Gambler's Ruin Problem in Several Dimensions

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Abstract

We give an explicit solution of the Gambler's Ruin Problem when the players use two or more currencies. We also determine the asymptotics of the expected duration of the game when both players have equal amounts of each currency.

1 Introduction

In the one-dimensional Gambler's Ruin Problem two players start out with i and $N - i$ dollars, respectively. At each step they toss a fair coin to decide who wins a dollar from the opponent. The game is over when one of them goes bankrupt. It is well known that the expected duration of the game is $i(N - i)$ (see [3], or almost any other textbook on probability).

In the two-dimensional variant [6] the players use two different currencies, say dollars and euros. They start out with (i dollars, j euros) and ($N - i$ dollars, $M - j$ euros), respectively. At each step they toss fair coins to decide the currency and the winner. The game is over when one of them runs out of either currency. What is the expected duration of the game? According to [6], no closed-form solution of this problem is known to exist, and probably doesn't exist.

Denote by game(i, j) the game with the first player's initial assets equal to (i, j). Assume that $1 \le i \le N - 1$ and $1 \le j \le M - 1$. Then after the first step, game(i, j) turns into one of game$(i + 1, j)$, game$(i - 1, j)$, game$(i, j + 1)$, or game$(i, j - 1)$, each with probability $1/4$. It follows that the expected duration $a_{i,j}$ of game(i, j) satisfies the recurrence equation

$$a_{i,j} = \frac{a_{i+1,j} + a_{i-1,j} + a_{i,j+1} + a_{i,j-1}}{4} + 1 \qquad (1 \le i \le N - 1,\ 1 \le j \le M - 1) \qquad (1)$$

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and the boundary conditions

$$a_{0,j} = a_{N,j} = a_{i,0} = a_{i,M} = 0 \qquad (0 \le i \le N,\ 0 \le j \le M). \tag{2}$$

The unknown $a_{i,j}$, $1 \le i \le N-1, 1 \le j \le M-1$, can be obtained from (1) by straightforward linear algebra. Instead of solving this linear system of $(N-1)(M-1)$ equations, Orr and Zeilberger [6] have shown how to obtain the values $a_{1,j} = a_{N-1,j}$ $(1 \le j \le M-1)$ and $a_{i,1} = a_{i,M-1}$ $(1 \le i \le N-1)$ from a system containing $O(N+M)$ equations only. The remaining $a_{i,j}$ can then be computed directly from the recurrence

$$a_{i+1,j} = 4\,a_{i,j} - (a_{i-1,j} + a_{i,j+1} + a_{i,j-1}) - 4 \qquad (1 \le i \le N-3,\ 2 \le j \le M-2). \tag{3}$$

Because of the obvious symmetries $a_{i,j} = a_{N-i,j} = a_{i,M-j} = a_{N-i,M-j}$, it suffices to compute a quarter of these numbers only. Note however that in floating-point arithmetic, this computation is numerically unstable due to the coefficient 4 in front of $a_{i,j}$ in (3).

In this paper we give an explicit solution of the two-dimensional Gambler's Ruin Problem (Sec. 2) as well as of its obvious generalization to larger numbers of currencies (Sec. 3). Although our solution is not in closed form because it contains a double sum, it provides a direct way to compute the expected duration of the game without having to solve linear systems or use recursive computations. In Section 4 we express the middle element $a_{N/2,\,N/2}$ as a single sum when N is a power of two. In Section 5 we determine the asymptotics of the middle element $a_{N/2,\,N/2,\,\ldots,\,N/2}$ (when N is even) for any number of currencies.

