A method for computing all values λ such that $A + \lambda B$ has a multiple eigenvalue

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Abstract

Given a pair of $n \times n$ matrices A and B, we consider the problem of finding values λ such that the matrix $A + \lambda B$ has a multiple eigenvalue. Our approach solves the problem using only the standard matrix computation tools. By formulating the problem as a singular two-parameter eigenvalue problem, we construct matrices Δ_1 and Δ_0 of size $3n^2 \times 3n^2$ with the property that the regular eigenvalues of the singular pencil $\Delta_1 - \lambda\Delta_0$ are the values λ, such that $A + \lambda B$ has a multiple eigenvalue. We show that these values can be computed numerically from Δ_1 and Δ_0 by the staircase algorithm.

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1. Introduction

For a given pair of $n \times n$ complex matrices A and B we would like to find all points (λ, μ) such that the matrix $A + \lambda B$ has a multiple eigenvalue μ. For a nice introduction to the problem, its properties, and applications, see [6]. We are motivated by a note in [6] that there appears to be no globally convergent numerical method for the problem of finding all such values λ. In this paper we present a method that finds all the solutions using only standard eigenvalue computation tools. The method is very sensitive and can be applied for small matrices only. In this way the method is not competitive to the numerical method from [6], yet it is quite elegant and provides new insight into the problem.

In the next section we introduce some basic properties of the problem. Section 3 contains a brief overview of the method MFRD from [6]. In Section 4 we introduce our main tool, two-parameter eigenvalue problems. Then we present some auxiliary results in Section 5. In Section 6 we present an approach that finds only solutions such that $A + \lambda B$ has an eigenvalue of geometric

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multiplicity at least two, while in Section 7 we present a method that returns all solutions and is appropriate for generic matrices A and B. In Section 8 we present the staircase algorithm for the extraction of the regular eigenvalues from a singular matrix pencil. In Section 9 we provide some numerical examples and give some conclusions in Section 10.

2. Basic properties

If we take the characteristic polynomial $p(\lambda,\mu) = \det(A + \lambda B - \mu I)$, then points (λ,μ) such that μ is a multiple eigenvalue of $A + \lambda B$ are solutions of the polynomial system

$$\begin{aligned} p(\lambda,\mu) &= 0 \\ p_\mu(\lambda,\mu) &= 0. \end{aligned}$$

A generic situation for a multiple eigenvalue is a double eigenvalue. From Bézout theorem it follows that for generic matrices A and B there are $n(n-1)$ values of λ such that $A + \lambda B$ has a double eigenvalue.

A double eigenvalue μ of $A + \lambda B$ can be semisimple or non-semisimple. In the semisimple case the geometric multiplicity of eigenvalue μ is 2 and there exist linearly independent vectors x_1 and x_2 such that

$$\begin{aligned} (A + \lambda B - \mu I)x_1 &= 0 \\ (A + \lambda B - \mu I)x_2 &= 0. \end{aligned}$$

In the non-semisimple case geometric multiplicity of μ is 1 and there exist linearly independent vectors x and y such that

$$\begin{aligned} (A + \lambda B - \mu I)x &= 0 \\ (A + \lambda B - \mu I)y &= x. \end{aligned}$$

For generic matrices A and B, there exist $n(n-1)$ values of λ such that $A+\lambda B$ has a non-semisimple double eigenvalue.

3. The method of fixed relative distance (MFRD)

Jarlebring, Kvaal, and Michiels use the following idea in [6]. If $A+\lambda B$ has a double eigenvalue μ, then for a small perturbation $\delta\lambda$ the matrix $A + (\lambda + \delta\lambda)B$ has two eigenvalues close to μ.

Suppose that λ is such that μ and $(1 + \epsilon)\mu$, where ϵ is small, are eigenvalues of $A + \lambda B$. Such pair (λ,μ) could be used as an initial approximation for the local iterative (Newton's) method. We can write this as the following: there exist nonzero vectors x and y, such that

$$\begin{aligned} (A + \lambda B - \mu I)x &= 0 \\ (A + \lambda B - \mu(1 + \epsilon)I)y &= 0. \end{aligned} \tag{1}$$

This is a two-parameter eigenvalue problem (see Section 4), which is nonsingular for $\epsilon \neq 0$.

By solving the above two-parameter eigenvalue problem by one of the existing numerical methods we get initial approximations for Newton's method that computes the final solutions. This combination is a globally convergent method that computes all of the solutions.

Parameter ϵ in (1) is a regularization parameter. If ϵ is close to 0, then the problem is close to being singular and the methods for nonsingular two-parameter eigenvalue problems have difficulties. On the other hand, if ϵ is far away from 0, the obtained approximations can be very poor.

In our approach we take $\epsilon = 0$ and study the obtained singular two-parameter eigenvalue problem. As a result we obtain a method that can compute all pairs (λ, μ) without a need for local iterative methods and is, up to our knowledge, the first method with such properties.

4. Two-parameter eigenvalue problems

The algebraic two-parameter eigenvalue problem has the form

$$\begin{aligned} (A_1 + \lambda B_1 + \mu C_1)x_1 &= 0, \\ (A_2 + \lambda B_2 + \mu C_2)x_2 &= 0, \end{aligned} \tag{2}$$

where A_i, B_i, and C_i are $n_i \times n_i$ complex matrices, $\lambda, \mu \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$. A pair (λ, μ) is an eigenvalue if it satisfies (2) for nonzero vectors x_1, x_2, and the tensor product $x_1 \otimes x_2$ is the corresponding eigenvector. If we introduce matrices

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2, \\ \Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2, \end{aligned} \tag{3}$$

then the problem (2) is related to a coupled pair of generalized eigenvalue problems

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \tag{4}$$

for a decomposable tensor $z = x_1 \otimes x_2$ (for details see, e.g., [1]).

We say that a two-parameter eigenvalue problem is nonsingular if there exists a nonsingular linear combination of matrices Δ_0, Δ_1, and Δ_2. For an overview of available numerical methods for nonsingular problems, see, e.g., [4] and references therein.

