D'Alembertian Solutions of Linear Differential and Difference Equations

Sergei A. Abramov
Computer Center of
the Russian Academy of Science,
Vavilova 40, Moscow 117967, Russia.
abramov@sms.ccas.msk.su

Marko Petkovšek
Department of Mathematics and Mechanics,
University of Ljubljana,
Jadranska 19, 61111 Ljubljana, Slovenia.
marko.petkovsek@uni-lj.si

Abstract

D'Alembertian solutions of differential (resp. difference) equations are those expressible as nested indefinite integrals (resp. sums) of hyperexponential functions. They are a subclass of Liouvillian solutions, and can be constructed by recursively finding hyperexponential solutions and reducing the order. Knowing d'Alembertian solutions of $Ly = 0$, one can write down the corresponding solutions of $Ly = f$ and of $L^*y = 0$.

1 Introduction

Functions with rational logarithmic derivative play an important role in manipulation of differential equations, notably in finding Liouvillian solutions of linear differential equations and in factorization of linear differential operators. As any algorithm for finding solutions of linear differential equations in a certain class of functions, the algorithm for finding solutions with rational logarithmic derivative can be combined with the standard technique of reduction of order (also known as d'Alembert substitution) to yield solutions in a (possibly) larger class of functions which, in this case, we call *d'Alembertian functions.* They can be described as nested indefinite integrals of functions with rational logarithmic derivative, or equivalently, as functions that are annihilated by products of first-order differential operators. Similar observations can be made in the case of linear difference equations and difference operators where d'Alembertian functions can be described as nested indefinite sums of hypergeometric terms.

Let k be a differential field of characteristic 0 with derivation D. Denote by $\mathcal{H}(k)$ the set of all nonzero elements h belonging to some differential extension field K of k and such that $Dh/h \in k$.

Similarly, if k is a difference field of characteristic 0 with shift operator E, we denote by $\mathcal{H}(k)$ the set of all nonzero elements h belonging to some difference extension field K of k and such that $Eh/h \in k$.

We use $\int u$ to denote the set of all v such that $Dv = u$. If $\mathcal{U}$ is a set of functions, $\int \mathcal{U}$ denotes the set of all v such that $Dv \in \mathcal{U}$. Similarly, $\sum u$ denotes the set of all v such that $\Delta v = u$, and $\sum \mathcal{U}$ denotes the set of all v such that $\Delta v \in \mathcal{U}$, where $\Delta = E - 1$. In particular, $\int 0$ resp. $\sum 0$ denotes the set of constants.

Consider the homogeneous equation

$$Ly = 0 \tag{1}$$

where $L \in k[D]$ is a linear differential operator with coefficients in k. Let $\operatorname{ord} L = n$. If $\varphi \in \mathcal{H}(k)$ solves (1) then the substitution

$$y = \varphi \int u \tag{2}$$

leads to an $(n-1)$-st order equation for u with coefficients in k. Similarly, if $L \in k[E]$ is a linear difference operator with coefficients in k the substitution

$$y = \varphi \sum u \tag{3}$$

leads to an $(n-1)$-st order equation for u with coefficients in k.

Assume that the new equation is reduced again using its solution $\varphi_2 \in \mathcal{H}(k)$ and so on, until the last d'Alembert substitution connected with $\varphi_r \in \mathcal{H}(k)$ produces an equation with no solution in $\mathcal{H}(k)$. Then a subspace of solutions of (1) has been found, namely

$$\varphi_1 \int \varphi_2 \ldots \int \varphi_r \int 0 \tag{4}$$

in the differential case, and

$$\varphi_1 \sum \varphi_2 \ldots \sum \varphi_r \sum 0 \tag{5}$$

in the difference case. In [Abr93] it was shown that this subspace is independent of the choice of solutions φ_i, and that it is equal to the space of all d'Alembertian solutions of (1). Furthermore, (4) and (5) correspond to a factorization

$$L = L_r R_r R_{r-1} \cdots R_1 \tag{6}$$

of L where L_r has no first-order right factor, R_i are monic first-order operators, and if

$$h_i = \varphi_1 \varphi_2 \cdots \varphi_i \tag{7}$$

in the differential case, or

$$h_i = E^{i-1}\varphi_1 E^{i-2}\varphi_2 \cdots \varphi_i \tag{8}$$

in the difference case, then $R_i h_i = 0$.

