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Generating the inverse limit of free groups

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ABSTRACT

We study the relation between two uncountable groups with remarkable properties (cf. [15]): the *topological free product* of infinite cyclic groups $\mathcal{G}$ (the fundamental group of the Hawaiian Earring), and the inverse limit of finitely generated free groups $\widehat{F}$. The former has a canonical embedding as a proper subgroup of the latter and we examine when $\mathcal{G}$, together with certain naturally defined normal subgroups of $\widehat{F}$ generate the entire group $\widehat{F}$. We are interested in particular in normal subgroups

$$\mathrm{Ker}_T(\widehat{F}) = \bigcap \{\mathrm{Ker}\varphi \mid \varphi \in \hom(\widehat{F}, T)\},$$

where T is some finitely-presented n-slender group.
Our main results state that if T is the infinite cyclic group or the free nilpotent class 2 group on 2 generators, then $\mathcal{G}$ and $\mathrm{Ker}_T(\widehat{F})$ generate $\widehat{F}$. On the other hand, if T is the free

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nilpotent class 3 group or a Baumslag-Solitar group, then the product of subgroups $\mathcal{G} \cdot \mathrm{Ker}_T \widehat{F}$ is a proper subgroup of $\widehat{F}$. In the last section, we provide an interesting geometric interpretation of the above results in terms of path-connectedness of certain fibrations arising as inverse limits of covering spaces over the Hawaiian earring space.

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1. Introduction

In this paper we study certain residual properties of the limit of an inverse sequence of free groups. Let F_k denote the free group on generators $x_1, \ldots, x_k$ and, let $p_k\colon F_{k+1} \to F_k$ be the obvious epimorphism, that sends x_{k+1} to 1. Then we define $\widehat{F}$ as

$$\widehat{F} := \varprojlim (F_1 \xleftarrow{p_1} F_2 \xleftarrow{p_2} F_3 \longleftarrow \cdots)$$

By analogy with the free group on countably many generators, whose elements are finite words on a countable alphabet (generators and their inverses), one could expect that the elements of $\widehat{F}$ may be represented as infinite words of the form $w = a_1a_2a_3\ldots$ where $a_i \in \{x_1, x_1^{-1}, x_2, x_2^{-1}\ldots\}$. Surprisingly, such infinite words make only a tiny fraction of $\widehat{F}$. Indeed, observe that the inverse of $w = a_1a_2a_3\ldots$ is $w^{-1} = \ldots a_2^{-1}a_1^{-1}$, and that infinite concatenations of such words lead to complicated linearly ordered sequences of letters, so the elements of $\widehat{F}$ have to be described in terms of linear order types (cf. [1]). The group $\widehat{F}$ was studied by Higman [15] under the name of *unrestricted free products*.

Higman studied, in the same paper, a subgroup of $\widehat{F}$ which he named P, consisting of elements in which every letter appears only finitely many times, and which we will denote as $\mathcal{G}$. A few years later Griffiths [13] showed that $\mathcal{G}$ can be naturally identified as the fundamental group of a peculiar topological space called the *Hawaiian earring*; hence we will refer to $\mathcal{G}$ as the *Hawaiian earring group*. Even more is true: every fundamental group of a 1-dimensional Peano continuum (compact, connected and locally path connected metric space) embeds as a subgroup of $\widehat{F}$ and the embedding reflects much of the geometry of the space. See [1, Introduction] for a historical overview of these results.

The groups $\mathcal{G}$ and $\widehat{F}$ are not isomorphic, as can be deduced e.g. from [10, Theorem 1.2]. Yet $\mathcal{G}$ is with many aspects similar to its supergroup $\widehat{F}$, but has the advantage of a straightforward geometric interpretation. We have recently discovered that $\widehat{F}$ is generated by $\mathcal{G}$ together with the so called $\mathbb{Z}$*-kernel* of $\widehat{F}$, denoted $\mathrm{Ker}_{\mathbb{Z}}(\widehat{F})$ and defined as the intersection of all kernels of homomorphisms from $\widehat{F}$ to $\mathbb{Z}$. In other words we may represent $\widehat{F}$ as a product of two subgroups

$$\widehat{F} = \mathcal{G} \cdot \mathrm{Ker}_{\mathbb{Z}}(\widehat{F}).$$

Note that the $\mathbb{Z}$-kernel of a free group is exactly its commutator subgroup, while for $\widehat{F}$, which is not free but only locally free (cf. [15]) the commutator subgroup is properly contained in $\mathrm{Ker}_{\mathbb{Z}}(\widehat{F})$. We will show in the last section that $\mathbb{Z}$-kernels have a nice geometric interpretation in terms of inverse limits of infinite cyclic coverings. The intersection of $\mathcal{G}$ with $\mathrm{Ker}_{\mathbb{Z}}(\widehat{F})$ is quite big, so it is natural to ask if there are other groups T for which the formula $\widehat{F} = \mathcal{G} \cdot \mathrm{Ker}_T(\widehat{F})$ holds but where the T-kernel, $\mathrm{Ker}_T(\widehat{F})$, is as small as possible.

To explain what kind of groups we are looking for, we need another classical concept. An abelian group A is called *slender* if every homomorphism φ from the Cartesian product $\mathbb{Z}^{\mathbb{N}}$ to A factors as

$$\begin{tikzcd}
\mathbb{Z}^{\mathbb{N}} \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}"'] & & A \\
& \mathbb{Z}^n \arrow[ur, dashed, "\varphi_n"'] &
\end{tikzcd}$$

for some n, in other words, if the value φ on some $g \in \mathbb{Z}^{\mathbb{N}}$ is completely determined by the values it takes on a finite number of coordinates of g (cf [11, p. 489]). K. Eda [9, Definition 3.1] introduced a non-commutative version of slenderness by replacing $\mathbb{Z}^{\mathbb{N}}$ with the Hawaiian earring group $\mathcal{G}$. Thus, a group T is *non-commutatively slender* (*n-slender* for short) if every homomorphism $\varphi \colon \mathcal{G} \to T$ factors as

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "\varphi"] \arrow[dr, "p_n"'] & & T \\
& F_n \arrow[ur, dashed, "\varphi_n"'] &
\end{tikzcd}$$

where p_n is the canonical projection from $\mathcal{G}$ to the free group of rank n. Eda proved [9, Corollary 3.7] that every free group (and in particular the group $\mathbb{Z}$) is n-slender. Thus we may state the main question that we consider in this work:

Question 1.1. *For which n-slender groups T does the following equality hold*

$$\widehat{F} = \mathcal{G} \cdot \mathrm{Ker}_T(\widehat{F})?$$

Note that we may equivalently ask for which T is the following composition

$$\mathcal{G} \longrightarrow \widehat{F} \longrightarrow \widehat{F}/\mathrm{Ker}_T(\widehat{F})$$

surjective? This alternative formulation provides a link to properties of inverse sequences of coverings over the Hawaiian earring.

Natural candidates for T are groups that are close to abelian and are nilpotent or soluble. Thus we considered two types of groups, the free nilpotent groups $F_c(x, y)$ of nilpotency class c, and the Baumslag-Solitar groups $B(1,n)$. Our main results (Theorem 3.2 and Theorem 3.3) may be summarized as follows.

Theorem 1.2. *If $T=\mathbb{Z}$ or $T=F_2(x,y)$, then $\widehat{F}=\mathcal{G}\cdot\mathrm{Ker}_T(\widehat{F})$. On the other hand, if $T=F_3(x,y)$ or $T=B(1,n)$, then $\mathcal{G}\cdot\mathrm{Ker}_T(\widehat{F})$ is a proper subgroup of $\widehat{F}$.*

We believe that the above results and the methods of proof are sufficiently interesting for their algebraic content but let us also mention briefly their geometric interpretation (see Section 4). Denote by X the Hawaiian earring space, so that $\mathcal{G}=\pi_1(X)$. We will see that for every homomorphism $\varphi\colon\mathcal{G}\to T$ to a n-slender group T there is a covering space $p\colon\widetilde{X}\to X$, such that $\mathrm{Ker}\varphi=\mathrm{Im}\,p_*$. As a consequence, the set of all homomorphisms from $\mathcal{G}$ to T induces an inverse system of covering spaces over X, whose inverse limit is a fibration with unique path-lifting property (cf. [16, Chapter 2]) that we denote as $\overline{p}\colon\overline{X}_T\to X$. A remarkable fact [3, Section 4] is that the space $\overline{X}_T$ is path-connected if, and only if, $\widehat{F}=\mathcal{G}\cdot\mathrm{Ker}_T(\widehat{F})$. Thus, the above theorem immediately implies the following

Corollary 1.3. *For $T=\mathbb{Z}$ or $T=F_2(x,y)$ the inverse limit of all T-coverings over the Hawaiian earring is path-connected, while for $T=F_3(x,y)$ or $T=B(1,n)$ the inverse limit of all T-coverings over the Hawaiian earring is not path-connected.*

The rest of the paper is split in three sections. Section 2 contains the preliminary results about slender groups, T-kernels, uniform normal forms and alternating tensors. Section 3 contains the main results of the paper. We first prove that for $T=\mathbb{Z}$ or $T=F_2(x,y)$ we have $\widehat{F}=\mathcal{G}\cdot\mathrm{Ker}_T(\widehat{F})$. Then we turn to a (much harder) negative result, that for $T=F_3(x,y)$ or $T=B(1,n)$ the product $\mathcal{G}\cdot\mathrm{Ker}_T(\widehat{F})$ is a proper subgroup of $\widehat{F}$. In the final section we give an application of our algebraic results to fibrations over the Hawaiian earring.

1.1. Notation

For a group G, we will denote the commutator subgroup by $G'=[G,G]$, which is generated by the set of all commutators $[x,y]=xyx^{-1}y^{-1}$ for x and y in G. The abelianization G/G' will be denoted by G^{ab}. For iterated commutators we use the abbreviation $[x,y,z]=[x,[y,z]]$.

For any group G let $\gamma_c(G)$ denote the c-th term of the lower central series

$$\gamma_1(G)=G\geq\gamma_2(G)=[G,\gamma_1(G)]\geq\gamma_3(G)=[G,\gamma_2(G)]\geq\cdots$$

The *free nilpotent group* of class c freely generated by a set X is the factor group $F_c(X):=F(X)/\gamma_{c+1}(F(X))$. In particular, $F_1(x,y)$ is the free abelian group on two generators, while $F_2(x,y)$ and $F_3(x,y)$ are respectively class 2 and class 3 free nilpotent groups on two generators.

Another two-generator group that will play an important role in this paper is $B(1,n)$, the *Baumslag-Solitar group*, defined by the presentation $\langle x, y \mid yxy^{-1} = x^n \rangle$. To avoid trivial exceptions we will always assume that $n \notin \{-1, 0, 1\}$.

2. Preliminaries

2.1. Slender groups

As we mentioned in the Introduction, an abelian group A is said to be *slender* if every homomorphism $\varphi\colon \mathbb{Z}^{\mathbb{N}} \to A$ factors through some $\mathbb{Z}_n$. This is a classical and very useful concept with several natural extensions. For example, K. Eda defined an arbitrary group T to be *non-commutatively slender* or *n-slender* if every homomorphism φ from the Hawaiian Earring group $\mathcal{G}$, a non-abelian analogue to $\mathbb{Z}^{\mathbb{N}}$, to T factors through some F_n. He also proved that an abelian group is slender if, and only if it is n-slender (see [9]). Slender groups are related to automatic continuity of homomorphisms. S. Corson proved the following characterization of n-slender groups. A group T is n-slender if, and only if, every homomorphism from the fundamental group of any locally path-connected metric space to T has open kernel (with respect to the natural quotient topology, see [6, Lemma 5.2]). This result motivated another definition: a group T is *cm-slender* (short for 'completely metrizable') if every homomorphism $\varphi\colon G \to T$, where G is a completely metrizable group, has open kernel.

