



Uncountable groups and the geometry of inverse limits of coverings



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ABSTRACT

In this paper we develop a new approach to the study of uncountable fundamental groups by using Hurewicz fibrations with the unique path-lifting property (*lifting spaces* for short) as a replacement for covering spaces. In particular, we consider the inverse limit of a sequence of covering spaces of X . It is known that the path-connectivity of the inverse limit can be expressed by means of the derived inverse limit functor \varprojlim^1 , which is, however, notoriously difficult to compute when the fundamental group, $\pi_1(X)$, is uncountable. To circumvent this difficulty, we express the set of path-components of the inverse limit \tilde{X} of a sequence of covering spaces in terms of the functors \varprojlim and \varprojlim^1 applied to sequences of countable groups arising from polyhedral approximations of X .

A consequence of our computation is that path-connectedness of a lifting space, \tilde{X} , implies that $\pi_1(\tilde{X})$ supplements $\pi_1(X)$ in $\tilde{\pi}_1(X)$ where $\tilde{\pi}_1(X)$ is the inverse limit of fundamental groups of polyhedral approximations of X . As an application we show that $\mathcal{G} \cdot \text{Ker}_{\mathbb{Z}}(\hat{F}) = \hat{F} \neq \mathcal{G} \cdot \text{Ker}_{B(1,n)}(\hat{F})$, where \hat{F} is the canonical inverse limit of finite rank free groups, \mathcal{G} is the fundamental group of the Hawaiian Earring, $B(1, n)$ is the Baumslag-Solitar group, and $\text{Ker}_A(\hat{F})$ is the intersection of kernels of homomorphisms from \hat{F} to A .

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1. Introduction

A famous theorem of Shelah [25] states that the fundamental groups of Peano continua present a striking dichotomy: they are either finitely presented or uncountable. The first case corresponds to fundamental

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groups of finite polyhedra and covering space theory has traditionally been an effective geometric approach to the study of the structure of these groups. The second case is by no means exotic either: Peano continua with uncountable fundamental group arise naturally as attractors of dynamical systems [17], as fractal spaces [22,27], as boundaries of non-positively curved groups [20], and in many other situations. The fundamental difference between the two cases is of a local nature. If the fundamental group of a Peano continuum X is uncountable, then by [25] X is not semilocally simply connected at some point. As a consequence most of the subgroups of $\pi_1(X)$ do not correspond to a covering space over X , which represents a major obstacle for a geometric study of these groups. This work is part of a wider program to study fundamental groups of ‘wild’ Peano continua where the role of covering spaces is taken by more general fibrations with the unique path-lifting property. These fibrations were introduced by Spanier [26, Chapter 2] who developed much of the theory of covering spaces within this more general setting. Since the term ‘Hurewicz fibration with the unique path-lifting property’ is somewhat impractical we call them *lifting spaces* and the corresponding maps from the total space to the base are called *lifting projections*. For some examples of lifting projections, see [7].

The main advantage of lifting spaces over covering spaces is that the former are closed with respect to composition and arbitrary inverse limits - see [26, Section II.2]. Most notably, the inverse limit of a sequence of covering spaces over X is always a lifting space over X (and is not a covering projection, unless the sequence is eventually constant). If X is semilocally simply-connected (e.g., a CW-complex), and the sequence is not eventually constant, then the limit lifting space is path-disconnected (see Corollary 2.3).

This is a geometric consequence of the fact that the fundamental group of the base is countable while the fibre of the lifting projection is uncountable. However, if X is not semilocally simply-connected, then its fundamental group is uncountable by Shelah’s theorem and the limit space may or may not be path-connected, depending on the interplay between $\pi_1(X)$ and the sequence of subgroups corresponding to the coverings in the inverse sequence.

The path-components of an inverse limit of covering spaces are classically determined by the derived inverse limit functor \varprojlim^1 applied to a sequence of subgroups of $\pi_1(X)$. Difficulties arise in the computation of \varprojlim^1 for inverse sequences of uncountable groups, which make this an ineffective approach to determining path-connectivity (see the discussion at the end of Section 2 and Examples 3.7-3.11 in Section 3). In Section 3, we consider inverse sequences of coverings over some polyhedral expansion of the base space X and study the path-connectedness of the limit. We prove the following result, which allows one to work with inverse sequences of countable groups and thus avoid the difficulties of computing \varprojlim^1 for uncountable groups.

Theorem 1 (Theorem 3.6). *Every inverse limit of covering projections over a Peano continuum X is homeomorphic to an inverse limit of covering projections over a polyhedral expansion of X .*

The main result of Section 3, Theorem 3.7, completely describes the fundamental group and the set of path components of an inverse limit of coverings over some polyhedral expansions of a Peano continuum. The statement is quite technical, but it leads to the following important consequence which characterizes path-connectedness of the limit space.

Corollary 2 (Corollary 3.8). *Let $p: \tilde{X} \rightarrow X$ be the inverse limit of covering maps over a Peano continuum X . Then \tilde{X} is path-connected if, and only if, the inverse sequence of fundamental groups, $\pi_1(\tilde{X}_i)$, satisfies the Mittag-Leffler property and the natural homomorphism*

$$\pi_1(X) \longrightarrow \varprojlim \left(\frac{\pi_1(X_i)}{\pi_1(\tilde{X}_i)} \right)$$

is surjective, where $\tilde{X}_i \rightarrow X_i$ are covering projections of a polyhedral expansion $\{X_i\}$ of X .

The Hawaiian earring group \mathcal{G} naturally embeds into its shape group \widehat{F} , which is the inverse limit of finite rank free groups. However, apart from the fact that the embedding of \mathcal{G} contains the free group of countable rank and is thus dense in \widehat{F} , very little is known about the size of \mathcal{G} in \widehat{F} . The above corollary leads to interesting algebraic applications concerning the image of the fundamental group into the shape group which we consider in Section 4.

Given groups G and A , we let $\text{Ker}_A(G)$ be the intersection of kernels of all homomorphisms from G to A . If G is a free group and $B(1, n)$ is a Baumslag-Solitar group, then $\text{Ker}_{\mathbb{Z}}(G)$ is exactly the commutator subgroup of G and $\text{Ker}_{B(1,n)}(G)$ is exactly the second derived subgroup of G , see [8]. However, \widehat{F} is locally free but non-free and the corresponding $\text{Ker}_{\mathbb{Z}}(\widehat{F})$ properly contains the commutator subgroup of \widehat{F} . In fact, $\text{Ker}_{\mathbb{Z}}(\widehat{F})$ is sufficiently large to be a supplement of \mathcal{G} in \widehat{F} while $\text{Ker}_{B(1,n)}(\widehat{F})$ is not. These results will be made precise in our next theorem which has been proved by purely group theoretic means in [8, Theorem 1.2]. The proof we provide in the present paper is completely different as it exploits the geometry of the Hawaiian Earring.

Theorem 3 (Theorem 4.5 and Theorem 4.6). *The group \widehat{F} is equal to the internal product of its subgroups \mathcal{G} and $\text{Ker}_{\mathbb{Z}}(\widehat{F})$, while the internal product of \mathcal{G} and $\text{Ker}_{B(1,n)}(\widehat{F})$ is a proper subgroup of \widehat{F} .*

2. Preliminaries on inverse limits of spaces and groups

Conventions: Throughout the paper, we will assume that all base spaces of covering maps are path-connected. (However, the inverse limits of path-connected spaces will not necessarily be path-connected.) All spaces will be pointed and maps will be assumed to take base points to base points.

The main object of our study are lifting spaces that arise as inverse limits of covering spaces. Let us consider the following sequence of regular covering maps $p_i: \widetilde{X}_i \rightarrow X$, together with the inverse limit map $p: \widetilde{X} \rightarrow X$:

$$\begin{array}{ccccccc}
 \widetilde{X}_1 & \xleftarrow{f_1} & \widetilde{X}_2 & \xleftarrow{f_2} & \widetilde{X}_3 & \xleftarrow{\dots} & \widetilde{X} \\
 p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & & \downarrow p \\
 X & \xlongequal{\quad} & X & \xlongequal{\quad} & X & \xlongequal{\quad} & \dots \xlongequal{\quad} X
 \end{array}$$

For each i , we identify $\pi_1(\widetilde{X}_i)$ with its image $p_{i*}(\pi_1(\widetilde{X}_i))$ which is a normal subgroup of $\pi_1(X)$. Under this identification $\pi_1(\widetilde{X}_i)$ is called a covering subgroup of $\pi_1(X)$. We will use the more descriptive notation $p_{i*}(\pi_1(\widetilde{X}_i))$, when necessary to avoid confusion. By standard covering space theory the bonding maps $f_i: \widetilde{X}_{i+1} \rightarrow \widetilde{X}_i$ are covering maps and we have a decreasing sequence of normal subgroups of $\pi_1(X)$

$$\pi_1(X) \supseteq \pi_1(\widetilde{X}_1) \supseteq \pi_1(\widetilde{X}_2) \supseteq \dots$$

Not every subgroup of $\pi_1(X)$ is a covering subgroup. However, if X is locally path-connected and $H_2 \subset H_1$ are normal covering subgroups, then by [26, Theorem II, 5.2] there exist unique regular covering maps $p_i: \widetilde{X}_i \rightarrow X$ with $\pi_1(\widetilde{X}_i) = H_i$ and a unique covering map $f: \widetilde{X}_2 \rightarrow \widetilde{X}_1$ such that $p_2 = p_1 \circ f$. Thus for locally path-connected spaces, a decreasing sequence of normal covering subgroups of $\pi_1(X)$ corresponds uniquely to an inverse sequence of regular coverings over X .

