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MONOTONICITY OF THE SCHWARZ GENUS

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Abstract. The *Schwarz genus* $\mathtt{g}(\xi)$ of a fibration $\xi\colon E\to B$ is defined as the minimal integer n such that there exists a cover of B by n open sets that admit partial sections to ξ. Many important concepts, including the Lusternik–Schnirelmann category, Farber's topological complexity, and Smale–Vassiliev's complexity of algorithms can be naturally expressed as Schwarz genera of suitably chosen fibrations. In this paper we study Schwarz genus in relation with certain types of morphisms between fibrations. Our main result is the following: if there exists a fibrewise map $f\colon E\to E'$ between fibrations $\xi\colon E\to B$ and $\xi'\colon E'\to B$ which induces an n-equivalence between respective fibres for a sufficiently big n, then $\mathtt{g}(\xi)=\mathtt{g}(\xi')$. From this we derive several interesting results relating the topological complexity of a space with the topological complexities of its skeleta and subspaces (and similarly for the category). For example, we show that if a CW-complex has high topological complexity (with respect to its dimension and connectivity), then the topological complexity of its skeleta is an increasing function of the dimension.

1. Introduction

In this article we will use the standard definitions and results of the Lusternik–Schnirelmann category and the topological complexity. Interested readers can refer to [2] for the Lusternik–Schnirelmann category, to [3] for topological complexity of a space, and to [8] for topological complexity of a map. Note, however, that we follow the "non-normalized" convention, so the category and the topological complexity of a contractible space are both equal to 1.

Our main objective is to explain the relation between the topological complexity of a space and the topological complexity of its skeleta. A well-known result (see [4, Corollary to Theorem 1] or [2, Theorem 1.66]) states that if X is a noncontractible CW-complex, then $\mathrm{cat}(X)\geq\mathrm{cat}(X^{(r)})$, where $X^{(r)}$ denotes the r-skeleton of X. The result is in a sense surprising because $\mathrm{cat}(X)$ is a homotopy invariant of X, while the homotopy type of the skeleton can vary for different CW-decompositions of X. As a consequence, the above result restricts the homotopy type of skeleta. For example, it implies that every skeleton of a (noncontractible) coH-space is itself a coH-space.

The relation between the category of a space and the category of its skeleta does not extend directly to topological complexity. For example, the topological

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complexity is 2 for odd-dimensional spheres and 3 for even-dimensional spheres (see [3, Proposition 4.41]). Therefore, if we consider the standard CW-decomposition of S^∞, whose skeleta are finite-dimensional spheres, then the topological complexity of skeleta is an alternating sequence of 2's and 3's, while $\mathrm{TC}(S^\infty) = 1$.

In order to understand the causes for different behaviours of two closely related concepts we study certain properties of the Schwarz genus of a fibration (see [10] or [2, Section 9.3]). In fact, the category and the topological complexity can both be described in terms of the Schwarz genus of suitably chosen maps. Section 2 is dedicated to the study of the relations between genera of fibrations induced by morphisms of fibrations. The main result is Theorem 2.8: it gives sufficient conditions on a morphism between fibrations ξ and ξ' over a common base space which imply that $\mathsf{g}(\xi) = \mathsf{g}(\xi')$. Under similar assumptions but for a morphism between fibrations ξ and ξ' over different base spaces, we then obtain an inequality $\mathsf{g}(\xi) \leq \mathsf{g}(\xi')$. Section 3 is split into three subsections in which we apply the general theory to obtain a series of results that compare topological complexity or the category of a space to the topological complexity/category of its subspaces.

We will assume throughout this paper that the spaces under consideration are of the homotopy type of a CW-complex and have base-points, and that all maps are base-point preserving. Nevertheless we will systematically omit the base-points from the notation and we will not distinguish notationally between a map and its homotopy class.

2. Comparison of Schwarz genera

Let us recall some basic terminology about fibrations (see [9] for more details). A (Hurewicz) *fibration* is a triple (E, ξ, B), where the space B is the *base*, E is the *total space*, and $\xi\colon E \to B$ is a (continuous) map that has the homotopy lifting property for maps from arbitrary spaces. A *morphism* of fibrations is a pair $(f, \bar{f})\colon (E, \xi, B) \to (E', \xi', B')$, where $f\colon E \to E'$ and $\bar{f}\colon B \to B'$ are maps such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & E' \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\xi'} \\ B & \xrightarrow[\;\bar{f}\;]{} & B' \end{array}$$

