Walks confined in a quadrant are not always D-finite

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Abstract

We consider planar lattice walks that start from a prescribed position, take their steps in a given finite subset of $\mathbb{Z}^2$, and always stay in the quadrant $x \geq 0, y \geq 0$. We first give a criterion which guarantees that the length generating function of these walks is D-finite, that is, satisfies a linear differential equation with polynomial coefficients. This criterion applies, among others, to the ordinary square lattice walks. Then, we prove that walks that start from $(1, 1)$, take their steps in $\{(2, -1), (-1, 2)\}$ and stay in the first quadrant have a non-D-finite generating function. Our proof relies on a functional equation satisfied by this generating function, and on elementary complex analysis.

1 Introduction

The enumeration of lattice walks is one of the most venerable topics in enumerative combinatorics, which has numerous applications in probabilities [12, 22, 34]. These walks take their steps in a finite subset $\mathfrak{S}$ of $\mathbb{Z}^d$, and might be constrained in various ways. One can only cite a small percentage of the relevant litterature, which dates back at least to the next-to-last century [1, 11, 14, 19, 27, 28]. Many recent publications show that the topic is still active [4, 6, 8, 16, 17, 29, 30].

After the solution of many explicit problems, certain patterns have emerged, and a more recent trend consists in developing methods that are valid for generic sets of steps. Special attention is being paid to the *nature* of the generating function of the walks under consideration. For instance, the generating function for unconstrained walks on the line $\mathbb{Z}$ is rational, while the generating function for walks constrained to stay in the half-line $\mathbb{N}$ is always algebraic [3]. This result has often been described in terms of *partially directed* 2-dimensional walks confined in a quadrant (or *generalized Dyck walks* [10, 15, 20, 21]), but is, essentially, of a 1-dimensional nature.

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Similar questions can be addressed for *real* 2-dimensional walks. Again, the generating function for unconstrained walks starting from a given point is clearly rational. Moreover, the argument used for 1-dimensional walks confined in $\mathbb{N}$ can be recycled to prove that the generating function for the walks that stay in the half-plane $x \geq 0$ is always algebraic. What about doubly-restricted walks, that is, walks that are confined in the quadrant $x \geq 0, y \geq 0$? It would be satisfactory if the hierarchy unconstrained walks / mono-constrained walks / bi-constrained walks could match the classical hierarchy of generating functions: rational series, algebraic series, D-finite series[1] (also called holonomic series). A rapid inspection of the most standard cases only corroborates this hope. For instance, the generating function for walks on the square lattice (with North, East, South and West steps) that start from the origin and stay in the first quadrant is

$$\sum_{m,n\geq 0} \binom{m+n}{m} \binom{m}{\lfloor m/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor} t^{m+n} = \sum_{n\geq 0} \binom{n}{\lfloor n/2 \rfloor} \binom{n+1}{\lceil n/2 \rceil} t^n,$$

which is a D-finite series. The first expression comes from the fact that these walks are shuffles of two prefixes of Dyck walks, and the Chu-Vandermonde identity transforms it into the second simpler expression.

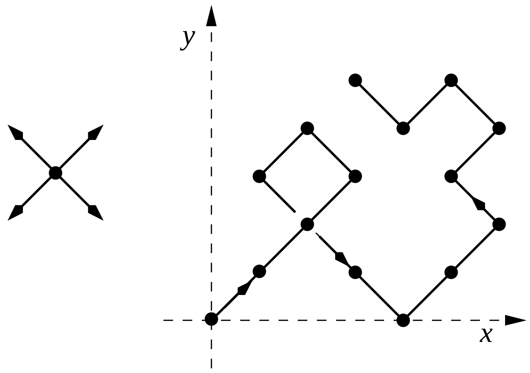

Figure 1: A walk on the diagonal square lattice confined in the first quadrant.

The case of the diagonal square lattice, where the steps are North-East, South-East, North-West and South-West (Figure 1) is even simpler: by projecting the walks on the x- and y-axes, we obtain two decoupled prefixes of Dyck paths, so that the generating function for walks in the first quadrant is now

$$\sum_{n\geq 0} \binom{n}{\lfloor n/2 \rfloor}^2 t^n,$$

another D-finite series. In both cases, the number of n-step walks can be shown to grow asymptotically like $4^n/n$, which prevents the corresponding generating function from being algebraic (see [13] for the possible asymptotic behaviours of coefficients of algebraic series).

[1] A series $F(t)$ is D-finite if it satisfies a linear differential equation with polynomial coefficients in t.

In Section 2 of this paper, we shall generalize this result by proving that, if the set of steps $\mathfrak{S}$ is symmetric with respect to the x-axis and satisfies a *small height variation* condition, then the generating function for walks with steps in $\mathfrak{S}$, starting from any given point (i_0, j_0), is D-finite. This result covers the above two cases.

However, and most importantly, we shall also prove in Section 3 that this holonomy result does not hold for *any* set of steps: walks that start from $(1,1)$, take their steps in $\mathfrak{S} = \{(2,-1), (-1,2)\}$ and always stay in the first quadrant have a non-D-finite generating function. The central point of our proof is the study of a series $G(x)$, defined by an equation of the form

$$G(x) + G(\xi(x)) = A(x),$$

where $\xi(x)$ and $A(x)$ are explicit algebraic series in x. We consider the solution $G(x)$ to this equation as a function of a complex variable x, and prove that it has *infinitely many singularities*, which prevents it from being D-finite. Hence our proof is based on complex analysis. To our knowledge, there is no classification of the solutions to this type of equation. In some very specific cases (like $\xi(x) = x^p$ or $\xi(x) = cx$) some hypertranscendence results[2] have been obtained, either by some *ad hoc* methods [18, 26, 31], or via general results about "very" lacunary series [25].

These two sections raise the question of a classification of the sets $\mathfrak{S}$ according to the nature of the generating function for walks in a quadrant that take their steps in $\mathfrak{S}$. Let us mention that some sets of steps, like $\mathfrak{S} = \{(1,1), (0,-1), (-1,0)\}$ yield, for non-trivial reasons, algebraic generating functions [5, 14, 19, 29].