Since the group $\widehat{F}$ is an inverse limit of discrete groups, it is clearly completely metrizable. We say that a group T is *$\widehat{F}$-slender* provided every homomorphism $\varphi\colon \widehat{F} \to T$ has open kernel, i.e., factors through a canonical projection $p_n : \widehat{F} \to F_n$ for some $n \geq 1$. Note that every cm-slender group is $\widehat{F}$-slender.

Lemma 2.1. *The groups $F_3(x,y)$ and $B(1,n)$ $(n \neq 0)$ are $\widehat{F}$-slender.*

Proof. It suffices to prove that they are cm-slender. If $T = F_3(x,y)$, then the factors of the lower central series are $T/\gamma_2(T) \cong \mathbb{Z} \times \mathbb{Z}$, $\gamma_2(T)/\gamma_3(T) \cong \mathbb{Z}$ and $\gamma_3(T) \cong \mathbb{Z} \times \mathbb{Z}$. Free abelian groups are cm-slender by Theorem A and Theorem B(f) in [2]. Moreover, by [2, Lemma 3.2] cm-slenderness is preserved by group extensions, therefore $F_3(x,y)$ is cm-slender.

The group $B(1,n)$ is cm-slender by [2, Theorem B(g)]. □

2.2. The T-kernel of a group

Let us begin with a formal definition of the concept that we mentioned in the introduction. For arbitrary groups G and T we define the *T-kernel* of G as the intersection of the kernels of all homomorphisms from G to T:

$$\mathrm{Ker}_T(G) := \bigcap_{\varphi\colon G \to T} \mathrm{Ker}(\varphi).$$

In other words $\mathrm{Ker}_T(G)$ is the intersection of all normal subgroups $N \triangleleft G$ such that G/N can be embedded in T. Clearly $\mathrm{Ker}_T(G)$ is a fully characteristic subgroup of G.

More generally, given a homomorphism $f\colon G \to H$ and an element $x \in \mathrm{Ker}_T(G)$, then $\varphi(f(x)) = 0$ for every $\varphi\colon H \to T$, therefore $f(x) \in \mathrm{Ker}_T(H)$, so the restriction of f maps $\mathrm{Ker}_T(G)$ into $\mathrm{Ker}_T(H)$. In categorical language, $\mathrm{Ker}_T(\cdot)$ assigns to every group G a subgroup $\mathrm{Ker}_T(G)$, and to every homomorphism $f\colon G \to H$ a homomorphism $f_T\colon \mathrm{Ker}_T(G) \to \mathrm{Ker}_T(H)$, and is thus a covariant functor. Some of its basic properties are given in the following proposition.

Proposition 2.2. *Let T be any group, and let $f_T\colon \mathrm{Ker}_T(G) \to \mathrm{Ker}_T(H)$ be induced by a homomorphism $f\colon G \to H$. Then*

(1) $\mathrm{Ker}(f_T) = \mathrm{Ker}(f) \cap \mathrm{Ker}_T(G)$;
(2) $\mathrm{Im}\,(f_T) \cong \mathrm{Ker}_T(G)/\mathrm{Ker}(f_T) \cong \mathrm{Ker}_T(G/\mathrm{Ker}(f_T))$.
(3) If $f^*\colon \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)$ *is surjective, then* $\mathrm{Ker}_T(\mathrm{Im}\, f) = \mathrm{Im}\, f \cap \mathrm{Ker}_T(H)$.
(4) If f is a split epimorphism, then so is f_T. In other words, if $G \cong H \ltimes \mathrm{Ker} f$, then $\mathrm{Ker}_T(G) \cong \mathrm{Ker}_T(H) \ltimes \mathrm{Ker}_T(\mathrm{Ker} f)$.

Proof. The first statement is obvious, because f_T is just a restriction of f. Observe, that in particular, the functor Ker_T preserves monomorphisms.

Toward the proof of (2), we apply the functor Ker_T to the factorization

$$\begin{tikzcd}
G \arrow[r, two heads] \arrow[rr, bend left, "f"] & G/\mathrm{Ker} f \arrow[r, tail] & H
\end{tikzcd}$$

and obtain

$$\begin{tikzcd}
\mathrm{Ker}_T(G) \arrow[rr] \arrow[rrr, bend left, "f_T"] \arrow[dr] & & \mathrm{Ker}_T(G/\mathrm{Ker} f) \arrow[r, tail] & \mathrm{Ker}_T(H) \\
& \mathrm{Ker}_T(G)/\mathrm{Ker}(f) \cap \mathrm{Ker}_T(G) \arrow[ur, tail] & &
\end{tikzcd}$$

Clearly $\mathrm{Im}\,(f_T) \cong \mathrm{Ker}_T(G)/\mathrm{Ker}(f) \cap \mathrm{Ker}_T(G) = \mathrm{Ker}_T(G)/\mathrm{Ker}(f_T)$.

By applying Ker_T to the epimorphism $G \to G/\mathrm{Ker}(f_T)$ we obtain a homomorphism $\mathrm{Ker}_T(G) \to \mathrm{Ker}_T(G/\mathrm{Ker}(f_T))$, whose kernel is clearly $\mathrm{Ker}(f_T)$, by (1). To show that $\mathrm{Ker}_T(G) \to \mathrm{Ker}_T(G/\mathrm{Ker}(f_T))$ is surjective, let $[x] \in G/\mathrm{Ker}(f_T)$ be a coset satisfying $\varphi([x]) = 0$ for every $\varphi\colon G/\mathrm{Ker}(f_T) \to T$. Every homomorphism $\psi\colon G \to T$ becomes trivial when restricted to $\mathrm{Ker}(f_T) \subseteq \mathrm{Ker}_T(G)$. Therefore ψ factors as $\psi = \overline{\psi} \circ q$ for the quotient homomorphism $q\colon G \to G/\mathrm{Ker}(f_T)$ and some (uniquely determined) homomorphism $\overline{\psi}\colon G/\mathrm{Ker}(f_T) \to T$. As a consequence $x \in \mathrm{Ker}_T(G)$ and $\mathrm{Ker}_T(G) \to \mathrm{Ker}_T(G/\mathrm{Ker}(f_T))$ is surjective.

To prove (3) note that $G/\mathrm{Ker} f \cong \mathrm{Im}\, f \subseteq H$, therefore $\mathrm{Ker}_T(\mathrm{Im}\, f) \subseteq \mathrm{Im}\, f \cap \mathrm{Ker}_T(H)$. On the other hand, if $f^*\colon \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)$ is surjective, then so is the restriction map $\mathrm{Hom}(H,T) \to \mathrm{Hom}(\mathrm{Im}\, f, T)$. In other words, every $\varphi\colon \mathrm{Im}\, f \to T$ is a restriction of some $\overline{\varphi}\colon H \to T$. As a consequence $\varphi(x) = 0$ for every $x \in \mathrm{Im}\, f \cap \mathrm{Ker}_T(H)$ and $\varphi\colon \mathrm{Im}\, f \to T$, therefore $\mathrm{Ker}_T(\mathrm{Im}\, f) \supseteq \mathrm{Im}\, f \cap \mathrm{Ker}_T(H)$.

Finally, claim (4) follows by functoriality. □

The formula $\mathrm{Ker}_T(G)/\mathrm{Ker}(f_T) \cong \mathrm{Ker}_T(G/\mathrm{Ker}(f_T))$ in (2) above has several interesting consequences.

Corollary 2.3.

(1) If N is a normal subgroup of G, then

$$\mathrm{Ker}_T\big(G/\mathrm{Ker}_T(N)\big) \cong \mathrm{Ker}_T(G)/\mathrm{Ker}_T(N).$$

(2) Moreover, $\mathrm{Ker}_T(G) = N$ *if, and only if,* $N \subseteq \mathrm{Ker}_T(G)$ *and* $\mathrm{Ker}_T(G/N) = 1$.

(3) $\mathrm{Ker}_T(G/\mathrm{Ker}_T(G)) = 1$.

Proof. Since $\mathrm{Ker}_T(N) \subseteq \mathrm{Ker}_T(G)$, we have that any homomorphism $G \to T$ factors through $G \to G/\mathrm{Ker}_T(N)$. Thus the induced homomorphism $\mathrm{Ker}_T(G) \to \mathrm{Ker}_T\big(G/\mathrm{Ker}_T(N)\big)$ is surjective. Then to prove (1) of the corollary, we may simply apply (1) of Proposition 2.2 to the homomorphism $G \to G/\mathrm{Ker}_T(N)$ taking into account that $\mathrm{Ker}_T(N) \subseteq \mathrm{Ker}_T(G)$.

For (2) assume $N \subseteq \mathrm{Ker}_T(G)$ and apply Proposition 2.2(2) to the quotient homomorphism $f\colon G \to G/N$. Then $N \subseteq \mathrm{Ker}_T(G)$ implies $\mathrm{Ker}(f_T) = N \cap \mathrm{Ker}_T(G) = N$, and so $\mathrm{Ker}_T(G/N) \cong \mathrm{Ker}_T(G)/N$. Clearly, triviality of $\mathrm{Ker}_T(G/N)$ implies $\mathrm{Ker}_T(G) = N$, and vice-versa. (3) is just (an important) special case of (2). □

Remark 2.4. Statement (3) of Corollary 2.3 is sometimes called a *residual property*. In fact, for a given group T we may consider the variety $\mathfrak{V}_T$ of all groups H satisfying the same identities as T. For a group G and variety $\mathfrak{V}$, we may define the $\mathfrak{V}$*-residual* $\mathfrak{R}_{\mathfrak{V}}(G)$ of G, obtained as the intersection of all kernels of homomorphisms from G to some group in $\mathfrak{V}$. As a consequence of the definitions

$$\mathfrak{R}_{\mathfrak{V}_T}(G) \leq \mathrm{Ker}_T(G).$$

Let us note that this containment is in general proper: if we set $G := \mathbb{Q}$ and $T := \mathbb{Z}$, then the variety corresponding to $T = \mathbb{Z}$ is $\mathfrak{A}$, the class of abelian groups and then $\mathfrak{R}_{\mathfrak{A}}(\mathbb{Q}) = \{0\}$ while $\mathrm{Ker}_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}$.

We will also need to understand how T-kernels are affected by changing T. The following properties will be needed in the sequel.

Lemma 2.5. *Let $f\colon S \to T$ be a homomorphism, and G an arbitrary group.*

(1) If f is injective, then $\operatorname{Ker}_T(G) \leq \operatorname{Ker}_S(G)$.
(2) If the function $\operatorname{Hom}(G,S) \to \operatorname{Hom}(G,T)$ given by $\varphi \mapsto f \circ \varphi$ is surjective, then $\operatorname{Ker}_S(G) \leq \operatorname{Ker}_T(G)$.
(3) If $S \triangleleft T$ and $\operatorname{Hom}(G, T/S) = \{*\}$, then $\operatorname{Ker}_S(G) = \operatorname{Ker}_T(G)$.

Proof. (1) If $x \in \operatorname{Ker}_T(G)$, then $f(\varphi(x)) = 0$ for every $\varphi\colon G \to S$. Since f is injective, it follows that $x \in \operatorname{Ker}_S(G)$.

(2) Let $x \in \operatorname{Ker}_S(G)$, and consider any homomorphism $\varphi\colon G \to T$. By the assumption, there exists $\varphi'\colon G \to S$, such that $\varphi = f \circ \varphi'$. As a consequence $\varphi(x) = f(\varphi'(x)) = 0$, therefore $x \in \operatorname{Ker}_T(G)$.