Observe that the map f completely determines the map $\bar{f}$. If $B = B'$ and $\bar{f}$ is the identity map, we usually abbreviate the notation and write f instead of $(f, 1_B)$. For every $b \in B$ the preimage $\xi^{-1}(b) \subset E$ is called the *fibre* of ξ over b. In a morphism $(f, \bar{f})$ of fibrations the map f clearly sends the fibre over $b \in B$ to the fibre over $\bar{f}(b) \in B'$, so we will occasionally refer to f as a *fibrewise* map. A map $\sigma\colon B \to E$ is a *section* of ξ if $\xi \circ \sigma = 1_B$. More generally, if for some $A \subseteq B$ there exists a map $\sigma\colon A \to E$ such that $\xi \circ \sigma$ equals the inclusion of A in B, we will call it a *partial section* of ξ over A.

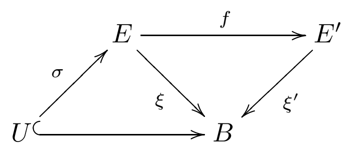
In a fibration (E, ξ, B) we will always assume that the base B is path-connected, which in turn implies (cf. [9, Proposition 1.12]) that all fibres of ξ have the same homotopy type. The fibre over the base-point of B will be called *the fibre of* ξ. If $(f, \bar{f})\colon (E, \xi, B) \to (E', \xi', B')$ is a morphism of fibrations, we will denote by $\tilde{f}\colon F \to F'$ the induced map between the respective fibres.

Following [10] we define the *genus* $\mathsf{g}(\xi)$ of the fibration (E, ξ, B) as the minimal integer n for which there exists a cover of B by n open sets that admit partial sections of ξ. In the context of the Lusternik–Schnirelmann category the genus is often called the *sectional category* of ξ (see [2, Section 9.3]).

Observe that the genus can be defined for arbitrary maps $\xi\colon E \to B$ by requiring that B have a cover by n open sets that admit *homotopy* sections to ξ. If ξ is a fibration, then every homotopy section can be replaced by a strict section, so the two definitions agree. As a matter of fact, most of our results could be easily generalized from fibrations to arbitrary maps. Our goal in this section is to show that certain kinds of morphisms between fibrations induce equality between the respective Schwarz genera (see Theorem 2.8). To this end we prove several preparatory lemmas.

Lemma 2.1. *If there exists a morphism of fibrations $f\colon (E, \xi, B) \to (E', \xi', B)$, then $\mathsf{g}(\xi) \geq \mathsf{g}(\xi')$.*

Proof. Consider the following diagram:



If σ is a partial section to ξ over U, then $f\sigma$ is a partial section to ξ' over U, and therefore $\mathsf{g}(\xi) \geq \mathsf{g}(\xi')$. ☐

Lemma 2.2. *Let $f\colon (E, \xi, B) \to (E', \xi', B)$ be a morphism of fibrations, and let $\sigma'\colon B \to E'$ be a section of ξ'. If a map $\sigma\colon B \to E$ satisfies $f\sigma \simeq \sigma'$, then σ is a homotopy section of ξ.*

Proof. Straightforward, since $\xi\sigma = \xi' f\sigma \simeq \xi'\sigma' = 1_B$. ☐

Following [11, Section VII,6.] we will say that a map $f\colon X \to Y$ is an n-*equivalence* for $n \geq 1$ if f induces a bijection between the path components of X and Y, and if the induced homomorphism on the homotopy groups $f_\sharp\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $0 < i < n$ and an epimorphism for $i = n$ (for any choice of base-points in X and Y). A canonical example of an n-equivalence is the inclusion map $X^{(n)} \hookrightarrow X$ of the n-skeleton of a CW-complex X. By [11, Corollary VII,6.23] if $f\colon X \to Y$ is an n-equivalence, then the induced function $f_*\colon [P, X] \to [P, Y]$ is bijective for every CW-complex P of dimension $\dim(P) \leq n-1$, and is surjective if $\dim(P) \leq n$.

By analogy, let us say that $f\colon X \to Y$ is a *homology n-equivalence* if the induced homomorphism on the integral homology groups $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for $0 \leq i < n$ and an epimorphism for $i = n$. By [11, Theorem 7.5.4] an n-equivalence is always a homology n-equivalence, and the converse holds if X and Y are simply connected.