2 Solution of the 2-dimensional problem

For the sake of simplicity we henceforth assume that $M = N$, but it is straightforward to generalize our results to the case $M \ne N$. Let $A = [a_{i,j}]_{i,j=1}^{N-1}$ be the matrix of unknown values $a_{i,j}$. Writing (1) in the form

$$(a_{i,j-1} - 2a_{i,j} + a_{i,j+1}) + (a_{i-1,j} - 2a_{i,j} + a_{i+1,j}) \;=\; -4 \qquad (1 \le i,j \le N-1)$$

and using (2) we see that A satisfies the matrix equation

$$AD + DA = -4J \tag{4}$$

where

$$D \;=\; \begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & -2 & 1 \\ 0 & 0 & \ldots & 0 & 1 & -2 \end{bmatrix}$$

is an $(N-1)\times(N-1)$ symmetric tridiagonal Toeplitz matrix, and J is the $(N-1)\times(N-1)$ matrix of ones. This equation can be solved explicitly in terms of the eigenvalues and eigenvectors of D.

Indeed, let $\Lambda = \mathrm{diag}(\lambda_k)$ be the diagonal matrix of the eigenvalues of D, and P the orthogonal matrix whose columns are the corresponding eigenvectors of D. Then $D = P\Lambda P^T$ and $P^T P = I$. Multiplying (4) with P^T on the left and with P on the right gives

$$B\Lambda + \Lambda B = R$$

where $B = P^T AP$, and

$$R = -4\, P^T JP. \tag{5}$$

Then

$$b_{i,j} = \frac{r_{i,j}}{\lambda_i + \lambda_j} \quad (1 \le i, j \le N-1), \tag{6}$$

so, using (5) and (6), the unknown matrix

$$A = PBP^T \tag{7}$$

can be expressed explicitly in terms of Λ and P. These, however, are well known, namely,

$$\lambda_k = -4\, \sin^2 \frac{k\pi}{2N} \quad (1 \le k \le N-1), \tag{8}$$

$$P_{i,j} = \sqrt{\frac{2}{N}} \sin \frac{\pi ij}{N} \quad (1 \le i, j \le N-1) \tag{9}$$

(cf. [2, Sec. 2.6, item 7^0]). Therefore by (5),

$$r_{i,j} = -4 \sum_{l=1}^{N-1} \sum_{k=1}^{N-1} P_{l,i} P_{k,j} = -\frac{8}{N} \sum_{l=1}^{N-1} \sin \frac{\pi il}{N} \sum_{k=1}^{N-1} \sin \frac{\pi jk}{N} \quad (1 \le i, j \le N-1).$$

This sum is easy to evaluate:

$$\sum_{k=1}^{N-1} \sin \frac{\pi jk}{N} = \Im \sum_{k=0}^{N-1} e^{\frac{kj\pi\imath}{N}} = \Im\, \frac{e^{j\pi\imath} - 1}{e^{\frac{j\pi\imath}{N}} - 1} = \begin{cases} 0, & j \text{ even}, \\ \cot \frac{j\pi}{2N}, & j \text{ odd}, \end{cases} \tag{10}$$

so

$$r_{i,j} = \begin{cases} 0, & ij \text{ even}, \\ -\frac{8}{N} \cot \frac{i\pi}{2N} \cot \frac{j\pi}{2N}, & ij \text{ odd}. \end{cases}$$

From (6) and (8) it follows that

$$b_{i,j} = \begin{cases} 0, & ij \text{ even}, \\ \frac{2 \cot \frac{i\pi}{2N} \cot \frac{j\pi}{2N}}{N\left(\sin^2 \frac{i\pi}{2N} + \sin^2 \frac{j\pi}{2N}\right)}, & ij \text{ odd}. \end{cases}$$

Finally we obtain from (7) and (9)

$$a_{i,j} = \frac{4}{N^2} \sum_{\substack{k=1 \\ k \,\text{odd}}}^{N-1} \sin \frac{jk\pi}{N} \cot \frac{k\pi}{2N} \sum_{\substack{l=1 \\ l \,\text{odd}}}^{N-1} \frac{\sin \frac{il\pi}{N} \cot \frac{l\pi}{2N}}{\sin^2 \frac{k\pi}{2N} + \sin^2 \frac{l\pi}{2N}} \quad (0 \le i, j \le N). \tag{11}$$

This is the promised explicit expression for the expected duration of game(i, j). It can also be obtained by applying the Discrete Fourier Transform to (1) (see, e.g., [5, Thm. 3a]). Note that it requires only $O(N^2)$ time to compute $a_{i,j}$ given i and j.

