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# The surface-to-volume ratio in thermal physics: from cheese cube physics to animal metabolism

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**Abstract**
The surface-to-volume ratio is an important quantity in thermal physics. For example it governs the behaviour of heating or cooling of physical objects as a function of size like, e.g. cubes or spheres made of different material. The starting point in our paper is the simple physics problem of how cheese cubes of different sizes behave if heated either in a conventional oven or in a microwave oven. The outcome of these experiments depends on a balance between heating and cooling with the surface-to-volume ratio (S/V) as the key parameter. The role of (S/V) becomes most obvious in studying cooling curves of differently sized objects like cubes or spheres, alone. Besides problems in thermal physics, the surface-to-volume ratio has many important applications in biochemistry, chemistry and biology. It allows us to draw general conclusions concerning the thermal and mechanical properties of different-sized animals, in particular their metabolism. Hence, this topic offers rich contexts for interdisciplinary teaching. An example presented in this paper starts in physics while studying thermal properties of cheese cubes and ends up in biology by discussing the differences in food intake of animals from small mice to huge elephants.

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## 1. Introduction

It is a trend today to make connections between knowledge obtained in different subjects (so-called cross-curricular connections), particularly between science subjects in secondary school and at university introductory level. In physics we would ideally like to show the general validity of physics concepts using interdisciplinary examples. Though most of the problems in practise are interdisciplinary, it is not easy to find the examples suitable for teaching at introductory level. The question of how warm-blooded animals are able to keep their body temperature constant in spite of the large differences in their sizes, is one example that is often mentioned in biology classes, but not so often in physics. A very useful paper written for a high school level that gives simple physical explanations of some characteristics of biological systems appeared some time ago [1]. Maybe the experiment-based approach presented in this paper along with some examples of basic theoretical treatment will encourage lecturers of introductory physics courses to include this and other interdisciplinary problems more often in their repertoire.

In the following, an interdisciplinary teaching unit will be presented. The unit starts by discussing experimental results for the various ways of heating cheese cubes. In this part, extended verbal interpretation of the simple derivation is included in the way we present this topic to students in introductory courses. The cheese cube section is followed by experiments on the cooling of hot solid bodies. Combined with the heating sequence, the respective results will emphasize the general role of the surface-to-volume ratio that can be observed in various thermal processes. Since cheese is said to be a favourite food of mice, this offers a nice entry to biology and animal metabolism which is the central theme of the last part of the paper. In addition, students' ideas about principles involved in cheese cube experiments presented in this paper have been tested with 29 first year physics majors. The short survey of the results is presented in the appendix.

## 2. Heating up: cheese cubes

The teaching unit is presented as a predict–observe–explain (POE) activity [2]. These type of activities help teachers to follow students' reasoning and to identify misconceptions that students may hold about the topic.

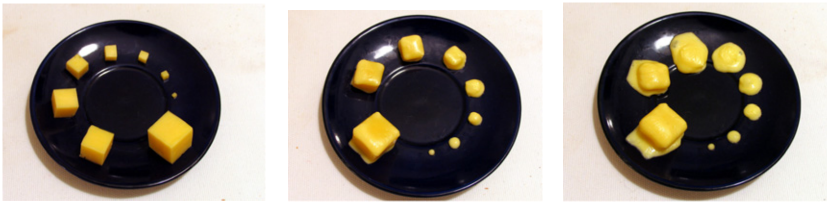
### *2.1. Cheese heated in conventional preheated oven*

*2.1.1. Experiment.* Take a piece of cheese with no holes, such as Cheddar or Gouda. Cut two sets of cheese cubes with lengths ranging from about 2 mm to 15 mm. Arrange one set of cubes on a heat and microwave proof ceramic plate. Ask students to predict which cubes will start to melt first, if we put the plate with cheese cubes into the classic electric oven, preheated to 200 °C. Will it be the smallest first, the largest first or will all melt at about the same time?

Based on experiences from several everyday situations, the majority of students will correctly predict that the smallest cubes will start to melt first (see also the appendix).

*2.1.2. Explanation.* The outcome of the experiment (figure 1) can be explained as follows. The temperature inside the oven is much higher than the initial temperature of the cheese cubes. Therefore heat flows from the oven to the cheese cubes. The cheese cubes are also heated by the thermal radiation emitted by the oven walls; however, the penetration depth of this near-IR radiation is assumed to be much smaller than the size of the cubes. This means that heat is transferred to the internal parts of the cubes mainly by conduction. The smaller cubes get heated throughout their interior first and therefore melt first.

Note that the cube corners will melt first, because they are exposed to the heat transfer from three sides and will reach the melting point before the rest of the cube does. This causes the rounding of the corners, followed by the edges.



(a) (b) (c)

**Figure 1.** (a) Cheese cubes before being placed into the oven. Cheese cubes after being heated at 200 °C in a conventional preheated electric oven (b) for 70 s and (c) after additional heating for another 40 s (i.e. total heating time 110 s).

(a) (b) (c)

**Figure 2.** (a) Cheese cubes before being placed into the microwave oven. Cheese cubes after being heated in microwave oven (with rotating table, at 800 W): (b) for 8 s and (c) for additional 25 s. Note that the largest cube starts to melt first at the bottom where the cooling due to the presence of the plate is less efficient.

### *2.2. Cheese heated in a microwave oven*

*2.2.1. Experiment.* Arrange a second set of cheese cubes on the same plate as before (at the same initial temperature) and ask students to predict which cubes will start to melt first, if we put them into a microwave oven, on a rotating plate. The cubes are arranged in a circle, the centre of which is placed in the centre of the microwave oven.

