On polynomial solutions of linear partial differential and (q-)difference equations

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Abstract. We prove that the question of whether a given linear partial differential or difference equation with polynomial coefficients has non-zero polynomial solutions is algorithmically undecidable. However, for equations with constant coefficients this question can be decided very easily since such an equation has a non-zero polynomial solution iff its constant term is zero. We give a simple combinatorial proof of the fact that in this case the equation has polynomial solutions of *all* degrees. For linear partial q-difference equations with polynomial coefficients, the question of decidability of existence of non-zero polynomial solutions remains open. Nevertheless, for such equations with constant coefficients we show that the space of polynomial solutions can be described algorithmically. We present examples which demonstrate that, in contrast with the differential and difference cases where the dimension of this space is either infinite or zero, in the q-difference case it can also be finite and non-zero.

1 Introduction

Polynomial solutions of linear differential and (q-)difference equations often serve as a building block in algorithms for finding other types of closed-form solutions. Computer algebra algorithms for finding polynomial (see, for example, [4]) and rational (see [1, 2, 7, 10, 8] etc.) solutions of linear *ordinary* differential and difference equations with polynomial coefficients are well known. Note, however, that relatively few results about rational solutions of *partial* linear differential and (q-)difference equations can be found in the literature. Only recently, M. Kauers and C. Schneider [11, 12] have started work on the algorithmic aspects of finding universal denominators for rational solutions in the difference case. Once such a denominator is obtained, one needs to find polynomial solutions of the equation satisfied by the numerators of the rational solutions of the original equation. This is our motivation for considering polynomial solutions of linear partial differential and (q-)difference equations with polynomial coefficients in the present paper.

Let K be a field of characteristic 0, and let $x_1, \ldots, x_m$ be independent variables where $m \geq 2$. In Section 2, using an argument similar to the one given in [9, Thm. 4.11], we show that there is no algorithm which, for an arbitrary linear differential or difference operator L with coefficients from $K[x_1, \ldots, x_m]$, determines whether or not there is a non-zero polynomial $y \in K[x_1, \ldots, x_m]$ such that $L(y) = 0$ (Theorem 1). The proof is based on the Davis-Matiyasevich-Putnam-Robinson (DMPR) theorem which states that the problem of solvability of Diophantine equations is algorithmically undecidable, i.e., that there is no algorithm which, for an arbitrary polynomial $P(t_1, \ldots, t_m)$ with integral coefficients, determines whether or not the

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equation $P(t_1, \ldots, t_m) = 0$ has an integral solution [14, 17]. In fact, we use the equivalent form which states that existence of *non-negative* integral solutions of $P(t_1, \ldots, t_m) = 0$ is undecidable as well.

Of course, by limiting the class of operators considered, the corresponding problem may become decidable. For example, it is well known that a partial linear differential or difference operator L with coefficients in K (a.k.a. an operator with *constant* coefficients) has a non-zero polynomial solution iff $L(1) = 0$ (see, for example, [20, Lemma 2.3]). In addition, in Section 3 we show that in this case, the equation $L(y) = 0$ has polynomial solutions of degree d *for all* $d \in \mathbb{N}$ (Theorem 2). This is contrasted with the univariate case $m = 1$, where the degree of a polynomial solution cannot exceed $\operatorname{ord} L$ (but note that, when a univariate L is considered to be m-variate with $m \geq 2$, and $L(1) = 0$, equation $L(y) = 0$ does have solutions of all degrees). In the differential case, when the affine algebraic variety defined by $\sigma(L) = 0$ (where $\sigma : K[\partial/\partial x_1, \ldots, \partial/\partial x_n] \to K[x_1, \ldots, x_n]$ is the ring homomorphism given by $\sigma|_K = \mathrm{id}_K$, $\sigma(\partial/\partial x_j) = x_j$) is not singular at 0, and for d large enough, Theorem 2 follows from [20, Prop. 3.3(e)]. Here we present a short direct proof based on a simple counting argument. For a given $d \in \mathbb{N}$, all solutions of degree d of such an equation can be found, e.g., by the method of undetermined coefficients. Of course, there exist more efficient ways to do that: in [19], the application of Janet bases to the computation of (formal) power series and polynomial solutions is considered; in [19, Ex. 4.6], the command `PolySol` for computing polynomial solutions from the *Janet* Maple package is illustrated. Computing polynomial solutions using Gröbner bases is described in [22, Sect. 10.3, 10.4] and [19, Sect. 10.8]. The more general problem of finding polynomial solutions of holonomic systems with polynomial coefficients (if they exist) is treated in [16, 21], and the resulting algorithms are implemented in Macaulay2 [13].

