Minimal Decomposition of Indefinite Hypergeometric Sums

S.Abramov, M.Petkovšek

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Abstract

We present an algorithm which, given a hypergeometric term $T(n)$, constructs hypergeometric terms $T_1(n)$ and $T_2(n)$ such that $T(n) = T_1(n+1) - T_1(n) + T_2(n)$, and $T_2(n)$ is minimal in some sense. This solves the decomposition problem for indefinite sums of hypergeometric terms: $T_1(n+1) - T_1(n)$ is the "summable part" and $T_2(n)$ the "non-summable part" of $T(n)$.

1 Introduction

A sequence $T(n)$ is a *hypergeometric term* (or simply a *term*) if the ratio $T(n+1)/T(n)$ is a rational function of n. We call this function the *certificate* of T.

The well-known Gosper's algorithm [5] solves the *problem of indefinite hypergeometric summation*: Given a hypergeometric term $T(n)$, find another hypergeometric term $T_1(n)$ such that

$$T(n) = T_1(n+1) - T_1(n), \tag{1}$$

provided that such a term exists. If it does, and if $T(k)$ and $T_1(k)$ are defined for $k = n_0, n_0+1, \ldots, n$, then we obtain from (1) the summation identity

$$\sum_{k=n_0}^{n-1} T(k) = T_1(n) - T_1(n_0).$$

If no hypergeometric term $T_1(n)$ satisfies (1), we can ask for *two* hypergeometric terms $T_1(n)$ and $T_2(n)$ such that

$$T(n) = T_1(n+1) - T_1(n) + T_2(n) \tag{2}$$

and $T_2(n)$ is minimal in some sense. We call this the *decomposition problem for indefinite hypergeometric sums*. In this case

$$\sum_{k=n_0}^{n-1} T(k) = T_1(n) - T_1(n_0) + \sum_{k=n_0}^{n-1} T_2(k). \tag{3}$$

As a special case when $T(n)$ is a rational function of n, the algorithms given in [1, 3, 8] yield rational functions $T_1(n)$ and $T_2(n)$ such that (2), (3) hold and the denominator of T_2 is of the least possible degree. These approaches are unified in [6].

Throughout the paper K is a field of characteristic zero, n an indeterminate, and E the shift operator on sequences with elements in K defined by $ET(n) = T(n+1)$. We write $p \perp q$ to indicate that the polynomials $p, q \in K[n]$ are relatively prime. The *leading coefficient* of a rational function is the quotient of the leading coefficients of its numerator and denominator. A rational function is *monic* if its leading coefficient is 1.

Following [6] we introduce the notion of shift-equivalence among polynomials.

Definition 1 *Polynomials $p, q \in K[n]$ are* shift-equivalent *if $p = E^k q$ for some $k \in \mathbb{Z}$. In this case we write $p \overset{\text{sh}}{\sim} q$. A monic rational function is* shift-homogeneous *if all nonconstant irreducible monic factors of its numerator and denominator belong to the same shift-equivalence class.*

By grouping together shift-equivalent irreducible monic factors of its numerator and denominator every rational function can be written in the form

$$R(n) = z\, R_1(n) R_2(n) \cdots R_k(n) \tag{4}$$

where $z \in K$, $k \geq 0$, each R_i is a shift-homogeneous rational function, and any two nonconstant monic irreducible factors p of R_i and q of R_j are pairwise shift-inequivalent whenever $i \neq j$. We call (4) a *shift-homogeneous factorization* of R. It is clear that the shift-homogeneous factorization is unique up to the order of the factors.

Definition 2 *Let $r, s \in K[n]$. If $r \perp E^k s$ for all $k \in \mathbb{Z}$ then the rational function r/s is* shift-reduced*. If $r \perp E^k r$ for all $k \in \mathbb{Z} \setminus \{0\}$ then the polynomial r is* shift-free*.*

Now we define a normal form for rational functions which reveals the shift structure of the factors. We need this normal form to define a measure of complexity of a hypergeometric term.

Definition 3 *Let $R \in K(n)$ be a nonzero rational function. If there exist $z \in K$ and monic polynomials $r, s, u, v \in K[n]$ such that $R = z \cdot \frac{r}{s} \cdot \frac{EF}{F}$ where r/s is shift-reduced, $F = u/v$ and $u \perp v$, then (z, r, s, u, v) is a* rational normal form (RNF) *of R. The rational function zr/s is the* kernel *of this RNF.*

In [4] it is shown that every nonzero rational function has an RNF, and that regarding uniqueness we have the following results:

Lemma 1 *Let $R \in K(n)$ be shift-homogeneous. If $(1, r, s, u, v)$ and $(1, r_1, s_1, u_1, v_1)$ are two RNF's of R then $r = r_1 = 1$ and $\deg s = \deg s_1$, or $s = s_1 = 1$ and $\deg r = \deg r_1$.*

Theorem 1 *Let (z, r, s, u, v) and (z', r', s', u', v') be two RNF's of $R \in K(n) \setminus \{0\}$. Then*

(i) $z = z'$,

(ii) $\deg r = \deg r'$ *and* $\deg s = \deg s'$,

(iii) *there is a one-to-one correspondence f between the multisets of nonconstant irreducible monic factors of r and r' such that $p \overset{\text{sh}}{\sim} f(p)$ for all $p \,|\, r$,*

(iv) *there is a one-to-one correspondence g between the multisets of nonconstant irreducible monic factors of s and s' such that $q \overset{\text{sh}}{\sim} g(q)$ for all $q \,|\, s$.*

Now we can formulate the decomposition problem for indefinite hypergeometric sums in the following way:

Given a hypergeometric term T, find hypergeometric terms T_1, T_2 such that

$$T = (E - 1)T_1 + T_2 \tag{5}$$

and the certificate ET_2/T_2 has an RNF

$$(z, r, s, u, v) \tag{6}$$

with v of minimal degree.

This formulation agrees with the decomposition problem for indefinite sums of rational functions [1, 3, 8] because if $T_2 \in K(n)$ then $r = s = 1$ and v is the denominator of T_2.

Definition 4 *If (z, r_1, s_1, u_1, v_1) and (z, r_2, s_2, u_2, v_2) are two RNF's for the same rational function we state that the former is* simpler *than the latter if $\deg v_1 < \deg v_2$.*

If the terms T, T_1, T_2 satisfy (5) then they are *similar*, i.e., there exist $R_1, R_2 \in K(n)$ such that $T_1 = R_1 T$ and $T_2 = R_2 T$ (cf. [7, Prop. 5.6.2]).

Definition 5 *A rational function $F \in K(n)$ is* adequate *to a hypergeometric term T if the certificate ET/T has an RNF with F as its kernel.*

Let T, T_1, T_2 satisfy (5). By the similarity of these terms each rational function that is adequate to one is also adequate to the other two.

We will describe a solution to the decomposition problem in three stages:

(A) We describe an algorithm which, given a term T, constructs terms T_1, T_2 similar to T, and a rational function F adequate to T, T_1, T_2, such that if one restricts attention to RNF's with kernel F then T_1, T_2 solve the decomposition problem.