Proposition 2.1 ([5, Proposition 2.2]). *The inverse limit of fibrations with the unique path-lifting property is a fibration with the unique path-lifting property.*

Let $F(p)$ denote the fibre of the lifting projection p . Then $F(p)$ is the inverse limit of the fibres $F(p_i)$, and so by [26, Theorem II, 6.2] it can be naturally identified with the inverse limit of the quotients

$$F(p) = \varprojlim (\pi_1(X)/\pi_1(\tilde{X}_i)).$$

Since all bonding maps in the inverse sequence are fibrations, it is well-known (see [3,29,19]) that the homotopy groups of \tilde{X} can be expressed in terms of the homotopy groups of \tilde{X}_i :

$$\pi_1(\tilde{X}) = \varprojlim \pi_1(\tilde{X}_i) = \bigcap_i \pi_1(\tilde{X}_i), \quad \pi_0(\tilde{X}) = \varprojlim^1 \pi_1(\tilde{X}_i)$$

and $\pi_n(\tilde{X}) = \pi_n(X)$ for $n > 1$. In the above equalities, we are identifying $\pi_1(\tilde{X})$ with its image in $\pi_1(X)$. Here \varprojlim denotes the inverse limit functor on groups and \varprojlim^1 is its first derived functor (see [16, Section 11.3]). For a sequence of commutative groups $\{A_i\}$ with bonding maps $\theta_i : A_{i+1} \rightarrow A_i$ these functors can be defined as kernel and cokernel of the homomorphism $\varphi : \prod A_i \rightarrow \prod A_i$, given as

$$\varphi(u_1, u_2, u_3, \dots) = (u_1 - \theta_1(u_2), u_2 - \theta_2(u_3), \dots)$$

and so they fit in the exact sequence

$$0 \rightarrow \varprojlim A_i \rightarrow \prod A_i \xrightarrow{\varphi} \prod A_i \rightarrow \varprojlim^1 A_i \rightarrow 0.$$

For a sequence of non-commutative groups $\{G_i\}$ with bonding maps $\theta_i : G_{i+1} \rightarrow G_i$ the situation is more delicate: in this case $\varprojlim^1 G_i$ is a pointed set defined as the quotient of $\prod G_i$ under the equivalence relation given by

$$(u_i) \sim (v_i) \Leftrightarrow (v_i) = (x_i u_i \theta_i(x_{i+1})^{-1}) \text{ for some } (x_i) \in \prod G_i$$

(see [15] for detailed treatment of the non-commutative case).

The values of \varprojlim^1 are notoriously hard to compute. Here we will be only interested whether \varprojlim^1 of a sequence is trivial, and this can be settled if we can show that the inverse sequence satisfies the *Mittag-Leffler condition*, which we now define. In an inverse sequence of groups

$$G_1 \longleftarrow G_2 \longleftarrow G_3 \longleftarrow \dots$$

for a fixed j the image of the homomorphism $G_i \rightarrow G_j$ decreases as i goes toward infinity. The inverse sequence is said to satisfy the *Mittag-Leffler condition* if for every j the sequence $\{\text{Im}(G_i \rightarrow G_j) \mid i = j, j+1, \dots\}$ stabilizes. Clearly, sequences with epimorphic bonding maps satisfy the Mittag-Leffler condition. On the other hand, if the bonding maps are monomorphisms, then the sequence satisfies the Mittag-Leffler condition if, and only if, it is eventually constant. The following result is proved in [15] (see also [16, Theorem 11.3.2]):

Proposition 2.2. *If an inverse sequence $\{G_i\}$, of groups, satisfies the Mittag-Leffler condition, then $\varprojlim^1 G_i = \{1\}$. Conversely, if $\varprojlim^1 G_i = \{1\}$ and each G_i is countable, then $\{G_i\}$ satisfies the Mittag-Leffler condition.*

As we explained before, the sequence of fundamental groups induced by an inverse sequence of coverings over polyhedra satisfies Mittag-Leffler condition only if it is constant from some point on. Thus we get immediately the following corollary.

Corollary 2.3. *Let $p : \tilde{X} \rightarrow X$ be the inverse limit of an inverse sequence of covering maps $p_i : \tilde{X}_i \rightarrow X$ and assume that $\pi_1(\tilde{X}_i)$ is countable, for some i . Then either the sequence is eventually constant or \tilde{X} is not path-connected.*

In particular, the inverse limit of coverings over a countable CW-complex is either a covering projection or its total space is not path-connected.

This last statement is not surprising if one considers the tail of the exact homotopy sequence of the fibration $p: \tilde{X} \rightarrow X$, shown in the following diagram together with the above mentioned identifications:

$$\begin{array}{ccccccccccc}
 1 & \longrightarrow & \pi_1(\tilde{X}) & \xrightarrow{p_*} & \pi_1(X) & \xrightarrow{\partial} & \pi_0(F(p)) & \longrightarrow & \pi_0(\tilde{X}) & \longrightarrow & * \\
 & & \parallel & & \parallel & & \parallel & & \parallel & & \\
 1 & \longrightarrow & \varprojlim \pi_1(\tilde{X}_i) & \longrightarrow & \pi_1(X) & \longrightarrow & \varprojlim (\pi_1(X)/\pi_1(\tilde{X}_i)) & \longrightarrow & \lim^1 \pi_1(\tilde{X}_i) & \longrightarrow & *
 \end{array}$$

If the sequence is not eventually constant, then $\pi_0(F(p))$ is uncountable. But if X is a countable CW-complex, then $\pi_1(X)$ is a countable group and so ∂ cannot be surjective.

However, if $\pi_1(X)$ is uncountable, then ∂ can be surjective. For example, the lifting projection over the infinite product of circles, obtained by ‘unwrapping’ one circle at a time, as in

$$(S^1)^\infty \leftarrow (\mathbb{R} \times S^1 \times S^1 \times \dots) \leftarrow (\mathbb{R} \times \mathbb{R} \times S^1 \times \dots) \leftarrow \dots \leftarrow \mathbb{R}^\infty$$

has a path-connected total space in spite of the fact that it is not eventually constant. Another, less obvious example is a sequence of 2-fold coverings over the Hawaiian earring whose limit space is path-connected (see [6, Section 2.7]). In both cases we have decreasing sequences of uncountable fundamental groups that do not satisfy the Mittag-Leffler condition but their derived inverse limits are nonetheless trivial.

Of course, one can easily find examples of inverse limits of coverings of the Hawaiian ring whose total space has uncountably many path-components. Distinguishing various covering spaces is hindered by the difficulties in the explicit computation of the derived limit functor for decreasing sequences of uncountable groups. We present one approach to this problem in the following section, while in the last section we consider the algebraic implications of the path-(dis)connectedness of the limit space.

3. Inverse limits of coverings over an expansion of X

Our approach to reducing the complexity of computing the derived limit functor to the more manageable countable case is to consider a polyhedral expansion $\{X_i\}$ of X and represent a lifting projection $p: \tilde{X} \rightarrow X$ as an inverse limit of covering maps over a polyhedral expansion, i.e., find covering maps $p_i: \tilde{X}_i \rightarrow X_i$ that satisfy the following diagram.

$$\begin{array}{ccccccc}
 \tilde{X}_1 & \xleftarrow{f_1} & \tilde{X}_2 & \xleftarrow{f_2} & \tilde{X}_3 & \longleftarrow \dots & \longleftarrow \tilde{X} \\
 p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & & \downarrow p \\
 X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & X_3 & \longleftarrow \dots & \longleftarrow X
 \end{array}$$

There are several standard methods by which a compact metric space can be represented as a limit of an inverse sequence of polyhedra (see Mardesic-Segal [24]). What best suits our purposes is the one given by nerves of coverings, which we briefly recall (see [24, Appendix 1] for details). Every metric compactum X admits arbitrarily fine finite open coverings. Moreover, for every finite open covering $\mathcal{U} = \{U_1, \dots, U_n\}$ of X there exists a partition of unity subordinate to \mathcal{U} , which we will denote by $\{\rho_i: X \rightarrow [0, 1] \mid i = 1, \dots, n\}$. Let $N(\mathcal{U})$ be the *nerve* of the covering \mathcal{U} , i.e., the simplicial complex whose vertices are elements of \mathcal{U} , and whose simplices are spanned by elements of \mathcal{U} with non-empty intersection. Then the formula

$$f(x) := \sum_{i=1}^n \rho_i(x) \cdot U_i \tag{1}$$

defines a map $f: X \rightarrow |N(\mathcal{U})|$ (where $|N(\mathcal{U})|$ is the geometric realization of the nerve). It is well known that the choice of the partition of unity does not affect the homotopy class of f , so the induced homomorphism $f_*: \pi_1(X) \rightarrow \pi_1(|N(\mathcal{U})|)$ depends only on the cover \mathcal{U} .

Lemma 3.1. *If each element of \mathcal{U} is path-connected, then $f_*: \pi_1(X) \rightarrow \pi_1(|N(\mathcal{U})|)$ is surjective.*

Proof. Without loss of generality we may assume that the partition of unity subordinate to \mathcal{U} is reduced in the sense that for every i there exists $x_i \in U_i$, such that $\rho_i(x_i) = 1$. For every pair of intersecting sets $U_i, U_j \in \mathcal{U}$ we may choose a path in $U_i \cup U_j$ between x_i and x_j . These paths determine a map $g: |N(\mathcal{U})^{(1)}| \rightarrow X$ from the 1-skeleton of the nerve to X . One can check that, for every 1-simplex σ in $N(\mathcal{U})^{(1)}$, the image $f(g(\sigma))$ is contained in the open star of σ in $N(\mathcal{U})$, which implies that $f \circ g$ is homotopic to the inclusion $i: |N(\mathcal{U})^{(1)}| \hookrightarrow |N(\mathcal{U})|$. Since $i_*: \pi_1(|N(\mathcal{U})^{(1)}|) \rightarrow \pi_1(|N(\mathcal{U})|)$ is surjective, f_* is also surjective. \square

For any compact metric space X , there exists a sequence of finite open coverings (\mathcal{U}_i) and piecewise linear bonding maps $f_i: \mathcal{U}_{i+1} \rightarrow \mathcal{U}_i$ such that the inverse limit of the system of polyhedra

$$|N(\mathcal{U}_1)| \xleftarrow{f_2} |N(\mathcal{U}_2)| \xleftarrow{f_3} |N(\mathcal{U}_3)| \longleftarrow \dots$$

is homeomorphic to X . After identifying the inverse limit with X via this homeomorphism, the projection map from $X = \varprojlim |N(\mathcal{U}_i)|$ to $|N(\mathcal{U}_n)|$ is contiguous to a map given by a partition of unity subordinate to \mathcal{U}_n , i.e., the map given by equation (1) above. Since contiguous maps are homotopic, if the map given by a partition of unity induces a surjective map on fundamental groups so does the projection from $\varprojlim |N(\mathcal{U}_i)|$ to $|N(\mathcal{U}_n)|$. As well, if X is locally path-connected, then we may choose the open covers \mathcal{U}_i to be path-connected. (See [28, Proof of Theorem 4.10.10].)