Finally, in Section 4, we say a few words about the closely related topic of multidimensional linear recurrences with constant coefficients, and prove a non-holonomy result that was announced (but not proven) in [7].

Let us conclude this introduction with a few more formal definitions on walks and power series.

Let $\mathfrak{S}$ be a finite subset of $\mathbb{Z}^2$. A walk with steps in $\mathfrak{S}$ is a finite sequence $w = (w_0, w_1, \ldots, w_n)$ of vertices of $\mathbb{Z}^2$ such that $w_i - w_{i-1} \in \mathfrak{S}$ for $1 \leq i \leq n$. The number of steps, n, is the *length* of w. The starting point of w is w_0, and its endpoint is w_n. The *complete generating function* for a set $\mathcal{A}$ of walks starting from a given point w_0 is the series

$$A(x, y; t) = \sum_{n \geq 0} t^n \sum_{i,j \in \mathbb{Z}} a_{i,j}(n) x^i y^j,$$

where $a_{i,j}(n)$ is the number of walks of $\mathcal{A}$ that have length n and end at (i,j). This series is a formal power series in t whose coefficients are polynomials in $x, y, 1/x, 1/y$. We shall often denote $\bar{x} = 1/x$ and $\bar{y} = 1/y$. The length generating function for walks of A is simply

$$A(t) = \sum_{n \geq 0} a(n) t^n$$

where $a(n)$ is the number of walks of $\mathcal{A}$ that have length n. Note that $A(t) = A(1,1;t)$.

[2] A series is hypertranscendental if it does not satisfy any polynomial differential equation of the type $P(x, F(x), F'(x), \ldots, F^{(k)}(x)) = 0$, where P is a polynomial.

Given a ring $\mathbb{L}$ and k indeterminates $x_1, \ldots, x_k$, we denote by $\mathbb{L}[x_1, \ldots, x_k]$ (resp. $\mathbb{L}[[x_1, \ldots, x_k]]$) the ring of polynomials (resp. formal power series) in $x_1, \ldots, x_k$ with coefficients in $\mathbb{L}$. If $\mathbb{L}$ is a field, we denote by $\mathbb{L}(x_1, \ldots, x_k)$ the field of rational functions in $x_1, \ldots, x_k$ with coefficients in $\mathbb{L}$.

Assume $\mathbb{L}$ is a field. A series F in $\mathbb{L}[[x_1, \ldots, x_k]]$ is *rational* if there exist polynomials P and Q in $\mathbb{L}[x_1, \ldots, x_k]$, with $Q \neq 0$, such that $QF = P$. It is *algebraic* (over the field $\mathbb{L}(x_1, \ldots, x_k)$) if there exists a non-trivial polynomial P with coefficients in $\mathbb{L}$ such that $P(F, x_1, \ldots, x_k) = 0$. The sum and product of algebraic series is algebraic.

The series F is *D-finite* (or *holonomic*) if the partial derivatives of F span a finite dimensional vector space over the field $\mathbb{L}(x_1, \ldots, x_k)$ (this vector space is a subspace of the fraction field of $\mathbb{L}[[x_1, \ldots, x_k]]$); see [35] for the one-variable case, and [23, 24] otherwise. In other words, for $1 \leq i \leq k$, the series F satisfies a non-trivial partial differential equation of the form

$$\sum_{\ell=0}^{d_i} P_{\ell,i} \frac{\partial^\ell F}{\partial x_i^\ell} = 0,$$

where $P_{\ell,i}$ is a polynomial in the x_j. Any algebraic series is holonomic. The sum and product of two holonomic series is still holonomic. The specializations of an holonomic series (obtained by giving values from $\mathbb{L}$ to some of the variables) are holonomic, if well-defined. Moreover, if F is an *algebraic* series and $G(t)$ is a holonomic series of one variable, then the substitution $G(F)$ (if well-defined) is holonomic [24, Prop. 2.3].

2 A sufficient condition for holonomy

Let $\mathfrak{S}$ be a finite subset of $\mathbb{Z}^2$. We say that $\mathfrak{S}$ is symmetric with respect to the x-axis if

$$(i, j) \in \mathfrak{S} \Rightarrow (i, -j) \in \mathfrak{S}.$$

We say that $\mathfrak{S}$ has small height variation if

$$(i, j) \in \mathfrak{S} \Rightarrow |j| \leq 1.$$

The usual square lattice steps satisfy these two conditions. So do the steps of the diagonal square lattice (Figure 1).

Theorem 1 *Let $\mathfrak{S}$ be a finite subset of $\mathbb{Z}^2$ that is symmetric with respect to the x-axis and has small height variations. Let $(i_0, j_0) \in \mathbb{N}^2$. Then the length generating function for walks that start from (i_0, j_0), take their steps in $\mathfrak{S}$ and stay in the first quadrant is D-finite.*

We shall need the following preliminary result, which does not require any property on $\mathfrak{S}$.

Proposition 2 *Let $\mathfrak{S}$ be a finite subset of $\mathbb{Z}^2$. Let $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$. Then the complete generating function for walks that start from (i_0, j_0), take their steps in $\mathfrak{S}$ and stay in the right half-plane $x \geq 0$ is algebraic.*

Proof. The problem being invariant by any vertical translation, we can assume $j_0 = 0$. Let us also assume that $i_0 = 0$. The argument is easily adapted when $i_0 > 0$.

Projecting the walks on the x-axis reduces the problem to the enumeration of 1-dimensional walks on the half-line $\mathbb{N}$, starting from 0, in which each step of size i is weighted by a Laurent polynomial in y:

$$\sum_{j:(i,j)\in\mathfrak{S}} y^j.$$

The weight of a walk is taken to be the product of the weights of its steps. We can now invoke some 1-dimensional results, like those of [3], and conclude that the complete generating function for walks in the right-half plane is algebraic over $\mathbb{Q}(x, y, t)$.