(B) We prove that for any other rational function F' adequate to T and for any term T_1' similar to T, the RNF (with kernel F') of ET_2'/T_2' where $T_2' = T - (E-1)T_1'$ is not simpler than the RNF of the certificate of T_2 constructed by the algorithm in (A).

(C) Finally we show how sometimes one can "improve" T_2 without changing $\deg v$ in such a way that $\deg u$ in (6) is not too large. In particular, we show how the decomposition algorithm also covers the case when T is summable.

For notational convenience, we sometimes write an RNF of a rational function R as $G\frac{EF}{F}$ where $G = z\frac{r}{s}$, $F = \frac{u}{v}$, instead of as (z, r, s, u, v).

2 Regular descriptions of terms

Let D be a rational function and $n_0 \in \mathbb{Z}$. Set $\prod_{k=n_0}^{n_0-1} D(k) = 1$.

Definition 6 *Let $n_0 \in \mathbb{Z}$ and $D, U \in K(n)$. The triple (D, U, n_0)* regularly describes *a term $T(n)$ (in other words this triple is a* regular description *of $T(n)$) if for all integer $n \geq n_0$*

- *D, U have neither a pole nor a zero,*
- *$T(n) = U(n) \prod_{k=n_0}^{n-1} D(k)$.*

If $U(n) \prod_{k=n_0}^{n-1} D(k)$ is a regular description of $T(n)$ then the certificate of $T(n)$ is equal to $D\frac{EU}{U}$. On the other hand, if $F\frac{EV}{V}$ is an RNF of the certificate of $T(n)$ and $n_0 \in \mathbb{Z}$ is such that for all $n \geq n_0$ the rational functions F, V have neither a pole nor a zero, and $T(n)$ is defined, then the triple

$$(F,\ \frac{T(n_0)}{V(n_0)}V,\ n_0)$$

regularly describes $T(n)$, and F is shift-reduced in this regular description.

Lemma 2 *Let (D, U, n_0) regularly describe a term T, $n_1 \geq n_0$, and*

$$V(n) = U(n)\frac{T(n_1)}{U(n_1)} = U(n)\prod_{k=n_0}^{n_1-1} D(k).$$

Then (D, V, n_1) regularly describes T.

Proof: A direct check. □

Theorem 2 *Given regular descriptions (D_1, U_1, n_0), (D_2, U_2, n_1) of similar terms T_1, T_2, one can construct*

(a) regular descriptions of T_1, T_2 that have the form (D, W_1, n_2), (D, W_2, n_2);

(b) a regular description of $T_1 + T_2$.

Proof: Since T_1 and T_2 are similar, the ratio of their certificates is of the form ER/R, $R \in K(n)$ where R satisfies the equalities $T_1 = RT_2$ and

$$\frac{D_1}{D_2} \cdot \frac{E(U_1/U_2)}{(U_1/U_2)} = \frac{ER}{R}.$$

This implies that

$$\frac{D_1}{D_2} = \frac{ES}{S}$$

for $S = RU_2/U_1$. The construction of an RNF of D_1/D_2 yields

$$\frac{D_1}{D_2} = \frac{ES'}{S'},$$

where S' is equal to S up to a constant factor. Setting $V = S'U_1/U_2$ results in $T_1 = \mu V T_2$, $\mu \in K$. Take $n_2 \geq max\{n_0, n_1\}$ such that V has neither a pole nor a zero at any integer $n \geq n_2$. Then by Lemma 2 there exist $\alpha_1, \alpha_2 \in K$ such that $(D_1, \alpha_1 U_1, n_2)$, $(D_2, \alpha_2 U_2, n_2)$ regularly describe T_1 and T_2. This implies

$$\alpha_1 U_1(n_2) = \mu \alpha_2 V(n_2) U_2(n_2),$$

and we can find μ. Set $D = D_2$, $W_2 = \alpha_2 U_2$, $W_1 = \mu V W_2$. This gives (a). Now take n_3 such that $W_1 + W_2$ has neither a pole nor a zero at any integer $n \geq n_3$. Then by Lemma 2 there exist regular descriptions of the form $(D, \beta W_1, n_3)$, $(D, \beta W_2, n_3)$, $\beta = \prod_{k=n_2}^{n_3-1} D(k)$, of T_1, T_2. Hence, $(D, \beta(W_1 + W_2), n_3)$ is a regular description of $T_1 + T_2$. This gives (b). □

Lemma 3 *Let the triples* (D, U, n_0) *and* (D, U_1, n_0) *regularly describe (similar) terms* T *and* T_1*. Then the certificate of the term* $T_2 = T - (E-1)T_1$ *is*

$$D\frac{EU_2}{U_2} \tag{7}$$

where

$$U_2 = U - D(EU_1) + U_1. \tag{8}$$

Proof: For all integer $n \geq n_0$ we have

$$\begin{aligned} T_2(n) &= U(n) \prod_{k=n_0}^{n-1} D(k) - (E-1)\left(U_1(n) \prod_{k=n_0}^{n-1} D(k)\right) \\ &= U(n) \prod_{k=n_0}^{n-1} D(k) - (EU_1(n)) \prod_{k=n_0}^{n} D(k) \\ &+ U_1(n) \prod_{k=n_0}^{n-1} D(k) \\ &= (U(n) - (EU_1(n))D(n) + U_1(n)) \prod_{k=n_0}^{n-1} D(k). \end{aligned}$$

It follows that the values of ET_2/T_2 are equal to the values of (7) for all integer $n \geq n_0$. This proves the claim. □

Theorem 3 *Let the triple* (D, U, n_0) *regularly describe a term* T*. Let* $U_1, U_2 \in K(n)$ *be such that equality (8) holds. Then there exist terms* T_1, T_2 *such that* $T = (E-1)T_1 + T_2$*. If* $U_i = 0$ *then* $T_i = 0$*, otherwise* T_i *has a regular description of the form* $(D, \beta U_i, n_1)$*,* $n_1 \geq n_0$*,* $i = 1, 2$*,* $\beta \in K$*.*

Proof: Choose $n_1 \geq n_0$ such that if $U_i \neq 0$ then U_i has neither a pole nor a zero for all $n \geq n_1$, $i = 1, 2$. Multiplying (8) by $\prod_{k=n_1}^{n-1} D(k)$ we get

$$U_2(n) \prod_{k=n_1}^{n-1} D(k) = U(n) \prod_{k=n_1}^{n-1} D(k) - (E-1) \left(U_1(n) \prod_{k=n_1}^{n-1} D(k) \right).$$

But

$$U(n) \prod_{k=n_1}^{n-1} D(k) = \frac{1}{\prod_{k=n_0}^{n_1-1} D(k)} T(n).$$

Hence we can take

$$T_1 = \beta U_1(n) \prod_{k=n_1}^{n-1} D(k), \tag{9}$$

$$T_2 = \beta U_2(n) \prod_{k=n_1}^{n-1} D(k), \tag{10}$$

where $\beta = \prod_{k=n_0}^{n_1-1} D(k)$. □

3 General approach and an algorithm

Lemma 4 *Let $D, U \in K(n)$, $D = d_1/d_2$, $U = u_1/u_2$ where $d_1, d_2, u_1, u_2 \in K[n]$, and D is shift-reduced. Then there exist $U_1, U_2 \in K(n)$ such that (8) hold and*

$$U_2 = \frac{v_1}{(E^{-1}d_1)^i d_2^j v_2}, \tag{11}$$

where $i, j \in \{0, 1\}$, $v_1, v_2 \in K[n]$, and for any irreducible $p \in K[n]$ dividing v_2 and any $h \in \mathbb{Z}$ the following conditions are satisfied:

$$E^h p \,|\, v_2 \Rightarrow h = 0, \tag{12}$$

$$E^h p \,|\, d_1 \Rightarrow h < 0, \tag{13}$$

$$E^h p \,|\, d_2 \Rightarrow h > 0 \tag{14}$$

(in words, (12) means that v_2 is shift-free).