If the equation (1) was reduced by means of $\varphi_1, \varphi_2, \ldots, \varphi_r$ and the last d'Alembert substitution connected with φ_r produced an equation $M_r y = 0$, then the general solution of (1) can be written as

$$\varphi_1 \int \varphi_2 \ldots \int \varphi_r \int g \quad \text{resp.} \quad \varphi_1 \sum \varphi_2 \ldots \sum \varphi_r \sum g \tag{9}$$

where g is the general solution of equation $M_r y = 0$.

The relationship between L_r and M_r is as follows: Write $L \sim M$ when there is an element $c \in k \setminus \{0\}$ such that $L = cM$. Then $M_r \sim L_r h_r$ (in the differential case), resp. $M_r \sim L_r E h_r$ (in the difference case). Here M_r, L_r, h_r and Eh_r are regarded as elements of $K[X]$ where K is an extension ring of k containing h_r.

In this paper we give new proofs for some of the results from [Abr93] in the more general setting of linear operators. We attempt to treat differential and difference equations in a unified way. In addition, we show how d'Alembertian solutions of the adjoint equation can be used to construct further solutions of the original equation.

In Section 2 we cite some results on skew polynomial rings. The main result of Section 3 is Theorem 3 which says that whenever an equation has a nonzero d'Alembertian solution then it has a hyperexponential solution (*i.e.*, one satisfying a first-order equation). This is a (modest) analogue of Singer's theorem saying that when a differential equation has a nonzero Liouvillian solution then it has a solution whose logarithmic derivative is algebraic over the coefficient field [Sin81]. We also show that d'Alembertian functions form a linear space, and give an algorithm to find a basis of all d'Alembertian solutions for the given equation. D'Alembertian solutions of $Ly = 0$ are used to find solutions of $Ly = f$ in Section 4, and to find those of $L^* y = 0$ in Section 5.

We will need explicit solutions of first-order nonhomogeneous equations. Let $R_1 \in k[D]$ be a monic differential operator of order 1. Then the general solution of equation $R_1 y = f$ can be written as

$$y = h_1 \int \frac{f}{h_1} \tag{10}$$

where $h_1 \in \mathcal{H}(k)$ is such that $R_1 h_1 = 0$. Using (10) repeatedly we see that the general solution of an equation $R_n R_{n-1} \cdots R_1 y = f$ where $R_n, R_{n-1}, \ldots, R_1 \in k[D]$ are monic first-order operators can be written as

$$y = h_1 \int \frac{h_2}{h_1} \int \frac{h_3}{h_2} \cdots \int \frac{f}{h_n} \tag{11}$$

where $h_i \in \mathcal{H}(k)$ are such that $R_i h_i = 0$.

Similarly, if $R_1 \in k[E]$ is a monic difference operator of order 1, the general solution of equation $R_1 y = f$ can be written as

$$y = h_1 \sum \frac{f}{Eh_1} \tag{12}$$

where $h_1 \in \mathcal{H}(k)$ is such that $R_1 h_1 = 0$. Using (12) repeatedly we see that the general solution of an equation $R_n R_{n-1} \cdots R_1 y = f$ where $R_n, R_{n-1}, \ldots, R_1 \in k[E]$ are monic first-order operators can be written as

$$y = h_1 \sum \frac{h_2}{Eh_1} \sum \frac{h_3}{Eh_2} \cdots \sum \frac{f}{Eh_n} \tag{13}$$

where $h_i \in \mathcal{H}(k)$ are such that $R_i h_i = 0$.

2 Algebraic preliminaries

Let k be a field of characteristic zero, X an indeterminate over k, σ a nonzero field-endomorphism of k, and $\delta : k \to k$ a map satisfying

$$\delta(a+b) = \delta a + \delta b \quad \text{and} \quad \delta(ab) = \sigma(a)\,\delta b + \delta a\, b \tag{14}$$

for any $a, b \in k$. The *left skew polynomial ring* given by σ and δ is the ring $(k[X], +, \cdot)$ of polynomials in X over k with the usual polynomial addition, and the multiplication given by

$$Xa = \sigma(a)X + \delta a \qquad \text{for any } a \in k.$$

To avoid confusing it with the usual commutative polynomial ring $k[X]$, the left skew polynomial ring is denoted $k[X;\sigma,\delta]$. We will denote by $\mathrm{Const}_{\sigma,\delta}(k)$ the *constant subfield* of k (*i.e.*, the set of all $a \in k$ such that $\sigma(a) = a$ and $\delta a = 0$).