The heating should take place during an integer number of revolutions of the turntable: this ensures that all cubes see the same microwave standing fields (which are usually not homogeneously distributed within the oven [3]). The question again is: which cubes melt first, the smallest, the largest or do they all melt at about the same time? Then perform the experiment. In our case the power was set to maximum value (800 W) and the microwave oven was switched on for about half a minute. You may want to put a small glass of water in the corner of the microwave oven. It will absorb excess microwave energy and thus prevent damage to your oven. After half a minute of cooking, take the plate out and show the result to the students.

The result is surprising for many students (figure 2). The largest cube starts to melt first. The larger the cube, the faster it melts. Observing the results from repeated experiments one can conclude that there exists a certain size of the cube for which all cubes smaller than that size will not melt even after repeating the experiment for extended times at maximum power. (Note: if you repeat the experiment several times you will need to replace the water in the glass with cold water after a few minutes of cooking. This helps to keep the temperature of the air in the microwave oven constant.) Ask students to suggest possible explanations for the result of the experiment. Encourage different and alternative ideas but stay neutral and do not reveal the correct answer too early even if it was suggested (see the appendix for some alternative student ideas).

*2.2.2. Explanation.* The 'surprising' result of the second experiment needs a more detailed explanation. For our purpose students only need to know that the microwaves heat the food by shaking the water molecules inside it (if you want to learn more about physics of a microwave oven, see, for example, [3]). For the size of all cubes used in our experiment we can assume that microwaves penetrate throughout the cubes. The cheese that we have chosen is homogeneous, so that every second, each volume element of the cheese absorbs approximately an equal amount of energy from the microwave radiation. This means that the power absorbed by each cheese cube of size $a$ is proportional to its volume,

$$\frac{\mathrm{d}W_{\mathrm{abs}}}{\mathrm{d}t} = P_{\mathrm{absorb}} \propto V = k_1 a^3, \tag{1}$$

where $k_1$ is a constant which depends on the absorption coefficient of microwaves in cheese. It is easier to think of the power as the energy that flows into (or out of) the body in 1 s. As the cheese absorbs the microwave radiation its temperature $T$ rises.

However, as soon as the cheese cube's temperature becomes higher than the temperature of the surroundings, some energy starts to flow out from the cube, thus cooling it. The energy flow balance for the cheese cube during the heating can be written in the following form:

$$P_{\mathrm{eff.\,heating}} = P_{\mathrm{absorb}} - P_{\mathrm{cool}}, \tag{2}$$

where $P_{\mathrm{absorb}}$ is part of the energy flow that raises the cheese temperature (increasing its internal energy according to (1)) and $P_{\mathrm{cool}}$ is the energy flow that 'leaks' from the cheese and thus cools it.

In general, $P_{\mathrm{cool}}$ is due to the three usual heat transfer contributions conduction, convection and radiation (e.g. [4, 5]). In addition, there may be latent heats of transformation due to phase changes at certain temperatures.

Students are most familiar with heat conduction in solids, but the same heat transfer mechanism also applies for fluid (gas or liquid), in which there is no bulk motion. Most students know that the heat flow is proportional to the difference between body temperature and surrounding temperature and to the surface area of the body

$$P_{\mathrm{cond}} \propto S(T_{\mathrm{body}} - T_{\mathrm{surr}}). \tag{3}$$

Convection of heat occurs if a body is surrounded by fluid which is allowed to take bulk motion. It is usually divided into free convection (where the flow is due to buoyancy forces) and forced convection (where flows are due to external forces). Despite this distinction, the heat exchange due to convection is usually approximated by the same form as (3) [6]. For natural convection alone, i.e. if any external air flows are suppressed, one would instead find a power law with temperature difference to the power 1.25 rather than 1.0 [7]. This situation is, however, not fulfilled in our experiments.

Finally, the radiative heat exchange between a body of temperature $T$ and surrounding temperature $T_0$ is given by $P \propto S(T^4 - T_0^4)$. The latter expression can be simplified for $(T - T_0) \ll T_0$ as $4T_0^3(T - T_0)$, i.e. $P \propto S(T - T_0)$, which again is of the same form as (3).

In this case, the combination of conduction, convection and radiation leads to the so-called Newton law of cooling (for a critical assessment, see [4, 5]). Applying it to our problem, cooling power of the cheese cube while staying in the microwave oven can be expressed as

$$P_{\mathrm{cool}} = k \cdot S(T_{\mathrm{cheese}} - T_{\mathrm{oven}}), \tag{4}$$

where $k$ is a constant which depends, e.g. on the thermal conductivity of the cheese, $S$ is the surface area of the cube (i.e. $6a^2$), $T_{\mathrm{cheese}}$ is the average temperature of the cheese cube and

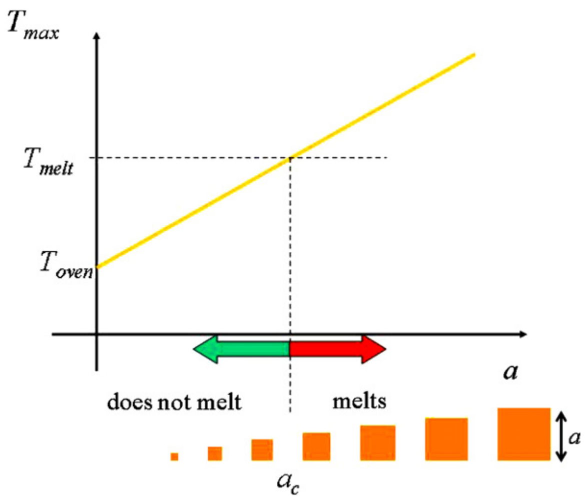



**Figure 3.** The simple model predicts that the equilibrium temperature of the cheese cubes in a microwave oven is linearly proportional to the cube size.

