spaces and their properties, with particular emphasis to inverse limits of covering spaces, universal lifting spaces and their fundamental groups. In the following section we give an alternative definition of lifting spaces and derive their basic properties, which include stability under arbitrary products, compositions and inverse limits. Then we study the structure of the category of lifting spaces over a given space X and prove the existence of the universal lifting space over X. In Section 3 we study subgroups of the fundamental group $\pi_1(X)$ that correspond to covering spaces over X (without the assumption that X is semi-locally simply connected). One of the main consequences is identification of the fundamental group of the universal lifting space over X as the shape kernel of X (i.e. the kernel of the homomorphism from the fundamental group of X to its shape fundamental group). In Section 4 we restrict our attention to a special class of lifting spaces over X that can be constructed as limits of inverse systems of covering spaces over polyhedral approximations of X. We show that much of the theory of covering spaces can be extended to this more general class. At this point one may expect that every lifting space over a given base space X can be obtained as a limit of an inverse system of covering spaces over X or its approximations. However, a closer look reveals that this a subtle question, that there exists an immense variety of lifting spaces, and that even the detection of those that are inverse limits of coverings is a non-trivial problem. We will give a detailed analysis of this and related question in a forthcoming paper. Nevertheless, we can construct a universal lifting spaces for the class of lifting spaces studied in Section 4. This is achieved in Section 5 by taking a polyhedral expansion of the base space and considering the corresponding inverse system of universal coverings. It turns out that the so obtained universal lifting space is in many aspects analogous to the universal covering space, especially when the base space is locally path-connected. In order to improve our grasp on spaces that are not locally path-connected we dedicate the final section to the study of the fibration properties of the hat construction, which to a non-locally path-connected space assigns the 'closest' locally path-connected one. The main result is that by applying the hat construction on a locally compact metric space we obtain a fibration (in fact, a lifting projection) if and only if the hat space is locally compact.

2. Fundamental groups of lifting spaces

Much of the exposition of covering spaces in [10] is done in a more general setting of (Hurewicz) fibrations with unique path lifting property. To work with Hurewicz fibrations we will use the following standard characterization in terms of lifting functions. Every map $p\colon L \to X$ induces a map $\overline{p}\colon L^I \to X^I \times L$, $\overline{p}\colon \gamma \mapsto (p \circ \gamma, \gamma(0))$. In general $\overline{p}$ is not surjective, in fact its image is the subspace

$$X^I \sqcap L := \{(\gamma, l) \in X^I \times L \mid p(l) = \gamma(0)\} \subset X^I \times L.$$

A *lifting function* for p is a section of $\overline{p}$, that is, a map $\Gamma\colon X^I \sqcap L \to L^I$ such that $\overline{p} \circ \Gamma$ is the identity map on $X^I \sqcap L$. Then we have the following basic result:

Theorem 2.1. *(cf.* [7, Theorem 1.1]*) A map $p\colon L \to X$ is a Hurewicz fibration if and only if it admits a continuous lifting function* Γ.

Moreover, unique path-lifting property for p means that for $(\gamma, l), (\gamma', l') \in X^I \sqcap L$ the equality $\Gamma(\gamma, l) = \Gamma(\gamma', l')$ implies that $(\gamma, l) = (\gamma', l')$. This condition is clearly is equivalent to the injectivity of $\overline{p}$, which leads to the following definition.

A map $p\colon L \to X$ is a *lifting projection* if $\overline{p}\colon L^I \to X^I \sqcap L$ is a homeomorphism, or equivalently, if the following diagram

$$\begin{array}{ccc} L^I & \xrightarrow{\mathrm{ev}_0} & L \\ {\scriptstyle p\circ -}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X^I & \xrightarrow[\mathrm{ev}_0]{} & X \end{array}$$

is a pull-back in the category of topological spaces. Given a path $\gamma\colon I \to X$ and an element $l \in L$ with $p(l) = \gamma(0)$ we will denote by $\langle \gamma, l \rangle$ the unique path in L which starts at l and covers γ (i.e. $\overline{p}(\langle \gamma, l \rangle) = (\gamma, l)$). The *lifting space* is the triple (L, p, X) where $p\colon L \to X$ is a lifting projection. We will occasionally abuse the notation and refer to the space L or the map $p\colon L \to X$ itself as a lifting space over X.

Clearly, every covering map is a lifting projection. Before giving further examples we list some basic properties of lifting spaces (cf. [10], Section 2.2., short proofs are included here to illustrate the efficiency of the alternative definition).

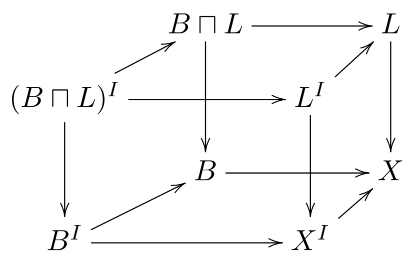
Proposition 2.2. (1) *Arbitrary pull-backs, compositions, products, fibred products and inverse limits of lifting spaces are lifting spaces.*

(2) *In a lifting space $p\colon L \to X$, given a path $\gamma\colon I \to X$ the formula $f_\gamma\colon l \mapsto \langle \gamma, l\rangle(1)$ determines a homeomorphism $f_\gamma\colon p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ between the fibres. In particular, if X is path-connected then any two fibres of p are homeomorphic.*

(3) *A fibration $p\colon L \to X$ is a lifting space if, and only if its fibres are totally path-disconnnected (i.e. admit only constant paths).*

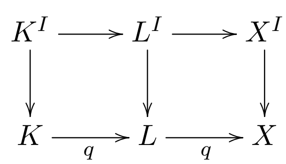
Proof. (1) All claims follow from general facts about pull-backs as we now show.

Pull-backs: Let $p\colon L \to X$ be a lifting projection and let $f\colon B \to X$ be any map. Then in the following commutative cube



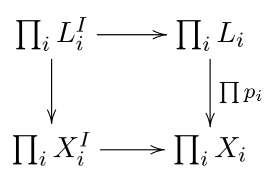
the front, back and right vertical face are pull-backs, which by abstract nonsense implies that the left vertical square is also a pull-back. Therefore the pull-back projection $B \sqcap L \to B$ is a lifting projection.

Compositions: If $p\colon L \to X$ and $q\colon K \to L$ are lifting projections then in the following diagram



the two inner squares are pull-backs, which implies that the outer square is a pull-back, hence $p \circ q$ is also lifting projection.

Products: If $\{p_i\colon L_i \to X_i\}$ is a family of lifting projections, then the diagram



is a product of pull-back diagrams, hence a pull-back diagram itself. It follows that $\prod p_i$ is a lifting projection.

Fibred products: The fibred product of a family $\{p_i\colon L_i \to X\}_{i\in\mathcal{I}}$ of lifting spaces over X is obtained by pulling back their product along the diagonal map $X \to X^{\mathcal{I}}$, hence is a lifting space by the above.

Inverse limits: An inverse system of lifting spaces is given by a directed set $\mathcal{I}$, two $\mathcal{I}$-indexed inverse systems $\mathbf{L} = (L_i, u_{ij}\colon L_j \to L_i)$ and $\mathbf{X} = (X_i, v_{ij}\colon X_j \to X_i)$, and a morphism of systems $\mathbf{p}\colon \mathbf{L} \to \mathbf{X}$, such that $p_i\colon L_i \to X_i$ is a lifting projection for all $i \in \mathcal{I}$. In order to prove that the limit map

$$\varprojlim p_i\colon \varprojlim L_i \to \varprojlim X_i$$

is a lifting projection it is sufficient to observe that we have the natural identifications

$$\Big(\varprojlim L_i\Big)^I = \varprojlim L_i^I, \qquad \Big(\varprojlim X_i\Big)^I \sqcap \Big(\varprojlim L_i\Big) = \varprojlim \big(X_i^I \sqcap L_i\big)\ ,$$

and that the projection $\Big(\varprojlim L_i\Big)^I \longrightarrow \Big(\varprojlim X_i\Big)^I \sqcap \Big(\varprojlim L_i\Big)$ is a homeomorphism because it is the inverse limit of homeomorphisms $\overline{p}_i\colon L_i^I \xrightarrow{\approx} X_i^I \sqcap L_i$.

(2) Let $\overline{\gamma}$ denote the inverse path of the path γ. We claim that the map $f_{\overline{\gamma}}\colon p^{-1}(y) \to p^{-1}(x),\ \ l \mapsto \langle\overline{\gamma}, l\rangle(1)$ is the inverse of f_γ. Indeed, since $\overline{p}\big(\overline{\langle\gamma, l\rangle}\big) = \big(\overline{\gamma}, \langle\gamma, l\rangle(1)\big)$ we get the equality $\overline{\langle\gamma, l\rangle} = \langle\overline{\gamma}, \langle\gamma, l\rangle(1)\rangle$ and so

$$f_{\overline{\gamma}}(f_\gamma(l)) = \langle\overline{\gamma}, \langle\gamma, l\rangle(1)\rangle(1) = \langle\overline{\gamma}, \langle\gamma, l\rangle(1)\rangle(1) = \overline{\langle\gamma, l\rangle}(1) = \langle\gamma, l\rangle(0) = l.$$

That $f_\gamma f_{\overline{\gamma}}$ is also the identity map is proved analogously.

(3) Assume $p\colon L \to X$ is a lifting space and let γ be a path in $p^{-1}(x) \subset L$. Then $\overline{p}(\gamma) = (\mathrm{const}_x, \gamma(0)) = \overline{p}(\mathrm{const}_{\gamma(0)})$, hence $\gamma = \mathrm{const}_{\gamma(0)}$.

Conversely, assume that all fibres of p admit only constant paths. Since p is a fibration there is a map $\Gamma\colon X^I \sqcap L \to L^I$ such that $\overline{p} \circ \Gamma = \mathrm{Id}$, and we only need to prove that $\gamma = \Gamma(p \circ \gamma, \gamma(0))$ for all $\gamma\colon I \to L$. For $s \in I$ let γ_s denote the path $\gamma_s(t) := \gamma(st)$, and let H be the standard homotopy starting at $\overline{(p\gamma)_s} \cdot (p\gamma)_s$ and ending at $\mathrm{const}_{p\gamma(0)}$. Let moreover $\widetilde{H}\colon I \times I \to L$ be a lifting of H starting at $\widetilde{H}|_{0\times I} = \overline{\gamma_s} \cdot \Gamma(p\gamma, \gamma(0))_s$. It is easy to check that the restriction of $\widetilde{H}$ to $I \times 0 \cup 1 \times I \cup I \times 1$ determines a path in the fibre $p^{-1}(p\gamma(t))$ from $\gamma(s)$ to $\Gamma(p\gamma, \gamma(0))(s)$, so by the assumption $\gamma(s) = \Gamma(p\gamma, \gamma(0))(s)$. □

We have recently proved in [8, Theorem 3.2] that, under very general assumptions, lifting spaces are preserved by the mapping space construction, which yields a host of examples of lifting spaces that are very far from being coverings. The following examples illustrate typical ways how a lifting space can fail to be a covering space.

Example 2.3. *Let $p\colon \mathbb{R} \to S^1$ be the usual covering of the circle. Then the countable product $p^{\mathbb{N}}\colon \mathbb{R}^{\mathbb{N}} \to (S^1)^{\mathbb{N}}$ is a lifting space by Proposition 2.2, but is not a covering space. In fact, the fibre of p is not a discrete space, being an infinite product of $\mathbb{Z}$. Even more drastically, one can easily verify that the* *infinite product of circles is not semi-locally simply connected at any point, which means that it cannot have at all a simply connected covering space.*

Example 2.4. *Another basic example is given by the following inverse limit of 2^n-fold coverings*

$$\begin{array}{ccccccccc} S^1 & \xleftarrow{2} & S^1 & \xleftarrow{2} & S^1 & \longleftarrow & \cdots & \longleftarrow & \mathbb{S}_2 \\ {\scriptstyle 2}\downarrow & & {\scriptstyle 4}\downarrow & & {\scriptstyle 8}\downarrow & & & & \downarrow{\scriptstyle p} \\ S^1 & = & S^1 & = & S^1 & = & \cdots & = & S^1 \end{array}$$

which presents the dyadic solenoid $\mathbb{S}_2$ as a lifting space over the circle. By varying the choice of coverings we obtain an entire family of non-equivalent lifting spaces over the circle leading to the following interesting problem: is it possible to classify all lifting spaces over the circle? One should keep in mind that this necessarily require the study of non-locally path-connected total spaces. In fact, Spanier [10, Proposition 2.4.10] *proved that a lifting space $p\colon L \to X$ over a locally path-connected and semi-locally simply-connected base X is a covering space if, and only if L is locally path-connected.*

Example 2.5. *An important role in the theory of non-locally-path-connected spaces is played by the so called* hat construction*: given any space X let $\widehat{X}$ denote the same set re-topologized by taking the minimal topology that contains all path components of open sets in X. Obviously, if X is locally path-connected then $\widehat{X} = X$, but for non locally path-connected spaces we obtain a strictly stronger topology, so the identity map $\iota\colon \widehat{X} \to X$ is a continuous bijection but not a homeomorphism. For example, if W is the standard Warsaw circle, then one can easily check that $\widehat{W}$ is homeomorphic to the interval $[0, 1)$. Observe that the hat-construction is functorial (in fact, together with the projection to the original space it forms an idempotent augmented functor), so that for every map $f\colon Y \to X$ we obtain a commutative diagram*

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\widehat{f}} & \widehat{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow[f]{} & X \end{array}$$

It follows that every map from a locally path-connected space to X lifts uniquely to a map to $\widehat{X}$, so in particular, the projection from the hat space admits unique path liftings. Even more, it is always a Serre fibration, but it is not in general a Hurewicz fibration (and hence not a lifting space). We are going to study this question in detail in the last section of the paper.