■

Proof of Theorem 1. Let $\mathcal{Q}$ denote the set of walks that start from (i_0, j_0), take their steps in $\mathfrak{S}$ and stay in the first quadrant. We shall prove that walks of $\mathcal{Q}$ are, roughly speaking, "equivalent to" walks in the right half-plane ending on the x-axis.

We claim that it suffices to prove that the subset of $\mathcal{Q}$ consisting of the walks that hit the x-axis at some point has a D-finite generating function. Indeed, the remaining walks are, by a vertical translation, in one-to-one correspondence with walks that start from $(i_0, j_0 - k)$, for some $k \in [1, j_0]$, stay in the first quadrant and hit the x-axis.

Let us first focus on the set $\mathcal{Q}_e^{(0)}$ of walks in the first quadrant that start from (i_0, j_0), hit the x-axis and end at an even ordinate. These walks are in bijection with the set $\mathcal{H}_0$ of walks that start from (i_0, j_0), stay in the right half-plane and end on the x-axis. This bijection, illustrated by Figure 2, is a mere adaptation of a classical 1-dimensional correspondence, which establishes that Dyck prefixes ending at an even ordinate are equivalent to bilateral Dyck walks (see the Catalan factorisation in [9].) Starting from a walk w of $\mathcal{Q}_e^{(0)}$, ending at level $2k$, we denote by $s_1, \ldots, s_k$ the steps that follow the last visit of w to a point of level (ordinate) $0, \ldots, k-1$. Replacing these k steps by their symmetric steps with respect to the x-axis yields a walk $\bar{w}$ that belongs to $\mathcal{H}_0$. The ordinate of the lowest point(s) visited by $\bar{w}$ is $-k$. Conversely, the steps $\bar{s}_1, \ldots, \bar{s}_k$ of $\bar{w}$ that we have to flip back to recover w are the first steps of w that lead to level $-j$, for $1 \leq j \leq k$.

Let $H(x, y; t)$ be the complete generating function for walks in the right half-plane. The length generating function for walks of $\mathcal{H}_0$ is obtained by extracting the coefficient of y^0 in the generating function $H(1, y; t)$. Extracting the constant term of a D-finite series is known to give another D-finite series [23]: thus the generating function for walks of $\mathcal{H}_0$, and hence of $\mathcal{Q}_e^{(0)}$, is D-finite.

A similar argument holds for the set $\mathcal{Q}_o^{(0)}$ of walks in the first quadrant that start from (i_0, j_0), hit the x-axis and end at an odd ordinate: they are in one-to-one correspondence with the set $\mathcal{H}_{-1}$ of walks that start from (i_0, j_0), stay in the right half-plane and end at level -1. The generating function for walks of $\mathcal{H}_{-1}$ is the coefficient of y^0 in $yH(1, y; t)$, and hence is D-finite. Given that the sum of D-finite series is D-finite, this concludes the proof of Theorem 1.

■

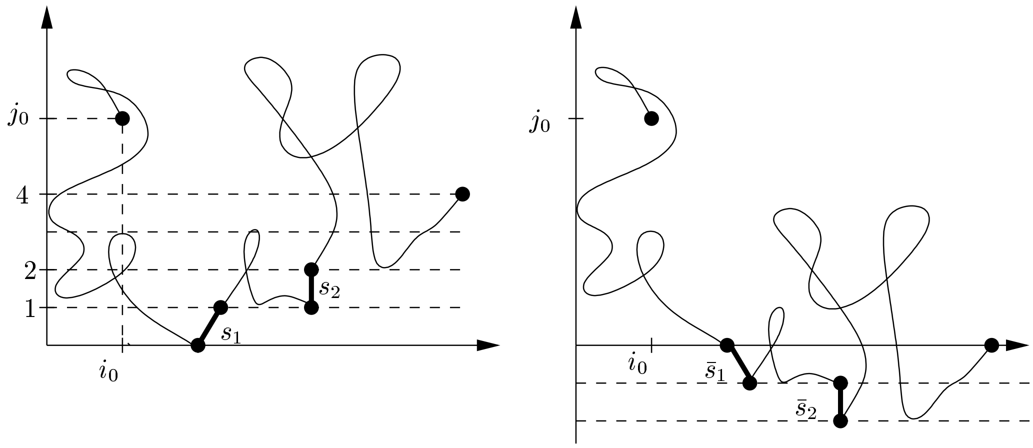

Figure 2: The bijection between some walks in the quadrant and walks in the right half-plane ending on the x-axis.

3 The knight walk is not holonomic

3.1 The main result

We study walks that start at $(1, 1)$, take their steps in $\{(-1, 2), (2, -1)\}$ and stay in the first quadrant. We call them *knight walks*, since their steps correspond to two of the knight moves on a chessboard (Figure 3). We note that a walk ending at (i, j) has always $i + j - 2$ steps: hence the information contained in the complete generating function is actually already contained in the following bivariate series:

$$Q(x, y) = \sum_{i \geq 0, j \geq 0} Q_{i,j}\, x^i y^j,$$

where $Q_{i,j}$ denotes the number of knight walks ending at (i, j).

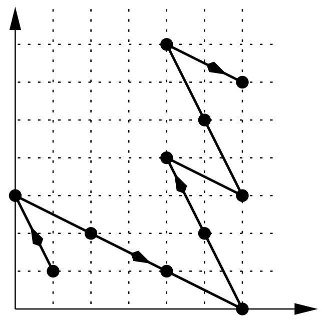

Figure 3: A knight walk.

The coefficients $Q_{i,j}$ satisfy the following recurrence relation:

$$Q_{i,j} = \begin{cases} 0 & \text{if } i < 0 \text{ or } j < 0, \\ 1 & \text{if } i = j = 1, \\ Q_{i+1,j-2} + Q_{i-2,j+1} & \text{otherwise.} \end{cases} \tag{1}$$

This recurrence allows us to compute the numbers $Q_{i,j}$ inductively, for instance diagonal by diagonal. The first few values are given in Table 1 below, in which the zero entries are left out. The non-zero entries lie on the lines $i = j \bmod 3$.

j ↑														
12	24													
11			108											
10		24			312									
9	6			84			720							
8			24			204			1440					
7		6			60			408						
6	2			18			120			720				
5			6			36			204					
4		2			12			60			312			
3	1			4			18			84				
2			2			6			24			108		
1		1			2			6			24			
0				1			2			6			24	
	0	1	2	3	4	5	6	7	8	9	10	11	12	→ i

Table 1: The number $Q_{i,j}$ of knight walks ending at (i,j).

