A Note on Enumeration of One-Vertex Maps*

Alen Orbanić   Alen.Orbanic@fmf.uni-lj.si,
Marko Petkovšek   Marko.Petkovsek@fmf.uni-lj.si,
Tomaž Pisanski† Tomaz.Pisanski@fmf.uni-lj.si, and
Primož Potočnik   Primoz.Potocnik@fmf.uni-lj.si

Institute of Mathematics, Physics and Mechanics
and
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19, 1000 Ljubljana, SLOVENIJA

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Abstract

The number of distinct maps with a single vertex and valence d is computed for any value of d. The question of what precisely is meant by a map can be answered in several different ways. We enumerate 16 different types of objects where each type is obtained by selecting whether the object is a pre-map or a map, is oriented or general, whether the underlying graph or pre-graph is signed or unsigned, is directed or undirected.

1 Introduction

It may seem that enumerating one-vertex maps looks rather sterile. On the other hand, a natural approach to vertex-transitive maps, such as Cayley maps (see Malnič, Nedela and Škoviera [17] and Richter, Širáň, Jajcay, Tucker and Watkins [21]), is to take the quotient by the automorphism group, or a transitive subgroup of the automorphism group, which automatically gives a one-vertex map. Thus any understanding of vertex-transitive maps depends on an understanding of one-vertex maps. Furthermore, taking quotients by group actions leads to a variety of "degenerate" structures, like semi-edges, which might not arise from an intuitive idea of a map and which have not been treated consistently and uniformly in the study of maps. The purpose of this paper is to provide a general setting for maps allowing such degenerate structures from the outset, and then using that setting to analyse and enumerate one-vertex maps.

We should make it clear that the theory of combinatorial maps has a rich history. Unfortunately, there were several equivalent, but not identical, approaches to maps introduced in the past. In particular, treatment of oriented maps (on orientable surfaces) is much simpler than the treatment of general maps that may result in orientable or non-orientable surfaces. When considering maps as graphs, embedded in surfaces, the approach by Heffter [11], later formalized by Stahl [26], with the so-called generalized embedding scheme for general maps (orientable or not) is quite useful. The reader is referred to the three classical textbooks in topological graph theory; see [22, 7, 31]. Unfortunately, this approach was not developed for pre-maps, i.e., maps with underlying pre-graphs admitting semi-edges. An alternative way of presenting maps is via specifying three involutions [6].

Let us also point out that the field of enumeration of maps, even one-vertex maps, is very rich and well-established. Certain problems of enumerating isomorphism classes of one-vertex maps were addressed by Gross, Robbins and Tucker in [8], by Cori and Marcus in [4], by Stoimenow in [28], by Khruzin in [13][1], by Kwak and Shim in [14], and by Chapuy in [2]. An approach using advanced techniques, such as moduli spaces, was suggested in a well-known work by Harer and Zagier [10], where they compute the number of one-vertex maps, not only according to the number of edges but according to the number of edges AND the number of faces (i.e., the genus of the surface). We would like to make it clear that maps in this paper are not being counted by genus; the formulæ give the total number of maps of orientable genus g over all possible g. In the case of pre-maps, we also count the maps on surfaces with boundary. In the case of general maps, the enumeration also applies to the non-orientable genus.

Also, there are now several bijective interpretations of the enumerative formulæ found by Harer and Zagier, in particular by Goulden and Nica [5]. The reader is referred to the work of Liskovets such as [15] where he develops techniques similar to ours. Standard counting techniques to enumerate simple structures (like dissections of a polygon) up to symmetry can be found, for instance, in the book by Harary and Palmer [9]. Similarly, check the work of Walsh and Lehman [29] and Robinson [23].

In this article, we consider three distinct boolean parameters giving rise to eight enumeration problems:

First, we distinguish between maps and pre-maps. This parameter is needed since we are mostly interested in counting pre-maps, but topologically maps are more natural objects. Note that maps correspond to Cayley graphs with no involutory generators.