Every covering space and every locally trivial fibration is an open map. In view of the above examples it would be interesting to know whether all lifting projections over a locally path-connected base are open maps.

In the solenoid example above the total space is not path-connected. Clearly, if $p\colon L \to X$ is a lifting space then the restriction of p to any path-component of L is a lifting space, too. In order to study the fundamental groups of lifting spaces, we now restrict our attention to based path-connected spaces. Let Lift_X denote the category whose objects are path-connected open lifting spaces over X, and morphisms are fibre-preserving maps between them. All spaces have base-points and all maps are base-point preserving, but we systematically omit the base-points from the notation. The category Lift_X shares many properties with its full subcategory of covering spaces Cov_X but is in some aspects more flexible.

Proposition 2.6. *Morphisms in* Lift_X *are lifting projections and* Lift_X *is an ordered category (i.e. there is at most one morphism between any two objects).*

Proof. Let $f\colon L \to K$ be a morphism between lifting spaces $p\colon L \to X$ and $q\colon K \to X$. Then we have the natural identification $K^I \sqcap L = (X^I \sqcap K) \sqcap L = X^I \sqcap L = L^I$ induced by f, therefore f is a lifting projection.

If $f, g\colon L \to K$ are morphisms in Lift_X then the unique path-lifting property imply that f and g coincide on path-components. Since f and g coincide on the base-point, and since L is path-connected, we have $f = g$. □

Clearly the category Lift_X has equalizers as there are no parallel pairs of distinct maps. It also has products: one can easily check that the categorical product of a set of lifting spaces $\{L_i \to X\}_{i\in\mathcal{I}}$ is obtained by taking the path-component of the fibred product of $\{L_i \to X\}_{i\in\mathcal{I}}$ containing the base-point. Since categorical products and equalizers suffice for the construction of any set-indexed categorical limit we obtain the following fact.

Proposition 2.7. *Category* Lift_X *has arbitrary small (i.e. set-indexed) limits.*

Corollary 2.8. *For every path-connected space* X *the category* Lift_X *has the universal (initial) object* $\widetilde{X}$. *The correspondence* $X \mapsto \widetilde{X}$ *determines an idempotent augmented functor, as* $f\colon X \to Y$ *induce the commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Proof. We first observe that the isomorphism classes of objects in Lift_X form a set. In fact by [10, Theorem 2.3.9] the points on any fibre of a lifting space $p\colon L \to X$ are in bijection with the set cosets of the subgroup $p_\sharp(\pi_1(L))$ in $\pi_1(X)$. This means that every lifting space over X corresponds to a choice of a subgroup of $\pi_1(X)$, together with a choice of a topology on the cartesian product of the set X with the set of cosets of $p_\sharp(\pi_1(L))$ in $\pi_1(X)$. We conclude that the class of possible lifting spaces over X whose total spaces is path-connected, forms a set. By Proposition 2.7 the categorical product of a set of representatives of all objects in Lift_X exists and is clearly the initial object of the category. The other properties of the universal lifting space follow from general properties of initial objects. □

As for covering spaces, it is of crucial importance to determine the fundamental group of the universal lifting space. The following result is a step in that direction.

Proposition 2.9. *Fundamental group of a categorical product in* Lift_X *is the intersection of the fundamental groups of its factors.*

Proof. Let $p\colon L \to X$ be the categorical product of the family of lifting spaces $\{p_i\colon L_i \to X\}$. By Proposition 2.6 the projection maps $q_i\colon L \to L_i$ are lifting projections, so by [10, Theorem 2.3.4] they induce monomorphisms $(q_i)_\sharp\colon \pi_1(L) \to \pi_1(L_i)$. It follows that $\pi_1(L) \cong \operatorname{Im} p_\sharp \leq \pi_1(X)$ is contained in $\bigcap_i \operatorname{Im}(p_i)_\sharp$. For the converse implication, let the loop $\alpha\colon S^1 \to X$ represent an element of $\bigcap_i \pi_1(L_i) \cong \bigcap_i \operatorname{Im}(p_i)_\sharp \leq \pi_1(X)$. By the unique path lifting property there are unique lifts $\alpha_i\colon S^1 \to L_i$ for the loop α, so they define an element $\widetilde{\alpha} \in \pi_1(L)$. We therefore conclude that $\pi_1(L) \cong \bigcap_i \pi_1(L_i)$. □

In order to achieve a more precise identification of the fundamental group of $\widetilde{X}$ we need a better understanding of subgroups of the $\pi_1(X)$ that correspond to covering spaces of X.

3. Covering subgroups

In this section we consider the question which subgroups of the fundamental group of X correspond to coverings of X and relate them to the shape kernel of X.

Let (X, x_0) be a based space. A subgroup $G \leq \pi_1(X, x_0)$ is a *covering subgroup* if there is a covering space $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$, such that $\operatorname{Im} p_\sharp = G$. Spanier gave a simple characterization of covering subgroups in terms of $\mathcal{U}$-small loops. Given a covering $\mathcal{U}$ of X a based loop in (X, x_0) is said to be *$\mathcal{U}$-small* if it is of the form $\gamma \cdot \alpha \cdot \overline{\gamma}$ where α is a (non-based) loop whose image is contained in some element of $\mathcal{U}$ and γ is a path in X connecting x_0 and $\alpha(0)$. We denote by $\pi_1(X, x_0; \mathcal{U})$ the subgroup of $\pi_1(X, x_0)$ generated by classes of $\mathcal{U}$-small loops. It is clear that $\pi_1(X, x_0; \mathcal{U})$ is always a normal subgroup of $\pi_1(X, x_0)$, and that $\pi_1(X, x_0; \mathcal{V})$ is contained in $\pi_1(X, x_0; \mathcal{U})$ whenever $\mathcal{V}$ is a covering of X that refines $\mathcal{U}$.

The covering subgroups can be characterized as follows:

Theorem 3.1 ([10] Lemma 2.5.11 and Theorem 2.5.13)**.** *Let X be connected and locally path-connected. Then $G \leq \pi_1(X, x_0)$ is a covering subgroup if, and only if G contains a subgroup of the form $\pi_1(X, x_0; \mathcal{U})$ for some cover $\mathcal{U}$ of X.*

A natural source of covering subgroups are continuous maps into polyhedra (or more generally, into semi-locally simply connected spaces).

Corollary 3.2. *Let $f\colon X \to K$ be a map from X to a semi-locally simply connected space K. Then the kernel of the induced homomorphism $f_\sharp\colon \pi_1(X, x_0) \to \pi_1(K, f(x_0))$ is a covering subgroup of $\pi_1(X, x_0)$.*

Proof. Let $\mathcal{U}$ be a cover of K, such that for all $U \in \mathcal{U}$ the inclusion $U \hookrightarrow K$ induces a trivial homomorphism on the fundamental group. Then the group $\pi_1(X, x_0; f^{-1}\mathcal{U})$ is contained in the kernel of $f_\sharp$ because f maps every $f^{-1}\mathcal{U}$-small loop in X to a homotopically trivial loop in K. Theorem 3.1 then implies that $\operatorname{Ker} f_\sharp$ is a covering subgroup of $\pi_1(X, x_0)$. □

For a partial converse to the above result assume that X has a numerable cover $\mathcal{U} = \{U\}$ with a subordinated locally finite partition of unity $\{\rho_U\}$, and let $|\mathcal{U}|$ denote the geometric realization of the nerve of $\mathcal{U}$. Then the formula $f(x) := \sum_{U \in \mathcal{U}} \rho_U(x) \cdot U$ defines a map $f\colon X \to |\mathcal{U}|$. It is well known that the choice of the partition of unity does not affect the homotopy class of f, so the induced homomorphism $f_\sharp$ depends only on the cover $\mathcal{U}$.

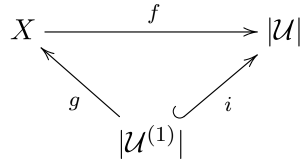
Lemma 3.3. *Let X be a path-connected space with a numerable cover $\mathcal{U}$ consisting of path-connected open sets. Then there is a short exact sequence*

$$1 \longrightarrow \pi_1(X, x_0; 2\mathcal{U}) \longrightarrow \pi_1(X, x_0) \xrightarrow{f_\sharp} \pi_1(|\mathcal{U}|, f(x_0)) \longrightarrow 1,$$

where $2\mathcal{U}$ is the cover of X consisting of all unions of pairs of intersecting sets in $\mathcal{U}$.

Proof. By a suitable modification of the partition of unity we may assume without loss of generality that every $U \in \mathcal{U}$ contains some point $x_U \in U$ such that $\rho_U(x_U) = 1$.

For every intersecting pair of sets $U, V \in \mathcal{U}$ choose a path in $U \cup V$ between x_U and x_V. These paths determine a map $g\colon |\mathcal{U}^{(1)}| \to X$. For every 1-simplex σ in $|\mathcal{U}^{(1)}|$ the image $f(g(\sigma))$ is contained in the open star of σ, which implies that the map $f \circ g$ is contiguous to the inclusion $i\colon |\mathcal{U}^{(1)}| \hookrightarrow |\mathcal{U}|$ so the following diagram homotopy commutes:



By applying the fundamental group functor we obtain the diagram

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{\quad f_\sharp \quad} & \pi_1(|\mathcal{U}|, f(x_0)) \\ & {\scriptstyle g_\sharp}\nwarrow \qquad \nearrow{\scriptstyle i_\sharp} & \\ & \pi_1(|\mathcal{U}^{(1)}|, f(x_0)) & \end{array}$$

Since the homomorphism $i_\sharp$ is surjective, so must be $f_\sharp$.

If $U, V \in \mathcal{U}$ intersect then every loop, whose image is contained in $U \cup V$ is mapped by f to the star of the simplex in $|\mathcal{U}|$ spanned by the vertices U and V. It follows that $f_\sharp$ is trivial on $2\mathcal{U}$-small loops, therefore $\pi_1(X, x_0; 2\mathcal{U}) \subseteq \operatorname{Ker} f_\sharp$.

For the converse we extend the above diagram to obtain the following one:

$$\begin{array}{ccccc} \operatorname{Ker} i_\sharp & \longrightarrow & \pi_1(|\mathcal{U}^{(1)}|, u_0) & \xrightarrow{\quad i_\sharp \quad} & \pi_1(|\mathcal{U}|, u_0) \cong \pi_1(X,x_0)/\operatorname{Ker} f_\sharp \\ {\scriptstyle g_\sharp}\downarrow & & {\scriptstyle g_\sharp}\downarrow & \nearrow{\scriptstyle f_\sharp} & \downarrow \\ \pi_1(X, x_0; 2\mathcal{U}) & \longrightarrow & \pi_1(X, x_0) & \longrightarrow & \pi_1(X,x_0)/\pi_1(X,x_0;2\mathcal{U}) \end{array}$$

Since $\operatorname{Ker} i_\sharp$ is generated by loops given by the boundaries of 2-simplexes in the nerve of $\mathcal{U}$ we may use the same method as in the proof of Theorem 7.3(2) in [2] to show that every such loop is mapped by $g_\sharp$ to a sum of $2\mathcal{U}$-small loops, so $g_\sharp(\operatorname{Ker} i_\sharp) \subseteq \pi_1(X, x_0; 2\mathcal{U})$. Thus we obtain the induced natural map $\pi_1(X,x_0)/\operatorname{Ker} f_\sharp \to \pi_1(X,x_0)/\pi_1(X,x_0;2\mathcal{U})$, and so $\operatorname{Ker} f_\sharp \subseteq \pi_1(X, x_0; 2\mathcal{U})$. □

Lemma 3.4. *For every cover $\mathcal{U}$ of a paracompact space X there is a numerable cover $\mathcal{V}$ such that its double $2\mathcal{V}$ is a refinement of $\mathcal{U}$. Moreover, if X is locally path-connected, then $\mathcal{V}$ can be chosen so that its elements are path connected.*

Proof. Let $\mathcal{U}'$ be a locally finite refinement of $\mathcal{U}$ with a subordinated partition of unity. Then we have a map $f\colon X \to |\mathcal{U}'|$ defined as before. Let $\mathcal{V}$ be the cover of X obtained by taking preimages of open stars of vertexes in the barycentric subdivision of the nerve of $\mathcal{U}'$. Clearly, two elements of $\mathcal{V}$ can have a non-empty intersection only if they are both contained in some element of $\mathcal{U}'$, which means that $2\mathcal{V}$ refines $\mathcal{U}'$, and hence $\mathcal{U}$.

If X is locally path-connected, then we can further refine $\mathcal{V}$ by taking the cover formed by the path-components of elements of $\mathcal{V}$. □

Assume that a pointed space (X, x_0) can be represented as a limit of an inverse system of pointed polyhedra $(X, x_0) = \varprojlim \big((K_i, k_i), p_{ij}, \mathcal{I}\big)$ (or more generally, that X has a polyhedral resolution in the sense of [6]). Then the homomorphisms $(p_i)_\sharp\colon \pi_1(X, x_0) \to \pi_1(K_i, k_i)$ induce a homomorphism $\partial\colon \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$, where $\check{\pi}_1(X, x_0)$ is the so called *first shape group* of X, and is defined as the limit of the inverse system of fundamental groups

$\big(\pi_1(K_i,k_i),(p_{ij})_\sharp,\mathcal{I}\big)$. Although the definitions are based on a specific resolution of X, it is a standard fact (see [6]) that both the first shape group of X and the homomorphism $\partial\colon \pi_1(X,x_0)\to\check{\pi}_1(X,x_0)$ are independent of the chosen resolution for X.

We are now ready to prove the main theorem of this section which relates various subgroups of the fundamental group.