(3) It is sufficient to observe that the image of every homomorphism $\varphi\colon G \to T$ is contained in S, therefore the claim follows by (1) and (2). □

2.3. T-kernels of free groups

We can view $\operatorname{Ker}_T(G)$ as the part of G that cannot be represented in a product of copies of the group T. For example $\operatorname{Ker}_{\mathbb{Z}}(G) = 0$ if, and only if, G is residually free-abelian. Moreover, T-kernels give an alternative description for some important characteristic subgroups. The first claim in the next proposition is well-known, while the second is proved in [4].

Proposition 2.6. *Let F be the free group on any generating set. Then the $\mathbb{Z}$-kernel of F is F', the commutator subgroup of F (coinciding with the set of words x with exponent sum zero for every generator), and the $B(1,n)$-kernel of F is F'', the second commutator subgroup of F.*

One of the main results in this section is the following description of terms of the central lower series of free groups as certain T-kernels.

Theorem 2.7. *Let F be a free group and $T := F_c(x,y)$, the free nilpotent group of class $c \leq 3$. Then $\operatorname{Ker}_T(F) = \gamma_{c+1}(F)$.*

Proof. For $c = 1$ this is just Proposition 2.6.

If $c = 2$, then T is nilpotent of class 2. Therefore, taking Remark 2.4 into account, it follows that $\operatorname{Ker}_{F_2(x,y)}(F)$ must contain the nilpotent class 2 residual of F, i.e. $\gamma_3(F)$. By Corollary 2.3(2) we need to show that $\operatorname{Ker}_{F_2(x,y)}(F/\gamma_3(F)) = 1$.

Let us denote $G := F/\gamma_3(F)$, so that G is a free nilpotent group of class 2. If $g \in G \setminus G'$, then $F_2(x,y)' = \langle [x,y] \rangle \cong \mathbb{Z}$ implies that we can find a homomorphism $\varphi\colon G \to F_2(x,y)' \subset F_2(x,y)$ such that $\varphi(g) \neq 1$. As a consequence $g \notin \mathrm{Ker}_{F_2(x,y)}(G)$.

Thus assume $g \in G'$, and apply the Hall commutator collecting process as described e.g. in [5, p. 78]. Let $\{x_i : i \in I\}$ be a linearly ordered base of the free group F. Then the set of all commutators

$$\{[x_i, x_j] : i, j \in I, \quad i > j\}$$

is a $\mathbb{Z}$-module basis of F' and has an induced linear ordering. Therefore there is a unique presentation of g modulo $\gamma_3(F)$ as a finite product of powers of basic commutators

$$\prod_{i,j \in I, i>j} [x_i, x_j]^{\alpha_{ij}}, \quad \alpha_{ij} \in \mathbb{Z}.$$

There is at least one pair (i,j) such that $\alpha_{ij} \neq 0$. Since G is free nilpotent of class 2, in order to define a suitable homomorphism ϕ, it suffices to prescribe the image of the base $\{x_i : i \in I\}$ of G in $F_2(x,y)$:

For $k \notin \{i,j\}$ send $x_k \mapsto 1$, $x_i \mapsto x$ and $x_j \mapsto y$. Use the freeness of G to extend this to a homomorphism $\varphi\colon G \to F_2(x,y)$ for which

$$\varphi(g) = [x,y]^{\alpha_{ij}} \neq 1,$$

and thus show that $g \notin \mathrm{Ker}_{F_2(x,y)}(G)$. Therefore $\mathrm{Ker}_{F_2(x,y)}(F/\gamma_3(F)) = 1$, so that $\mathrm{Ker}_{F_2(x,y)}(F) = \gamma_3(F)$ by Corollary 2.3(2).

Finally, let $c = 3$. Since T has nilpotency class 3 Remark 2.4 and the fact that $\gamma_4(F)$ is the nilpotent class 3 residual of F imply that $\gamma_4(F) \subseteq \mathrm{Ker}_T(F)$. Similarly as before, we set $G := F/\gamma_4(F)$ and show that $\mathrm{Ker}_T(G) = 1$. Given a $g \in \gamma_3(F)$ we need to find $\varphi\colon G \to T$, such that $\varphi(g) \neq 1$. Again we make use of a linear ordering of the free basis $\{x_i : i \in I\}$ of the free group F. Therefore the Hall commutator collecting process (see [5, p. 78]) shows that g modulo $\gamma_4(F)$ can be written in a unique way as a finite product of powers of basic commutators

$$[x_i, x_j, x_k]^{\alpha_{ijk}}, \quad i \geq j, \quad k > j, \quad \alpha_{ijk} \in \mathbb{Z}.$$

Suppose now that (i,j,k) is such that it appears with nontrivial exponent in this presentation of g. We consider cases:

(A) $i = j < k$; and
(B) $i = k > j$; and
(C) $i > j$ and $k > j$ while $i \neq k$.

(A) Thus g contains $[x_i, [x_i, x_k]]^{\alpha_{iik}}$ with $\alpha_{iik} \neq 0$. Send $x_m \mapsto 1$ for $m \notin \{i, k\}$, $x_i \mapsto x$, and, $x_k \mapsto y$. Then extend this map to an element $\varphi \in \hom(G, T)$ with $\varphi(g) = [x, [x, y]]^{\alpha_{iik}} \neq 1$.

(B) is handled similarly by sending $x_m \mapsto 1$ for $m \notin \{j, k\}$ while $x_i \mapsto x$ and $x_j \mapsto y$.

(C) For defining $\varphi \in \hom(G, T)$ with $\varphi(g) \neq 1$ it suffices to send the generators $(x_m)_{m \in I}$ to elements in $T = F_3(x, y)$. Sending $x_m \mapsto 1$ for all $m \notin \{i, j, k\}$ implies that only the sub-expression

$$[x_i, x_j, x_k]^{\alpha_{ijk}} [x_k, x_j, x_i]^{\alpha_{kji}}$$

of g needs to be considered. Sending $x_i \mapsto x$, $x_j \mapsto x$, and $x_k \mapsto y$ results in

$$\varphi(g) = [x, x, y]^{\alpha_{ijk}} \neq 1.$$

Altogether, making use of Corollary 2.3(2) we showed that $\mathrm{Ker}_{F_3(x,y)}(F) = \gamma_4(F)$. □

It looks plausible that one can continue in a similar fashion to prove that $\gamma_{c+1}(F) = \mathrm{Ker}_{F_c(x,y)}(F)$ also for all $c \geq 4$, but we will not need such a result in what follows.

One of the interesting features of the factor group $\widehat{F}/\mathrm{Ker}_{\mathbb{Z}}(\widehat{F})$ is that it allows "infinite commutation" as it is isomorphic to $\mathbb{Z}^{\mathbb{N}} = \prod_{n \geq 1} \langle x_i \rangle$. In fact, the normal form connected with the $\mathbb{Z}$-kernel rewrites every element $\hat{f} \in \widehat{F}$ into the form

$$\widehat{R}(\hat{f}) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

Here the "generators" x_i of $\widehat{F}$ can be viewed as commutators of weight 1 and the normal forms in F_n are achieved by the Hall commutator collecting process where the generators are ordered as

$$x_1 < x_2 < \ldots < x_n.$$

This leads to considering the Hall's commutator collecting procedure first for the free group F on countably many generators $(x_i)_{i \geq 1}$ for nilpotent groups of class $c \leq 3$. Up to class 3 the basic commutators are as follows (see e.g. in [5, Chapter 3]):

Weight 1: $(x_n)_{n \geq 1}$ and $x_i < x_j$ if, and only if, $i < j$.

Weight 2: $\{[x_i, x_j] : 1 \leq i < j\}$ and $[x_i, x_j] < [x_{i'}, x_{j'}]$ if, and only if, either $j < j'$ or $j = j'$ and $i < i'$.

Weight 3: $\{[x_i, x_j, x_k] : i \geq j, k > j\}$ and $[x_i, x_j, x_k] < [x_{i'}, x_{j'}, x_{k'}]$ if, and only if, one of the following holds:

- $\max\{i, j, k\} < \max\{i', j', k'\}$, or
- $m_1 := \max\{i, j, k\} = \max\{i', j', k'\}$ and either

- $\max(\{i,j,k\} \setminus \{m_1\}) < \max(\{i',j',k'\} \setminus \{m_1\})$ or
- $m_2 := \max(\{i,j,k\} \setminus \{m_1\}) < \max(\{i',j',k'\} \setminus \{m_1\})$ and $\min\{i,j,k\} < \min\{i',j',k'\}$.

It can be seen that our ordering is *coherent* with the inverse system (F_n, p_n).

Lemma 2.8. *For any group G,*

$$\mathrm{Ker}_{F_3(x,y)}(G) \leq \mathrm{Ker}_{F_2(x,y)}(G) \leq \mathrm{Ker}_{\mathbb{Z}}(G).$$

Proof. Both containments follow from Lemma 2.5(1). For the second observe that the subgroup of $F_2(x,y)$ generated by the commutator $[x,y]$ is isomorphic to $\mathbb{Z}$. The first containment follows similarly, because $F_2(x,y)$ is isomorphic to the subgroup of $F_3(x,y)$ generated by x and $[x,[x,y]]$. □

Lemma 2.9. *For every free group F, we have*

$$\mathrm{Ker}_{B(1,n)}(F) \leq \mathrm{Ker}_{F_3(x,y)}(F).$$

Proof. By Proposition 2.6 $\mathrm{Ker}_{B(1,n)}(F) = F''$, while by Theorem 2.7 $\mathrm{Ker}_{F_3(x,y))}(F) = \gamma_4(F)$. The lemma follows from the well-known relation $F'' \leq \gamma_4(F)$. □

2.4. *T-kernels of $\widehat{F}$*

In order to extend the above lemma to the inverse limit of free groups we need the following result.

Lemma 2.10. *If T is an $\widehat{F}$-slender group, then*

$$\mathrm{Ker}_T(\widehat{F}) = \varprojlim \mathrm{Ker}_T(F_n).$$

Proof. By applying the functor Ker_T to the inverse sequence of free groups with limit $\widehat{F}$

$$\begin{tikzcd} F_1 & F_2 \arrow[l, "p_1"'] & F_3 \arrow[l, "p_2"'] & \cdots \arrow[l] & \widehat{F} \arrow[l] \end{tikzcd}$$

we obtain the following diagram

$$\begin{tikzcd} \mathrm{Ker}_T(F_1) & \mathrm{Ker}_T(F_2) \arrow[l, "(p_1)_T"'] & \mathrm{Ker}_T(F_3) \arrow[l, "(p_2)_T"'] & \ldots \arrow[l] & \varprojlim \mathrm{Ker}_T(F_n) \arrow[l] \\ & & & \mathrm{Ker}_T(\widehat{F}) \arrow[ulll] \arrow[ull] \arrow[ul] \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

where f is uniquely determined by the universal property of the inverse limit. Both $\mathrm{Ker}_T(\widehat{F})$ and $\varprojlim \mathrm{Ker}_T(F_n)$ can be naturally viewed as subgroups of $\widehat{F}$, and then f is simply the inclusion. It remains to show that $\varprojlim \mathrm{Ker}_T(F_n) \subseteq \mathrm{Ker}_T(\widehat{F})$. Since T is $\widehat{F}$-slender, every homomorphism $\varphi\colon \widehat{F} \to T$ factors through some F_n. Thus, for a given φ we get a commutative diagram

$$\begin{tikzcd}
 & T & \\
\widehat{F} \arrow[ur, "\varphi"] \arrow[r] & F_n \arrow[u, "\varphi_n"] & \\
\varprojlim \mathrm{Ker}_T(F_n) \arrow[u, tail] \arrow[r] & \mathrm{Ker}_T(F_n) \arrow[u, tail] \arrow[uu, dashed, bend right=60, "0"'] &
\end{tikzcd}$$

which implies that $\varprojlim \mathrm{Ker}_T(F_n) \subseteq \mathrm{Ker}\varphi$. Since φ was arbitrary, we conclude that $\varprojlim \mathrm{Ker}_T(F_n)$ is contained in the T-kernel, as claimed. □

We have seen in Section 2.1 that the groups $\mathbb{Z}, F_2(x,y), F_3(x,y)$ and $B(1,n)$ are all $\widehat{F}$-slender, so Lemmata 2.8-2.10 immediately imply the following result.