Our attention was drawn to these problems by M. Kauers. In a letter to the first author he presented a proof of undecidability of existence of non-zero polynomial solutions of partial differential equations with polynomial coefficients, and attributed it to mathematical folklore. In our paper a simple common proof for the differential and difference cases is proposed. The situation when coefficients are constant is clarified as well.

In Section 4 we consider the q-difference case, assuming that $K = k(q)$ where k is a subfield of K and q is transcendental over k (q-calculus, as well as the theory and algorithms for q-difference equations, are of interest in combinatorics, especially in the theory of partitions [5, Sect. 8.4], [6]). The question of decidability of existence of non-zero polynomial solutions of an arbitrary q-difference equation with polynomial coefficients is still open. As for the equations with constant coefficients, we formulate and prove a necessary condition for existence of a non-zero polynomial solution: if $L(1) = p(q) \in K[q]$, then $p(1) = 0$, or, more succinctly: $(L(1))(1) = 0$. We also show that the dimension of the space of polynomial solutions of a linear q-difference equation with constant coefficients can be, in contrast with the differential and difference cases, not only zero or infinite, but also finite positive. An explicit description of this space can be obtained algorithmically. We consider this as one of the first steps in the program to find wider classes of closed-form solutions of multivariate q-difference equations.

Terminology and notation. We write $x = (x_1, \ldots, x_m)$ for the variables, $D = (D_1, \ldots, D_m)$ for partial derivatives ($D_i = \frac{\partial}{\partial x_i}$), and $\Delta = (\Delta_1, \ldots, \Delta_m)$ for partial differences ($\Delta_i = E_i - 1$ where $E_i f(x) = f(x_1, \ldots, x_i + 1, \ldots, x_m)$). Multiindices from $\mathbb{N}^m$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$) are denoted by lower-case Greek letters, so that a partial linear operator of order at most r with polynomial coefficients is written as

$$L = \sum_{|\mu| \leq r} a_\mu(x) D^\mu \tag{1}$$

in the differential case, and

$$L = \sum_{|\mu| \le r} a_\mu(x) \Delta^\mu \tag{2}$$

in the difference case, with $a_\mu(x) \in K[x_1, \ldots, x_m]$ in both cases. We denote the dot product of multiindices $\mu, \alpha \in \mathbb{N}^m$ by $\mu \cdot \alpha = \mu_1 \alpha_1 + \cdots + \mu_m \alpha_m$.

We call $y(x) \in K[x_1, \ldots, x_m]$ a *solution* of L if $L(y) = 0$.

Let $c \in K \setminus \{0\}$. As usual, we define

$$\deg_{x_i}(c x_1^{n_1} \cdots x_m^{n_m}) = n_i$$

for $i = 1, \ldots, m$, and

$$\deg(c x_1^{n_1} \cdots x_m^{n_m}) = n_1 + \cdots + n_m.$$

For $p(x) \in K[x_1, \ldots, x_m] \setminus \{0\}$ we set $\deg_{x_i} p(x)$ for $i = 1, \ldots, m$ to be equal to $\max \deg_{x_i} t$, and $\deg p(x)$ to be equal to $\max \deg t$ where the maximum is taken over all the terms t of the polynomial $p(x)$. We define $\deg_{x_i} 0 = \deg 0 = -\infty$ for $i = 1, \ldots, m$.