Example 1 Let k be a differential field of characteristic 0 with derivation D. Let σ be the identity on k and take $\delta = D$. Then $k[X;\sigma,\delta] = k[D]$, the ring of linear differential operators with coefficients in k. □

Example 2 Let k be a difference field of characteristic 0 with transform (*i.e.*, shift operator) E. Take $\sigma = E$ and $\delta = 0$. Then $k[X;\sigma,\delta] = k[E]$, the ring of linear difference operators with coefficients in k. □

The ring $k[X;\sigma,\delta]$ has no zero divisors and possesses a right Euclidean division algorithm. For any two nonzero polynomials $R, S \in k[X;\sigma,\delta]$, one can compute their greatest common right divisor $\mathrm{gcrd}(R,S)$ and their least common left multiple $\mathrm{lclm}(R,S)$. We define $\mathrm{gcrd}(R,S)$ and $\mathrm{lclm}(R,S)$ to be monic. For details, see [Ore33] or [Bro&Pet94].

Definition 1 For $R, S \in k[X;\sigma,\delta] \setminus \{0\}$, let R/S denote the right quotient of $\mathrm{lclm}(R,S)$ by S.

Note that $(R/S)S = \mathrm{lclm}(R,S) = (S/R)R$.

Proposition 1 *Let $R, S \in k[X;\sigma,\delta] \setminus \{0\}$.*

1. $\deg R/S = \deg R - \deg \mathrm{gcrd}(R,S)$.
2. *If R is irreducible then so is R/S.*
3. *If $R = R_1 R_2 \cdots R_m$ where $R_1, R_2, \ldots, R_m \in k[X;\sigma,\delta]$ are monic then $R/S = (R_1/S_1)(R_2/S_2)\cdots(R_m/S_m)$ where $S_j = S/(R_{j+1}R_{j+2}\cdots R_m)$, for $j = 1, 2, \ldots, m$.*

For a proof, see [Ore33, Thm. I/16] and [Koo91, pp. 16–20].

Definition 2 A polynomial $R \in k[X;\sigma,\delta]$ *splits into first-order factors* if $\deg R = 0$, or $\deg R = d \geq 1$ and there are polynomials $R_1, R_2, \ldots, R_d \in k[X;\sigma,\delta]$ such that $\deg R_i = 1$ and $R = R_1 R_2 \cdots R_d$.

Corollary 1 *Let $R, S \in k[X;\sigma,\delta] \setminus \{0\}$. If R splits into first-order factors then so does R/S.*

Definition 3 Let $R, S \in k[X;\sigma,\delta]$. R is *similar* to S if $R = S/T$ for some $T \in k[X;\sigma,\delta]$ such that $\deg \mathrm{gcrd}(S,T) = 0$.

Note that similar polynomials are of the same degree, and that similarity is a symmetric relation.

Theorem 1 [Ore33, Thm. II/1] *Let $L \in k[X;\sigma,\delta]$ be monic. If $R_1R_2\cdots R_m$ and $S_1S_2\cdots S_n$ are two factorizations of L into irreducible monic factors, then $m = n$ and the factors are similar in pairs.*

Corollary 2 *Let $L \in k[X;\sigma,\delta]$ split into first-order factors. If L_1 is a right factor of L then L_1 splits into first-order factors.*

In particular, if $L, M \in k[X;\sigma,\delta]$ and L splits into first-order factors then so does $gcrd(L,M)$. An algorithm which computes $gcrd(L,M)$ in factored form (given L in factored form) is described in [Abr93].

3 D'Alembertian subspace

Henceforth we limit our attention to the rings of differential (resp. difference) operators $k[X]$ where X is D (resp. E). Let K be a differential (resp. difference) extension ring of k such that the constant subring $\mathrm{Const}_X(K)$ is a field. Then any element $R \in k[X]$ can be viewed as acting on K, and its kernel $\mathrm{Ker}\,R$ is a linear space over $\mathrm{Const}_X(K)$. Call K *adequate* for R if $\dim \mathrm{Ker}\,R = \mathrm{ord}\,R$. It is well known that for any finite set of operators $\mathcal{R} \subseteq k[X]$ there is an extension field K of k which is adequate for all $R \in \mathcal{R}$. If K is adequate for R and S then $\mathrm{Ker}\,R/S = S\,\mathrm{Ker}\,R$ (cf. [Koo91]).