In the case $i = 1$, the method of [7] yields the explicit formula

$$a_{1,j} = \frac{4}{N} \sum_{k=0}^{N-1} \sin \frac{(2k+1)j\pi}{N} \cot \frac{(2k+1)\pi}{2N} \frac{\xi^N - \xi}{(\xi^N + 1)(\xi - 1)}$$

where

$$\xi = \left(\sin \frac{(2k+1)\pi}{2N} - \sqrt{1 + \sin^2 \frac{(2k+1)\pi}{2N}} \right)^2 .$$

This indicates the possibility that $a_{i,j}$ could be given in the form of a single sum.

3 Generalization to higher dimensions

In d dimensions, the two players use d different currencies. Their initial assets are $(i_1, i_2, \ldots, i_d)$ and $(N_1 - i_1, N_2 - i_2, \ldots, N_d - i_d)$, respectively. At each step they first decide the currency, then the winner, both uniformly at random. The game is over as soon as one of them runs out of any of the currencies. Let $a_{i_1,i_2,\ldots,i_d}$ denote the expected duration of the game. Again we assume for simplicity that $N_1 = N_2 = \cdots = N_d = N$.

The solution of the general problem can be expressed explicitly by means of tensor product of matrices. If A is any matrix and B is a matrix of order $m \times n$, their tensor product is defined as

$$A \otimes B = \begin{bmatrix} b_{1,1}A & b_{1,2}A & \cdots & b_{1,n}A \\ b_{2,1}A & b_{2,2}A & \cdots & b_{2,n}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1}A & b_{m,2}A & \cdots & b_{m,n}A \end{bmatrix} .$$

In particular, if I denotes the identity matrix,

$$A \otimes I = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} .$$

Now consider *Sylvester's equation*

$$AD + CA = R$$

where C, D, R are known matrices, A is an unknown matrix, A and R are of order $p \times q$, C is of order $p \times p$, and D is of order $q \times q$ (cf. [4, Sec. 7.6.3]). This is a system of linear equations

$$\sum_{s=1}^{n} a_{i,s} d_{s,j} + \sum_{t=1}^{n} c_{i,t} a_{t,j} = r_{i,j} \quad (1 \le i, j \le n) \tag{12}$$

for the unknown $a_{i,j}$. Let a resp. r be the vectors obtained by stacking the columns of A resp. R one above another. If the equations are ordered lexicographically first by j and then by i, we can rewrite (12) as a vector equation

$$(C \otimes I_q + I_p \otimes D^T)\, a = r \tag{13}$$

where I_n denotes the identity matrix of order n. Using this on Eqn. (4) which is a special case of Sylvester's equation with $C = D$, $D = D^T$, and $p = q = N - 1$, we have

$$D_1 a = -4\, r \tag{14}$$

where $D_1 = D \otimes I_{N-1} + I_{N-1} \otimes D$ and r is a vector with all components equal to 1. Because $D_1 = (P \otimes P)(\Lambda \otimes I_{N-1} + I_{N-1} \otimes \Lambda)(P \otimes P)^T$ is the spectral decomposition of D_1, we obtain a compact form of solution of the two-dimensional problem,

$$a = -4\,(P \otimes P)(\Lambda \otimes I_{N-1} + I_{N-1} \otimes \Lambda)^{-1}(P \otimes P)^T\, r.$$

In the three-dimensional case, we have the recurrence

$$\begin{array}{rlcl} & (a_{i,j,k-1} - 2a_{i,j,k} + a_{i,j,k+1}) & & \\ + & (a_{i,j-1,k} - 2a_{i,j,k} + a_{i,j+1,k}) & & \\ + & (a_{i-1,j,k} - 2a_{i,j,k} + a_{i+1,j,k}) & = & -6 \qquad (1 \le i,j,k \le N-1). \end{array} \tag{15}$$