Thus Lemma 3.1 implies the following well-known fact. (See also Hahn-Mazurkiewicz theorem and [21, Theorems 4.1 and 4.2].)

Corollary 3.2. *Every Peano continuum X can be represented as the limit of an inverse system of finite polyhedra*

$$X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow \dots \longleftarrow \varprojlim X_i = X$$

such that the homomorphisms $\pi_1(X) \rightarrow \pi_1(X_i)$ induced by the projection maps are surjective.

Remark 3.3. If \mathcal{U}' is a covering of X that refines \mathcal{U} (i.e. every element of \mathcal{U}' is contained in some element of \mathcal{U}), then there is a natural simplicial map $|N(\mathcal{U}')| \rightarrow |N(\mathcal{U})|$. Thus for a sequence of covers (\mathcal{U}_i) such that \mathcal{U}_{i+1} refines \mathcal{U}_i , we can obtain an inverse system

$$|N(\mathcal{U}_1)| \longleftarrow |N(\mathcal{U}_2)| \longleftarrow |N(\mathcal{U}_3)| \longleftarrow \dots$$

where the bonding maps are simplicial and not just piecewise linear. However, it is not necessarily the case that the inverse limit is homeomorphic to X (even if the mesh converges to 0), see [11,23]. We thank the referee for pointing this out.

Let \mathcal{U} be an open cover of X . We will use $\pi_1(X; \mathcal{U})$ to denote the \mathcal{U} -Spanier subgroup, i.e. $\pi_1(X; \mathcal{U})$ is the subgroup of $\pi_1(X, x_0)$ generated by

$$\{[\alpha * \beta * \bar{\alpha}] \mid \alpha: (I, 0) \rightarrow (X, x_0) \text{ and } \text{Im}(\beta) \subset U \text{ for some } U \in \mathcal{U}\}.$$

For an open cover \mathcal{U} of X , we will use $2\mathcal{U}$ to denote the cover of X consisting of all unions of two elements of \mathcal{U} .

We do not know whether every inverse limit of coverings over a path-connected and locally path-connected base X can be derived from an inverse limit of coverings over some expansion of X . However, if we assume that X is compact, we may use the approximation by nerves of coverings to prove the following theorem.

Lemma 3.4. *Let X be a Peano continuum and $g : X \rightarrow Y$ be a continuous map into a polyhedron Y . If \mathcal{U} is a cover of X by path-connected open sets such that the preimage of the open star of any vertex of Y is contained in an element of \mathcal{U} , then $\text{Ker}(g_*) \subset \pi_1(X; 2\mathcal{U})$.*

Proof. If we denote by $f : X \rightarrow |N(\mathcal{U})|$ the map from X to the realization of the nerve of \mathcal{U} , then the kernel of the homomorphism $f_* : \pi_1(X) \rightarrow \pi_1(|N(\mathcal{U})|)$ is $\pi_1(X; 2\mathcal{U})$, see [5, Lemma 3.3].

For every vertex v of Y , we may choose $U_v \in \mathcal{U}$ such that $g^{-1}(O(v)) \subset U_v$, where $O(v)$ is the open star of the vertex v . For each simplex σ of Y with vertices v_1, \dots, v_r , we have that $\bigcap_i g^{-1}(O(v_i)) \subset \bigcap_i U_{v_i}$ is nonempty and thus U_{v_1}, \dots, U_{v_r} span a simplex of $N(\mathcal{U})$. Thus we can define a simplicial map $h : Y \rightarrow |N(\mathcal{U})|$ by sending the vertex v of Y to the vertex U_v of $N(\mathcal{U})$. Hence $h \circ g(x)$ and $f(x)$ are in the same simplex of $|N(\mathcal{U})|$ for each $x \in X$. Thus f is homotopic to $h \circ g$, which implies that $\text{Ker}(g_*) \subset \text{Ker}(f_*) = \pi_1(X; 2\mathcal{U})$. \square

Lemma 3.5. *Let X be a Peano continuum. Suppose that $X = \varprojlim X_i$ where each X_i is a compact polyhedron and the projection maps $f_i : X \rightarrow X_i$ induce surjective homomorphisms of fundamental groups. Then for any covering map $p : \tilde{X} \rightarrow X$, there exists an i and a covering map $\tilde{p}_i : \tilde{X}_i \rightarrow X_i$ such that p is isomorphic to the pull-back of \tilde{p}_i by f_i .*

Proof. Let

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \dots \xleftarrow{\quad} \varprojlim X_i = X$$

be a polyhedral expansion of X , such that the projection maps $f_i : X \rightarrow X_i$ induce surjective homomorphisms of fundamental groups.

Let $p : \tilde{X} \rightarrow X$ be an arbitrary covering map. Then p evenly covers elements of some open cover \mathcal{U} of X and $\pi_1(\tilde{X}) \supseteq \pi_1(X; \mathcal{U})$, where the latter denotes the \mathcal{U} -Spanier subgroup of $\pi_1(X)$. Since X is compact we may assume that \mathcal{U} is finite. Choose another finite cover \mathcal{V} of X by path-connected open sets whose double $2\mathcal{V}$ (set of all unions of two elements of \mathcal{V}) refines \mathcal{U} .

By Theorem 5 of [24] there exists an i such that the preimage of the open star of any vertex of X_i is contained in an element of \mathcal{V} . Then by Lemma 3.4, we have

$$\pi_1(\tilde{X}) \supseteq \pi_1(X; \mathcal{U}) \supseteq \pi_1(X; 2\mathcal{V}) \supseteq \text{Ker}(\pi_1(X) \rightarrow \pi_1(X_i)).$$

Denote $K_i := \text{Ker}(\pi_1(X) \rightarrow \pi_1(X_i))$ and let $\tilde{p}_i : \tilde{X}_i \rightarrow X_i$ be the covering map that corresponds to the subgroup $\text{Im}(\pi_1(\tilde{X}) \rightarrow \pi_1(X_i))$. By the lifting theorem, there is a fibre-preserving map $\tilde{f}_i : \tilde{X} \rightarrow \tilde{X}_i$ for which the following diagram commutes

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}_i} & \tilde{X}_i \\ \downarrow & & \downarrow \tilde{p}_i \\ X & \xrightarrow{f_i} & X_i \end{array}$$

To compute the restriction of \tilde{f}_i to a fibre, we examine the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(\tilde{X})/K_i & \longrightarrow & \pi_1(X)/K_i & \longrightarrow & \pi_1(X)/\pi_1(\tilde{X}) \longrightarrow 1 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \\
 1 & \longrightarrow & \pi_1(\tilde{X}_i) & \longrightarrow & \pi_1(X_i) & \longrightarrow & \pi_1(X_i)/\pi_1(\tilde{X}_i) \longrightarrow 1
 \end{array}$$

By the short five-lemma we conclude that \tilde{f}_i induces a bijection

$$\pi_1(X)/\pi_1(\tilde{X}) \rightarrow \pi_1(X_i)/\pi_1(\tilde{X}_i),$$

and hence a bijection between the fibres of respective covering maps. In other words, the above diagram is a pull-back of covering maps. \square

By repeating the above argument for all terms in an inverse sequence of covering maps over X we obtain an inverse sequence of covering maps over the given expansion of X with the same limit lifting projection.

Theorem 3.6. *If X is a Peano continuum, then the inverse limit of any inverse sequence of covering spaces over X is homeomorphic to an inverse limit of covering spaces over some polyhedral expansion of X .*

Note that our proof will show that the theorem will hold for any polyhedral expansion $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots \leftarrow \varprojlim X_i = X$ such that the projection maps $X \rightarrow X_n$ induce surjective maps on fundamental groups.

Proof. Let

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \dots \xleftarrow{\quad} \varprojlim X_i = X$$

be a polyhedral expansion of X , such that the projection maps $X \rightarrow X_i$ induce surjective homomorphisms of fundamental groups and suppose that

$$X \longleftarrow \tilde{X}_1 \longleftarrow \tilde{X}_2 \longleftarrow \tilde{X}_3 \longleftarrow \dots \longleftarrow \varprojlim \tilde{X}_i$$

is an inverse sequence of coverings over X . By applying Lemma 3.5, we may find, for each i , an index $j(i)$ and covering map $\bar{p}_i : \bar{X}_{j(i)} \rightarrow X_{j(i)}$ such that the following diagram is a pull-back.

$$\begin{array}{ccc}
 \tilde{X}_i & \xrightarrow{\tilde{f}_i} & \bar{X}_{j(i)} \\
 \downarrow & & \downarrow \bar{p}_i \\
 X & \longrightarrow & X_{j(i)}
 \end{array}$$

Since the diagram is a pull-back, \tilde{f}_i is a bijection on fibres. Thus we have the following diagram on fundamental groups, where the right most vertical map is a bijection.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(\tilde{X}_i) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X)/\pi_1(\tilde{X}_i) \longrightarrow * \\
 & & \downarrow \tilde{f}_{i*} & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \pi_1(\bar{X}_{j(i)}) & \xrightarrow{\bar{p}_{i*}} & \pi_1(X_{j(i)}) & \longrightarrow & \pi_1(X_{j(i)})/\pi_1(\bar{X}_{j(i)}) \longrightarrow *
 \end{array}$$

The homomorphism \tilde{f}_{i*} is a surjective map on fundamental groups, since the middle vertical map is surjective and the right most is bijective. Thus we have the following commutative diagram.