The second parameter is the choice of directed vs. undirected loops. In our setting this corresponds to the choice whether we consider Cayley graphs as directed or not, or equivalently, whether we select in each pair consisting of a generator and its inverse one element as the "preferred one".

Our final parameter is the possibility to give a sign to some loops. Such a sign defines a "twist" in the regular neighborhood of the corresponding non-involutory generator of the Cayley graph, thereby extending the definition of the Cayley map to non-orientable surfaces. Note that the actual Cayley map may still reside on an orientable surface (if the order of the generator is even, and some other conditions are met; see for instance [7].) The fact that the graph

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[1]Note that the value given in [13, p. 8] for $d_{11} = \delta_{\bar{S}\bar{D}\bar{P}}(22)$ is incorrect.

is signed or not is not a big problem in enumeration of one-vertex maps. It just consists in giving one of two possible colors (say red or blue) to each loop. This makes it equivalent with enumeration of loop-bicolored (oriented!) maps.

We introduce a structure that we call a pre-graph; see also [17]. A *pre-graph* G is a quadruple $G = (V, A, i, r)$ where V is the set of vertices, A is the set of *arcs* (also known as semi-edges, darts, sides, ...), i is the *initial mapping* $i : A \to V$ specifying the origin or initial vertex for each arc, while r is the *reversal involution*: $r : A \to A$, $r^2 = 1$. We may also define the *terminal mapping* $t : A \to V$ as $t(s) := i(r(s))$, specifying the terminal vertex for each arc. An arc s forms an edge $e = \{s, r(s)\}$, which is called *proper* if $|e| = 2$ and is called a *half-edge* if $|e| = 1$. Define $\partial(e) = \{i(s), t(s)\}$. A pre-graph without half-edges is called a *(general) graph*. Note that G is a graph if and only if the reversal involution has no fixed points. A proper edge e with $|\partial(e)| = 1$ is called a *loop* and two edges e, e' are *parallel* if $\partial(e) = \partial(e')$. A graph without loops and parallel edges is called *simple*. The *valence* of a vertex v is defined as $val(v) = |\{s \in A | i(s) = v\}|$. All pre-graphs in this note are connected unless stated otherwise.

Topologically, an oriented map is a 2-cell embedding of a graph into an oriented closed surface. However, in this paper we will operate with the following combinatorial description. For us an *oriented pre-map* is a triple (A, r, R) where A is a finite non-empty set of *darts* and r, R are permutations on A such that $r^2 = 1$ and $\langle r, R \rangle$ acts transitively on A (see [17], but the description goes back to Cori [3]). Note that the vertices of the pre-map correspond to the cycles in the cyclic decomposition of R. An isomorphism between two pre-maps (A, r, R) and (A', r', R') is any bijection $\pi : A \to A'$ for which $\pi R = R' \pi$ and $\pi r = r' \pi$ holds. An automorphism of a map (A, r, R) is thus a permutation of A which commutes with both R and r. If r is fixed-point free, the structure is called an oriented *map* rather than a pre-map. Combinatorial oriented maps correspond to topological maps on oriented closed surfaces while oriented pre-maps correspond to topological maps on surfaces with boundary (with a natural orbifold structure). To each oriented pre-map $M = (A, r, R)$ we may associate a unique map with reversed orientation $M^{-1} = (A, r, R^{-1})$. Each oriented map gives rise to the automorphism group $\mathrm{Aut}_0(M)$ of all (orientation-preserving) automorphisms, which can be extended to a larger group $\mathrm{Aut}(M)$ by allowing the orientation-reversing automorphisms, i.e., isomorphisms from (A, r, R) to (A, r, R^{-1}). Note that the index of $\mathrm{Aut}_0(M)$ in $\mathrm{Aut}(M)$ is either 1 or 2. If it is 1 then M is *chiral*.