Let $A = [a_{\cdot,\cdot,1} \,|\, a_{\cdot,\cdot,2} \,|\, \cdots \,|\, a_{\cdot,\cdot,N-1}]$ be the $(N-1)^2 \times (N-1)$ matrix whose k-th column $a_{\cdot,\cdot,k}$ is obtained by stacking the columns of the $(N-1) \times (N-1)$ matrix $[a_{i,j,k}]_{i,j=1}^{N-1}$ one above another. Then (15) is equivalent to Sylvester's equation

$$AD + D_1 A = -6J.$$

Let a be the vector obtained by stacking the columns of A one above another. Using (13) we have

$$D_2\, a = -6\, r,$$

where $D_2 = D_1 \otimes I_{N-1} + I_{(N-1)^2} \otimes D$, and r is a vector of ones.

In the d-dimensional case we recursively define three sequences of matrices by setting $D_0 = D$, $\Lambda_0 = \Lambda$, $P_0 = P$, and, for $k = 1, 2, \ldots, d-1$,

$$\begin{array}{rcl} D_k & = & D_{k-1} \otimes I_{N-1} + I_{(N-1)^k} \otimes D, \\ \Lambda_k & = & \Lambda_{k-1} \otimes I_{N-1} + I_{(N-1)^k} \otimes \Lambda, \\ P_k & = & P_{k-1} \otimes P. \end{array}$$

Then for all k we have $P_k^T P_k = I_{(N-1)^{k+1}}$, $D_k = P_k \Lambda_k P_k^T$, and

$$D_{d-1}\, a = -2d\, r$$

where a is the vector with components $a_{i_1,i_2,\ldots,i_d}$, ordered lexicographically first by i_d, then by i_{d-1}, $\ldots$, last by i_1, and r is a vector of ones. Thus the solution in the d-dimensional case is given by

$$a = -2d\, P_{d-1} \Lambda_{d-1}^{-1} P_{d-1}^T\, r. \tag{16}$$

4 A faster way to solution for $N = 2^p$

Let $d = 2$ again, and $N = 2^p$ for some integer $p > 0$. In this case the system (14) can be solved efficiently using the *block cyclic reduction* technique or *Buneman's algorithm* (see [1], [8], or [4, Sec. 4.5.4]). Recall that $D_1 = D \otimes I_{N-1} + I_{N-1} \otimes D$, so (14) has the $(N-1)\times(N-1)$ block-tridiagonal form

$$\begin{bmatrix} G & I & 0 & 0 & \dots & 0 \\ I & G & I & 0 & \dots & 0 \\ 0 & I & G & I & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & I & G & I \\ 0 & 0 & \dots & 0 & I & G \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} b \\ b \\ b \\ \vdots \\ b \\ b \end{bmatrix}$$

where $G = D - 2I$ is of size $(N-1)\times(N-1)$, a_j is the j-th column of A, and b is a vector with all components equal to -4.

Multiplying every other row by $-G$ and adding the two neighboring rows to it we eliminate the odd-numbered blocks and obtain the $(N/2-1)\times(N/2-1)$ block-tridiagonal system

$$\begin{bmatrix} G_2 & I & 0 & 0 & \dots & 0 \\ I & G_2 & I & 0 & \dots & 0 \\ 0 & I & G_2 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & I & G_2 & I \\ 0 & 0 & \dots & 0 & I & G_2 \end{bmatrix} \begin{bmatrix} a_2 \\ a_4 \\ a_6 \\ \vdots \\ a_{N-4} \\ a_{N-2} \end{bmatrix} = \begin{bmatrix} b_2 \\ b_2 \\ b_2 \\ \vdots \\ b_2 \\ b_2 \end{bmatrix}$$

where $G_2 = 2I - G^2$ and $b_2 = (2I - G)b$. Repeating this procedure $p-1$ times we come to a single-block system