Lemma 2.3. *Given a morphism of fibrations $f\colon (E, \xi, B) \to (E', \xi', B)$ the fibre-wise map $f\colon E \to E'$ is an n-equivalence if and only if the induced map between the respective fibres $\tilde{f}\colon F \to F'$ is an n-equivalence.*

Proof. Consider the following commutative ladder of exact sequences of homotopy groups of fibrations (E,ξ,B) and (E',ξ',B):

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \pi_{i+1}(E) & \longrightarrow & \pi_{i+1}(B) & \longrightarrow & \pi_i(F) & \longrightarrow & \pi_i(E) & \longrightarrow & \pi_i(B) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle f_\sharp} & & \| & & \downarrow{\scriptstyle \tilde f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \| & \\
\cdots \longrightarrow & \pi_{i+1}(E') & \longrightarrow & \pi_{i+1}(B) & \longrightarrow & \pi_i(F') & \longrightarrow & \pi_i(E') & \longrightarrow & \pi_i(B) & \longrightarrow \cdots
\end{array}$$

If $\tilde f_\sharp\colon \pi_i(F) \to \pi_i(F')$ is an isomorphism for $i < n$, then by the five lemma $f_\sharp\colon \pi_i(E) \to \pi_i(E')$ is also an isomorphism for $i < n$. Moreover, if $\tilde f_\sharp\colon \pi_{n-1}(F) \to \pi_{n-1}(F')$ is an isomorphism and $\tilde f_\sharp\colon \pi_n(F) \to \pi_n(F')$ is an epimorphism, then by the four lemma $f_\sharp\colon \pi_n(E) \to \pi_n(E')$ is an epimorphism. The converse implication is similarly proved. □

Proposition 2.4. *Let B be a CW-complex, and let $f\colon (E,\xi,B) \to (E',\xi',B)$ be a morphism of fibrations such that the induced map $\tilde f\colon F \to F'$ is an n-equivalence for some $n \geq \dim(B)$. Then for every section σ' of ξ' there exists a section σ of ξ for which $f\sigma \simeq \sigma'$.*

Proof. By Lemma 2.2 it is sufficient to find a map σ that is a lifting of the map σ' along f as in the diagram:

$$\begin{array}{ccc}
 & & E \\
 & {\scriptstyle \sigma}\nearrow & \downarrow{\scriptstyle f} \\
B & \xrightarrow[\sigma']{} & E'
\end{array}$$

By Lemma 2.3 the map $f\colon E \to E'$ is an n-equivalence. Since $n \geq \dim(B)$, the induced function $f_*\colon [B,E] \to [B,E']$ is surjective, which implies that there exists $\bar\sigma\colon B \to E$ such that $f\bar\sigma \simeq \sigma'$. By Lemma 2.2 $\bar\sigma$ is a homotopy section of ξ, and $\bar\sigma$ can be deformed to a strict section $\sigma\colon B \to E$ because ξ is a fibration. □

Schwarz [10, II.1] introduced a very useful construction that essentially reduces the computation of the genus to a problem in obstruction theory. Given a fibration (E,ξ,B) with fibre F, Schwarz defined the *n-fold fibrewise join* construction (actually called a *sum* in [10]) as a fibration $(*^n_B E, \xi_n, B)$, where $*^n_B E$ is a suitable subspace of the n-fold join $E*\cdots*E$, and the projection map ξ_n is a fibration whose fibre is the n-fold join of fibres F, which we denote as $*^n F$. The main property of Schwarz's construction is stated in the following theorem.

Theorem 2.5 (Schwarz, Theorem 3 in [10])**.** *The fibration (E,ξ,B) has genus $\mathsf{g}(\xi) \leq n$ if and only if the n-fold fibrewise join $(*^n_B E, \xi_n, B)$ admits a section.*

In other words, $\mathsf{g}(\xi)$ equals the minimal n for which ξ_n admits a section. Observe that the fibrewise join operation $*^n_B$ is functorial, i.e., a morphism

$$f\colon (E,\xi,B) \to (E',\xi',B)$$

induces a morphism

$$*^n_B f\colon (*^n_B E, \xi_n, B) \to (*^n_B E', \xi'_n, B),$$

whose restriction to the fibres is the usual n-fold join of maps $*^n \tilde f\colon\ *^n F \to *^n F'$.

Lemma 2.6. *Let X be a $(c-1)$-connected space, and let the map $f\colon X \to X'$ be a $(c+k)$-equivalence $(c, k \geq 0, c+k \geq 1)$. Moreover, let Y be a $(d-1)$-connected space, and let the map $g\colon Y \to Y'$ be a $(d+k)$-equivalence $(d \geq 0, d+k \geq 1)$. Then $X * Y$ is $(c+d)$-connected, and the map*

$$f * g\colon X * Y \to X' * Y'$$

is a $(c+d+k+1)$-equivalence.