$$\begin{array}{ccccc}
 & & \pi_1(\bar{X}_{j(i+1)}) & \longleftarrow & \pi_1(\tilde{X}_{i+1}) \\
 & & \downarrow & & \swarrow \tilde{f}_{(i+1)*} \\
 \pi_1(\bar{X}_{j(i)}) & \longleftarrow & \pi_1(\tilde{X}_i) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 \pi_1(X_{j(i)}) & \longleftarrow & \pi_1(X_{j(i+1)}) & \longleftarrow & \pi_1(X)
 \end{array}$$

which implies that the image of $\pi_1(\bar{X}_{j(i+1)})$ in $\pi_1(X_{j(i)})$ is contained in the image of $\pi_1(\bar{X}_{j(i)})$ in $\pi_1(X_{j(i)})$. By the lifting criteria there is a unique based map from $\bar{X}_{j(i+1)}$ to $\bar{X}_{j(i)}$ which is a lift of the map $\bar{X}_{j(i+1)} \rightarrow X_{j(i+1)} \rightarrow X_{j(i)}$. Thus we have

$$\begin{array}{ccccc}
 & & \bar{X}_{j(i+1)} & \longleftarrow & \tilde{X}_{i+1} \\
 & & \downarrow & & \swarrow \\
 \bar{X}_{j(i)} & \longleftarrow & \tilde{X}_i & & \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{j(i)} & \longleftarrow & X_{j(i+1)} & \longleftarrow & X
 \end{array}$$

The maps \tilde{f}_i induce a map $\tilde{f} : \varprojlim \tilde{X}_i \rightarrow \varprojlim \bar{X}_{j(i)}$ and the maps \bar{p}_i induce a map from $\varprojlim \bar{X}_{j(i)} \rightarrow \varprojlim X_{j(i)} = X$. Then, for each i , we have the following commutative diagram where g_i is induced by the universal property of pullbacks.

$$\begin{array}{ccc}
 \varprojlim \bar{X}_{j(i)} & & \\
 \downarrow \tilde{f} & \searrow g_i & \\
 \tilde{X}_i & \longrightarrow & \bar{X}_{j(i)} \\
 \downarrow & & \downarrow \bar{p}_i \\
 X & \longrightarrow & X_{j(i)}
 \end{array}$$

Using the universal property of inverse limits the maps g_i induce a map $g : \varprojlim \bar{X}_{j(i)} \rightarrow \varprojlim \tilde{X}_i$. Since both g and \tilde{f} are defined using universal properties, it is a standard diagram chase argument to show that g is the continuous inverse to \tilde{f} . Thus \tilde{f} is a homeomorphism. \square

The above theorem shows that for most cases of interest it is sufficient to consider lifting projections over X that are inverse limits of covering maps over some expansion of X as in

$$\begin{array}{ccccccc}
 \tilde{X}_1 & \xleftarrow{\tilde{f}_1} & \tilde{X}_2 & \xleftarrow{\tilde{f}_2} & \tilde{X}_3 & \xleftarrow{\dots} & \tilde{X} \\
 p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & & \downarrow p \\
 X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & X_3 & \xleftarrow{\dots} & X
 \end{array}$$

Suppose that X_i are polyhedra and that the maps $X \rightarrow X_i$ induce epimorphisms between respective fundamental groups (e.g. if X is locally path-connected). As in Section 2, we may compare the tail of the exact homotopy sequence of the fibration p with the derived exact sequence of the inverse limit functor to obtain the following diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(\tilde{X}) & \longrightarrow & \pi_1(X) & \longrightarrow & \varprojlim(\pi_1(X_i)/\pi_1(\tilde{X}_i)) \longrightarrow \pi_0(\tilde{X}) \longrightarrow * \\
 & & \varphi \downarrow & & \iota \downarrow & & \parallel & & \downarrow \psi \\
 1 & \longrightarrow & \varprojlim \pi_1(\tilde{X}_i) & \longrightarrow & \varprojlim \pi_1(X_i) & \longrightarrow & \varprojlim(\pi_1(X_i)/\pi_1(\tilde{X}_i)) & \longrightarrow & \varprojlim^1 \pi_1(\tilde{X}_i) \longrightarrow *
 \end{array}$$

The right bottom term is a point since $\{\pi_1(X_i), f_{i*}\}$ is a Mittag-Leffler inverse sequence of groups. Mardesic-Segal [24, Ch 2] prove that the limit $\varprojlim \pi_1(X_i)$ is independent of the choice of a polyhedral expansion $\{X_i\}$ of X . For any two polyhedral expansions $\{X_i\}$ and $\{Y_i\}$ of X , where the projection maps induce surjective maps of fundamental groups for both expansions, the pro-isomorphism between $\varprojlim \pi_1(X_i)$ and $\varprojlim \pi_1(Y_i)$ can be lifted to a pro-isomorphism between $\varprojlim \pi_1(\tilde{X}_i)$ and $\varprojlim \pi_1(\tilde{Y}_i)$ via the homotopy lifting criterion. The limit $\varprojlim \pi_1(X_i)$ is usually called the *Čech fundamental group* (or the *first shape group*) of X , and is denoted by $\check{\pi}_1(X)$. Since \tilde{X} is not necessarily compact, $\varprojlim \pi_1(\tilde{X}_i)$ is not necessarily the first shape group of \tilde{X} . Regardless, we will still denote $\varprojlim \pi_1(\tilde{X}_i)$ by $\check{\pi}_1(\tilde{X})$. The kernel of the induced map $\iota: \pi_1(X) \rightarrow \check{\pi}_1(X)$ is called the *shape kernel* of X and denoted $\text{ShKer}(X)$. Note that by the exactness of the above diagram we have that

$$\text{Ker}(\pi_1(\tilde{X}) \rightarrow \varprojlim \pi_1(\tilde{X}_i)) = \text{Ker}(\pi_1(X) \rightarrow \varprojlim \pi_1(X_i)) = \text{ShKer}(X).$$

Clearly, $\psi: \pi_0(\tilde{X}) \rightarrow \varprojlim^1 \pi_1(\tilde{X}_i)$ is a surjective function between homogeneous sets. Thus, in order to determine $\pi_0(\tilde{X})$ we need to compute the fibres of ψ . From the exactness of the second row we deduce that $\varprojlim^1 \pi_1(\tilde{X}_i)$ is the set of cosets of the action of the group $\check{\pi}_1(X)/\check{\pi}_1(\tilde{X})$ on $\varprojlim(\pi_1(X_i)/\pi_1(\tilde{X}_i))$. Then a straightforward diagram chasing shows that the fibres of ψ can be naturally identified with the cosets of the action of $\iota(\pi_1(X))$ on $\check{\pi}_1(X)/\check{\pi}_1(\tilde{X})$.

Theorem 3.7. *Let X be a path-connected space that admits a polyhedral expansion $f_i: X \rightarrow X_i$ such that the induced homomorphisms f_{i*} are surjective and let $p: \tilde{X} \rightarrow X$ be the inverse limit of a sequence of coverings $p_i: \tilde{X}_i \rightarrow X_i$ as described above. Then the fundamental group of \tilde{X} fits in the exact sequence of groups*

$$1 \rightarrow \text{ShKer}(X) \rightarrow \pi_1(\tilde{X}) \rightarrow \check{\pi}_1(\tilde{X}),$$

and the set of path-components of \tilde{X} fits in the exact sequence of groups and pointed sets

$$\pi_1(X) \xrightarrow{\iota} \check{\pi}_1(X)/\check{\pi}_1(\tilde{X}) \longrightarrow \pi_0(\tilde{X}) \longrightarrow \varprojlim^1 \pi_1(\tilde{X}_i) \longrightarrow *$$

The main advantage of Theorem 3.7 with respect to the description of $\pi_0(\tilde{X})$ over the one given in Section 2 is that all limits and derived limits are taken over inverse sequences of countable groups.

Corollary 3.8. *Let $p: \tilde{X} \rightarrow X$ be the inverse limit of covering maps as in the above theorem. Then \tilde{X} is path-connected if, and only if, the inverse sequence $\pi_1(\tilde{X}_1) \leftarrow \pi_1(\tilde{X}_2) \leftarrow \pi_1(\tilde{X}_3) \leftarrow \dots$ satisfies the Mittag-Leffler property and the natural homomorphism $\pi_1(X) \rightarrow \frac{\tilde{\pi}_1(X)}{\tilde{\pi}_1(\tilde{X})}$ is surjective.*

Furthermore, if either of the above conditions holds, then $\frac{\tilde{\pi}_1(X)}{\tilde{\pi}_1(\tilde{X})}$ is isomorphic to $\varprojlim \left(\frac{\pi_1(X_i)}{\pi_1(\tilde{X}_i)} \right)$.

Proof. Theorem 3.7 implies that $\pi_0(\tilde{X})$ is trivial if, and only if $\varprojlim^1 \pi_1(\tilde{X}_i)$ is trivial and $\pi_1(X) \rightarrow \tilde{\pi}_1(X)/\tilde{\pi}_1(\tilde{X})$ is surjective. Since the fundamental groups of polyhedra are countable, then by Proposition 2.2 the inverse sequence is Mittag-Leffler if and only if $\varprojlim^1 \pi_1(\tilde{X}_i)$ is trivial.

Observe that the isomorphism between $\tilde{\pi}_1(X)/\tilde{\pi}_1(\tilde{X})$ and $\varprojlim (\pi_1(X_i)/\pi_1(\tilde{X}_i))$ follows from the Mittag-Leffler property and does not hold in general. \square

Let us consider a couple of examples.

