Pascal-like Determinants are Recursive

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Abstract

Let $P = [p_{i,j}]_{i,j\geq 0}$ be an infinite matrix whose entries satisfy $p_{i,j} = \mu p_{i,j-1} + \lambda p_{i-1,j} + \nu p_{i-1,j-1}$ for $i, j \geq 1$, and whose first column resp. row satisfy linear recurrences with constant coefficients of orders ρ resp. σ. Then we show that its principal minors d_n satisfy $d_n = \sum_{j=1}^{\delta} c_j \omega^{jn} d_{n-j}$ where c_j are constants, $\omega = \lambda\mu + \nu$, and $\delta = \binom{\rho+\sigma-2}{\rho-1}$. This implies a recent conjecture of Bacher [2].

1 Introduction

The problem of symbolic evaluation of determinants has rightly received considerable attention in the literature (see [5] for an excellent survey). Nevertheless, we are still lacking a uniform mechanism for determinant evaluation akin to the well-known Zeilberger's algorithm for evaluating hypergeometric sums [9]. Given the summand, Zeilberger's algorithm constructs a recurrence (w.r.t. a parameter) which is satisfied by the sum in question. This recurrence is very useful as it can often be used to evaluate the sum in closed form, or to prove its properties by induction.

There have been attempts to emulate this approach in the determinant calculus, notably by using Dodgson's recurrence satisfied by the minors of a matrix (see [10] and [1]). However, this recurrence is nonlinear, and furthermore, once unfolded, it involves *all* the contiguous minors of the starting matrix, which is unfortunate if only some of them (e.g., the principal ones) have a nice evaluation.

For matrices defined by simple linear recurrences with constant coefficients, R. Bacher has recently conjectured ([2, Conjecture 3.3]; see also [6]) that the sequence of their principal minors satisfies such a recurrence as well. More precisely, let $P = [p_{i,j}]_{i,j\geq 0}$ be an infinite matrix with the following properties:

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1. The elements of the first column $(p_{i,0})_{i\geq 0}$ satisfy a homogeneous linear recurrence with constant coefficients of order at most ρ, and the elements of the first row $(p_{0,j})_{j\geq 0}$ satisfy a homogeneous linear recurrence with constant coefficients of order at most σ.

2. The remaining elements of P satisfy *Pascal's rule* $p_{i,j} = p_{i-1,j} + p_{i,j-1}$.

Let d_n be the principal minor of P consisting of the elements in its first $n+1$ rows and columns. Then Bacher conjectured that *the sequence* $(d_n)_{n\geq 0}$ *satisfies a homogeneous linear recurrence with constant coefficients whose order is at most* $\binom{\rho+\sigma-2}{\rho-1}$. In [2], he proved this conjecture in the case $\rho = \sigma = 2$. In the same paper, Bacher conjectured some evaluations of determinants whose entries satisfy the modified Pascal's rule

$$p_{i,j} = p_{i-1,j} + p_{i,j-1} + xp_{i-1,j-1}.$$

These conjectured evaluations were later proved by Krattenthaler [6].

In this paper we prove a generalization of Bacher's conjecture on principal minors of Pascal-like matrices. We allow that the matrix entries are defined recursively by

$$p_{i,j} = \mu p_{i,j-1} + \lambda p_{i-1,j} + \nu p_{i-1,j-1}.$$

We show that when the elements of the first row and column (except, maybe, a few starting elements) satisfy homogeneous linear recurrence equations with constant coefficients then the sequence of the principal minors

1. is eventually zero if $\lambda\mu + \nu = 0$,

2. satisfies a homogeneous linear recurrence with constant coefficients if $\lambda\mu+\nu = 1$ (which implies Bacher's conjecture where $\mu = \lambda = 1, \nu = 0$),

3. satisfies a homogeneous linear recurrence with exponential coefficients if $\lambda\mu+\nu \neq 0, 1$.

In the latter case, a simple substitution converts the recurrence into one with constant coefficients, so in all three cases it is possible to obtain a closed-form evaluation of the n-th principal minor.

Our proof, like Bacher's in the case $\rho = \sigma = 2$, proceeds in two main steps: first we transform our Pascal-like matrix by determinant-preserving transformations to band-diagonal form. Then we show that the principal minors of every band-diagonal matrix satisfy a nontrivial linear recurrence. To avoid explicit summations in matrix products, we use generating functions. In Section 2 we show how to compute the generating function of a matrix product from the generating functions of its factors (Theorem 1), and characterize Pascal-like matrices through the type of their generating function. We also define (r, s)-*banded matrices* as band-diagonal matrices of bandwidth $r + s + 1$, with r subdiagonals and s superdiagonals. In Section 3 we show that the principal minors of any (r, s)-banded matrix M satisfy a homogeneous linear recurrence of order at most $\binom{r+s}{r}$ (Theorem 2). If M is constant along diagonals this recurrence has constant coefficients. In Section 4 we transform a Pascal-like matrix to a $(\rho-1, \sigma-1)$-banded matrix while preserving the values of its principal minors (Theorem 3). Finally, in Section 5, we give our main result (Corollary 4) which implies an algorithm for evaluation of Pascal-like determinants in closed form (Remark 3), and present several examples of its use, and of applications of our methods to some other determinant evaluations.

2 Preliminaries

We work with infinite matrices of the form $A = [a_{i,j}]_{i,j\geq 0}$ and their generating functions $F_A(x,y) = \sum_{i,j\geq 0} a_{i,j}x^i y^j$, regarded as formal power series in x and y. First we show how to compute the generating function of a matrix product from the generating functions of its factors.

Definition 1 *Let $f(u,v) = \sum_{i,j\geq 0} a_{i,j}u^i v^j$ be a formal power series in u and v. Its* diagonal *is the univariate series* $\operatorname{diag} f = (\operatorname{diag} f)(t) = \sum_{k\geq 0} a_{k,k}t^k$.

Theorem 1 *Let $A = [a_{i,j}]_{i,j\geq 0}$ and $B = [b_{i,j}]_{i,j\geq 0}$ be such that the sum $\sum_{k\geq 0} a_{i,k}b_{k,j}$ has only finitely many nonzero terms for all $i, j \geq 0$. Then $F_{AB}(x,y) = g(1)$ where $g(t) = (\operatorname{diag} f)(t)$ and $f(u,v) = F_A(x,u)F_B(v,y)$.*

Proof: Write $a^k(x) = \sum_{i\geq 0} a_{i,k}x^i$ and $b_k(y) = \sum_{j\geq 0} b_{k,j}y^j$. Then

$$F_{AB}(x,y) \;=\; \sum_{i,j,k\geq 0} a_{i,k}b_{k,j}x^i y^j \;=\; \sum_{k\geq 0}\sum_{i\geq 0} a_{i,k}x^i \sum_{j\geq 0} b_{k,j}y^j \;=\; \sum_{k\geq 0} a^k(x)b_k(y).$$

As

$$\begin{aligned} f(u,v) &= F_A(x,u)F_B(v,y) \;=\; \sum_{i,k\geq 0} a_{i,k}x^i u^k \sum_{m,j\geq 0} b_{m,j}v^m y^j \\ &= \sum_{k,m\geq 0} u^k v^m \sum_{i\geq 0} a_{i,k}x^i \sum_{j\geq 0} b_{m,j}y^j \;=\; \sum_{k,m\geq 0} a^k(x)b_m(y)u^k v^m, \end{aligned}$$

it follows that

$$(\operatorname{diag} f)(t) \;=\; \sum_{k\geq 0} a^k(x)b_k(y)t^k,$$

hence $F_{AB}(x,y) = (\operatorname{diag} f)(1)$ as claimed. $\square$

Corollary 1 *Let A and B be infinite matrices satisfying the assumptions of Theorem 1 and having rational generating functions. Write $G(t,z) = F_A(x,t/z)F_B(z,y)/z$. Then*

(i) $F_{AB}(x,y) = g(1)$ where $g(t) = \sum_i \operatorname{Res}_{z=z_i(t)} G(t,z)$, and the sum of residues ranges over those poles $z = z_i(t)$ of $G(t,z)$ for which $\lim_{t\to 0} z_i(t) = 0$,

(ii) $F_{AB}(x,y)$ is algebraic.

