On a Conjecture of Ira Gessel

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Abstract

Let $F(m;\, n_1, n_2)$ denote the number of lattice walks from $(0,0)$ to (n_1, n_2), always staying in the first quadrant $\{(n_1, n_2);\ n_1 \ge 0, n_2 \ge 0\}$ and having exactly m steps, each of which belongs to the set $\{E = (1,0), W = (-1,0), NE = (1,1), SW = (-1,-1)\}$. Ira Gessel conjectured that $F(2n;\, 0,0) = 16^n \frac{(1/2)_n (5/6)_n}{(2)_n (5/3)_n}$. We pose similar conjectures for some other values of (n_1, n_2), and give closed-form formulas for $F(n_1;\, n_1, n_2)$ when $n_1 \ge n_2$ as well as for $F(2n_2 - n_1;\, n_1, n_2)$ when $n_1 \le n_2$. In the main part of the paper, we derive a functional equation satisfied by the generating function of $F(m;\, n_1, n_2)$, use the kernel method to turn it into an infinite lower-triangular system of linear equations satisfied by the values of $F(m;\, n_1, 0)$ and $F(m;\, 0, n_2) + F(m;\, 0, n_2 - 1)$, and express these values explicitly as determinants of lower-Hessenberg matrices with unit superdiagonals whose non-zero entries are products of two binomial coefficients.

1 Introduction

Let $F(m;\, n_1, n_2)$ denote the number of lattice walks from $(0,0)$ to (n_1, n_2), always staying in the first quadrant $\{(n_1, n_2);\ n_1 \ge 0, n_2 \ge 0\}$ and having exactly m steps, each of which belongs to the set $\{E = (1,0), W = (-1,0), NE = (1,1), SW = (-1,-1)\}$. From the obvious recurrence

$$\begin{aligned} F(m;\, n_1, n_2) &= F(m-1;\, n_1+1, n_2) + F(m-1;\, n_1-1, n_2) \\ &+ F(m-1;\, n_1+1, n_2+1) + F(m-1;\, n_1-1, n_2-1) \end{aligned} \tag{1}$$

valid for $m \ge 1, n_1, n_2 \ge 0$, the initial conditions

$$F(0;\, n_1, n_2) = \begin{cases} 1, & n_1 = n_2 = 0, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

(so called because we'll think of m as being the time variable), and the boundary conditions

$$F(m;\, n_1, n_2) = 0, \quad \text{for } n_1 < 0 \text{ or } n_2 < 0, \tag{3}$$

we can calculate many values of $F(m;\, n_1, n_2)$. For example, the sequence $F(2n;\, 0,0)_{n=0}^{\infty}$ of the numbers of lattice walks returning to $(0,0)$ after $2n$ steps starts out as

1, 2, 11, 85, 782, 8004, 88044, 1020162, 12294260, 152787976, 1946310467, 25302036071, 334560525538, 4488007049900, 60955295750460, 836838395382645, 11597595644244186, ...

Based on such empirical evidence, Ira Gessel conjectures (cf. [1, p. 4]) that

$$F(2n;\,0,0) = 16^n \frac{(1/2)_n (5/6)_n}{(2)_n (5/3)_n}. \tag{4}$$

Therefore we will call the numbers $F(2n;\,0,0)$ *Gessel numbers*.

2 Similar conjectures

Analogous conjectures can be made about other points. For example, we conjecture that

$$F(2n;\,0,1) \;=\; 16^n \frac{(1/2)_n}{(3)_n}\left(\frac{5}{27}\frac{(7/6)_n}{(7/3)_n} + \frac{(111n^2+183n-50)}{270}\frac{(5/6)_n}{(8/3)_n}\right).$$

More generally, looking at the points $(0,k)$ it seems that

$$F(2n;\,0,k) \;=\; 16^n \frac{(1/2)_n}{(k+2)_n}\left(\frac{(7/6)_n}{((3k+4)/3)_n} p_k(n) + \frac{(5/6)_n}{((3k+5)/3)_n} q_k(n)\right)$$

where $p_k(n)$ is a polynomial of degree $2k-2$ and $q_k(n)$ is a polynomial of degree $2k$. These two polynomial sequences seem to be non-holonomic.