Example 3.9. We have already mentioned the inverse sequence of covering maps over the infinite product of circles

$$(S^1)^\infty \leftarrow (\mathbb{R} \times S^1 \times S^1 \times \dots) \leftarrow (\mathbb{R} \times \mathbb{R} \times S^1 \times \dots) \leftarrow \dots$$

It can be replaced by the following inverse sequence of covering maps over polyhedral approximations of $(S^1)^\infty$ (with horizontal maps the usual projections)

$$\begin{array}{ccccccc} \mathbb{R} & \longleftarrow & \mathbb{R} \times \mathbb{R} & \longleftarrow & \mathbb{R} \times \mathbb{R} \times \mathbb{R} & \longleftarrow & \dots \longleftarrow \mathbb{R}^{\mathbb{N}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^1 & \longleftarrow & S^1 \times S^1 & \longleftarrow & S^1 \times S^1 \times S^1 & \longleftarrow & \dots \longleftarrow (S^1)^{\mathbb{N}} \end{array}$$

The fundamental groups of the products of copies of \mathbb{R} are trivial so the corresponding sequence is Mittag-Leffler. By Corollary 3.8 the path-connectedness of $\mathbb{R}^{\mathbb{N}}$ is equivalent to the equality $\pi_1((S^1)^{\mathbb{N}}) = \varprojlim \pi_1((S^1)^n) = \tilde{\pi}_1((S^1)^{\mathbb{N}})$.

Example 3.10. Let T^∞ be the infinite product of circles endowed with the CW-topology, i.e. the direct limit of the sequence of spaces

$$S^1 \hookrightarrow S^1 \times S^1 \hookrightarrow S^1 \times S^1 \times S^1 \hookrightarrow \dots T^\infty$$

Clearly, the fundamental group of T^∞ is $\bigoplus_{k=1}^\infty \mathbb{Z}$. For each $i \in \mathbb{N}$ let $p_i: \tilde{X}_i \rightarrow T^\infty$ be the covering map that corresponds to the subgroup $\pi_1(\tilde{X}_i) = \bigoplus_{k=i}^\infty \mathbb{Z}$. By Corollary 2.3 the inverse limit $\tilde{X} := \varprojlim \tilde{X}_i$ is not path-connected.

Remark 3.11. The last result is somewhat counter-intuitive, as one could argue that $\tilde{X}_i \approx \mathbb{R}^{i-1} \times T^\infty$ and that the inverse limit of the coverings is simply the product \mathbb{R}^∞ . However, Example 3.9 shows that \tilde{X} is not path-connected and thus not homeomorphic to \mathbb{R}^∞ .

Example 3.12. Consider the squaring lifting projection $p: \tilde{H}_\infty \rightarrow H$ over the Hawaiian earring H described in [6, Section 2.7]. The map p is obtained as a limit of a sequence of 2-fold covering projections and, although

it resembles at first sight the construction of the dyadic solenoid, we have been able to prove by a geometric argument that its total space \tilde{H}_∞ is path-connected. We are going to show how this result is reflected in the algebraic computation of the set of path-components of \tilde{H}_∞ .

As explained in [6, Section 3.3] we can obtain the projection p as follows. Let H_i be a wedge of i circles, so that $H_1 \leftarrow H_2 \leftarrow \dots \leftarrow H$ is the standard polyhedral expansion of the Hawaiian earring. For each i let $p_i: \tilde{H}_i \rightarrow H_i$ be the 2^i -fold covering projection, obtained as the restriction to $H_i \subset (S^1)^i$ of the squaring map

$$(S^1)^i \rightarrow (S^1)^i, \quad (z_1, \dots, z_i) \mapsto (z_1^2, \dots, z_i^2).$$

It is then easy to check that p is the inverse limit of the sequence of coverings p_i , in particular $\tilde{H}_\infty = \varprojlim \tilde{H}_i$.

The sequence of groups $(\pi_1(\tilde{H}_i))$ has surjective bonding maps, so by Corollary 3.8 the path-connectedness of \tilde{H}_∞ is equivalent to the surjectivity of the homomorphism $\pi_1(H) \rightarrow (\mathbb{Z}_2)^\mathbb{N}$. But the latter is obvious, because any sequence $(a_i) \in (\mathbb{Z}_2)^\mathbb{N}$, where $a_i \in \{0, 1\}$ can be obtained as the image of the loop that winds around the i -th circle exactly when $a_i = 1$.

Example 3.13. We are going to show that a minor modification of covering maps in the inverse sequence described in Example 3.9 yields a completely different limit space. This is surprising and very difficult to see geometrically. Let us define two covering projections with fibre \mathbb{Z} . The first is the standard covering exponential map from the real line to the circle

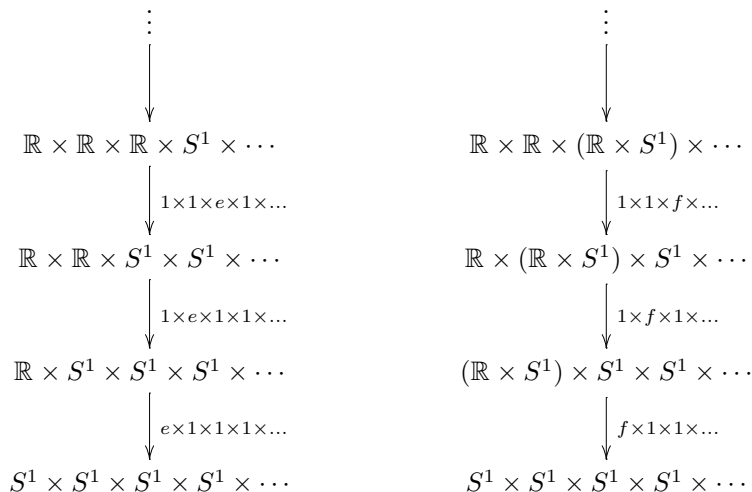
$$e: \mathbb{R} \rightarrow S^1, \quad e(t) := e^{2\pi it}$$

and the second combines the exponential covering with the two-fold covering of the circle given by the squaring map

$$f: \mathbb{R} \times S^1 \rightarrow S^1 \times S^1, \quad f(t, z) := (e^{2\pi it} z^2, z).$$

Note that the induced homomorphism $f_*: \pi_1(\mathbb{R} \times S^1) \rightarrow \pi_1(S^1 \times S^1)$ sends the generator of $\pi_1(\mathbb{R} \times S^1) \cong \mathbb{Z}$ to the element $(2, 1) \in \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, therefore f can be characterized as the covering of $S^1 \times S^1$ corresponding to the cyclic subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $(2, 1)$.

The following diagram depicts two inverse sequences of covering maps over $(S^1)^\mathbb{N}$.



The first sequence was already considered in Example 3.9 where we showed that its inverse limit is $\mathbb{R}^{\mathbb{N}}$, which is, of course, path-connected. The second is also a sequence of \mathbb{Z} -covering maps and the form of the total spaces suggest that its limit is $\mathbb{R}^{\mathbb{N}}$ as well. However, it is not difficult to check that there does not exist a map $p : \mathbb{R}^{\mathbb{N}} \rightarrow (S^1)^{\mathbb{N}}$ such that, for each i , we have the following commutative diagram where the diagonal arrow is given by the composition of the bonding maps of the second sequence.

$$\begin{array}{ccc} \mathbb{R}^{\mathbb{N}} & \longrightarrow & \mathbb{R} \times \cdots \times \mathbb{R} \times (\mathbb{R} \times S^1) \times S^1 \times \cdots \\ \downarrow p & \swarrow & \\ (S^1)^{\mathbb{N}} & & \end{array}$$

However, in place of proving this, we will simply compute the inverse limit using a polyhedral expansion for $(S^1)^{\mathbb{N}}$.

To compute the set of path components of the inverse limit of the second sequence we compare it with the following sequence over a polyhedral expansion of $(S^1)^{\mathbb{N}}$:

$$\begin{array}{ccccccc} \mathbb{R} \times S^1 & \xleftarrow{f_2} & \mathbb{R} \times \mathbb{R} \times S^1 & \xleftarrow{f_3} & \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S^1 & \xleftarrow{\quad} & \cdots \\ p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & & \\ S^1 \times S^1 & \xleftarrow{\quad} & S^1 \times S^1 \times S^1 & \xleftarrow{\quad} & S^1 \times S^1 \times S^1 \times S^1 & \xleftarrow{\quad} & \cdots \end{array}$$

The maps $f_i : \mathbb{R}^i \times S^1 \rightarrow \mathbb{R}^{i-1} \times S^1$ are defined as a product of a projection on the first i -components with the square map (i.e. two-fold covering map) on S^1 . Furthermore, the maps $p_i : \mathbb{R}^i \times S^1 \rightarrow (S^1)^{i+1}$ are covering maps corresponding to respectively the cyclic subgroup generated by the element $(2^i, 2^{i-1}, \dots, 2, 1) \in \mathbb{Z}^{i+1} = \pi_1((S^1)^{i+1})$. Note that p_1 is exactly the map f and

$$p_i \circ f_{i+1} = \underbrace{(1 \times \cdots \times 1 \times f)}_{i \text{ times}} \circ \underbrace{(1 \times \cdots \times 1 \times f \times 1)}_{(i-1) \text{ times}} \circ \cdots \circ (f \times \underbrace{1 \times \cdots \times 1}_{i \text{ times}}).$$

Thus the diagram commutes and the i -th term in the original sequence is the pull-back of p_i along the projection map as in the diagram

$$\begin{array}{ccc} \mathbb{R}^i \times S^1 \times \cdots & \longrightarrow & \mathbb{R}^i \times S^1 \\ \downarrow & & \downarrow p_i \\ (S^1)^{\mathbb{N}} & \longrightarrow & (S^1)^{i+1} \end{array}$$

It follows that the inverse limit of the original sequence and the inverse limit of the sequence of covering over the polyhedral expansion coincide. We may now apply Theorem 3.7 to determine the set of components of the total space of the limit. In fact, the inverse sequence of the fundamental groups of the covering spaces is

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{\quad} \cdots$$

which is not Mittag-Leffler, therefore its derived inverse limit is non-trivial (it is actually an uncountable abelian group). As a consequence, the total space of the inverse limit has uncountably many components. Indeed, the inverse sequence

$$\mathbb{R} \times S^1 \xleftarrow{f_2} \mathbb{R} \times \mathbb{R} \times S^1 \xleftarrow{f_3} \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S^1 \xleftarrow{\dots}$$

splits as

$$\mathbb{R} \times S^1 \xleftarrow{h_1 \times q} \mathbb{R}^2 \times S^1 \xleftarrow{h_2 \times q} \mathbb{R}^3 \times S^1 \xleftarrow{\dots}$$

where $h_i : \mathbb{R}^{i+1} \rightarrow \mathbb{R}^i$ is projection map onto the first i factors and $q : S^1 \rightarrow S^1$ is a double cover map. Thus the inverse limit is homeomorphic to the product of $\mathbb{R}^{\mathbb{N}}$ with the dyadic solenoid Sol_2 .