Proof: Item (i) follows from Theorem 1 and [4, Theorem 1] (see also [8, p. 182]), and (ii) is an immediate consequence of (i). $\square$

Remark 1 *Note that since the diagonals of $f(u,v)$ and $f(v,u)$ coincide, we may also take $G(t,z) = F_A(x,z)F_B(t/z,y)/z$ in Corollary 1.*

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 1: Matrices L and R

Next we assign to each linear recurrence operator with constant coefficients a lower triangular matrix in a natural way. Let E denote the shift operator acting on sequences $x = (x_n)_{n\geq 0}$ by $(Ex)_n = x_{n+1}$. Define E^{-1} by $(E^{-1}x)_n = x_{n-1}$ for $n \geq 1$, $(E^{-1}x)_0 = 0$. Clearly E^{-1} is a right inverse of E.

Let $p(t) = \sum_{k=0}^{r} c_k t^k$ be a polynomial. Then

$$p(E^{-1}) \;=\; \sum_{k=0}^{r} c_k E^{-k}$$

is a linear recurrence operator acting on the space of sequences. Define $\mathcal{M}(p(E^{-1})) = [m_{i,j}]_{i,j\geq 0}$ to be the lower triangular, band-diagonal matrix with elements

$$m_{i,j} \;=\; \begin{cases} c_{i-j}, & 0 \leq i-j \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathcal{M}(p(E^{-1}))$ is the matrix representing $p(E^{-1})$ in the canonical basis $(e^{(j)})_{j\geq 0}$ where $e_n^{(j)} = \delta_{j,n}$ for all $j, n \geq 0$. It is straightforward to compute its generating function

$$F_{\mathcal{M}(p(E^{-1}))}(x,y) \;=\; \frac{p(x)}{1-xy}.$$

Proposition 1 *Denote $L = \mathcal{M}(E^{-1})$ and $R = L^T$ (see Fig. 1). Then $F_{L^k}(x,y) = x^k/(1-xy)$, $F_{R^k}(x,y) = y^k/(1-xy)$, and $RL = I$. Let A be an infinite matrix. Then LA is A with a new first row of zeros, AL is A without its first column, AR is A with a new first column of zeros, RA is A without its first row, $F_{L^kA}(x,y) = x^k F_A(x,y)$, and $F_{AR^k}(x,y) = y^k F_A(x,y)$. Also, $LA = AL$ iff A is lower triangular and constant along diagonals (i.e., $a_{i,j}$ depends only on $i-j$), and $AR = RA$ iff A is upper triangular and constant along diagonals.*

We omit the straightforward proof.

Definition 2 *An infinite matrix $P = [p_{i,j}]_{i,j\geq 0}$ is* Pascal-like *with parameters λ, μ, ν if its entries satisfy the recurrence*

$$p_{i,j} \;=\; \lambda p_{i-1,j} + \mu p_{i,j-1} + \nu p_{i-1,j-1} \qquad \text{for } i,j \geq 1, \tag{1}$$

and the generating functions of its first column and row, denoted $\alpha(x) = \sum_{i\geq 0} p_{i,0} x^i$ and $\beta(y) = \sum_{j\geq 0} p_{0,j} y^j$, respectively, are rational functions of x and y.

Proposition 2 *A matrix P is Pascal-like with parameters λ, μ, ν if and only if its generating function is of the form*

$$F_P(x, y) \; = \; \frac{f(x) + g(y)}{1 - \lambda x - \mu y - \nu xy} \tag{2}$$

where $f(x)$ and $g(y)$ are univariate rational functions.

Proof: Let P be Pascal-like with parameters λ, μ, ν. Recurrence (1) is clearly equivalent to the matrix equation

$$P = \lambda LP + \mu PR + \nu LPR + C \tag{3}$$

where $c_{0,0} = p_{0,0}$, $c_{i,0} = p_{i,0} - \lambda p_{i-1,0}$ for $i \geq 1$, $c_{0,j} = p_{0,j} - \mu p_{0,j-1}$ for $j \geq 1$, and $c_{i,j} = 0$ for $i, j \geq 1$. From (3) we obtain, by Proposition 1,

$$F_P(x, y) = \lambda x F_P(x, y) + \mu y F_P(x, y) + \nu xy F_P(x, y) + F_C(x, y)$$

which implies

$$F_P(x, y) = \frac{F_C(x, y)}{1 - \lambda x - \mu y - \nu xy} \tag{4}$$

where

$$F_C(x, y) \; = \; (1 - \lambda x)\alpha(x) + (1 - \mu y)\beta(y) - p_{0,0}. \tag{5}$$

Conversely, if $F_P(x, y) = \sum_{i,j \geq 0} p_{i,j} x^i y^j$ is of the form (2) then, by equating the coefficients of like powers of x and y in (2), we obtain (1). The generating functions $F_P(x, 0) = (f(x) + g(0))/(1 - \lambda x)$ and $F_P(0, y) = (f(0) + g(y))/(1 - \mu y)$ of the first column and row of P, respectively, are obviously rational. $\Box$

Note that (4) also immediately follows from [3, Equation (18)].

Corollary 2 *Any submatrix $S = [p_{i,j}]_{i \geq i_0, j \geq j_0}$ of a Pascal-like matrix $P = [p_{i,j}]_{i,j \geq 0}$ with parameters λ, μ, ν is again Pascal-like with the same parameters.*

Proof: The entries of S clearly satisfy (1). By Proposition 2, the generating function $F_P(x, y)$ is rational, hence so are the generating functions of any of its rows and columns. In particular, the generating functions of the first row and column of S, which are truncations of row i_0 and column j_0 of P, respectively, are rational. $\Box$

Unless noted otherwise, we use the following notation in the rest of the paper:

- $P = [p_{i,j}]_{i,j \geq 0}$ is a Pascal-like matrix with parameters λ, μ, ν,
- $\alpha(x) = a_1(x)/a(x) = \sum_{i \geq 0} p_{i,0} x^i$ where a_1 and a are polynomials and $a(0) = 1$,
- $\beta(y) = b_1(y)/b(y) = \sum_{j \geq 0} p_{0,j} y^j$ where b_1 and b are polynomials and $b(0) = 1$,
- $\rho = \deg a$, $\sigma = \deg b$,
- $d_n = \det\, [p_{i,j}]_{0 \leq i,j \leq n}$,

- $\omega = \lambda\mu + \nu$.