Proof. Recall the standard homotopy equivalence $X*Y \simeq \Sigma(X \wedge Y)$, which together with the properties of the smash product immediately implies that $X*Y$ is $(c+d)$-connected.

Let us first consider the case where at least one of the spaces X, Y is path-connected, i.e., $c+d \geq 1$. Then $X*Y$ and $X'*Y'$ are 1-connected, so $f*g$ is a $(c+d+k+1)$-equivalence if and only if it is a homology $(c+d+k+1)$-equivalence. To prove the latter it is sufficient to show that $f \wedge g\colon X \wedge Y \to X' \wedge Y'$ is a homology $(c+d+k)$-equivalence. Consider the morphism of the Künneth short exact sequences for the reduced homology of the smash product (see [11, Theorem 5.3.10]):

$$\begin{array}{ccccc}
\bigoplus\limits_{i+j=l}(\widetilde{H}_i(X)\otimes\widetilde{H}_j(Y)) & \rightarrowtail & \widetilde{H}_l(X\wedge Y) & \twoheadrightarrow & \bigoplus\limits_{i+j=l-1}(\widetilde{H}_i(X)*\widetilde{H}_j(Y)) \\
{\scriptstyle \bigoplus(f_*\otimes g_*)}\downarrow & & {\scriptstyle (f\wedge g)_*}\downarrow & & \downarrow{\scriptstyle \bigoplus(f_* * g_*)} \\
\bigoplus\limits_{i+j=l}(\widetilde{H}_i(X')\otimes\widetilde{H}_j(Y')) & \rightarrowtail & \widetilde{H}_l(X'\wedge Y') & \twoheadrightarrow & \bigoplus\limits_{i+j=l-1}(\widetilde{H}_i(X')*\widetilde{H}_j(Y'))
\end{array}$$

Note that $\widetilde{H}_i(X) = \widetilde{H}_i(X') = 0$ for $i < c$ and that $\widetilde{H}_j(Y) = \widetilde{H}_j(Y') = 0$ for $i < d$. Therefore, if $l < c+d+k$, then $\widetilde{H}_i(X) \otimes \widetilde{H}_j(Y) \neq 0$ only if $c \leq i < c+k$ and $d \leq j < d+k$. Since f_* and g_* are isomorphisms in that range, we conclude that $\bigoplus(f_* \otimes g_*)$ is also an isomorphism. A similar argument shows that $\bigoplus(f_* * g_*)$ is an isomorphism, and therefore the middle map $(f \wedge g)_*$ is also an isomorphism.

If $l = c+d+k$, then we may follow the same line of reasoning to conclude that the first and the last summand in $\bigoplus(f_* \otimes g_*)$ are epimorphisms, while the remaining summands are isomorphisms, and therefore $\bigoplus(f_* \otimes g_*)$ is an epimorphism. The argument for $\bigoplus(f_* * g_*)$ is simpler, because $i+j = l-1$, which implies that all summands are isomorphisms and thus $\bigoplus(f_* * g_*)$ is an isomorphism. The four lemma then implies that $(f \wedge g)_*$ is an epimorphism.

If $c = d = 0$ (i.e., X, X', Y, and Y' are disconnected), then $k \geq 1$ and $X * Y, X' * Y'$ are path-connected but not necessarily simply connected. To simplify the notation we assume that all spaces have at most countably many components, so let $X_0, X_1, \ldots$ be the components of X, and let $Y_0, Y_1, \ldots$ be the components of Y. Moreover, assume that the base-point x_0 of X is contained in X_0 and that the base point y_0 of Y is contained in Y_0. By the definition of the smash product we have

$$X \wedge Y = (X_0 \wedge Y_0) \amalg \left(\coprod_{i>0} \frac{X_i \times Y_0}{X_i \times y_0}\right) \amalg \left(\coprod_{j>0} \frac{X_0 \times Y_j}{x_0 \times Y_j}\right) \amalg \left(\coprod_{i,j>0} X_i \times Y_j\right),$$

and there is an analogous description of $X' \wedge Y'$. Since $f\colon X \to X'$ induces a bijection between the components of X and X', and similarly for $g\colon Y \to Y'$, it follows that $f \wedge g$ induces a bijection between the components of $X \wedge Y$ and

$X' \wedge Y'$. By Seifert–van Kampen's theorem, $f * g\colon X * Y \to X' * Y'$ induces an isomorphism of respective fundamental groups. For higher homotopy groups one has to examine the map induced by $f * g$ between the respective universal covering spaces (or equivalently, consider the induced homomorphisms as homomorphisms of free $\mathbb{Z}G$-modules, where $G = \pi_1(X * Y)$). The details are straightforward but tedious, so we omit them. □

Note that the above lemma essentially states that the join operation preserves the property that a map is an equivalence in the k successive dimensions above the connectivity. Thus we have the following proposition.