$$G_p a_{N/2} = b_p \tag{17}$$

where we recursively defined $G_1 = G$, $b_1 = b$, $G_{k+1} = 2I - G_k^2$, and $b_{k+1} = (2I - G_k)b_k$. Let

$$\begin{aligned} \Lambda_1 &= \Lambda - 2I, \\ \Lambda_{k+1} &= 2I - \Lambda_k^2 \quad \text{for } k \geq 0 \end{aligned} \tag{18}$$

where $D = P\Lambda P^T$, $P^T P = I$, and the elements of Λ and P are given in (8) resp. (9). By induction on k it follows that $G_k = P\Lambda_k P^T$ and $b_k = P(2I + \Lambda_k)\Lambda^{-1}P^T b$. Hence from (17)

$$a_{N/2} = P(I + 2\Lambda_p^{-1})\Lambda^{-1}P^T b. \tag{19}$$

The remaining a_j can now be found by back-substitution from the intermediate systems.

For the middle element $a_{N/2,N/2} = a_{2^{p-1},2^{p-1}}$ we obtain from (19), (8), (9), and (10)

$$a_{N/2,N/2} = \frac{2}{N}\sum_{k=0}^{N/2-1} (-1)^k \left(1 + \frac{2}{\lambda_{2k+1}^{(p)}}\right) \frac{\cos\frac{(2k+1)\pi}{2N}}{\sin^3\frac{(2k+1)\pi}{2N}} \tag{20}$$

where $\lambda_j^{(k)}$, the j-th element of Λ_k, satisfies

$$\begin{aligned} \lambda_j^{(1)} &= \lambda_j - 2, \\ \lambda_j^{(k+1)} &= 2 - \left(\lambda_j^{(k)}\right)^2 \qquad \text{for } k \geq 0. \end{aligned}$$

Using the substitution $\lambda_j^{(k)} = -2\cosh a_k$ we find $a_{k+1} = 2a_k$ and $a_1 = \cosh^{-1}(1 - \lambda_j/2)$, so $a_k = 2^{k-1}a_1$ and $\lambda_j^{(k)} = -2\cosh\left(2^{k-1}\cosh^{-1}(1-\lambda_j/2)\right)$. With (8) and (20) we finally obtain

$$a_{N/2,N/2} = \frac{2}{N}\sum_{k=0}^{N/2-1}(-1)^k\left(1 - \frac{1}{\cosh\left(\frac{N}{2}\cosh^{-1}\left(1+2\sin^2\frac{(2k+1)\pi}{2N}\right)\right)}\right)\frac{\cos\frac{(2k+1)\pi}{2N}}{\sin^3\frac{(2k+1)\pi}{2N}}.$$

Note that this formula requires only $O(N)$ operations to compute $a_{N/2,N/2}$.

5 Some asymptotics

Clearly the expected duration of the game is largest when the initial assets of the two players in each currency are as close to each other as possible. Here we assume that $N_1 = N_2 = \cdots = 2n$, and find an asymptotic formula for the middle element $a_{n,n,\ldots,n}$.

Theorem 1 *Let $d = 2$. If $N_1 = N_2 = 2n$ then*

$$a_{n,n} = c_2\, n^2 + O(\log^2 n)$$

where

$$c_2 = \frac{256}{\pi^4}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}}{(2k+1)(2l+1)\left((2k+1)^2+(2l+1)^2\right)} \approx 1.17874. \tag{21}$$

Proof: From (11) we have

$$\begin{aligned} a_{n,n} &= \frac{1}{n^2}\sum_{\substack{k=1\\ k\text{ odd}}}^{2n-1}\sum_{\substack{l=1\\ l\text{ odd}}}^{2n-1}\frac{\sin\frac{k\pi}{2}\sin\frac{l\pi}{2}\cot\frac{k\pi}{4n}\cot\frac{l\pi}{4n}}{\sin^2\frac{k\pi}{4n}+\sin^2\frac{l\pi}{4n}} \\ &= \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\frac{(-1)^{k+l}\cot\frac{(2k+1)\pi}{4n}\cot\frac{(2l+1)\pi}{4n}}{\sin^2\frac{(2k+1)\pi}{4n}+\sin^2\frac{(2l+1)\pi}{4n}}. \end{aligned} \tag{22}$$