Definition 3 *Let r and s be nonnegative integers. An infinite matrix $A = [a_{i,j}]_{i,j\geq 0}$ is* (r,s)-banded *if $a_{i,j} = 0$ unless $-s \leq i - j \leq r$.*

Note that an (r,s)-banded matrix is band-diagonal with bandwidth at most $r+s+1$. In particular, an (r,r)-banded matrix is $(2r+1)$-diagonal.

Definition 4 *Infinite matrices $A = [a_{i,j}]_{i,j\geq 0}$ and $B = [b_{i,j}]_{i,j\geq 0}$ are* equimodular *if the sequences of their principal minors agree, i.e.,* $\det[a_{i,j}]_{0\leq i,j\leq n} = \det[b_{i,j}]_{0\leq i,j\leq n}$ *for all $n \geq 0$.*

For example, if A is lower triangular with unit diagonal, then P, AP, and PA^T are equimodular matrices.

3 A recurrence for band-diagonal determinants

In this section we show that the sequence of principal minors of an (r,s)-banded matrix satisfies a homogeneous linear recurrence of order $\delta = \binom{r+s}{r}$. This is a generalization of the well-known three-term recurrence satisfied by the principal minors of tridiagonal matrices, and also of [2, Theorem 7.1] which proves this result for matrices M satisfying $m_{i,j} = m_{i-p,j-p}$ (*p-periodic matrices*).

Definition 5 *A rational function $f(x_1,\ldots,x_n)$ is* homogeneous of order k *if $f(tx_1,\ldots,tx_n) = t^k f(x_1,\ldots,x_n)$ where t is another indeterminate.*

Theorem 2 *Let $A = [a_{i,j}]_{i,j\geq 0}$ be an (r,s)-banded matrix and $\delta = \binom{r+s}{r}$. Denote by d_n the principal minor of A consisting of the elements in rows $0,1,\ldots,n$ and columns $0,1,\ldots,n$ of A. Then for $n \geq \delta$ the sequence $(d_n)_{n\geq 0}$ satisfies a nontrivial homogeneous linear recurrence of the form $d_n = \sum_{k=1}^{\delta} R_k d_{n-k}$ where each R_k is a homogeneous rational function of order k of entries $a_{n-i,n-j}$ $(0 \leq i \leq \delta - 1, -s \leq j \leq r + \delta - 1)$.*

Proof: Denote by

$$A\begin{pmatrix} c_0, c_1, \ldots, c_n \\ r_0, r_1, \ldots, r_n \end{pmatrix}$$

the submatrix of A consisting of the elements in rows $r_0, r_1, \ldots, r_n$ and columns $c_0, c_1, \ldots, c_n$. For $1 \leq j_1 < j_2 < \cdots < j_r \leq r+s$, define

$$A_n^{(j_1,j_2,\ldots,j_r)} = \begin{cases} A\begin{pmatrix} {\scriptstyle 0,1,\ldots,n-r,n-r+j_1,n-r+j_2,\ldots,n-r+j_r} \\ {\scriptstyle 0,1,\ldots,n} \end{pmatrix}, & n \geq r, \\ A\begin{pmatrix} {\scriptstyle n-r+j_{r-n},n-r+j_{r-n+1},\ldots,n-r+j_r} \\ {\scriptstyle 0,1,\ldots,n} \end{pmatrix}, & n < r, \end{cases}$$

and

$$d_n^{(j_1,j_2,\ldots,j_r)} = \det A_n^{(j_1,j_2,\ldots,j_r)}.$$

Note that $j_s \geq s$, so $n - r + j_{r-n} \geq 0$. Also,

$$d_n^{(1,2,\ldots,r)} = d_n.$$

Let $n \geq \delta$. To obtain a system of recurrences satisfied by $d_n^{(j_1,j_2,\ldots,j_r)}$, expand $d_n^{(j_1,j_2,\ldots,j_r)}$ w.r.t. its last row. There are two cases to distinguish:

a) If $j_r = r+s$, the expansion has only a single nonzero term:

$$d_n^{(j_1,j_2,\ldots,j_r)} \;=\; a_{n,n+s}\, d_{n-1}^{(1,j_1+1,j_2+1,\ldots,j_{r-1}+1)}. \tag{6}$$

b) If $j_r < r+s$, we obtain the expansion

$$\begin{aligned} d_n^{(j_1,j_2,\ldots,j_r)} \;&=\; (-1)^r a_{n,n-r}\, d_{n-1}^{(j_1+1,j_2+1,\ldots,j_r+1)} \\ &\quad+\; \sum_{i=1}^{r}(-1)^{r+i} a_{n,n-r+j_i}\, d_{n-1}^{(1,j_1+1,\ldots,j_{i-1}+1,j_{i+1}+1,\ldots,j_r+1)} \end{aligned} \tag{7}$$

where $a_{n,j} = 0$ if $j < 0$.

Let $S_1, S_2, \ldots, S_\delta$ be an enumeration of all r-subsets of $\{1,2,\ldots,r+s\}$, such that $S_1 = \{1,2,\ldots,r\}$. If $S_i = \{j_1,j_2,\ldots,j_r\}$, denote

$$d_n^{(i)} \;=\; d_n^{(j_1,j_2,\ldots,j_r)}.$$

Then (6) and (7) can be written uniformly as

$$d_n^{(i)} \;=\; \sum_{k=1}^{\delta} c_{i,k}(n)\, d_{n-1}^{(k)} \quad (1 \le i \le \delta) \tag{8}$$

where each nonzero $c_{i,k}(n)$ is one of $a_{n,n-j}$ $(-s \le j \le r)$ or its negative. By shifting (8) down $\delta - 1$ times w.r.t. n we obtain δ^2 linear equations

$$d_{n-j+1}^{(i)} \;=\; \sum_{k=1}^{\delta} c_{i,k}(n-j+1)\, d_{n-j}^{(k)} \quad (1 \le i \le \delta,\ 1 \le j \le \delta). \tag{9}$$

The entries of A involved are $a_{n-i,n-j}$ $(0 \le i \le \delta-1, -s \le j \le r+\delta-1)$, where entries with a negative index are taken to be zero. We claim that these equations for the unknowns $d_{n-j}^{(i)}$, $1 \le i \le \delta$, $0 \le j \le \delta$, are linearly independent. To prove this, assume that a linear combination $\mathcal{L}$ of them vanishes identically. As each $d_n^{(i)}$, $1 \le i \le \delta$, appears in a single equation, and its coefficient in that equation equals 1, all the coefficients in $\mathcal{L}$ corresponding to these δ equations (which have $j = 1$) must vanish. But now each $d_{n-1}^{(i)}$, $1 \le i \le \delta$, appears in a single equation as well, and its coefficient in that equation equals 1, so again all the coefficients in $\mathcal{L}$ corresponding to these δ equations (which have $j = 2$) must vanish. Inductively, we see that all the coefficients in $\mathcal{L}$ vanish, which proves the claim.

It follows that from (9) we can eliminate (by Gaussian elimination, say) the $\delta^2 - 1$ unknowns $d_{n-j}^{(i)}$, $2 \le i \le \delta$, $0 \le j \le \delta$. This leaves us with a nontrivial linear equation involving the unknowns $d_{n-j}^{(1)}$, $0 \le j \le \delta$, which is the desired recurrence satisfied by the sequence $(d_n^{(1)})_{n\ge0} = (d_n)_{n\ge0}$.