Proof: Let q be an irreducible in $K[n]$ and $u_2 = u_2' q^k$ where $q \perp u_2'$ and $k > 0$. Then there are $a, b \in K[n]$ such that

$$U = \frac{a}{u_2'} + \frac{b}{q^k}. \tag{15}$$

We distinguish two cases.

a) There is an integer $h \leq 0$ such that $E^h q \,|\, d_2$. Let $U_1{}' = E^{-1}\left(b/(Dq^k)\right)$. Then $D(EU_1{}') = b/q^k$, so $U - D(EU_1{}') + U_1{}'$ can be written as

$$\frac{c_0}{u_2'} + \frac{c_1}{E^{-1}d_1} + \frac{c_2}{(E^{-1}q)^l}$$

where $l \leq k$, $c_0, c_1, c_2 \in K[n]$.

b) There is an integer $h \geq 0$ such that $E^h q \,|\, d_1$. Let $U_1{}' = -b/q^k$. Then $U - D(EU_1{}') + U_1{}'$ can be written as

$$\frac{c_0}{u_2'} + \frac{c_1}{d_2} + \frac{c_2}{(Eq)^l}$$

where $l \leq k,\ c_0, c_1, c_2 \in K[n]$.

Since D is shift-reduced, at most one of the cases a), b) can occur. Repeating these steps if necessary (using $U_1'', U_1''', \ldots$) we obtain a rational function $U - DE(U_1' + U_1'' + \cdots) + (U_1' + U_1'' + \cdots)$ whose denominator is divisible by at most one polynomial of the form $E^\gamma q$. If such a γ exists then for $p = E^\gamma q$ we have (12), (13), and (14). Similarly we can proceed with the irreducible factors of u_2 that are different from $E^\sigma q, \sigma \in \mathbb{Z}$.

Suppose that u_2 is not shift-free. If there is an integer $h > 0$ such that $E^h q | u_2$ for some irreducible q and if (15) holds, then we can transform U as it is described in b). Similarly we can use the way described in a) if we have $h < 0$. □

Using Lemmas 4 and 3 one obtains the terms T_1 and $T_2 = T - (E-1)T_1$ with certificates in the form $D\frac{EU_1}{U_1}$ and $D\frac{EU_2}{U_2}$, respectively. The certificate of T_2 can be rewritten in a simpler form: suppose that U_2 has the form (11), then we can remove the factors $(E^{-1}d_1)^i, d_2^j,\ i, j \in \{0, 1\}$ from the denominator of U_2. Since

$$D\frac{EU_2}{U_2} = \frac{d_1 \left(\frac{E^{-1}d_1}{d_1}\right)^i}{d_2 \left(\frac{Ed_2}{d_2}\right)^j} \frac{E\left(\frac{v_1}{v_2}\right)}{\frac{v_1}{v_2}},$$

by setting

$$F = \frac{d_1 \left(\frac{E^{-1}d_1}{d_1}\right)^i}{d_2 \left(\frac{Ed_2}{d_2}\right)^j}, \quad V = \frac{v_1}{v_2},$$

we get the certificate of T_2 in the form $F\frac{EV}{V}$. Reformulating properties (12), (13), (14) in terms of F and V, we have the following theorem.

Theorem 4 *Let T be a term. Then it is possible to find a term T_1 similar to T, and a shift-reduced rational function $F = f_1/f_2$ that is adequate to T, T_1, such that the certificate of the term $T_2 = T - (E-1)T_1$, written in the RNF with F as the kernel:*

$$F\frac{EV}{V},\ V = \frac{v_1}{v_2},\ v_1 \perp v_2,$$

has the following properties **Pa**, **Pb**:

Pa : *v_2 is shift-free;*

Pb : *if p is an irreducible from $K[n]$ such that $p \mid v_2$, then*

$$E^h p \mid f_1 \quad \Rightarrow \quad h < 0, \tag{16}$$
$$E^h p \mid f_2 \quad \Rightarrow \quad h > 0. \tag{17}$$

If we use (10) and thus find a regular description $(D, \beta U_2, n_1)$ of T_2 and if n_1 is such that F, V have neither a pole nor a zero for all $n \geq n_1$, then we can get the regular description of T_2

$$(F,\ \frac{\beta U_2(n_1)}{V(n_1)}V,\ n_1) \tag{18}$$

as well; in so doing we can get $U_2(n_1)$ from (8):

$$U_2(n_1) = U(n_1) - D(n_1)(U(n_1+1) - U_1(n_1)) + U_1(n_1). \tag{19}$$

As the proofs of Lemmas 4 and 3 are constructive, we now describe an algorithm to compute the terms T_1, T_2 mentioned in Theorem 4. In case a) of the proof of Lemma 4 we considered the irreducibles q's and the integers h's such that $q|u_2$, $E^h q|d_2$, $h \leq 0$. All the q's (say $q_1, \ldots, q_\kappa$) that relate to the minimal possible h can be considered together. Using resultant and gcd techniques, we find the minimal value of h along with

$q' = q_1^{\nu_1} \dots q_\kappa^{\nu_\kappa}$, $q'|u_2$, $\nu_1, \dots, \nu_\kappa > 0$, then compute $\tilde{q} = q_1^{\mu_1} \dots q_\kappa^{\mu_\kappa}$, where $\mu_1, \dots, \mu_\kappa$ are maximal possible values such that $q_1^{\mu_1} \dots q_\kappa^{\mu_\kappa} | u_2$. We can use the following simple algorithm *pump*:

`algorithm` *pump*;
`input:` $f, g \in K[n],\ f|g$;
`output:` $\tilde{f}, \tilde{g} \in K[n],\ f|\tilde{f},\ \tilde{f}\tilde{g} = g,\ \tilde{f} \perp \tilde{g}$;

$\tilde{f} := f;\ \tilde{g} := g/\tilde{f}$;
`repeat` $d = \gcd(\tilde{f}, \tilde{g})$;
$\tilde{f} := \tilde{f}d;\ \tilde{g} := \tilde{g}/d$;
`until` $\deg d = 0$.
return $(\tilde{f},\ \tilde{g})$.