At $(2n+2k;\,0,n)$ we seem to have

$$\begin{aligned}
F(2n;\,0,n) &= 4^n \tfrac{(3/2)_n}{(2n+1)(2)_n} = \tfrac{4^n (1/2)_n}{(2)_n},\\
F(2n+2;\,0,n) &= \tfrac{2^{2n+1}(n+1)(3/2)_n}{(3)_n},\\
F(2n+4;\,0,n) &= \tfrac{4^n (n+1)\left(8n^2+32n+33\right)(3/2)_n}{3\,(4)_n},\\
F(2n+6;\,0,n) &= \tfrac{4^{n-1}(n+1)\left(64n^4+672n^3+2648n^2+4641n+3060\right)(3/2)_n}{9\,(5)_n}, \text{ etc.,}
\end{aligned}$$

from which we conjecture that

$$F(2n+2k;\,0,n) \;=\; 4^n \frac{(3/2)_n}{(k+2)_n}\, r_k(n) \tag{5}$$

where $r_k(n)$ is a polynomial of degree $2k-1$ divisible by $n+1$ for $k \geq 1$, and $r_0(n) = 1/(2n+1)$. It seems that this polynomial sequence is non-holonomic.

Another empirical observation is that $g(n) = F(2n+1;\,1,0)$ likely satisfies the second-order recurrence

$$\begin{aligned}
&(n+3)(3n+7)(3n+8)\;\; g(n+1)\\
-\;&8\,(2n+3)(18n^2+54n+35)\;\; g(n)\\
+\;&256\,n\,(3n+1)(3n+2)\;\; g(n-1) \;=\; 0.
\end{aligned} \tag{6}$$

According to algorithm *Hyper*, this recurrence has no hypergeometric solutions. But $F(2n;\,2,0)$ seems to be non-holonomic.

On the other hand, we seem to have

$$\begin{aligned}
F(n;\,n,0) &= 1,\\
F(n+2;\,n,0) &= \tfrac{1}{2}(n+1)(n+4),\\
F(n+4;\,n,0) &= \tfrac{1}{12}(n+1)\left(n^3+15n^2+74n+132\right),\\
F(n+6;\,n,0) &= \tfrac{1}{144}(n+1)\left(n^5+32n^4+407n^3+2620n^2+8604n+12240\right), \text{ etc.,}
\end{aligned}$$

from which we conjecture that

$$F(n+2k;\, n, 0) \;=\; s_k(n) \tag{7}$$

where $s_k(n)$ is a polynomial of degree $2k$ with leading coefficient $\frac{1}{k!(k+1)!}$, which is divisible by $n+1$ when $k \ge 1$. This polynomial sequence seems to be non-holonomic. While $s_k(0) = F(2k; 0, 0)$ is hypergeometric by Gessel's conjecture, and $s_k(1)$ seems to be holonomic of order two as per (6), the sequences $s_k(2)$, $s_k(3)$, ... all seem to be non-holonomic. On the other hand, the coefficient sequences $[n^{2k}]s_k(n)$, $[n^{2k-1}]s_k(n)$, $[n^{2k-2}]s_k(n)$, ... all seem to be hypergeometric, again harboring a polynomial sequence of increasing degrees.

3 Some values of the numbers $F(m; n_1, n_2)$

Proposition 1 *$F(m; n_1, n_2) \neq 0$ only if*

(i) $m \equiv n_1 \pmod 2$,

(ii) $n_1 \le m$,

(iii) $n_2 \le \frac{1}{2}(n_1 + m)$.

Proof: As $F(m; n_1, n_2) \neq 0$, there exists a walk w from $(0, 0)$ to (n_1, n_2) having m steps. Assume that out of these m steps, a, b, c resp. d are E, W, NE resp. SW steps. Then

$$\begin{aligned} a+b+c+d &= m, \\ a-b+c-d &= n_1, \\ c-d &= n_2, \end{aligned}$$

hence $2a + 2c = m + n_1$, so $m \equiv n_1 \pmod 2$. Also, $n_1 = a - b + c - d \le a + b + c + d = m$, and $n_2 = c - d \le a + c = (m + n_1)/2$. □

Theorem 1 *Let $s(n_1, n_2)$ be the length of a shortest walk w from $(0, 0)$ to (n_1, n_2).*

(i) *If $n_1 \ge n_2$ then w uses E, NE steps only, and*

$$F(n_1;\, n_1, n_2) \;=\; s(n_1, n_2) \;=\; \binom{n_1}{n_2}.$$

(ii) *If $n_1 \le n_2$ then w uses W, NE steps only, and*

$$F(2n_2 - n_1;\, n_1, n_2) \;=\; s(n_1, n_2) \;=\; \frac{n_1 + 1}{2n_2 - n_1 + 1}\binom{2n_2 - n_1 + 1}{n_2 + 1}.$$

Proof: Denote by a, b, c resp. d the numbers of E, W, NE resp. SW steps in w, and by m the length of w.