It remains to show that this recurrence has the promised form. Call a linear equation $\mathcal{E}$ of the form $\sum_{k=1}^{\delta}\sum_{j=0}^{\sigma} R_{j,k} d_{n-j}^{(k)} = 0$ *uniform* if each coefficient $R_{j,k}$ is a rational function of the matrix entries, homogeneous of degree $j + t_{\mathcal{E}}$ where $t_{\mathcal{E}}$ depends only on $\mathcal{E}$. Equations (8) are clearly uniform in this sense, and Gaussian elimination preserves uniformity of equations. Hence the final equation (involving only d_{n-j} for $0 \le j \le \delta$) is uniform as well. Division by the highest nonzero coefficient, and some shifts if necessary, put it into the desired form. □

Remark 2 *The asymmetry in the ranges of row and column indices of entries $a_{n-i,n-j}$ which appear in the coefficients of the recurrence obtained in Theorem 2 ($0 \le i \le \delta - 1$ vs. $-s \le j \le r + \delta - 1$) seems to be an artefact of our proof. Had we expanded $d_n^{(j_1,j_2,\ldots,j_r)}$ w.r.t. its last column rather than row (or equivalently, if we wrote down the recurrence for the transposed matrix A^T, then rewrote the coefficients in terms of the entries of A), the ranges would be $-r \le i \le s + \delta - 1$ and $0 \le j \le \delta - 1$. But since elimination can be performed in such a way that the obtained recurrence is of minimal order, both recurrences should be the same. This argument, if made rigorous (which we believe could be done), would imply that the actual ranges are $0 \le i, j \le \delta - 1$, and that the recurrence is invariant under transposition $a_{i,j} \leftrightarrow a_{j,i}$. Both claims are supported by the example below.*

Example 1 *We wrote a* Mathematica *program based on the proof of Theorem 2 which computes a recurrence satisfied by the sequence of principal minors of an (r, s)-banded matrix. For a generic tridiagonal matrix ($r = s = 1$), we obtain for $n \ge 2$*

$$d_n \;=\; a_{n,n}\, d_{n-1} - a_{n-1,n}\, a_{n,n-1}\, d_{n-2}. \tag{10}$$

This recurrence is well known in numerical linear algebra where it is used in certain methods for computing eigenvalues of symmetric matrices.

For a generic pentadiagonal matrix ($r = s = 2$), we obtain for $n \ge 6$

$$\begin{aligned}
&d_{n-6}\, a_{n-5,n-3}\, a_{n-4,n-2}\, a_{n-3,n-5}\, a_{n-3,n-1}\, a_{n-2,n-4}\, a_{n-1,n-3}\, (a_{n-2,n-1}\, a_{n-1,n}\, a_{n,n-2} - \\
&\quad a_{n-2,n}\, a_{n-1,n-2}\, a_{n,n-1}) + \\
&d_{n-5}\, a_{n-4,n-2}\, a_{n-3,n-1}\, a_{n-2,n-4}\, a_{n-1,n-3}\, (a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-1,n}\, a_{n,n-2} - \\
&\quad a_{n-3,n-3}\, a_{n-2,n-1}\, a_{n-1,n}\, a_{n,n-2} - a_{n-3,n-2}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n,n-1} + \\
&\quad a_{n-3,n-3}\, a_{n-2,n}\, a_{n-1,n-2}\, a_{n,n-1}) - \\
&d_{n-4}\, a_{n-3,n-1}\, a_{n-1,n-3}\, (a_{n-3,n-2}\, a_{n-2,n-1}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n,n-2} - \\
&\quad a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-2,n}\, a_{n-1,n-2}\, a_{n,n-2} + a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-2,n-2}\, a_{n-1,n}\, a_{n,n-2} - \\
&\quad a_{n-3,n-2}\, a_{n-2,n-3}\, a_{n-2,n-1}\, a_{n-1,n}\, a_{n,n-2} - a_{n-3,n-2}\, a_{n-2,n-2}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n,n-1} + \\
&\quad a_{n-3,n-2}\, a_{n-2,n-3}\, a_{n-2,n}\, a_{n-1,n-2}\, a_{n,n-1}) - \\
&d_{n-3}\, (a_{n-3,n-2}\, a_{n-1,n-3}\, a_{n-1,n}\, a_{n,n-2}\, a_{n-2,n-1}^2 - \\
&\quad a_{n-3,n-2}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n-1,n-1}\, a_{n,n-2}\, a_{n-2,n-1} - \\
&\quad a_{n-3,n-1}\, a_{n-2,n-2}\, a_{n-1,n-3}\, a_{n-1,n}\, a_{n,n-2}\, a_{n-2,n-1} + \\
&\quad a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-2,n}\, a_{n-1,n-2}\, a_{n-1,n-1}\, a_{n,n-2} - \\
&\quad a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-2,n}\, a_{n-1,n-2}^2\, a_{n,n-1} + \\
&\quad a_{n-3,n-1}\, a_{n-2,n-2}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n-1,n-2}\, a_{n,n-1}) - \\
&d_{n-2}\, (a_{n-3,n-1}\, a_{n-2,n-1}\, a_{n-1,n-3}\, a_{n-1,n}\, a_{n,n-2} - a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-1,n-1}\, a_{n-1,n}\, a_{n,n-2} - \\
&\quad a_{n-3,n-1}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n-1,n-2}\, a_{n,n-1} + a_{n-3,n-2}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n-1,n-1}\, a_{n,n-1} - \\
&\quad a_{n-3,n-2}\, a_{n-2,n-1}\, a_{n-1,n-3}\, a_{n-1,n}\, a_{n,n-1} + a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-1,n-2}\, a_{n-1,n}\, a_{n,n-1}) - \\
&d_{n-1}\, (a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-1,n}\, a_{n,n-2} - a_{n-3,n-2}\, a_{n-2,n}\, a_{n-1,n-3}\, a_{n,n-1} + \\
&\quad a_{n-3,n-2}\, a_{n-2,n-1}\, a_{n-1,n-3}\, a_{n,n} - a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-1,n-2}\, a_{n,n}) + \\
&d_n\, (a_{n-3,n-2}\, a_{n-2,n-1}\, a_{n-1,n-3} - a_{n-3,n-1}\, a_{n-2,n-3}\, a_{n-1,n-2}) \;=\; 0.
\end{aligned}$$

The recurrence is simpler when the matrix A is constant along diagonals. Writing $a_{n+2,n} = a$, $a_{n+1,n} = b$, $a_{n,n} = c$, $a_{n,n+1} = d$, and $a_{n,n+2} = e$, we obtain for $n \geq 6$

$$\begin{aligned} d_{n-6}\, a^3\, e^3 - d_{n-5}\, a^2\, c\, e^2 - d_{n-4}\, a\, e\, (a\, e - b\, d) - d_{n-3}\, (e\, b^2 + a\, d^2 - 2\, a\, c\, e) + \\ d_{n-2}\, (b\, d - a\, e) - d_{n-1}\, c + d_n \;=\; 0. \end{aligned} \tag{11}$$

If, in addition, the matrix A is symmetric (i.e., $a = e$ and $b = d$), the order of the recurrence decreases by one. In this case we obtain for $n \geq 5$

$$d_{n-5}\, a^5 + d_{n-4}\, a^3\, (a - c) + d_{n-3}\, a\, (b^2 - a\, c) - d_{n-2}\, (b^2 - a\, c) - d_{n-1}\, (a - c) - d_n \;=\; 0. \tag{12}$$

4 Transformation to band-diagonal form

In this section we show how to transform a Pascal-like matrix P to a band-diagonal matrix by a sequence of determinant-preserving elementary row and column operations. Following Bacher's proof of his conjecture in the case $\rho = \sigma = 2$, we do this in two steps. At the first step, we multiply P with operator matrices corresponding to annihilators of its first column and row from the left and from the right, respectively.