$T_{\text{oven}}$ is the temperature inside the oven during the experiment, assumed to be constant (of course, one side of the cube sits on the plate, leading to higher conduction, this will, however be neglected for the moment since we are not interested in absolute quantitative results).

Equation (4) tells us that the higher the temperature of the cheese, the larger the heat flow that cools the cheese cube. Since the in-flow of the energy from the microwave radiation is assumed to be constant for the given cheese cube, the two flows ($P_{\text{absorb}}$ and $P_{\text{cool}}$) will eventually become equal. This happens when the cheese cube temperature reaches a certain value $T_{\text{cheese}} = T_{\text{max}}$. At this point the energy flows reach dynamic equilibrium. When the cheese temperature reaches $T_{\text{max}}$, all absorbed energy flows into the surroundings and there is no energy left for heating the cheese further ($P_{\text{eff.heating}} = 0$). From this moment on the temperature of the cheese cube does not change any more.

Combining (1), (2) and (4) and setting $P_{\text{eff.heating}} = 0$, one obtains the following expression:

$$k_1 \cdot a^3 = k_2 \cdot a^2 \cdot (T_{\text{max}} - T_{\text{oven}}), \tag{5}$$

where $k_2$ equals $6k$. We emphasize again, that $a^3$ and $a^2$ are the characteristics for the volume and surface of the cheese cubes, respectively. In all calculations, the ratio of these two quantities, i.e. $(S/V) \propto \frac{1}{a}$ or $(S/V)^{-1} \propto V/S \propto a$ reflects the influence of the surface-to-volume ratio. Solving (5) for $T_{\text{max}}$ gives

$$T_{\text{max}} = T_{\text{oven}} + \frac{k_1}{k_2} \cdot a. \tag{6}$$

It follows from (6) that the equilibrium temperature of the cheese cubes in a microwave oven is linearly proportional to the cube size. Since cheese starts to melt at a certain temperature $T_{\text{melt}}$, cheese cubes that are smaller than some critical size $a_c$ will never melt in a microwave, no matter how long they have been exposed to the microwave radiation (as long as the temperature inside the microwave remains constant). The cheese cubes with the sides larger than $a_c$ will sooner or later start to melt (figure 3). This conclusion qualitatively agrees with the outcome of the simple experiment (figure 2).

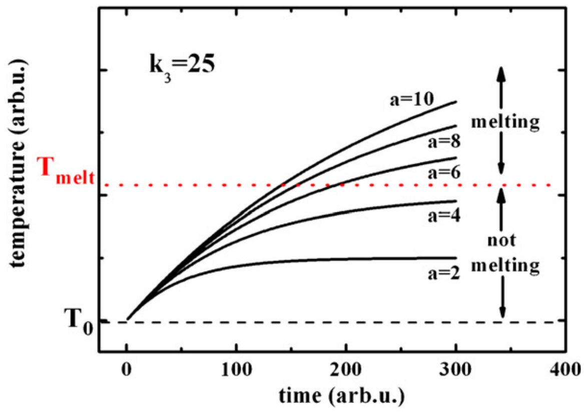



**Figure 4.** Temperature of cheese cubes of different sizes according to (9) for a given value of $k_3 = 25$ (i.e., e.g., $\tau = 50$ for $a = 2$). See the text for details.

### *2.3. Cheese cube temperatures as a function of time*

The process of permanent heating and simultaneous cooling can also be easily used as an exercise in calculus. Rewriting the energy flows from (2), we find a differential equation for temperature $T(t)$ as a function of time,

$$P_{\text{eff .heating}} = c \cdot m \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = k_1 \cdot a^3 - k_2 \cdot a^2 \cdot (T(t) - T_0), \tag{7}$$

where $c$ is the specific heat (in J kg$^{-1}$·K$^{-1}$), $m$ is the mass of the cubes, d$T$ denotes the temperature change during time interval d$t$ due to the effectively absorbed energy and $T_0$ is the temperature of the air within the oven. $T$ is the average cheese temperature, which is assumed to be nearly constant across the cubes. This holds if the heat transfer within the cube due to heat conductivity is larger than the heat transfer at the surface. This can be rearranged into the inhomogeneous differential equation of the following form:

$$\dot{T} + AT = B \quad \text{with} \quad A = \frac{k_2 \cdot a^2}{c \cdot m} = \frac{k_2}{c \cdot \rho} \cdot \frac{1}{a} = \frac{1}{k_3 \cdot a} \quad \text{and} \quad B = \frac{k_1}{c \cdot \rho} + \frac{k_2 \cdot T_0}{c \cdot \rho} \cdot \frac{1}{a}. \tag{8}$$

Here, $A$ is a quantity which is proportional to the surface-to-volume ratio, i.e. proportional to $1/a$. With boundary conditions $\mathrm{d}T/\mathrm{d}t = 0$ for $t \to \infty$, and $T(t_0) = T_0$, the solution can easily be found to be

$$T(t) = T_0 + \frac{k_1}{k_2} a \left[1 - \mathrm{e}^{-\frac{(t-t_0)}{\tau}}\right], \tag{9}$$

where the time constant $\tau = 1/A = k_3 a$. Despite not knowing exact values for the constants, it is possible to plot the general form of $T(t)$ as shown in figure 4 for different values of cube size $a$. It follows from (9) (and has been shown earlier in figure 3) that the temperature of each cube eventually reaches the asymptotic value $T_0 + \frac{k_1}{k_2} a$. If this temperature is below the melting temperature, the cheese will never melt. An interesting feature of (9) is, that since the time constant $\tau$ is proportional to the cube size, the time until the maximum temperature is reached at equilibrium conditions is shortest for small cubes. As shown in figure 4, only the smallest cubes have reached the maximum possible temperature, whereas the largest cubes are still far away from equilibrium.