We can compute $(\tilde{q}, \tilde{u}_2) = pump(q, u_2)$ and then use the partial-fraction decomposition

$$U = \frac{\tilde{a}}{\tilde{u}_2} + \frac{\tilde{b}}{\tilde{q}}, \tag{20}$$

where $\tilde{a}, \tilde{b} \in K[n]$ for the decomposition (15) (the polynomial part of U can be arbitrarily shared between the terms in the right hand side of (20)). We can similarly proceed in case b) of the proof of Lemma 4 and arrive to the following algorithm *dcert*:

`algorithm` *dcert*;
`input:` $D, U \in K(n)$ where D is shift-reduced;
`output:` $U_1, F, V \in K(n)$ such that the term T with certificate $D\frac{EU}{U}$ has decomposition $T = T_2 + (E-1)T_1$, where

- if $U_1 = 0$ then $T_1 = 0$, otherwise T_1 has the certificate $D\frac{EU_1}{U_1}$;
- if $V = 0$ then $T_2 = 0$, otherwise T_2 has the certificate $F\frac{EV}{V}$, with F, V having properties **Pa**, **Pb** of Theorem 4;

$U_1 := 0;\ U_2 := U$;
$u_2 := denominator(U_2)$;
$N_1 := -1;\ M := -1$;
$R_1(m) := Res_n(u_2(n+m), d_1(n))$;
$R(m) := Res_n(u_2(n+m), u_2(n))$;
`if` $R_1(m)$ has some nonnegative integer root
`then` $N_1 := \max\{m :\ m \in \mathbb{Z}, R_1(m) = 0\}$
`fi;`
`fi;`
`if` $R(m)$ has some positive integer root
`then` $M := \max\{m :\ m \in \mathbb{Z}, R(m) = 0\}$
`fi;`
$N_1 = \max\{N_1, M\}$;
`for` $h = N_1$ `downto` 0 `do`
$q := \gcd(u_2, E^{-h}d_1)$;
`if` $h > 0$
`then`
$t := u_2/q$;
$q := q\gcd(t, E^{-h}t)$
`fi;`
$(\tilde{q}, \tilde{u}_2) := pump(q, u_2)$;
represent U_2 in the form $U_2 = \frac{\tilde{a}}{\tilde{u}_2} + \frac{\tilde{b}}{\tilde{q}}$,

$\tilde{a}, \tilde{b} \in K[n]$;
$U_1{}' := -\tilde{b}/\tilde{q}$;
$U_2 := U_2 - D(EU_1{}') + U_1{}'$; $U_1 := U_1 + U_1{}'$
$u_2 := \mathit{denominator}(U_2)$;
`od;`
$R_2(m) := \mathit{Res}_n(u_2(n+m), d_2(n))$;
$N_2 := 1$;
`if` $R_2(m)$ has some nonpositive integer root
`then` $N_2 := \min\{m : m \in \mathbb{Z}, R_2(m) = 0\}$
`fi;`
`for` $h = N_2$ `to` 0 `do`
$q := \gcd(u_2, E^{-h} d_2)$;
$(\tilde{q}, \tilde{u}_2) := \mathit{pump}(q, u_2)$;
represent U_2 in the form $U_2 = \frac{\tilde{a}}{\tilde{u}_2} + \frac{\tilde{b}}{\tilde{q}}$,
$\tilde{a}, \tilde{b} \in K[n]$;
$U_1{}' := E^{-1}(\tilde{b}/(D\tilde{q}))$;
$U_2 := U_2 - D(EU_1{}') + U_1{}'$; $U_1 := U_1 + U_1{}'$
$u_2 := \mathit{denominator}(U_2)$;
`od;`
$v_1 := \mathit{numerator}(U_2)$; $v_2 := u_2$;
$w_1 := \gcd(E^{-1} d_1, v_2)$;
$v_2 := v_2/w_1; f_1 := (d_1/Ew_1)w_1$;
$w_2 := \mathit{gcd}(d_2, v_2)$;
$v_2 := v_2/w_2$; $f_2 := (d_2/w_2)Ew_2$;
$F := \frac{f_1}{f_2}$; $V := \frac{v_1}{v_2}$.
return $(U_1,\ F,\ V)$.

It is now easy to write down algorithm *dterm* whose input and output are terms:

`algorithm` *dterm*;
`input:` a term T that is regularly described by the triple (D, U, n_0) with the shift-reduced D;
`output:` terms T_1, T_2 such that $T = (E-1)T_1 + T_2$. The term T_2 either equals zero, or has the certificate $F\frac{EV}{V}$, with F, V having properties **Pa**, **Pb** of Theorem 4;

$(U_1, F, V) := \mathit{dcert}(D, U)$;
`if` $U_1 = 0$
`then` *return* $(0,\ T)$
`fi;`
find integer $n_1 \geq n_0$ s.t. $U_1(n)$, and additionally $F(n), V(n)$ in the case $V \neq 0$, have neither a pole nor a zero for all $n \geq n_1$;
using (9) to obtain the regular description of the term T_1;
`if` $V = 0$
`then` *return* $(T_1,\ 0)$
`fi;`
using (18), (19) to obtain the regular description of the term T_2;
return $(T_1,\ T_2)$.

We now prove that if one restricts attention to RNF's with kernel F then T_1, T_2 solve the decomposition problem (see Theorem 5 below). We will prove in section 4 that these terms give a complete solution to the decomposition problem.

Theorem 5 *Let terms T, T_1, T_1' be such that $T_2 = T - (E-1)T_1$, $T_2' = T - (E-1)T_1'$, and let $F = f_1/f_2$*

be a shift-reduced rational function adequate to these terms. Let $ET_2/T_2 = F\frac{EV}{V}$ *where* $F, V \in K(n)$ *have properties* **Pa** *and* **Pb** *of Theorem 4, and* $ET_2'/T_2' = F\frac{EV'}{V'}$*. If* $V = v_1/v_2$ *and* $V' = v_1'/v_2'$ *where* $v_1, v_2, v_1', v_2' \in K[n]$ *and* $v_1 \perp v_2$ *then* $\deg v_2 \leq \deg v_2'$*.*

Proof: We have

$$T_2' = T_2 - (E-1)(T_1' - T_1).$$

Suppose that the certificate of $T_1' - T_1$ is equal to $F\frac{EW}{W}$ where $W = w_1/w_2$ and $w_1 \perp w_2$. Then, by Lemma 3,

$$\frac{v_1'}{v_2'} = \frac{v_1}{v_2} - \frac{f_1}{f_2}\frac{Ew_1}{Ew_2} + \frac{w_1}{w_2}. \tag{21}$$

Consider an arbitrary irreducible $p \in K[n]$ such that $p \mid v_2$. We set

$$k = \max\{\alpha : p^\alpha \mid v_2\}$$

and claim that $E^l p^k \mid v_2'$ for some $l \in \mathbb{Z}$. Since the pair F, V has property **Pa**, this claim implies the statement of the theorem. Suppose that p^k does not divide v_2'. Equation (21) implies that v_2 and hence p^k divides the lcm of v_2', $f_2 E w_2$, and w_2. By (17) we have $p \perp f_2$, therefore $p^k \mid Ew_2$ or $p^k \mid w_2$.