Theorem 3.5. *Let X be a connected, locally path-connected and paracompact space. Then the the following subgroups of $\pi_1(X,x_0)$ coincide:*

(1) *Intersection of all groups of $\mathcal{U}$-small loops $\pi_1(X,x_0;\mathcal{U})$ for $\mathcal{U}$ a cover of X.*
(2) *Intersection of all covering subgroups of X.*
(3) *Intersection of all kernels* Ker $f_\sharp$ *for maps f from X to a polyhedron.*
(4) *The* shape kernel *of X, defined as* ShKer(X) $:=$ Ker$(\partial\colon \pi_1(X)\to\check{\pi}_1(X))$.

Proof. The inclusion (1)$\subseteq$(2) follows from Theorem 3.1 because every covering subgroup contains some subgroup of $\mathcal{U}$-small loops. Corollary 3.2 implies that (2)$\subseteq$(3).

To prove the inclusion (3)$\subseteq$(1) observe that by Lemma 3.3 (3) is contained in the intersection of all groups of the form $\pi_1(X,x_0;2\mathcal{U})$, while by Lemma 3.4 the latter is contained in (4).

Finally the equality (3)=(4) amounts to the standard description of the shape kernel. □

As a consequence we obtain the following description of the fundamental group of the universal lifting space:

Theorem 3.6. *If X is a connected, locally path-connected and paracompact then $\pi_1(\widetilde{X})$ coincides with the shape kernel of X.*

Proof. By Corollary 2.8 and Proposition 2.9 the fundamental group of the universal lifting space is contained in the intersection of all covering subgroups of $\pi_1(X)$, which by Theorem 3.5 coincides with the shape kernel of X. For the converse, take a loop α representing an element of the intersection $\bigcap \pi_1(X;\mathcal{U})$, and approximate α by a sequence of homotopically trivial loops α_i, such that $\alpha\simeq\alpha_i (\mathrm{mod}\,\mathcal{U}_i)$. As each α_i lift to a loop in the universal space, we may apply the fibration property to show that α also lift to a loop, hence $\alpha\in\pi_1(\widetilde{X})$. □

4. Inverse limits of coverings

Inverse limits of coverings are in many aspects the most tractable class of lifting spaces (with the exception of coverings, of course). To simplify the notation we agree that all spaces have base-points which are omitted from the notation whenever they are not explicitly used, and all maps are base-point preserving.

Let $\mathcal{I}$ be a directed set and $\mathbf{X} = (X_i, u_{ij} \colon X_j \to X_i, i, j \in \mathcal{I})$ an $\mathcal{I}$-indexed inverse system of path-connected and semi-locally simply-connected spaces. For each $i \in \mathcal{I}$ let $q_i \colon \widetilde{X}_i \to X_i$ be the universal cover of X_i. By the standard lifting criterion for maps between covering spaces (see [10, Theorem 2.4.5]) there are unique maps $\tilde{u}_{i,j} \colon \widetilde{X}_j \to \widetilde{X}_i$ such that the diagram

$$\begin{array}{ccc} \widetilde{X}_j & \xrightarrow{\tilde{u}_{ij}} & \widetilde{X}_i \\ {\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle q_i} \\ X_j & \xrightarrow[u_{ij}]{} & X_i \end{array}$$

commutes. Furthermore, $\widetilde{\mathbf{X}} := \big(\widetilde{X}_i, \tilde{u}_{i,j}, \mathcal{I}\big)$ is an inverse system of spaces and $\mathbf{q} := (q_i) \colon \widetilde{\mathbf{X}} \to \mathbf{X}$ is a level-preserving mapping of inverse systems.

More generally, let us for every $i \in \mathcal{I}$ choose a subgroup $G_i \leq \pi_1(X_i)$ and consider maps $q_i \colon \widetilde{X}_i \to \widetilde{X}_i/G_i$ and $p_i \colon \widetilde{X}_i/G_i \to X_i$, where p_i is the covering projection corresponding to the group G_i. If $(u_{ij})_\sharp(G_j) \subseteq G_i$, then again by the lifting theorem for covering spaces there exist unique maps $\bar{u}_{ij} \colon \widetilde{X}_j/G_j \to \widetilde{X}_i/G_i$ such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{X}_j & \xrightarrow{\tilde{u}_{ij}} & \widetilde{X}_i \\ {\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle q_i} \\ \widetilde{X}_j/G_j & \xrightarrow{\bar{u}_{ij}} & \widetilde{X}_i/G_i \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle p_i} \\ X_j & \xrightarrow[u_{ij}]{} & X_i \end{array}$$

We will say that the an $\mathcal{I}$-indexed family of groups $G_i \leq \pi_1(X_i)$ form a *coherent thread* with respect to the inverse system $\mathbf{X}$ if $(u_{ij})_\sharp(G_j) \subseteq G_i$ for all $i \leq j$ or, in other words, if $\mathbf{G} = \big(G_i, (u_{ij})_\sharp, \mathcal{I}\big)$ is an inverse system of groups.

Clearly, every coherent thread $\mathbf{G}$ for $\mathbf{X}$ induces an inverse system of spaces $\widetilde{\mathbf{X}}/\mathbf{G} := (\widetilde{X}_i/G_i, \bar{u}_{ij}, \mathcal{I})$. Moreover, the inverse systems $\mathbf{X}$, $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{X}}/\mathbf{G}$ are related by level preserving morphisms of inverse systems $\mathbf{p} := (p_i) \colon \widetilde{\mathbf{X}}/\mathbf{G} \to \mathbf{X}$ and $\mathbf{q} := (q_i) \colon \widetilde{\mathbf{X}} \to \mathbf{X}/\mathbf{G}$. Observe that the systems $\widetilde{\mathbf{X}}$ and $\mathbf{X}$ may be viewed as special instances of $\widetilde{\mathbf{X}}/\mathbf{G}$ with respect to coherent threads consisting of trivial groups or of groups $G_i = \pi_1(X_i)$ respectively.

Since the bonding maps are base-point preserving, the inverse limits $X := \varprojlim \mathbf{X}$, $\widetilde{X} := \varprojlim \widetilde{\mathbf{X}}$ and $\widetilde{X}_{\mathbf{G}} := \varprojlim \widetilde{\mathbf{X}}/\mathbf{G}$ are non-empty. We will denote by $u_i \colon X \to X_i$, $\tilde{u}_i \colon \widetilde{X} \to \widetilde{X}_i$ and $\bar{u}_i \colon \widetilde{X}_{\mathbf{G}} \to \widetilde{X}_i/G_i$ the projections from the limit spaces to the system.

Proposition 4.1. *The limit map* $p_{\mathbf{G}} := \varprojlim \mathbf{p}\colon \widetilde{X}_{\mathbf{G}} \to X$ *is a lifting projection. Moreover, if* X *is locally path-connected then* $p_{\mathbf{G}}$ *is an open map.*

Proof. The first claim follows directly from the fact that inverse limits of lifting projections is a lifting projection, as proved in Proposition 2.2.

Toward the proof of the second claim observe that X and $\widetilde{X}_{\mathbf{G}}$ may be viewed as subspaces of the topological products $\prod_{i\in\mathcal{I}} X_i$ and $\prod_{i\in\mathcal{I}} X_i/G_i$, respectively. Therefore, it is sufficient to show that $p_{\mathbf{G}}(\widetilde{U})$ is open in X for every sub-basic open set of the form $\widetilde{U} = X_{\mathbf{G}} \cap (\widetilde{U}_i \times \prod_{j\neq i} X_j/G_j)$, where $\widetilde{U}_i$ is an open subset of X_i/G_i that is homeomorphically projected to an elementary open subset $U_i = p_i(\widetilde{U}_i) \subset X_i$. Let $x = (x_i) \in p_{\mathbf{G}}(\widetilde{U})$ be the projection of the point $\tilde{x} = (\tilde{x}_i) \in \widetilde{U}$. Since X is locally path-connected, there exists a path-connected open set $V \subset X$ such that $x \in V \subset X \cap (U_i \times \prod_{j\neq i} X_j)$. For every $y \in V$ we can find a path $\alpha\colon (I,0,1) \to (V,x,y)$. Then there is a unique lifting $\widetilde{\alpha}\colon (I,0) \to (\widetilde{U},\tilde{x})$ of α along the lifting projection p. As $p_i\colon \widetilde{U}_i \approx U_i$ the construction of the lifting function implies that $\widetilde{\alpha}(1)_i \in \widetilde{U}_i$, therefore $y = p(\widetilde{\alpha}(1)) \in p(\widetilde{U})$. We may therefore conclude that $V \subset p(\widetilde{U})$, and consequently, that $p(\widetilde{U})$ is open in X. □

The same argument can be used to prove a more general statement that if $\mathbf{G}$ and $\mathbf{G}'$ are coherent threads such that $G_i \leq G'_i \leq \pi_1(X_i)$ for every $i \in \mathcal{I}$, then we obtain a lifting projection $\widetilde{X}_{\mathbf{G}} \to \widetilde{X}_{\mathbf{G}'}$, which is an open map, whenever $\widetilde{X}_{\mathbf{G}'}$ is locally path-connected.

In the following proposition we give an explicit description of the fibre of $p_{\mathbf{G}}$ in terms of the fundamental groups of the spaces in the inverse system $\mathbf{X}$ and the coherent thread $\mathbf{G}$:

Proposition 4.2. *The fibre of* $p_{\mathbf{G}}$ *is naturally homeomorphic to the inverse limit of the system of cosets* $\big(\pi_1(X_i)/G_i, (u_{ij})_\sharp, \mathcal{I}\big)$.

Proof. For each $i \in \mathcal{I}$ let x_i and $\bar{x}_i$ denote respectively the base-points of X_i and $\widetilde{X}_i/G_i$, and let $x = (x_i)$ be the base-point of X. The corresponding fibres over x_i and x are $F_i := p_i^{-1}(x_i) \subset \widetilde{X}_i/G_i$ and $F := p_{\mathbf{G}}^{-1}(x_i)$. If $i \leq j$ then the restriction of $\bar{u}_{ij}$ maps F_j to F_i so we obtain the inverse system $\big(F_i, \bar{u}_{ij}, \mathcal{I}\big)$ whose limit is precisely F, and the projection maps $F \to F_i$ may be identified with the restrictions of the projections $\bar{u}_i\colon \widetilde{X}_{\mathbf{G}} \to \widetilde{X}_i/G_i$.

It is well-known that the function $\partial\colon \pi_1(X_i) \to F_i$, that to every loop $\alpha \in \pi_1(X_i)$ assigns the end point of the lifting of α to $\widetilde{X}_i$, $\partial(\alpha) := \langle\widetilde{\alpha}, \bar{x}_i\rangle(1) \in F_i$, induces a bijection $l_i\colon \pi_1(X_i)/G_i \to F_i$. The bijections l_i are compatible with the bonding homomorphisms in the inverse system, as for each $i \leq j$ we have a commutative diagram

$$\begin{array}{ccc} \pi_1(X_j)/G_j & \xrightarrow{(\bar{u}_{ij})_\sharp} & \pi_1(X_i)/G_i \\ {\scriptstyle l_j}\downarrow & & \downarrow{\scriptstyle l_i} \\ F_j & \xrightarrow[\bar{u}_{ij}]{} & F_i \end{array}$$

Observe that the bijections l_i are actually homeomorphisms, as the fibres of covering spaces are discrete topological spaces. We conclude that the morphisms of inverse systems (l_i) induces a natural homeomorphism between $F = \varprojlim F_i$ and $\varprojlim \pi_1(X_i)/G_i$. □

We may be lead to expect that $p_{\mathbf{G}}$ is never a covering projection but that is not the case. Indeed, to determine if $p_{\mathbf{G}}$ is a covering projection it is sufficient to consider the topology on its fibre. The product topology on the limit of an inverse system of discrete spaces is not discrete, unless almost all bonding maps in the system are injective. Thus we have the following corollary (where we assume, for simplicity, that $\mathcal{I} = \mathbb{N}$, so that the inverse system is in fact an inverse sequence).

Corollary 4.3. *$p_{\mathbf{G}}$ is a covering projection if and only if the connecting morphisms in the inverse system $\big(\pi_1(X_i)/G_i, (u_{ij})_\sharp, \mathcal{I}\big)$ are eventually injective.*

Proof. If there exists N such that $(u_{i\,i-1})_\sharp$ are injective for $i > N$ then $p_{\mathbf{G}}$ is the pullback of the covering projection $p_N \colon \widetilde{X}_N/G_N \to X_N$ and so it is itself a covering projection. Conversely, if infinitely many bonding maps in the sequence are non-injective then the limit space is not discrete, hence $p_{\mathbf{G}}$ is a lifting projection but not a covering. □

Observe that Examples 2.3 (infinite product of coverings) and 2.4 (dyadic solenoid) are inverse limits of coverings whose fibres are not discrete, so they are not covering spaces over the circle. On the other hand, the fibre of the hat construction described in Example 2.5 is discrete but the total space is usually disconnected, so it is not a covering space in the usual sense, but rather a disjoint union of coverings. Here is another interesting lifting space:

Example 4.4. *Let W be the Warsaw circle, and let $(W_i, u_{ij}, \mathbb{N})$ be the usual sequence of approximations of W by topological annuli. Then $\pi_1(W_i) \cong \mathbb{Z}$ and $(u_{ij})_\sharp$ are isomorphisms. By choosing a coherent thread $\mathbf{G}$ with $G_i := 2^i\mathbb{Z}$ we obtain an inverse sequence which is at group level analogous to that of example 2.4. The limit $p_{\mathbb{G}} \colon \widetilde{W}_{\mathbb{G}} \to W$ is an interesting lifting space that resembles a dyadic solenoid over W, so we may call it a* dyadic Warsawonoid.