Proposition 2.7. *Assume that F is $(c-1)$-connected $(c \geq 0)$ and that $\tilde{f}\colon F \to F'$ is a $(c+k)$-equivalence for some $k \geq 0$. Then $*^n \tilde{f}\colon *^n F \to *^n F'$ is an $(n(c+1)+k-1)$-equivalence.*

Proof. By inductive application of Lemma 2.6 the n-fold join $*^n F$ is $(n(c+1)-2)$-connected, hence $\pi_i(*^n \tilde{f})$ is an isomorphism in the following $(k-1)$ dimensions and an epimorphism in dimension $(n(c+1)+k-1)$. □

We are now ready to prove the main theorem of this section.

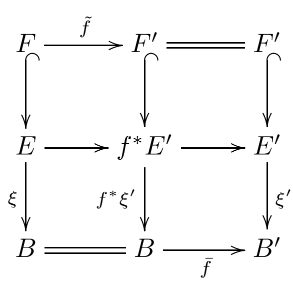
Theorem 2.8. *Let (E', ξ', B) be a fibration with a $(c-1)$-connected fibre F'. If there exists a morphism of fibrations $f\colon (E, \xi, B) \to (E', \xi', B)$ such that $\tilde{f}$ is a $(c+k)$-equivalence for some $k > \dim(B) - (c+1) \cdot \mathsf{g}(\xi')$, then $\mathsf{g}(\xi) = \mathsf{g}(\xi')$.*

Proof. By Lemma 2.1 it is always the case that $\mathsf{g}(\xi) \geq \mathsf{g}(\xi')$, so we only need to prove the converse inequality. By Schwarz's Theorem $n = \mathsf{g}(\xi')$ implies that the fibration ξ'_n admits a global section. By naturality of the Schwarz's construction there is a morphism of fibrations $*^n_B f\colon (*^n_B E, \xi_n, B) \to (*^n_B E', \xi'_n, B)$, whose restriction to the fibres is an $(n(c+1)+k-1)$-equivalence by Proposition 2.7. By assumptions of the theorem $\dim B \leq n(c+1)+k-1$, so by Proposition 2.4 the fibration ξ_n admits a section, and hence by Schwarz's Theorem $\mathsf{g}(\xi) \leq n$. □

We obtain as an immediate consequence the following comparison of genera of fibrations with different base spaces.

Corollary 2.9. *Let (E', ξ', B') be a fibration with a $(c-1)$-connected fibre F'. If there exists a morphism of fibrations $(f, \bar{f})\colon (E, \xi, B) \to (E', \xi', B')$ such that $\tilde{f}$ is a $(c+k)$-equivalence for some $k > \dim(B) - (c+1) \cdot \mathsf{g}(\xi')$, then $\mathsf{g}(\xi) \leq \mathsf{g}(\xi')$.*

Proof. We may decompose the morphism $(f, \bar{f})$ as in the following diagram, where the middle column represents the pullback of ξ' along $\bar{f}$, and f is equal to the composition $E \to f^*E' \to E'$:



The fibrewise map $E \to f^*E'$ satisfies the assumptions of Theorem 2.8; therefore $\mathsf{g}(\xi) = \mathsf{g}(f^*\xi')$. On the other hand, the genus of a pullback of ξ' is clearly smaller than or equal to the genus of ξ', and hence $\mathsf{g}(\xi) = \mathsf{g}(f^*\xi') \leq \mathsf{g}(\xi')$. □

3. Applications

In this section we are going to compare the values that invariants such as the topological complexity or category assume on a space X to the values that the same invariants assume on the skeleta and other subspaces of X. Recall that the usual approach to the computation of category and topological complexity is to find suitable upper and lower estimates. General upper estimates are based on dimension and connectivity of the spaces involved, while the lower estimates usually rely on the multiplicative structures in cohomology. Although the interval between the upper and lower estimates can be large, in most cases when the values of topological complexity or category are known exactly they are equal (or differ by one) to the general upper bound. Thus, in what follows we will normally begin with a general result and then consider the most interesting special case, when the value of the invariant is close to the general upper bound. Let us first discuss the topological complexity of maps.