Applying the transformation $(i,j) \to ((2i+j)/3, (i+2j)/3)$ shows that $Q_{i,j}$ is also the number of walks made of North and East steps, that start from $(1,1)$, end at $((2i+j)/3, (i+2j)/3)$ and always stay above the line $2y = x$ and below the line $2x = y$ (see Figure 4). Ignoring these two conditions gives the following simple bound:

$$Q_{i,j} \le \binom{i+j-2}{(2i+j-3)/3}.$$

In particular,

$$Q_{3i,0} \le \binom{3i-2}{i-1}. \tag{2}$$

This bound could be sharpened by counting walks that stay above the line $2y = x$ (see e.g. [11] or [7, Example 4]), but the above bound will be enough for our purpose.

By summing the recurrence relation we obtain:

$$\begin{aligned} Q(x,y) = \sum_{i,j \ge 0} Q_{i,j} x^i y^j &= xy + \sum_{i,j \ge 0} (Q_{i+1,j-2} + Q_{i-2,j+1}) x^i y^j \\ &= xy + y^2/x(Q(x,y) - Q(0,y)) + x^2/y(Q(x,y) - Q(x,0)), \end{aligned}$$

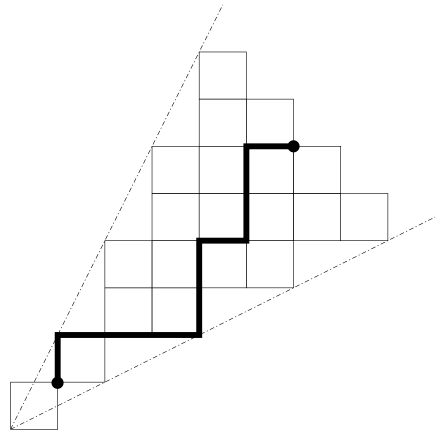

Figure 4: Another description of the knight walk.

that is

$$(xy - x^3 - y^3)Q(x, y) = x^2y^2 - G(x) - G(y) \tag{3}$$

where

$$G(x) = x^3 \sum_{i \geq 0} Q_{i,0}x^i = x^3 Q(x, 0) \tag{4}$$

counts knight walks ending on the x-axis. (We have used the symmetry of the problem in x and y.) Note that the length generating function for all knight walks is $t^{-2}Q(t, t)$. The above equation, combined with the elementary properties of D-finite series, imply that the following three statements are equivalent:

— the bivariate generating function $Q(x, y)$ for knight walks is D-finite,

— the generating function $Q(x, 0)$ for knight walks ending on the x-axis is D-finite,

— the length generating function $t^{-2}Q(t, t)$ for knight walks is D-finite.

Our main result asserts that none of these statements hold.

Theorem 3 *The length generating function for walks that start from* $(1, 1)$*, take their steps in* $\{(-1, 2), (2, -1)\}$ *and always stay in the first quadrant (knight walks) is not D-finite. Nor is the generating function for knight walks that end on the* x*-axis.*

Looking at Table 1, one might still have some hope that the numbers $Q_{i,j}$ are not so bad. In particular, they seem to have small prime factors. This pattern actually does not go on, and, in case there would still be a doubt, the following proposition reinforces the non-holonomic character of these numbers.

Proposition 4 *The length generating function for knight walks ending on the main diagonal* $x = y$ *is not D-finite.*

3.2 The kernel method

The so-called *kernel method* solves completely the functional equation (3). This method has been around since, at least, the 70's, and is currently the subject of a certain rebirth (see the references in [2, 3, 7]).

Applied to our equation, this method consists in coupling the variables x and y so as to cancel the *kernel* $xy - x^3 - y^3$; this yields the missing information about the series $G(x)$. More precisely, let $\xi(x)$ be the unique formal power series in x satisfying

$$x\xi - x^3 - \xi^3 = 0.$$

The Lagrange inversion formula, applied to $\xi(x)/x$, provides an explicit expression for $\xi(x)$:

$$\xi(x) = x^2 \sum_{m\geq 0} \frac{x^{3m}}{2m+1}\binom{3m}{m} = O(x^2). \tag{5}$$

Replacing y by $\xi(x)$ in (3) gives a functional equation that defines the power series $G(x)$:

$$G(x) + G(\xi(x)) = x^2\xi(x)^2. \tag{6}$$

Indeed we obtain, after iterating this equation infinitely many times:

$$G(x) = \sum_{i\geq 0}(-1)^i \left(\xi^{(i)}(x)\xi^{(i+1)}(x)\right)^2$$

where $\xi^{(i)} = \xi \circ \cdots \circ \xi$ is the ith iterate of ξ. Note that $\xi^{(i)}(x) = O(x^{2^i})$, so that the sum is convergent in the ring $\mathbb{C}[[x]]$. Replacing $G(x)$ by the above explicit value in (3) would give an expression for $Q(x,y)$.

However, we shall not exploit these expressions, but rather the functional equation (6), to prove that $G(x)$, hence $Q(x,y)$, is not D-finite. Our proof will be of an analytic nature. The idea is to consider $G(x)$ as a function of a complex variable x and study its singularities. Using functional equations like (6), we shall build new singularities of G from old ones – and end up with infinitely many singularities, thus proving that G cannot be holonomic.

However, even though (6) defines $G(x)$ uniquely, this equation itself is not sufficient for our purpose: we shall introduce the other two roots $\xi_1(x)$ and $\xi_2(x)$ of the kernel, and the corresponding analogues of (6), to obtain enough singularities.