Corollary 2.11.

$$\mathrm{Ker}_{B(1,n)}(\widehat{F}) \leq \mathrm{Ker}_{F_3(x,y)}(\widehat{F}) \leq \mathrm{Ker}_{F_2(x,y)}(\widehat{F}) \leq \mathrm{Ker}_{\mathbb{Z}}(\widehat{F})$$

Eda in [9] proved the following interesting analogy of Proposition 2.6: the $\mathbb{Z}$-kernel of the Hawaiian Earring group $\mathcal{G}$ coincides with the set of all elements $x \in \mathrm{Ker}_{\mathbb{Z}}(G)$, such that for every k and every generator of $F_k = F(x_1, \ldots, x_k)$ the sum of exponents of x_i in the projection $p_k(x) \in F_k$ is zero. In other words, $x \in \mathrm{Ker}_{\mathbb{Z}}(\mathcal{G})$ if $p_k(x) \in F_k'$ for every k. We are now able to extend Eda's result to T-kernels with respect to an arbitrary n-slender group T.

Lemma 2.12. *If the group T is n-slender, then the restriction map*

$$r : \mathrm{hom}(\widehat{F}, T) \to \mathrm{hom}(\mathcal{G}, T)$$

is surjective.

Proof. Let us denote by $p_k\colon \widehat{F} \to F_k$ and $q_k\colon \mathcal{G} \to F_k$ canonical projections to the terms of the inverse sequence of free groups, whose limit is $\widehat{F}$. Since T is n-slender, for every $\varphi\colon \mathcal{G} \to T$, there exists $k \geq 1$ and $\overline{\varphi}\colon F_k \to T$ such that $\varphi = \overline{\varphi} \circ q_k$. We may fit these data in a commutative diagram

$$\begin{tikzcd}
\mathcal{G} \arrow[dr, "q_k"'] \arrow[drr, bend left, "\varphi"] \arrow[dd, hook] & & \\
& F_k \arrow[r, "\overline{\varphi}"] & T \\
\widehat{F} \arrow[ur, "p_k"] \arrow[urr, bend right, "\overline{\varphi}\circ p_k"'] & &
\end{tikzcd}$$

Clearly, φ is equal to the restriction of $\overline{\varphi} \circ p_k$ to $\mathcal{G}$. $\square$

By Lemma 2.12 the inclusion $\mathcal{G} \hookrightarrow \widehat{F}$ satisfies the assumptions of Proposition 2.2(3), so we get the following equality.

Proposition 2.13. *If the group T is n-slender, then*

$$\mathrm{Ker}_T(\mathcal{G}) = \mathrm{Ker}_T(\widehat{F}) \cap \mathcal{G}.$$

2.5. Coherent normal forms

Let us again consider the inverse sequence (F_n, p_n) of free rank n groups with inverse limit $\widehat{F}$, and let us choose for each n, a normal subgroup $K_n \lhd F_n$ in a coherent way, i.e. such that $p_n(K_n) = K_{n-1}$. Thus we have inverse sequences $(K_n, p_n|_{K_n})$ and $(F_n/K_n, \bar{p}_n)$ whose limits we denote respectively as $\widehat{K}$ and $\widehat{G}$. It is not difficult to prove directly that $\widehat{K}$ can be naturally identified with a normal subgroup of $\widehat{F}$, and that $\widehat{G} \cong \widehat{F}/\widehat{K}$. For a more sophisticated proof see Geoghegan [12, Section 11.3] and observe that the derived inverse limit $\varprojlim^1 K_n$ is trivial because the bonding homomorphisms are surjective.

In order to perform explicit computations in $\widehat{G}$, we will need normal forms relative to an arbitrary normal subgroup.

Definition 2.14. Given a normal subgroup K of a free group F, we say that the map (not necessarily a homomorphism) $R : F \to F$ is a *normal form* provided $R(1) = 1$ and $R(F)$ is a complete set of coset representatives in F for K (i.e., each coset in F/K is represented exactly once).

One can easily check the following properties of normal forms:

Lemma 2.15. *Given a normal subgroup K of F, let R be a normal form on F. The following statements hold:*

(a) *$R(a_1) = R(a_2)$ if, and only if, $a_1K = a_2K$.*
(b) *$R(K) = 1$.*
(c) *$RR = R$.*

(d) *We may define an operation* $\circledcirc$ *on* $R(F)$ *by setting* $R(a_1) \circledcirc R(a_2) := R(a_1 \cdot a_2)$. *Then* $R\colon (F, \cdot) \to (R(F), \circledcirc)$ *is a homomorphism of groups with kernel* K. *Moreover,* $\varphi\colon (R(F), \circledcirc) \to (G/K, \cdot)$, *given as* $\varphi(R(a)) := aK$ *is an isomorphism.*

Definition 2.16. Let (F_n, p_n) be the inverse sequence of free rank n groups with limit $\widehat{F}$, and let $(K_n, p_n|_{K_n})$ be an inverse sequence of normal subgroups as discussed at the beginning of this subsection. A sequence $R_n\colon F_n \to F_n,\ n = 1, 2, \ldots$ of normal forms is said to be *coherent* if $p_n R_n = R_{n-1} p_n$ holds for every $n \geq 2$, i.e. if there is a commutative ladder

$$\begin{tikzcd}
F_1 \arrow[d, "R_1"'] & F_2 \arrow[l, "p_1"'] \arrow[d, "R_2"'] & F_3 \arrow[l, "p_2"'] \arrow[d, "R_3"'] & \cdots \arrow[l] \\
F_1 & F_2 \arrow[l, "p_1"'] & F_3 \arrow[l, "p_2"'] & \cdots \arrow[l]
\end{tikzcd}$$

Given a coherent sequence of normal forms $(R_n), n = 1, 2, \ldots$, we will denote their inverse limit as $\widehat{R}\colon \widehat{F} \to \widehat{F}$. Clearly, $\widehat{R}$ is a normal form on $\widehat{F}$ for the normal subgroup $\widehat{K}$, and one can easily check the following properties:

(1) $\widehat{R}(\widehat{F})$ is the set-theoretic inverse limit of sets $R_n(F_n)$. Moreover, $\widehat{F} = \widehat{R}(\widehat{F}) \cdot \widehat{K}$ and $\widehat{R}(\widehat{F}) \cap \widehat{K} = \{1\}$.

(2) Coherence of the sequence of normal forms implies that $\big((R_n(F_n), \circledcirc), p_n\big)$ is actually an inverse system of groups, so that $\widehat{R}(\widehat{F})$ inherits group operation which we denote with the same symbol. Clearly, $\big(\widehat{R}(\widehat{F}), \circledcirc\big) \cong \widehat{F}/\widehat{K}$.

Example 2.17. We let $K_n := F_n' = \gamma_2(F_n)$ and note that $R_n : F_n \to F_n$ sending $g \mapsto R_n(g) := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ if and only if $g \equiv R_n(g) \pmod{F_n'}$ defines a normal form on F_n. The inverse system (F_n, p_n) satisfies $p_n(F_{n+1}') = F_n'$ and $p_n(R_{n+1}(g)) = R_n(p_n(g))$ holds for all $g \in F_{n+1}$ and all $n \geq 1$, i.e., the sequence $(R_n)_{n\geq 1}$ is coherent. Therefore the inverse limit $\hat{R} = \varprojlim_n R_n$ is a normal form on $\hat{F}$:

$$\hat{R} : \hat{F} \to \hat{F}, \quad g \mapsto \hat{R}(g) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$$

for $g \equiv \hat{R}(g) \pmod{\hat{K}}$ where $\hat{K} = \varprojlim_n F_n'$.

Next consider $K_n := \gamma_3(F_n)$ and note that R_n sending

$$g \mapsto R_n(g) := x_1^{\alpha_1} \ldots x_n^{\alpha_n} \prod_{1\leq i<j\leq n} [x_i, x_j]^{\alpha_{ij}}$$

whenever $g \equiv R_n(g) \pmod{\gamma_3(F_n)}$ provides a normal form (see [5, page 17]). A quick inspection shows the coherence of the sequence $(R_n)_{n\geq 1}$ and therefore the inverse limit $\hat{R}$ exists. Thus, setting $\hat{K} := \varprojlim_n \gamma_3(F_n)$, it turns out that

$$\hat{R}(g) = (x_1^{\alpha_1} x_2^{\alpha_2} \cdots) \cdot ([x_1, x_2]^{\alpha_{12}} [x_1, x_3]^{\alpha_{13}} [x_2, x_3]^{\alpha_{23}} [x_1, x_4]^{\alpha_{14}} [x_2, x_4]^{\alpha_{24}} [x_3, x_4]^{\alpha_{34}} \cdots)$$

Let us remark that if $\hat{R}^1$ denotes the normal form with respect to $\varprojlim_n F_n'$, then

$$\hat{R}(g) = \hat{R}^1(g) \prod_{1 \le i < j} [x_i, x_j]^{\alpha_{ij}}.$$

Finally we consider $K_n := \gamma_4(F_n)$ and denote the previous normal form for $\gamma_3(F_n)$ by R_n^2. Every element $g \in G$ can be uniquely written in the form

$$g = R_n^2(g) u$$

for $u \in \gamma_3(F_n)$. Observing that by [5, page 17] u is in a unique way presentable as a product

$$u = \left(\prod_{n \ge i \ge j \ge 1, n \ge k > j \ge 1} [x_i, x_j, x_k]^{\alpha_{ijk}} \right) \cdot v,$$

with $v \in \gamma_4(F_4)$, we may define

$$R_n(g) := R_n^2(g) \cdot \left(\prod_{n \ge i \ge j \ge 1, n \ge k > j \ge 1} [x_i, x_j, x_k]^{\alpha_{ijk}} \right).$$

As in the previous cases the sequence $(R_n)_{n \ge 1}$ turns out to be coherent and thus every element in $\varprojlim_n \gamma_3(F_n)$ has modulo $\varprojlim_n \gamma_4(F_n)$ a presentation as a possibly infinite product of weight 3 commutators $[x_i, x_j, x_k]$ for $i \ge j$ and $k > j$.

2.6. Alternating tensors

We will reprove a well-known result about alternating 2-tensors over a field K, see e.g. [14, 8. Kriterium]. An alternating 2-tensor ω has *rank* r if it can be decomposed

$$\omega = \sum_{i=1}^{r} x_i \wedge y_i,$$

but does not allow a decomposition of this sort with fewer than r summands.

Proposition 2.18. *The following for a 2-vector* $\omega = \sum_{j=1}^{t} x_i \wedge y_i$ *are equivalent.*

(a) *The subset* $S := \{x_i : 1 \le i \le t\} \cup \{y_i : 1 \le i \le t\}$ *of* K^n *is* K*-linear independent.*
(b) ω *has rank* t*.*

Proof. Assume (b) and let t be minimal such that (a) fails to hold. Without losing generality we can assume

$$y_t = \sum_{i=1}^{t} \lambda_i x_i + \sum_{i=1}^{t-1} \mu_i y_i$$

for coefficients λ_i and μ_i in K. Inserting this term into ω results in

$$\begin{aligned} \omega &= \sum_{i=1}^{t} x_i \wedge y_i \\ &= \sum_{i=1}^{t-1} \omega_i, \quad \text{where } \omega_i := x_i \wedge y_i + \lambda_i x_t \wedge x_i + \mu_i x_t \wedge y_i. \end{aligned}$$

Rearranging terms in ω_i displays them to be of the form

$$\omega_i = \begin{cases} (x_t - \frac{1}{\lambda_i} y_i) \wedge (\lambda_i x_i + \mu_i y_i), & \text{for } \lambda_i \neq 0, \\ (x_i + \mu_i x_t) \wedge y_i. & \text{for } \lambda_i = 0. \end{cases}$$

Altogether we have found that ω has rank less than t, contrary to the assumptions. Thus (a) holds.