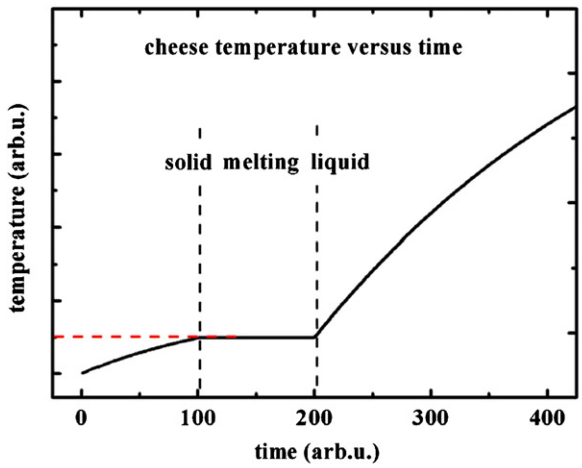



**Figure 5.** Temperature as a function of time for cheese cubes that may completely melt during the continuous heating (for instance in a microwave oven).

The physics behind this is quite simple: the smallest cubes have the largest surface-to-volume ratio and hence, the cooling for small cubes is more efficient than for large cubes. Therefore thermal equilibrium is reached much faster.

Additional insight into the heating process of the cheese cube, which may melt during the heating is depicted in figure 5.

While melting, the temperature stays constant (this is again an approximation, in reality the melting starts in the corners and there are temperature gradients across the cube). When being melted, the liquid temperature continues to rise with the following differences with regard to the heating of the solid phase: first, the liquid phase will most probably have a much larger absorption coefficient (ice and water in a microwave oven have quite different absorption behaviour: in the liquid, the absorption is several orders of magnitude larger, see [3]); second, the melted cheese will have a different specific heat (e.g. water has a specific heat which is about twice as large as for ice). Third, heat conduction of the liquid phase may be different (water has a thermal conductivity which is about a factor of 3 smaller than the one for ice). The combination of all effects will result in a longer time constant, but much larger total temperature rise and hence a more rapid increase in temperature than during the heating of the solid phase. In principle, there could be another second maximum temperature and again, if this would be above the boiling point, a second phase change could take place and afterwards, the cheese would be vaporized or carbonated in the oven (which would mean quite a mess!).

### *2.4. Measuring cheese temperatures with IR imaging*

In order to get more quantitative information about the temperatures of the cheese cubes right after they have been irradiated by microwaves some measurements with IR imaging have been done. This contact-less method is elegant and pretty accurate (though still somehow expensive). Here, we used an LW infrared camera [8, 9], operating in the range of 8–14 $\mu$m. The IR photos of cheese cubes before the experiment, after 10 s and after an additional 20 s of irradiation with microwaves are presented in figure 6.

Inspection of the cubes shows that after the first irradiation the largest cube was just about to melt and after the second irradiation the largest three cubes were melted.

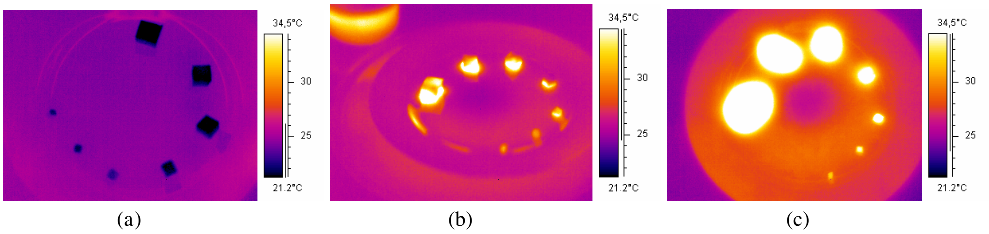



**Figure 6.** IR photos of cheese cubes (a) before the experiment (at room temperature), (b) after 10 s and (c) after an additional 20 s of irradiation in a microwave oven (mW power was set to 360 W and the turntable was on).

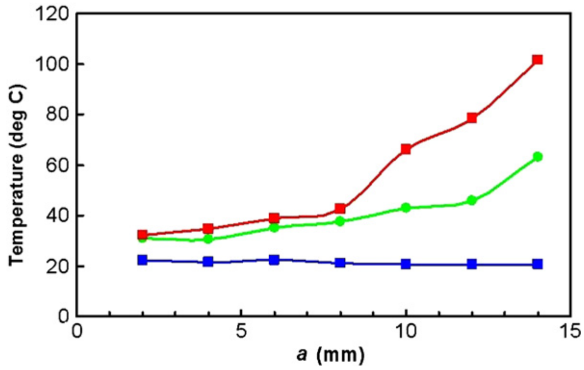



**Figure 7.** Cheese cube temperature versus cube size: at room temperature (lower curve), after 10 s (middle curve) and after an additional 20 s of irradiation with microwaves (upper curve). Temperature values were obtained from IR photos shown in figure 6.