4.1. **Homotopy exact sequence.** Next we consider the long homotopy exact sequence of the lifting space $p_{\mathbf{G}}$. As the fibres of a lifting space are totally path-disconnected (cf. [10, Theorem 2.2.5]), we conclude that $\pi_n(\widetilde{X}_{\mathbf{G}}) \cong \pi_n(X)$ for $n \geq 2$, and that $\pi_0(F)$ may be identified with F.

Furthermore, we have the following exact sequence of groups and pointed sets

$$1 \longrightarrow \pi_1(\widetilde{X}_{\mathbf{G}}) \xrightarrow{p_{\mathbf{G}\sharp}} \pi_1(X) \xrightarrow{\partial} \pi_0(F) \longrightarrow \pi_0(\widetilde{X}_{\mathbf{G}}) \longrightarrow \pi_0(X) \longrightarrow *$$

where the function ∂ is determined by the action of $\pi_1(X)$ on the fibre F. We will normally assume that X is path-connected in which case the above exact sequence ends at the term $\pi_0(\widetilde{X}_{\mathbf{G}})$. The following theorem identifies $\pi_1(\widetilde{X}_{\mathbf{G}})$ and $\pi_0(\widetilde{X}_{\mathbf{G}})$. Observe that the fibre F is a closed subspace of $\widetilde{X}_{\mathbf{G}}$, which by Proposition 4.2 may be identified with the inverse limit of the system of cosets $F = \varprojlim \pi_1(X_i)/G_i$.

Theorem 4.5. *Assume X is path-connected. Then there is an exact sequence of groups and pointed sets*

$$1 \longrightarrow \pi_1(\widetilde{X}_{\mathbf{G}}) \xrightarrow{p_{\mathbf{G}\sharp}} \pi_1(X) \xrightarrow{\partial} \varprojlim \pi_1(X_i)/G_i \longrightarrow \pi_0(\widetilde{X}_{\mathbf{G}}) \longrightarrow *$$

where the connecting function ∂ is given by the composition

$$\pi_1(X) \xrightarrow{\varprojlim (u_i)_\sharp} \varprojlim \pi_1(X_i) \longrightarrow \varprojlim \pi_1(X_i)/G_i$$

Consequently, $p_{\mathbf{G}}$ induces an isomorphism

$$\pi_1(\widetilde{X}_{\mathbf{G}}) \cong \bigcap_{i\in\mathcal{I}} (u_i)_\sharp^{-1}(G_i).$$

Moreover, if G_i is a normal subgroup of $\pi_1(X_i)$ for each i, then $\varprojlim \pi_1(X_i)/G_i$ is a group, ∂ is a homomorphism and $\pi_0(\widetilde{X}_{\mathbf{G}})$ may be identified with the set of cosets

$$\big(\varprojlim \pi_1(X_i)/G_i\big)/\partial(\pi_1(X)).$$

Proof. By its definition, ∂ maps every $\alpha \in \pi_1(X)$ to the end point of the lifting to $\widetilde{X}_{\mathbf{G}}$ of any representative of α, $\partial(\alpha) = \widetilde{\alpha}(1) \in F$. The uniqueness of liftings in covering spaces imply the commutativity of the following diagram

$$\begin{array}{ccc}
\pi_1(X) & \xrightarrow{\partial} & F \\
\downarrow{\scriptstyle (u_i)_\sharp} & & \downarrow{\scriptstyle u_i} \\
\pi_1(X_i) & \xrightarrow{\partial} & F_i \\
\downarrow & & \parallel \\
\pi_1(X_i)/G_i & \xrightarrow[l_i]{} & F_i
\end{array}$$

which, together with Proposition 4.2, leads to the above description of the connecting map ∂.

The description of ∂ implies that $(u_i)_\sharp(p_{\mathbf{G}}(\pi_1(\widetilde{X}_{\mathbf{G}}))) \subseteq G_i$ for all $i \in \mathcal{I}$, therefore $\operatorname{Im} p_{\mathbf{G}} \subseteq \bigcap_i (u_i)_\sharp^{-1}(G_i)$. Conversely, let α be a loop representing an element of $\bigcap_i (u_i)_\sharp^{-1}(G_i)$. Then for every i the loop $u_i \circ \alpha$ represents an element of $G_i \leq \pi_1(X_i)$, so it lifts to a loop $\widetilde{\alpha}_i$ in $\widetilde{X}_i/G_i$. The uniqueness of liftings imply that $\bar{u}_{ij} \circ \widetilde{\alpha}_j = \widetilde{\alpha}_i$ whenever $i \leq j$, hence we obtain a loop $\widetilde{\alpha} := \varprojlim \widetilde{\alpha}_i$ in $\widetilde{X}_{\mathbf{G}}$. Clearly, $p_{\mathbf{G}} \circ \widetilde{\alpha} = \alpha$, therefore $\alpha \in \operatorname{Im} p_{\mathbf{G}}$.

The proof of the last claim is straightforward. □

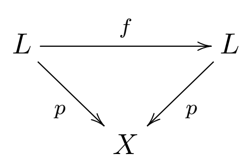
There are two special cases of the Theorem worth mentioning:

1. For a subgroup $G \leq \pi_1(X)$ one may define a coherent thread $\mathbf{G}$ by letting $G_i := (u_i)_\sharp(G)$. Then $(u_i)_\sharp^{-1}(G_i) = G \cdot \operatorname{Ker}(u_i)_\sharp$, and so
$$\pi_1(\widetilde{X}_G) \cong G \cdot \bigcap_{\mathcal{I}} \operatorname{Ker}(u_i)_\sharp.$$
Moreover if G is a normal subgroup of $\pi_1(X)$, and if all homomorphisms $(u_i)_\sharp$ are surjective, then $\varprojlim \pi_1(X_i)/G_i$ is a group and $\pi_0(\widetilde{X})$ is a set of cosets.

2. If $\mathbf{G}$ is a thread of trivial groups then clearly $\widetilde{X}_{\mathbf{G}} = \widetilde{X}$. In this case
$$\pi_1(\widetilde{X}) \cong \bigcap_{\mathcal{I}} \operatorname{Ker}(u_i)_\sharp,$$
while $\pi_0(\widetilde{X})$ may be identified with the set of cosets $\varprojlim \pi_1(X_i)/\partial(\pi_1(X))$.

4.2. **Deck transformations.** In the theory of covering spaces deck transformations provide a crucial connection between the algebra of the fundamental group and the geometry of the covering space. Their role is even more important in the theory of lifting spaces because they enclose the information about the interleaving of the path-components of the total space and induce a topology on the fundamental group of the base.

A *deck transformation* of a lifting space $p\colon L \to X$ is a homeomorphism $f\colon L \to L$, such that $p \circ f = p$:



The set of all deck transformations of p clearly form a group, which we denote by $A(p)$. We will consider two basic questions: is the action of $A(p)$ free and is it transitive on the fibres of p. Our first result is valid for general lifting spaces.

Proposition 4.6. *Let $p\colon L \to X$ be a lifting space such that each path-component of L is dense in L. Then the action of $A(p)$ on L is free.*

Proof. Assume that $f(x) = x$ for some $f \in A(p)$ and $x \in L$. For every path $\alpha\colon (I, 0) \to (L, x)$ the equality $p \circ f \circ \alpha = p \circ \alpha$ implies that $f \circ \alpha$ and α are lifts of the path $p \circ \alpha\colon I \to X$. Since $f(\alpha(0)) = f(x) = x = \alpha(0)$ and since p has unique path liftings, we conclude that $f(\alpha(1)) = \alpha(1)$ and so f fixes all elements of the path-component of L that contains x. But each path-component of L is dense in L and so the deck transformation f must be the identity. $\square$

We must therefore determine when are the path-components of L dense. For inverse limits of coverings it is sufficient to consider the function ∂ that we introduced earlier.

Proposition 4.7. *Let* $\mathbf{X}$ *be an inverse system of spaces,* $\mathbf{G}$ *a coherent thread of groups and and let* F *be the fibre of* $p_{\mathbf{G}}$.

If the image of the function $\partial\colon \pi_1(X) \to \pi_0(F) = \varprojlim \pi_1(X_i)/G_i$ *is dense in* $\pi_0(F)$ *(with respect to the inverse limit topology) then every path-component of* $\widetilde{X}_{\mathbf{G}}$ *is dense in* $\widetilde{X}_{\mathbf{G}}$.

Conversely, if X *is locally path-connected and every path-component of* $\widetilde{X}_{\mathbf{G}}$ *is dense, then the image of* ∂ *is dense in* F.

Proof. Recall that we have identified F with $\pi_0(F)$ and that the function ∂ is determined by the liftings of representatives of $\pi_1(X)$ in $\widetilde{X}_{\mathbf{G}}$. This in particular means that all elements of the image of ∂ belong to the same path-component L_0 of $\widetilde{X}_{\mathbf{G}}$.

For a given $\tilde{x} \in \widetilde{X}_{\mathbf{G}}$ let α be a path in X from x_0 to $x := p_{\mathbf{G}}(\tilde{x})$. Then the formula $g(\tilde{y}) := \langle \alpha, \tilde{y} \rangle(1)$ determines a homeomorphism between F and the fibre $p_{\mathbf{G}}^{-1}(x)$. This implies that $\tilde{x}$ is in the closure of $g(\operatorname{Im}\partial)$, and hence in the closure of the path-component L_0, because clearly $g(\operatorname{Im}\partial) \subset L_0$.

To show that some other path-component L_1 of $\widetilde{X}_{\mathbf{G}}$ is also dense, it is sufficient to choose a base-point for $\widetilde{X}_{\mathbf{G}}$ in L_1 and repeat the above argument. Observe that a different choice of a base-point simply conjugates the function ∂, so that its image is still dense in F.

To show the converse statement, assume that some $\tilde{x} \in F = p_{\mathbf{G}}^{-1}(x_0)$ is in the closure of the path component L_0 and let $\widetilde{U} := \prod_{i \in \mathcal{F}} \widetilde{U}_i \times \prod_{i \notin \mathcal{F}} \widetilde{X}_i/G_i$ be a product neighbourhood of $\tilde{x}$. Here $\mathcal{F}$ is some finite subset of $\mathcal{I}$ and each $\widetilde{U}_i$ is homeomorphically projected by p_i to some elementary neighbourhood in X_i. We must show that $\widetilde{U}$ intersects $\operatorname{Im}\partial = L_0 \cap F$. By Proposition 4.1 the projection $p_{\mathbf{G}}$ is an open map, so we may find a path-connected open set V such that $x_0 \in V \subseteq p_{\mathbf{G}}(\widetilde{U})$. Since $\tilde{x}$ is in the closure of L_0 there exists $\tilde{y} \in \widetilde{U} \cap p_{\mathbf{G}}^{-1}(V) \cap L_0$. Let α be a path in V from $p_{\mathbf{G}}(\tilde{y})$ to x_0. Then $\langle \alpha, \tilde{y} \rangle(1)$ is by construction an element of $\operatorname{Im}\partial \cap \widetilde{U}$, which proves that $\operatorname{Im}\partial$ is dense in F $\square$

For arbitrary lifting spaces it can be sometimes hard to determine whether the image of ∂ is dense in the fibre, but for inverse limits of coverings we may rely on the following algebraic criterion. Observe that if in an inverse system of sets $(A_i, u_{ij}, \mathcal{I})$ we replace each A_i by the corresponding *stable image* $A_i^{\text{Stab}} := \bigcap_{j \geq i} u_{ij}(A_j)$ (and the bonding maps by their restrictions) then $\varprojlim(A_i, u_{ij}, \mathcal{I}) = \varprojlim(A_i^{\mathrm{Stab}}, u_{ij}, \mathcal{I})$, i.e. the inverse limit depends only on stable images of bonding maps. In fact, the converse is also true, as the stable image is precisely the image of the projection from the inverse limit, $A_i^{\mathrm{Stab}} = \varphi_i(\varprojlim A_i)$. It is clear that for every morphism from a set A into the system, $(\varphi_i \colon A \to A_i, i \in \mathcal{I})$, we have $\varphi_i(A) \subseteq A_i^{\mathrm{Stab}}$ for all $i \in \mathcal{I}$. We will say that such a morphism is *stably surjective* if $\varphi_i(A) = A_i^{\mathrm{Stab}}$ for all $i \in \mathcal{I}$. Obviously, every morphism that consists of surjective maps is stably surjective.

Lemma 4.8. *Let $(\varphi_i \colon A \to A_i, \mathcal{I})$ be a stably surjective morphism from A to an inverse system of sets $(A_i, u_{ij}, \mathcal{I})$. Then the image of the limit map $\varphi := \varprojlim \varphi_i \colon A \to \varprojlim A_i$ is dense in $\varprojlim A_i$ (with respect to the product topology of discrete spaces).*

Proof. The statement is trivial when the index set $\mathcal{I}$ is finite, so we will assume that $\mathcal{I}$ is infinite. Let (a_i) be an element of $\varprojlim A_i$. By definition of a product topology, a local basis of neighbourhoods at (a_i) is given by the sets of the form $U_{\mathcal{F}} := \prod_{i\in\mathcal{F}}\{a_i\} \times \prod_{i\notin\mathcal{F}} A_i$, where $\mathcal{F}$ is a finite subset of $\mathcal{I}$. Given any such $\mathcal{F}$, let $j \in \mathcal{I}$ be bigger then all elements of $\mathcal{F}$. By the assumptions, there exists $a \in A$ such that $\varphi_j(a) = a_j$, but then $\varphi(a)_i = a_i$ for every $i \leq j$, and so $\varphi(a) \in U_{\mathcal{F}}$. We conclude that $\varphi(A)$ is dense in $\varprojlim A_i$. □

We may finally formulate our main result about the free action of the group of deck transformations of an inverse limit of coverings.