Conversely, assume (a), i.e., for linearly independent vectors x_i, y_i

$$\omega = \sum_{i=1}^{t} x_i \wedge y_i = \sum_{j=1}^{s} u_j \wedge v_j$$

for vectors $u_j, v_j \in \operatorname{span}\{x_i, y_i : 1 \leq i \leq t\}$ and $s < t$ and also s as small as possible. Note that $t > 1$ must hold and therefore, for $1 \leq j \leq s$, there are $u'_j, v'_j \in \operatorname{span}\{x_i, y_i : 1 \leq i \leq t-1\}$ and coefficients $\alpha_j, \beta_j, \gamma_j, \delta_j \in K$ such that

$$u_j = u'_j + \alpha_j x_t + \beta_j y_t, \quad \text{and} \quad v_j = v'_j + \gamma_j x_t + \delta_j y_t. \tag{*}$$

First let us show that we may assume $\delta_j = 0$ for all $j = 1, \ldots, s$: Indeed, if $\gamma_j = 0$ then replace u_j by v_j and v_j by $-u_j$. Otherwise replace v_j by $v_j - \frac{\gamma_j}{\delta_t} u'_j$ and note that ω is not altered. Thus from now on we can assume $\delta_j = 0$ in Eq. (*) for $1 \leq j \leq s$. Taking the bilinearity of the alternating tensor ω into account one finds:

$$\begin{aligned} \omega &= \sum_{j=1}^{s} (u'_j + \alpha_j x_t + \beta_j y_t) \wedge (v'_j + \gamma_j x_t) \\ &= \sum_{j=1}^{s} u'_j \wedge v'_j + \underbrace{x_t \wedge (\sum_{j=1}^{s} \alpha_j v'_j - \gamma_j u'_j)}_{T_1} - \underbrace{(\sum_{j=1}^{s} \beta_j v'_j) \wedge y_t}_{T_2} - \underbrace{(\sum_{j=1}^{s} \beta_j \gamma_j) x_t \wedge y_t}_{T_3} . \end{aligned} \tag{†}$$

Equating the presentation of ω in Eq. (†) with $\omega = \sum_{j=1}^{s-1} x_i \wedge y_i + x_t \wedge y_t$ one observes that the coefficient of $x_t \wedge y_t$ in Term T_3 of Eq. (†) must be equal to -1 and therefore there exists j_0 such that $\beta_{j_0} \neq 0$. The term T_2 in Eq. (†) must vanish so that $\sum_{j=1}^{s} \beta_j v'_j \in \operatorname{span}\{y_t\}$ follows. Since $v'_j \in \operatorname{span}\{x_i, y_i : i < t\}$ we may conclude $\sum_{j=1}^{s} \beta_j v'_j = 0$ and therefore, setting $\beta'_j := \frac{\beta_j}{\beta_{j_0}}$ we deduce

$$v'_{j_0} = -\sum_{j \neq j_0} \beta'_j v'_j.$$

Introducing this expression into the right hand side of Eq. (†) and noting that the term T_1 must be zero and $T_3 = x_t \wedge y_t$, one finds

$$\sum_{i=1}^{t-1} x_i \wedge y_i = \sum_{j \neq j_0} (u'_j - \beta'_j u'_{j_0}) \wedge v'_j.$$

On the right hand side we have a sum of $s-1$ wedge products and by the minimality assumption on s and t we conclude $s-1 = t-1$, i.e. $s = t$, a final contradiction. Thus (b) holds. □

2.7. Identities in nilpotent class 3 groups

In this subsection we derive several formulas which are valid in nilpotent groups of nilpotency class 3 and that will be of great help in later computations. We begin with some well-known identities (compare e.g. with [5, Section 1.2]). Recall that for $a, b, c \in G$ we abbreviate the commutator $[a, [b, c]]$ as $[a, b, c]$. We also remind the reader that the commutator subgroup G' of any nilpotent class 3 group is *abelian*.

Lemma 2.19. *Let G be a nilpotency class 3 group. For $a, b, c \in G$ and $k \in \mathbb{N}$ the following identities hold:*

(a) $a^b = bab^{-1} = [b, a]a$.
(b) $ab = [a, b]ba$.
(c) $[ab, c] = [a, c][b, c][a, b, c]$.
(d) $[a, bc] = [a, b][a, c][b, a, c]$.
(e) $[a^{-1}, b] = [a, b]^{-1}[a, a, b]$ *and* $[a, b, c]^{-1} = [a, c, b]$.
(f) $[a^k, b] = [a, b]^k [a, a, b]^{\binom{k}{2}}$.
(g) $[a, b^k] = [a^k, b][a, b, a]^{\binom{k}{2}}[b, a, b]^{\binom{k}{2}}$.
(h) *If, in addition, $c \in G'$ then*

$$[ab, c] = [a, c][b, c], \quad [a, bc] = [a, b][a, c].$$

Proof. (a), (b) and (c) follow immediately from the definition of the commutator. (d) and (e) are consequences of (c). For $k \in \mathbb{N}$, (f) follows by induction from (c) and for negative k one needs to invoke (e) as well.

(g) follows from (f) by the computation

$$\begin{aligned}[a, b^k] &= [b^k, a]^{-1} \\ &= ([b, a]^k [b, b, a]^{\binom{k}{2}})^{-1} \\ &= [a, b]^k [b, a, b]^{\binom{k}{2}} \\ &= [a^k, b][a, b, a]^{\binom{k}{2}} [b, a, b]^{\binom{k}{2}}.\end{aligned}$$

Finally (h) follows from the statements in (c) and (d) thereby taking $c \in G'$ into account. □

Lemma 2.20. *Let $x_i, y_i,\ i = 1, 2, \dots$ be any sequence of elements in a nilpotency class 3 group G. Define $u_i := x_i y_i$, $c_i := [x_i, y_i]$ and for every $k \geq 1$ set*

$$U_k := u_1 \cdots u_k, \quad X_k := x_1 \cdots x_k, \quad Y_k := y_1 \cdots y_k, \ \textit{and}\ Q_k := U_k X_k^{-1} Y_k^{-1}.$$

Then the following relations hold:

$$Q_1 = [x_1, y_1], \ \textit{and}\ Q_k = \prod_{j=1}^{k-1} [Y_j, X_{j+1}, y_{j+1}] \prod_{j=1}^{k} [X_j, y_j], \ \textit{for}\ k \geq 2.$$

Proof. By definition $U_1 = x_1 y_1$, $X_1 = x_1$, $Y_1 = y_1$, therefore $Q_1 = U_1 X_1^{-1} Y_1^{-1} = x_1 y_1 x_1^{-1} y_1^{-1} = [x_1, y_1]$.

The following formula will be used as the inductive step:

$$Q_{k+1} = Q_k [Y_k, X_{k+1}, y_{k+1}][X_{k+1}, y_{k+1}].$$

Toward the proof of this formula we use the identities from Lemma 2.19 and the obvious relations

$$X_{k+1} = X_k x_{k+1}, \ Y_{k+1} = Y_k y_{k+1}, \ U_{k+1} = U_k x_{k+1} y_{k+1}.$$

By taking into account that γ_3 is central we have the following computation:

$$\begin{aligned}Q_{k+1} &= U_{k+1} X_{k+1}^{-1} Y_{k+1}^{-1} \\ &= Q_k Y_k X_k \, x_{k+1} y_{k+1} \, x_{k+1}^{-1} X_k^{-1} \, y_{k+1}^{-1} Y_k^{-1} \\ &= Q_k Y_k X_k c_{k+1} X_k^{-1} [X_k, y_{k+1}] Y_k^{-1} \\ &= Q_k Y_k [X_k, c_{k+1}] c_{k+1} [X_k, y_{k+1}] Y_k^{-1} \\ &= Q_k [X_k, c_{k+1}] Y_k \, \underline{c_{k+1} [X_k, y_{k+1}]} \, Y_k^{-1}\end{aligned}$$

The underlined term can be transformed by applying Lemma 2.19:

$$\begin{aligned} c_{k+1}[X_k, y_{k+1}] &= [X_k, y_{k+1}]c_{k+1} \\ &= [X_k, y_{k+1}][x_{k+1}, y_{k+1}] \\ &= [X_k x_{k+1}, y_{k+1}][X_k, x_{k+1}, y_{k+1}]^{-1} \\ &= [X_{k+1}, y_{k+1}][X_k, c_{k+1}]^{-1} \end{aligned}$$

and so we obtain

$$\begin{aligned} Q_{k+1} &= Q_k[X_k, c_{k+1}]Y_k\, \underline{[X_{k+1}, y_{k+1}][X_k, c_{k+1}]^{-1}}\, Y_k^{-1} \\ &= Q_k Y_k [X_{k+1}, y_{k+1}] Y_k^{-1} \\ &= Q_k[Y_k, [X_{k+1}, y_{k+1}]][X_{k+1}, y_{k+1}], \end{aligned}$$

as has been claimed. The stated result now follows by straightforward induction. □

Lemma 2.21. *Let $z_i, i = 1, 2, \ldots$ be a sequence of elements in a nilpotency class 3 group G and let $Z_k := z_1 \cdots z_k$ for every $k \geq 1$. Then for every $t \in G$ and $k \geq 1$ we have*

$$\prod_{j=1}^{k}[z_j, t] = [Z_k, t]\prod_{j=1}^{k-1}[Z_j, t, z_{j+1}].$$

Proof. For $k = 1$ the lemma states $[Z_1, t] = [z_1, t]$ which is clearly true. Making use of Lemma 2.19(c) one obtains first $[Z_{k+1}, t] = [Z_k z_{k+1}, t] = [Z_k, t][z_{k+1}, t][Z_k, z_{k+1}, t]$, i.e.,

$$[Z_k, t][z_{k+1}, t] = [Z_{k+1}, t][Z_k, t, z_{k+1}].$$

Using the latter equality, induction and taking Lemma 2.19(h) into account, yields

$$\begin{aligned} \prod_{j=1}^{k+1}[z_j, t] = \prod_{j=1}^{k}[z_j, t][z_{k+1}, t] &= [Z_k, t]\prod_{j=1}^{k-1}[Z_j, t, z_{j+1}][z_{k+1}, t] \\ &= [Z_{k+1}, t]\prod_{j=1}^{k}[Z_j, t, z_{j+1}]. \quad \square \end{aligned}$$

3. Main results

In this section we will answer Question 1.1 for different choices of the group T and prove our main result, Theorem 1.2. Its statements when respectively $\widehat{F} = \mathcal{G} \cdot \mathrm{Ker}_T(\widehat{F})$ and $\mathcal{G} \cdot \mathrm{Ker}_T(\widehat{F})$ is a *proper* subgroup of $\widehat{F}$ will be dealt separately in Theorems 3.2 and 3.3. The proof of Theorem 1.2 will be given at the end of the section.

Recall from the Introduction that $\widehat{F} = \varprojlim F_n$ where (F_n, p_n) is an inverse system of free groups $F_n = F(x_1, \dots, x_n)$ and $p_n : F_{n+1} \to F_n$ is the unique epimorphism sending $x_i \mapsto x_i$ for $1 \leq i \leq n$ and $x_{n+1} \mapsto 1$. We begin with $T = F_2(x, y)$ (which will also settle the case $T = \mathbb{Z}$).