Lemma 1 *Let $A = \mathcal{M}(a(E^{-1}))$ and $B = \mathcal{M}(b(E^{-1}))$. Then*

$$F_{APB^T}(x, y) = \frac{F_{ACB^T}(x, y)}{1 - \lambda x - \mu y - \nu xy} \tag{13}$$

where

$$F_{ACB^T}(x, y) \;=\; (1 - \lambda x) a_1(x) b(y) + (1 - \mu y) a(x) b_1(y) - p_{0,0} a(x) b(y) \tag{14}$$

is a polynomial in x and y.

Proof: Multiply (3) by A from the left and by B^T from the right. Proposition 1 implies that $AL = LA$ and $RB^T = B^T R$, hence

$$APB^T = \lambda LAPB^T + \mu APB^T R + \nu LAPB^T R + ACB^T,$$

which gives (13). Now we compute $F_{ACB^T}(x, y)$ using Corollary 1, Remark 1, and Equation (5). The rational function

$$\frac{F_A(x, t/z) F_C(z, y)}{z} = \frac{a(x)}{z - xt}((1 - \lambda z)\alpha(z) + (1 - \mu y)\beta(y) - p_{0,0})$$

has a single pole $z = xt$ satisfying $z \to 0$ as $t \to 0$, hence $F_{AC}(x, y) = a(x) F_C(x, y)$. Similarly, from the residue of $F_{AC}(x, z) F_{B^T}(t/z, y)/z$ at $z = yt$ we obtain $F_{ACB^T}(x, y) = a(x) b(y) F_C(x, y)$ which implies (14). $\Box$

It turns out that for each j there is a polynomial q_j with $\deg q_j \leq j - 1$ such that for $i \geq j + \rho$, the i-th element of the j-th column of APB^T equals $q_j(i)\lambda^i$. An analogous statement holds for the rows of APB^T. Thus, at the second step, we annihilate these "out-of-the-band" entries by multiplying APB^T with suitable lower resp. upper triangular matrices from the left resp. from the right.

Let $S(c) = [s_{i,j}]_{i,j\geq 0}$ denote the infinite matrix with entries

$$s_{i,j} = \begin{cases} (-c)^{i-j}\binom{i}{j}, & i \geq j, \\ 0, & i < j, \end{cases}$$

whose generating function is

$$F_{S(c)}(x,y) = \frac{1}{1+cx-xy}. \tag{15}$$

Further, let $S_r(c)$ denote the block-diagonal matrix

$$S_r(c) = \begin{bmatrix} I_r & 0 \\ 0 & S(c) \end{bmatrix}$$

where I_r is the identity matrix of order r. Note that $S_r(c)$ is lower triangular with unit diagonal, and that $F_{S_r(c)}(x,y) = (1-x^r y^r)/(1-xy) + x^r y^r/(1+cx-xy)$. In particular,

$$F_{S_1(c)}(x,y) = \frac{1+cx}{1+cx-xy}. \tag{16}$$

Lemma 2 *Let P be a Pascal-like matrix with parameters λ, μ, ν whose first column is zero. Then $S(\lambda)P$ is an upper triangular matrix with zero diagonal.*

Proof: By assumption, $\alpha(x) = 0$. We use Corollary 1, (15), and (4). From the residue of

$$F_{S(\lambda)}(x, t/z)F_P(z,y)/z = \frac{1}{z(1+\lambda x) - xt} \cdot \frac{(1-\mu y)\beta(y)}{1-\lambda z - \mu y - \nu yz}$$

at $z = xt/(1+\lambda x)$ we obtain

$$F_{S(\lambda)P}(x,y) = \frac{(1-\mu y)\beta(y)}{1-\mu y - \omega xy}.$$

Hence $F_{S(\lambda)P}(x,y) = f(y, xy)$ where

$$f(u,v) := \frac{(1-\mu u)\beta(u)}{1-\mu u - \omega v}.$$

Thus $F_{S(\lambda)P}(x,y)$ is a power series in y and xy, showing that $S(\lambda)P$ is upper triangular, and the generating function of its diagonal is $f(0,t) = \beta(0)/(1-\omega t) = p_{0,0}/(1-\omega t) = 0$. $\square$

Lemma 3 *Let P be a Pascal-like matrix with parameters λ, μ, ν whose first column and row have generating functions of the form $\alpha(x) = a_1(x)/(1-\lambda x)^\rho$ and $\beta(y) = b_1(y)/(1-\mu y)^\sigma$, respectively, where $\deg a_1 < \rho$ and $\deg b_1 < \sigma$. Then*

$$F_{S(\lambda)PS(\mu)^T}(x,y) = \frac{(1+\lambda x)^{\rho-1}a_1(x/(1+\lambda x)) + (1+\mu y)^{\sigma-1}b_1(y/(1+\mu y)) - p_{0,0}}{1-\omega xy} \tag{17}$$

where $\omega = \lambda\mu + \nu$.

Proof: We use Corollary 1, Remark 1, (15), and (4). From the residue of $F_{S(\lambda)}(x,t/z)F_P(z,y)/z$ at $z = xt/(1+\lambda x)$ we have

$$F_{S(\lambda)P}(x,y) = \frac{(1+\lambda x)^{\rho-1}a_1(x/(1+\lambda x)) + (1-\mu y)^{1-\sigma}b_1(y) - p_{0,0}}{1-\mu y-\omega xy}.$$

From the residue of $F_{S(\lambda)P}(x,z)F_{S(\mu)^T}(t/z,y)/z$ at $z = yt/(1+\mu y)$ we obtain (17). $\square$

Corollary 3 *Let P satisfy the assumptions of Lemma 3. Then $Q = S(\lambda)PS(\mu)^T$ is an $(\rho-1,\sigma-1)$-banded matrix, and the sequence of elements along each diagonal is geometric with quotient ω.*

Proof: This follows from the shape of the denominator of (17), and from the fact that the numerator of (17) is a sum of two polynomials, one of degree $\rho-1$ and depending only on x, the other of degree $\sigma-1$ and depending only on y. $\square$

Now we have everything that we need to transform a general Pascal-like matrix to band-diagonal form.