We denote the rising factorial by

$$a^{\overline{n}} = \prod_{i=0}^{n-1} (a + i).$$

2 Equations with polynomial coefficients

Theorem 1 *There is no algorithm to decide whether an arbitrary linear partial differential resp. difference operator L with polynomial coefficients in an arbitrary number m of variables, of the form (1) resp. (2), has a non-zero polynomial solution.*

Proof. Let $P(t_1, \ldots, t_m) \in \mathbb{Z}[t_1, \ldots, t_m]$ be arbitrary. For $i = 1, \ldots, m$ write $\theta_i = x_i D_i$ and $\sigma_i = x_i \Delta_i$. Then

$$\theta_i(x_1^{n_1} \cdots x_m^{n_m}) = n_i x_1^{n_1} \cdots x_m^{n_m} \tag{3}$$

and

$$\sigma_i(x_1^{\overline{n}_1} \cdots x_m^{\overline{n}_m}) = n_i x_1^{\overline{n}_1} \cdots x_m^{\overline{n}_m}, \tag{4}$$

for $i = 1, \ldots, m$. Define an operator L of the form (1) resp. (2) by setting $L = P(\theta_1, \ldots, \theta_m)$ in the differential case, and $L = P(\sigma_1, \ldots, \sigma_m)$ in the difference case. Let $f(x_1, \ldots, x_m) \in K[x_1, \ldots, x_m]$ be a polynomial over K. From (3) and (4) it follows that L annihilates f iff it annihilates each term of f separately, so L has a non-zero polynomial solution iff it has a monomial solution (where in the difference case we assume that the polynomial f is expanded in terms of the rising factorial basis). But we have

$$L(x_1^{n_1} \cdots x_m^{n_m}) = P(n_1, \ldots, n_m)\, x_1^{n_1} \cdots x_m^{n_m}$$

in the differential case, and

$$L(x_1^{\overline{n}_1} \cdots x_m^{\overline{n}_m}) = P(n_1, \ldots, n_m)\, x_1^{\overline{n}_1} \cdots x_m^{\overline{n}_m}$$

in the difference case. So L has a monomial solution iff there exist $n_1, \ldots, n_m \in \mathbb{N}$ such that $P(n_1, \ldots, n_m) = 0$. Hence an algorithm for deciding existence of non-zero polynomial solutions of linear partial differential or difference operators with polynomial coefficients would give rise to an algorithm for deciding existence of non-negative integral solutions of polynomial equations with integral coefficients, in contradiction to the DMPR theorem.

Remark 1 *In [9, Thm. 4.11], it is shown that there is no algorithm for deciding existence of formal power series solutions of an inhomogeneous partial differential equations with polynomial coefficients and right-hand side equal to* 1 *(see also Problem 13 in [15, p. 62] and Problem 3 in [18, p. 27]). Even though the same polynomial* P *in* θ_i *is used in the proof of Theorem 1 as in the proof of [9, Thm. 4.11], it is not at all clear whether the former follows from the latter.*

Remark 2 *Since the DMPR theorem holds for any fixed number* $m \geq 9$ *of variables as well (cf. [17]), the same is true of Theorem 1.*

3 Equations with constant coefficients

In this section we assume that L is an operator of the form (1), (2) with coefficients $a_\mu \in K$.

For $i = 1, \ldots, m$, let

$$\delta_i = \begin{cases} D_i, \text{ in the differential case,} \\ \Delta_i, \text{ in the difference case.} \end{cases}$$

Lemma 1 *Let* $L \in K[\delta_1, \ldots, \delta_m]$ *and let the equation*

$$L(y) = 0 \tag{5}$$

have a polynomial solution of degree $k \geq 0$. *Then this equation has a polynomial solution of degree* j *for* $j = 0, 1, \ldots, k$.