In a singular two-parameter eigenvalue problem both matrix pencils (4) are singular. There exists a numerical method, presented in [9], that computes the regular eigenvalues of (2) from the common regular part of (4). For the generic singular case it is shown in [8] that the regular eigenvalues of (2) and (4) do agree. For other types of singular two-parameter eigenvalue problems the relation between the regular eigenvalues of (4) and the regular eigenvalues of (2) is not completely known, but the numerical examples indicate that the method from [9] can be successfully applied to such problems as well.

5. Auxiliary results

In this section we briefly review the Kronecker canonical form. More about the subject can be found in, e.g., [2], [3], and [11].

Definition 1. Let $S - \lambda T \in \mathbb{C}^{m\times n}$ be a matrix pencil. Then there exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that

$$P^{-1}(S - \lambda T)Q = \widetilde{S} - \lambda\widetilde{T} = \operatorname{diag}(S_1 - \lambda T_1, \ldots, S_k - \lambda T_k)$$

is the *Kronecker canonical form (KCF)*. Each block $S_i - \lambda T_i$ for $i = 1, \ldots, k$ must be of one of the following forms: $J_d(\alpha), N_d, L_d$, or L_d^T, where blocks

$$J_d(\alpha) = \begin{bmatrix} \alpha-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha-\lambda \end{bmatrix} \in \mathbb{C}^{d\times d}, \quad N_d = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{d\times d},$$

$$L_d = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix} \in \mathbb{C}^{d\times(d+1)}, \quad L_d^T = \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix} \in \mathbb{C}^{(d+1)\times d},$$

represent a finite regular block, an infinite regular block, a right singular block, and a left singular block, respectively. To each Kronecker block we associate a *Kronecker chain* of linearly independent vectors as follows:

a) A finite regular block $J_d(\alpha)$ is associated with vectors $u_1, \ldots, u_d$ that satisfy

$$\begin{aligned} (S - \alpha T)u_1 &= 0, \\ (S - \alpha T)u_{i+1} &= Tu_i, \quad i = 1, \ldots, d-1. \end{aligned}$$

b) An infinite regular block N_d is associated with vectors $u_1, \ldots, u_d$ that satisfy

$$\begin{aligned} Tu_1 &= 0, \\ Tu_{i+1} &= Su_i, \quad i = 1, \ldots, d-1. \end{aligned}$$

c) A right singular block L_d is associated with vectors $u_1, \ldots, u_{d+1}$ that satisfy

$$\begin{aligned} Tu_1 &= 0, \\ Tu_{i+1} &= Su_i, \quad i = 1, \ldots, d, \\ 0 &= Su_{d+1}. \end{aligned}$$

d) A left singular block L_d^T, $d \geq 1$, is associated with vectors $u_1, \ldots, u_d$ that satisfy

$$Tu_i = Su_{i+1}, \quad i = 1, \ldots, d-1.$$

Definition 2. The *normal rank* of a square matrix pencil $S - \lambda T$ is

$$\operatorname{nrank}(S - \lambda T) = \max_{\lambda\in\mathbb{C}} \operatorname{rank}(S - \lambda T).$$

A value $\lambda_0 \in \mathbb{C}$, such that $\operatorname{rank}(S - \lambda_0 T) < \operatorname{nrank}(S - \lambda T)$, is a *finite regular eigenvalue*.

Košir shows in [7] that the kernel of $\Delta = A \otimes D - B \otimes C$ can be constructed from the Kronecker chains of matrix pencils $A - \lambda B$ and $C - \mu D$. This construction will be very important in the following sections as Δ-matrices related to two-parameter eigenvalue problems have this structure.

Theorem 3 ([7, Theorem 4]). *A basis for the kernel of $\Delta = A \otimes D - B \otimes C$ is the union of sets of linearly independent vectors associated with the following types of pairs of Kronecker blocks:*

a) $(J_{d_1}(\alpha_1),\ J_{d_2}(\alpha_2))$, *where* $\alpha_1 = \alpha_2$,
b) $(N_{d_1},\ N_{d_2})$, $(N_{d_1},\ L_{d_2})$, $(L_{d_1},\ N_{d_2})$, $(L_{d_1},\ J_{d_2}(\alpha))$, $(J_{d_1}(\alpha),\ L_{d_2})$, *and* $(L_{d_1},\ L_{d_2})$,
c) $(L_{d_1},\ L_{d_2}^T)$, *and* $(L_{d_2}^T,\ L_{d_1})$, *where* $d_1 < d_2$,

where the left block of each pair belongs to the pencil $A - \lambda B$ and the right block belongs to the pencil $C - \mu D$.

For each pair of Kronecker blocks from a) we can construct an associated set of linearly independent vectors $z_1, \ldots, z_d$ in the kernel of Δ as follows. Let $u_1, \ldots, u_{d_1}$ form a Kronecker chain associated with the block $J_{d_1}(\alpha_1)$ of the pencil $A - \lambda B$ and let $v_1, \ldots, v_{d_2}$ form a Kronecker chain associated with the block $J_{d_2}(\alpha_2)$ of the pencil $C - \mu D$. Then $d = \min(d_1, d_2)$ and

$$z_j = \sum_{i=1}^{j} u_i \otimes v_{j+1-i}, \quad j = 1, \ldots, d.$$

In the above theorem we omitted the constructions of vectors in the kernel for all pairs of Kronecker blocks from b) and c) because they are not relevant to our case. For a complete description, see [7].