3.3 The roots of $x^3 + y^3 = xy$

As a polynomial in y, the kernel $xy - x^3 - y^3$ has three roots. Only one of them, given by (5), is a power series in x. We shall denote it, from now on, by ξ_0:

$$\xi_0(x) \quad = \quad x^2 + x^5 + 3x^8 + 12x^{11} + 55x^{14} + 273x^{17} + \cdots$$

The other two roots are power series in $\sqrt{x}$, and their expansion can be computed inductively:

$$\begin{aligned}\xi_1(x) &= +\sqrt{x} - \frac{x^2}{2} - \frac{3}{8}x^3\sqrt{x} - \frac{x^5}{2} - \frac{105}{128}x^6\sqrt{x} - \cdots \\ \xi_2(x) &= -\sqrt{x} - \frac{x^2}{2} + \frac{3}{8}x^3\sqrt{x} - \frac{x^5}{2} + \frac{105}{128}x^6\sqrt{x} - \cdots\end{aligned}$$

Of course, $\xi_2(x)$ is derived from $\xi_1(x)$ by replacing $\sqrt{x}$ by $-\sqrt{x}$. Guided by the above expressions, let us write

$$\begin{aligned}\xi_1(x) &= +\sqrt{x}\,\psi(x) - \phi(x) \\ \xi_2(x) &= -\sqrt{x}\,\psi(x) - \phi(x)\end{aligned} \tag{7}$$

where ψ and ϕ are formal power series in x. As the three roots sum to zero, one has

$$\phi(x) = \xi_0(x)/2. \tag{8}$$

In order to compute the coefficients of ψ, we shall use again the Lagrange inversion formula (LIF). Let $\chi(x)$ be defined by

$$\xi_1(x) = \sqrt{x}(1 - \chi(x)). \tag{9}$$

Then

$$\chi = \frac{x\sqrt{x}}{(1-\chi)(2-\chi)},$$

so that the LIF gives, for $n \geq 1$,

$$a_n := [x^{3n/2}]\chi(x) = \frac{1}{n2^n}\sum_{k=0}^{n-1}\frac{1}{2^k}\binom{n+k-1}{k}\binom{2n-k-2}{n-1}. \tag{10}$$

Using the package EKHAD [33], we prove that the sequence a_n satisfies, for $n \geq 1$,

$$a_{n+2} = \frac{3(3n+1)(3n-1)}{4(n+2)(n+1)}\,a_n.$$

When n is odd, say $n = 2m+1$, we derive from (9) and (7) that the coefficient a_n is $[x^{3m+2}]\phi(x)$, and the above recursion yields $a_n = \binom{3m}{m}/(4m+2)$. We thus recover (8). Similarly, when $n = 2m$, the coefficient a_n is $-[x^{3m}]\psi(x)$ and the above recursion now gives

$$\psi(x) = -\sum_{m\geq 0}\frac{m!(6m)!}{(6m-1)(2m)!^2(3m)!}\frac{x^{3m}}{16^m}.$$

Let us summarize the results we have obtained.

Lemma 5 *Let ξ and ψ be the following power series in x:*

$$\begin{aligned}\xi &= x^2 \sum_{m\geq 0} \frac{x^{3m}}{2m+1}\binom{3m}{m},\\ \psi &= -\sum_{m\geq 0} \frac{m!(6m)!}{(6m-1)(2m)!^2(3m)!}\frac{x^{3m}}{16^m}.\end{aligned}$$

Then the three roots of $x^3+y^3=xy$ are

$$\begin{aligned}\xi_0(x) &= \xi\\ \xi_1(x) &= \sqrt{x}\,\psi - \xi/2\\ \xi_2(x) &= -\sqrt{x}\,\psi - \xi/2.\end{aligned}$$

Both $\xi(x)$ and $\psi(x)$ have radius of convergence $x_c = 4^{1/3}/3$.

The last statement is obtained using the Stirling formula. Note that one also has the following closed form expressions:

$$\begin{aligned}\xi(x) &= 2\sqrt{\frac{x}{3}}\,\sin\left(\frac{\arcsin\left((3x)^{3/2}/2\right)}{3}\right),\\ \psi(x) &= \cos\left(\frac{\arcsin\left((3x)^{3/2}/2\right)}{3}\right).\end{aligned}$$

We shall now consider the $\xi_i(x)$ as functions of a complex variable x. We choose a determination of the square root that coincides with the usual determination on $\mathbb{R}^+$:

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2} \quad \text{for} \quad -\pi < \theta < \pi.$$

Figure 5 shows the (real) values of the functions ξ_i for real values of x. When x is positive, the plots show, from bottom to top, $\xi_2(x)$, $\xi_0(x)$ and $\xi_1(x)$. When x is negative, the only real branch is ξ_0.

The implicit function theorem implies that the singularities of the functions ξ_i are to be found among the complex numbers x such that the pair $(x,y)\equiv(x,\xi_i(x))$ satisfies $3y^2=x$ (and, of course, $x^3+y^3=xy$); that is,

$$(x,y) \in \{(0,0),(x_c,y_c),(jx_c,j^2y_c),(j^2x_c,jy_c)\} \tag{11}$$

where $x_c = 2^{2/3}/3$, $y_c = 2^{1/3}/3$ and $j=\exp(2i\pi/3)$. A more detailed investigation gives the following result.

Lemma 6 *The singularities of the functions ξ_i are given by:*

$$Sing(\xi_0) = \{x_c, jx_c, j^2x_c\}, \quad Sing(\xi_1) = \{0, x_c\} \quad \textit{and} \quad Sing(\xi_2) = \{0, jx_c, j^2x_c\}.$$

In particular, ξ_2 is not singular at x_c, as suggested by Figure 5. All the above singularities are of the square root type.

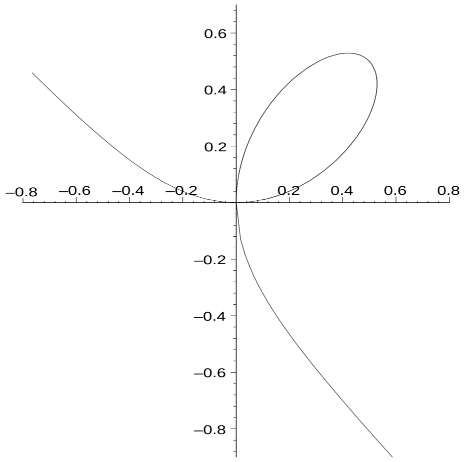

Figure 5: The real values of $\xi_i(x)$ for a real x.