4. Algebraic applications

We begin by describing a construction of inverse sequences of covering projections whose limits correspond to meaningful subgroups of the fundamental group.

Recall that for groups G and A , we defined the A -kernel of G to be

$$\text{Ker}_A(G) = \bigcap_{\phi \in \text{Hom}(G,A)} \ker \phi.$$

If F is any free group and $B(1, n)$ is a Baumslag-Solitar group, then $\text{Ker}_{\mathbb{Z}}(F)$ is exactly the commutator subgroup of F and $\text{Ker}_{B(1,n)}(F)$ is exactly the second derived subgroup of F , see [8].

Theorem 4.1. *Let X be a countable CW complex. Then there exists an inverse sequence*

$$X \xleftarrow{\dots} X_1 \xleftarrow{\dots} X_2 \xleftarrow{\dots} \varprojlim X_i = \widehat{X}$$

of covering projections such that the inverse limit $\widehat{p} : \widehat{X} \rightarrow X$ is a unique path-lifting fibration such that $\widehat{p}_*(\pi_1(\widehat{X})) = \text{Ker}_{\mathbb{Z}}(\pi_1(X))$. If additionally, $\pi_1(X)$ is finitely generated then \widehat{p} is a covering projection.

Proof. Let X be a countable CW-complex. Suppose that $f : X \rightarrow S^1$ is a continuous map. Let $p : \widetilde{X} \rightarrow X$ be the covering map obtained as a pullback of the universal covering $e : \mathbb{R} \rightarrow S^1$ of S^1 along f , as in the following diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \mathbb{R} \\ p \downarrow & & \downarrow e \\ X & \xrightarrow{f} & S^1 \end{array}$$

It is easy to check that p is the covering map that corresponds to the kernel of f_* , that is to say

$$\pi_1(\widetilde{X}) \cong \text{Im}(p_* : \pi_1(\widetilde{X}) \rightarrow \pi_1(X)) = \text{Ker}(f_* : \pi_1(X) \rightarrow \pi_1(S^1)).$$

Clearly, p is a regular covering whose fibres can be naturally identified with the infinite cyclic group \mathbb{Z} , so we often say that p is the \mathbb{Z} -covering, corresponding to the map $f : X \rightarrow S^1$ (or rather, to its homotopy class). Since S^1 is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 1)$, there is a bijection

$$[X, S^1] = \text{Hom}(\pi_1(X), \mathbb{Z}).$$

Thus, we may also say that p is the \mathbb{Z} -covering corresponding to a given homomorphism $\varphi: \pi_1(X) \rightarrow \mathbb{Z}$, in the sense that $\pi_1(\tilde{X}) = \text{Ker } \varphi$. Note, that every non-trivial subgroup of \mathbb{Z} is isomorphic to \mathbb{Z} , so we may assume without loss of generality that φ is surjective. Since $\text{Hom}(\pi_1(X), \mathbb{Z})$ is countable, we may enumerate it by $\{\varphi_1, \varphi_2, \varphi_3, \dots\} = \text{Hom}(\pi_1(X), \mathbb{Z})$. For $n \in \mathbb{N}$, let

$$\Phi_n := (\varphi_1, \dots, \varphi_n): \pi_1(X) \rightarrow \mathbb{Z}^n$$

and let $p_n: \tilde{X}_n \rightarrow X$ be the covering projection whose fundamental group is

$$K_n := \text{Ker } \Phi_n = \text{Ker } \varphi_1 \cap \dots \cap \text{Ker } \varphi_n.$$

Thus we obtain an inverse sequence of coverings

$$\begin{array}{ccccccc} \tilde{X}_1 & \xleftarrow{f_1} & \tilde{X}_2 & \xleftarrow{f_2} & \tilde{X}_3 & \xleftarrow{\quad} & \dots \\ p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & & \\ X & \xlongequal{\quad} & X & \xlongequal{\quad} & X & \xlongequal{\quad} & \dots \end{array}$$

where each f_n is a covering projection. To compute the fibre of f_n , note that we have a short exact sequence

$$0 \rightarrow K_{n-1}/K_n \rightarrow \pi_1(X)/K_n \rightarrow \pi_1(X)/K_{n-1} \rightarrow 0$$

where the second and third group are subgroups of \mathbb{Z}^n and \mathbb{Z}^{n-1} respectively. From this we deduce that K_{n-1}/K_n is either trivial or isomorphic to \mathbb{Z} , therefore all f_n are either trivial coverings (identity maps) or \mathbb{Z} -coverings.

The limit of the resulting inverse sequence of coverings is a lifting projection $\hat{p}: \hat{X} \rightarrow X$ whose fundamental group is

$$\pi_1(\hat{X}) = \bigcap_{\varphi: \pi_1(X) \rightarrow \mathbb{Z}} \text{Ker } \varphi.$$

If $\pi_1(X)$ is finitely generated (e.g., if the 1-skeleton of X is finite), then \hat{p} is a covering projection. In fact, in that case $\text{Hom}(\pi_1(X), \mathbb{Z}) = \text{Hom}(H_1(X), \mathbb{Z})$ is also finitely generated and the tower of coverings is actually finite (i.e., all but finitely many coverings in the sequence are trivial). Actually, \hat{p} can be obtained as a covering of X that corresponds to the kernel of the natural homomorphism $\pi_1(X) \rightarrow FH_1(X)$, where $FH_1(X)$ denotes the maximal free abelian quotient of $H_1(X)$. \square

Let us introduce the following functor for arbitrary groups G and A :

$$\text{Ker}_A(G) := \bigcap_{f: G \rightarrow A} \text{Ker}(f).$$

Note that $\text{Ker}_A(G)$ is a fully characteristic subgroup of G . More generally, if $\varphi: G \rightarrow H$ is a homomorphism and if $x \in \text{Ker}_A(G)$, then for every $\psi: H \rightarrow A$ we have that $\psi \circ \varphi(x) = 0$, therefore $\varphi(x) \in \text{Ker}_A(H)$. In categorical terms the correspondence $G \mapsto \text{Ker}_A(G)$ is a covariant functor. It turns out that many characteristic subgroups can be described as intersection of kernels to some group A . In fact we may view $\text{Ker}_A(G)$ as the part of G that cannot be represented in a product of copies of the group A . For example $\text{Ker}_{\mathbb{Z}}(G) = 0$ if, and only if, G is residually free-abelian.

What is the algebraic meaning of the intersection of kernels of homomorphisms to some group? It is well-known that the commutator subgroup $F'_n = [F_n, F_n]$ of the free group on n generators F_n consists of

all words in F_n for which the sum of exponents of each letter equals 0. This description is not intrinsic, as it requires to choose a basis for the free group. An equivalent description without reference to a basis is the following: if F is a free group (on any set of generators), then

$$F' = \text{Ker}_{\mathbb{Z}}(G).$$

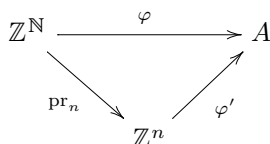
This approach was used in Cannon-Conner [1, Section 4] to describe the *big commutator subgroup* of the fundamental group of the Hawaiian earring $\mathcal{G} = \pi_1(H)$:

$$BC(\mathcal{G}) = \text{Ker}_{\mathbb{Z}}(\mathcal{G}).$$

The big commutator subgroup is much larger than the usual commutator subgroup \mathcal{G}' but shares many interesting properties with the latter.

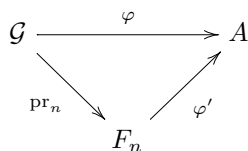
One can consider intersections of homomorphisms to other groups as well, for example finite or torsion-free groups. In [8] we gave an intrinsic description of the second commutator subgroup F'' by showing that $F'' = \text{Ker}_{B(1,n)}(F)$. In [9], we consider $\text{Ker}_A(\widehat{F})$ where A is a free nilpotent group.

Recall that a torsion-free abelian group A is *slender* if every homomorphism $\varphi: \mathbb{Z}^{\mathbb{N}} \rightarrow A$ can be factored through some finite rank free abelian group, i.e., there exists a homomorphism $\varphi': \mathbb{Z}^n \rightarrow A$ so that the following diagram commutes



(see Fuchs [14, Ch. XIII]) where pr_n is the projection map onto the first n factors of $\mathbb{Z}^{\mathbb{N}}$. Free abelian groups are slender, subgroups and extensions of slender groups are also slender. A reduced abelian group is slender if, and only if, it does not contain a subgroup isomorphic to the group $\mathbb{Z}^{\mathbb{N}}$ or to a group of p -adic integers for some prime p (cf. Fuchs [14]).

Katsuya Eda [12] extended this concept to arbitrary groups by defining a group A to be *non-commutatively slender* (nc-slender) if every homomorphism $\varphi: \mathcal{G} \rightarrow A$ from the Hawaiian earring group \mathcal{G} to A can be factored through some free (non-commutative) group of finite rank as in



where pr_n is the homomorphism induced by collapsing all but the first n loops of the Hawaiian earring.