In addition to oriented maps we will also consider general (possibly non-orientable) (pre-)maps, which can be defined as $M = (A, r, R, \lambda)$, where A, r and R are as in the definition of oriented maps, and λ is a *sign mapping* assigning either 1 or −1 to each proper edge of the underlying pre-graph of the map M. Recall that each cycle $C = (s_1, \ldots, s_k)$ of R corresponds to a vertex of the map. Substituting the cycle C in R with its reverse cycle and inverting the λ-value of proper edges underlying the darts $s_1, \ldots, s_k$ other than loops, results in a new map M', which is said to be obtained from M by a *local orientation change*. Two maps $M_1 = (A_1, r_1, R_1, \lambda_1)$ and $M_2 = (A_2, r_2, R_2, \lambda_2)$ are *isomorphic* if there exists a map $M' = (A_1, r_1, R', \lambda')$ obtained from M_1 by a series of local orientation changes, and a bijection $\pi : A_1 \to A_2$, such that $\pi r_1 = r_2 \pi$, $\pi R' = R_2 \pi$ and $\lambda' = \lambda_2 \pi$. One of the early definitions of general maps goes back to Tutte [30], his definition is equivalent to ours.

Note that in the topological interpretation a general pre-map resides in a surface that may be orientable or non-orientable. There is a well-known condition for testing the orientability of this surface. Namely the underlying surface is non-orientable if and only if there is a cycle with the product of signs λ along its edges equal to -1.

There is a well-known relationship between oriented maps and maps that reside on orientable surfaces, namely each pair of oriented maps (A, r, R) and (A, r, R^{-1}) gives rise to the same map (A, r, R, λ) where λ is the constant 1, while each orientable map admits a representation of such form and gives rise to two oriented maps: (A, r, R) and (A, r, R^{-1}).

In case of embeddings of directed graphs or pre-graphs we may speak, respectively, of *directed maps* or *directed pre-maps*.

We refer the reader to [20] for further details on the theory of maps and embeddings of graphs and pre-graphs into surfaces. For the theory of Cayley maps, see for instance [21]. An overview of a combinatorial approach to coverings of graphs and pre-graphs was given in [19].

The main combinatorial tool that we use in this paper is the classical orbit-counting lemma of Cauchy and Frobenius, which can also be found in Burnside's book [1].

Lemma 1.1. Let α be an action of a finite group G on a finite set A, $\sim_\alpha$ the associated equivalence relation on A, $|A/\!\sim_\alpha|$ the number of orbits of α, and $\mathrm{Fix}_\alpha(g)$ the set of elements of A fixed by $g \in G$ under α. Then

$$|A/\!\sim_\alpha| \;=\; \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}_\alpha(g)|.$$

For a proof, see, e.g., [27, Lemma 7.24.5].

Denote by $\gamma_\tau(d)$ the number of non-isomorphic one-vertex *oriented* (pre-)maps of type τ and valence d, and by $\delta_\tau(d)$ the number of non-isomorphic one-vertex (*orientable or non-orientable*) (pre-)maps of type τ and valence d. The types of (pre-)maps that we consider are $\bar{S}\bar{D}\bar{P}$, $S\bar{D}\bar{P}$, $\bar{S}\bar{D}P$, $\bar{S}D\bar{P}$, $SD\bar{P}$, $S\bar{D}P$, $\bar{S}DP$, and SDP, indicating whether the underlying (pre-)graphs are signed or unsigned (S resp. $\bar{S}$), directed or undirected (D resp. $\bar{D}$), pre-graphs or graphs (P resp. $\bar{P}$). We write $\tau = \tau_1\tau_2\tau_3$ in order to be able to refer to the individual symbols which constitute the type τ. As it turns out, $\gamma_\tau(d)$, $\delta_\tau(d)$ and the various auxiliary functions can be written in the same general form for all τ, but with different values of parameters (see Eqn. (6) for $\gamma_\tau(d)$, Eqns. (7), (5), (9), and (8) for $\delta_\tau(d)$, and Table 1 for the values of parameters).