Proof. For each of the values of x given by (11), we first have to compute the values $\xi_i(x)$, $0 \le i \le 2$, in order to determine which pairs $(x, \xi_i(x))$ are actually critical, and then, to check the existence and nature of the singularity.

At $x = 0$, all the $\xi_i(x)$ are zero. The explicit expansion of Lemma 5, combined with the fact that ψ and ξ have a positive radius of convergence, shows that only ξ_1 and ξ_2 are singular — their singularity being obviously of a square root type.

When $x = x_c$, factoring the polynomial $x^3 + y^3 - xy$ shows that the multiset $\{\xi_0(x_c), \xi_1(x_c), \xi_2(x_c)\}$ equals $\{y_c, y_c, -2y_c\}$. The functions $\xi_i(x)$ are real and continuous on $[0, x_c]$, and the expansions of Lemma 5 show that $\xi_0(x)$ and $\xi_1(x)$ are positive as $x \to 0^+$, while $\xi_2(x)$ is negative. As the ξ_i can only vanish at 0, this sign pattern must go on until x_c, so that

$$\xi_0(x_c) = \xi_1(x_c) = y_c \quad \text{and} \quad \xi_2(x_c) = -2y_c. \tag{12}$$

In view of (11), x_c cannot be a singularity of ξ_2. Now a local expansion of $x^3 + y^3 - xy$ around (x_c, y_c) shows that as x approaches x_c^-,

$$\xi_{0,1}(x) = y_c \mp \frac{1}{\sqrt{3}}\sqrt{x_c - x}(1 + o(1)), \tag{13}$$

which confirms that the singularity of ξ_0 and ξ_1 at x_c is of the square root type.

The values (12) imply in particular that

$$\sqrt{x_c}\,\psi(x_c) = 3y_c/2 = 2^{-2/3}.$$

Using this result, the fact that ψ and ξ are essentially functions of x^3, and the values

$$\sqrt{j} = -j^2 \quad \text{and} \quad \sqrt{j^2} = -j,$$

we now compute

$$\begin{array}{rclcrcl}
\xi_0(jx_c) & = & j^2 y_c & & \xi_0(j^2 x_c) & = & j y_c \\
\xi_1(jx_c) & = & -2j^2 y_c & & \xi_1(j^2 x_c) & = & -2j y_c \\
\xi_2(jx_c) & = & j^2 y_c & & \xi_2(j^2 x_c) & = & j y_c
\end{array}$$

so that ξ_1 cannot be singular at jx_c or j^2x_c. Finally, local expansions of x^3+y^3-xy confirm as above the existence of square roots singularities of ξ_0 and ξ_2 at jx_c and j^2x_c.

■

Each of the functions ξ_i has a unique analytic continuation on any simply connected domain avoiding $Sing(\xi_i)$, for instance the domain obtained by removing from $\mathbb{C}$ the four half-lines of Figure 6.

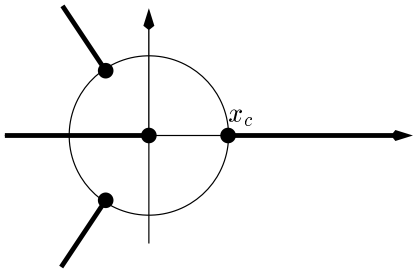

Figure 6: A domain on which all the functions ξ_i are holomorphic.

Observe that the series ξ_1 and ξ_2 can *also* be substituted for y in the functional equation (3); thus for $i \in \{0,1,2\}$, the following equation holds

$$G(x) + G(\xi_i(x)) = x^2\xi_i(x)^2, \tag{14}$$

at least as an identity between power series.

We end this subsection with a lemma that will be useful to build large singularities from small ones.

Lemma 7 *Let $x \in \mathbb{C}$, $x \neq 0$. Then one of the roots of $x^3+y^3-xy=0$ has modulus larger than $|x|$.*

Proof. Let y_0, y_1, y_2 denote the three roots, and assume none of then has modulus larger than $|x|$. The relation $y_0y_1y_2 = -x^3$ forces $|y_0| = |y_1| = |y_2| = |x|$. Then, the relation $y_0+y_1+y_2 = 0$ implies $\{y_1, y_2\} = \{jy_0, j^2y_0\}$. Finally, the relation $y_0y_1+y_0y_2+y_1y_2 = -x$ yields $x = 0$. ■

3.4 The series $G(x)$ is not D-finite

We now turn our attention to the series $G(x)$ defined by (4).

Proposition 8 *The series $G(x)$ has radius of convergence $x_c = 4^{1/3}/3$. It is singular at x_c, with*

$$G(x) = A - B\sqrt{1-x/x_c}(1+o(1)),$$

where A and B are non-zero real numbers.

Proof. We start from the functional equation defining G:

$$G(x) + G(\xi_0(x)) = x^2 \xi_0(x)^2.$$

Recall that $G(x)$ and $\xi_0(x)$ have nonnegative coefficients ($G(x)$ counts walks, and $\xi_0(x) \equiv \xi(x)$ is given explicitly by (5)). Hence $G(\xi_0(x))$ and $x^2\xi_0(x)^2$ also have nonnegative coefficients. The radius of convergence of $\xi_0(x)$ being x_c, the series $G(x)$ and $G(\xi_0(x))$ have radius *at least* x_c. The fact that $G(x)$ has radius at least x_c can also be directly derived from the upper bound (2).