Theorem 4.9. *Let $\mathbf{X}$ be an inverse system of spaces and let $\mathbf{G}$ be a coherent thread of groups. If the morphism $((u_i)_\sharp \colon \pi_1(X) \to \pi_1(X_i)/G_i, \rangle)$ is stably surjective, then $A(p_{\mathbf{G}})$ acts freely on $\widetilde{X}_{\mathbf{G}}$.*

Proof. By Theorem 4.5 the function ∂ can be identified with the inverse limit $\varprojlim(u_i)_\sharp$. By Lemma 4.8 the image of ∂ is dense in the $\varprojlim \pi_1(X_i)/G_i$, which is by Proposition 4.2 homeomorphic to the fibre of $p_{\mathbf{G}}$. Proposition 4.7 implies that the path-components of $\widetilde{X}_{\mathbf{G}}$ are dense, and finally Proposition 4.6 implies that the action of $A(p_{\mathbf{G}})$ on $\widetilde{X}_{\mathbf{G}}$ is free. □

Our next objective is to examine the transitivity of the action of $A(p)$ on the fibre of p. It is well-known that for a covering space $p\colon \widetilde{X}/G \to X$ the action of $A(p)$ on the fibre is transitive if and only if G is a normal subgroup of $\pi_1(X)$. It is therefore reasonable to restrict our attention to coherent threads of normal subgroups. Let $(X_i, u_{ij}, \mathcal{I})$ be an inverse system of spaces and $\mathbf{G} = (G_i, \mathcal{I})$ a coherent thread of normal subgroups, i.e. $G_i \triangleleft \pi_1(X_i)$ for every $i \in \mathcal{I}$. Then we have group isomorphisms $l_i \colon A(p_i) \to \pi_1(X_i)/G_i$, explicitly given as $l_i(f) := [p\circ\tilde{\alpha}]$, where $\tilde{\alpha}$ is any path in $\widetilde{X}_i/G_i$ from the base-point $\tilde{x}_i$ to its image $f(\tilde{x}_i)$, and $[p\circ\tilde{\alpha}]$ is the coset in $\pi_1(X_i)/G_i$ determined by the loop $p \circ \tilde{\alpha}$ (cf. the description in [10, Section 2.6]). Moreover, whenever $j \geq i$ there is a homomorphism $\hat{u}_{ij} \colon A(p_j) \to A(p_i)$, where $\hat{u}_{ij}(f)$ is defined to be the unique deck transformation of $p_i\colon \widetilde{X}_i/G_i \to X_i$ that maps the base-point $\tilde{x}_i$ to $\hat{u}_{ij}(f(\tilde{x}_j))$. It is easy to check that whenever $i \le j$ we have a commutative diagram

$$\begin{array}{ccc} A(p_j) & \xrightarrow{\hat{u}_{ij}} & A(p_i) \\ {\scriptstyle l_j}\downarrow{\scriptstyle \cong} & & {\scriptstyle l_i}\downarrow{\scriptstyle \cong} \\ \pi_1(X_j)/G_j & \xrightarrow[(u_{ij})_\sharp]{} & \pi_1(X_i)/G_i \end{array}$$

Thus the homomorphisms l_i determine an isomorphism of inverse systems

$$(l_i)\colon (A(p_i), \hat{u}_{ij}, \mathcal{I}) \to (\pi_1(X_i)/G_i, (u_{ij})_\sharp, \mathcal{I}).$$

Theorem 4.10. *There is an isomorphism of groups*

$$\varprojlim l_i\colon \varprojlim A(p_i) \xrightarrow{\cong} \varprojlim \pi_1(X_i)/G_i.$$

The group $\varprojlim A(p_i)$ acts freely and transitively on the fibre of $p_{\mathbf{G}}$. Since $\varprojlim A(p_i)$ is a subgroup of $A(p_{\mathbf{G}})$ it follows that $A(p_{\mathbf{G}})$ also acts transitively on the fibre of $p_{\mathbf{G}}$.

Proof. That $\varprojlim l_i$ is an isomorphism follows from the above discussion. Toward the proof of transitivity, let $(\tilde{x}_i), (\tilde{x}'_i)$ be elements of the fibre of $p_{\mathbf{G}}$: then for every $i \in \mathcal{I}$ there exists a unique deck transformation $f_i \in A(p_i)$ such that $f_i(\tilde{x}_i) = \tilde{x}'_i$. In order to prove that transformations f_i represent an element of the inverse limit, consider an element $\tilde{y} \in \widetilde{X}_j/G_j$ and a path $\widetilde{\alpha}\colon (I,0,1) \to (\widetilde{X}_j/G_j, \tilde{x}_j, \tilde{y})$. Then

$$p_i f_i \bar{u}_{ij}\widetilde{\alpha} = p_i \bar{u}_{ij}\widetilde{\alpha} = u_{ij} p_j \widetilde{\alpha} = u_{ij} p_j f_j \widetilde{\alpha} = p_i \bar{u}_{ij} f_j \widetilde{\alpha},$$

which means that $f_i\bar{u}_{ij}\widetilde{\alpha}$ and $\bar{u}_{ij}f_j\widetilde{\alpha}$ are paths with the same initial point and the same projection in $\widetilde{X}_i/G_i$, so by the monodromy theorem they coincide. In particular $f_i(\bar{u}_{ij}(\tilde{y})) = \bar{u}_{ij}(f_j(\tilde{y}))$ for every $\tilde{y} \in \widetilde{X}_j/G_j$, therefore (f_i) is an element of $\varprojlim A(p_i)$ that maps $(\tilde{x}_i)$ to $(\tilde{x}'_i)$.

On the other side, if $\varprojlim f_i(\tilde{x}_i) = (\tilde{x}_i)$ for some $(f_i) \in \varprojlim A_(p_i)$ and $(x_i) \in \widetilde{X}_{\mathbf{G}}$ then $f_i(\widetilde{x}_i) = x_i$ for each $i \in \mathcal{I}$. It follows that all f_i are identity deck transformations on their respective domains, therefore the action of $\varprojlim A(p_i)$ is free. □

One could expect that $A(p)$ coincides with $\varprojlim A_(p_i)$ but we don't know if that is true in general. In fact, there are inverse limits of coverings where the path-components are not dense (like the Warsawonoid from Example 4.4), and so it is conceivable that the action of $A(p)$ may not be free. In view of the above Theorem, this would imply that $\varprojlim A(p_i)$ is a proper subgroup of $A(p)$. This problem disappears, if the fundamental group of the limit maps surjectively to the fundamental groups of its approximations, so we have the following result.

Corollary 4.11. *Let* $\mathbf{X}$ *be an inverse system of spaces and let* $\mathbf{G}$ *be a coherent thread of normal subgroups. If the morphism* $((u_i)_\sharp\colon \pi_1(X) \to \pi_1(X_i)/G_i, \mathcal{I})$ *is stably surjective, then* $A(p_\mathbf{G})$ *acts freely and transitively on the fibre of* $p_\mathbf{G}$ *and is therefore isomorphic to* $\varprojlim A(p_i)$*.*

Furthermore, the map $p = \varprojlim p_i \colon \widetilde{X}_\mathbf{G} \to X$ *induces a continuous bijection* $\overline{p}\colon \widetilde{X}_\mathbf{G}/A(p_\mathbf{G}) \to X$*. In addition, if* X *is locally path-connected then* $\overline{p}$ *is a homeomorphism.*

Proof. The action of $A(p_\mathbf{G})$ on the fibre F is free and transitive by the assumptions and Theorems 4.9 and 4.10. Thus we may conclude that there is a bijection $A(p_\mathbf{G}) \to F \cong \varprojlim A(p_i)$, which is clearly compatible with the group structures.

Map $\overline{p}$ is induced by $p\colon \widetilde{X}_\mathbf{G} \to X$ as in the following diagram

$$\begin{array}{ccc} \widetilde{X}_\mathbf{G} & \xrightarrow{\;p\;} & X \\ \downarrow & \nearrow_{\overline{p}} & \\ \widetilde{X}_\mathbf{G}/A(p_\mathbf{G}) & & \end{array}$$

and is clearly a continuous bijection. If X is locally connected then Proposition 4.1 implies that the map p is open. Then by the definition of quotient topology $\overline{p}$ is also an open map and hence a homeomorphism. □

5. Universal lifting spaces

We are now going to apply methods developed in previous sections to construct certain lifting spaces that in many aspect behave as the universal covering spaces. We will use freely terminology and constructions that are standard in shape theory and are described, for example, in [6]. Recall that all spaces are based (even if the base-points are normally omitted from the notation) and the maps are base-point preserving.

Throughout this section X will be a path-connected metric compactum. We can choose an embedding $X \hookrightarrow M$ into an absolute retract (for metric spaces) M, and consider the inverse system $\mathbf{X}$ of open neighbourhoods of X in M, ordered by the inclusion. Then $\mathbf{X}$ is an *ANR expansion* of X in the sense of [6], and moreover $X = \varprojlim \mathbf{X}$. Observe that each space in the expansion is semi-locally simply-connected and therefore admit all covering spaces, including the universal one. As in the previous section, we may define $\widetilde{\mathbf{X}}$ to be the associated inverse system of universal coverings, and let $\widetilde{X} := \varprojlim \widetilde{\mathbf{X}}$. We are going to prove that the definition of $\widetilde{X}$ is independent of the embedding $X \hookrightarrow M$.

To this end we will first show that the above construction is functorial. Let $Y \hookrightarrow N$ be another metric compactum embedded into an absolute retract

N, and let $f\colon X \to Y$ be any map. By the standard properties of the ANR expansions, f induces a morphism of systems $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$. Note that the morphism $\mathbf{f}$ is in general not level-preserving, and commutes with bonding maps only up to homotopy. Nevertheless, when composed with projections from X, it commutes exactly, so we may write $f = \varprojlim \mathbf{f}$. Every map between base-spaces admits a unique lifting to a map between respective universal coverings, so we have a morphism of systems $\tilde{\mathbf{f}}\colon \widetilde{\mathbf{X}} \to \widetilde{\mathbf{Y}}$ that is the unique lifting of $\mathbf{f}$. Here again, the morphism $\tilde{\mathbf{f}}$ commutes with the bonding maps only up to homotopy, but it commutes exactly when composed with projections from $\widetilde{X}$, so we are justified to define $\tilde{f} := \varprojlim \tilde{\mathbf{f}}$. Clearly, $\widetilde{1}_X = 1_{\widetilde{X}}$ and $\widetilde{g \circ f} = \widetilde{g} \circ \widetilde{f}$.

Proposition 5.1. *The definition of $\widetilde{X}$ is independent of the choice of embedding for X. The correspondence $X \mapsto \widetilde{X}$ and $f \mapsto \widetilde{f}$ determines a functor from compact metric spaces to metric spaces. This functor is augmented in the sense that the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Proof. Let $i\colon X \hookrightarrow M$ and $j\colon X \hookrightarrow N$ be embeddings of X into absolute retracts M and N. Then we have corresponding ANR expansions $\mathbf{X}_M$ and $\mathbf{X}_N$. Let $\mathbf{f}\colon \mathbf{X}_M \to \mathbf{X}_N$ and $\mathbf{g}\colon \mathbf{X}_N \to \mathbf{X}_M$ morphisms of systems induced by the identity map on X. By the above discussion $\mathbf{f}$ and $\mathbf{g}$ induce maps $\tilde{f}, \tilde{g}\colon \widetilde{X} \to \widetilde{X}$ which are inverse one to the other, which implies that $\widetilde{X}$ is uniquely defined up to a natural homeomorphism. The functoriality and the relation between f and $\tilde{f}$ follow directly from the definitions. □

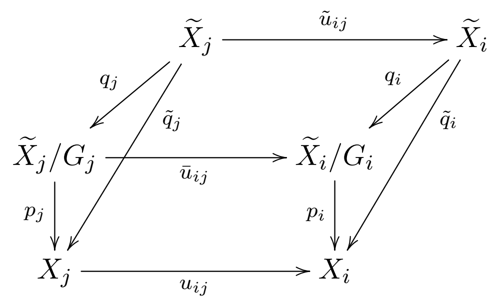
It is a standard fact of shape theory (see for example [6, I, 5.1]) that for metric compacta every inverse system of polyhedra whose limit is X is an expansion and that for every such expansion we can choose a cofinal sequence. Thus we obtain a more manageable description of $\widetilde{X}$:

Corollary 5.2. *If X is a metric compactum then $\widetilde{X} = \varprojlim \widetilde{X}_i$ for any inverse sequence of polyhedra, converging to X.*

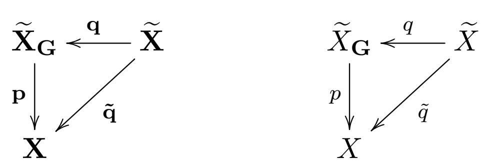
Clearly, if X is a compact polyhedron, then $\widetilde{X}$ is just the universal covering space of X. For a less trivial example, if W denotes the Warsaw circle, then it is easy to see that $\widetilde{W}$ is the *'Warsaw line'*, which we may describe as the real line in which every segment $[n, n+1)$ is replaced by the topologist's sine curve. As another illustration, if H is the Hawaiian earring, then $\widetilde{H}$ is the inverse limit of trees that are universal covering spaces for finite wedges of circles.