Definition 4 Let k be a differential (resp. difference) field of characteristic 0 and y an element of a differential (resp. difference) extension ring K of k. Then y is *hyperexponential* over k if $y \neq 0$ and $Ry = 0$ for some first-order operator $R \in k[X]$; and *d'Alembertian* over k if $Ry = 0$ for some $R \in k[X]$ which splits into first-order factors. We will denote the set of all hyperexponential elements over k by $\mathcal{H}(k)$, and the set of all d'Alembertian elements over k by $\mathcal{A}(k)$.

Theorem 2 *Let K be a differential (resp. difference) extension ring of k such that $\mathrm{Const}_X(K)$ is a field. Then $\mathcal{A}(k)$ is a linear space over $\mathrm{Const}_X(K)$.*

Proof: Let $a, b \in \mathcal{A}(k)$ and $\lambda, \mu \in \mathrm{Const}_X(K)$. Then there are monic operators R and S which split into first-order factors, and satisfy $Ra = Sb = 0$. Since $\mathrm{lclm}(R,S)\,(\lambda a + \mu b) = 0$ and $\mathrm{lclm}(R,S) = (R/S)S$ splits into first-order factors by Corollary 1, it follows that $\lambda a + \mu b \in \mathcal{A}(k)$. □

Lemma 1 *Let $L \in k[X]$. If $y \in \mathcal{H}(k)$ then $Ly = qy$ for some $q \in k$, and if $Ly \neq 0$ then $Ly \in \mathcal{H}(k)$. If $y \in \mathcal{A}(k)$ then $Ly \in \mathcal{A}(k)$.*

Proof: If $Ry = 0$ then $(R/L)Ly = (L/R)Ry = 0$. As $\mathrm{ord}\,R/L \leq \mathrm{ord}\,R$, and since by Corollary 1 R/L splits into linear factors when R does, the lemma follows. □

Theorem 3 *Let $L \in k[X]$. If the equation $Ly = 0$ has a nonzero d'Alembertian solution then it also has a hyperexponential solution.*

Proof: Let $a \neq 0$ and $La = Ra = 0$ where $R \in k[X]$ splits into linear factors. If M is a minimal operator annihilating a then M is a right factor of both L and R. By Corollary 2, M splits into linear factors. It follows that L has a first-order right factor, or equivalently, that the equation $Ly = 0$ has a hyperexponential solution. □

Theorem 4 *Let $L \in k[X]$. There are unique operators $L_1, L_2 \in k[X]$ such that $L = L_1L_2$, L_1 has no first-order right factor, and L_2 is monic and splits into first-order factors.*

Proof: To prove existence, let L_2 be a monic right factor of L of largest order which splits into first-order factors. Let L_1 be the right quotient of L by L_2. Then $L = L_1L_2$ and L_1 has no first-order right factor.

To prove uniqueness, assume that additionally $L = M_1M_2$ where M_1 has no first-order right factor, and M_2 is monic and splits into first-order factors. If $a \in \mathrm{Ker}\,L_2 \setminus \mathrm{Ker}\,M_2$ then $y = M_2a$ is a nonzero d'Alembertian solution of $M_1y = 0$. Then by Theorem 3, M_1 has a first-order right factor, contrary to the assumption. It follows that $\mathrm{Ker}\,L_2 = \mathrm{Ker}\,M_2$, hence $L_2 = M_2$ and also $L_1 = M_1$. □

Assume that there is an algorithm **H** which takes an operator $L \in k[X]$ as input and returns a hyperexponential solution of the equation $Ly = 0$ if it exists. In the case when k is the field of rational functions $F(x)$ over some field F of characteristic 0 such algorithms are given, e.g., in [Sin91] or [Bro92b] for differential operators, and in [Pet92] for difference operators. We show that algorithm **H**, used recursively in combination with reduction of order, will construct a basis for the space of all d'Alembertian solutions of $Ly = 0$. We call this algorithm **A**.

Algorithm A

INPUT: A nonzero linear operator $L \in k[X]$.

OUTPUT: A basis for the space $\mathrm{Ker}\,L \cap \mathcal{A}(k)$.