Proof. By induction on j from k down to 0.

$j = k$: This holds by assumption.

$0 \leq j \leq k-1$: By inductive hypothesis, equation (5) has a polynomial solution $y(x) = p(x_1, \ldots, x_m)$ of degree $j+1$. Let $t = cx_1^{n_1} \cdots x_m^{n_m}$ be a term of the polynomial p such that $\deg t = j+1$, and let $i \in \{1, \ldots, m\}$ be such that $\deg_{x_i} t > 0$. Then $\delta_i(p)$ has the desired properties. Indeed, $\deg \delta_i(p) = \deg p - 1 = j$ and, since operators with constant coefficients commute, $L(\delta_i(p)) = \delta_i(L(p)) = \delta_i(0) = 0$.

Theorem 2 *Let* $m \geq 2$, *and let* $L \in K[\delta_1, \ldots, \delta_m]$ *be a linear partial differential or difference operator with constant coefficients. The following assertions are equivalent:*

(a) For each $k \in \mathbb{N}$, L *has a polynomial solution of degree* k.
(b) L *has a non-zero polynomial solution.*
(c) $L(1) = 0$.

Proof. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Assume that L has a non-zero polynomial solution $p(x)$. Then $\deg p \geq 0$, and by Lemma 1, L has a solution of degree 0. Hence $L(1) = 0$ as well.

(c) $\Rightarrow$ (a): It is well known that, in m variables, the number of monomials of degree d is $\binom{d+m-1}{m-1}$, and the number of monomials of degree at most d is $\binom{d+m}{m}$. Set

$$d = \binom{k+1}{2}$$

and denote by $\mathcal{M}$ the set of all monomials in the variables $x_1, \ldots, x_m$ of degrees k, $k+1, \ldots, d$. Then

$$|\mathcal{M}| = \binom{d+m}{m} - \binom{k-1+m}{m}.$$

Let $\mathcal{P} = L(\mathcal{M})$. From (c) it follows that the free term c_0 of L is equal to 0, hence $\deg L(t) < \deg t$ for any $t \in \mathcal{M}$, and so the degrees of polynomials in $\mathcal{P}$ do not exceed $d-1$.

If $\mathcal{M}$ contains two distinct monomials m_1 and m_2 such that $L(m_1) = L(m_2)$ then $p = m_1 - m_2$ is a non-zero polynomial solution of L of degree at least k.

Otherwise, L is injective on $\mathcal{M}$, and so $|\mathcal{P}| = |\mathcal{M}|$. From $d+1 > k(k+1)/2$, $d \geq k$ and $m \geq 2$ it follows that

$$\begin{aligned}(d+1)^{\overline{m}} - d^{\overline{m}} &= m(d+1)^{\overline{m-1}} \\ &= m(d+1)\ (d+2)^{\overline{m-2}} \\ &> m\,\frac{k(k+1)}{2}\ (k+2)^{\overline{m-2}} \\ &\geq k^{\overline{m}},\end{aligned}$$

hence $(d+1)^{\overline{m}} - k^{\overline{m}} > d^{\overline{m}}$. Dividing this by $m!$ we see that

$$\begin{aligned}|\mathcal{P}| \ = \ |\mathcal{M}| &= \binom{d+m}{m} - \binom{k-1+m}{m} \\ &> \binom{d-1+m}{m}.\end{aligned}$$

Since the dimension of the space of polynomials of degrees at most $d-1$ is $\binom{d-1+m}{m}$, it follows that the set $\mathcal{P}$ is linearly dependent. Hence there is a nontrivial linear combination p of the monomials in $\mathcal{M}$ such that $L(p) = 0$. Clearly, p is a non-zero polynomial solution of L of degree at least k.

In either case (if L is injective on $\mathcal{M}$ or not) we have obtained a non-zero polynomial solution of L of degree at least k. By Lemma 1 it follows that L has a non-zero polynomial solution of degree k.