6. Eigenvalues of geometric multiplicity at least two

In the following we assume that A and B are generic matrices in a sense that for all but a finite number of values λ the matrix $A + \lambda B$ has n simple eigenvalues. If we set $\epsilon = 0$ in (1), then we obtain a two-parameter eigenvalue problem

$$\begin{aligned} (A + \lambda B - \mu I)x &= 0 \\ (A + \lambda B - \mu I)y &= 0. \end{aligned} \tag{5}$$

The corresponding Δ-matrices of size $n^2 \times n^2$ are

$$\begin{aligned} \Delta_0 &= I \otimes B - B \otimes I, \\ \Delta_1 &= A \otimes I - I \otimes A. \end{aligned}$$

Theorem 4. *The normal rank of pencil $\Delta_1 - \lambda\Delta_0$ is $n^2 - n$. A value $\lambda_0 \in \mathbb{C}$ is a regular eigenvalue of pencil $\Delta_1 - \lambda\Delta_0$ if and only if the matrix $A + \lambda_0 B$ has a multiple eigenvalue with geometric multiplicity at least two.*

Proof. We can write

$$\Delta_1 - \lambda\Delta_0 = (A + \lambda B) \otimes I - I \otimes (A + \lambda B) \tag{6}$$

and apply Theorem 3, which states that a basis for the null space of $\Delta_1 - \lambda_0\Delta_0$ can be constructed from the Kronecker chains of the matrix pencil $A + \lambda_0 B - \mu I$.

For a generic λ_0 the matrix $A + \lambda_0 B$ has n algebraically simple eigenvalues $\mu_1, \ldots, \mu_n$ with the corresponding n linearly independent eigenvectors $x_1, \ldots, x_n$. The eigenvalues of $\Delta_1 - \lambda_0\Delta_0$ are $\mu_i - \mu_j$ for $i, j = 1, \ldots, n$. It follows from (6) that $\dim(\mathrm{Ker}(\Delta_1 - \lambda_0\Delta_0)) = n$, where the linearly independent vectors in the kernel are $x_i \otimes x_i$ for $i = 1, \ldots, n$. We see that each algebraically simple eigenvalue adds one linearly independent vector to the null space. As the matrices Δ_0 and Δ_1 are of size $n^2 \times n^2$, the normal rank of $\Delta_1 - \lambda\Delta_0$ is $n^2 - n$.

Let now λ_0 be such that the matrix $A + \lambda_0 B$ has an eigenvalue μ_0 of multiplicity $d > 1$.

a) If μ_0 is geometrically simple then the pencil $A + \lambda_0 B - \mu I$ has $J_d(\mu_0)$ block in its KCF. It follows from Theorem 3 that μ_0 contributes d linearly independent vectors to the null space of $\Delta_1 - \lambda_0\Delta_0$, therefore $\mathrm{rank}(\Delta_1 - \lambda_0\Delta_0) = \mathrm{nrank}(\Delta_1 - \lambda\Delta_0)$.
b) If μ_0 has geometric multiplicity g more than one, then there are g Jordan blocks $J_{d_i}(\mu_0)$ in the KCF of $A + \lambda_0 B - \mu I$, such that $d_1 + \cdots + d_g = d$. One can see from the construction in Theorem 3 that μ_0 now contributes more than d linearly independent vectors to the null space of $\Delta_1 - \lambda_0\Delta_0$. The rank of $\Delta_1 - \lambda\Delta_0$ drops at λ_0 and λ_0 is a regular eigenvalue. □

To make things clear, let us review the situation from Theorem 4 when μ_0 is a double eigenvalue of $A + \lambda_0 B$.

a) Let μ_0 be a non-semisimple eigenvalue with the eigenvector x and root vector y. The linearly independent vectors from the null space of $\Delta_1 - \lambda_0\Delta_0$ are $x \otimes x$ and $x \otimes y + y \otimes x$. A double eigenvalue λ_0 contributes two linearly independent vectors to the null space and the rank of the pencil $\Delta_1 - \lambda\Delta_0$ does not change at $\lambda = \lambda_0$.
b) Let μ_0 be a semisimple eigenvalue with linearly independent eigenvectors x_1 and x_2. The linearly independent vectors from the null space of $\Delta_1 - \lambda_0\Delta_0$ are $x_1 \otimes x_1$, $x_2 \otimes x_2$, $x_1 \otimes x_2$, and $x_2 \otimes x_1$. A double eigenvalue λ_0 contributes four linearly independent vectors to the null space and the rank of the pencil $\Delta_1 - \lambda\Delta_0$ drops for two at $\lambda = \lambda_0$.

From the above we can conclude that the rank of $\Delta_1 - \lambda\Delta_0$ drops only at points $\lambda = \lambda_0$ such that the matrix $A + \lambda_0 B$ has a multiple eigenvalue with geometric multiplicity at least two. The smallest such example is a double semisimple eigenvalue. If we apply the numerical algorithm from Section 8 on pencil $\Delta_1 - \lambda\Delta_0$, then we get all such solutions.

However, this approach does not detect values λ_0 that give multiple eigenvalues of geometric multiplicity one. In particular, it does not detect double non-semisimple eigenvalues, which are the generic multiple eigenvalues. For these solutions we use a different two-parameter eigenvalue problem in the next section.

7. Second version

In order to obtain also the values λ_0 such that $A + \lambda_0 B$ has a multiple eigenvalue of geometric multiplicity one, we take the following nonlinear two-parameter eigenvalue problem

$$\begin{aligned} (A + \lambda B - \mu I)x &= 0 \\ (A + \lambda B - \mu I)^2 y &= 0. \end{aligned} \tag{7}$$

If $A + \lambda_0 B$ has a double eigenvalue μ_0 with an eigenvector x, then it is easy to see that (7) has a solution, where y is either the second linearly independent eigenvector in the case of a semisimple eigenvalue or the corresponding root vector if the eigenvalue μ_0 is non-semisimple.

The second equation in (7) is a quadratic bivariate polynomial with matrix coefficients. Its full expansion is

$$(A^2 + \lambda(AB + BA) - 2\mu A + \lambda^2 B^2 - 2\lambda\mu B + \mu^2 I)y = 0.$$

We linearize the above equation as

$$\left(\underbrace{\begin{bmatrix} A^2 & AB + BA & -2A \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}_{P} + \lambda \underbrace{\begin{bmatrix} 0 & B^2 & -B \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{Q} + \mu \underbrace{\begin{bmatrix} 0 & -B & I \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{bmatrix}}_{R} \right) \begin{bmatrix} y \\ \lambda y \\ \mu y \end{bmatrix} = 0. \tag{8}$$

This is one of the possible linearizations of two-parameter polynomial eigenvalue problems, for more details on linearizations, see [5].