Let $p^k \mid Ew_2$. Then

$$E^{-1}p^k \mid w_2. \tag{22}$$

Set $l = \min\{m : E^m p^k \mid w_2\}$. Evidently $E^l p^k$ does not divide Ew_2. It follows from (22) that $l \leq -1$; together with (17) this gives $E^l p \perp f_2$. As v_2 is shift-free and $p \mid v_2$, it follows that $E^l p^k$ does not divide v_2. Therefore (21) implies

$$E^l p^k \mid v_2'. \tag{23}$$

Let $p^k \mid w_2$. Then

$$Ep^k \mid Ew_2. \tag{24}$$

Set $l = \max\{m : E^m p^k \mid Ew_2\}$. Evidently $E^l p^k$ does not divide w_2. It follows from (24) that $l \geq 1$; together with (16) this gives $E^l p \perp f_1$. Therefore (21) implies (23) in this case as well. □

Corollary 1 *Let* $F, U, S_1, S_2 \in K(n)$ *where* F *is shift-reduced. Let the rational functions*

$$V_1 = U - FES_1 + S_1,\ V_2 = U - FES_2 + S_2$$

be such that the pairs F, V_1 *and* F, V_2 *have properties* **Pa** *and* **Pb** *of Theorem 4. Then the degrees of the denominators of* V_1 *and* V_2 *are the same.*

4 A verification

If both rational functions F_1, F_2 are adequate to a term T then there exists $G \in K(n)$ such that

$$\frac{F_1}{F_2} = \frac{EG}{G}. \tag{25}$$

Indeed, for some $U_1, U_2 \in K(n)$ we have

$$F_1\frac{EU_1}{U_1} = F_2\frac{EU_2}{U_2},$$

and therefore $G = U_1^{-1}U_2$. The case where $G \in K[n]$ is of special interest.

Theorem 6 *Let* F_1, F_2 *be rational functions that are adequate to a term* T*. Let (25) hold with* $G \in K[n]$*. Let* V *be such that the pair* F_1, V *has properties* **Pa** *and* **Pb***. Then the denominator of* V *is coprime with* G *and the pair* F_2, GV *has properties* **Pa** *and* **Pb***.*

Proof: First we prove that the denominator of V is coprime with G. If they have a common irreducible factor p then the set $\{\nu : E^\nu p \,|\, G\}$ is a non-empty finite set. Suppose that m, M are, resp., the minimal and the maximal elements of this set. Write

$$W = \frac{G}{EG} = \frac{F_2}{F_1} = \frac{w_1}{w_2}, \quad w_1 \perp w_2.$$

Then $E^{M+1}p \,|\, w_2$ and $E^m p \,|\, w_1$. We have $F_2 = WF_1$. As p divides the denominator of V and the pair F_1, V has properties **Pa** and **Pb**, the numerator of F_1 is not divisible by $E^{M+1}p$ since $M+1>0$. Similarly the denominator of F_1 is not divisible by $E^m p$ since $m \leq 0$. Therefore the numerator of F_2 is divisible by $E^m p$ while the denominator of F_2 is divisible by $E^{M+1}p$. But F_2 is shift-reduced by Definition 3(ii), a contradiction.

Now we prove that the pair F_2, GV has properties **Pa** and **Pb**. We have

$$F_2 = \frac{G}{EG} F_1$$

and the pair F_2, GV has property **Pa** because the denominator of GV divides the denominator of V. Now we shall be concerned with **Pb**. Let p be an irreducible from $K[n]$ that divides the denominator of GV and thereby divides the denominator of V. Let $E^h p$, $h \leq 0$, divide the denominator of F_2. Then $E^h p$ does not divide the denominator of F_1 since the pair F_1, V has properties **Pa** and **Pb**. The equality $(EG)F_2 = GF_1$ implies that $E^h p \,|\, EG$. Set $h_0 = \min\{\nu : E^\nu p \,|\, EG\}$. Then $h_0 \leq h \leq 0$ and $E^{h_0 - 1}p \,|\, G$, but $E^{h_0-1}p$ does not divide EG. The denominator of F_1 is not divisible by $E^{h_0-1}p$ since the pair F_1, V has properties **Pa** and **Pb**. Therefore $E^{h_0-1}p$ divides the numerator of F_2. But as $E^h p$ divides the denominator of F_2, this contradicts the fact that F_2 is shift-reduced.

Similarly it can be shown that $E^h p$, $h \geq 0$, cannot divide the numerator of F_2. □

Lemma 5 *Let $F, F_1, U, U_1 \in K(n)$, $G \in K[n]$ be such that $F/F_1 = EG/G$, $G \in K[n]$ and $F\frac{EU}{U} = F_1 \frac{EU_1}{U_1}$. Then there exists $\overline{G} \in K[n]$ such that $\overline{G}U = U_1$ and for any $S \in K(n)$ we have*

$$\overline{G}(U - FES + S) = U_1 - F_1 E(\overline{G}S) + \overline{G}S.$$

Proof: We have

$$\frac{E(U^{-1}U_1)}{U^{-1}U_1} = \frac{EG}{G}.$$

It follows from this that there exists $\alpha \in K$ such that $U^{-1}U_1 = \alpha G$. Set $\overline{G} = \alpha G$. We get

$$\frac{E\overline{G}}{\overline{G}} F_1 = F, \quad U_1 = \overline{G}U.$$

Substituting U_1 for $\overline{G}U$ and $(E\overline{G}/\overline{G})F_1$ for F in $\overline{G}U - (\overline{G}F)ES + \overline{G}S$ gives $U_1 - F_1E(\overline{G}S) + \overline{G}S$. □

Theorem 7 *Let F_1, F_2 be rational functions that are adequate to a term T. Let $U_1, U_2, R \in K(n)$ be such that*

$$F_1 \frac{EU_1}{U_1} = F_2 \frac{EU_2}{U_2} = R. \tag{26}$$

For $S_1, S_2 \in K(n)$, let

$$V_1 = U_1 - F_1 ES_1 + S_1, \; V_2 = U_2 - F_2 ES_2 + S_2 \tag{27}$$

be such that the pairs F_1, V_1 and F_2, V_2 have properties **Pa** *and* **Pb**. *Then the denominators of V_1 and V_2 are of the same degree.*