4 q-Difference equations with constant coefficients

The question of decidability of the existence of non-zero polynomial solutions of an arbitrary q-difference equation with polynomial coefficients is still open. In this section we consider equations with coefficients from K, assuming that $K = k(q)$ where k is a subfield of K and q is transcendental over k.

We write $Q = (Q_1, \ldots, Q_m)$ for partial q-shift operators where

$$Q_i f(x) = f(x_1, \ldots, qx_i, \ldots, x_m),$$

so that a partial linear q-difference operator with constant coefficients of order at most r is written as

$$L \ = \ \sum_{|\mu| \leq r} a_\mu Q^\mu \tag{6}$$

with $a_\mu \in K$. Clearly, for multiindices μ and α,

$$\begin{aligned}Q^\mu x^\alpha &= Q_1^{\mu_1} \cdots Q_m^{\mu_m}\, x_1^{\alpha_1} \cdots x_m^{\alpha_m} \\ &= Q_1^{\mu_1} x_1^{\alpha_1} \cdots Q_m^{\mu_m} x_m^{\alpha_m} \\ &= (q^{\mu_1} x_1)^{\alpha_1} \cdots (q^{\mu_m} x_m)^{\alpha_m} \\ &= q^{\mu_1\alpha_1 + \cdots + \mu_m\alpha_m} x_1^{\alpha_1} \cdots x_m^{\alpha_m} \\ &= q^{\mu\cdot\alpha} x^\alpha.\end{aligned} \tag{7}$$

Lemma 2 *An operator L of the form (6) has a nonzero polynomial solution iff it has a monomial solution.*

Proof. If L has a monomial solution x^α, then x^α is also a non-zero polynomial solution of L.

Conversely, assume that $p(x) \in K[x]$ is a non-zero polynomial solution of L. Write

$$p(x) \;=\; \sum_\alpha c_\alpha x^\alpha$$

where only finitely many c_α are non-zero, and define its support by

$$\operatorname{supp} p \;=\; \{\alpha \in \mathbb{N}^m;\; c_\alpha \neq 0\}.$$

Then

$$\begin{aligned} L(p) &= \sum_\mu a_\mu \sum_\alpha c_\alpha Q^\mu x^\alpha \\ &= \sum_\mu a_\mu \sum_\alpha c_\alpha q^{\mu\cdot\alpha} x^\alpha \quad \text{(by (7))} \\ &= \sum_\alpha c_\alpha \left(\sum_\mu a_\mu q^{\mu\cdot\alpha} \right) x^\alpha, \end{aligned}$$

hence from $L(p) = 0$ it follows that

$$\sum_\mu a_\mu q^{\mu\cdot\alpha} \;=\; 0$$

whenever $c_\alpha \neq 0$. Therefore, by (7),

$$L(x^\alpha) \;=\; \sum_\mu a_\mu Q^\mu x^\alpha \;=\; \sum_\mu a_\mu q^{\mu\cdot\alpha} x^\alpha \;=\; 0$$

for all such α, so x^α is a monomial solution of L for each $\alpha \in \operatorname{supp} p$.

By clearing denominators in the equation $L(y) = 0$, we can assume that the coefficients of L are in $k[q]$, hence we can rewrite

$$L \;=\; \sum_\mu \sum_i a_{\mu,i} q^i Q^\mu \tag{8}$$

where only finitely many $a_{\mu,i} \in k$ are non-zero. Define

$$\operatorname{supp} L \;=\; \{(\mu, i) \in \mathbb{N}^{m+1};\; a_{\mu,i} \neq 0\}.$$

Let P be a partition of $\operatorname{supp} L$. We call such a partition *balanced* if

$$\sum_{(\mu,i)\in B} a_{\mu,i} \;=\; 0$$

for every block $B \in P$. To any $\alpha \in \mathbb{N}^m$ we assign the partition $P_{L,\alpha}$ of $\operatorname{supp} L$ induced by the equivalence relation

$$(\mu, i) \;\sim\; (\nu, j) \quad \text{iff} \quad \mu\cdot\alpha + i = \nu\cdot\alpha + j.$$