Theorem 3 *Let P be a Pascal-like matrix with* $\deg a_1 < \rho+k$ *and* $\deg b_1 < \sigma+l$ *where k and l are nonnegative integers. If $A = \mathcal{M}(a(E^{-1}))$ and $B = \mathcal{M}(b(E^{-1}))$ then $M := S_{\rho+k}(\lambda)APB^TS_{\sigma+l}(\mu)^T$ is an $(\rho+k-1,\sigma+l-1)$-banded matrix, and each diagonal sequence $(m_{i,i+d})_{i\geq\max\{\rho+k,\sigma+l-d\}}$ where $-(\rho+k)\leq d\leq\sigma+l$ is geometric with quotient ω.*

Proof: By Lemma 1,

$$F_{APB^T}(x,y) = \frac{F_{ACB^T}(x,y)}{1-\lambda x-\mu y-\nu xy}$$

where $F_{ACB^T}(x,y)$ is given in (14). The matrix $Q := APB^T$ is, in general, not Pascal-like. But it is only in the upper-left $((\rho+k)\times(\sigma+l))$-corner that recurrence (1) is not satisfied, therefore we divide Q in blocks as follows:

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_3 & Q_4 \end{bmatrix} = \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix},$$

where Q_1 has $\rho+k$ rows, Q_3 has $\sigma+l$ columns, and $Q_{1,1}$ is of order $(\rho+k)\times(\sigma+l)$. Recalling that

$$S_{\rho+k}(\lambda) = \begin{bmatrix} I_{\rho+k} & 0 \\ 0 & S(\lambda) \end{bmatrix}, \quad S_{\sigma+l}(\mu)^T = \begin{bmatrix} I_{\sigma+l} & 0 \\ 0 & S(\mu)^T \end{bmatrix},$$

we have

$$M = S_{\rho+k}(\lambda)QS_{\sigma+l}(\mu)^T = \begin{bmatrix} Q_{1,1} & Q_{1,2}S(\mu)^T \\ S(\lambda)Q_{2,1} & S(\lambda)Q_{2,2}S(\mu)^T \end{bmatrix}.$$

We are going to show that $S(\lambda)Q_{2,1}$ is upper triangular with zero diagonal, $Q_{1,2}S(\mu)^T$ is lower triangular with zero diagonal, and $N := S(\lambda)Q_{2,2}S(\mu)^T$ is $(\sigma+l-1,\rho+k-1)$-banded. As $n_{i,j} = m_{i+\rho+k,j+\sigma+l}$, this will imply that M itself is $(\rho+k-1,\sigma+l-1)$-banded.

Clearly $Q_2 = R^{\rho+k}Q$, $Q_4 = QL^{\sigma+l}$, and $Q_{2,2} = Q_2L^{\sigma+l} = R^{\rho+k}Q_4$. From the residue of

$$\frac{z^{\rho+k}F_{ACB^T}(t/z, y)}{(1-xz)((1-\mu y)z-(\lambda+\nu y)t)}$$

at $z = t(\lambda + \nu y)/(1-\mu y)$ we have

$$F_{Q_2}(x, y) = \left(\frac{\lambda+\nu y}{1-\mu y}\right)^{\rho+k} \frac{F_{ACB^T}((1-\mu y)/(\lambda+\nu y), y)}{1-\lambda x-\mu y-\nu xy}.$$

Similarly we obtain

$$F_{Q_4}(x, y) = \left(\frac{\mu+\nu x}{1-\lambda x}\right)^{\sigma+l} \frac{F_{ACB^T}(x, (1-\lambda x)/(\mu+\nu x))}{1-\lambda x-\mu y-\nu xy}.$$

By Proposition 2, both Q_2 and Q_4 are Pascal-like matrices with parameters λ, μ, ν, and by Corollary 2, so is their common submatrix $Q_{2,2}$.

If $\lambda \neq 0$ then the generating function of the first column of Q_2 is, by (14) and (5),

$$F_{Q_2}(x, 0) = \lambda^{\rho+k}\frac{F_{ACB^T}(1/\lambda, 0)}{1-\lambda x} = \lambda^{\rho+k}\frac{a(1/\lambda)F_C(1/\lambda, 0)}{1-\lambda x} = 0.$$

It can be seen that $F_{Q_2}(x, 0) = 0$ also when $\lambda = 0$. By Lemma 2, it follows that $S(\lambda)Q_2$ is upper triangular with zero diagonal. The same is then also true of $S(\lambda)Q_{2,1}$, which is composed of the first $\sigma + l$ columns of $S(\lambda)Q_2$. In an analogous way, it can be shown that $Q_{1,2}S(\mu)^T$ is lower triangular with zero diagonal.

The generating function of the first row of Q_2 is

$$F_{Q_2}(0, y) = \left(\frac{\lambda+\nu y}{1-\mu y}\right)^{\rho+k} \frac{F_{ACB^T}((1-\mu y)/(\lambda+\nu y), y)}{1-\mu y}.$$

From (14) it follows that $f(y) := (\lambda+\nu y)^{\rho+k}F_{ACB^T}((1-\mu y)/(\lambda+\nu y), y)$ is a polynomial in y of degree at most $\rho + k + \sigma + l$. If $\mu \neq 0$ then $f(1/\mu) = (\omega/\mu)^{\rho+k}F_{ACB^T}(0, 1/\mu) = (\omega/\mu)^{\rho+k}a(0)b(1/\mu)F_C(0, 1/\mu) = 0$, so $f(y)$ is divisible by $1 - \mu y$. Hence $F_{Q_2}(0, y) = f_1(y)/(1-\mu y)^{\rho+k}$ where $f_1(y)$ is a polynomial in y of degree less than $\rho + k + \sigma + l$. This can be seen to hold also when $\mu = 0$.

By applying Extended Euclidean Algorithm to the coprime polynomials $(1 - \mu y)^{\rho+k}$ and $y^{\sigma+l}$ we can find polynomials $p(y)$ and $f_2(y)$ such that $\deg p < \sigma + l$, $\deg f_2 < \rho + k$, and $f_1(y) = (1-\mu y)^{\rho+k}p(y) + y^{\sigma+l}f_2(y)$. Hence $F_{Q_2}(0, y) = p(y) + y^{\sigma+l}f_2(y)/(1-\mu y)^{\rho+k}$ where $\deg p < \sigma + l$, so the generating function $F_{Q_{2,2}}(0, y)$ of the first row of $Q_{2,2}$ equals $f_2(y)/(1-\mu y)^{\rho+k}$ where $\deg f_2 < \rho + k$.

Similarly we can see that the generating function $F_{Q_{2,2}}(x, 0)$ of the first column of $Q_{2,2}$ is $g_2(x)/(1-\lambda x)^{\sigma+l}$ where $\deg g_2 < \sigma + l$. Therefore, by Corollary 3, $S(\lambda)Q_{2,2}S(\mu)^T$ is an $(\sigma + l - 1, \rho + k - 1)$-banded matrix whose diagonals are geometric sequences with quotient ω. As already mentioned, this implies that M is $(\rho + k - 1, \sigma + l - 1)$-banded. $\square$

5 Results and examples

Corollary 4 *Let P be a Pascal-like matrix with $\deg a_1 < \rho + k$ and $\deg b_1 < \sigma + l$ where k and l are nonnegative integers, and let $\delta = \binom{\rho+k+\sigma+l-2}{\rho+k-1}$. Then the sequence of its principal minors $(d_n)_{n\geq 0}$ for $n \geq \delta + \max(\rho + k, \sigma + l)$ satisfies a nontrivial linear recurrence of the form $d_n = \sum_{j=1}^{\delta} c_j \omega^{jn} d_{n-j}$ where c_j are constants.*