Now we form a linear two-parameter eigenvalue problem by taking the first equation from (7) and (8). The corresponding Δ-matrices of size $3n^2 \times 3n^2$ are

$$\begin{aligned} \Delta_0 &= B \otimes R + I \otimes Q \\ \Delta_1 &= -I \otimes P - A \otimes R. \end{aligned}$$

In order to simplify some of the following proofs, we introduce the Tracy–Singh product of partitioned matrices [10].

Definition 5. *Let an $m \times n$ matrix A be partitioned into the $m_i \times n_j$ blocks A_{ij} and a $p \times q$ matrix B into the $p_k \times q_l$ blocks B_{kl} such that $m = \sum_{i=1}^r m_i$, $n = \sum_{j=1}^s n_j$, $p = \sum_{k=1}^t p_k$, and $q = \sum_{l=1}^u q_l$. The Tracy–Singh product $A \circledast B$ is a $mp \times nq$ matrix, defined as*

$$A \circledast B = (A_{ij} \circledast B)_{ij} = ((A_{ij} \otimes B_{kl})_{kl})_{ij},$$

where the (ij)th block of the product is the $m_i p \times n_j q$ matrix $A_{ij} \circledast B$, of which the (kl)th subblock equals the $m_i p_k \times n_j q_l$ matrix $A_{ij} \otimes B_{kl}$.

Theorem 6 ([10, Theorem 5]). *In the case of balanced partitioning, where in both matrices A and B all blocks are of the same size, the Tracy–Singh product $A \circledast B$ and the Kronecker product $A \otimes B$ are permutation equivalent.*

In some cases it is more convenient to work with the Tracy-Singh product instead of the Kronecker product as some properties are easier to prove. All our block matrices have balanced partition, so this is just a reordering of columns and rows. We denote by TS the map that reorders the elements of $A \otimes B$ so that $TS(A \otimes B) = A \circledast B$. Also, we denote $\widetilde{\Delta}_i = TS(\Delta_i)$ for $i = 0, 1$.

Lemma 7. *For generic $n \times n$ matrices A and B, the normal rank of matrix pencil $\Delta_1 - \lambda \Delta_0$ is $3n^2 - n$.*

Proof. First we show that for a generic matrix A the rank of Δ_1 is $3n^2 - n$. If we apply the Tracy-Singh reordering, we get

$$\widetilde{\Delta}_0 = TS(\Delta_0) = \begin{bmatrix} 0 & I \otimes B^2 - B \otimes B & B \otimes I - I \otimes B \\ -I \otimes I & 0 & 0 \\ -B \otimes I & 0 & 0 \end{bmatrix} \tag{9}$$

and

$$\widetilde{\Delta}_1 = TS(\Delta_1) = \begin{bmatrix} -I \otimes A^2 & -I \otimes (AB + BA) + A \otimes B & 2I \otimes A - A \otimes I \\ 0 & -I \otimes I & 0 \\ A \otimes I & 0 & -I \otimes I \end{bmatrix}. \tag{10}$$

From the second block row in the equation

$$\widetilde{\Delta}_1 \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0 \tag{11}$$

it follows that $w_2 = 0$, while the third row gives $w_3 = (A \otimes I)w_1$. When we insert w_2 and w_3 in the first row of (11), we obtain

$$-(I \otimes A - A \otimes I)^2 w_1 = 0.$$

There are n linearly independent solutions of the above equation that have the form $x_i \otimes x_i$, where x_i is the eigenvector of matrix A for $i = 1, \ldots, n$. Therefore, $\operatorname{rank}(\Delta_1) = \operatorname{rank}(\widetilde{\Delta}_1) = 3n^2 - n$.

On the other hand, if (μ_0, x) is an eigenpair of $A + \lambda_0 B$, then clearly $(A + \lambda_0 B - \mu_0)^2 x = 0$ and

$$x \otimes \begin{bmatrix} x \\ \lambda_0 x \\ \mu_0 x \end{bmatrix} \in \operatorname{Ker}(\widetilde{\Delta}_1 - \lambda_0 \widetilde{\Delta}_0).$$

Since we have n such eigenpairs in the generic case, $\operatorname{rank}(\widetilde{\Delta}_1 - \lambda_0 \widetilde{\Delta}_0) \leq 3n^2 - n$. Together with $\operatorname{rank}(\widetilde{\Delta}_1) = 3n^2 - n$ this shows that in the generic case the normal rank of the matrix pencil $\Delta_1 - \lambda \Delta_0$ is indeed $3n^2 - n$. □

We can write

$$-(\Delta_1 - \lambda\Delta_0) = (A + \lambda B) \otimes R + I \otimes (P + \lambda Q) \tag{12}$$

and apply Theorem 3. It follows that the null space of $\Delta_1 - \lambda_0\Delta_0$ depends on the KCF of pencils $A + \lambda_0 B - \mu(-I)$ and $P + \lambda_0 Q - \mu R$. The first pencil $A + \lambda_0 B + \mu I$ is clearly regular and has only regular Jordan blocks in its KCF. The following lemma shows that the second pencil $P + \lambda_0 Q - \mu R$ is regular as well, but does contain some infinite blocks in addition.