In order to explain in what sense is $\widetilde{X}$ universal with respect to the lifting spaces that are inverse limits of coverings, let us consider an expansion

$\mathbf{X} = (X_i, u_{ij}, \mathcal{I})$ of a metric compactum X. Then any inverse system of coverings over $\mathbf{X}$ uniquely corresponds to a choice of a coherent thread $\mathbf{G} = (G_i, \mathcal{I})$, i.e. is of the form $(\widetilde{X}_i/G_i, \bar{u}_{ij}, \mathcal{I})$. By the lifting properties of covering spaces, for every $i \leq j$ in $\mathcal{I}$ there are unique maps q_i, q_j for which the following diagram commutes



Thus we obtain a commutative diagram of inverse systems and the corresponding limits



Theorem 5.3. *Let $\tilde{q}\colon \widetilde{X} \to X$ be the universal lifting space for X and let $p\colon \overline{X} \to X$ be a lifting space obtained as a limit of an inverse system of coverings over some expansion of X. Then $\overline{X} = \widetilde{X}_{\mathbf{G}}$ for some (essentially unique) coherent thread $\mathbf{G}$, and there is a unique (base-point preserving) lifting space projection $q\colon \widetilde{X} \to \widetilde{X}_{\mathbf{G}}$ for which $p \circ q = \widetilde{q}$.*

Furthermore, let $\pi_1(X_i)^{\mathrm{Stab}}$ and $(\pi_1(X_i)/G_i)^{\mathrm{Stab}}$ be the stable images of the systems of groups and cosets induced by the inverse systems for X and $\overline{X}$. Then $q(\widetilde{X})$ is dense in $\widetilde{X}_{\mathbf{G}}$ if and only if the induced map

$$\pi_1(X_i)^{\mathrm{Stab}} \longrightarrow (\pi_1(X_i)/G_i)^{\mathrm{Stab}}$$

is surjective for every $i \in \mathcal{I}$.

Proof. The first assertion follows from the above discussion, so it remains to prove the characterization of density. Assume that the map between stable images is surjective and consider an element $\bar{x} = (\tilde{x}_i G_i) \in \overline{X}$ (with $\tilde{x}_i \in \widetilde{X}_i$ as representatives of the orbits). By choosing a path α in X connecting the base-point to $p(\bar{x})$ we obtain a coherent sequence of paths $(\alpha_i\colon I \to X_i)$, where α_i connects the base point of X_i to $u_i(p(\bar{x}))$. The unique path-lifting along α_i yields a commutative diagram

$$\begin{array}{ccc} \pi_1(X_i) & \hookrightarrow & \widetilde{X}_i \\ \downarrow & & \downarrow{\scriptstyle q_i} \\ \pi_1(X_i)/G_i & \hookrightarrow & \widetilde{X}_i/G_i \\ \downarrow & & \downarrow{\scriptstyle p_i} \\ * & \hookrightarrow & X_i \end{array}$$

where the horizontal maps are $\pi_1(X_i)$-equivariant bijections onto the fibres over $u_i(p(\bar{x}))$ of the lifting spaces p_i and $\tilde{q}_i$. By construction, these diagrams commute with the bonding maps of the respective inverse systems so they form a commutative diagram of inverse systems

$$\begin{array}{ccc} (\pi_1(X_i), (\tilde{u}_{ij})_\sharp, \mathcal{I}) & \longrightarrow & (\widetilde{X}_i, \tilde{u}_{ij}, \mathcal{I}) \\ \downarrow & & \downarrow{\scriptstyle q_i} \\ (\pi_1(X_i)/G_i, (\bar{u}_{ij})_\sharp, \mathcal{I}) & \longrightarrow & (\widetilde{X}_i/G_i, \bar{u}_{ij}, \mathcal{I}) \\ \downarrow & & \downarrow{\scriptstyle p_i} \\ * & \longrightarrow & (X_i, u_{ij}, \mathcal{I}) \end{array}$$

As a consequence, we obtain bijections $\pi_1(X_i)^{\mathrm{Stab}} \to \widetilde{X}_i^{\mathrm{Stab}} \cap \tilde{q}_i^{-1}(u_i(p(\bar{x})))$ and $(\pi_1(X_i)/G_i)^{\mathrm{Stab}} \to (\widetilde{X}_i/G_i)^{\mathrm{Stab}} \cap \tilde{p}_i^{-1}(u_i(p(\bar{x})))$.

A local basis of neighbourhoods at $(\tilde{x}_i G_i)$ is given by the sets of the form $U_{\mathcal{F}} := \prod_{i\in\mathcal{F}}\{x_i G_i\} \times \prod_{i\notin\mathcal{F}} \widetilde{X}_i/G_i$, where $\mathcal{F}$ is any finite subset of $\mathcal{I}$. Let $j \in \mathcal{I}$ be bigger than all elements of $\mathcal{F}$. Since $x_j G_j \in (\widetilde{X}_j/G_j)^{\mathrm{Stab}}$, it corresponds to to some $g_j G_j$, where by our assumption $g_j \in \pi_1(X_j)^{\mathrm{Stab}}$. It follows that there is an $\tilde{x} \in \widetilde{X}$ such that $\tilde{u}_j(\tilde{x})G_j = \tilde{x}_j G_j$, and consequently $\tilde{u}_i(\tilde{x})G_i = \tilde{x}_i G_i$ for all $i \in \mathcal{F}$. We have thus proved that $q(\widetilde{X})$ intersects every open set in $\widetilde{X}_{\mathbf{G}}$, hence the image of q is dense in $\widetilde{X}_{\mathbf{G}}$.

Conversely, if for some $i \in \mathcal{I}$ the map $\pi_1(X_i)^{\mathrm{Stab}} \longrightarrow (\pi_1(X_i)/G_i)^{\mathrm{Stab}}$ is not surjective, then we may use the previously described correspondence between $\pi_1(X_i)$ and $\pi_1(X_i)/G_i$ with the fibres of q_i and p_i to find an element $\bar{x} \in \widetilde{X}_{\mathbf{G}}$, such that $q(\widetilde{X})$ does not intersect its open neighbourhood $\{\bar{u}_i(\bar{x})\} \times \prod_{j\neq i} \widetilde{X}_j/G_j$. □

In various practical situations it may be hard to verify whether the image of $\widetilde{X}$ is dense in $\overline{X}$ directly from the theorem. The following corollary provides several sufficient conditions.

Corollary 5.4. *Any of the following conditions imply that* $q(\widetilde{X})$ *is dense in* $\widetilde{X}_{\mathbf{G}}$.

(1) *The inverse system $(\pi_1(X_i), (u_{ij})_\sharp, \mathcal{I})$ has the Mittag-Leffler property (this holds in particular, if all bonding homomorphisms in the system are surjective) and* $\mathbf{G}$ *is arbitrary.*
(2) G_i *is a subgroup of* $\pi_1(X_i)^{\mathrm{Stab}}$ *for every* $i \in \mathcal{I}$.
(3) $G_i \cdot \pi_1(X_i)^{\mathrm{Stab}} = \pi_1(X_i)$ *for every* $i \in \mathcal{I}$.
(4) *There exists a cofinal subsequence* $\mathcal{C} \subseteq \mathcal{I}$*, and all groups in the thread* $\mathbf{G}$ *are finite.*

Proof. (1) Recall that $(\pi_1(X_i), (u_{ij})_\sharp, \mathcal{I})$ satisfy the Mittag-Leffler condition if the stable images are achieved at some finite stage, i.e. for every $i \in \mathcal{I}$ there is a $j \geq i$ such that $\pi_1(X_i)^{\mathrm{Stab}} = (u_{ik})_\sharp(\pi_1(X_k))$ for every $k \geq j$. Given an element $g_iG_i \in \pi_1(X_i)^{\mathrm{Stab}}$ there exists $g_j \in \pi_1(X_j)$ such that $g_iG_i = (u_{ij})_\sharp(g_jG_j) = (u_{ij})_\sharp(g_j)G_i$, so g_iG_i is the image of $(u_{ij})_\sharp(g_j)$, which is by the Mittag-Leffler condition an element of $\pi_1(X_i)^{\mathrm{Stab}}$.

(2) Assume that $g_iG_i \in (\pi_1(X_i)/G_i)^{\mathrm{Stab}}$, so that for every $j \geq i$ there exists some $g_j \in \pi_1(X_j)$ satisfying $g_iG_i = (u_{ij})_\sharp(g_jG_j) = (u_{ij})_\sharp(g_j)G_i$. It follows that $(u_{ij})_\sharp(g_j) = g_ig$ for some $g \in G_i$. Since $G_i \subseteq \pi_1(X_i)^{\mathrm{Stab}}$ there exists some $h \in \pi_1(X_j)$ such that $(u_{ij})_\sharp(h) = g$, and so $g_i = (u_{ij})_\sharp(g_jh^{-1})$, which shows that g_i is in the image of $(u_{ij})_\sharp$ for every $j \geq i$, therefore it is in the stable image.

(3) The condition implies that every element of $\pi_1(X_i)/G_i$, and hence every element of $(\pi_1(X_i)/G_i)^{\mathrm{Stab}}$, is in the image of $\pi_1(X_i)^{\mathrm{Stab}}$.

(4) If $g_iG_i \in (\pi_1(X_i)/G_i)^{\mathrm{Stab}}$, then for every $j \in \mathcal{C}, j \geq i$ there exists $g_j \in \pi_1(X_j)$ such that $(u_{ij})_\sharp(g_jG_j) = g_iG_i$. This means that $(u_{ij})(g_j)$ is one of the finitely many elements of the coset x_iG_i and so at least one of them must appear infinitely many times as value of $(u_{ij})_\sharp$ for $j \in \mathcal{C}, j \geq i$. By the cofinality of $\mathcal{C}$ this element must be in $\pi_1(X_i)^{\mathrm{Stab}}$.

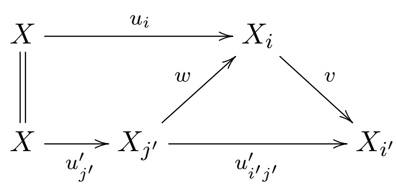
□

Example 5.5. *If X is a polyhedron, then we may always take the trivial expansion so the bonding maps on the associated system of fundamental groups are identity homomorphisms. If $p\colon \overline{X} \to X$ is a limit of an inverse system coverings over X then by Theorem 5.3 there exists a unique lifting projection $q\colon \widetilde{X} \to \overline{X}$ such that $p \circ q = \tilde{q}\colon \widetilde{X} \to X$, the standard universal covering space projection. Moreover, by Corollary 5.4 the image $q(\widetilde{X})$ is dense in $\overline{X}$.*

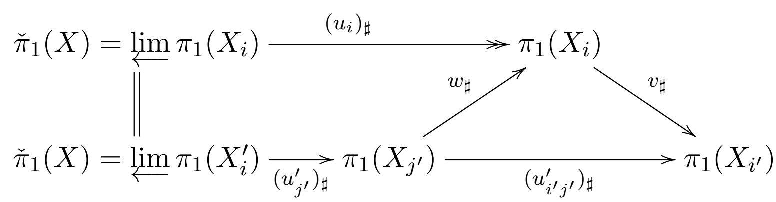
Observe that in principle the density of $q(\widetilde{X})$ in $\overline{X}$ depends on the properties of the expansion used in the construction of $\overline{X}$. However, we have the following result that allows to avoid this objection. Our argument is based on the fact that the limit of the inverse system of fundamental groups associated to an expansion of X actually depends only on the space itself: the *shape fundamental group* is defined as $\check{\pi}_1(X) := \varprojlim(\pi_1(X_i), (u_{ij})_\sharp, \mathcal{I})$, where $(X_i, u_{ij}, \mathcal{I})$ is any expansion of X (cf. [6, II, 3.3]).

Lemma 5.6. *Let $(X_i, u_{ij}, \mathcal{I})$ and $(X'_i, u'_{ij}, \mathcal{I}')$ be two polyhedral expansions for X, and assume that the bonding homomorphisms in the associated system of fundamental groups $(\pi_1(X_i), (u_{ij})_\sharp, \mathcal{I})$ are surjective. Then the system $(\pi_1(X'_i), (u'_{ij})_\sharp, \mathcal{I}')$ satisfies the Mittag-Leffler condition.*

Proof. By the properties of expansions, for a given $i' \in \mathcal{I}'$ we may find an $i \in \mathcal{I}$ and a map $v\colon X_i \to X_{i'}$ such that $v \circ u_i = u'_{i'}$. Similarly, we may find some $j' \in \mathcal{I}'$ and a map $w\colon X_{j'} \to X_i$ satisfying $w \circ u'_{j'} = u_i$.



Then, by applying fundamental groups we obtain the following diagram



Since $(u_i)_\sharp$ is surjective, so must be $w_\sharp$, thus we have

$$\pi_1(X'_{i'})^{\mathrm{Stab}} = (u_{i'})_\sharp(\check{\pi}_1(X)) = v_\sharp(\pi_1(X_i)) = (u'_{i'j'})_\sharp(\pi_1(X_{j'}))$$

which proves that the stable image coincides with the image of the bonding map $(u'_{i'j'})_\sharp$, as required by the Mittag-Leffler condition. □

Example 5.7. *We have already mentioned that the universal lifting space of the Warsaw circle W is the Warsaw line $\widetilde{W}$. Since we may obtain W as a limit of shrinking annuli, where the induced homomorphisms on the fundamental group are identities, the above results imply that $\widetilde{W}$ is mapped densely into every inverse limit of coverings over any expansion of W. This applies in particular to all Warsawonoids that we described in Example 4.4.*

In general one cannot expect to find an expansion of X for which the bonding homomorphisms in the induced system of fundamental groups are surjective. Nevertheless, this important property can be always achieved for expansions of locally path-connected spaces.