1. Call **H** on $Ly = 0$.
   If no hyperexponential solution was found then return $\emptyset$ and stop.
   Otherwise let y_1 be a hyperexponential solution of $Ly = 0$.
2. Let R_1 be the monic first-order operator such that $R_1y_1 = 0$.
   Divide L by R_1 to obtain $L = L_1R_1$.
3. Recursively call **A** on L_1. Let the output be $\{z_2, z_3, \ldots, z_m\}$.
4. For $i = 2, 3, \ldots, m$ construct solutions y_i of $R_1y_i = z_i$ using (10) and (12).
   Return $\{y_1, y_2, \ldots, y_m\}$ and stop.

In step 4, Gosper's algorithm [Gos78] applied to (12) (resp. its continuous analogue [Alm&Zei90] applied to (10)) may be able to eliminate some summation (resp. integration) signs from the final result.

Theorem 5 *Algorithm **A** returns a basis for the space $Ker\,L \cap \mathcal{A}(k)$.*

Proof: Obviously algorithm **A** finds a factorization $L = L_mR_mR_{m-1}\cdots R_1$ where $\mathrm{ord}\,R_i = 1$ and the equation $L_my = 0$ has no hyperexponential solution, hence by Theorem 3 no nonzero d'Alembertian solution.

We claim that $\mathrm{Ker}\,L \cap \mathcal{A}(k) = \mathrm{Ker}\,R_mR_{m-1}\cdots R_1$. If $R_mR_{m-1}\cdots R_1a = 0$ then $La = 0$. Conversely, assume that $a \in \mathcal{A}(k)$ and $La = 0$ but $b := R_mR_{m-1}\cdots R_1a \neq 0$. By Lemma 1, $b \in \mathcal{A}(k)$. Since $L_mb = La = 0$, it follows from the previous paragraph that $b = 0$, proving the claim.

It is easy to show that $R_j R_{j-1} \cdots R_1 y_j = 0$ and $R_{j-1} \cdots R_1 y_j \neq 0$ for $1 \leq j \leq m$. Hence $y_j \in \operatorname{Ker} R_m R_{m-1} \cdots R_1 = \operatorname{Ker} L \cap \mathcal{A}(k)$ for $1 \leq j \leq m$. Let $\sum_{i=1}^m \alpha_i y_i = 0$ where $\alpha_i \in \operatorname{Const}_X(K)$ for some extension ring K of k. Successively applying $R_j R_{j-1} \cdots R_1$ for $j = m-1, \ldots, 1$ to both sides shows that $\alpha_m = \ldots = \alpha_1 = 0$. Since $\dim \operatorname{Ker} L \cap \mathcal{A}(k) = \dim \operatorname{Ker} R_m R_{m-1} \cdots R_1 = m$, the set $\{y_1, y_2, \ldots, y_m\}$ is a basis for $\operatorname{Ker} L \cap \mathcal{A}(k)$. □

Example 3 Let n be a discrete variable and E the shift operator with respect to n. Let the operators from $k[E]$ act on the quotient ring of sequences over k by the ideal of the eventually zero sequences (in other words, two sequences are identified if they agree from some point on). Consider the equation

$$\begin{aligned} &2(1+n)^3(2+n)^2(3+n)^2(4+n)^2(5+n)y_n - \\ &(2+n)^2(3+n)^2(4+n)^2(5+n)(2+6n+5n^2)y_{n+1} + \\ &n(3+n)^2(4+n)^2(5+n)(1+6n+4n^2)y_{n+2} + \\ &n(1+n)(4+n)^2(5+n)(17+4n-n^2)y_{n+3} - \\ &n(1+n)(2+n)(5+n)(19+5n)y_{n+4} + \\ &n(1+n)(2+n)(3+n)y_{n+5} \ = \ 0\,. \end{aligned} \tag{15}$$

Algorithm **A**, or repeated d'Alembert substitution, produces the general solution of (15) in the form

$$n! \sum n \sum g_n \tag{16}$$

where g_n is the general solution of the third-order equation

$$2(1+n)y_n - ny_{n+1} - 3y_{n+2} + y_{n+3} = 0 \tag{17}$$

which has no solution in $\mathcal{H}(k)$. Nevertheless, we will show in Section 5 that the general solution of (17), and hence of (15), can be expressed in terms of solutions of a second-order equation. □

4 Nonhomogeneous equations

Consider the equation

$$Ly = f \tag{18}$$

where $L \in k[X]$ and $\operatorname{ord} L = n$. Let c be the leading coefficient of L.