The microwave power was set to 360 W and the plate with the cheese cubes was placed on the turntable. Temperature readings from the IR measurements are presented in the graph in figure 7.

From the diagram in figure 4 alone, one could not expect such behaviour, since the longer time constants would slow down the temperature rise for the larger cubes. However, the argument must include the effect of phase change on temperature behaviour, i.e. one must compare the situation in figure 5. After melting, the cheese cubes will have on one hand a much higher absorption coefficient (as discussed above) and on the other hand, the time constant will also increase due to an increase of the specific heat and decrease of heat conductivity.

According to figure 4, one would therefore qualitatively expect small nearly linear increases in temperature for the cubes as long as they are not melted and much larger temperature increases after they are melted. This expectation is supported by the experimental results in figure 7.

From figure 7, we also see that the critical temperature, where the change happens, is around 50 °C. This temperature should be in the region, where the cheese melts (we do not know the exact melting point of the cheese that we used but it is generally known that at about 50 °C albumen starts to decompose).

It is very instructive to encourage students to discover these observations and comment on the graphs like this. Brighter students may realize that the theory behind the graph in figure 4 is not sufficient and that changing the model after the phase transition is needed to explain the data. In particular, the above discussion shows that it is very important to make students

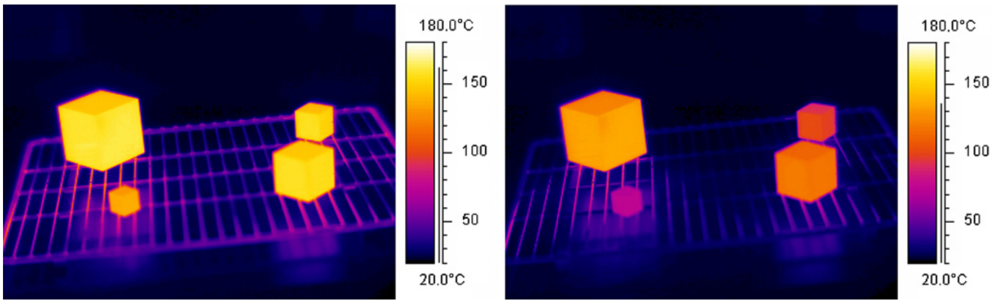



**Figure 8.** IR images of four aluminium cubes of 20 mm, 30 mm, 40 mm and 60 mm length near the beginning (left) and during the cooling process (right). Small cubes cool down fastest.

realize the limitations of the most simple model. Then they need to master the next step, i.e. to find which part of the initial theoretical model should be changed if the cubes start to melt.

Other complications—not dealt with so far—include the evaporation rate of water for the hottest cubes, which in turn may affect the absorption coefficient. In addition, there are also shape changes. Melted cheese cubes look more like a pancake. This changes the amount of surface of the cheese which is in thermal contact with the plate and hence has an effect on conductive cooling.

Obviously, such a simple experiment as the heating of cheese cubes can become very complicated. Still we are convinced that the general principles which govern its behaviour are easy to understand. The physics behind it may, however, become more obvious to students, if the cooling process is isolated from the heating as presented in the following section.

## 3. Cooling down: metal cubes

In the cheese cube problem, the change of temperature was due to heating and cooling simultaneously, eventually leading to dynamic thermal equilibrium with steady-state temperatures. The simpler problem of cooling alone can help to isolate the role of surface-to-volume ratio in thermal processes. Four metal cubes of sizes between 20 mm and 60 mm were heated in a conventional oven up to temperatures of about 170 °C. The side faces were covered with black paint of a high emissivity to allow for a quantitative IR analysis. After thermal equilibrium had been established, the cubes (on a grid) were placed on top of some Styrofoam blocks on a table. The IR camera immediately started recording temperatures at intervals of about 10 s. Figure 8 depicts two images of the time series, clearly demonstrating the much faster cooling of the smaller cubes. An avi video is available as online supplementary material (stacks.iop.org/EJP/29/369).

The cube temperatures were measured by averaging the measured temperatures over areas covering about half the size of the smallest cube. Due to the high thermal conductivities of the metal the surface temperatures are assumed to be a good measure for the average temperatures of the cubes. Figure 9 depicts the result.

The smallest cubes cool down most rapidly. This behaviour was expected from the cheese experiments, since there, also, the smallest cubes reached the thermal equilibrium fastest. The theoretical description is very similar to the cheese cube problem, the main difference being that the cubes were initially hot and were no longer heated after the start of the experiment. The boundary conditions are therefore given by $T(t \rightarrow \infty) = T_0$ with $T_0$ being room temperature and $T(t = 0) = T_{\text{start}}$, where $T_{\text{start}}$ is the initial cube temperature.