Lemma 5.8. *Every locally path-connected space X admits an expansion $(X_i, u_{ij}, \mathcal{I})$ such that the induced homomorphisms $(u_i)_\sharp\colon \pi_1(X) \to \pi_1(X_i)$ are surjective for all $i \in \mathcal{I}$.*

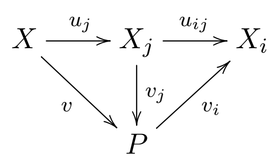
Proof. We are going to exploit the technique used in the proof of Lemma 3.3. In fact, let $\mathcal{U}$ be a finite, non-redundant open cover of X and let $f\colon X \to |\mathcal{U}|$ the map induced by some choice of a partition of unity subordinated to $\mathcal{U}$. Then by Lemma 3.3 the induced homomorphism $f_\sharp\colon \pi_1(X) \to \pi_1(|\mathcal{U}|$

is surjective. Therefore, if we take the standard Čech expansion of X by polyhedra X_i that are nerves of covers of X by metric balls of radius $1/i$ for $i = 1, 2, 3, \ldots$, then the resulting inverse sequence satisfy the desired surjectivity property. □

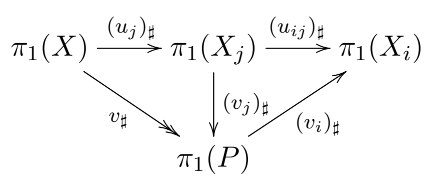
As a consequence we obtain strong surjectivity properties of inverse systems of fundamental groups associated to expansions of locally path-connected spaces.

Corollary 5.9. *Let $(X_i, u_{ij}, \mathcal{I})$ be any expansion of a connected and locally path-connected compact metric space X. Then the induced morphism $(\pi_1(X) \to \pi_1(X_i), i \in \mathcal{I})$ is stably surjective and the associated inverse system $(\pi_1(X_i), (u_{ij})_\sharp, \mathcal{I})$ satisfies the Mittag-Leffler condition.*

Proof. By Lemma 5.8 and the properties of an expansion we may find for each $i \in \mathcal{I}$ a polyhedron P and maps $v\colon X \to P$ and $v_i\colon P \to X_i$ so that $v_i \circ v = u_i$ and $v_\sharp\colon \pi_1(X) \to \pi_1(P)$ is a surjection. On the other side, we may also find a $j \geq i$ and a map $v_j\colon X_j \to P$ so that $v_j \circ u_j = v$ and $v \circ v_j \simeq u_{ij}$. Thus we obtain the following diagram



which induces a commutative diagram of fundamental groups



Since $(u_{ij})_\sharp$ factors through $(v_i)_\sharp$, it follows that $(v_i)_\sharp(\pi_1(P)$ contains the stable image $\pi_1(X_i)^{\mathrm{Stab}}$. But then the surjectivity of $v_\sharp$ implies that the homomorphism $(u_i)_\sharp\colon \pi_1(X) \to \pi_1(X_i)^{\mathrm{Stab}}$ is also surjective.

The second claim follows immediately by Lemma 5.6. □

Connected and locally path-connected compact metric spaced form a large class of spaces that include all finite polyhedra, compact manifolds and many other important spaces. They are in fact more commonly known as *Peano continua*. This name is a distant echo of the Peano space-filling curves, consolidated by the famous Hahn-Mazurkiewicz Theorem that characterizes Peano continua as Hausdorff spaces that can be obtained as a continuous image of an arc. The following theorem summarizes the main properties of lifting spaces over Peano continua.

Theorem 5.10. *Let X be a Peano continuum, $\tilde{q}\colon \widetilde{X} \to X$ its universal lifting space and $p\colon \overline{X} \to X$ any lifting space that can be obtained as a limit* *of an inverse system of coverings* $p_i\colon \overline{X}_i \to X_i$ *over some expansion of* X. *Then:*

(1) *There exists a unique lifting projection* $q\colon \widetilde{X} \to \overline{X}$ *such that* $p \circ q = \tilde{q}$. *Moreover, the image* $q(\widetilde{X})$ *is dense in* $\overline{X}$.
(2) *The group of deck transformations* $A(p)$ *acts freely on* $\overline{X}$.
(3) *If* $\overline{X}$ *is an inverse limit of normal coverings then* $A(p)$ *acts freely and transitively on the fibres of* p, *and there is an isomorphism* $A(p) \cong \varprojlim A(p_i)$. *In particular,* $A(\tilde{q})$ *is naturally isomorphic with the shape fundamental group* $\check{\pi}_1(X)$.
(4) *If* $\overline{X}$ *is an inverse limit of normal coverings then* p *induces a homeomorphism* $\bar{p}\colon \overline{X}/A(p) \to X$.
(5) *There is an exact sequence of groups and sets*

$$1 \longrightarrow \pi_1(\widetilde{X}) \xrightarrow{(\tilde{q})_\sharp} \pi_1(X) \xrightarrow{\partial} \check{\pi}_1(X) \longrightarrow \pi_0(\widetilde{X}) \longrightarrow *$$

In particular, $\pi_1(\widetilde{X})$ *may be identified with the kernel of the natural homomorphism* $\partial\colon \pi_1(X) \to \check{\pi}_1(X)$, *also known as* shape kernel *of* X. *Similarly,* $\pi_0(\widetilde{X})$ *may be identified with the* shape cokernel *of* X, *namely the set of cosets* $\check{\pi}_1(X)/\partial(\pi_1(X))$.
(6) *Let* $\mathbf{G} = (G_i)$ *be the coherent thread of groups, determined by* $\overline{X}$. *Then the map* $q\colon \widetilde{X} \to \overline{X}$ *induces a commutative ladder:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\widetilde{X}) & \xrightarrow{\tilde{q}_\sharp} & \pi_1(X) & \xrightarrow{\partial} & \check{\pi}_1(X) & \longrightarrow & \pi_0(\widetilde{X}) & \longrightarrow & * \\
 & & \downarrow{\scriptstyle q_\sharp} & & \| & & \downarrow & & \downarrow{\scriptstyle q_\sharp} & & \\
1 & \longrightarrow & \pi_1(\overline{X}) & \xrightarrow[p_\sharp]{} & \pi_1(X) & \xrightarrow[\partial]{} & \varprojlim \pi_1(X_i)/G_i & \longrightarrow & \pi_0(\overline{X}) & \longrightarrow & *
\end{array}$$

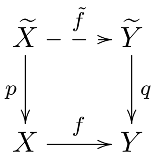
Proof. The existence of the map $q\colon \widetilde{X} \to \overline{X}$ was proved in Theorem 5.3. As for the second claim, observe that by Lemma 5.8 X admits an expansion such that the induced homomorphisms between fundamental groups are surjective. By Corollary 5.9 the induced system satisfies the Mittag-Leffler condition, and then by Corollary 5.4 it follows that $q(\widetilde{X})$ is dense in X.

The statements 2., 3. and 4. also follow from Corollary 5.9, combined with Theorem 4.9 and Corollary 4.11.

Finally, 5. and 6. follow from Theorem 4.5, in particular from the naturality of the exact sequence of a fibration. □

Observe that the shape kernel and shape cokernel are not shape invariants: in fact they are more like 'anti-invariants' as they measure the variation of the structure of the universal lifting space within shape-equivalent spaces.

We conclude the section with a lifting theorem for inverse limits of covering spaces. Given a map $f\colon X \to Y$ and lifting spaces $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ we would like to know if there exists a map $\tilde{f}$ for which the following diagram commutes.

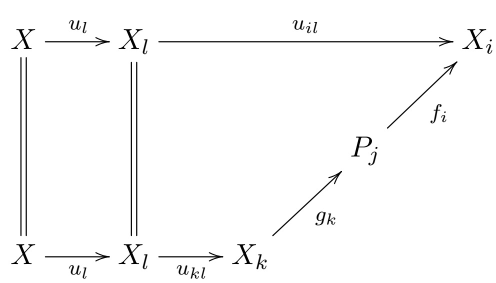


If $\widetilde{X}$ is connected and locally path connected then the answer is given by the classical lifting theorem [10, Theorem II.4.5]: $\widetilde{f}$ exists if, and only if $f_\sharp(p_\sharp(\pi_1(\widetilde{X})) \subseteq q_\sharp(\pi_1(\widetilde{Y}))$. We are going to extend this result to more general lifting spaces. Observe that a very special case was already considered as a part of Proposition 5.1.

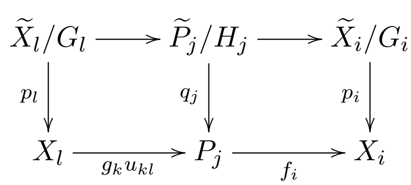
Let us first show that the property of being the limit of a sequence of covering spaces over a polyhedral expansion for X is independent on the choice of the expansion.

Proposition 5.11. *Let X be the limit of an polyhedral expansion $\mathbf{X} = (X_i, u_{ij} \colon X_j \to X_i, i, j \in \mathbb{N})$ and let Let $p_{\mathbf{G}} \colon \widetilde{X}_{\mathbf{G}} \to X$ be the lifting space determined by a coherent thread $\mathbf{G}$ of subgroups of $\pi_1(X_i)$. Then for every polyhedral expansion $\mathbf{P} = (P_i, v_{ij} \colon P_j \to P_i, i, j \in \mathbb{N})$ for X there exists a coherent thread $\mathbf{H}$, such that $\widetilde{X}_{\mathbf{H}} = \widetilde{X}_{\mathbf{G}}$ and $p_{\mathbf{H}} = p_{\mathbf{G}}$.*

Proof. We will use properties of polyhedral expansions to obtain approximations of the identity map of X with respect to the expansions $\mathbf{X}$ and $\mathbf{P}$ (cf. [6]). For every i there exists j and a map $f_i \colon P_j \to X_i$ such that $f_i v_j = u_i$. Furthermore, there exists a k and a map $g_j \colon X_k \to P_j$ such that $g_j u_k = v_j$. Finally, since f_i and g_j are both approximations of the identity on X there exists some index l such that $u_{il} \simeq f_i g_j u_{kl}$.



Let $H_j := (f_{i\sharp})^{-1}(G_i) \leq \pi_1(P_j)$. Then $(g_k u_{kl})_\sharp(G_l) \subseteq H_j$ and we obtain a sequence of covering projections



which shows that we may refine the system of coverings over X_i by coverings over P_i. It follows that $\widetilde{X}_{\mathbf{G}}$ can be represented as an inverse limit of coverings over the system $\mathbf{P}$ with respect to some coherent thread of groups $\mathbf{H}$. □

Theorem 5.12. *Let $\mathbf{X}$ be a polyhedral expansion for X and $p_{\mathbf{G}}\colon \widetilde{X}_{\mathbf{G}} \to X$ the lifting space determined by a coherent thread $\mathbf{G}$. Similarly, let $\mathbf{Y}$ be a polyhedral expansion for Y and $q_{\mathbf{H}}\colon \widetilde{Y}_{\mathbf{H}} \to Y$ the lifting space determined by a coherent thread $\mathbf{H}$.*

Then a map $f\colon X \to Y$ can be lifted to a map $\tilde{f}\colon \widetilde{X}_{\mathbf{G}} \to \widetilde{Y}_{\mathbf{H}}$ if, and only if for a morphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ induced by f we have that for every index i there exists some index j such that $(f_i u_{ij})_\sharp(G_j) \leq H_i$.

The condition for the existence of a lifting is independent from the choices of expansions for X and Y.

Proof. The assumption that for every i there is some j, such that $(f_i v_{ij})_\sharp(H_j) \leq G_i$ implies that there exist maps $\tilde{f}_i$ so that the following diagram commutes

$$\begin{array}{ccc} \widetilde{X}_j/G_j & \xrightarrow{\tilde{f}_i} & \widetilde{Y}_i/H_i \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle q_i} \\ X_j & \xrightarrow[f_i u_{ij}]{} & Y_i \end{array}$$

After suitable reindexing we see that maps $\tilde{f}_i$ determine a morphism of systems $\tilde{\mathbf{f}}\colon \widetilde{\mathbf{X}}/\mathbf{G} \to \widetilde{\mathbf{Y}}/\mathbf{H}$, and hence define the lifting $\tilde{f}\colon \widetilde{X}_{\mathbf{G}} \to \widetilde{Y}_{\mathbf{H}}$ for f.

That the existence of the lifting is independent from the choice of the expansions for X and Y follows from Proposition 5.11. We prefer to omit technical details. The converse implication is obvious. □

6. When is the hat space a lifting space?

In the preceding sections we have mostly assumed that the spaces under consideration are locally path-connected. For more general spaces we may first construct the hat space mentioned in Example 2.5. Since the unique path-lifting property of the map $\iota\colon \widehat{X} \to X$ is obvious, ι is a lifting space if and only if it is a fibration. When this happens, every lifting space over $\widehat{X}$ is automatically a lifting space over X. Moreover, every lifting projection $p\colon Y \to X$ with Y locally path-connected, factors through $\widehat{X}$ as a composition of two lifting projections.

In this section we discuss in detail the fibration properties of the hat construction. In particular we prove that the hat construction over a metric space yields a fibration for the class of all metric (in fact 1-countable) spaces if and only if the hat space is locally compact.

Recall that the *hat space* $\widehat{X}$ is obtained from a space X generated by taking the path components of open sets in the topology of X as a sub-basis. The identity map $\iota\colon \widehat{X} \to X$ is continuous and bijective. If $f\colon Y \to X$ is a continuous map and C a path-component of an open set $U \subseteq X$, then $f^{-1}(C)$ is a union of components of the open set $f^{-1}(U)$. Therefore, if Y is locally path-connected, then $f\colon Y \to \widehat{X}$ is also continuous. In particular the paths in X correspond precisely to paths in $\widehat{X}$ and the same holds for path-components of subsets as well. This implies that $\widehat{X}$ is locally path-connected, so $\widehat{(\widehat{X})} = \widehat{X}$. In addition, if $f\colon Y \to X$ is continuous, then $\hat{f}\colon \widehat{Y} \to \widehat{X}$ (where $\hat{f} = f$ as a function between sets) is also continuous, and the following diagram commutes

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\hat{f}} & \widehat{X} \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota} \\ Y & \xrightarrow[f]{} & X \end{array}$$

We may summarize these facts in categorical terms by saying that the hat construction is an idempotent augmented functor.