Denote by s_n the expression obtained by replacing $\sin x$ with x and $\cos x$ with 1 everywhere in (22):

$$s_n = \frac{256}{\pi^4}n^2\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\frac{(-1)^{k+l}}{(2k+1)(2l+1)\left((2k+1)^2+(2l+1)^2\right)}. \tag{23}$$

For $0 < x, y < \pi/2$ we have

$$\frac{\left(1-\frac{x^2}{2}\right)\left(1-\frac{y^2}{2}\right)}{xy(x^2+y^2)} < \frac{\cot x\cot y}{\sin^2 x+\sin^2 y} < \frac{1}{xy(x^2+y^2)}.$$

The first inequality follows from $\cos x > 1 - x^2/2$ and $\sin x < x$, the second from the fact that the Taylor coefficients of $f(x,y) = (\sin^2 x + \sin^2 y)\tan x \tan y - xy(x^2+y^2)$ are nonnegative at $x = y = 0$. Thus

$$0 < \frac{1}{xy(x^2+y^2)} - \frac{\cot x \cot y}{\sin^2 x + \sin^2 y} < \frac{1 - \left(1 - \frac{x^2}{2}\right)\left(1 - \frac{y^2}{2}\right)}{xy(x^2+y^2)} = \frac{x^2 + y^2 - \frac{x^2y^2}{2}}{2xy(x^2+y^2)} < \frac{1}{2xy}$$

which shows that the error committed in replacing $\frac{\cot x \cot y}{\sin^2 x + \sin^2 y}$ with $\frac{1}{xy(x^2+y^2)}$ is bounded by $\frac{1}{2xy}$. It follows that

$$|a_{n,n} - s_n| < \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \frac{1}{2\frac{(2k+1)\pi}{4n}\frac{(2l+1)\pi}{4n}} = \frac{8}{\pi^2}\sum_{k=0}^{n-1}\frac{1}{2k+1}\sum_{l=0}^{n-1}\frac{1}{2l+1} = \frac{8}{\pi^2}(H_{2n} - \frac{1}{2}H_n)^2$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number. As $\ln n + \gamma < H_n \le \ln n + 1$, it follows that $a_{n,n} - s_n = O(\log^2 n)$.

Next replace the sum in (23) by the infinite sum

$$t_n = \frac{256}{\pi^4}n^2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l} r_{k,l} = c_2\, n^2 \tag{24}$$

where

$$r_{k,l} = \frac{1}{(2k+1)(2l+1)\left((2k+1)^2 + (2l+1)^2\right)}.$$

The sum in (24) is absolutely convergent, therefore it can be rearranged into

$$t_n = \frac{256}{\pi^4}n^2 \sum_{j=0}^{\infty}(-1)^j d_j \tag{25}$$

where

$$d_j = \sum_{k=0}^{j} r_{k,j-k}$$

is the absolute value of the sum of the j-th diagonal of the original array. We transform s_n into t_n in two steps: first we approximate s_n by

$$u_n = \frac{256}{\pi^4}n^2 \sum_{j=0}^{n}(-1)^j d_j,$$

then we replace u_n by t_n.

It is possible to show that $d_j > d_{j+1}$ for all $j \ge 0$. Therefore the error in truncating the alternating sum (24) after the n-th diagonal does not exceed

$$\frac{256}{\pi^4}n^2 d_{n+1} = \frac{256}{\pi^4}n^2 \sum_{k=0}^{n+1} r_{k,n+1-k} \le \frac{256}{\pi^4}n^2(n+2) r_{0,n+1} = \frac{256\, n^2(n+2)}{\pi^4(2n+3)(4n^2+12n+10)} < \frac{32}{\pi^4}.$$

Thus both differences $s_n - u_n$ and $u_n - t_n$ are $O(1)$, and the theorem is proved. □