For $|x| < x_c$, one has $|\xi_0(x)| < \xi_0(x_c) = y_c$, and y_c is smaller than x_c. This means that the above functional equation now holds as an identity between analytic functions of x in the disk $|x| < x_c$. As x approaches x_c inside this disk, a local expansion gives, thanks to (13),

$$G(x) = x_c^2 y_c^2 - G(y_c) - \frac{1}{\sqrt{3}}\sqrt{x_c - x}\,\big(2x_c^2 y_c - G'(y_c)\big) + O(x_c - x).$$

The upper bound (2) yields

$$G'(y_c) \leq \sum_{i\geq 1} \binom{3i-2}{i-1} y_c^{3(i+1)}.$$

This sum can be evaluated numerically, and is found to be smaller than 0.16. Comparing with $2x_c^2 y_c \approx 0.23$ gives the announced result.

∎

Observe that (14) now holds as an identity between *functions* of x, as long as $|x| < x_c$ and $|\xi_i(x)| < x_c$ (and $x \notin \mathbb{R}^-$ if $i = 1$ or 2).

Proposition 9 *The series $G(x)$ that counts knight walks ending on the x-axis is not D-finite.*

Proof. Assume G is D-finite. Then, it has a finite number of singularities, and has a unique analytic continuation on any simply connected domain avoiding these singularities. For $i \in \{0, 1, 2\}$, the series $G(\xi_i)$ are also D-finite (since ξ_i is algebraic). Moreover, by analytic continuation of (14),

$$G(x) + G(\xi_i(x)) = x^2 \xi_i(x)^2 \qquad (15)$$

for all x in any domain where all our functions are analytically defined. Let us consider the identity (15) for $i = 2$. As x approaches x_c, the function $G(x)$ becomes singular, while $\xi_2(x)$ does not: this shows that $\xi_2(x_c) = -2y_c$ is a singularity of G.

Now let x_s be a singularity of G of maximal modulus. According to Lemma 7, there exists i such that $|\xi_i(x_s)| > |x_s|$. By assumption, $G(x)$ is singular at x_s. But $|x_s| \geq 2y_c > x_c$, so that x_s is not a singularity of ξ_i. Hence (15) implies that G is singular at $\xi_i(x_s)$. But $|\xi_i(x_s)| > |x_s|$, which contradicts the maximality of $|x_s|$.

∎

Remarks

1. The only property of D-finite series we have really used is the fact that these series have finitely many singularities. Hence, what we have actually proved is that G *cannot have finitely many singularities.*

2. The principle of the proof can be applied to other functional equations of the same type. We apply it in the next section to a (minor) variation of the knight walks.

There remains to prove Proposition 4: the generating function for knight walks ending on the diagonal is not D-finite.

Proof of Proposition 4. The length generating function for walks ending on the diagonal is the coefficient of x^0 in the series $t^{-2}Q(tx, t\bar{x})$, where $\bar{x} = 1/x$. This series is a power series in t whose coefficients are Laurent polynomials in x. Let us denote $u = x^3$ and $\bar{u} = 1/u$. Then, by Eq. (3),

$$t^2 Q(tx, t\bar{x}) = \frac{t^4 - S(t^3 u) - S(t^3 \bar{u})}{1 - t(u + \bar{u})}, \tag{16}$$

where the series $S(z)$ is defined by $S(z^3) = G(z)$. Let us convert $1/(1 - t(u + \bar{u}))$ in partial fractions of u. We obtain

$$\frac{1}{1 - t(u + \bar{u})} = \frac{1}{\sqrt{1 - 4t^2}} \left(\frac{1}{1 - uU(t)} + \frac{1}{1 - \bar{u}U(t)} - 1 \right),$$

where

$$U(t) = \frac{1 - \sqrt{1 - 4t^2}}{2t}.$$

We can now extract the constant term in x from (16):

$$\begin{aligned} [x^0]t^2 Q(tx, t\bar{x}) &= \frac{1}{\sqrt{1 - 4t^2}} \left(t^4 - [u^0] \frac{S(t^3 u)}{1 - \bar{u}U(t)} - [u^0] \frac{S(t^3 \bar{u})}{1 - uU(t)} \right) \\ &= \frac{1}{\sqrt{1 - 4t^2}} \left(t^4 - 2S(t^3 U(t)) \right). \end{aligned}$$

Assume this series is D-finite: then the series $D(t)$ defined by $D(t) = S(t^3 U(t))$ is D-finite in t too. Let $T(s)$ be the unique power series in s such that $T(0) = 0$, $T'(0) = 1$ and $T^6 - s^4 T^2 + s^8 = 0$. Then the fact that D-finite series are stable by any algebraic substitution tells us that $D(T(s)) = S(T^3 U(T))$ is D-finite. But $T^3 U(T) = s^4$, so that $S(s^4)$ itself, and hence $S(s)$ and $G(s)$, are D-finite too, which we have proved to be wrong.

■

4 A link with multidimensional linear recurrences with constant coefficients

In [7], we considered d-dimensional sequences of complex numbers, denoted $a_{\boldsymbol{n}} = a_{n_1, n_2, \ldots n_d}$, defined by recurrence relations of the following form:

$$a_{\boldsymbol{n}} = \sum_{\boldsymbol{h} \in H} c_{\boldsymbol{h}} a_{\boldsymbol{n}+\boldsymbol{h}} \qquad \text{for } \boldsymbol{n} \geq \boldsymbol{s}, \tag{17}$$

where $H = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_k\} \subseteq \mathbb{Z}^d$ is the set of *shifts*, $(c_{\boldsymbol{h}})_{\boldsymbol{h}\in H}$ are given nonzero constants, and $\boldsymbol{s} \in \mathbb{N}^d$ is the *starting point* satisfying $\boldsymbol{s} + H \subseteq \mathbb{N}^d$. We think of the $\boldsymbol{h}_i$ as having mostly (but not necessarily only) negative coordinates, and of the point $\boldsymbol{n}$ as depending on the points $\boldsymbol{n}+\boldsymbol{h}_1, \boldsymbol{n}+\boldsymbol{h}_2, \dots, \boldsymbol{n}+\boldsymbol{h}_k$ as far as the value of $a_{\boldsymbol{n}}$ is concerned. A given function φ specifies the initial conditions:

$$a_{\boldsymbol{n}} = \varphi(\boldsymbol{n}) \qquad \text{for } \boldsymbol{n} \geq \boldsymbol{0},\ \boldsymbol{n} \not\geq \boldsymbol{s}\,. \tag{18}$$

The convex hull of the set H is assumed not to intersect the first orthant (*i.e.*, $\boldsymbol{n} \geq \boldsymbol{0}$). This condition, as shown in [7], guarantees that the numbers $a_{\boldsymbol{n}}$ can be computed recursively using (17).