Lemma 4.2. *If A is slender, then*

$$\text{Ker}_A(\mathbb{Z}^{\mathbb{N}}) = \varprojlim \text{Ker}_A(\mathbb{Z}^n).$$

Similarly, if A is nc-slender, then

$$\text{Ker}_A(\widehat{F}) = \varprojlim \text{Ker}_A(F_n) \quad \text{and} \quad \text{Ker}_A(\mathcal{G}) = \mathcal{G} \cap \varprojlim \text{Ker}_A(F_n).$$

Proof. The natural projections $\text{pr}_n: \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z}^n$ induce an isomorphism $\mathbb{Z}^{\mathbb{N}} \rightarrow \varprojlim \mathbb{Z}^n$. By functoriality of Ker_A , the homomorphisms $\text{Ker}_A(\mathbb{Z}^{\mathbb{N}}) \rightarrow \text{Ker}_A(\mathbb{Z}^n)$ induced by pr_n are coherent and induce a homomorphism $\text{Ker}_A(\mathbb{Z}^{\mathbb{N}}) \rightarrow \varprojlim \text{Ker}_A(\mathbb{Z}^n)$. This homomorphism is injective, because it is a restriction of the isomorphism $\mathbb{Z}^{\mathbb{N}} \rightarrow \varprojlim \mathbb{Z}^n$. To prove surjectivity, consider an element $(x_i) \in \varprojlim \text{Ker}_A(\mathbb{Z}^n) = \mathbb{Z}^{\mathbb{N}}$. We need only show that $(x_i) \in \text{Ker}_A(\mathbb{Z}^{\mathbb{N}})$. Fix a homomorphism $\varphi: \mathbb{Z}^{\mathbb{N}} \rightarrow A$. Since A is slender, φ can be factored as $\varphi = \varphi' \circ \text{pr}_n$ for some n and some $\varphi': \mathbb{Z}^n \rightarrow A$. Then $\varphi((x_i)) = \varphi'(x_n)$ which is trivial since $x_n \in \text{Ker}_A(\mathbb{Z}^n)$. Thus $\text{Ker}_A(\mathbb{Z}^{\mathbb{N}}) \rightarrow \varprojlim \text{Ker}_A(\mathbb{Z}^n)$ is surjective as well.

In [10], the authors show that if A is nc-slender then every homomorphism from \widehat{F} to A also factors through a projection to F_n for some n . Then the proof for the non-commutatively slender cases follows analogously. \square

As a consequence, A -kernels of \widehat{F} with respect to a slender group A give rise to inverse limits of coverings over the Hawaiian earring. Indeed, recall that the Hawaiian earring H can be represented as inverse limit of a sequence

$$\cdots \leftarrow X_n \leftarrow X_{n+1} \leftarrow \cdots \leftarrow H,$$

where X_n is a wedge of n circles and the bonding maps are the obvious projections. For each n the fundamental group of X_n is F_n , the free group on n generators and there is the universal covering projection $q_n: \widehat{X}_n \rightarrow X_n = \widehat{X}_n/F_n$. The limit of the resulting inverse sequence of coverings is a fibration with unique path lifting property

$$q: \widehat{H} = \varprojlim \widehat{X}_n \rightarrow H = \varprojlim X_n.$$

By [5, Section 4] the group \widehat{F} acts freely and transitively on the fibres of q so we may consider the quotient of \widehat{H} with respect to any (normal) subgroup of \widehat{F} .

For each n we have a commutative diagram of the form

$$\begin{array}{ccccc} \widehat{H} & \longrightarrow & \widehat{X}_n & \longrightarrow & \widehat{X}_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \varprojlim (\widehat{X}_n/\text{Ker}_A(F_n)) & \longrightarrow & \widehat{X}_n/\text{Ker}_A(F_n) & \longrightarrow & \widehat{X}_{n-1}/\text{Ker}_A(F_{n-1}) \\ q' \downarrow & & \downarrow q'_n & & \downarrow q'_{n-1} \\ H & \longrightarrow & X_n & \longrightarrow & X_{n-1} \end{array}$$

where for each n the map q'_n is a covering projection with fibre $F_n/\text{Ker}_A(F_n)$.

Proposition 4.3. *If A is any nc-slender group, then the projection*

$$q_A: \widehat{H}/\text{Ker}_A(\widehat{F}) \rightarrow H$$

is the inverse limit of coverings $q'_n: \widehat{X}_n/\text{Ker}_A(F_n) \rightarrow X_n$, and is therefore a fibration with unique path-lifting property.

Proof. These coverings form an inverse sequence and by naturality we get a mapping to the inverse limit

$$\begin{array}{ccc}
 \widehat{H} / \text{Ker}_A(\widehat{F}) & \xrightarrow{f} & \varprojlim (\widehat{X}_n / \text{Ker}_A(F_n)) \\
 q_A \downarrow & & \downarrow q' \\
 H & \xlongequal{\quad\quad\quad} & \varprojlim X_n
 \end{array}$$

To prove our claim it is sufficient to show that the restriction of f to the fibres

$$f: \widehat{F} / \text{Ker}_A(\widehat{F}) \rightarrow \varprojlim (F_n / \text{Ker}_A(F_n))$$

is an isomorphism. Consider the following inverse sequence of short exact sequences

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \text{Ker}_A(F_1) & \longrightarrow & F_1 & \longrightarrow & F_1 / \text{Ker}_A(F_1) \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 1 & \longrightarrow & \text{Ker}_A(F_2) & \longrightarrow & F_2 & \longrightarrow & F_2 / \text{Ker}_A(F_2) \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 1 & \longrightarrow & \text{Ker}_A(F_3) & \longrightarrow & F_3 & \longrightarrow & F_3 / \text{Ker}_A(F_3) \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Each projection $F_{n+1} \rightarrow F_n$ is a split surjection, so by functoriality of Ker_A all bonding homomorphisms in the first column are surjective. By [16, Section 11.3] the resulting sequence of inverse limits is also exact. Moreover, by Lemma 4.2 $\varprojlim \text{Ker}_A(F_n) = \text{Ker}_A \widehat{F}$ so we have the following short exact sequence

$$1 \longrightarrow \text{Ker}_A(\widehat{F}) \longrightarrow \widehat{F} \longrightarrow \varprojlim (F_n / \text{Ker}_A(F_n)) \longrightarrow 1,$$

which implies that the projections $\widehat{F} \rightarrow F_n$ induce the isomorphism

$$\widehat{F} / \text{Ker}_A(\widehat{F}) \cong \varprojlim (F_n / \text{Ker}_A(F_n))$$

as claimed. \square

In order to prove the main results of this section we will need an algebraic lemma that is reminiscent of Theorem 3.7. Let G be a subgroup of the inverse limit of a sequence of groups $\{G_i\}$ such that all projections $G \rightarrow G_i$ are surjective. Furthermore, let H_i be a decreasing sequence of subgroups of G such that H_i contains the kernel of the homomorphism $G \rightarrow G_i$ and, for each i , let \overline{H}_i be the image of H_i in G_i . Observe that $G/H_i \cong G_i/\overline{H}_i$ (a bijection for arbitrary groups H_i , and a group isomorphism if H_i are normal subgroups of G). We can fit the above data in a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \bigcap H_i & \longrightarrow & G & \longrightarrow & \varprojlim G/H_i \longrightarrow \varprojlim^1 H_i \longrightarrow \bullet \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \varprojlim \overline{H}_i & \longrightarrow & \varprojlim G_i & \longrightarrow & \varprojlim G_i/\overline{H}_i \longrightarrow \varprojlim^1 \overline{H}_i \longrightarrow \bullet
 \end{array}$$

The following lemma is proved by elementary diagram chasing.

Lemma 4.4. $\text{Ker}(\bigcap H_i \rightarrow \varprojlim \overline{H}_i) = \text{Ker}(G \rightarrow \varprojlim G_i)$, thus we have an exact sequence

$$1 \longrightarrow \text{Ker}(G \rightarrow \varprojlim G_i) \longrightarrow \bigcap H_i \longrightarrow \varprojlim \overline{H}_i$$

Moreover $\varprojlim^1 H_i \rightarrow \varprojlim^1 \overline{H}_i$ is a surjection of pointed sets whose fibres can be naturally identified with the cokernel of the homomorphism

$$G \rightarrow (\varprojlim G_i)/(\varprojlim \overline{H}_i)$$

As a consequence, $\varprojlim^1 H_i$ is trivial if and only if $\varprojlim^1 \overline{H}_i$ is trivial and $G \rightarrow (\varprojlim G_i)/(\varprojlim \overline{H}_i)$ is surjective.

We have previously considered the inverse sequence of wedges of circles

$$\cdots \leftarrow X_n \leftarrow X_{n+1} \leftarrow \cdots \leftarrow H,$$

converging to the Hawaiian earring. Since $\pi_1(X_n) = F_n$, the free group on n generators F_n , the inverse sequence of spaces gives rise to a homomorphism from the Hawaiian earring group $\mathcal{G} := \pi_1(H)$ to the inverse limit of finite rank free groups $\widehat{F} := \varprojlim F_n$. By a result of Higman [18], see also [2] and [13], the homomorphism $\mathcal{G} \rightarrow \widehat{F}$ is injective, so the Hawaiian earring group can be viewed as a subgroup of \widehat{F} . It is known that both the group \mathcal{G} and its index in \widehat{F} are uncountable, and that \mathcal{G} can be viewed as a dense subgroup of \widehat{F} with respect to the inverse limit topology on \widehat{F} . The following theorem gives a more precise description of the relation between the two groups. The proof is a combination of algebraic and geometric techniques developed in the previous sections.