Theorem 3.1. *There is a normal form $\widehat{R}$ for $\widehat{F}/\mathrm{Ker}_T(\widehat{F})$ which assigns to every $x \in \widehat{F}$ a unique expression*

$$\widehat{R}(x) = \left(x_1^{\alpha_1} x_2^{\alpha_2} \cdots\right) \cdot \left([x_1, x_2]^{\alpha_{12}} [x_1, x_3]^{\alpha_{13}} [x_2, x_3]^{\alpha_{23}} [x_1, x_4]^{\alpha_{14}} [x_2, x_4]^{\alpha_{24}} [x_3, x_4]^{\alpha_{34}} \cdots\right)$$

Moreover, replacing the expression in the second set of parenthesis of the righthand side of the equation by

$$W(x) := \prod_{1 \leq i} [x_i, \prod_{i<j} x_j^{\alpha_{ij}}]$$

yields another coset representative modulo $\mathrm{Ker}_T(\widehat{F})$ of $\widehat{R}(x)$. Here the product is meant to be the unique element mapping under canonical projection from $\widehat{F}$ onto F_n onto the partial products

$$\prod_{1 \leq i < n} [x_i, \prod_{i < j \leq n} x_j^{\alpha_{ij}}].$$

In particular, $W(x)$ belongs to $\mathcal{G}$.

Proof. The discussion in Example 2.17 shows that the coherent ordering of the basic commutators from the previous section yields a well-defined $\widehat{R}$. Lemma 2.10 shows that $\mathrm{Ker}_T(\widehat{F}) = \varprojlim \gamma_3(F_n)$. Therefore it is a consequence of the nilpotent class 2 commutation rules in $F_n/\gamma_3(F_n)$ that we can give $\widehat{R}(x)$ the second form up to a factor from $\mathrm{Ker}_T(\widehat{F})$. Certainly in $W(x)$ for fixed i the element x_i appears only finitely often. Thus $W(x)$ is in $\mathcal{G}$ (see [8, Section 2] for a combinatorial description of $\mathcal{G} \subset \widehat{F}$) and is a *legal word* in $\mathcal{G}$ in the sense of [1]. □

Theorem 3.2. *When T is either $\mathbb{Z}$ or $F_2(x, y)$ then the restriction to $\mathcal{G}$ of the canonical epimorphism $\varphi_T : \widehat{F} \to \widehat{F}/\mathrm{Ker}_T(\widehat{F})$ is itself an epimorphism. In other words,*

$$\widehat{F} = \mathcal{G} \cdot \mathrm{Ker}_{F_2(x,y)}(\widehat{F}) = \mathcal{G} \cdot \mathrm{Ker}_{\mathbb{Z}}(\widehat{F}).$$

Proof. Let $T := F_2(x, y)$. By Theorem 3.1 there is a normal form $\widehat{R} : \widehat{F}/\mathrm{Ker}_{F_2(x,y)}(\widehat{F}) \to \mathcal{G}$. It follows that

$$\varphi_{F_2(x,y)}(\mathcal{G}) \leq \varphi_{F_2(x,y)}(\widehat{R}(\widehat{F})) = \widehat{F}/\mathrm{Ker}_{F_2(x,y)}(\widehat{F}).$$

For $T = \mathbb{Z}$ the result follows from Corollary 2.11. □

The description of the normal form in Theorem 3.1 suggests that one could hope for a similar representation of the group $\widehat{F}/\mathrm{Ker}_{F_3(x,y)}(\widehat{F})$. Unfortunately, this is not the case and we will dedicate the rest of the section to the proof of the following theorem.

Theorem 3.3. *The epimorphism $\varphi : \widehat{F} \to \widehat{F}/\mathrm{Ker}_{F_3(x,y)}(\widehat{F})$ does not restrict to an epimorphism from the embedded $\mathcal{G}$ to $\widehat{F}/\mathrm{Ker}_{F_3(x,y)}(\widehat{F})$.*

We will actually prove a slightly more general result, Proposition 3.5 below. In order to formulate the statement we adopt the following notation.

Notation 3.4. We rename the generators of the free groups $F_n = F(x_1, \dots, x_n)$ as follows: Let t denote x_1, and, for $k \geq 1$ set $a_k := x_{2k}$ and $b_k := x_{2k+1}$. Then, setting

$$A_k := F(a_1, b_1, \dots, a_k, b_k), \quad B_k := A_k^{\mathrm{ab}},$$

one notes

$$\Phi_k := F_{2k+1} = \langle t \rangle * A_k$$

and one has the inverse system (Φ_k, q_k) for $q_k = p_{2k}p_{2k+1}$ with inverse limit $\widehat{F}$. On the other hand, there is also an induced inverse system (A_k, r_k) with r_k the restriction of q_k to A_k and we let $\widehat{A}$ denote the inverse limit. By the left exactness of the $\varprojlim$-functor we may identify $\widehat{A}$ with a subgroup of $\widehat{F}$ and hence there is a natural embedding of $\langle t \rangle * \widehat{A}$ in $\widehat{F}$. Our characterization of $\mathrm{Ker}_{F_3(x,y)}(\Phi_n)$ in Theorem 2.7 as the nilpotent class 3 residual $\gamma_4(\Phi_n)$ implies that

$$\mathrm{Ker}_{F_3(x,y)}(\widehat{F}) = \varprojlim \mathrm{Ker}_{F_3(x,y)}(\Phi_n) = \varprojlim \gamma_4(\Phi_n).$$

The discussion in Example 2.17 shows that $\widehat{R} = \varprojlim R_n$ is the inverse limit of a coherent sequence of normal forms (R_n) associated to the quotient homomorphisms $\Phi_n \to \Phi_n/\gamma_4(\Phi_n)$.

Finally, let

$$G_n := \Phi_n/\gamma_4(\Phi_n), \quad \text{and} \quad L_n := (\langle t \rangle * B_n)/\gamma_4(\langle t \rangle * B_n)$$

and note that

$$\widehat{G} := \varprojlim G_n \cong \widehat{F}/\mathrm{Ker}_{F_3(x,y)}(\widehat{F}).$$

We also note that L_n can be understood to be the free product $\langle t \rangle *^{\mathfrak{N}_3} B_n$ in the variety $\mathfrak{N}_3$ of nilpotent class 3 groups. We will use this notation throughout the remainder of this section and state a more general form of Theorem 3.3:

Proposition 3.5. *The epimorphism $\varphi : \widehat{F} \to \widehat{G} = \widehat{F}/\mathrm{Ker}_{F_3(x,y)}(\widehat{F})$ does not restrict to an epimorphism on $\langle t\rangle * \widehat{A} \leq \widehat{F}$.*

The main idea for the proof is that every element in $\langle t\rangle * \widehat{A}$, can be written, up to a conjugation in $\langle t\rangle * \widehat{A}$, as a word

$$P_{k+1} := x_1 y_1 \cdots x_{k+1} y_{k+1} \tag{*}$$

with $x_i \in \widehat{A}$ and $y_i = t^{\alpha_i}$ for $\alpha_i \in \mathbb{Z}$. In Subsection 3.1 we will be able to estimate the commutator length of P_{k+1} from above. On the other hand we will show that the formal expression

$$\hat{g} := \prod_{i=1}^{\infty} [a_i, t, b_i]$$

is a "legal infinite word" in $\widehat{G}$, and we will be able to estimate the commutator length of $\hat{g}$ from below (see Subsection 3.2). Since $\widehat{G}$ is of nilpotency class 3, all iterated commutators belong to the center of $\widehat{G}$ and so $\hat{g}$ would be of the form $\hat{g} = \varphi(P_{k+1})$ for P_{k+1} as above. We will use the above estimates to produce an element that is not in the image of φ.

The proofs need some technical preparation that will be split in several subsections.

3.1. The commutator length of elements of $\gamma_3(L_n)$

Recall the various groups defined in Notation 3.4, in particular $G_n = (\langle t\rangle * A_n)/\gamma_4(\langle t\rangle * A_n)$ and $L_n = (\langle t\rangle * B_n)/\gamma_4(\langle t\rangle * B_n)$, where $B_n = A_n^{\mathrm{ab}}$. Then L_n is a proper factor group of G_n.

Observe, that B_n is naturally a subgroup of L_n and, abusing notation, we shall denote by a_i and b_i for $1 \leq i \leq n$ the images in L_n of the respective free generators of $A_n = F(a_1, b_1, \ldots, a_n, b_n)$.

The factor group $M_n := L_n/\gamma_3(L_n)$ is nilpotent of class 2 and agrees with the free nilpotent product $\langle t\rangle *^{\mathfrak{N}_2} B_n$. One can provide an explicit description of M_n as follows:

- Consider the abelian groups $\langle t\rangle \cong \mathbb{Z}$ and $B_n \cong \mathbb{Z}^{2n}$ as $\mathbb{Z}$-modules and form their skew symmetric tensor product $\langle t\rangle \wedge B_n$.
- Next form the direct sum $(\langle t\rangle \wedge B_n) \oplus B_n$, an abelian group.
- Define an action of $\langle t\rangle$ by letting $\langle t\rangle$ act trivially on $\langle t\rangle \wedge B_n$ and, for any $b \in B_n$ set $b^{t^\alpha} := \alpha t \wedge b + b$.
- Then $M_n \cong (\langle t\rangle \wedge B_n \oplus B_n) \rtimes \langle t\rangle$.
- Setting $g_{2k-1} := a_k$ and $g_{2k} := b_k$ for $1 \leq k \leq n$ the set $\{t \wedge g_k : 1 \leq k \leq n\}$ turns out to be a basis of the free abelian group $M_n' = \langle t\rangle \wedge B_n$. In group theoretic terms this is the set of commutators $[t, g_k]$ for $1 \leq k \leq 2n$.

Lemma 3.6. *If for some $Z \in B_n$ the commutator $[Z, t]$ belongs to $\gamma_3(L_n)$, then $Z = 1$.*

Proof. It suffices to consider $M_n = L_n/\gamma_3(L_n)$ and to conclude from $[Z,t] = 1$ that $Z = 1$. By definition the set $\{g_1, \dots, g_{2n}\}$ freely generates B_n. Hence there are uniquely determined integers α_i with

$$Z = \prod_{i=1}^{2n} g_i^{\alpha_i}.$$

Then

$$1 = [Z,t] = \prod_{k=1}^{n} [g_i, t]^{\alpha_k}$$

and, as observed above, M_n' is freely generated by the set $\{[g_k, t] : 1 \leq k \leq 2n\}$. Hence $\alpha_k = 0$ must hold for all $1 \leq k \leq 2n$ and hence $Z = 1$. □

Lemma 3.7. *Assume that some element P_{k+1} in $\gamma_3(L_n)$ can be written as*

$$P_{k+1} = x_1 y_1 \cdots x_{k+1} y_{k+1} \in \gamma_3(G_n),$$

where $x_i \in B_n$ and $y_i = t^{\alpha_i} \in \langle t \rangle$. For $j \geq 1$ set $u_j := x_j y_j$, $U_j := u_1 \dots u_j$, $X_j := x_1 \dots x_j$, $Y_j := y_1 \dots y_j$, $z_j := X_j^{\alpha_j}$, and, $Z_j := z_1 \dots z_j$.