3.1. **Topological complexity of maps.** Given a continuous map $u\colon X \to Y$ we consider the space X^I of all paths $\alpha\colon I \to X$, and a map $\xi_u\colon X^I \to X \times Y$ defined as $\xi(\alpha) := (\alpha(0), u(\alpha(1))$. The *topological complexity of the map* u, denoted $\mathrm{TC}(u)$, is defined as the minimal integer n such that there exists an increasing sequence of open subsets

$$\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = X \times Y$$

such that each difference $U_i - U_{i-1}$, $i = 1, \ldots, n$, admits a continuous partial section to the projection ξ_u. This concept appeared in [7] where it was used in order to measure the manipulation complexity of a robotic device: X and Y were, respectively, the configuration space and the working space of a mechanical system (like a robot arm) and $u\colon X \to Y$ was interpreted as the forward kinematic map of the system. The theory was further developed in [8]. In particular, if u is a fibration, then by [8, Lemma 4.1] the map $\xi_u\colon X^I \to X \times Y$ is also a fibration. As a consequence, if u is a fibration, then its topological complexity can be expressed in terms of Schwarz's genus:

$$\mathrm{TC}(u) = \mathsf{g}(\xi_u)$$

(see [8, Corollary 4.2]).

Topological complexity of a single space X is clearly equal to the topological complexity of the identity map 1_X. On the other hand, by [8, Corollary 4.8] the category of X can be retrieved as the topological complexity of the path fibration $\mathrm{ev}_1\colon PX \to X$, where PX denotes the space of all based paths in X, and ev_1 is the evaluation map that sends a based path α to its end-point $\alpha(1)$.

Theorem 4.9 in [8] allows further simplification, as it shows that $\mathrm{TC}(u) = \mathsf{g}(\eta_u)$, where $\eta_u\colon X \sqcap Y^I \to X \times Y$ is a fibration, whose total space is

$$X \sqcap Y^I = \{(x, \alpha) \in X \times Y^I \mid f(x) = \alpha(0)\}$$

and $\eta_u(x, \alpha) = (x, \alpha(1))$. It is easy to see that the fibre of η_u is the loop space ΩY.

Theorem 3.1. *Let* $(f, \bar{f})\colon (X, u, Y) \to (X', u', Y')$ *be a morphism of fibrations, and assume that* Y *is* $(c-1)$*-connected and that the map* $\bar{f}\colon Y \to Y'$ *is an* n*-equivalence, for some* $n \geq c \geq 1$. *If* $\dim(X \times Y) < c \cdot (\mathrm{TC}(u') - 1) + n$, *then* $\mathrm{TC}(u) \leq \mathrm{TC}(u')$.

Proof. First of all, observe that $(f, \bar{f})$ determines a morphism of fibration

$$\begin{array}{ccc} X \sqcap Y^I & \xrightarrow{f \times (\bar{f} \circ -)} & X' \sqcap (Y')^I \\ \Big\downarrow{\scriptstyle \eta_u} & & \Big\downarrow{\scriptstyle \eta_{u'}} \\ X \times Y & \xrightarrow[f \times \bar{f}]{} & X' \times Y' \end{array}$$

It is easy to check that the induced map between the fibres of η_u and $\eta_{u'}$ is $\Omega\bar{f}\colon \Omega Y \to \Omega Y'$. By the assumptions of the theorem ΩY is $(c-2)$-connected, and $\Omega\bar{f}$ is an $(n-1)$-equivalence. We may apply Corollary 2.9 with $B = X \times Y$ and $k = n - c$ to conclude that $c \cdot (\mathsf{g}(\eta_{u'}) - 1) + n > \dim(X \times Y)$ implies $\mathsf{g}(\eta_{u'}) \geq \mathsf{g}(\eta_u)$. The statement of the theorem follows immediately from the fact that $\mathrm{TC}(u) = \mathsf{g}(\eta_u)$ and $\mathrm{TC}(u') = \mathsf{g}(\eta_{u'})$. □

3.2. **Topological complexity of spaces.** Topological complexity of a space is the topological complexity of its identity map $\mathrm{TC}(X) = \mathrm{TC}(1_X)$. More traditionally (cf. [3, Definition 4.11, Proposition 4.12]), the *topological complexity* of X is the minimal integer n for which there exists an increasing sequence of open subsets

$$\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = X \times Y$$

such that each difference $U_i - U_{i-1}$, $i = 1, \ldots, n$, admits a continuous partial section to the projection $\xi\colon X^I \to X \times X$, $\xi(\alpha) := (\alpha(0), \alpha(1))$.