(i) $n_1 \ge n_2$: By Proposition 1(ii), $m \ge n_1$. The walk consisting of n_2 NE steps followed by $n_1 - n_2$ E steps ends at (n_1, n_2) and has length n_1, so it is the shortest such walk. Hence $s(n_1, n_2) = F(n_1; n_1, n_2)$. From $a + b + c + d = m = n_1 = a - b + c - d$ it follows that $b = d = 0$, so w uses E and NE steps only. From $a + c = m = n_1$ and $c = n_2$ it follows that $a = n_1 - n_2$. Thus w must contain $n_1 - n_2$ E steps and n_2 NE steps. There is no restriction on the order of these steps, so there are $\binom{n_1}{n_2}$ such walks.

(ii) $n_1 \le n_2$: By Proposition 1(iii), $m \ge 2n_2 - n_1$. The walk consisting of n_2 NE steps followed by $n_2 - n_1$ W steps ends at (n_1, n_2) and has length $2n_2 - n_1$, so it is the shortest such walk. Hence $s(n_1, n_2) = F(2n_2 - n_1;\, n_1, n_2)$. In the inequality $m = a+b+c+d \ge -a+b+c-d = 2(c-d) - (a-b+c-d) = 2n_2 - n_1$ equality holds when $a = d = 0$, so w uses W and NE steps only. Conversely, any walk w using W and NE steps only satisfies $-b + c = n_1$ and $c = n_2$, so $b = n_2 - n_1$ and $m = b + c = 2n_2 - n_1$, implying that w is a shortest such walk. Thus to compute the number $s(n_1, n_2)$ of these walks it suffices to enumerate lattice walks from $(0,0)$ to (n_1, n_2) staying in the octant $0 \le n_1 \le n_2$ and using W and NE steps only. We have the recurrence

$$s(n_1, n_2) \;=\; s(n_1 + 1, n_2) + s(n_1 - 1, n_2 - 1) \quad \text{for } n_2 \ge n_1 + 1 \ge 1$$

with boundary conditions

$$\begin{aligned} s(-1, n_2) &= 0, \quad \text{for } n_2 \ge 0, \\ s(n_1, n_1) &= 1, \quad \text{for } n_1 \ge 0. \end{aligned}$$

To this end, we transform the octant $0 \le n_1 \le n_2$ into the first quadrant by the linear map

$$\mu:\; (1,1) \mapsto (1,0),\; (0,1) \mapsto (0,1).$$

The matrix corresponding to

$$\mu^{-1}:\; (1,0) \mapsto (1,1),\; (0,1) \mapsto (0,1)$$

in the standard basis of $\mathbb{R}^2$ is

$$M^{-1} \;=\; \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

so the matrix corresponding to μ is

$$M \;=\; \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Writing $\mathbf{n} = (n_1, n_2)$, define

$$u(n_1, n_2) \;=\; s(M^{-1}\mathbf{n}) = s(n_1, n_1 + n_2).$$

Then $s(n_1, n_2) = u(M\mathbf{n}) = u(n_1, n_2 - n_1)$, and u satisfies the recurrence

$$\begin{aligned} u(n_1, n_2) &= s(n_1, n_1 + n_2) \\ &= s(n_1 + 1, n_1 + n_2) + s(n_1 - 1, n_1 + n_2 - 1) \\ &= u(n_1 + 1, n_2 - 1) + u(n_1 - 1, n_2), \end{aligned} \tag{8}$$

for $n_1 \ge 0 \wedge n_2 \ge 1$, with boundary conditions

$$u(-1, n_2) \;=\; s(-1, n_2 - 1) \;=\; 0, \quad \text{for } n_2 \ge 1, \tag{9}$$