Proof: By Theorem 3, P is equimodular with an $(\rho + k - 1, \sigma + l - 1)$-banded matrix M whose nonzero entries are of the form $m_{n,n+d} = m_d \cdot \omega^n$ for $n \geq \max(\rho + k, \sigma + l - d)$ and $-(\rho + k) \leq d \leq \sigma + l$, where m_d depends only on d. By Theorem 2 and Remark 2, the sequence $(d_n)_{n\geq 0}$ for $n \geq \delta$ satisfies a nontrivial recurrence of the form $d_n = \sum_{j=1}^{\delta} R_j d_{n-j}$ where each R_j is a rational function of $m_{n-u,n-v}$ $(0 \leq u, v \leq \delta - 1)$. As R_j is homogeneous of order j and $m_{n-u,n-v} = \omega^n \cdot m_{u-v}/\omega^u$ for $n - u \geq \rho + k$ and $n - v \geq \sigma + l$, this can be written in the announced form. □

Corollary 5 *Let P and δ be as in Corollary 4. Then there are a nonnegative integer $m \leq \delta$, polynomials $p_1(n), \ldots, p_m(n)$, and algebraic numbers $\gamma_1, \ldots, \gamma_m$ of degree at most δ such that $d_n = \omega^{\binom{n}{2}} \sum_{i=0}^{m} p_i(n)\gamma_i^n$, for all $n \geq \max(\rho + k, \sigma + l)$.*

Proof: Let $n \geq \delta + \max(\rho + k, \sigma + l)$. By Corollary 4, $d_n = \sum_{j=1}^{\delta} c_j \omega^{jn} d_{n-j}$. Write $d_n = \omega^{\binom{n}{2}} e_n$. Then $e_n = \sum_{j=1}^{\delta} b_j e_{n-j}$ where $b_j = c_j \omega^{\binom{j+1}{2}}$. The assertion now follows from the well-known theory of linear recurrences with constant coefficients. □

Remark 3 *Let P be Pascal-like with $\deg a_1 < \rho + k$ and $\deg b_1 < \sigma + l$. Denote $\delta = \binom{\rho+k+\sigma+l-2}{\rho+k-1}$ and $m = \max(\rho + k, \sigma + l)$. The closed-form expression for its n-th principal minor d_n as described in Corollary 5 can be obtained by the following algorithm:*

1. *For $n = m, m + 1, \ldots, m + 2\delta - 1$ evaluate $e_n = d_n/\omega^{\binom{n}{2}}$.*

2. *Solve the Hankel system of δ linear algebraic equations*

$$e_n = \sum_{j=1}^{\delta} b_j e_{n-j} \quad (m + \delta \leq n \leq m + 2\delta - 1)$$

   for the unknown coefficients $b_1, b_2, \ldots, b_\delta$.

3. *Find a closed-form representation of the sequence η_n which satisfies the recurrence*

$$\eta_n = \sum_{j=1}^{\delta} b_j \eta_{n-j} \quad (n \geq m + \delta)$$

   and initial conditions $\eta_n = e_n$, for $n = m, m + 1, \ldots, m + \delta - 1$. This step involves computation with algebraic numbers.

4. *If $n < m$ then compute d_n directly, else $d_n = \eta_n \omega^{\binom{n}{2}}$.*

Example 2 *We illustrate the algorithm of Remark 3 on a "random" example. Let* $P = [p_{i,j}]_{i,j\geq 0}$ *be the matrix defined by* $p_{0,0} = 1$, $p_{1,0} = 2$, $p_{0,1} = 1$,

$$\begin{aligned} p_{i,0} &= 3p_{i-1,0} - p_{i-2,0} \quad (i \geq 2), \\ p_{0,j} &= p_{0,j-1} + p_{0,j-2} \quad (j \geq 2), \\ p_{i,j} &= 2p_{i-1,j} + 3p_{i,j-1} - p_{i-1,j-1} \quad (i,j \geq 1). \end{aligned}$$

This is a Pascal-like matrix with $\lambda = 2$, $\mu = 3$, $\nu = -1$, $\omega = 5$, $\rho = \sigma = 2$, $k = l = 0$, $\delta = \binom{2}{1} = 2$. *Let* d_n *be its principal minor of order* $n+1$ *and let* $e_n = d_n/5^{\binom{n}{2}}$. *Then by Corollary 4,* e_n *for* $n \geq 4$ *satisfies a recurrence of order 2 with constant coefficients. From its initial values* $1, 5, 24, 111, 495, 2124, 8721, 33885, 121824, 384831, 905175, \ldots$ *we find the recurrence* $e_n = 9e_{n-1} - 21e_{n-2}$ *which happens to be valid for all* $n \geq 2$. *We solve it and obtain the evaluation*

$$d_n = \frac{2}{\sqrt{3}}\left(\frac{21}{5}\right)^{n/2} 5^{n^2/2} \cos\left(\frac{\pi}{6} - n \arctan\left(\frac{\sqrt{3}}{9}\right)\right) \quad (n \geq 0).$$

We conclude by giving several determinant evaluations which imply or generalize some results and examples from the literature.

Example 3 *As a special case, Lemma 3 implies [6, Theorem 3] where P is given by*

$$p_{i,j} = p_{i,j-1} + p_{i-1,j} + \nu p_{i-1,j-1}, \qquad i,j \geq 1,$$

and $p_{i,0} = -p_{0,i} = i$ *for* $i \geq 0$. *Here* $\rho = \sigma = 2$, $\lambda = \mu = 1$, $\omega = 1+\nu$, $a(x) = b(x) = (1-x)^2$, *and* $a_1(x) = -b_1(x) = x$. *By Lemma 3,* $F_{S(\lambda)PS(\mu)^T}(x,y) = (x-y)/(1-\omega xy)$, *so P is equimodular with the tridiagonal matrix*

$$\begin{bmatrix} 0 & -1 & 0 & 0 & \ldots \\ 1 & 0 & -\omega & 0 & \ldots \\ 0 & \omega & 0 & -\omega^2 & \ldots \\ 0 & 0 & \omega^2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Hence, by (10), the sequence $(d_n)_{n\geq 0}$ *of principal minors of P satisfies the recurrence* $d_n = \omega^{2n-2} d_{n-2}$ *for* $n \geq 2$. *Together with* $d_0 = 0$ *and* $d_1 = 1$ *this gives* $d_{2n} = 0$, $d_{2n-1} = \omega^{2n(n-1)}$.

Example 4 *Let P be given by*

$$p_{i,j} = p_{i,j-1} + p_{i-1,j} + \nu p_{i-1,j-1}, \qquad i,j \geq 1,$$

$p_{0,0} = 0$, *and* $p_{i,0} = -p_{0,i} = q^{i-1}$ *for* $i \geq 1$ *where q is a constant. Here* $\mu = \lambda = 1$, $\alpha(x) = -\beta(x) = x/(1-qx)$, $\rho = \sigma = k = l = 1$, $\omega = 1+\nu$, *hence Theorem 3 implies that P is equimodular with a tridiagonal matrix whose diagonals (with the first two elements* *deleted) are geometric with quotient ω. In fact, following the proof of Theorem 3, this matrix turns out to be*

$$\begin{bmatrix} 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & -(\nu+q) & 0 & \dots \\ 0 & \nu+q & 0 & -(\nu+q)\omega & \dots \\ 0 & 0 & (\nu+q)\omega & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Hence, by (10), the sequence $(d_n)_{n\geq 0}$ of principal minors of P satisfies the recurrence $d_n = (\nu+q)^2\omega^{2n-4}d_{n-2}$. Together with initial conditions $d_0 = 0,\ d_1 = 1$ this implies $d_{2n} = 0$ and $d_{2n-1} = (\nu+q)^{2n-2}\omega^{2(n-1)^2}$, in agreement with [6, Theorem 2].