The enumeration of walks in a quadrant fits exactly in this framework, with $d = 3$. Indeed, denoting $Q_{i,j}(n)$ the number of walks that start from a given point (i_0, j_0), end at (i, j) and have length n, we have

$$Q_{i,j}(n) = \begin{cases} 1 & \text{if } (i,j,n) = (i_0, j_0, 0) \\ 0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0 \\ \displaystyle\sum_{(h,k)\in\mathfrak{S}} Q_{i-h,j-k}(n-1) & \text{otherwise,} \end{cases}$$

where $\mathfrak{S}$ is the set of steps. A translation of the indices i and j transforms this recursion into one of the above type, with $d = 3$ and $H = \{(-h, -k, -1), (h, k) \in \mathfrak{S}\}$.

The paper [7] mostly dealt with the algebraic nature of the generating function of the solution of such recurrences, and we started a classification, based on the *apex* of the recurrence, defined as the componentwise maximum of the points in $H \cup \{\boldsymbol{0}\}$. We proved that when the initial conditions have rational generating functions and the apex is $\boldsymbol{0}$, the generating function of the solution is rational. Next, when the initial conditions have algebraic generating functions and the apex has at most one positive coordinate, the generating function of the solution is algebraic.

When the apex has two positive coordinates, and $d = 3$, our study of walks in a quadrant shows that the solution might be, or not, D-finite. For 2-dimensional sequences, the "simplest" example with apex $(1,1)$ was introduced in [32]: For $i, j \geq 0$, let

$$a_{i,j} = \begin{cases} a_{i+1,j-2} + a_{i-2,j+1} & \text{if } i, j \geq 2, \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

This recurrence is obviously closely connected to the knight walk. Again, it has a unique solution whose terms can be computed inductively. The first few values are given in the following array.

j								
$\uparrow$								
6	1	1	5	7	$\cdot$	$\cdot$	$\cdot$	
5	1	1	3	5	10	14	$\cdot$	
4	1	1	3	4	6	10	$\cdot$	
3	1	1	2	2	4	5	7	
2	1	1	2	2	3	3	5	
1	1	1	1	1	1	1	1	
0	1	1	1	1	1	1	1	
	0	1	2	3	4	5	6	$\rightarrow i$

Defining the generating function

$$A(x,y) = \sum_{i,j \geq 2} a_{i,j} x^{i-2} y^{j-2},$$

we obtain, by summing the recurrence relation over $i,j \geq 2$, the following functional equation:

$$(xy - x^3 - y^3)A(x,y) = R(x,y) - F(x) - F(y) \tag{20}$$

where

$$R(x,y) = xy\left(\frac{1+y}{1-x} + \frac{1+x}{1-y}\right) \qquad \text{and} \qquad F(x) = \sum_{i \geq 2} a_{i,2} x^{i+1} = x^3 A(x,0).$$

We have used the symmetry of the problem in i and j. It was first proved in [32] that $F(x)$ and $A(x,y)$ are irrational. Then, we claimed in [7, p. 74] that they are even not D-finite, but without giving a proof. The tools developed above for the knight walk apply perfectly to this problem.

Proposition 10 *The series $F(x)$ and $A(x,y)$ are not D-finite.*

Proof. The argument is very close to the knight one. The kernel method first gives

$$F(x) + F(\xi_i(x)) = R(x, \xi_i(x)), \tag{21}$$

for any of the three roots ξ_i of the kernel.

The only difference with the knight treatment comes from the fact that $R(x,\xi_i(x))$ might have — and indeed, has — more singularities than ξ_i. They can be determined exactly, but we shall only use the following obvious information

$$Sing(R(x,\xi_i(x))) \subset Sing(\xi_i) \cup \{1\} \cup \{x : x^3 - x + 1 = 0\}. \tag{22}$$

In particular, all singularities of $R(x,\xi_i(x))$ have modulus at most 1.33 (an upper bound for the modulus of the largest root of $x^3 - x + 1$).

We start from Eq. (21), in the case $i = 0$. As $R(x,\xi_0(x))$ has radius x_c, we find again that the radius of $F(x)$ is at least x_c. Moreover, comparing the recurrence relations (1) and (19), and the corresponding tables, shows that for $i, j \geq 2$, one has $a_{i,j} \geq Q_{i-2,j-2}$. This implies that the radius of $F(x)$ is bounded from above by the radius of $G(x)$, which was proved to be exactly x_c. Hence $F(x)$ has radius x_c.

Assume $F(x)$ is D-finite. As in the proof of Proposition 9, we first construct large singularities of F (here, large means larger than 1.33). Let us start from $x_0 = x_c = 4^{1/3}/3 \approx 0.53$, which *is* a singularity of F. As it is not a singularity of $R(x, \xi_2(x))$, Eq. (21) implies that $x_1 := \xi_2(x_c) = -2y_c \approx -0.84$ is singular for F.

But x_1 is singular for none of the $R(x, \xi_i(x))$. Moreover,

$$\{\xi_0(x_1), \xi_1(x_1), \xi_2(x_1)\} = \{x_0, -0.26... \pm 1.02...i\}.$$

Using the same trick as above, we see that $x_2 \approx -0.26 - 1.02i$ is singular for F.

One more step: Among the $\xi_i(x_2)$, one is $x_3 \approx 0.92 - 1.02i$, which has modulus larger than 1.33. As x_2 is not singular for any of the $R(x, \xi_i(x))$, the $\xi_i(x_2)$ are singularities of F. In particular, x_3 is singular for F.

We conclude as above, by considering the largest singularity x_s of F (in modulus), showing that one of the $\xi_i(x_s)$ is singular for F *and* larger than $|x_s|$ in modulus.

∎

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