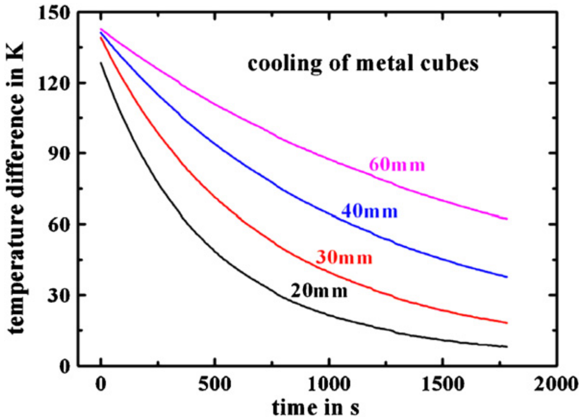



**Figure 9.** Measured temperatures during the cooling curves of metal cubes from figure 8, detected with IR imaging.

The hot cube loses energy via cooling of the cube, i.e., $\mathrm{d}Q = mc\mathrm{d}T$, and hence the thermal power, lost by the cube is $P = mc\mathrm{d}T/\mathrm{d}t = \rho ca^3\,\mathrm{d}T/\mathrm{d}t$. This thermal power flows away via conduction, convection and radiation. As before, conduction and convection are proportional to $S(T(t) - T_0)$, even if conduction may consist of two contributions from the grid and the air. As shown earlier, for small temperature differences the radiation term can be approximated with the same functional form. In the experiment, the initial cube temperature was about 430 K and $T_0 = 290$ K. If in this case linear approximation is used for cooling by radiation, the result is almost a factor 2 smaller than the correct value. For cube temperature 340 K, however, the correction factor already decreases to about 1.4. This should quite obviously lead to a deviation from the exponential cooling law. Whenever this is not observed, one may conclude that either radiation will not be the dominant cooling mechanism or the linearizing of the radiation power law is already a reasonable approximation (or both of these conditions apply). For the purpose of this work, it was assumed in the following model that radiation is also proportional to $S(T(t) - T_0)$.

Energy conservation yields the differential equation

$$c \cdot \rho \cdot a^3 \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = -k_2 \cdot a^2 \cdot (T(t) - T_0), \tag{10}$$

where $k_2$ includes all cooling mechanisms constants. Similar to the solution of (8), we find

$$\dot{T} + AT = B \quad \text{with} \quad A = \frac{k_2 \cdot a^2}{c \cdot m} = \frac{k_2}{c \cdot \rho} \cdot \frac{1}{a} \quad \text{and} \quad B = \frac{k_2 \cdot T_0}{c \cdot \rho \cdot a}. \tag{11}$$

The solution is again easily found to be

$$T(t) = T_0 + (T_{\mathrm{Start}} - T_0)\left[1 - \mathrm{e}^{-\frac{(t-t_0)}{\tau}}\right], \tag{12}$$

with $\tau = 1/A$ again depending on the surface-to-volume ratio. Theoretical curves based on (12) nicely fit the data for all four curves, indicating that the above approximation was valid. Figure 10 shows an example for a fit of (12) to the data for a cube size of 30 mm. The data and the fit are in excellent agreement giving a typical correlation coefficient $R^2$ for the fit of 0.9998 or better. Since deviations from theoretical prediction are smaller than the line widths, a strongly magnified part of the plot is shown in figure 10b, to make difference visible.

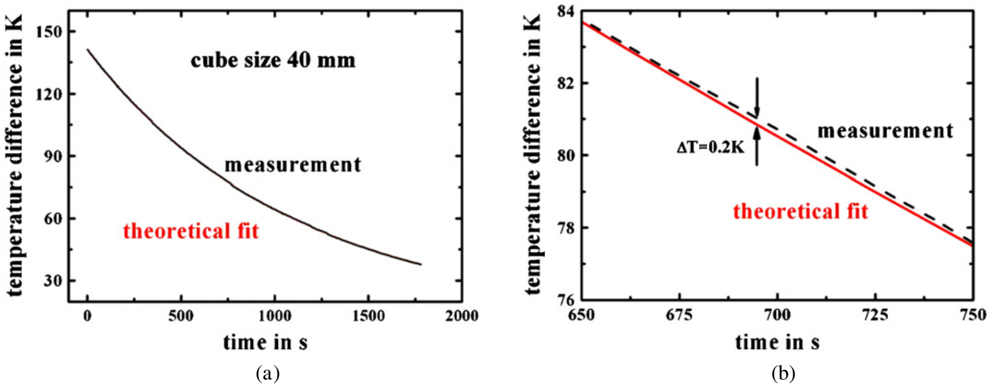



**Figure 10.** (a) Comparison of experimental cooling curve with the theoretical prediction of an exponential law given by (12). The agreement of the experiment with the exponential law is very good, even in the magnified part of the plot (b).

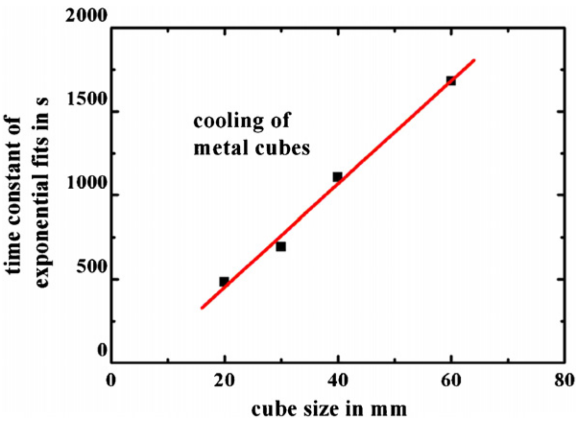



**Figure 11.** The time constants for the exponential fits are linearly proportional to the size of the cubes, as expected.

Figure 11 gives a comparison of the constant $A$ for all four measurements. As expected, it shows a linear dependence on cube size.

The linear relationship $\tau \propto a$ is direct proof of the influence of surface-to-volume ratio on the cooling of objects.