Proof: First of all we show that there exists a shift-reduced rational function a/b, such that for the rational functions

$$F_0 = \frac{a}{b}, \; F_{-1} = \frac{E^{-1}a}{b}, \; F_{-2} = \frac{a}{Eb}, \; F_{-3} = \frac{E^{-1}a}{Eb} \tag{28}$$

the equalities

$$\frac{F_i}{F_1} = \frac{EG'_i}{G'_i},\ \frac{F_i}{F_2} = \frac{EG''_i}{G''_i},\ G'_i, G''_i \in K[n], \tag{29}$$

hold for $i = -1, -2, -3$. It is sufficient to prove the theorem for shift-homogeneous F_1, F_2 which belong to the same shift-homogeneous class. Then, by Lemma 1, either both F_1 and F_2 are polynomials, or both F_1 and F_2 are reciprocals of polynomials. By Theorem 1(ii) we have

$$F_1 = \prod_{i=1}^{\tau} E^{h_i} p,\ F_2 = \prod_{i=1}^{\tau} E^{l_i} p, \tag{30}$$

in the former case, and

$$F_1 = \frac{1}{\prod_{i=1}^{\tau} E^{h_i} p},\ F_2 = \frac{1}{\prod_{i=1}^{\tau} E^{l_i} p}. \tag{31}$$

in the latter, where $p \in K[n]$ is irreducible. In case of (30), set

$$a = \prod_{i=1}^{\tau} E^{\max\{h_i, l_i\}+1} p,\ b = 1,$$

and in case of (31), set

$$a = 1,\ b = \prod_{i=1}^{\tau} E^{\min\{h_i, l_i\}-1} p.$$

It is easy to see that if $F_0, F_{-1}, F_{-2}, F_{-3}$ are defined as in (28) then the equalities (29) hold for some polynomials G'_i, G''_i.

Considering the RNF of R with the kernel a/b and using algorithm *hgad* we can get $i,\ -3 \le i \le 0$, and $F, U, V, S \in K[n]$ such that

- $F = F_i$;
- $R = F\frac{EU}{U},\ U = \frac{u_1}{u_2},\ u_1 \perp u_2$;
- $V = U - FES + S$;
- the pair F, V has properties **Pa** and **Pb**.

Set

$$G' = G'_i,\ G'' = G''_i$$

for the computed i. By Lemma 5 there exists a polynomial $\overline{G}'$ such that

$$\overline{G}' V = \overline{G}'(U - FES + S) = U_1 - F_1 E(\overline{G}' S) + \overline{G}' S.$$

By Theorem 6 the pair F_1, $U_1 - F_1 E(\overline{G}' S) + \overline{G}' S$ has properties **Pa** and **Pb** and the degree of denominator of $\overline{G}' V$ is equal to the degree of the denominator of V. By Corollary 1 the denominator of V is of the same degree as the denominator of V_1, and similarly for the degrees of the denominators of V and V_2. The claim follows. □

The following theorem is the main theorem of this section.

Theorem 8 *Let T, T_1, T'_1 be similar terms. Let the certificates of the terms $T_2 = T - (E-1)T_1, T'_2 = T - (E-1)T'_1$ be written in the form*

$$F\frac{EV}{V},\ F'\frac{EV'}{V'}$$

with shift-reduced F, F'. Let the pair F, V have properties **Pa** *and* **Pb***. Then the degree of the denominator of V is less than or equal to the degree of the denominator of V'.*

Proof: Since $ET_2'/T_2' = F'(EV'/V')$, where F' is shift-reduced, there exists $U \in K(n)$ such that $ET/T = F'(EU/U)$. Now applying *hgad* to the input F', U yields $\tilde{U}, \tilde{V} \in K(n)$ such that the term T has the decomposition $T = (E-1)\tilde{T}_1 + \tilde{T}_2$ where $\tilde{T}_1$, $\tilde{T}_2$ have the certificates $F'(E\tilde{U}/\tilde{U})$ and $F'(E\tilde{V}/\tilde{V})$, resp. with $F', \tilde{V}$ satisfying properties **Pa** and **Pb**. From Theorem 7, one concludes that the denominators of V and $\tilde{V}$ are of the same degree. The claim now follows since it is clear from Theorem 5 that the degree of the denominator of $\tilde{V}$ is less than or equal to the degree of the denominator of V'. □

5 Remarks on the algorithm and summability

Any algorithm to solve the decomposition problem for rational functions guarantees that if the input rational function T is rational-summable, then it will return a rational function T_1 such that

$$T = (E-1)T_1.$$

It would be logical if an algorithm to solve the same problem for hypergeometric terms follows the same concept. It is clear that by simply applying *dterm*, one will not achieve this goal. The following theorem will be useful.

Theorem 9 *Let T be a summable term, i.e., there exists a term T_1 such that $(E-1)T_1 = T$. Let the certificates of T, T_1 be written in the form $F\frac{EV}{V}$, $F\frac{ER}{R}$ with shift-reduced F. Let the pair F, V have properties* **Pa** *and* **Pb**. *Then V, R are polynomials.*

Proof: Set $V = v_1/v_2$, $F = f_1/f_2$, $v_1, v_2, f_1, f_2 \in K[n]$. If $(E-1)T_1 = T$ then there exists $\mu \in K$ such that $F\,E(\mu R) - \mu R = V$. Set $S = \mu R$ then

$$F E S - S = V, \tag{32}$$

or equivalently,

$$v_2\, f_1\, ES - v_2\, f_2\, S = f_2\, v_1. \tag{33}$$

Since the pair F, V has properties **Pa** and **Pb**, the dispersion of the polynomials $f_2 v_2$, $f_1 v_2$ cannot be a positive integer. Therefore there is no non-polynomial rational function S that satisfies (33) (see [2]). Consequently, $S, R \in K[n]$. It follows from (32) that $V \in K[n]$. □

Consider the algorithm *dpol* for the case where $V \in K[n]$.

`algorithm` *dpol*;
`input:` a term T that is regularly described by the triple (F, V, n_0) with F, V having properties **Pa**, **Pb**. D is shift-reduced, $V \in K[n]$;
`output:` a term T_1 such that $T = (E-1)T_1$ if such a term exists and 0 otherwise;

`if` the equation $FEy - y = V$ has a polynomial solution
    `then` set S to an arbitrary polynomial solution
    `else` *return* 0
`fi;`
find integer $n_1 \geq n_0$ s.t. $S(n)$ has neither a pole
 nor a zero for all $n \geq n_1$;
 using (9) with $D = F, U = S$ to obtain the regular description
 of the term T_1;
return T_1.

We now arrive to presenting algorithm *hgadddec* that solves the additive decomposition problem, and that also includes the recognition of summability.

`algorithm` *hgadddec*;
`input:` a term T that is regularly described by the triple (D, U, n_0) with the shift-reduced D;
`output:` terms T_1, T_2 such that $T = (E-1)T_1 + T_2$. The term T_2 either equals zero, or has the certificate $F\frac{EV}{V}$, with V having the denominator of minimal possible degree. If T is summable then $T_2 = 0$;

$(T_1, T_2) := dterm(T)$ (let (F, V, n_1) be the obtained
    regular description of T_2 in the case $T_2 \neq 0$);
`if` $T_2 = 0$ `or` $V \notin K[n]$
    `then` *return* $(T_1,\ T_2)$
`fi;`
$T_3 := dpol(T_2)$;
`if` $T_3 = 0$
    `then` *return* $(T_1,\ T_2)$
`fi;`
construct $T_1' = T_1 + T_3$ using Theorem 2(b);
return $(T_1',\ 0)$.