Lemma 8. *For each λ_0, the KCF of the matrix pencil $P + \lambda_0 Q - \mu R$ consists of n blocks N_1 and a regular part of dimension $2n$.*

Proof. The rank of matrix R is $2n$. Vectors in the null space of R have the form

$$\begin{bmatrix} 0 \\ z \\ Bz \end{bmatrix} \tag{13}$$

for an arbitrary nonzero vector z. Since the vector

$$(P + \lambda_0 Q) \begin{bmatrix} 0 \\ z \\ Bz \end{bmatrix} = \begin{bmatrix} * \\ z \\ * \end{bmatrix} \tag{14}$$

is clearly nonzero, the matrices $P + \lambda_0 Q$ and R do not have a common null space. It is also easy to see that (14) can not belong to the image of R, where the second block of all vectors is zero. As a result the pencil $P + \lambda_0 Q - \mu R$ has no Kronecker chains of length two or more for blocks N or L and we can conclude that the KCF has n infinite regular blocks N_1 and a regular part of dimension $2n$. □

In the next theorem we show that, on contrary to the situation in Section 7, the regular eigenvalues of $\Delta_1 - \lambda\Delta_0$ agree with the values λ_0 such that $A + \lambda_0 B$ has a multiple eigenvalue μ_0, regardless of the geometric multiplicity of μ_0. In order to show this, we need the following two lemmas.

Lemma 9. *If the KCF of the pencil $A + \lambda_0 B + \mu I$ contains a block $J_1(-\mu_0)$ with the corresponding eigenvector x, then the pencil $P + \lambda_0 Q - \mu R$ contains a block $J_k(-\mu_0)$ of size $k \geq 2$ whose first two vectors in the corresponding Kronecker chain are*

$$\begin{bmatrix} x \\ \lambda_0 x \\ \mu_0 x \end{bmatrix}, \quad \textit{and} \quad \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}. \tag{15}$$

Proof. Clearly, from $(A + \lambda_0 B - \mu_0 I)x = 0$ we get $(A + \lambda_0 B - \mu_0 I)^2 x = 0$ and after linearization (8) we obtain

$$(P + \lambda_0 Q + \mu_0 R) \begin{bmatrix} x \\ \lambda_0 x \\ \mu_0 x \end{bmatrix} = 0.$$

The following relation shows that vectors (15) indeed form the initial part of a Kronecker chain:

$$(P+\lambda_0 Q+\mu_0 R)\begin{bmatrix}0\\0\\x\end{bmatrix}=\begin{bmatrix}(-2A-\lambda_0 B+\mu_0 I)x\\0\\x\end{bmatrix}=\begin{bmatrix}(\lambda_0 B-\mu_0 I)x\\0\\x\end{bmatrix}=-R\begin{bmatrix}x\\\lambda_0 x\\\mu_0 x\end{bmatrix}. \qquad \square \quad (16)$$

Lemma 10. *If the KCF of the pencil $A+\lambda_0 B+\mu I$ contains a block $J_d(-\mu_0)$ of size $d \geq 2$ with the corresponding eigenvector x and the first root vector y, then the pencil $P+\lambda_0 Q-\mu R$ contains a block $J_{k_1}(-\mu_0)$ of size $k_1 \geq 1$ with the corresponding eigenvector*

$$\begin{bmatrix}y\\\lambda_0 y\\\mu_0 y\end{bmatrix}$$

and a block $J_{k_2}(-\mu_0)$ of size $k_2 \geq 3$ whose first three vectors in the corresponding Kronecker chain are

$$\begin{bmatrix}x\\\lambda_0 x\\\mu_0 x\end{bmatrix}, \quad \begin{bmatrix}0\\0\\x\end{bmatrix}-\frac{1}{2}\begin{bmatrix}y\\\lambda_0 y\\\mu_0 y\end{bmatrix}, \quad \textit{and} \quad -\frac{1}{2}\begin{bmatrix}0\\0\\y\end{bmatrix}.$$

Proof. For the root vector y we have $(A+\lambda_0 B-\mu_0 I)^2 y = 0$ and then, similar as in the previous lemma, it follows that

$$(P+\lambda_0 Q+\mu_0 R)\begin{bmatrix}y\\\lambda_0 y\\\mu_0 y\end{bmatrix}=0.$$

So, $\begin{bmatrix}y^T & \lambda_0 y^T & \mu_0 y^T\end{bmatrix}^T$ is an eigenvector corresponding to a block $J_{k_1}(-\mu_0)$ of size at least 1.

The second block $J_{k_2}(-\mu_0)$ starts with the eigenvector $\begin{bmatrix}x^T & \lambda_0 x^T & \mu_0 x^T\end{bmatrix}^T$. Equation (16) gives the second vector in the Kronecker chain. But, we also have the third vector. From $(A+\lambda_0 B-\mu_0 I)y = x$ we derive

$$(P+\lambda_0 Q+\mu_0 R)\begin{bmatrix}0\\0\\y\end{bmatrix}=\begin{bmatrix}(-2A-\lambda_0 B+\mu_0 I)y\\0\\y\end{bmatrix}=-R\left(\begin{bmatrix}y\\\lambda_0 y\\\mu_0 y\end{bmatrix}-2\begin{bmatrix}0\\0\\x\end{bmatrix}\right)$$

and therefore the size k_2 is at least 3. $\square$

Theorem 11. *For generic $n \times n$ matrices A and B, a value $\lambda_0 \in \mathbb{C}$ is a regular eigenvalue of the pencil $\Delta_1 - \lambda\Delta_0$ if and only if the matrix $A+\lambda_0 B$ has a multiple eigenvalue.*

Proof. We know from Lemma 7 that $\mathrm{nrank}(\Delta_1-\lambda\Delta_0) = 3n^2-n$. For a fixed λ_0 we can obtain the null space of $\Delta_1-\lambda_0\Delta_0$ by applying Theorem 3 on (12). As we know that none of the pencils $A+\lambda_0 B+\mu I$ and $P+\lambda_0 Q-\mu R$ contain singular blocks, all vectors from the null space of $\Delta_1-\lambda_0\Delta_0$ come from combinations of Jordan blocks.

