A Comparison of Degree Polynomials (Note)

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1 Introduction

All known algorithms for finding polynomial solutions of linear difference equations with polynomial coefficients start out by computing a *degree bound* for such solutions. This bound is obtained as the maximum element of a finite set of integers which includes the integer roots of a certain polynomial, called the *degree polynomial* of the equation. In this note we compare the various degree bounds and degree polynomials proposed in [1], [3], [2], and [4].

2 Degree bounds

We consider equations of the form

$$Ly = f \tag{1}$$

where L is a linear difference operator with polynomial coefficients, and f is a given polynomial. The variations in degree bounds found in the literature stem from the choice of bases in which to expand L and its coefficients. L is expanded in powers of the forward shift E in [3], while in [1], [2] and [4] it is expanded in powers of the forward difference Δ. The coefficients of L which are polynomials of n are expanded in the power basis $\langle n^k \rangle_{k=0}^{\infty}$ in [1], [3] and [4], while in [2] they are expanded in the binomial basis $\langle \binom{n}{k} \rangle_{k=0}^{\infty}$. Let

$$L = \sum_{k=0}^{r} p_k(n) E^k = \sum_{k=0}^{r} q_k(n) \Delta^k.$$

Then, as $\Delta = E - 1$ and $E = \Delta + 1$, we have

$$p_k(n) = \sum_{j=k}^{r} (-1)^{j-k} \binom{j}{k} q_j(n), \quad q_k(n) = \sum_{j=k}^{r} \binom{j}{k} p_j(n). \tag{2}$$

Denote

$$b = \max_{0 \le k \le r} \left(\deg q_k(n) - k \right),$$

$$d = \max_{0 \le k \le r} \deg p_k(n) = \max_{0 \le k \le r} \deg q_k(n),$$

$$p_k(n) = \sum_{j=0}^{d} a_{k,j} n^{d-j}, \quad q_k(n) = \sum_{j=0}^{d} c_{j,k} n^j.$$

Let $y(n)$ be a polynomial solution of (1).

a) According to [1] and [4],

$$\deg y \le \max\{M, \deg f - b, \, - b - 1\} \tag{3}$$

where M is the maximal integer root of the degree polynomial

$$A(x) = \sum_{k=0}^{r} c_{b+k,k} \, x^{\underline{k}}.$$

b) For $0 \le j \le t$ define

$$b_j^{(t)} = \sum_{i=0}^{r} a_{i,t-j} i^j, \quad D_t(x) = \sum_{j=0}^{t} \binom{x}{j} b_j^{(t)},$$

and let

$$s = \min\{t \ge 0; \; D_t(x) \not\equiv 0\}.$$

Then, by [3],

$$\deg y \le \max\{M, \deg f + s - d\} \tag{4}$$

where M is the maximal integer root of the degree polynomial

$$D(x) = D_s(x) = \sum_{j=0}^{s} \binom{x}{j} b_j^{(t)}.$$

c) Following [2],

$$\deg y \le \max\{M, \deg f - b, \, -b-1\}$$

where M is the maximal integer root of the degree polynomial

$$B(x) = \sum_{k=0}^{r} \sum_{j=b+k}^{d} \binom{x+b}{j} \binom{j}{b+k} \Delta^j q_k(a)$$

where a is arbitrary.

3 Relations among degree polynomials

Theorem 1 (i) *$s + b = d$,*

(ii) *$D(x) = A(x)$.*

Proof: From (2) we read off

$$a_{k,j} = \sum_{i=k}^{r} (-1)^{i-k} \binom{i}{k} c_{d-j,i}.$$

Therefore

$$\begin{aligned} b_j^{(t)} &= \sum_{k=0}^{r} \sum_{i=k}^{r} (-1)^{i-k} \binom{i}{k} c_{d-t+j,i} k^j = \sum_{i=0}^{r} c_{d-t+j,i} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} k^j \\ &= \sum_{i=0}^{r} c_{d-t+j,i} \Delta^i x^j|_{x=0} = \sum_{i=0}^{j} c_{d-t+j,i} \Delta^i x^j|_{x=0}. \end{aligned}$$

If $t < d - b$ then $d - t + j > b + j \ge b + i \ge \deg p_i$ by definition of b, hence $c_{d-t+j,i} = 0$ for $0 \le i \le j$. So $b_j^{(t)} = 0$ for $0 \le j \le t$ and $D_t(x) \equiv 0$. It follows that $s \ge d - b$.