Then $P_{k+1} = U_k X_k^{-1} Y_k^{-1}$ and, moreover,

$$P_{k+1} = \prod_{j=1}^{k-1} [Y_j, X_{j+1}, t^{\alpha_{j+1}}] \prod_{j=1}^{k} [X_j^{\binom{\alpha_j}{2}}, t, X_j] \prod_{j=1}^{k} [t^{\binom{\alpha_j}{2}}, X_j, t] \prod_{j=1}^{k-1} [Z_j, t, X_{j+1}^{\alpha_{j+1}}]. \qquad (\dagger)$$

Proof. Since $L_n = \langle t, B_n \rangle$ is nilpotent of class 3, the condition $P_{k+1} \in \gamma_3(L_n)$ implies $P_{k+1} \in \gamma_2(L_n) = [t, B_n]$. Modulo $\gamma_2(L_n)$, one has $P_{k+1} \equiv X_{k+1} Y_{k+1}$ which allows us to conclude that $X_{k+1} = Y_{k+1} = 1$. Therefore

$$x_{k+1} = X_k^{-1} \quad \text{and} \quad y_{k+1} = Y_k^{-1}.$$

Plugging these relations into the definition of P_{k+1} one finds, as claimed, $P_{k+1} = U_k X_k^{-1} Y_k^{-1}$.

Since $L_n = \langle t, B_n \rangle$ is nilpotent of class 3, using the notation from Lemma 2.20, we can see that $P_{k+1} = U_k X_k^{-1} Y_k^{-1} = Q_k$. Then Lemma 2.20 allows us to arrive at the formula

$$P_{k+1} = \prod_{j=1}^{k-1} [Y_j, X_{j+1}, y_{j+1}] \prod_{j=1}^{k} [X_j, y_j].$$

Recalling $y_j = t^{\alpha_j}$ and $z_j = X_j^{\alpha_j}$ one observes that the first product on the right hand side of this formula agrees with the first product in (†) and, making use of Lemma 2.19(g), the second one can be rewritten as

$$\prod_{j=1}^{k}[X_j, t^{\alpha_j}] = \prod_{j=1}^{k}[X_j^{\alpha_j}, t] \prod_{j=1}^{k}[X_j, t, X_j]^{\binom{\alpha_j}{2}} \prod_{j=1}^{k}[t, X_j, t]^{\binom{\alpha_j}{2}},$$

where the second and third term in (†) become visible. It remains to show that first term in the latter formula

$$W_k := \prod_{j=1}^{k}[z_j, t] = \prod_{j=1}^{k}[X_j^{\alpha_j}, t]$$

matches the last term in (†).

The condition $P_{k+1} \in \gamma_3(L_n)$ is equivalent to $W_k \in \gamma_3(L_n)$ and Lemma 2.21 shows that

$$W_k = [Z_k, t] \prod_{j=1}^{k-1}[Z_j, t, z_{j+1}],$$

and it follows immediately that the product agrees with the last term in (†). Finally, since $W_k \in \gamma_3(L_n)$, we have that $[Z_k, t] \in \gamma_3(L_n)$, so that $Z_k = 1$ by Lemma 3.6. □

We can now estimate from above the commutator length of P_{k+1} (i.e. the minimal number of factors needed to express P_{k+1} as a product of iterated commutators):

Corollary 3.8. *$P_{k+1} = x_1 t^{\alpha_1} \cdots x_{k+1} t^{\alpha_{k+1}}$ can be represented as the product of at most $4k - 1$ elements in $\gamma_3(L_n)$.*

Proof. While the first factor in the presentation of P_{k+1} of Lemma 3.7 contributes at most $k - 1$ factors in $\gamma_3(L_n)$, each of the remaining three factors are made up by at most k sub-factors. Thus there are at most $4k - 1$ factors in $\gamma_3(L_n)$ whose product is P_{k+1}. □

3.2. The commutator length of $[a_1, t, b_1] \cdots [a_n, t, b_n]$

The present subsection will use the group $L_n = \langle t\rangle *^{\mathfrak{N}_3} A_n^{\mathrm{ab}}$ and its subgroup $B_n = A_n^{\mathrm{ab}}$, defined in Subsection 3.1, to prove Lemma 3.15 which essentially states that in a free nilpotent group of nilpotency class 3 any element of the form $c = [a_1, t, b_1] \cdots [a_n, t, b_n]$ cannot be written as the product of less than n commutators of weight 3. For the proof of this fact it will suffice to consider a suitable factor group of L_n.

Notation 3.9. Denote by $\mathbb{F}_2$ the field with two elements. Let X and Y be finite-dimensional $\mathbb{F}_2$-vector spaces of the same dimension and fix an isomorphism $\psi \in \hom(X, Y)$. Later we shall fix a basis B_X of X. Let $\Omega := Y \wedge Y$ denote the alternating tensor product (which agrees, because of characteristic 2 with the symmetric tensor product) and let $M := \Omega \oplus Y$ be the direct sum of abelian groups. For every $x \in X$ define a map $\iota_x \in \hom(M, M)$

$$\iota_x(\omega + y) := \omega + \psi(x) \wedge y + y. \tag{1}$$

Lemma 3.10. *The map $x \mapsto \iota_x$ defines an isomorphism from X into the automorphism group of M. In the semidirect product*

$$N := M \rtimes X$$

conjugation of elements in M with $x \in X$ is expressed as $(\omega + y)^x = x \otimes y + y$. Letting $n := (\omega + y, x)$ and $n' := (\omega' + y', x')$, one has

$$[n, n'] = \psi(x') \wedge y + \psi(x) \wedge y' \in \Omega.$$

Moreover, N is nilpotent of class 2 and $[N, N] = \Omega = Y \wedge Y$.

Proof. The linearity of $y \mapsto \psi(x) \wedge y$ implies that ι_x is a homomorphism for every fixed element $x \in X$. The linearity of $x \mapsto \psi(x) \wedge y$ implies that $x \mapsto \iota_x$ is a homomorphism. Pick $m = \omega + y \in \ker(\iota_x)$. Then

$$0 = \iota_x(m) = \omega + \psi(x) \wedge y + y$$

implies $y = 0$ and therefore also $\omega = 0$. Hence $x \mapsto \iota_x$ maps from X into the automorphism group of M and thus the semidirect product $N = M \rtimes X$ is well-defined. Conjugation of elements in M by ones in X reads

$$(\omega + y)^x = \iota_x(\omega + y) = \psi(x) \wedge y + y$$

and one obtains the commutator

$$[x, \omega + y] = (\omega + y)^x + (\omega + y) = \psi(x) \wedge y.$$

For computing the commutator of elements in N we find, partially using multiplicative notation, and letting $n := (m, x)$, $n' := (m', x')$ where $m := \omega + y$ and $m' := \omega' + y'$,

$$[n, n'] = [mx, m'x'] = [m, x'] + [x, m'] = [x', m] + [x, m'] = \psi(x') \wedge y + \psi(x) \wedge y'.$$

From this formula one deduces $[N, N] \leq \Omega$ and since $\Omega \leq Z(N)$ it follows that N is nilpotent of class 2 and the last statement of the lemma is an immediate consequence of this just established fact. □

The group N, by construction, is the free product $X *^{\mathfrak{N}_2} Y$ in the variety of class 2 nilpotent groups of $\mathbb{F}_2$-vector spaces of the same dimension. In a moment we shall define a group Δ to be a semidirect product of N with a group $\langle t\rangle$ of order 2 with the help of a certain cocycle $\sigma : X \to M$ constructed with the aid of $\psi : X \to Y$ from Notation 3.9.

Lemma 3.11. *There is a unique cocycle $\sigma \in Z^1(N, M)$ that agrees with ψ when restricted to the base B_X and contains M in its kernel.*

Proof. Certainly $\sigma(0) := 0$. Order the finite base B_X linearly and set $X_x := \langle x' \in B_X : x' < x\rangle$. Suppose that σ is well-defined on X_x and accordingly satisfies for all $w, w' \in X_x$ the *cocycle condition*

$$\sigma(w + w') = \sigma(w) + \sigma(w') + \psi(w) \wedge \psi(w').$$

For extending σ one step further to all elements in $\langle X_x, x\rangle$ define

$$\sigma(w + x) := \sigma(w) + \psi(x) + \psi(w) \wedge \psi(x).$$

For proving the cocycle condition pick $v := w + \epsilon x$, $v' := w' + \epsilon' x$ where $\epsilon, \epsilon' \in \{0, 1\}$. Then, observing that the above formulas imply that for all $u \in \langle X_x, x\rangle$ one has $\sigma(u) \equiv \psi(u)$ modulo Ω and that therefore for all $y \in Y$ one has $\sigma(u) \wedge y = \psi(u) \wedge y$, we obtain

$$\begin{aligned}\sigma(v + v') &= \sigma((w + w') + (\epsilon + \epsilon')x)\\ &= \sigma(w + w') + (\epsilon + \epsilon')\psi(x) + \psi(w + w') \wedge \psi((\epsilon + \epsilon')x)\\ &= \sigma(w + \epsilon x) + \sigma(w' + \epsilon' x) + \psi(w + \epsilon x) \wedge \psi(w' + \epsilon' x)\\ &= \sigma(v) + \sigma(v') + \psi(v) \wedge \psi(v'),\end{aligned}$$

i.e., the cocycle condition holds for v and v'. Thus, σ is well-defined on X. Now, for any $m \in M$, it is natural to set

$$\sigma((m, x)) := \sigma(x)$$

and elementary computation yields the cocycle condition of all of $n, n' \in N$:

$$\begin{aligned}\sigma(n, n') = \sigma((m, x)(m', x')) &= \sigma((m, x)) + \sigma((m', x')) + \psi(x) \wedge \psi(x')\\ &= \sigma(n) + \sigma(n') + \sigma(n) \wedge \sigma(n'). \quad \square\end{aligned}$$

Notation 3.12. Given $N = M \rtimes X$, a base B_X of X and a cocycle σ as in Lemma 3.11, we first define an action of the two element group $\mathbb{Z}(2)$ generated by an element t upon N by

$$n^t := \sigma(n)n.$$

The cocycle condition ensures that this action is well-defined. Therefore we may form the semidirect product

$$\Delta := N \rtimes \langle t\rangle = ((M \rtimes X) \rtimes \langle t\rangle) = ((\Omega \oplus Y) \rtimes X) \rtimes \langle t\rangle.$$

Lemma 3.13. *The group Δ is nilpotent of class 3. Its commutator subgroup $[\Delta, \Delta]$ agrees with $M = \Omega \oplus Y$ and $\gamma_3(\Delta) = [N, N] = \Omega$. Moreover, every triple commutator $[u, u', u''] = [u, [u', u'']]$ for elements $u, u', u'' \in \Delta$ can be written in the form $[x, [t, x']]$ for suitable elements $x, x' \in X$.*

Proof. When x runs through the basis B_X of X then the commutators $[t, x] = \sigma(x) = \psi(x)$ run through a set of generators of Y since $\psi : X \to Y$ is an isomorphism (see Notation 3.12). Hence $Y \leq [\Delta, \Delta]$. Moreover, by Lemma 3.10 $[N, N] = \Omega = Y \wedge Y$ and hence $M = \Omega + Y \leq [\Delta, \Delta]$. Since $M/N \cong X$ is abelian $[\Delta, \Delta] = M$ follows. One deduces from this and the trivial action of t upon M that

$$\gamma_3(\Delta) = [M, \Delta] = [M, N\langle t\rangle] = [M, N] = \Omega.$$

Since $\Omega \leq Z(\Delta)$ by our constructions, it follows that Δ is nilpotent of class 3.