6 A control on the degree of the numerator

So, the denominator of V has the minimal possible degree. How to reduce the degree of the numerator of V to a tolerable size? Recall that when one solves the decomposition problem for indefinite sums of rational functions, it is always possible to have the degree of the numerator less than the degree of the denominator (this is because any rational function is the sum of a polynomial and a proper rational function, and the equation

$$(E-1)y = u \tag{34}$$

has a polynomial solution for any $u \in K[n]$).

In the hypergeometric case the situation is not so simple. Consider a term which is regularly described by a triple (F, V, n_0), $V \in K[n]$, $D = f_1/f_2$, where D is shift-reduced. If $S \in K[n]$ then the rational function

$$V - \frac{f_1}{f_2}ES + S$$

can be represented in the form P/f_2, $P = f_2V - f_1ES + f_2S \in K[n]$. We will take into account the denominator f_2 in P/f_2 later in this section. Let us first try to find a polynomial S such that the polynomial

$$f_2V - f_1ES + f_2S \tag{35}$$

has a "reasonable" degree. We considered in Section 5 the case where there exists S such that (35) is equal to zero, i.e., the equation

$$f_1Ey - f_2y = f_2V \tag{36}$$

(an analogue of equation (34)) has a polynomial solution. This is the best case. But in any case, we can start a process of searching for a polynomial solution of (36), trying to find the degree and the leading coefficient of $y(x)$, then its next coefficient and so on. This process decreases step-by-step the degree of the right hand side of (36), and stops if either we get a polynomial solution of (35) or the construction of the next coefficient is not possible. So, even if there does not exist a polynomial S such that (35) equals 0, it is possible that we are able to find S such that the degree of (35) is less than the degree of f_2V. To be more precise, rewrite (35) as

$$P = f_2V - f_1(E-1)S + (f_2 - f_1)S \tag{37}$$

and set

$$M = f_2V, \quad T = -f_1(E-1)S + (f_2 - f_1)S.$$

Evidently $\deg(E-1)S = \deg S - 1$, $\mathrm{lc}\,((E-1)S) = \deg S \cdot \mathrm{lc}\, S$. By a judicious choice of S we can cancel out some leading terms of M if the degree of M is large enough. The number of those terms depends first of all on the relation between $\deg(f_2 - f_1)$ and $\deg f_1$:

1. $\deg(f_2 - f_1) > \deg f_1$. Then $\deg f_2 > \deg f_1$ and $\deg T = \deg f_2 + \deg S$, $\mathrm{lc}\, T = \mathrm{lc}\, f_2 \cdot \mathrm{lc}\, S$. We can transform M to P of degree $< \deg f_2$.

2. $\deg(f_2 - f_1) = \deg f_1$. Then $\deg f_2 \le \deg f_1$ and $\deg T = \deg f_1 + \deg S$, $\operatorname{lc} T = \operatorname{lc}(f_2 - f_1) \cdot \operatorname{lc} S$. We can transform M to P of degree $< \deg f_1$.

3. $\deg(f_2 - f_1) < \deg f_1$. Then $\deg f_2 = \deg f_1$, $\operatorname{lc} f_1 = \operatorname{lc} f_2$.

   3a. $\deg(f_2 - f_1) < \deg f_1 - 1$. Then the coefficients of $x^{\deg f_1 - 1}$ in f_1 and f_2 are equal. We have $\deg T = \deg f_1 + \deg S - 1$, $\operatorname{lc} T = -\operatorname{lc} f_1 \cdot \operatorname{lc} S \cdot \deg S$. We can transform M to P of degree $\sigma < \deg f_1$. (Moreover, if $\sigma = \deg(f_2 - f_1)$, then by applying an additional elimination we can transform P to P' of degree $< \deg(f_2 - f_1)$.)

   3b. $\deg(f_2 - f_1) = \deg f_1 - 1$. Then the coefficients of $x^{\deg f_1 - 1}$ in f_1 and f_2 are not equal. If

$$-\operatorname{lc} f_1 \cdot \deg S + \operatorname{lc}(f_2 - f_1) \neq 0$$

   then

$$\deg T = \deg f_1 + \deg S - 1 = \deg(f_2 - f_1) + \deg S,$$
$$\operatorname{lc} T = \operatorname{lc} S \cdot (-\operatorname{lc} f_1 \cdot \deg S + \operatorname{lc}(f_2 - f_1)).$$

   In such a case if the equation

$$-\operatorname{lc} f_1 \cdot X + \operatorname{lc}(f_2 - f_1) = 0$$

   has no integer root on the segment $[0; \deg M - \deg(f_2 - f_1)]$ or, equivalently, on the segment $[0; \deg M - \deg f_1 + 1]$, then we can transform M to P of degree $< \deg f_1 - 1$. If

$$-\operatorname{lc} f_1 \cdot \tau + \operatorname{lc}(f_2 - f_1) = 0$$

   for an integer τ from the segment $[0; \deg M - \deg(f_2 - f_1)]$, then we can transform M to P of degree $< \deg(f_2 - f_1) + \tau + 1$, or, equivalently, $< \deg f_1 + \tau$.

Therefore we have the following

Theorem 10 *Let $M, f_1, f_2 \in K[n]$, $f_2 \neq 0$. Then it is possible to find $S \in K[n]$ such that the degree of $P = M - f_1 ES + f_2 S$ is less than*

$$\lambda = \begin{cases} \deg f_2 & \text{if } \deg(f_2 - f_1) > \deg f_1, \\ \deg f_1 & \text{if } \deg(f_2 - f_1) = \deg f_1 \\ & \text{or } \deg(f_2 - f_1) < \deg f_1 - 1, \\ \deg f_1 + \tau & \text{if } \deg(f_2 - f_1) = \deg f_1 - 1, \end{cases}$$

where in the last case τ is equal to $\operatorname{lc}(f_2 - f_1)/\operatorname{lc} f_1$ if this is a nonnegative integer and -1 otherwise.

Observe that if $f_1 = f_2 = 1$, then $\lambda = 0$ and $P = 0$. This is in agreement with the fact that the equation $(E - 1)S = V$ has a polynomial solution when $V \in K[n]$.

Thus we can find S such that (35) is equal to a polynomial P whose degree is bounded from above as described in Theorem 10. This implies that

$$V(n) \prod_{k=n_0}^{n-1} F(k) - (E - 1)\left(S(n) \prod_{k=n_0}^{n-1} F(k)\right)$$
$$= \frac{P(n)}{f_2(n)} \prod_{k=n_0}^{n-1} F(k) = \frac{P(n)}{f_2(n_0)} \prod_{k=n_0}^{n-1} \frac{f_1(k)}{f_2(k+1)}.$$

Now suppose that V is a rational function of the form v_1/v_2. This leads to the expression

$$f_2 v_1 - v_2 f_1 ES + v_2 f_2 S$$

instead of (35). We can use the described techniques by considering $v_2 f_1, v_2 f_2$ instead of f_1, f_2, and obtain S, P such that

$$f_2 v_1 - v_2 f_1 ES + v_2 f_2 S = P.$$

In other words,

$$\begin{aligned}
&\frac{v_1(n)}{v_2(n)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k)}-(E-1)\left(S(n)\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k)}\right)\\
&= \frac{P(n)}{v_2(n)f_2(n)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k)} = \frac{P(n)}{v_2(n)f_2(n_0)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k+1)}.
\end{aligned}$$