What can be said about the fibration properties of the hat construction? Since maps and homotopies from cubes have unique liftings, the projection $\iota\colon \widehat{X} \to X$ is a Serre fibration with the unique path-lifting property. In particular, this shows that X and $\widehat{X}$ have isomorphic homotopy groups. However, the following example show that it is not in general a Hurewicz fibration.

Example 6.1. *Let us denote by $\mathbb{S} := \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$ the 'model' convergent sequence, and let $C\mathbb{S}$ be the cone over $\mathbb{S}$, which we view as a subspace of the plane, e.g. $C\mathbb{S} := \{(t, tu) \in \mathbb{R}^2 \mid t \in [0,1], u \in \mathbb{S}\}$. It is easy to see that $\widehat{C\mathbb{S}}$ can be identified with a countable one-point union of intervals. Consider the homotopy $H\colon C\mathbb{S} \times I \to C\mathbb{S}$, given by $H\big((x,y),t\big) := (tx, ty)$. Then the constant map H_0 lifts to $\widehat{C\mathbb{S}}$ but the entire homotopy H cannot be lifted, because its final stage would give a continuous inverse to $\iota\colon \widehat{C\mathbb{S}} \to C\mathbb{S}$, which would imply that $C\mathbb{S}$ is locally path-connected, a contradiction.*

Is there some natural condition that would imply that $\iota\colon \widehat{X} \to X$ is a fibration for a sufficiently large class of spaces, like the metric spaces? Toward an answer to this question we prove that in order to check the fibration property for metric spaces it is sufficient to check that ι has the covering homotopy property for maps from the model sequence $\mathbb{S}$.

Lemma 6.2. *The canonical map $\iota\colon \widehat{X} \to X$ is a fibration for the class of first countable spaces if and only if it has the homotopy lifting property for maps from $\mathbb{S}$ to X.*

Proof. One direction is immediate. For the other implication assume that Y is a 1-countable space and that we are given a homotopy $H\colon Y\times I\to X$ such that the lifting $\widehat{H}_0\colon Y\times 0\to\widehat{X}$ is continuous. In order to prove that $\widehat{H}\colon Y\times I\to\widehat{X}$ is continuous we take a sequence $\{(y_i,t_i)\}$ in $Y\times I$ converging to (y,t). The sequence $\{y_i\}$ converges to y and hence determines a map $g\colon\mathbb{S}\to Y$. The initial stage of the homotopy $F:=H\circ(g\times 1)\colon\mathbb{S}\times I\to X$ lifts to the continuous map $\widehat{F}_0=\widehat{H}\circ g\colon Y\times 0\to\widehat{X}$, so by the assumption $\widehat{F}\colon\mathbb{S}\times I\to\widehat{X}$ is continuous as well. In particular the sequence $\widehat{F}(y_i,t_i)$ converges to $\widehat{F}(y,t)$. □

Recall that a space is *sequentially compact* if every sequence in it has a convergent subsequence. In general the compactness and the sequential compactness are not directly related as none of them implies the other. However, they coincide for the class of metric spaces and for 1-countable spaces (or more generally, for *sequential spaces*) the compactness implies sequential compactness. Moreover, a space is *locally sequentially compact* if every point has a neigborhood whose closure is sequentially compact.

Theorem 6.3. *If X is a Hausdorff space and if $\widehat{X}$ is locally sequentially compact, then the canonical map $\iota\colon\widehat{X}\to X$ has the homotopy lifting property for maps from $\mathbb{S}$.*

Proof. Let $F\colon\mathbb{S}\times I\to X$ be a map, such that the restriction of the unique lift $\widehat{F}\colon\mathbb{S}\times I\to\widehat{X}$ is continuous when restricted to $\mathbb{S}\times 0$. We are going to show that $\widehat{F}$ is also continuous.

First observe that the local path-connectedness of $(\mathbb{S}-0)\times I$ imply that $\widehat{F}$ is continuous on $(\mathbb{S}-0)\times I$, so it only remains to prove the continuity of $\widehat{F}$ on $0\times I$. Since $\widehat{F}$ is continuous on $\mathbb{S}\times 0$ we may define

$$t:=\sup\{s\in I\mid \widehat{F}|_{\mathbb{S}\times[0,s]}\text{ is continuous}\}.$$

By the local sequential compactness of $\widehat{X}$ and the definition of the hat-topology we may choose a neighborhood $U\subset\widehat{X}$ of $\widehat{F}(0,t)$ that is a path-component of an open set $V\subset X$, and whose closure is sequentially compact. By the continuity of F there exist $\varepsilon,\delta>0$ such that the box neighborhood $B:=(\mathbb{S}\cap[0,\delta))\times(t-\varepsilon,t+\varepsilon)$ is contained in $F^{-1}(V)$. Moreover, since the restriction $\widehat{F}|_{0\times I}$ is continuous, we may assume (by decreasing ε, if necessary) that $\widehat{F}(0\times(t-\varepsilon,t])\subset U$. Furthermore, by the definition of t, there is an $s\in(t-\varepsilon,t]$, such that $\widehat{F}|_{\mathbb{S}\times s}$ is continuous, so we may assume (by adjusting δ if necessary) that $\widehat{F}\big((\mathbb{S}\cap[0,\delta))\times s\big)\subset U$. Since U is path-connected, it follows that $\widehat{F}(B)\subset U$. We claim that $\widehat{F}$ is continuous on $0\times(t-\varepsilon,t+\varepsilon)$.

Take an $x\in(t-\varepsilon,t+\varepsilon)$ and let $\{x_i\}$ be a sequence in B converging to $(0,x)$. To show that $\{\widehat{F}(x_i)\}$ converges to $\widehat{F}(0,x)$ we must show that every open neighborhood W of $\widehat{F}(0,x)$ contains all but finitely many elements of the sequence. Indeed, otherwise the elements of $\{\widehat{F}(x_i)\}$ outside W would have an accumulation point u in the compact set $\overline{U} - W$. This would imply that $F(0,x)$ and $\iota(u)$ are distinct accumulation points of the convergent sequence $\{F(x_i)\}$, a contradiction.

Our argument shows that the assumption $t < 1$ would lead to a contradiction, so we conclude that $t = 1$, and therefore $\widetilde{F}\colon \mathbb{S} \times I \to \widehat{X}$ is continuous. $\square$

Example 6.4. *The obvious map from $[0,1)$ to the Warsaw circle is a fibration for the class of 1-countable spaces, while the projection from the countable one-point union of intervals to $C\mathbb{S}$ is not.*

If X is first countable and Hausdorff then so is $\widehat{X}$: the Hausdorff property is obvious, for the other simply observe that if we take a countable local basis around the point x in X, then the path-components of the basic sets containing the point x constitute a countable local basis for x in $\widehat{X}$. If in addition $\widehat{X}$ is locally compact (and therefore locally sequentially compact), then we have just proved that $\iota\colon \widehat{X} \to X$ is a fibration for the class of all first countable spaces. But a map between first countable spaces that has the covering homotopy property for maps between from first countable has automatically the covering homotopy property for maps from arbitrary spaces, so we have the following

Theorem 6.5. *Let X be a first countable, Hausdorff space. If $\widehat{X}$ is locally compact then $\iota\colon \widehat{X} \to X$ is a lifting space.*

Proof. By 6.2 and 6.3 we already know that ι has the covering homotopy property for 1-countable spaces. Since we assumed that X is 1-countable that $\widehat{X}$ and X^I are also 1-countable, and therefore the space $(\widehat{X} \sqcap X^I) \times I$ is 1-countable, because it is a subspace of $\widehat{X} \times X^I$. It follows that there exists the unique map H that makes commutative the following diagram

$$\begin{array}{ccc}
(\widehat{X} \sqcap X^I) \times 0 & \xrightarrow{(\hat{x},\alpha,0)\mapsto\hat{x}} & \widehat{X} \\
\big\downarrow & \overset{H}{\nearrow} & \big\downarrow \iota \\
(\widehat{X} \sqcap X^I) \times I & \xrightarrow[(\hat{x},\alpha,t)\mapsto\alpha(t)]{} & X
\end{array}$$

Its adjoint map $\widetilde{H}\colon \widehat{X} \sqcap X^I \to \widehat{X}^I$ is the continuous inverse for the continuous bijection $\bar{\iota}\colon \widehat{X}^I \to \widehat{X} \sqcap X^I$, therefore ι is a lifting space. $\square$

Remark 6.6. *The above results can be easily extended to more general spaces. Indeed, let $\aleph$ be any cardinal number, and let $\omega = \omega(\aleph)$ be the corresponding initial ordinal. Then we may consider spaces for which every point has local bases of cardinality at most $\aleph$ and spaces that are ω-compact, i.e. each 'sequence' indexed by ω has an accumulation point. Then we can repeat the above proofs word-by-word to show that if X has local basis of cardinality at most $\aleph$ and if $\widehat{X}$ is locally ω-compact then $\iota\widehat{X} \to X$ is a lifting space.*

To prove the converse of Theorem 6.3 we need to assume that X is a metric space. Let us first show that if (X, d) is a path-connected metric space then $\widehat{X}$ is also metrizable. Indeed, a suitable metric on $\widehat{X}$ can be defined by taking into account the path structure on X. Let the *breadth* of a path $\alpha\colon I \to X$ be defined as

$$\mathrm{br}(\alpha) := \sup\big\{d\big(\alpha(0), \alpha(t)\big) + d\big(\alpha(t), \alpha(1)\big) \mid t \in [0,1]\big\}.$$

Then we get a metric ρ on $\widehat{X}$ by

$$\rho(x, x') = \inf\big\{\mathrm{br}(\alpha) \mid \alpha\colon (I, 0, 1) \to (X, x, x')\big\}.$$

Note that always $d(x, x') \leq \rho(x, x')$, and that for a path-connected set $A \subseteq X$ we have $\mathrm{diam}_\rho(A) \leq 2 \cdot \mathrm{diam}_d(A)$.

We can easily verify that the topology of $\widehat{X}$ is induced by ρ. In fact, let C be a component of an open set $U \subseteq X$. For every $x \in C$ there is an ε-ball $B_d(x, \varepsilon) \subseteq U$, and since d-distance does not exceed ρ-distance, we also have that $B_\rho(x, \varepsilon) \subseteq U$. It follows that all points of $B_\rho(x, \varepsilon)$ can be connected by a path in U, therefore $B_\rho(x, \varepsilon) \subseteq C$, and so C is open with respect to the metric ρ. On the other hand, the ball $B_\rho(x, \epsilon)$ is clearly open with respect to the metric d, so the path-component of $B_\rho(x, \epsilon)$ containing x is an open set in $\widehat{X}$ contained in that ball.

Theorem 6.7. *Suppose X is a locally compact, path-connected metric space. If $\iota\colon \widehat{X} \to X$ has the homotopy lifting property for maps from $\mathbb{S}$ then $\widehat{X}$ is locally compact.*

Proof. Assume by contradiction that $\widehat{X}$ is not locally compact, so that there is a point $x \in X$ that does not possess any compact neighborhood in $\widehat{X}$. Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots$ be a strictly decreasing sequence converging to zero, such that all $B_d(x, \varepsilon_i)$ are relatively compact. For every i the corresponding ball $B_\rho(x, \varepsilon_i) \subset \widehat{X}$ is path-connected, contained in $B_d(x, \varepsilon_i)$ and, by the assumption, not relatively compact. Therefore, we can choose for each i a sequence $x^i_1, x^i_2, \ldots$ of points in $B_\rho(x, \varepsilon_i)$ that converges in X to a point in the closure of $B_d(x, \varepsilon_i)$ but does not have any accumulation points in $\widehat{X}$. For every j, we have $\rho(x^i_j, x^{i+1}_j) < \varepsilon_i + \varepsilon_{i+1}$, so we can find a path of breadth less then $\varepsilon_i + \varepsilon_{i+1}$ connecting x^i_j and x^{i+1}_j. We can concatenate these paths to obtain a path $\alpha_j\colon I \to X$, running from x^1_j to x^2_j on $[0, 1/2]$, from to x^2_j to x^3_j on $[1/2, 3/4]$, and so on. The paths α_j (taken in reverse direction) together with the constant path in x define a homotopy $H\colon \mathbb{S} \times I \to X$, given by the formula

$$H(u, t) = \begin{cases} x & \text{if } u = 0 \\ \alpha_{\frac{1}{u}}(1 - t) & \text{otherwise.} \end{cases}$$

Since H_0 is a constant map it can be lifted to $\widehat{X}$, but we cannot lift the entire homotopy H, because then $\widehat{H}_1$ would send the convergent sequence $\mathbb{S}$ to the sequence $x^1_1, x^1_2, x^1_3, \ldots$ that does not converge in $\widehat{X}$. $\square$

We may now combine Theorem 6.7 with Theorems 6.3 and 6.5 to obtain the following result.

Theorem 6.8. *Let X be a path-connected, locally compact metric space. Then $\iota\colon \widehat{X} \to X$ is a lifting space if and only if $\widehat{X}$ is locally compact.*

We do not know, whether the last result can be extended to all first-countable spaces.

Let $p\colon L \to X$ be a lifting space whose total space L is locally path-connected. Then one can easily check that the induced map $\hat{p}\colon L \to \widehat{X}$ is also a lifting space.

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