## 4. From physics to biology: if there is cheese, there must be a mouse

The surface-to-volume ratio is not only important in thermal physics, but also in other scientific disciplines. Examples in physics are the influence on the melting temperature of particles on their size or the influence of optical properties on the size of nanoparticles. In chemistry, one of the most important examples is catalysis. Catalysts work best if dispersed in very fine particles with a large surface-to-volume ratio. The increased chemical reactivity of small particles is well known and sometimes represents a big threat, think, e.g., of explosions of flour mills. Similarly, biophysical processes also benefit from small sizes. The surface area

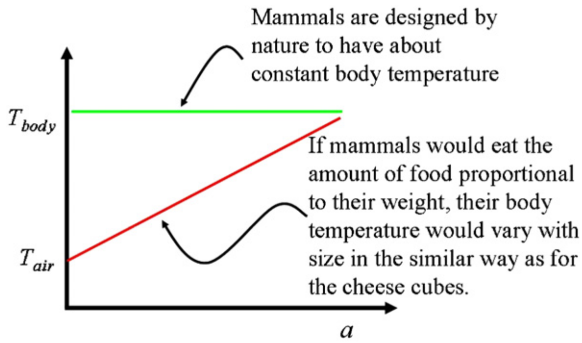



**Figure 12.** Connecting the cheese cube problem with the case of warm blooded animals.

of cells determines the rate of exchange processes with the external environment and hence, small cells can exchange more relative to their size, than larger cells. An even more obvious example from biology shall be discussed below: the role of surface-to-volume ratio and animal metabolism.

The moral drawn from the previous sections can be stated in the following way: cooling of warm bodies works best for small ones. If, in addition, the bodies absorb (or produce) heat in proportion to their volumes and cool through their surfaces, they will eventually reach an equilibrium temperature that is proportional to body size. Smaller bodies will be colder, larger bodies will be hotter and time constants to reach equilibrium will depend on surface-to-volume ratio.

Now, think of warm-blooded animals. They manage to keep their body temperature constant (at about 37 °C) within a few degrees Celsius though their sizes span several orders of magnitude. For simplicity we will focus on land mammals (some physical aspects of aquatic mammals are analysed for instance in [10]). The smallest mammal, a Pygmy white-toothed shrew weigh only about 2.5 g while the largest mammal, an African elephant, weigh about 10 tons which is about seven orders of magnitude more! Since all mammals have similar density the same comparison holds also for volumes.

Mammals (and other warm-blooded animals) consume food, which changes into heat due to metabolic processes that occur in the entire body (we shall look at animals at rest). If the amount of food that these animals eat per day was proportional to their masses, then their body temperature would be proportional to their size, just like for the cheese cubes in the microwave oven (figure 12). However, almost all mammals have about the same body temperature within the range from 35 °C to 38 °C [11].

This brings us to the following conclusion. If we measure the amount of the food eaten by the animal in animal mass, then small animals have to eat more such units of food per day than large animals. Using a simple model based on the reasoning given above one can derive an expression that relates the daily food consumption of an animal with the animal's mass. In thermal equilibrium, the thermal power generated in an animal's body due to daily food consumption $P_{\text{metabol}}$ is equal to the power of cooling through the surface of the body $S$,

$$P_{\text{metabol}} = K \cdot S \cdot (T_{\text{body}} - T_{\text{surr}}) = K \cdot S \cdot \Delta T, \tag{13}$$

where $T_{\text{body}}$ is the animal's body temperature, $T_{\text{surr}}$ is the temperature of the surroundings (both assumed to be constant) and $K$ is a unknown constant assumed to be equal for all animals in

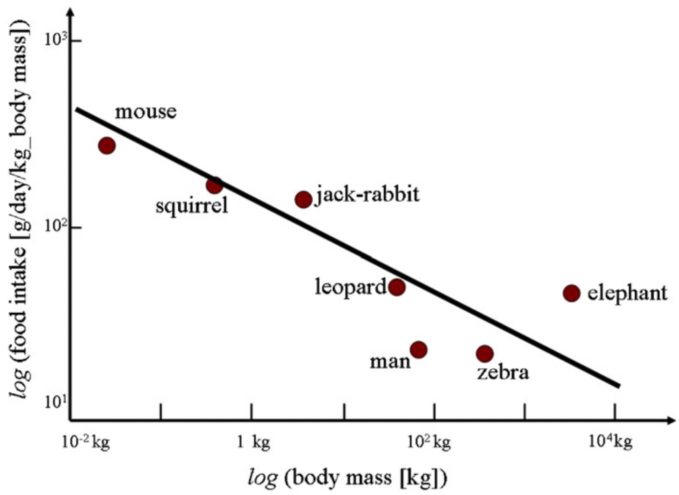



**Figure 13.** Log–log diagram of daily food consumption versus body mass for several mammals (reproduced from [12]).

our study. Assuming that mass and volume of the animal are proportional to $a^3$ and its surface to $a^2$ ($a$ being the typical size of the animal) then (13) can be rewritten in the following form:

$$P_{\text{metabol}} = Ka^2\Delta T = K(\sqrt[3]{V})^2\Delta T = K'm^{\frac{2}{3}}\Delta T, \tag{14}$$

which gives the following expression for the daily food consumption per body mass

$$\frac{P_{\text{metabol}}}{m} = K'\Delta T m^{-\frac{1}{3}}. \tag{15}$$

We note again that since the mass is proportional to $a^3$, $P/m \propto 1/a$, i.e., the surface-to-volume ratio of the animal.