Lemma 3 $L(x^\alpha) = 0$ *iff* $P_{L,\alpha}$ *is balanced.*

Proof.

$$L(x^\alpha) = \sum_{(\mu,i)\in \operatorname{supp} L} a_{\mu,i} q^i Q^\mu x^\alpha = \sum_{(\mu,i)\in \operatorname{supp} L} a_{\mu,i} q^{\mu\cdot\alpha+i} x^\alpha,$$

hence $L(x^\alpha) = 0$ iff $\sum_{(\mu,i)\in \operatorname{supp} L} a_{\mu,i} q^{\mu\cdot\alpha+i} = 0$. Since q is transcendental over k, the latter equality holds iff $\sum_{(\mu,i)\in B} a_{\mu,i} = 0$ for every block $B \in P_{L,\alpha}$, i.e., iff $P_{L,\alpha}$ is balanced.

Corollary 1 *L in (8) has a non-zero polynomial solution iff there is an $\alpha \in \mathbb{N}^m$ such that $P_{L,\alpha}$ is balanced.*

Proof. This follows from Lemmas 2 and 3.

Corollary 2 *If L in (8) has a non-zero polynomial solution then $\sum_\mu a_\mu = 0$.*

Proof. This follows from Corollary 1 since if $P_{L,\alpha}$ is balanced then $\sum_\mu a_\mu = 0$.

From Corollary 1 we obtain the following algorithm for deciding existence of non-zero polynomial solutions of L in (8):

for each balanced partition P of $\operatorname{supp} L$ **do**
  let S be the system of $|\operatorname{supp} L|$ linear equations

$$\mu\cdot\alpha + i \;=\; v_B, \quad (\mu, i) \in B \in P$$

  for the unknown vectors α and $v = (v_B)_{B\in P}$
  if S has a solution (α, v) with $\alpha \in \mathbb{N}^m$ **then**
    return "yes" and stop
return "no".

Corollary 3 *The problem of the existence of non-zero polynomial solutions of partial linear q-difference operators with constant coefficients is decidable.*

Note that one can convert the above decision algorithm into a procedure for providing a finite description of a (possibly infinite) basis for the space of all polynomial solutions of equation $L(y) = 0$.

The following simple examples demonstrate that, in contrast with the differential and difference cases, there are partial linear q-difference equations with constant coefficients such that the dimension of their space of polynomial solutions is: a) infinite, b) finite positive, c) zero.

Example 1 *Let $L_1 = Q_1^2 Q_2 + qQ_1Q_2^2 - 2q^2Q_2^3$. Then*

$$L_1(x_1^{\alpha_1} x_2^{\alpha_2}) \;=\; (q^{2\alpha_1+\alpha_2} + q^{\alpha_1+2\alpha_2+1} - 2q^{3\alpha_2+2}) x_1^{\alpha_1} x_2^{\alpha_2}$$

and $\operatorname{supp} L_1 = \{(2, 1, 0), (1, 2, 1), (0, 3, 2)\}$. *The only balanced partition of this set is the single-block partition $P = \{\operatorname{supp} L_1\}$, and we obtain the system of linear equations*

$$2\alpha_1 + \alpha_2 \;=\; \alpha_1 + 2\alpha_2 + 1 \;=\; 3\alpha_2 + 2$$

for α_1 and α_2. This system has infinitely many non-negative integer solutions of the form $\alpha_1 = t + 1$, $\alpha_2 = t$ where $t \in \mathbb{N}$. Therefore every non-zero linear combination of monomials of the form $x_1^{t+1}x_2^t$ where $t \in \mathbb{N}$, is a non-zero polynomial solution of the operator L_1.