Each oriented one-vertex pre-map is isomorphic to one of the form (A, r, R) where $A = \{1, 2, \ldots, d\}$, $R = (1, 2, \ldots, d)$. Such a pre-map can be represented by a matching in the complete graph K_d (possibly an empty one) in which two vertices $i, j \in \{1, 2, \ldots, d\}$ are matched whenever $r(i) = j$. Hence the number of all one-vertex pre-maps (A, r, R) with a given rotation R is the same as the number of all matchings (including the empty one) in K_d. This number is easily computed to be

$$i(d) = \sum_{0 \le 2k \le d} \binom{d}{2k} (2k-1)!!.$$

Of course, many of the above pre-maps are isomorphic. To compute the number of non-isomorphic ones, let I_d denote the set of all permutations r

on A with $r^2 = 1$, and recall that two oriented pre-maps $M = (A, r, R)$ and $M' = (A, r', R)$ are isomorphic if and only if there exists a permutation on A which centralizes (= commutes with) R and conjugates r to r'. Note that the permutations of A that centralize R (the centralizer of R) form the cyclic group C_n, generated by R. However, if orientable maps are considered as general maps that reside on orientable surfaces, then the maps (A, r, R) and (A, r, R^{-1}) are isomorphic. In this case, an automorphism of the map is any permutation of A which either commutes with R, or maps R to R^{-1}. Such permutations form the dihedral group D_d of order $2d$, therefore the number of orientable non-isomorphic one-vertex pre-maps of valence d equals the number of orbits of D_d in its action on the set I_d by conjugation. If the pre-maps are represented by the matchings in K_d, as described above, then the action of D_d on I_d corresponds to the natural action of D_d on the set of all matchings in K_d. In general, the number $\delta_\tau(d)$ of non-isomorphic one-vertex pre-maps of type τ and valence d equals the number of orbits of D_d in its natural action on the set of matchings of type τ in K_d. By Lemma 1.1 we thus have:

$$\gamma_\tau(d) = \frac{1}{|C_d|} \sum_{\sigma \in C_d} |\mathrm{Fix}_\tau(\sigma)|, \tag{1}$$

$$\delta_\tau(d) = \frac{1}{|D_d|} \sum_{\sigma \in D_d} |\mathrm{Fix}_\tau(\sigma)| \tag{2}$$

where $\mathrm{Fix}_\tau(\sigma)$ denotes the set of matchings of type τ in K_d fixed by σ.

In all our formulæ we use the convention that $0^0 = 1$, and define the double-factorial function recursively: $n!! = n \times (n-2)!!$ for $n \geq 1$, $0!! = (-1)!! = 1$.

2 Cyclic symmetry: Enumeration of oriented maps and pre-maps

Before we consider maps and pre-maps, we present simple formulæ for counting one-vertex pre-graphs and graphs.

Let $p(d)$ denote the number of one-vertex pre-graphs of valence d. Since each of them is determined by the number of loops, $p(d) = \lfloor d/2 \rfloor + 1$. This gives rise to the generating function $P(x) = 1/((1-x)^2(1+x)) = 1/((1-x^2)(1-x))$.

If there are no pending edges, the situation becomes much simpler. Let $g(d)$ denote the number of one-vertex graphs. Then $g(d) = 0$, for d odd, and $g(d) = 1$, for d even. The corresponding generating function $G(x)$ is $G(x) = 1/(1-x^2)$.

When extending our enumeration exercise we note immediately that the number of one-vertex digraphs is the same as the number of one-vertex graphs. Also, the number of one-vertex directed pre-graphs is the same as the number of one-vertex pre-graphs. Similarly, the number of one-vertex signed digraphs is the same as the number of one-vertex signed graphs, and the number of one-vertex signed directed pre-graphs is the same as the number of one-vertex signed pre-graphs. It is not hard to verify that the number of one-vertex signed graphs of valence d is 0 if d is odd and $d/2+1$ if d is even. The corresponding generating function is given by $1/(1-x^2)^2$. The number of one-vertex signed pre-graphs is given by $1 + 2 + \cdots + (\lfloor d/2 \rfloor + 1) = (\lfloor d/2 \rfloor + 1)(\lfloor d/2 \rfloor + 2)/2$. with the corresponding generating function $1/((1-x)^3(1+x)^2) = 1/((1-x^2)^2(1-x))$.