The expression tells us that smaller animals have to consume more food per day in proportion to their masses than larger animals in order to keep their body temperature constant, which qualitatively agrees with experimental observations that can be found in the literature [12] (figure 13). A more accurate derivation for land mammals, which takes into account details of their body geometry but is otherwise based on the same reasoning, gives the expression for daily food consumption per body mass with $m$ to the power $-0.25$ (see, for instance, [12, 13]).

### *4.1. Extreme conditions*

The above derivation went smoothly because we have made several crucial assumptions. We have assumed that all mammals have similar shape, are of the same average density, are covered with the same thermal insulation, are all at rest and live at constant ambient temperature (some physical aspects of energy consumption for animals in motion are discussed in [14]). It is almost surprising that in spite of all these assumptions, which obviously do not hold in reality, the presented model still gives a qualitatively correct prediction.

Let us now go only one step further and see what physical principles may help mammals in extreme conditions, such as very low or very high ambient temperatures, to keep their body temperature constant.

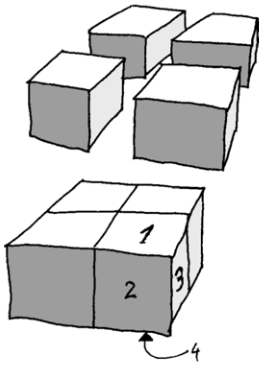



**Figure 14.** If animals huddle together in a group, the surface area per animal is reduced (here animals are approximated with cubes).

As we have shown before, the largest animals are less efficient in cooling their bodies than smaller animals. Therefore in hot climates the largest animals, such as elephants, have problems in reducing their body temperature. They cannot eat less (this would endanger their existence) so they had to find a way how to make their cooling more efficient. Elephants 'developed' disproportionally large ears with a rich blood circulating system. Another example are dogs for instance, which cool by rapid breathing and increased evaporation through their long tongues. On the other hand, in very cold climates small animals such as mice have the problem of how to reduce their heat loss (remember, small animals are efficient 'coolers'). They cannot eat more (actually they eat almost all the time they are awake), so they had to 'develop' other ways how to reduce heat loss. One way how to do this is to improve the thermal insulation of their bodies. That is why small mammals have fur and birds have feathers. Another way is to decrease the surface area through which the heat is leaving their bodies. One way of doing this is to huddle together in group. For example penguins in a rookery huddle together in large groups that may number several thousands [11]. Huddling together is also of particular importance for newborn mammals and bird nestlings. The basic physics behind huddling can be easily explained. Take for instance four animals approximated with cubes (figure 14). When separated each 'animal' has surface area $6a^2$. When grouped together as shown in the figure the surface area per 'animal' is only $4a^2$. These 'animals' could theoretically survive 1.5 times larger temperature difference at the same daily food consumption.

## 5. Conclusions

Teaching an interdisciplinary course on thermal physics could benefit from a discussion of the surface-to-volume ratio. This quantity does not only govern heating and cooling rates of objects in physics, but for example the animal metabolism of mammals in biology as well.

## Appendix. Students' ideas about the melting of cheese cubes: learning about misconceptions

Students' and physics teachers' ideas about the principles involved in melting cheese cubes in electric and microwave ovens have been observed by one of us on several occasions. In one particular case a short survey has been done with 29 first year physics major students, leading to the following results obtained from the written response of the students.

For predictions for the first experiment (cheese cubes in the classical electric oven): out of 29 students, 27 made the correct prediction.

Predictions for the second experiment (cheese cubes in the microwave oven) and the corresponding prevailing explanations could be split in the following three categories:

(1) Nineteen students predicted that all cubes will melt at about the same time because cubes absorb heat through the whole volume.
(2) Six students predicted that the smallest cubes will melt first. Two types of answers were identified: (a) because smaller cubes have smaller mass and (b) because microwaves penetrate only to a certain depth.
(3) For students predicted that the largest cubes will melt first. Again two types of answers were identified: (a) because larger cubes have more molecules at the surface and it is easier to shake them and (b) because all cubes are heated through the whole volume but for smaller cubes the cooling is more efficient.

The predictions on other occasions (including those for physics teachers) have been similar to the numbers presented above. The predictions for the second experiment show that most of the students forget to take into account the cooling of the cheese cubes. The reason may be that they forget (or do not know) that microwaves do not heat the air inside the oven. As often experienced in introductory courses, there are always some hardworking students who may know even more than what is required by the curriculum but fail to make a correct judgement of what are the major physical effects involved in the problem and what can be neglected. Students who gave answer (b) in the second category and answer (a) in the third category probably belong to this group.

When all students had written their predictions, the experiments were shown to them. After that, students were asked to give in written form the explanation of the outcome of the experiment that they have just witnessed (no explanation or comments has been given by the instructor so far).

About half of the students now gave the correct explanation for the experiment in a microwave oven but among alternative explanations the following one appeared on several occasions:

'Small cubes remain intact because their size is much smaller than the wavelength of the microwaves used in the experiment'. Obviously, these students confused absorption and diffraction of waves, which is not so surprising since this difference is usually not explicitly addressed in the introductory physics courses.

Finally, even seemingly well-prepared instructors may sometimes encounter a difficult time. One bright student from the group of 29 gave the following explanation: 'smaller cubes dry first because they have a larger surface-to-volume ratio. In dry cubes there is less water molecules and therefore microwaves cannot heat them any more. That is why they do not melt'. Our counter argument (after some sweating) was that the cheese cubes were cut just before the experiment and that the time to perform the experiment is much shorter than the time needed for the water content in the cubes to drop significantly, but this was a very good point and was accordingly acknowledged.

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