Hence we have proven

Theorem 11 *Let a term T be regularly described by a triple $(f_1/f_2, v_1/v_2, n_0)$, i.e.,*

$$T(n)=\frac{v_1(n)}{v_2(n)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k)}.$$

Then there exists a term T_1 of the form

$$T_1(n)=S(n)\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k)},$$

$S\in K[n]$, such that the term $T_2=T-(E-1)T_1$ is of the form

$$\frac{P(n)}{v_2(n)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k+1)},$$

where P is a polynomial whose degree is less than

$$\lambda=\begin{cases}\deg v_2+\deg f_2 & \text{if } \deg(f_2-f_1)>\deg f_1,\\ \deg v_2+\deg f_1 & \text{if } \deg(f_2-f_1)=\deg f_1\\ & \text{or } \deg(f_2-f_1)<\deg f_1-1,\\ \deg v_2+\deg f_1+\tau & \text{if } \deg(f_2-f_1)=\deg f_1-1,\end{cases}$$

where in the last case τ is equal to $\mathrm{lc}\,(f_2-f_1)/\mathrm{lc}\,f_1$ if this is a nonnegative integer, and -1 otherwise.

Observe that if $f_1=f_2=1$, then by this theorem $\deg P<\deg v_2$. This is in agreement with the properties of the decomposition of indefinite sums of rational functions. Considering the relation

$$\frac{P(n)}{v_2(n)f_2(n)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k)} = \frac{P(n)}{v_2(n)f_2(n_0)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k+1)},$$

we now take into account the possible common factors of $P(n)$ and $f_2(n)$. Note that if the pair $f_1/f_2, v_1/v_2$ has properties **Pa** and **Pb**, then the rational function P/v_2 is in reduced form (i.e., $P\perp v_2$) by Theorem 5. However we can try to reduce P/f_2. Suppose this gives P'/f_2', $f_2=f_2'f_2''$. Then

$$\begin{aligned}
&\frac{P(n)}{v_2(n)f_2(n)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2(k)}\\
&= \frac{P'(n)}{v_2(n)f_2'(n)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2'(k)f_2''(k)}\\
&= \frac{P'(n)}{v_2(n)f_2'(n_0)}\prod_{k=n_0}^{n-1}\frac{f_1(k)}{f_2''(k)f_2'(k+1)}.
\end{aligned}$$

Doing so as it is described in sections 3 and 5, we get a decomposition $T=(E-1)T_1+T_2$. If $T_2\neq 0$, then we can try to simplify T_2 as it was described in this section, applying the control on the degree of the numerator with respect to the term T_2: $T_2=(E-1)T'+T''$. After this we can combine the terms T_1 and T' together using Theorem 2(b).

These actions can be included into algorithm *hgadddec*. However, it is worth noting that Theorem 11 does not claim to decrease, or even to preserve, the degree of the numerator of V if it is already small enough.

7 Examples

Example 1 For the hypergeometric term

$$T = \left(\frac{1}{n+1} - \frac{1}{n}\right) \prod_{k=1}^{n-1} \frac{1}{k+2},$$

applying *hgadddec* results in

$$T_1 = \frac{1}{n} \prod_{k=1}^{n-1} \frac{1}{k+2},\ T_2 = \frac{1}{3} \prod_{k=1}^{n-1} \frac{1}{k+3}.$$

Example 2 For the hypergeometric term

$$T = \left(\frac{1}{n+1} - \frac{1}{n} - 1\right) \prod_{k=1}^{n-1} \frac{1}{k+2},$$

applying *dterm* results in

$$T_1 = \frac{1}{n} \prod_{k=1}^{n-1} \frac{1}{k+2},\ T_2 = -\frac{1}{3}(n+1) \prod_{k=1}^{n-1} \frac{1}{k+3}.$$

From the result shown above, T_2 is regularly described by the triple (F, V, n_0) where

$$F = \frac{f_1}{f_2},\ f_1 = 1,\ f_2 = n+3,\ V = -\frac{1}{3}(n+1).$$

Since $V \in K[n]$, we apply *dpol*. The equation

$$f_1\, Ey - f_2\, y = f_2\, V$$

has a polynomial solution

$$y = \frac{1}{3}(n+3).$$

Therefore,

$$T_2 = (E-1)\left(\frac{1}{3}(n+2) \prod_{k=1}^{n-1} \frac{1}{k+3}\right).$$

Consequently, T is summable:

$$T = (E-1)\left(\frac{n^2+3\,n+2}{3\,n} \prod_{k=1}^{n-1} \frac{1}{k+3}\right).$$

Example 3 Consider the hypergeometric term

$$T(n) = \frac{n^2}{\binom{2\,n}{n}}$$

which can be written as

$$\frac{n^2}{2} \prod_{k=1}^{n-1} \frac{1/4\,k + 1/4}{k+1/2}.$$

Applying *hgadddec* results in $T_1 = 0$, and

$$T_2 = \frac{v_1}{v_2} \prod_{k=n_0}^{n-1} \frac{f_1}{f_2} = \frac{n^2}{2} \prod_{k=1}^{n-1} \frac{1/4\,k + 1/4}{k+1/2}.$$

Since $\deg(f_2 - f_1) = \deg(f_1) = 1$, we are in case 2. An attempt to decrease the degree of v_1 results in

$$T = (E-1)\left(\left(-\frac{2}{3}n^2 - \frac{5}{9}n - \frac{2}{3}\right)\prod_{k=1}^{n-1}\frac{1/4\,k + 1/4}{k+1/2}\right) + \frac{5}{54}\prod_{k=1}^{n-1}\frac{k+1}{4\,k+6}.$$

Example 4 Consider the hypergeometric term

$$T(n) = \frac{1}{8}\,\frac{(n+3)\,(n+2)\,(n+4)\,(43\,n+35)}{(2\,n+1)\,(2\,n+3)\,(2\,n+5)\,(2\,n+7)}.$$

(Note that T is also a rational function of n.) Applying *hgadddec* results in

$$T_1 = -\frac{15}{256}\,\frac{168\,n^2 + 460\,n + 251}{(2\,n+1)\,(2\,n+3)\,(2\,n+5)},\quad T_2 = \frac{v_1}{v_2}\prod_{k=n_0}^{n-1}\frac{f_1}{f_2} = \frac{86\,n+457}{256\,n+896}\prod_{k=0}^{n-1}1.$$

Since $-\infty = \deg(f_2 - f_1) < \deg(f_1) - 1 = -1$, we are in case 3a. An attempt to decrease the degree of v_1 results in

$$T_2 = (E-1)\left(\frac{43}{128}\,n\right) + \frac{156}{256\,n+896}.$$

Therefore,

$$T = (E-1)\left(\frac{1}{256}\,\frac{688\,n^4 + 3096\,n^3 + 1436\,n^2 - 5610\,n - 3765}{(2\,n+1)(2\,n+3)(2\,n+5)}\right) + \frac{156}{256\,n+896}.$$

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