As argued in the Introduction, the number $\gamma_\tau(d)$ of oriented non-isomorphic one-vertex pre-maps of type τ and valence d equals the number of orbits of the cyclic group C_d in its natural action on the set of matchings of type τ in the complete graph K_d, represented as a regular d-gon with diagonals. By (1),

$$\gamma_\tau(d) = \frac{R_\tau(d)}{d} \qquad (3)$$

where $R_\tau(d)$ is the sum over all $\sigma \in C_d$ of the numbers of matchings of type τ in K_d which are fixed by σ. To compute this number, assume that $\sigma \in C_d$ is the counter-clockwise rotation by $2\pi k_\sigma/d$ where $k_\sigma \in \mathbb{Z}$ is such that $0 \leq k_\sigma < d$. In how many ways can one construct a matching M of K_d which is fixed by σ?

Let $r = \gcd(d, k_\sigma)$. Then σ has r orbits in $V(K_d)$, each containing d/r vertices. Let C denote a set of r consecutive vertices of K_d. Since C contains one representative from each orbit, it suffices to define M on C, and to extend it to $V(K_d) \setminus C$ by symmetry. Hence we can also think of M as a matching of orbits. Assume that $2j$ of the r orbits are matched in pairs, while the rest remain unmatched or are matched with themselves (the latter is possible only if antipodal vertices belong to the same orbit, i.e., if d is even and r divides $d/2$). There are $\binom{r}{2j}$ ways to select the $2j$ orbits, and $(2j-1)!!$ ways to group them into pairs. In each of the j pairs of orbits (α_i, β_i), $i = 1, 2, \ldots, j$, the vertex in $\alpha_i \cap C$ can be matched with any of the d/r vertices in β_i in t_τ ways, where the value of t_τ depends on the type τ of the problem considered, and is shown in Table 1. Each of the remaining $r - 2j$ orbits can be matched to themselves (or

τ	$\bar{S}\bar{D}\bar{P}$	$S\bar{D}\bar{P}$	$\bar{S}D\bar{P}$	$SD\bar{P}$	$\bar{S}\bar{D}P$	$S\bar{D}P$	$\bar{S}DP$	SDP
s_τ	1	2	0	0	2	3	1	1
t_τ	1	2	2	4	1	2	2	4
m_τ	1	2	2	4	2	3	3	5

Table 1: The values of s_τ, t_τ, m_τ for the types of (pre-)maps considered

be left unmatched) in $w_\tau(r, d)$ ways where

$$w_\tau(r, d) = \begin{cases} s_\tau, & r \mid (d/2), \\ 0, & r \nmid (d/2) \text{ and } \tau_3 = \bar{P}, \\ 1, & r \nmid (d/2) \text{ and } \tau_3 = P, \end{cases} \qquad (4)$$

and the value of s_τ is given in Table 1. For each divisor r of d, there are $\varphi(d/r)$ rotations σ in D_d having $\gcd(d, k_\sigma) = r$ where φ denotes Euler's totient function. Hence the total contribution $R_\tau(d)$ of the d rotations is

$$R_\tau(d) = \sum_{r \mid d} \varphi\left(\frac{d}{r}\right) \sum_{0 \leq 2j \leq r} \binom{r}{2j} (2j-1)!! \left(\frac{t_\tau d}{r}\right)^j w_\tau(r, d)^{r-2j}, \qquad (5)$$

and so by (3), we find that the number $\gamma_\tau(d)$ of oriented non-isomorphic one-vertex pre-maps resp. maps of type τ and valence d is

$$\gamma_\tau(d) = \frac{1}{d} \sum_{r \mid d} \varphi\left(\frac{d}{r}\right) \sum_{0 \leq 2j \leq r} \binom{r}{2j} (2j-1)!! \left(\frac{t_\tau d}{r}\right)^j w_\tau(r, d)^{r-2j} \qquad (6)$$

where t_τ is given in Table 1 and $w_\tau(r, d)$ is shown in (4). In Tables 2 and 3 we list the numbers $\gamma_\tau(d)$ for small values of d.