$$u(n_1, 0) \;=\; s(n_1, n_1) \;=\; 1, \quad \text{for } n_1 \ge 0. \tag{10}$$

Let

$$f(x,y) \;=\; \sum_{n_1,n_2\ge 0} u(n_1,n_2)x^{n_1}y^{n_2}$$

be the generating function of u. From (8) – (10) we obtain in the usual way the functional equation

$$(x - x^2 - y)f(x,y) \;=\; x - yf(0,y) \tag{11}$$

which can be solved by the kernel method. Since

$$x^2 - x + y \;=\; \left(x - \frac{1+\sqrt{1-4y}}{2}\right)\left(x - \frac{1-\sqrt{1-4y}}{2}\right),$$

substituting $x = x(y) = \frac{1-\sqrt{1-4y}}{2}$ in (11) yields

$$f(0,y) \;=\; \frac{x(y)}{y} \;=\; \frac{1-\sqrt{1-4y}}{2y} \;=\; C(y),$$

the generating function of Catalan numbers. Hence

$$f(x,y) \;=\; \frac{x - y\,C(y)}{x - x^2 - y} \;=\; -\frac{1}{x - \frac{1+\sqrt{1-4y}}{2}} \;=\; \frac{C(y)}{1 - x\,C(y)}.$$

Following [2, p. 154], this can be expanded into

$$f(x,y) \;=\; \sum_{n_1=0}^{\infty} x^{n_1}C(y)^{n_1+1} \;=\; \sum_{n_1,n_2\ge 0} \frac{n_1+1}{2n_2+n_1+1}\binom{2n_2+n_1+1}{n_2}x^{n_1}y^{n_2},$$

so we read off

$$u(n_1,n_2) \;=\; \frac{n_1+1}{2n_2+n_1+1}\binom{2n_2+n_1+1}{n_2}$$

and, finally,

$$F(2n_2-n_1;\, n_1,n_2) \;=\; s(n_1,n_2) \;=\; u(n_1,n_2-n_1) \;=\; \frac{n_1+1}{2n_2-n_1+1}\binom{2n_2-n_1+1}{n_2-n_1}.$$

□

Corollary 1 *For all $n \ge 0$, we have*

(i) $F(n;\, n,0) = 1$,

(ii) $F(2n;\, 0,n) = C_n$, *the n-th Catalan number.*

Proof: By Theorem 1(i), $F(n;\, n,0) = \binom{n}{0} = 1$. By Theorem 1(ii),

$$F(2n;\, 0,n) = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

□

4 A functional equation for the generating function

Let

$$G(x,y,z) \;=\; \sum_{m,n_1,n_2\ge 0} F(m;\,n_1,n_2)\,x^m y^{n_1} z^{n_2} \tag{12}$$

be the generating function of the numbers $F(m;\,n_1,n_2)$. From (1) – (3) we obtain in the usual way the following functional equation satisfied by $G(x,y,z)$:

$$K(x,y,z)G(x,y,z) \;=\; x(1+z)\,G(x,0,z) + x\,G(x,y,0) - x\,G(x,0,0) - y\,z. \tag{13}$$

Here the polynomial

$$K(x,y,z) \;=\; x(1+z)(1+y^2z) - y\,z \tag{14}$$

is called the *kernel* of equation (13).

In order to simplify (13), we introduce another generating function

$$H(x,y,z) \;=\; K(x,y,z)G(x,y,z) + y\,z.$$

Since

$$\begin{aligned} H(x,0,z) &= x(1+z)\,G(x,0,z), && (15)\\ H(x,y,0) &= x\,G(x,y,0), && (16)\\ H(x,0,0) &= x\,G(x,0,0), \end{aligned}$$

equation (13) becomes

$$H(x,y,z) \;=\; H(x,0,z) + H(x,y,0) - H(x,0,0). \tag{17}$$

This is the functional equation that we will work with in the sequel. Write

$$H(x,y,z) \;=\; \sum_{m,n_1,n_2\ge 0} \widetilde{F}(m;\,n_1,n_2)\,x^m y^{n_1} z^{n_2}.$$

It follows from (17), (16) and (15) that

$$\begin{aligned} \widetilde{F}(m;\,n_1,n_2) &= 0 \qquad \text{if } n_1n_2 \ne 0,\\ \widetilde{F}(m;\,n_1,0) &= F(m-1;\,n_1,0),\\ \widetilde{F}(m;\,0,n_2) &= F(m-1;\,0,n_2) + F(m-1;\,0,n_2-1). \end{aligned}$$