If λ_0 is such that $A+\lambda_0 B$ has n algebraically simple eigenvalues, then it follows from Lemma 9 that the KCF of $P+\lambda_0 Q-\mu R$ contains n Jordan blocks of size 2 or more. As we know from Lemma 8 that the regular part of $P+\lambda_0 Q-\mu R$ has dimension $2n$, the size of each Jordan block is exactly 2. It now follows from Theorem 3 that each eigenvalue μ_0 of $A+\lambda_0 B$ contributes one dimension to the null space of $\Delta_1-\lambda_0\Delta_0$ which has dimension n and $\mathrm{rank}(\Delta_1-\lambda_0\Delta_0)=\mathrm{nrank}(\Delta_1-\lambda\Delta_0)$.

We want to show that $\mathrm{rank}(\Delta_1-\lambda_0\Delta_0)<\mathrm{nrank}(\Delta_1-\lambda\Delta_0)$ if and only if $A+\lambda_0 B$ has a multiple eigenvalue. We first study the case when $A+\lambda_0 B$ has a double eigenvalue μ_0.

a) If μ_0 is semisimple, then the KCF of the pencil $A+\lambda_0 B+\mu I$ contains two blocks $J_1(-\mu_0)$. We know from Lemma 9 that to each of the blocks there corresponds a block $J_2(-\mu_0)$ in the KCF of the pencil $P+\lambda_0 Q-\mu R$. From Theorem 3 it follows that combinations of the above blocks contribute 4 linearly independent vectors to the null space of $\Delta_1-\lambda_0\Delta_0$. Thus, $\mathrm{rank}(\Delta_1-\lambda_0\Delta_0)=\mathrm{nrank}(\Delta_1-\lambda\Delta_0)-2$ and λ_0 is a regular eigenvalue.
b) Let μ_0 be non-semisimple with the corresponding eigenvector x and root vector y. Then, the KCF of the pencil $A+\lambda_0 B+\mu I$ contains a block $J_2(-\mu_0)$. We know from Lemma 10 that this block has corresponding blocks $J_1(-\mu_0)$ and $J_3(-\mu_0)$ in the KCF of the pencil $P+\lambda_0 Q-\mu R$. It follows from Theorem 3 that μ_0 contributes 3 linearly independent vectors to the null space of $\Delta_1-\lambda_0\Delta_0$. This gives $\mathrm{rank}(\Delta_1-\lambda_0\Delta_0)=\mathrm{nrank}(\Delta_1-\lambda\Delta_0)-1$ and λ_0 is a regular eigenvalue.

If $A+\lambda_0 B$ has an eigenvalue μ_0 of multiplicity d greater than 2, then we know from Lemma 9 and Lemma 10 that to each block $J_1(-\mu_0)$ of $A+\lambda_0 B+\mu I$ there corresponds a block $J_k(-\mu_0)$, $k\geq 2$, of $P+\lambda_0 Q-\mu R$, while to each block $J_d(-\mu_0)$, $d\geq 2$, of $A+\lambda_0 B+\mu I$ there correspond blocks $J_{k_1}(-\mu_0)$, $k_1\geq 1$, and $J_{k_2}(-\mu_0)$, $k_2\geq 3$, of $P+\lambda_0 Q-\mu R$. In addition, the sum of the dimension of all $J(-\mu_0)$ blocks of $P+\lambda_0 Q-\mu R$ is $2d$. It is easy to see from Theorem 3 that all possible combinations of blocks give enough linearly independent vectors in the null space that $\mathrm{rank}(\Delta_1-\lambda_0\Delta_0)<\mathrm{nrank}(\Delta_1-\lambda\Delta_0)$ and λ_0 is a regular eigenvalue. □

As before, let us review in detail the situation when μ_0 is a double eigenvalue of $A+\lambda_0 B$.

a) If μ_0 is semisimple with the corresponding linearly independent eigenvectors x_1 and x_2, then the linearly independent vectors from the null space of $\Delta_1-\lambda_0\Delta_0$ are

$$x_1\otimes\begin{bmatrix}x_1\\ \lambda_0 x_1\\ \mu_0 x_1\end{bmatrix},\ x_1\otimes\begin{bmatrix}x_2\\ \lambda_0 x_2\\ \mu_0 x_2\end{bmatrix},\ x_2\otimes\begin{bmatrix}x_1\\ \lambda_0 x_1\\ \mu_0 x_1\end{bmatrix},\ \text{and } x_2\otimes\begin{bmatrix}x_2\\ \lambda_0 x_2\\ \mu_0 x_2\end{bmatrix}.$$

A double eigenvalue λ_0 contributes four linearly independent vectors to the null space and the rank of the pencil $\Delta_1-\lambda\Delta_0$ drops for two at $\lambda=\lambda_0$.
b) If μ_0 is non-semisimple with the corresponding eigenvector x and root vector y, then the linearly independent vectors from the null space of $\Delta_1-\lambda_0\Delta_0$ are

$$x\otimes\begin{bmatrix}x\\ \lambda_0 x\\ \mu_0 x\end{bmatrix},\ x\otimes\begin{bmatrix}y\\ \lambda_0 y\\ \mu_0 y\end{bmatrix},\ \text{and } y\otimes\begin{bmatrix}x\\ \lambda_0 x\\ \mu_0 x\end{bmatrix}+x\otimes\begin{bmatrix}0\\ 0\\ x\end{bmatrix}.$$

A double eigenvalue λ_0 now contributes three linearly independent vectors to the null space and the rank of the pencil $\Delta_1-\lambda\Delta_0$ drops for one at $\lambda=\lambda_0$.

We know from the above that in the generic case the regular part of the pencil $\Delta_1 - \lambda\Delta_0$ has dimension $n^2 - n$, as each λ_0, such that $A + \lambda_0 B$ has a nonsemisimple double eigenvalue, is a regular eigenvalue of the pencil $\Delta_1 - \lambda\Delta_0$. We say more about the Kronecker structure of the pencil $\Delta_1 - \lambda\Delta_0$ in the following lemma.