Theorem 6 *If L splits into first-order factors, and $\varphi_1 \int \varphi_2 \ldots \int \varphi_n \int 0$ (in the differential case), resp. $\varphi_1 \sum \varphi_2 \ldots \sum \varphi_n \sum 0$ (in the difference case) is the general solution of the homogeneous equation $Ly = 0$ then*

$$\varphi_1 \int \varphi_2 \ldots \int \varphi_n \int \frac{f}{c\varphi_1 \cdots \varphi_n} \tag{19}$$

respectively

$$\varphi_1 \sum \varphi_2 \ldots \sum \varphi_n \sum \frac{f}{cE^n\varphi_1 E^{n-1}\varphi_2 \ldots E\varphi_n} \tag{20}$$

is the general solution of (18).

Proof: According to (11), the general solution of (18) is

$$y = h_1 \int \frac{h_2}{h_1} \int \frac{h_3}{h_2} \cdots \int \frac{f}{ch_n}$$

where, according to (7), $h_i = \varphi_1 \varphi_2 \cdots \varphi_i$. This implies (19). In the difference case, we use (13) instead of (11), and (8) instead of (7). □

This approach does not require solving systems of linear algebraic equations (as does the method of variation of constants).

Example 4 Let $(n+1)y_n - (2n+3)y_{n+1} + (n+2)y_{n+2} = 1$. The general solution

$$\sum \frac{1}{n+1} \sum 0$$

of the corresponding homogeneous equation leads by (20) immediately to the general solution of the original equation:

$$y_n = \sum \frac{1}{n+1} \sum 1 = \sum \left(\frac{C_1}{n+1} + 1\right) = n + C_1 H_n + C_2$$

where $H_n = \sum_{j=1}^n 1/j$ is the n-th harmonic number, and C_1, C_2 are arbitrary constants. □

Now consider the problem of finding all d'Alembertian solutions of (18) assuming that f satisfies $Mf = 0$ for some $M \in k[X]$. Then $MLy = 0$ for any solution y of (18). As in Theorem 4, write $ML = BA$ where A splits into first-order factors, and B has no first-order right factor.

Proposition 2 *With the above notation, equation (18) has a d'Alembertian solution if and only if $(A/L)f = 0$.*

Proof: If $La = f$ and $a \in \mathcal{A}(k)$ then $Aa = 0$. Hence $(A/L)f = (A/L)La = (L/A)Aa = 0$. Conversely, if $(A/L)f = 0$ then $f \in \operatorname{Ker} A/L = L \operatorname{Ker} A$, hence there is an $a \in \operatorname{Ker} A$ such that $f = La$. □

When $(A/L)f = 0$ it is possible to construct a particular d'Alembertian solution a of (18) by expanding it into a power series around some point which is not a singularity of the equation $MLy = 0$ (the details will appear in [Abr94b]).

5 Adjoint equations

The adjoint of a linear differential operator is defined by

$$\left(\sum_{i=0}^n c_i D^i\right)^* = \sum_{i=0}^n (-1)^i D^i c_i\,.$$

It can be shown that for any $L_1, L_2 \in k[D]$,

$$(L_1^*)^* = L_1, \quad (L_1 L_2)^* = L_2^* L_1^*\,. \tag{21}$$

The adjoint of a linear difference operator is defined by

$$\left(\sum_{i=0}^n c_i E^i\right)^* = \sum_{i=0}^n E^i c_{n-i}$$

when $c_n, c_0 \neq 0$. It can be shown that for any $L_1, L_2 \in k[E]$,

$$(L_1^*)^* = E^n L_1 E^{-n}, \quad (L_1 L_2)^* = (E^n L_2^* E^{-n}) L_1^* \tag{22}$$

where $n = \deg(L_1)$. Hence in both cases left factors of L correspond to right factors of L^*, and vice versa.

Let $R_1 \in k[X]$ be a monic operator of order 1, and $h_1 \in \mathcal{H}(k)$ such that $R_1 h_1 = 0$. Then it is easy to see that

$$R_1^* \frac{1}{h_1} = 0 \tag{23}$$

in the differential case, and

$$R_1^* \frac{1}{Eh_1} = 0 \tag{24}$$

in the difference case.