From the previous section we can immediately derive a result on topological complexity of spaces as follows. Let X be a $(c-1)$-connected space, and let $f\colon X \to X'$ be an n-equivalence for some $n \geq c \geq 1$. Then we have a morphism of trivial fibrations $(f, f)\colon (X, 1_X, X) \to (X', 1_{X'}, X')$, so by Theorem 3.1 we obtain

$$\mathrm{TC}(X) \leq \mathrm{TC}(X'),$$

provided that $2\dim(X) < c \cdot (\mathrm{TC}(X') - 1) + n$. In particular, if f is the inclusion of the n-skeleton $X^{(n)}$ into X we have the following result.

Theorem 3.2. *Let* X *be a finite-dimensional,* $(c-1)$*-connected CW-complex. Then* $\mathrm{TC}(X) \geq \mathrm{TC}(X^{(n)})$ *for all* $n < c \cdot (\mathrm{TC}(X) - 1)$.

If $X \simeq S^m$, then its n-skeleton is $(n-1)$-connected and at most n-dimensional, therefore it is homotopy equivalent to a (possibly empty) wedge of n-spheres. If m is odd, then $\mathrm{TC}(X) = 2$ and the assumptions of the corollary are not satisfied, because the dimension of the skeleton cannot be smaller than its connectivity. If m is even, then $\mathrm{TC}(X) = 3$ and the assumption of the corollary is that $n < 2n$, which holds for all positive n. Therefore, if $X \simeq S^m$ for m even, then $\mathrm{TC}(X^{(n)}) \leq 3$ for all $n \geq 1$.

By [6] we know that $\mathrm{TC}(X) = 2$ if and only if X is homotopy equivalent to an odd-dimensional sphere. Therefore, if X is $(c-1)$-connected and is not homotopy equivalent to a point or an odd-dimensional sphere, then there is always a nonempty range of dimensions, namely $c \leq n < c \cdot (\mathrm{TC}(X) - 1)$, for which $\mathrm{TC}(X) \geq \mathrm{TC}(X^{(n)})$. If the topological complexity of X is sufficiently large, then the above range can cover all skeleta and in some cases even all subcomplexes above the connectivity of X. Thus, by reformulating the above inequality we obtain the following estimates.

Corollary 3.3. *Let X be a $(c-1)$-connected CW-complex.*

(1) *If* $\mathrm{TC}(X) \geq \frac{\dim(X)}{c} + 1$, *then* $\mathrm{TC}(X) \geq \mathrm{TC}(X^{(n)})$ *for all* $n \geq c$.
(2) *Furthermore, if* $\mathrm{TC}(X) \geq \frac{2\dim(X)}{c}$, *then* $\mathrm{TC}(X) \geq \mathrm{TC}(A)$ *for every subcomplex A of X containing the $(c+1)$-skeleton of X.*

Proof. The first statement follows directly from Theorem 3.2. For the second statement note that by the cellular approximation theorem the inclusion of A in X is at least a $(c+1)$-equivalence. Then

$$c \cdot (\mathrm{TC}(X) - 1) + c + 1 > 2\dim(A),$$

so the discussion preceding Theorem 3.2 applies. □

Observe that for a $(c-1)$-connected complex there is a general upper estimate for topological complexity

$$\mathrm{TC}(X) \leq \frac{2\dim(X)}{c} + 1$$

(see [3, Theorem 4.16]). Thus, the assumption in Corollary 3.3(2) is that $\mathrm{TC}(X)$ differs at most by one from that upper bound. Many spaces are known to satisfy that assumption: examples include all closed surfaces with the exception of the torus, all complex and quaternionic projective spaces, most 3-dimensional lens spaces, configuration spaces, and many others (cf. [3], [5]). Further examples can be obtained by taking finite products of the above.

In spite of the above results, we are still not able to rule out possible anomalous behaviour, e.g., an increase in topological complexity caused by the removal of a single point. The following question would be of some interest to applications.

Question. Does there exist a closed manifold M such that $\mathrm{TC}(M) < \mathrm{TC}(M - x)$ for a single point $x \in M$?

Recall that the $(k+1)$-skeleton of a space X can be obtained as a mapping cone of a single attaching map to its k-skeleton, therefore $\mathrm{cat}(X^{(k+1)}) \leq \mathrm{cat}(X^{(k)}) + 1$. As a consequence the category of skeleta cannot "jump": if $\mathrm{cat}(X) = n$, then every integer $k = 1, 2, \ldots, n$ must appear as the category of some skeleton on X. The behaviour of the topological complexity can be much more complicated as shown by the sequence of $\mathrm{TC}(\mathbb{R}P^n)$ for $1 \leq n \leq 23$ in [3, p. 122]. In general we have only a coarse estimate [3, Prop. 4.28],

$$\mathrm{TC}(X^{(k+1)}) \leq \mathrm{TC}(X^{(k)}) + \mathrm{cat}(X^{(k)}) + 1.$$

Nevertheless, by assuming some control over the category, we are able to show that the topological complexity is an increasing function along the skeleta.