Theorem 2 *If $N_1 = N_2 = \cdots = N_d = N = 2n$ then*

$$a_{n,n,\ldots,n} \sim c_d\, n^2$$

where

$$c_d = d\left(1 - \frac{2^{2d+1}}{\pi^{d+1}} \sum_{k_1,k_2,\ldots,k_{d-1}\geq 0} \frac{(-1)^{\sum_{j=1}^{d-1} k_j} \prod_{j=1}^{d-1} \frac{1}{2k_j+1} \sum_{j=1}^{d-1} \frac{1}{(2k_j+1)^2}}{\cosh \frac{\pi}{2}\sqrt{\sum_{j=1}^{d-1}(2k_j+1)^2}}\right). \tag{26}$$

Proof: Consider the following Poisson partial differential equation in d dimensions

$$\Delta u = -2d \tag{27}$$

on the d-dimensional hypercube $Q_d = [0,1]^d$, with zero boundary conditions: $u|_{\partial Q_d} = 0$. Following the finite-difference method to obtain an approximate solution, we discretize the problem by introducing the d-dimensional sequence of values on the square grid

$$a_{i_1,i_2,\ldots,i_d} = \frac{1}{h^2}\, u\left(\frac{i_1}{N}, \frac{i_2}{N}, \ldots, \frac{i_d}{N}\right)$$

where $h = 1/N$ and $0 \leq i_1, i_2, \ldots, i_d \leq N$. Replacing second derivatives by second differences divided by h^2, Eqn. (27) turns into the d-dimensional Gambler's Ruin recurrence for $a_{i_1,i_2,\ldots,i_d}$. Thus, for large n,

$$a_{n,n,\ldots,n} \sim 4n^2\, u\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right). \tag{28}$$

It remains to solve Eqn. (27) exactly. First we homogenize the equation by introducing

$$v(x_1,x_2,\ldots,x_d) = u(x_1,x_2,\ldots,x_d) + \sum_{i=1}^{d} x_i(x_i-1) \tag{29}$$

which satisfies

$$\Delta v = 0, \quad v|_{x_i=0} = v|_{x_i=1} = \sum_{\substack{j=1\\ j\neq i}}^{d} x_j(x_j-1).$$

Then we solve this Dirichlet problem exactly by the standard Fourier-series method and obtain

$$v(x_1,x_2,\ldots,x_d) = -\frac{2^{2d-1}}{\pi^{d+1}} \sum_{\substack{k_1,\ldots,k_{d-1}\geq 0\\ k_1,\ldots,k_{d-1}\ \text{odd}}} \frac{\prod_{j=1}^{d-1}\frac{1}{k_j}\sum_{j=1}^{d-1}\frac{1}{k_j^2}}{\cosh\frac{\pi}{2}\sqrt{\sum_{j=1}^{d-1}k_j^2}} \sum_{i=1}^{d-1} f_i(x_1,\ldots,x_d;k_1,\ldots,k_{d-1}), \tag{30}$$

where

$$f_i(x_1,\ldots,x_d;k_1,\ldots,k_{d-1}) = \prod_{j=1}^{i-1}\sin\pi k_j x_j \prod_{j=i}^{d-1}\sin\pi k_j x_{j+1}\cosh\left(\pi\left(x_i-\frac{1}{2}\right)\sqrt{\sum_{j=1}^{d-1}k_j^2}\right).$$

Combining (28) with (29) and (30) yields (26). □

In particular, for $d = 2$ we obtain from (26) a single-sum expression for c_2,

$$c_2 = 2\left(1 - \frac{32}{\pi^3}\sum_{k\geq 0}\frac{(-1)^k}{(2k+1)^3\cosh\frac{\pi}{2}(2k+1)}\right)$$

(cf. Eqn. (21)). Some values of c_d are shown in Table 1.

d	1	2	3	4	5	6	7	8	9
c_d	1	1.17874	1.34911	1.51271	1.67071	1.82399	1.97321	2.11891	2.26149

Table 1: Approximate values of c_d

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