Note that $F(2n;\,0,0) = \widetilde{F}(2n+1;\,0,0)$, and Gessel's conjecture (4) can be stated as

$$G(x,0,0) \;=\; \frac{H(x,0,0)}{x} \;=\; {}_3F_2\left(\begin{matrix} 5/6,\,1/2,\,1 \\ 2,\,5/3 \end{matrix} \;\middle|\; 16x^2\right) \;=\; \frac{{}_2F_1\left(\begin{matrix} -1/2,\,-1/6 \\ 2/3 \end{matrix} \;\middle|\; 16x^2\right) - 1}{2x^2}.$$

In analogy to (5) we conjecture that

$$\widetilde{F}(2n+2k+1;\,0,n) \;=\; 4^n\,\frac{(1/2)_n}{(k+2)_n}\,\widetilde{r}_k(n)$$

where $\widetilde{r}_k(n)$ is a polynomial of degree $2k+1$, and the sequence of polynomials $\widetilde{r}_k(n)$ is not holonomic.

5 The kernel method

Equations (13) resp. (17) cannot be solved right away because they seem to contain other unknown functions beside the full generating functions $G(x,y,z)$ resp. $H(x,y,z)$ (the additional unknowns being just sections of $G(x,y,z)$ resp. $H(x,y,z)$, of course). To obtain more information, we look for roots of the kernel w.r.t. one of the variables which are power series in the remaining variables. Substituting such roots into (13) resp. (17) yields additional equations which are free of the term containing the full generating function $G(x,y,z)$ resp. $H(x,y,z)$.

In our case, the kernel (14) is linear in x, and quadratic in y and z. The roots of $K(x,y,z)=0$ w.r.t. y resp. z are not power series in x,z resp. x,y. But solving $K(x,y,z)=0$ for x yields

$$\begin{aligned} x \;=\; x(y,z) \;&=\; \frac{yz}{(1+z)(1+y^2z)} \;=\; \frac{y}{1-y}\left(\frac{z}{1+z}-\frac{yz}{1+yz}\right) \\ &=\; \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{n+1}y^k z^n \end{aligned} \tag{18}$$

which is a power series in y,z satisfying $x(0,0)=0$. So we can substitute it into (17) to obtain

$$H(x(y,z),0,z)+H(x(y,z),y,0)-H(x(y,z),0,0) \;=\; y\,z \tag{19}$$

where the rational function $x(y,z)$ is given in (18). This does not help us find $H(x,y,z)$ (or a non-trivial section of it) directly. However, (19) does determine all the coefficients of $H(x,y,0)$ and $H(x,0,z)$, and hence also of $H(x,y,z)$ and $G(x,y,z)$. How could we exploit this?

6 Gessel numbers as determinants

Expand the left-hand side of (19) into power series in y and z, and equate the coefficient of $y^u z^v$ to 0 (except for the coefficient of $y\,z$ which is equated to 1). This yields the following infinite system of linear equations for the values of $\widetilde{F}(m;\,n_1,n_2)$ on the planes $n_1=0$ and $n_2=0$:

$$\begin{aligned} &\sum_{\substack{m,n_2\ge 0\\ m\equiv u\ (\mathrm{mod}\ 2)}}\binom{-m}{\frac{u-m}{2}}\binom{-m}{v-n_2-\frac{u+m}{2}}\widetilde{F}(m;\,0,n_2)\\ &\quad+\sum_{\substack{m\ge 0,n_1\ge 1\\ m+n_1\equiv u\ (\mathrm{mod}\ 2)}}\binom{-m}{\frac{u-m-n_1}{2}}\binom{-m}{v-\frac{u+m-n_1}{2}}\widetilde{F}(m;\,n_1,0)\\ &\quad=\begin{cases}1, & u=v=1,\\ 0, & \text{otherwise,}\end{cases}\qquad \text{for all } u,v\ge 0. \end{aligned} \tag{20}$$