On the other hand,

$$b_j^{(d-b)} = \sum_{i=0}^{j} c_{b+j,i} \Delta^i x^j|_{x=0}.$$

If $i < j$ then $b + j > b + i \geq \deg p_i$ by definition of b, hence $c_{b+j,i} = 0$ for $0 \leq i < j$ and so

$$b_j^{(d-b)} = c_{b+j,j} \Delta^j x^j|_{x=0} = j!\, c_{b+j,j}.$$

Therefore

$$D_{d-b}(x) = \sum_{j=0}^{d-b} \binom{x}{j} b_j^{(d-b)} = \sum_{j=0}^{d-b} \binom{x}{j} j!\, c_{b+j,j} = \sum_{j=0}^{d-b} c_{b+j,j} x^{\underline{j}} = A(x) \not\equiv 0.$$

It follows that $s = d - b$ and $D(x) = A(x)$ as claimed. □

Theorem 2 $B(x) = \begin{cases} (x+b)^{\underline{b}} A(x), & b \geq 0, \\ A(x)/x^{\underline{-b}}, & b < 0. \end{cases}$

Proof: If $j > b+k$ then, as $b \geq \deg q_k - k$, we have $j > \deg q_k$ and consequently $\Delta^j q_k(a) = 0$. It follows that

$$\begin{aligned} B(x) &= \sum_{k=0}^{r} \sum_{j=b+k}^{d} \binom{x+b}{j} \binom{j}{b+k} \Delta^j q_k(a) = \sum_{k=0}^{r} \binom{x+b}{b+k} \Delta^{b+k} q_k(a) \\ &= \sum_{k=0}^{r} \binom{x+b}{b+k} (b+k)!\, c_{b+k,k} = \sum_{k=0}^{r} c_{b+k,k} (x+b)^{\underline{b+k}}. \end{aligned}$$

If $b \geq 0$ then $(x+b)^{\underline{b+k}} = (x+b)^{\underline{b}} x^{\underline{k}}$ and $B(x) = (x+b)^{\underline{b}} A(x)$. If $b < 0$ then $(x+b)^{\underline{b+k}} = x^{\underline{k}}/x^{\underline{-b}}$ and $B(x) = A(x)/x^{\underline{-b}}$, as claimed. □

4 Remarks on degree bounds

1. When using degree polynomial $A(x)$ it is not necessary to consider $-b-1$ when computing the degree bound (3). This is clear when $b \geq 0$. When $b < 0$ we can write

$$A(x) = \sum_{k=-b}^{r} c_{b+k,k}\, x^{\underline{k}},$$

so $A(x)$ is divisible by $x^{\underline{-b}}$ with roots $0, 1, \ldots, -b-1$, implying that $M \geq -b-1$. Thus we can use the same bound $\deg y \leq \max\{M, \deg f - b\}$ as in (4) (note that $b = d - s$ and $A(x) = D(x)$).

2. When $b \geq 0$ we have $B(x) = (x+b)^{\underline{b}} A(x)$. Clearly the factor $(x+b)^{\underline{b}}$ can be omitted because its roots $-b, -b+1, \ldots, -1$ are all negative. Thus to have a degree polynomial $P(x)$ of least degree we can take

$$P(x) = \begin{cases} A(x), & b \geq 0, \\ A(x)/x^{\underline{-b}}, & b < 0. \end{cases}$$

Then we have the bound

$$\deg y \leq \begin{cases} \max\{M, \deg f - b\}, & b \geq 0, \\ \max\{M, \deg f - b, -b-1\}, & b < 0, \end{cases}$$

where M is the maximal integer root of $P(x)$.

3. It is worth noting that in fact we know more than just an upper bound on the degrees of polynomial solutions of (1): these degrees must belong to the set consisting of the number $\deg f - b$ and the nonnegative integer roots of the degree polynomial $A(x)$.

References

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