d	$\gamma_{\bar{S}\bar{D}P}(d)$	$\gamma_{S\bar{D}P}(d)$	$\gamma_{\bar{S}DP}(d)$	$\gamma_{SDP}(d)$
1	1	1	1	1
2	2	3	2	3
3	2	3	3	5
4	5	11	8	21
5	6	17	17	57
6	18	69	60	299
7	34	187	187	1213
8	108	791	754	7189
9	294	2981	2981	36537
10	984	13575	13398	239027
11	3246	60851	60851	1412661
12	11810	301937	300982	10056109
13	43732	1513393	1513393	66657385
14	171218	8115273	8109584	510071643
15	689996	44293443	44293443	3711168173
16	2889970	254424929	254389122	30257794301
17	12458784	1490057537	1490057537	238049716833
18	55415816	9095634067	9095390802	2053738059275
19	253142182	56612205667	56612205667	17281059677029
20	1187979372	364845156207	364843436636	156892675044645

Table 2: The numbers of oriented non-isomorphic one-vertex pre-maps

d	$\gamma_{\bar{S}\bar{D}\bar{P}}(d)$	$\gamma_{S\bar{D}\bar{P}}(d)$	$\gamma_{\bar{S}D\bar{P}}(d)$	$\gamma_{SD\bar{P}}(d)$
2	1	2	1	2
4	2	6	4	14
6	5	28	22	164
8	18	234	218	3388
10	105	3112	3028	96776
12	902	55876	55540	3548876
14	9749	1237648	1235526	158146572
16	127072	32444680	32434108	8302721384
18	1915951	980254100	980179566	501851753292
20	32743182	33522615160	33522177088	34326661275896
22	624999093	1279939143784	1279935820810	2621308560998420
24	13176573910	53970651042864	53970628896500	221063688757926232

Table 3: The numbers of oriented non-isomorphic one-vertex maps

3 Dihedral symmetry: Enumeration of (non-) orientable maps and pre-maps

As argued in the Introduction, the number $\delta_\tau(d)$ of non-isomorphic one-vertex pre-maps of type τ and valence d equals the number of orbits of the dihedral group D_d in its natural action on the set of matchings of type τ in the complete graph K_d, represented as a regular d-gon with diagonals. By (2),

$$\delta_\tau(d) \;=\; \frac{R_\tau(d) + F_\tau(d)}{2d} \tag{7}$$

where $R_\tau(d)$ is the sum over all rotations $\sigma \in D_d$, and $F_\tau(d)$ the sum over all reflections $\sigma \in D_d$ of the numbers of matchings of type τ in K_d fixed by σ.

Since we have already computed $R_\tau(d)$ in (5), it only remains to compute $F_\tau(d)$. We distinguish two cases according to the parity of d.

If d is even there are two types of reflections of the regular d-gon: either across a median or across a main diagonal. Let σ be the reflection across a median, and let L denote the set of $n := d/2$ vertices of K_d on one side of the median. Denote $f_\tau(n) := |\mathrm{Fix}_\tau(\sigma)|$. For each $u \in L$, denote by u' its mirror image across the median. In a matching M which is fixed by σ, u can be either

- left unmatched, or

- matched with u', or

- matched with some $v \in L \setminus \{u\}$, provided that u' is matched with v', or

- matched with v' for some $v \in L \setminus \{u\}$, provided that u' is matched with v.

Denote by s_τ the number of ways in which u can be matched with u' (including leaving it unmatched), and by t_τ the number of ways in which u can be matched with $v \in L \setminus \{u\}$. Because of symmetry, the number of ways in which u can be matched with v' where $v \in L \setminus \{u\}$ is also t_τ. The values of s_τ and t_τ depend on the type τ of the problem considered, and are shown in Table 1.