Note that both sums are finite since a binomial coefficient vanishes when its lower symbol is negative. So $m\le u$ and $n_2\le v-u/2$ in the first sum, and $m+n_1\le u$ in the second. To put this system into a more compact form, pack its unknowns into an infinite matrix $[f(i,j)]_{i,j=0}^{\infty}$ defined by

$$f(i,j) \;=\; \begin{cases}\widetilde{F}(i;\,0,j-i), & i\le j,\\ \widetilde{F}(j;\,i-j,0), & i\ge j,\end{cases} \tag{21}$$

or graphically,

$$
f = \begin{bmatrix}
\widetilde{F}(0;0,0) & \widetilde{F}(0;0,1) & \widetilde{F}(0;0,2) & \widetilde{F}(0;0,3) & \widetilde{F}(0;0,4) & \widetilde{F}(0;0,5) & \cdots \\
\widetilde{F}(0;1,0) & \widetilde{F}(1;0,0) & \widetilde{F}(1;0,1) & \widetilde{F}(1;0,2) & \widetilde{F}(1;0,3) & \widetilde{F}(1;0,4) & \cdots \\
\widetilde{F}(0;2,0) & \widetilde{F}(1;1,0) & \widetilde{F}(2;0,0) & \widetilde{F}(2;0,1) & \widetilde{F}(2;0,2) & \widetilde{F}(2;0,3) & \cdots \\
\widetilde{F}(0;3,0) & \widetilde{F}(1;2,0) & \widetilde{F}(2;1,0) & \widetilde{F}(3;0,0) & \widetilde{F}(3;0,1) & \widetilde{F}(3;0,2) & \cdots \\
\widetilde{F}(0;4,0) & \widetilde{F}(1;3,0) & \widetilde{F}(2;2,0) & \widetilde{F}(3;1,0) & \widetilde{F}(4;0,0) & \widetilde{F}(4;0,1) & \cdots \\
\widetilde{F}(0;5,0) & \widetilde{F}(1;4,0) & \widetilde{F}(2;3,0) & \widetilde{F}(3;2,0) & \widetilde{F}(4;1,0) & \widetilde{F}(5;0,0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 5 & 11 & 19 & 10 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 9 & 37 & 85 & 158 & 103 & 35 & 5 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 14 & 87 & 332 & 782 & 1521 & 1126 & 499 & 126 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 20 & 172 & 911 & 3343 & 8004 & 16056 & 12941 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 27 & 305 & 2096 & 10147 & 36350 & 88044 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

Inverting this transformation, we clearly have

$$
\begin{aligned}
\widetilde{F}(m;0,n_2) &= f(m, m+n_2), \\
\widetilde{F}(m;n_1,0) &= f(m+n_1, m).
\end{aligned}
$$

Using this in (20), together with the change of variables $i = m$, $j = m + n_2$ in the first sum, and $i = m + n_1$, $j = m$ in the second, the left-hand side of (20) changes into

$$
\sum_{\substack{j \geq i \geq 0 \\ i \equiv u \ (\mathrm{mod}\ 2)}} \binom{-i}{\frac{u-i}{2}} \binom{-i}{v - j - \frac{u-i}{2}} f(i,j) \; + \sum_{\substack{i \geq j+1 \geq 1 \\ i \equiv u \ (\mathrm{mod}\ 2)}} \binom{-j}{\frac{u-i}{2}} \binom{-j}{v - j - \frac{u-i}{2}} f(i,j)
$$

which allows us to combine the two sums into a single one, and so to rewrite (20) as

$$
\sum_{\substack{i,j \geq 0 \\ i \equiv u \ (\mathrm{mod}\ 2)}} \binom{-\min\{i,j\}}{\frac{u-i}{2}} \binom{-\min\{i,j\}}{v - j - \frac{u-i}{2}} f(i,j) \;=\; \begin{cases} 1, & u = v = 1, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for all } u, v \geq 0. \tag{22}
$$

Denote the above equation by $E(u,v)$, and let $c(u,v,i,j)$ be the coefficient of $f(i,j)$ in $E(u,v)$:

$$
c(u,v,i,j) \;=\; \begin{cases} \dbinom{-\min\{i,j\}}{\frac{u-i}{2}} \dbinom{-\min\{i,j\}}{v - j - \frac{u-i}{2}}, & i \equiv u \ (\mathrm{mod}\ 2), \\ 0, & \text{otherwise}. \end{cases} \tag{23}
$$

Proposition 2 *Let $u, v, i, j \geq 0$. Then*

(i) $c(u, v, u, v) = 1$,

(ii) $c(u, v, i, j) = 0$ *if* $i > u$ *or* $j > v$.