Proposition 3.4. *If* $\mathrm{cat}(X) = \dim(X) + 1$, *then*

$$\mathrm{TC}(X^{(1)}) \leq \mathrm{TC}(X^{(2)}) \leq \cdots \leq \mathrm{TC}(X).$$

Proof. Since $\mathrm{cat}(X^{(n)}) \leq \mathrm{cat}(X^{(n-1)}) + 1$ for every n, it follows that $\mathrm{cat}(X^{(n)}) = n + 1$ for $0 \leq n \leq \dim(X)$. We may thus apply Corollary 3.3 at each stage of the CW-decomposition to show that $\mathrm{TC}(X^{(n)}) \geq \mathrm{TC}(X^{(n-1)})$. □

3.3. **Category of maps and spaces.** The *category of a map* $u\colon X \to Y$, denoted as $\mathrm{cat}(u)$, is defined as the minimal integer n for which there exists an open covering $U_1, \ldots, U_n$ of X such that the restrictions $f|_{U_i}\colon U_i \to Y$ are null-homotopic (cf. [2, p. 35]). Clearly, the category of a space X is equal to the category of the identity map 1_X. By [2, Proposition 9.18]

$$\mathrm{cat}(u) = \mathsf{g}(\mu_u),$$

where $\mu_u\colon X \sqcap PY \to X$ is the pullback of the path-fibration $PY \to Y$ along the map u, i.e.,

$$X \sqcap PY = \{(x, \alpha) \in X \times PY \mid u(x) = \alpha(1)\} \text{ and } \mu_u(x, \alpha) = x.$$

Note that we did not require that $u\colon X \to Y$ be a fibration, but it is easy to see that the category of a map is equal to the category of its fibrational substitute. As a consequence, we may assume without loss of generality that u is actually a fibration. The proof of the following theorem is analogous to that of Theorem 3.1, so we omit the details.

Theorem 3.5. *Let $(f, \bar{f})\colon (X, u, Y) \to (X', u', Y')$ be a morphism of fibrations, and assume that Y is $(c-1)$-connected and that the map $\bar{f}\colon Y \to Y'$ is an n-equivalence, for some $n \geq c \geq 1$. If $\dim(X) < c \cdot (\mathrm{cat}(u') - 1) + n$, then $\mathrm{cat}(u) \leq \mathrm{cat}(u')$.*

If u and u' are taken to be identity maps, we obtain a comparison between the category of a space and the categories of its skeleta and subspaces. The following theorem is a generalization of results proved by Felix, Halperin, and Thomas [4].

Theorem 3.6. *Let X be a $(c-1)$-connected CW-complex X $(c \geq 1)$.*

(1) *If A is a subcomplex of X containing $X^{(n)}$ $(n \geq c)$ and such that*

$$\dim(A) < n + c \cdot (\mathrm{cat}(X) - 1),$$

then $\mathrm{cat}(A) \leq \mathrm{cat}(X)$.

(2) *If $\mathrm{cat}(X) \geq \frac{\dim(X)}{c}$, then $\mathrm{cat}(A) \leq \mathrm{cat}(X)$ for every subcomplex $A \leq X$ containing the $(c+1)$-skeleton of X.*

Proof. If A contains $X^{(n)}$, then the inclusion $A \hookrightarrow X$ is an n-equivalence, therefore (1) follows directly from Theorem 3.5.

The inclusion of A into X is a $(c+1)$-equivalence and $c + 1 + c \cdot (\mathrm{cat}(X) - 1) > \dim(X) \geq \dim(A)$, so the statement follows from (1). Note that the statement remains valid even if X is infinite-dimensional. □

As a final remark, if $\mathrm{cat}(X^{(n)}) \geq 3$ for some n, then $\mathrm{cat}(X^{(n+1)}) \geq 2$ by [1, Proposition 2.6], and then [2, Theorem 1.66]) implies that $\mathrm{cat}(X^{(n+1)}) \geq \mathrm{cat}(X^{(n)}) \geq 3$. Therefore, as a function of dimension, the category of skeleta cannot decrease once it becomes bigger than 2. No such result is known for the topological complexity of skeleta.

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