Example 2 *Let* $L_2 = Q_1^4 Q_2 + Q_1^2 Q_2^3 - 2q^2 Q_1^3$. *Then*

$$L_2(x_1^{\alpha_1} x_2^{\alpha_2}) \;=\; (q^{4\alpha_1+\alpha_2} + q^{2\alpha_1+3\alpha_2} - 2q^{3\alpha_1+2}) x_1^{\alpha_1} x_2^{\alpha_2}$$

and $\operatorname{supp} L_2 = \{(4,1,0),(2,3,0),(3,0,2)\}$. *Again the only balanced partition of this set is the single-block partition, and we obtain the system of linear equations*

$$4\alpha_1 + \alpha_2 \;=\; 2\alpha_1 + 3\alpha_2 \;=\; 3\alpha_1 + 2$$

for α_1 *and* α_2. *The only solution of this system is* $\alpha_1 = \alpha_2 = 1$, *so the operator* L_2 *has a 1-dimensional space of polynomial solutions spanned by* $x_1 x_2$.

Example 3 *Let* $L_3 = Q_1^2 Q_2 + Q_1 Q_2^2 - 2q Q_2^3$. *Then*

$$L_3(x_1^{\alpha_1} x_2^{\alpha_2}) \;=\; (q^{2\alpha_1+\alpha_2} + q^{\alpha_1+2\alpha_2} - 2q^{3\alpha_2+1}) x_1^{\alpha_1} x_2^{\alpha_2}$$

and $\operatorname{supp} L_3 = \{(2,1,0),(1,2,0),(0,3,1)\}$. *Once again the only balanced partition of this set is the single-block partition, and we obtain the system of linear equations*

$$2\alpha_1 + \alpha_2 \;=\; \alpha_1 + 2\alpha_2 \;=\; 3\alpha_2 + 1$$

for α_1 *and* α_2. *Since this system has no solution, the operator* L_3 *has no non-zero polynomial solution.*

5 Conclusion

In this paper, we have investigated the computational problem of existence of non-zero polynomial solutions of linear partial differential and difference equations with polynomial coefficients. We have shown that the problem is algorithmically undecidable. This means that there is no hope of having a general algorithm for deciding existence of such solutions in a computer algebra system now or ever in the future.

However, we have shown that the existence problem is decidable in the case of partial linear differential or difference equations with constant coefficients: such an equation $L(y) = 0$ has non-zero polynomial solutions iff $L(1) = 0$. Moreover, when the latter condition is satisfied, this equation has polynomial solutions of any desired degree. A number of methods exist to search for such solutions efficiently (see, e.g., [19, 22]).

For partial equations with constant coefficients in the q-difference case which is of interest in combinatorics, we have formulated and proved a necessary condition for existence of non-zero polynomial solutions: $(L(1))(1) = 0$ (note that $L(1)$ is a polynomial in q). We have also shown that when the latter condition is satisfied, the dimension of the space of polynomial solutions in some particular cases can be finite and even zero (then no non-zero polynomial solutions exist). An explicit description of this space can be obtained algorithmically, and the corresponding algorithm is straightforward to implement in any computer algebra system.

The following interesting problems remain open:

1. (Un)decidability of existence of non-zero polynomial solutions of a given linear partial differential or difference equation with polynomial coefficients when the number of variables m is between 2 and 8.

2. (Un)decidability of existence of non-zero polynomial solutions of a given linear partial q-difference equation with polynomial coefficients (both the general problem when the number m of variables is arbitrary, and the problems related to particular numbers of variables).

Problem 1 seems to be very hard since the problem of solvability of Diophantine equations in m variables with m between 2 and 8 is still open (cf. [17]). Concerning

Problem 2, note that in the ordinary case ($m = 1$), certain existence problems in the q-difference case are decidable although the analogous problems in the differential and difference cases are not (see, e.g., [3]). An example of an open problem which might be easier than Problems 1 or 2 is the existence problem of non-zero polynomial solutions for *q-differential* equations.

We will continue to pursue this line of inquiry.

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