Proof: Assertion (i) is clear from (23). To prove (ii), assume that $i > u$ or $j > v$. If $i > u$ then $(u-i)/2 < 0$. Otherwise $i \leq u$. Then, by assumption, $j > v$, and so $v - j - (u-i)/2 < 0$. In either case, $c(u, v, i, j) = 0$. □

Proposition 2 implies that we can compute $f(u, v)$ from $E(u, v)$, provided that we have already computed $f(i, j)$ for all $(i, j) \neq (u, v)$ such that $i \leq u$ and $j \leq v$. In other words, the system (22) is a linear recurrence from which all the $f(i, j)$ can be computed one by one, in any order compatible with the standard componentwise partial order on $\mathbb{N} \times \mathbb{N}$. Nevertheless, we'll continue to regard (22) as an infinite system of linear equations, and will rewrite it in the form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A}$ is an infinite matrix and $\boldsymbol{x}, \boldsymbol{b}$ are infinite vectors. Then we can rephrase Proposition 2 as follows:

Corollary 2 *Let $\rho : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a monotonic bijection in the sense that $\rho(a, b) \leq \rho(c, d)$ whenever $a \leq c \ \wedge \ b \leq d$. Define $\boldsymbol{A} = [a(n, k)]_{n,k=0}^{\infty}$, $\boldsymbol{x} = [x(k)]_{k=0}^{\infty}$ and $\boldsymbol{b} = [b(n)]_{n=0}^{\infty}$ by*

$$\begin{aligned} a(n,k) &= c(u,v,i,j), \\ x(k) &= f(i,j), \\ b(n) &= \begin{cases} 1, & u = v = 1, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{24}$$

where c resp. f is given by (23) resp. (21), $(u, v) = \rho^{-1}(n)$, and $(i, j) = \rho^{-1}(k)$. Then

(i) $\boldsymbol{A}$ *is lower-triangular with unit diagonal,*

(ii) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Proof:

(i) By Proposition 2(i), $a(n, n) = c(u, v, u, v) = 1$, proving that $\boldsymbol{A}$ has unit diagonal. Now assume that $n < k$. Then $\rho(u, v) < \rho(i, j)$, hence by monotonicity of ρ, $i > u$ or $j > v$. By Proposition 2(ii), $a(n, k) = c(u, v, i, j) = 0$, proving that $\boldsymbol{A}$ is lower-triangular.

(ii) Let $n \in \mathbb{N}$ be arbitrary, and $(u, v) = \rho^{-1}(n)$. Then by (i), the sum $\sum_{k=0}^{\infty} a(n, k)\, x(k)$ exists, and by (23) and (22),

$$\sum_{k=0}^{\infty} a(n,k)\, x(k) = \sum_{i,j \geq 0} c(u,v,i,j)\, f(i,j) = \begin{cases} 1, & u = v = 1, \\ 0, & \text{otherwise}, \end{cases} = b(n),$$

proving that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. □

We can compute a particular component $x(k)$ of the solution vector $\boldsymbol{x}$ from the finite lower-triangular system with unit diagonal

$$\boldsymbol{A}^{(k)}\boldsymbol{x}^{(k)} = \boldsymbol{b}^{(k)}$$

where

$$
\begin{aligned}
\boldsymbol{A}^{(k)} &= [a(i,j)]_{i,j=0}^{k}\,, \\
\boldsymbol{x}^{(k)} &= (x(j))_{j=0}^{k}\,, \\
\boldsymbol{b}^{(k)} &= (b(i))_{i=0}^{k}\,.
\end{aligned}
$$

By Cramer's rule,

$$
x(k) \;=\; \frac{\det \widetilde{\boldsymbol{A}}^{(k)}}{\det \boldsymbol{A}^{(k)}} \;=\; \det \widetilde{\boldsymbol{A}}^{(k)}
$$

where $\widetilde{\boldsymbol{A}}^{(k)}$ is obtained from $\boldsymbol{A}^{(k)}$ by replacing its last column with $\boldsymbol{b}^{(k)}$.