Thus to construct all possible such matchings, select $2j$ vertices from among the n vertices in L, then construct a perfect matching on these $2j$ vertices. This can be done in $\binom{n}{2j}(2j-1)!!$ ways. There are $2t_\tau$ ways to match the two elements in each of the j pairs, yielding a factor of $(2t_\tau)^j$, and s_τ ways to match each of the remaining $n - 2j$ vertices to its mirror image, yielding a factor of s_τ^{n-2j}. Hence

$$f_\tau(n) \;=\; \sum_{0 \leq 2j \leq n} s_\tau^{n-2j}(2t_\tau)^j \binom{n}{2j}(2j-1)!! \;=\; n! \sum_{0 \leq 2j \leq n} \frac{s_\tau^{n-2j} t_\tau^j}{(n-2j)! j!}. \tag{8}$$

If σ is the reflection across a main diagonal, then $|\mathrm{Fix}_\tau(\sigma)| = m_\tau f_\tau(d/2-1)$ where m_τ is the number of ways in which it is possible to match the two vertices on the mirror with each other. The value of m_τ is shown in Table 1.

If d is odd there is only one type of reflections, namely across the bisector of a side of the d-gon, and $|\mathrm{Fix}_\tau(\sigma)| = f_\tau((d-1)/2)$ for pre-maps and 0 for

maps. So the total contribution $F_\tau(d)$ of the d reflections to the sum in (2) is

$$F_\tau(d) = \begin{cases} \dfrac{d}{2}\left(f_\tau\left(\dfrac{d}{2}\right) + m_\tau\, f_\tau\left(\dfrac{d}{2}-1\right)\right), & d \text{ even}, \\[2ex] d\, f_\tau\left(\dfrac{d-1}{2}\right), & d \text{ odd and } \tau_3 = P, \\[2ex] 0, & d \text{ odd and } \tau_3 = \bar{P}, \end{cases} \qquad (9)$$

where f_τ is given by (8), and m_τ is shown in Table 1.

In summary, the number $\delta_\tau(d)$ of non-isomorphic one-vertex pre-maps resp. maps of type τ and valence d is given by (7), where $R_\tau(d)$, $F_\tau(d)$, and $f_\tau(n)$ are given in (5), (9), and (8), respectively. In Tables 4 and 5 we list the numbers $\delta_\tau(d)$ for small values of d. For instance, $\delta_{\bar{S}\bar{D}P}(5) = 6$, so there are six pairwise non-isomorphic one-vertex undirected orientable pre-maps of valence five (which, incidentally, are discussed in detail in [24]).

4 Concluding remarks

4.1 Additional formulæ

In the special case when $w_\tau(r, d) = 0$ for all r, the double sum in (5) reduces to a single sum. For d even we thus have

$$R_{\bar{S}D\bar{P}}(d) = \sum_{r|d,\ r\text{ even}} \varphi\left(\frac{d}{r}\right)(r-1)!!\left(\frac{2d}{r}\right)^{r/2},$$

$$R_{SD\bar{P}}(d) = \sum_{r|d,\ r\text{ even}} \varphi\left(\frac{d}{r}\right)(r-1)!!\left(\frac{4d}{r}\right)^{r/2}.$$

It is not difficult to see that the sequence $f_\tau(n)$ from (8) satisfies the recurrence

$$f_\tau(n) = s_\tau\, f_\tau(n-1) + 2t_\tau\,(n-1)f_\tau(n-2), \quad \text{for } n \geq 2, \qquad (10)$$

with $f_\tau(0) = 1$, $f_\tau(1) = s_\tau$, and that its exponential generating function equals

$$\sum_{n=0}^{\infty} f_\tau(n)\frac{x^n}{n!} = \exp(s_\tau\, x + t_\tau\, x^2). \qquad (11)$$