Example 5 *Let P be given by*

$$p_{i,j} = p_{i,j-1} + p_{i-1,j}, \qquad i, j \geq 1$$

with a 3-periodic sequence

$$(p_0, p_1, p_2, \dots, p_k = p_{k-3}, \dots)$$

as its first row and column. It is conjectured in [2, Example 3.5] that the sequence $(d_n)_{n\geq 0}$ of principal minors of P satisfies a certain recurrence of order 5. Indeed, Theorem 3 implies that P is equimodular with a pentadiagonal matrix which, starting with the third element, is constant along each diagonal. In fact, following the proof of Theorem 3, this matrix turns out to be

$$\begin{bmatrix} p_0 & p_1 & p_2 & 0 & 0 & 0 & \dots \\ p_1 & 2p_1 & 2p_1+p_2 & a & 0 & 0 & \dots \\ p_2 & 2p_1+p_2 & 4p_1+2p_2 & p_0+6p_1+2p_2 & a & 0 & \dots \\ 0 & a & p_0+6p_1+2p_2 & c & b & a & \dots \\ 0 & 0 & a & b & c & b & \dots \\ 0 & 0 & 0 & a & b & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $a = p_0+p_1+p_2,\ b = 2p_0+7p_1+3p_2,\ c = p_0+12p_1+6p_2$. Hence the sequence $(d_n)_{n\geq 0}$ satisfies recurrence (12) for $n \geq 9$ (as it turns out, (12) holds for all $n \geq 5$).

The following is a generalization of [6, Theorem 1] (see also [2, Theorem 1.5]).

Lemma 4 *Let P be a matrix which satisfies (1), and whose first column has generating function of the form $\alpha(x) = p_{0,0}/(1-qx)$. If $A(q) = \mathcal{M}(1-qE^{-1})$ then $S_1(\lambda)A(q)P$ is an upper triangular matrix with diagonal elements*

$$p_{0,0},\ \xi,\ \xi\omega,\ \xi\omega^2,\ \xi\omega^3 \dots \tag{18}$$

where $\xi = p_{0,0}\omega + (p_{0,1} - \mu p_{0,0})(\lambda - q)$.

Proof: We use Corollary 1, Remark 1, (4), and (16). From the residue of

$$F_{A(q)}(x,t/z)F_P(z,y)/z = \frac{1-qx}{z-xt} \cdot \frac{F_C(z,y)}{1-\lambda z-\mu y-\nu zy}$$

at $z = xt$ we obtain

$$F_{A(q)P}(x,y) = (1-qx)F_P(x,y).$$

From the residue of

$$F_{S_1(\lambda)}(x,t/z)F_{A(q)P}(z,y)/z = \frac{(1+\lambda x)(1-qz)}{z+\lambda xz-xt}F_P(z,y)$$

at $z = xt/(1+\lambda x)$ we obtain

$$F_{S_1(\lambda)A(q)P}(x,y) = \frac{(1-\mu y)\beta(y)+(\lambda-q)x\,((1-\mu y)\beta(y)-p_{0,0})}{1-\mu y-\omega xy}.$$

The numerator can be rewritten as $(1-\mu y)\beta(y)+(\lambda-q)xy\left(\sum_{k\geq 0}p_{0,k+1}y^k-\mu\beta(y)\right)$, hence $F_{S_1(\lambda)A(q)P}(x,y) = f(y,xy)$ where

$$f(u,v) := \frac{(1-\mu u)\beta(u)+(\lambda-q)v\left(\sum_{k\geq 0}p_{0,k+1}u^k-\mu\beta(u)\right)}{1-\mu u-\omega v}.$$

Thus $F_{S_1(\lambda)A(q)P}(x,y)$ is a power series in y and xy, showing that $S_1(\lambda)A(q)P$ is upper triangular. The generating function of its main diagonal is

$$f(0,t) = \frac{p_{0,0}+(p_{0,1}-\mu p_{0,0})(\lambda-q)t}{1-\omega t}$$

which gives (18). $\square$

Remark 4 *Note that in Lemma 4, the matrix P need not be Pascal-like (i.e., the lemma holds regardless of the algebraic nature of the first row of P).*

Corollary 6 *Let P satisfy the assumptions of Lemma 4. Then $d_n = p_{0,0}\,\xi^n\,\omega^{\binom{n}{2}}$.*

Our methods can also be applied to evaluate some determinants whose entries do not necessarily satisfy recurrence (1).

Lemma 5 *Let M be a matrix whose i-th element of the j-th column is of the form $q_j(i)$, where q_j is a polynomial with $\deg q_j = j$. Then $S(1)M$ is an upper triangular matrix with diagonal elements*

$$(j!\,\mathrm{lc}(q_j))_{j\geq 0},$$

where $\mathrm{lc}(q_j)$ denotes the leading coefficient of q_j.

Proof: Let $N = S(1)M$. As row k of $S(1)$ consists of the coefficients of operator $(1-E^{-1})^k$, the entries $n_{k,0}, n_{k,1}, \ldots, n_{k,k-1}$ all vanish, and $n_{k,k} = (1-E^{-1})^k q_k(i)|_{i=k} = k!\,\mathrm{lc}(q_k)$. $\square$

Example 6 *Let*

$$M_n(a,b,x;k) = \left[\binom{ai+bj+x}{j-k}\right]_{0\le i,j\le n}, \quad a,b,x\in\mathbb{C},\ k\in\mathbb{N}.$$

In [6, Theorem 4] it is shown that

$$\det(M_n(2,2,x;0) - M_n(2,2,x;1)) = 2^{\binom{n+1}{2}}.$$

Using Lemma 5 this result can be extended. As for each $j\in\mathbb{N}$

$$\binom{ai+bj+x}{j-k}$$

is a polynomial in variable i of degree $j-k$ whose leading coefficient is $a^{j-k}/(j-k)!$ we obtain that

$$\det\ M_n(a,b,x;k) = \begin{cases} a^{\binom{n+1}{2}}, & k=0,\\ 0, & k>0.\end{cases}$$

Let $M = \sum_{m=0}^{l}\mu_m M_n(a_m,b_m,x_m;0)$ be a linear combination of matrices of this type. Obviously, column j of M is a polynomial in the row index of degree j with leading coefficient $\sum_{m=0}^{l}\mu_m a_m^j/j!$, therefore, by Lemma 5,

$$\det\left(\sum_{m=0}^{l}\mu_m M_n(a_m,b_m,x_m;0)\right) = \prod_{j=0}^{n}\left(\sum_{m=0}^{l}\mu_m a_m^j\right).$$

Similarly, if $M = \sum_{k=0}^{l}\mu_k M_n(a_k,b_k,x_k;k)$ then column j of M is a polynomial in the row index of degree j with leading coefficient $\mu_0 a_0^j/j!$, thus

$$\det\left(\sum_{k=0}^{l}\mu_k M_n(a_k,b_k,x_k;k)\right) = \mu_0^{n+1}a_0^{\binom{n+1}{2}}.$$

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