If $k < \rho(1,1)$ then $\boldsymbol{b}^{(k)} = \boldsymbol{0}$ and $x(k) = 0$. If $k \ge \rho(1,1)$ then the last column of $\widetilde{\boldsymbol{A}}^{(k)}$ has a single non-zero entry, 1, at position $\rho(1,1)$. Developing $\det \widetilde{\boldsymbol{A}}^{(k)}$ w.r.t. this column yields

$$
x(k) \;=\; \det \widetilde{\boldsymbol{A}}^{(k)} \;=\; \det \boldsymbol{T}^{(k)} \det \boldsymbol{H}^{(k)} \;=\; \det \boldsymbol{H}^{(k)}
$$

where $\boldsymbol{T}^{(k)}$ is a $\rho(1,1) \times \rho(1,1)$ lower-triangular matrix with unit diagonal, and $\boldsymbol{H}^{(k)}$ is the $(k - \rho(1,1)) \times (k - \rho(1,1))$ lower-Hessenberg matrix with unit superdiagonal, composed of the elements in rows $\rho(1,1)$ to k and columns $\rho(1,1) - 1$ to $k - 1$ of $\boldsymbol{A}^{(k)}$. Thus for Gessel numbers we have

$$
F(2n; 0,0) \;=\; \widetilde{F}(2n+1; 0,0) \;=\; f(2n+1, 2n+1) \;=\; x(\rho(2n+1, 2n+1)) \;=\; \det \boldsymbol{H}^{(\rho(2n+1,2n+1))}
$$

when $n \ge 0$. For example, if we use diagonal ordering to pack the unknown $f(i,j)$ into the vector $\boldsymbol{x}$, then

$$
\rho(i,j) \;=\; \binom{i+j+1}{2} + j,
$$

so $\rho(1,1) = 4$ and $\rho(3,3) = 24$. Hence, if $n = 1$, $F(2,0,0) = \det \boldsymbol{H}^{(24)} = 2$ where

$$
\boldsymbol{H}^{(24)} \;=\; \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

7 Gessel numbers as multiple sums

For any (finite or infinite) lower-triangular matrix $\boldsymbol{A} = [a(k,m)]_{k,m\geq 0}$ with unit diagonal entries, the lower-triangle elements of its inverse $\boldsymbol{A}^{-1} = [\bar{a}(k,m)]_{k,m\geq 0}$ are given by

$$\bar{a}(k,m) \;=\; \sum_{j=1}^{k-m}(-1)^j \sum_{m=\lambda_0<\lambda_1<\cdots<\lambda_j=k} \prod_{i=1}^{j} a(\lambda_i,\lambda_{i-1}) \tag{25}$$

when $k>m$. Therefore for any ordering ρ as described in Corollary 2,

$$F(2n;0,0) \;=\; \bar{a}(\rho(2n+1,2n+1),\rho(1,1))$$

where $\bar{a}, a$, and c are given in (25), (24), and (23), respectively.

8 The solution vector

Here we describe a few properties of the solution vector $\mathbf{x}^{(n)}$.

We'll think of the $n^2\times 1$ vector $\mathbf{x}^{(n)}$ as consisting of the concatenation of $\lfloor n^2/(2n+1)\rfloor$ vectors $\mathbf{u}_i$ $(i=1,2,\ldots)$, each of length $2n+1$, plus one more, of length $n^2 \bmod 2n+1$. Each of these vectors $\mathbf{u}_i$, except possibly the last, consists first of a certain universal sequence of length $2i$, followed by $(2n+1-2i)$ 0's. The last one consists of as much of the next universal sequence as there is room for. As n increases these vectors remain unchanged, and a new one appears at the end. Each of these universal sequences ends in a Catalan number.

The first several of these universal sequences are

```
1, 1
1, 2, 3, 1
1, 5, 11, 19, 10, 2
1, 9, 37, 85, 158, 103, 35, 5
1, 14, 87, 332, 782, 1521, 1126, 499, 126, 14
1, 20, 172, 911, 3343, 8004, 16056, 12941, 6765, 2296, 462, 42
1, 27, 305, 2096, 10147, 36350, 88044, 180621, 154750,
    90681, 37178, 10254, 1716, 132
1, 35, 501, 4300, 25927, 118472, 417565, 1020162,
    2128824, 1910006, 1217523, 570409, 193137, 44913, 6435, 429
```

The sequence of next-to-last members of the above is also holonomic, but the third-from-last sequence might or might not be.

Can we find these sequences by dealing only with the corresponding sections of the matrix?

References

[1] M. Kauers, D. Zeilberger, The quasi-holonomic ansatz and restricted lattice walks, to appear in *J. Difference Equations and Applications.*

[2] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, Inc., New York-London-Sydney 1968.