Theorem 4.5. *The group \widehat{F} is equal to the internal product of its subgroups \mathcal{G} and $\text{Ker}_{\mathbb{Z}}(\widehat{F})$, i.e., $\widehat{F} = \mathcal{G} \cdot \text{Ker}_{\mathbb{Z}}(\widehat{F})$.*

Proof. We are going to apply Lemma 4.4 to the following data: let G_i be the free group on i generators F_i , $G = \mathcal{G}$, and H_i the kernel of the epimorphism $\mathcal{G} \rightarrow \pi_1(X_i) \rightarrow H_1(X_i) \cong \mathbb{Z}^i$. Then $\varprojlim G_i = \widehat{F}$. Observe that the image of H_i in $\pi_1(X_i) = F_i$ is precisely the commutator subgroup $F'_i = \text{Ker}_{\mathbb{Z}}(F_i)$, since \mathcal{G} maps surjectively onto $\pi_1(X_i)$. The bonding maps in the inverse sequence of groups $\{\overline{H}_i\}$ are surjective, therefore $\varprojlim^1 \overline{H}_i$ is trivial. Moreover, by Lemma 4.2 $\varprojlim \overline{H}_i = \varprojlim \text{Ker}_{\mathbb{Z}}(F_i) = \text{Ker}_{\mathbb{Z}}(\widehat{F})$. Thus in this setting, Lemma 4.4 implies that if $\varprojlim^1 H_i$ is trivial, then the natural map from \mathcal{G} to $\widehat{F}/\text{Ker}_{\mathbb{Z}}(\widehat{F})$ is surjective and hence $\widehat{F} = \mathcal{G} \cdot \text{Ker}_{\mathbb{Z}}(\widehat{F})$. Thus to prove that $\widehat{F} = \mathcal{G} \cdot \text{Ker}_{\mathbb{Z}}(\widehat{F})$ it is sufficient to show that the derived inverse limit $\varprojlim^1 H_i$ is trivial.

It is classical that $\varprojlim^1 H_i$ is precisely the set of path-components of the inverse limit of covering maps that correspond to the sequence of kernels of the homomorphisms $\{\mathcal{G} \rightarrow H_1(X_i)\}$ (see [3, Theorem 2.1], [19, Theorem 2.1]). Analogously as in the examples in Section 3, we can represent this lifting projection as a limit of covering spaces over the approximations of the Hawaiian earrings by finite wedges of circles X_i . The covering space over X_i corresponding to the commutator subgroup F'_i can be identified with the integral 1-dimensional grid in \mathbb{R}^i , i.e.

$$\widetilde{X}_i = \{(x_1, \dots, x_i) \in \mathbb{R}^i \mid x_j \notin \mathbb{Z} \text{ for at most one index } j\}.$$

Observe that the bonding maps in the inverse system

$$\tilde{X}_1 \longleftarrow \tilde{X}_2 \longleftarrow \tilde{X}_3 \longleftarrow \dots$$

are retractions, and so their inverse limit \tilde{X} is the 1-dimensional integral grid in $\mathbb{R}^{\mathbb{N}}$. Alternatively, we may describe \tilde{X} by the following pullback diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathbb{R}^{\mathbb{N}} \\ \downarrow & & \downarrow e^{\mathbb{N}} \\ H & \hookrightarrow & (S^1)^{\mathbb{N}} \end{array}$$

Clearly, \tilde{X} is path-connected, which completes the proof of our claim. \square

Note that there one could also consider the commutator subgroup \widehat{F}' of \widehat{F} , which is however much smaller than $\varprojlim F'_n$. In fact, $\widehat{F} > \mathcal{G} \cdot \widehat{F}'$.

Continuing the previous line of thought we may ask whether \widehat{F} can be obtained by adding some other term of the derived series of \widehat{F} to the group \mathcal{G} . As before, we replace \widehat{F}'' with a suitable inverse limit group. We already mentioned that the second derived group of a free group F can be described as a kernel, $F'' = \text{Ker}_B(F)$ (see [8, Theorem 1]) for any solvable, deficiency 1 group B that is not virtually abelian. By work of Wilson [30], any solvable deficiency 1 group is isomorphic to a Baumslag-Solitar group $B(1, m)$ for some m . Since $B(1, m)$ is nc-slender (see [4]), Lemma 4.2 implies that $\text{Ker}_B(\widehat{F}) = \varprojlim \text{Ker}_B(F_n)$ for every group B that is solvable, of deficiency 1 and is not virtually abelian. As well, we have

$$\widehat{F}'' \leq \text{Ker}_B(\widehat{F}) \leq \text{Ker}_{\mathbb{Z}}(\widehat{F}).$$

In the next theorem we reverse the reasoning and use our methods to show that this subgroup of \widehat{F} is too small to generate, together with the Hawaiian earring group the entire group \widehat{F} .

Theorem 4.6. $\widehat{F} \neq \mathcal{G} \cdot \text{Ker}_B(\widehat{F})$ for every group B that is solvable, of deficiency 1 and is not virtually abelian.

Proof. Let B be a solvable deficiency 1 group that is not virtually abelian. Then B is isomorphic to a Baumslag-Solitar group $B(1, m)$ and, for a free group of rank n , we have $\text{Ker}_B(F_n) = F''_n = \text{Ker}_{\mathbb{Z}}(\text{Ker}_{\mathbb{Z}}(F_n))$. We may repeat almost verbatim the algebraic part of the proof of the previous theorem. We will use the same notation as in the proof of the previous theorem, except here H_i will be the kernel of homomorphisms $\mathcal{G} \rightarrow F_i/F''_i$ and we will consider \tilde{Y} , the inverse limit of covers of the Hawaiian earring corresponding to the subgroups H_i . As in the previous proof, by applying Lemma 4.4 we obtain that $\widehat{F} = \mathcal{G} \cdot \text{Ker}_B(\widehat{F})$ if and only if $\varprojlim^1 H_i$ is trivial. Again, $\varprojlim^1 H_i$ will be trivial if and only if the inverse limit \tilde{Y} is path-connected, (see [3, Theorem 2.1], [19, Theorem 2.1]). Thus, in order to prove our theorem, we must show that \tilde{Y} is not path-connected.

As before, we will describe \tilde{Y} in terms of a sequence of covers of a polyhedral expansion for H . Let \tilde{X} be the integral grid in $\mathbb{R}^{\mathbb{N}}$ and let $p: \tilde{X} \rightarrow H$ be the lifting projection obtained as the limit of the sequence

$$\begin{array}{ccccccc} \tilde{X}_1 & \longleftarrow & \tilde{X}_2 & \longleftarrow & \tilde{X}_3 & \longleftarrow & \dots & \longleftarrow & \tilde{X} \\ p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & & & & \downarrow p \\ X_1 & \longleftarrow & X_2 & \longleftarrow & X_3 & \longleftarrow & \dots & \longleftarrow & H \end{array}$$

where $\pi_1(\tilde{X}_n) = F'_n$. Let \tilde{Y}_n be a cover of \tilde{X}_n corresponding to the subgroup $(\pi_1(\tilde{X}_n))' = F''_n$ and let \tilde{Y} be the resulting inverse limit. We will construct a continuous surjection from \tilde{Y} to a path-disconnected

space. Since $\pi_1(\tilde{X}_2) = F'_2$ is a countably generated free group, we can find a sequence of homomorphisms $f_i : \pi_1(\tilde{X}_2) \rightarrow \mathbb{Z}$ such that

$$\bigcap_{i=1}^n \text{Ker}(f_i) \subsetneq \bigcap_{i=1}^{n-1} \text{Ker}(f_i) \quad \text{and} \quad \bigcap_{i=1}^{\infty} \text{Ker}(f_i) = F''_2.$$

Let \tilde{Z}'_n be the cover of \tilde{X}_2 corresponding to the subgroup $\bigcap_{i=1}^n \text{Ker}(\pi_1(\tilde{X}_2) \xrightarrow{f_i} \mathbb{Z})$. The covers \tilde{Z}'_n form an inverse sequence, whose limit $\tilde{Z}' := \varprojlim \tilde{Z}'_n$ is path-disconnected by Corollary 2.3. As a consequence, the pullback of \tilde{Z}' along the projection $p: \tilde{X} \rightarrow \tilde{X}_2$

$$\begin{array}{ccc} \tilde{Z}' & \longleftarrow & \tilde{Z} \\ \downarrow & & \downarrow \\ \tilde{X}_2 & \xleftarrow{p} & \tilde{X} \end{array}$$

yields a lifting projection $\tilde{Z} \rightarrow \tilde{X}$, whose total space is also path-disconnected. Note that \tilde{Z} can be alternatively obtained as inverse limit of a sequence $\tilde{Z} = \varprojlim \tilde{Z}_n$, where \tilde{Z}_n is the pull-back of \tilde{Z}'_n along the projection $p_n: \tilde{X}_n \rightarrow \tilde{X}_2$. Since commutator subgroups are characteristic we have $p_{n*}(\pi_1(\tilde{Y}_n)) \leq \pi_1(\tilde{Y}_2) \leq \pi_1(\tilde{Z}')$, thus $\pi_1(\tilde{Y}_n) \leq \pi_1(\tilde{Z}_n) = p_n^{-1}(\pi_1(\tilde{Z}'))$. It follows that \tilde{Y}_n covers \tilde{Z}_n for every n and we obtain the following diagram:

$$\begin{array}{ccccccccccc} \tilde{Y}_2 & \longleftarrow & \tilde{Y}_3 & \longleftarrow & \cdots & \longleftarrow & \tilde{Y}_{n-1} & \longleftarrow & \tilde{Y}_n & \longleftarrow & \tilde{Y}_{n+1} & \longleftarrow & \cdots & \longleftarrow & \tilde{Y} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \tilde{Z}_2 & \longleftarrow & \tilde{Z}_3 & \longleftarrow & \cdots & \longleftarrow & \tilde{Z}_{n-1} & \longleftarrow & \tilde{Z}_n & \longleftarrow & \tilde{Z}_{n+1} & \longleftarrow & \cdots & \longleftarrow & \tilde{Z} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \tilde{X}_2 & \longleftarrow & \tilde{X}_3 & \longleftarrow & \cdots & \longleftarrow & \tilde{X}_{n-1} & \longleftarrow & \tilde{X}_n & \longleftarrow & \tilde{X}_{n+1} & \longleftarrow & \cdots & \longleftarrow & \tilde{X} \end{array}$$

By construction, the fibre of the lifting projection $\tilde{Y} \rightarrow \tilde{X}$ maps surjectively to the fibre of $\tilde{Z} \rightarrow \tilde{X}$. Since every point in \tilde{Z} (respectively \tilde{Y}) is connected by a path to a point in the fibre, it follows, that the projection $\tilde{Y} \rightarrow \tilde{Z}$ is surjective, therefore \tilde{Y} is not path-connected. \square

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