Theorem 7 *Let $L \in k[X]$, $\operatorname{ord} L = n$, and let c be the leading coefficient of L. Let $\varphi_1 \int \varphi_2 \ldots \int \varphi_n \int 0$ (in the differential case), resp. $\varphi_1 \sum \varphi_2 \ldots \sum \varphi_n \sum 0$ (in the difference case) be the general solution of equation $Ly = 0$ in some extension ring K of k. If $\varphi_1, \varphi_2, \ldots, \varphi_n \in K$ are invertible, then*

$$\frac{1}{c\varphi_n \varphi_{n-1} \ldots \varphi_1} \int \varphi_n \int \varphi_{n-1} \ldots \int \varphi_2 \int 0 \tag{25}$$

in the differential case, respectively

$$\frac{1}{cE\varphi_n \ldots E^n \varphi_1} \sum E\varphi_n \ldots \sum E^{n-1}\varphi_2 \sum 0 \tag{26}$$

in the difference case, is the general solution of the adjoint equation $L^* y = 0$ in K.

Proof: According to (6),

$$L = cR_n R_{n-1} \cdots R_1$$

where R_i are monic first-order operators such that $R_i h_i = 0$, with h_i defined by (7) resp. (8).

In the differential case, let $h_i^* = 1/h_i$. Then by (23), $R_i^* h_i^* = 0$. Since $L^* = R_1^* R_2^* \cdots R_n^* c$, the general solution of $L^* y = 0$ is, by (11),

$$\frac{h_n^*}{c} \int \frac{h_{n-1}^*}{h_n^*} \int \ldots \frac{h_1^*}{h_2^*} \int 0 = \frac{1}{ch_n} \int \frac{h_n}{h_{n-1}} \int \ldots \frac{h_2}{h_1} \int 0$$

which, using (7), gives (25).

In the difference case, $L^* = S_1 S_2 \cdots S_n c$ where $S_j = E^{n-j} R_j^* E^{j-n}$. The operator S_j is not monic but rather has leading coefficient equal to $c_j = -E^{n-j+2}h_j / E^{n-j+1}h_j$. Let $h_j^* = 1/Eh_j$ and $g_j = E^{n-j}h_j^*$. Then by (24), $S_j g_j = 0$.

By a slight generalization of (13) to non-monic factors, the general solution of $L^* y = 0$ is given by

$$y = \frac{g_n}{c} \sum \frac{g_{n-1}}{c_n E g_n} \sum \frac{g_{n-2}}{c_{n-1} E g_{n-1}} \ldots \sum \frac{g_1}{c_2 E g_2} \sum 0\,.$$

Substitution of expressions given above for g_j and c_j together with (8), and absorption of the sign into $\sum 0$, now yields (26). □

Example 5 Consider equation (15) again. The adjoint equation of (17)

$$y_n - 3y_{n+1} - (2+n)y_{n+2} + 2(4+n)y_{n+3} = 0$$

has general solution

$$\frac{1}{2^n} \sum u_n \sum v_n \sum 0, \tag{27}$$

where $u_n \sum v_n \sum 0$ is the general solution of equation

$$4y_n - 2y_{n+1} - (4+n)y_{n+2} = 0 \tag{28}$$

which has no hypergeometric solution. Assume that u_n and v_n are nonzero for all large enough n. By Theorem 7,

$$g_n = \frac{2^n}{(n+4)v_{n+1}u_{n+2}} \sum v_{n+1} \sum u_{n+2} \sum 0$$

is the general solution of the second adjoint of (17), hence

$$n! \sum n \sum \frac{2^n}{(n+1)v_{n-2}u_{n-1}} E^{-3} \left(\sum v_{n+1} \sum u_{n+2} \sum 0 \right)$$

is the general solution of (15). □

Note that this approach is also suitable in the case of a nonhomogeneous equation.

We observe that adjoint equations offer an opportunity to parallelize the solving process. Consider the original equation $Ly = 0$ together with the adjoint equation $L^* y = 0$ and begin to search for particular solutions of both of them in parallel. If a particular solution of one of them is found then we reduce the order of the equation and begin to search for particular solutions of the new equation and its adjoint. If some branch of this process leads to an equation of order zero we use formulae (25) and (9) to obtain the general solution of the original equation.

6 Concluding remarks

Call an operator $L \in k[X]$ *rational* if $\operatorname{Ker} L$ has a basis in k. If in the definition of complete factorability the factors are required to be both first-order *and* rational, then all our results remain valid provided that hyperexponential solutions are everywhere replaced by solutions in k, and that in algorithm **A**, in place of algorithm **H** an algorithm for finding solutions in k is used. In the case when $k = F(x)$ such algorithms are given in [Abr89] both for differential and for difference equations, and in [Abr&Kva91] and [Bro92a] for differential equations.

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