Comparing this with the well-known exponential generating function of Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(t)\frac{x^n}{n!} = \exp(2tx - x^2),$$

we see that

$$f_\tau(n) = \left(i\sqrt{t_\tau}\right)^n H_n\left(\frac{s_\tau}{2i\sqrt{t_\tau}}\right) \qquad (12)$$

where $i^2 = -1$ and $H_n(t)$ is the n-th Hermite polynomial. In the special case when $s_\tau = 0$, the sum in (8) has a single term. Thus, for n even,

$$f_{\bar{S}D\bar{P}}(n) = 2^n(n-1)!!,$$
$$f_{SD\bar{P}}(n) = (2\sqrt{2})^n(n-1)!!.$$

d	$\delta_{\bar{S}\bar{D}P}(d)$	$\delta_{S\bar{D}P}(d)$	$\delta_{\bar{S}DP}(d)$	$\delta_{SDP}(d)$
1	1	1	1	1
2	2	3	2	3
3	2	3	2	3
4	5	11	6	14
5	6	15	11	33
6	17	60	37	167
7	27	125	100	619
8	83	529	405	3686
9	185	1663	1527	18389
10	608	7557	6824	120075
11	1779	31447	30566	706851
12	6407	155758	151137	5032026
13	22558	763211	757567	33334033
14	87929	4089438	4058219	255064335
15	348254	22190781	22150964	1855614411
16	1456341	127435846	127215233	15129137658
17	6245592	745343353	745057385	119025187809
18	27766356	4549465739	4547820514	1026870988199
19	126655587	28308456491	28306267210	8640532108675
20	594304478	182435301597	182422562168	78446356190934

Table 4: The numbers of non-isomorphic one-vertex pre-maps

d	$\delta_{\bar{S}\bar{D}\bar{P}}(d)$	$\delta_{S\bar{D}\bar{P}}(d)$	$\delta_{\bar{S}D\bar{P}}(d)$	$\delta_{SD\bar{P}}(d)$
2	1	2	1	2
4	2	6	3	9
6	5	26	13	90
8	17	173	121	1742
10	79	1844	1538	48580
12	554	29570	28010	1776358
14	5283	628680	618243	79080966
16	65346	16286084	16223774	4151468212
18	966156	490560202	490103223	250926306726
20	16411700	16764409276	16761330464	17163338379388
22	312700297	639992710196	639968394245	1310654311464970
24	6589356711	26985505589784	26985325092730	110531845060209836

Table 5: The numbers of non-isomorphic one-vertex maps

4.2 Identification of sequences in OEIS

Some of the sequences encountered in this paper can be found in the *The Online Encyclopedia of Integer Sequences* [25], to wit:

- $\langle \delta_{\bar{S}\bar{D}\bar{P}}(2n) \rangle_{n=1}^{\infty}$ is sequence A054499 in [25],

- $\langle \gamma_{\bar{S}\bar{D}\bar{P}}(2n) \rangle_{n=1}^{\infty}$ is sequence A007769 in [25],

- $\langle f_{\bar{S}\bar{D}\bar{P}}(n) \rangle_{n=0}^{\infty}$ is sequence A047974 in [25],

- $\langle f_{\bar{S}\bar{D}P}(n) \rangle_{n=0}^{\infty}$ is sequence A000898 in [25],

- $\langle f_{\bar{S}DP}(n) \rangle_{n=0}^{\infty}$ is sequence A115329 in [25],

- $\langle f_{\bar{S}D\bar{P}}(2n-2) \rangle_{n=1}^{\infty}$ is sequence A052714 in [25],

- $\langle f_{SD\bar{P}}(2n-2) \rangle_{n=1}^{\infty}$ is sequence A052734 in [25].

4.3 Possible extensions

Using the methods of [18] one can easily extend the counting to graphs with one-vertex connected components. For a more recent example of such methods, see also [16]. Motivated by [14], it would be worthwhile to extend this analysis to dipoles or any two-vertex graphs or pre-graphs.

Acknowledgements

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