Lemma 12. *For generic $n \times n$ matrices A and B, the KCF of $\Delta_1 - \lambda\Delta_0$ contains n^2 N_1 blocks, n L blocks of length at least one, n L^T blocks, and a regular part of dimension $n^2 - n$.*

Proof. From (9) one can find the structure of the vectors from the null space of $\widetilde{\Delta}_0$. It is easy to see that for an arbitrary pair of nonzero vectors x and y,

$$\widetilde{\Delta}_0 \begin{bmatrix} 0 \\ x \otimes y \\ x \otimes By \end{bmatrix} = 0. \tag{17}$$

This gives n^2 linearly independent vectors from $\operatorname{Ker}(\widetilde{\Delta}_0)$. In addition, if x_i is an eigenvector of matrix B for the eigevalue η_i, i.e, $Bx_i = \eta_i x_i$, then

$$\widetilde{\Delta}_0 \begin{bmatrix} 0 \\ x_i \otimes x_i \\ 0 \end{bmatrix} = \widetilde{\Delta}_0 \begin{bmatrix} 0 \\ 0 \\ x_i \otimes x_i \end{bmatrix}, \quad i = 1, \ldots, n. \tag{18}$$

Together with (17) this gives $n^2 + n$ linearly independent vectors from $\operatorname{Ker}(\widetilde{\Delta}_0)$.

We know from the proof of Lemma 7 that $\widetilde{\Delta}_1$ has n-dimensional null space. As all vectors in the null space of $\widetilde{\Delta}_1$ have nonzero first block it is obvious that the intersection of null spaces of $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0$ is trivial. Therefore, the pencil $\Delta_1 - \lambda\Delta_0$ has no L_0 blocks. We can conclude that the pencil $\Delta_1 - \lambda\Delta_0$ has n^2 N_k blocks and n L_k blocks, where the dimensions $k \geq 1$ of the blocks are yet to be determined.

For each L_k or N_{k+1} block for $k \geq 1$ there must exist a chain of linearly independent vectors that starts with $\widetilde{\Delta}_0 v_1 = 0$, continues by $\widetilde{\Delta}_0 v_{i+1} = \widetilde{\Delta}_1 v_i$ for $i = 1, \ldots, k$, and ends with $\widetilde{\Delta}_1 v_{k+1} = 0$ in case of an L_k block. Let us now show that all N blocks are of size one. Suppose that $\widetilde{\Delta}_0 v = 0$. If vector v belongs to an L_k or N_k block of size $k \geq 2$, then there exists a vector w, such that

$$\widetilde{\Delta}_0 w = \widetilde{\Delta}_1 v. \tag{19}$$

From

$$\widetilde{\Delta}_0 \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} (I \otimes B^2 - B \otimes B)w_2 + (B \otimes I - I \otimes B)w_3 \\ -w_1 \\ -(B \otimes I)w_1 \end{bmatrix} \tag{20}$$

and

$$\widetilde{\Delta}_1 \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} * \\ -v_2 \\ -v_3 \end{bmatrix}$$

we see that a necessary condition for (19) is that $v_3 = (B \otimes I)v_2$. A brief inspection of the vectors from (17) and (18) shows that only n linearly independent vectors from the null space of $\widetilde{\Delta}_0$ satisfy this condition. These are the vectors of the form

$$\begin{bmatrix} 0 \\ x_i \otimes x_i \\ \eta_i x_i \otimes x_i \end{bmatrix}, \tag{21}$$

where $Bx_i = \eta_i x_i$ for $i = 1, \ldots, n$.

With some additional computation it is possible to explicitly write down the vectors w that satisfy (19) if we take vector (21) for the initial vector v. From

$$\widetilde{\Delta}_1 \begin{bmatrix} 0 \\ x_i \otimes x_i \\ \eta_i x_i \otimes x_i \end{bmatrix} = \begin{bmatrix} \eta_i x_i \otimes Ax_i - x_i \otimes BAx_i \\ -x_i \otimes x_i \\ -\eta_i x_i \otimes x_i \end{bmatrix} = \begin{bmatrix} (B \otimes I - I \otimes B)x_i \otimes Ax_i \\ -x_i \otimes x_i \\ -\eta_i x_i \otimes x_i \end{bmatrix}$$

and the first block from (20), which we rewrite as

$$(I \otimes B^2 - B \otimes B)w_2 + (B \otimes I - I \otimes B)w_3 = (B \otimes I - I \otimes B)\big(w_3 - (I \otimes B)w_2\big),$$

it follows that the vector (the solution is not unique)

$$w = \begin{bmatrix} x_i \otimes x_i \\ 0 \\ x_i \otimes Ax_i \end{bmatrix}$$

satisfies (19).

Therefore, the n vectors from (21) are the initial vectors of L chains of length at least one, while the remaining n^2 vectors from the null space of $\widetilde{\Delta}_0$ belong to n^2 N_1 blocks. Finally, we know that the number of L^T blocks of the matrix pencil $\Delta_1 - \lambda\Delta_0$ is equal to the number of L blocks. □

The above lemma shows how difficult it is to prove anything about the KCF of the pencil $\Delta_1 - \lambda\Delta_0$. We did not manage to prove the complete structure for generic matrices A and B, but, based on numerical experiments in Matlab using the GUPTRI package [2, 3], we believe that the generic structure is as follows:

a) If $n = 2k + 1$, then the pencil contains n blocks L_{k+1}, n blocks L_k^T, n^2 blocks N_1, and a regular part of size $n(n-1)$.
b) If $n = 2k$, then the pencil contains k blocks L_k, k blocks L_{k+1}, k blocks L_{k-1}^T, k blocks L_k^T, n^2 blocks N_1, and a regular part of size $n(n-1)$.

8. The algorithm

For a pair of $n \times n$ matrices A and B, we would like to find all values λ such that the matrix $A+\lambda B$ has a multiple eigenvalue. We form a singular two-parameter eigenvalue problem (7), where we linearize the second equation as (8). This gives a singular two-parameter eigenvalue problem, whose regular eigenvalues (λ,μ) are such that the matrix $A+\lambda B$ has a double eigenvalue μ. One could apply the general numerical algorithm for singular two-parameter eigenvalue problems from [9] to compute these eigenvalues.