Let us compute a triple commutator $[u, u', u''] = [u, [u', u'']]$. Note the "linearity" of $[u, u', u'']$ in each of its three arguments and that its value in Ω is fully determined when its arguments are modified by elements in $\gamma_2(\Delta) = [\Delta, \Delta] = M$. Therefore we may assume $u = xs$, $u' = x's'$, and, $u'' = x''s''$ for $x, x', x'' \in X$ and $s, s', s'' \in \langle t\rangle$.

$$\begin{aligned}[u, u', u''] &= [xs, [u', u'']] = [x, [u', u'']][s, \underbrace{[u', u'']}_{\in M}] = [x, x's', x''s''] \\ &= \underbrace{[x, x', x'']}_{=1}[x, x', s''][x, s', x'']\underbrace{[x, s', s'']}_{=1} = [x, s'', x'][x, s', x''].\end{aligned}$$

If either $s'' = 1$ or $s' = 1$ then we have found a presentation of $[u, u', u'']$. Hence we may assume $s' = s'' = t$. Then

$$[u, u', u''] = [x, t, x'][x, t, x''] = [x, t, x'x''] = [x, [t, x'x'']]$$

is indeed of the desired form. □

Corollary 3.14. *Let $c \in \gamma_3(\Delta)$. Then c can be written as the product $c = \prod_{i=1}^{k}[u_i, v_i, w_i]$ if, and only if, there are elements $k_i, l_i \in X$ with $c = \prod_{i=1}^{k}[k_i, t, l_i]$ if, and only if, in $Y \wedge Y$ one has $c = \sum_{i=1}^{k} \psi(k_i) \wedge \psi(l_i)$.*

Let $n \geq 1$ and suppose the base of the $\mathbb{F}_2$ vector space X is given as $B_X = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. Denote the group of applying the construction in Notation 3.12

by Δ_n. As noted in Lemma 3.13 the group Δ_n is nilpotent of class 3. Recall from Notation 3.4 the group $L_n = \langle t\rangle * A_n^{\mathrm{ab}}$ for A_n freely generated by the set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ and observe that there is a unique canonical epimorphism from $L_n \to \Delta_n$ sending $t \mapsto t$, and for $1 \leq i \leq n$, every $a_i \mapsto a_i$ and $b_i \mapsto b_i$ (here we abuse notation by letting letters denote both, generators in L_n and their epimorphic images in Δ_n).

Lemma 3.15. *Let* $L_n = \langle t\rangle * A_n^{\mathrm{ab}}/\gamma_4(\langle t\rangle * A_n^{\mathrm{ab}}$ *be as in Notation 3.4 generated by*

$$t, a_1, b_1, \ldots, a_n, b_n.$$

Then

$$c := [a_1, t, b_1] \cdots [a_n, t, b_n]$$

cannot be written as a product of less than n *commutators of weight* 3.

Proof. Suppose $c = \prod_{i=1}^{k} [u_i, v_i, w_i]$ for elements $u_i, v_i, w_i \in G_n$ and k is chosen minimal with respect to this property. Then, using the canonical map from $L_n \to \Delta_n$, it follows that c, considered as an element of Δ_n, can be written as a product of k triple commutators. Therefore, taking Corollary 3.14 into account, it follows that x can be written as a product of k commutators of the form $[x_i, t, x_i']$ with $x_i, x_i' \in \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. Therefore, recalling from Notation 3.12 that $[t, b_n] = \psi(b_n) \in [\Delta_n, \Delta_n]$ Lemma 3.13 implies that $c = [a_1, [t, b_1]] \cdots [a_n, [t, b_n]] = [a_1, \psi(b_1)] \cdots [a_n, \psi(b_n)]$ can be written as product of k commutators in the nilpotent class 2 group $N_n = \langle a_1, \ldots, a_n, \psi(b_1), \ldots, \psi(b_n)\rangle$ and that $[N_n, N_n] = \Omega$. Since, by the construction in Notation 3.9, the group $\Omega \cong Y \wedge Y$ and c corresponds to the tensor $a_1 \wedge b_1 + \cdots + a_n \wedge b_n$, it follows from Proposition 2.18 that $k \geq n$ must hold. □

3.3. Proof of Proposition 3.5 and Theorem 3.3

We can now use the estimates derived in the previous sections to prove our main results.

Proof of Proposition 3.5. We will proceed by way of contradiction and suppose that we can write the element

$$\hat{g} = \prod_{i=1}^{\infty} [a_i, t, b_i] \in \widehat{G}$$

as the image of an element of $\langle t\rangle * \widehat{A}$, that can be put in the standard form

$$P_{k+1} = x_1 t^{\alpha_1} \cdots x_{k+1} t^{\alpha_{k+1}}$$

with $x_i \in \widehat{A}$ and $\alpha_i \in \mathbb{Z}$. Consider the projection of the above to L_n for $n > 4k-1$. Then $\hat{g} = \varphi(P_{k+1})$ becomes

$$x_1 t^{\alpha_1} \cdots x_{k+1} t^{\alpha_{k+1}} = \prod_{i=1}^{n} [a_i, t, b_i].$$

This is a contradiction, since Corollary 3.8 implies that the commutator length of the left hand side is at most $4k-1$, while Lemma 3.15 shows that the commutator length of the right-hand side is at least $n > 4k-1$. $\square$

Proof of Theorem 3.3. Observe that $\mathcal{G} = \circledast_{n\geq 1} \mathbb{Z} = \langle t\rangle * \circledast_{n\geq 1} F(a_n, b_n)$ is naturally a subgroup of $L = \langle t\rangle * \widehat{A}$. Then the result follows from Proposition 3.5. $\square$

Proof of Theorem 1.2. The first part of Theorem 1.2 is contained in Theorem 3.2.

On the other hand, Theorem 3.3 shows that $\widehat{F} \neq \mathcal{G} \cdot \mathrm{Ker}_{F_3(x,y)}(\widehat{F})$. Since $\mathrm{Ker}_{B(1,n)}(\widehat{F}) \leq \mathrm{Ker}_{F_3(x,y)}(\widehat{F})$ by Corollary 2.11, we also get $\widehat{F} \neq \mathcal{G} \cdot \mathrm{Ker}_{B(1,n)}(\widehat{F})$. $\square$

4. An application to fibration theory

As we already mentioned in the introduction our study was partly motivated by a strong relation between the size of the T-kernel of $\widehat{F}$ and geometrical properties of inverse limits of covering spaces induced by homomorphisms from $\mathcal{G}$ to T. In this section we will attempt a reasonably self-contained description of this relation in the most prominent case of the Hawaiian Earring. See also [3] for a more detailed and geometrically-oriented approach.

To fix the ideas, for $k = 1, 2, \ldots$ let $\alpha_k\colon [0, 2\pi] \to \mathbb{R}^2$ be a parametrization of the loop in the plane given by the formula $\alpha_k(t) := \frac{1}{k}(\sin t, 1 - \cos t)$. Then $X_n := \bigcup_{k=1}^{n} \alpha_k([0, 2\pi])$ is a union of n circles of decreasing radius and with common point $b = (0, 0)$, and $X := \bigcup_{k=1}^{\infty} \alpha_k([0, 2\pi])$ is the Hawaiian Earring. Note that X may be viewed as the inverse limit $X = \varprojlim X_n$ with respect to bonding maps $p_n\colon X_{n+1} \to X_n$ where p_n is the retraction of X_{n+1} onto X_n that sends the circle $\alpha_{n+1}([0, 2\pi])$ to the base-point b.

Given a sequence of letters $\{x_1, x_2, \ldots\}$, let F_n be the free group of rank n generated by the set $\{x_1, \ldots, x_n\}$. By assigning to each x_k the homotopy class of the loop α_k, we obtain an isomorphism $F_n \overset{\cong}{\to} \pi_1(X_n, b)$. Let $\varphi_n\colon F_{n+1} \to F_n$ be the homomorphism, which under the identification of F_n with $\pi_1(X_n, b)$, is induced by p_n. Then the kernel of φ_n is the normal closure of x_{n+1} in F_{n+1} and we obtain an inverse system of free groups whose limit is $\widehat{F}$.

Let $\widehat{X}_n$ be the Cayley graph of F_n with respect to $\{x_1, \ldots, x_n\}$. We view each $\widehat{X}_n$ as a topological space with a natural free action of F_n on $\widehat{X}_n$, and with X_n as a topological quotient $X_n = \widehat{X}_n / F_n$. The actions of F_n on $\widehat{X}_n$ for various n are compatible in the sense that there are canonical maps $\tilde{p}_n\colon \widehat{X}_{n+1} \to \widehat{X}_n$, such that

$$\tilde{p}_n(\tilde{x} \cdot g) = \tilde{p}_n(\tilde{x}) \cdot \varphi_n(g), \quad \forall \tilde{x} \in \widehat{X}_{n+1}, \quad \forall g \in F_{n+1}.$$

By passing to the quotients, we obtain commutative diagrams

$$\begin{tikzcd}
\widehat{X}_{n+1} \arrow[r, "\hat{p}_n"] \arrow[d] & \widehat{X}_n \arrow[d] \\
X_{n+1} = \widehat{X}_{n+1}/F_{n+1} \arrow[r, "p_n"'] & X_n = \widehat{X}_n/F_n
\end{tikzcd}$$

Thus, we may consider the action of $\widehat{F} = \varprojlim F_n$ on $\widehat{X} = \varprojlim \widehat{X}_n$. This action is clearly free, since F_n acts freely on $\widehat{X}_n$ for all n.

The quotient projection $q : \widehat{X} \to X = \widehat{X}/\widehat{F}$ is an inverse limit of covering projections, and is thus a (Hurewicz) fibration with unique path-lifting property (see [16, Chapter 2], [7, Section 2]). Liftings of paths determine an action of the fundamental group of the base $\mathcal{G} = \pi_1(X, b)$ on the fibre $\widehat{F}$. Under this action two elements of the fibre are related if, and only if they are connected by a path in $\widehat{X}$. Therefore, we may identify path-components of $\widehat{X}$ with the set of cosets $\widehat{F}/\mathcal{G}$.

More generally, for every normal subgroup $N \leq \widehat{F}$ we may consider the action of the group $\widehat{F}/N$ on $\widehat{X}/N$. Assume that N is such that the resulting projection $q_N\colon \widehat{X}/N \to X = (\widehat{X}/N)/(\widehat{F}/N)$ is a fibration with unique path-lifting property. As before, $\mathcal{G} = \pi_1(X, b)$ acts on the fibre $\widehat{F}/N$ and the space $\widehat{X}/N$ is path-connected if, and only if all points in $\widehat{F}/N$ are related by the action of $\mathcal{G}$. In other words:

Proposition 4.1. *Assume $q_N\colon \widehat{X}/N \to X$ is a fibration with unique path-lifting property. Then the fibration $\widehat{F}/N$ is path-connected if, and only if, $\widehat{F} = N \cdot \mathcal{G}$.*

Normal subgroups that arise as T-kernels satisfy the assumption of the above Proposition.

Proposition 4.2. *If T is an $\widehat{F}$-slender group, then the projection $q\colon \widehat{X}/Ker_T(\widehat{F}) \to X$ is a fibration with the unique path-lifting property. The total space $\widehat{X}/Ker_T(\widehat{F})$ is path-connected if $T = \mathbb{Z}$ or $T = F_2(x, y)$ and is path-disconnected if $T = F_3(x, y)$ or $T = B(1, n)$.*

Proof. That $q\colon \widehat{X}/Ker_T(\widehat{F}) \to X$ is a fibration with the unique path-lifting property follows from Proposition 4.2 in [3]. Proposition 4.2 in [3] is stated in terms of non-commutatively slender groups. However the only place that non-commutatively slender is needed is to show that $\mathrm{Ker}_T(\widehat{F}) = \varprojlim \mathrm{Ker}_T(F_n)$, which holds in the $\widehat{F}$-slender setting by Lemma 2.10.

If $T = F_2(x, y)$ or $T = F_3(x, y)$, the path-connected conclusion follows from Proposition 4.1 and Theorem 1.2. If $T = \mathbb{Z}$ or $T = B(1, n)$, the path-connected conclusion was proved in [3, Theorem 4.4] and [3, Theorem 4.5], respectively. □

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