As we only need the regular eigenvalues of the singular pencil $\Delta_1-\lambda\Delta_0$, we can modify and simplify the algorithm. It is enough to apply the staircase algorithm from [11] to extract the regular part of $\Delta_1-\lambda\Delta_0$. The basic steps of the staircase algorithm are column-row and row-column compressions.

Algorithm 13. *Row-column compression. Given an $m\times n$ matrix pencil $S-\lambda T$, the algorithm returns a compressed matrix pencil $S_1-\lambda T_1$.*

1. *Compute the singular value decomposition $T=U_1\Sigma_1V_1^*$.*
2. *Partition $U_1=[U_{1a}\ U_{1b}]$ such that U_{1a} has $r=\operatorname{rank}(S)$ columns.*
3. *If $r=m$ then exit and return $S_1=S$ and $T_1=T$ (no compression).*
4. *Compute the singular value decomposition $H=U_2\Sigma_2V_2^*$, where $H=U_{1b}^*S$.*
5. *Partition $V_2=[V_{2a}\ V_{2b}]$ such that V_{2a} has $c=\operatorname{rank}(H)$ columns.*
6. *Now we have*
$$U_1^*(S-\lambda T)V_2=\begin{bmatrix}\times & S_1\\ \times & 0\end{bmatrix}-\lambda\begin{bmatrix}\times & T_1\\ \times & 0\end{bmatrix},$$
where $S_1=U_{1a}^*SV_{2b}$ and $T_1=U_{1a}^*TV_{2b}$ are $r\times(n-c)$ matrices.

One step of Algorithm 13 returns the projected pencil $S_1-\lambda T_1$. The KCF of $S_1-\lambda T_1$ is related to the KCF of the initial pencil $S-\lambda T$ in the following way:

a) each L_k block of $S-\lambda T$ gives an L_k block of $S_1-\lambda T_1$,
b) each L_k^T block of $S-\lambda T$ gives an L_{k-1}^T block of $S_1-\lambda T_1$ for $k\ge 1$,
c) each N_k block of $S-\lambda T$ gives an N_{k-1} block of $S_1-\lambda T_1$ for $k\ge 2$,
d) blocks N_1 and L_0^T of $S-\lambda T$ vanish in $S_1-\lambda T_1$,
e) regular parts of both pencils are of the same size.

A dual form of Algoritm 13 is the column-row compression presented in Algorithm 14. It returns a compressed pencil $S_1-\lambda T_1$ with the KCF related to the KCF of $S-\lambda T$ in the same way as above, with the only exception that the roles of L_k and L_k^T blocks are exchanged.

Algorithm 14. *Column-row compression. Given an $m\times n$ matrix pencil $S-\lambda T$, the algorithm returns a compressed matrix pencil $S_1-\lambda T_1$.*

1. *Compute the singular value decomposition $T=U_1\Sigma_1V_1^*$.*
2. *Partition $V_1=[V_{1a}\ V_{1b}]$ such that V_{1a} has $r=\operatorname{rank}(S)$ columns.*
3. *If $r=n$ then exit and return $S_1=S$ and $T_1=T$ (no compression).*
4. *Compute the singular value decomposition $H=U_2\Sigma_2V_2^*$, where $H=SV_{1b}$.*
5. *Partition $U_2=[U_{2a}\ U_{2b}]$ such that U_{2a} has $c=\operatorname{rank}(H)$ columns.*

6. *Now we have*

$$U_2^*(S - \lambda T)V_1 = \begin{bmatrix} \times & \times \\ S_1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} \times & \times \\ T_1 & 0 \end{bmatrix},$$

where $S_1 = U_{2a}^* S V_{1b}$ and $T_1 = U_{2a}^* T V_{1b}$ are $(m - c) \times r$ matrices.

In order to extract the regular part from a singular pencil $\Delta_1 - \lambda\Delta_0$, we first apply as many row-column compressions as possible (Algorithm 13), followed by column-row compressions (Algorithm 14). We call this procedure the staircase algorithm. The final pencil contains only the regular part from which we can obtain the values λ as eigenvalues.

A vital part of the above procedure is a correct detection of ranks of submatrices in the algorithms for the column-row and row-column compression. If we know in advance the Kronecker structure of the initial singular pencil, which is the case if we use the approach from Section 7 for a generic pair of matrices A and B, then we can simplify the algorithm and use methods like rank revealing QR that are cheaper than the singular value decomposition.

9. Numerical results

The numerical results were obtained using Matlab R2007b, where the Kronecker structures were computed using the GUPTRI package [2, 3].

Example 15. We take matrices

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

and form a singular two-parameter eigenvalue problem (7), where we linearize the second equation as (8). The corresponding Δ-matrices are of size 27×27. The Kronecker structure of the pencil $\Delta_1 - \lambda\Delta_0$ consists of 3 blocks L_2, 3 blocks L_1^T, 9 blocks N_1 and a regular part of size 6.

The following table shows how the staircase algorithm from Section 7 extracts the regular part from the singular pencil. We start with row-column compressions and then, when they are not possible anymore, we switch to column-row compressions.

action	size	KCF
initial pencil	27×27	$3L_2,\ 3L_1^T,\ 9N_1,\ R(6)$
R-C compression	15×15	$3L_2,\ 3L_0^T,\ R(6)$
R-C compression	12×15	$3L_2,\ R(6)$
C-R compression	9×12	$3L_1,\ R(6)$
C-R compression	6×9	$3L_0,\ R(6)$
C-R compression	6×6	$R(6)$

In the end we obtain a 6×6 regular matrix pencil. Its eigenvalues are the six solutions of the initial problem. We write them together with the corresponding values μ, such that $A + \lambda B$ has a double eigenvalue μ.

λ	μ
1.5628	0.3651
-2.2078	-0.5000
$-1.1690 \pm 0.8436i$	$-2.9137 \pm 1.3289i$
$0.2735 \pm 0.0988i$	$2.3914 \pm 0.1452i$

Example 16. We take the matrix A from the previous example and $B = \mathrm{diag}(2, 2, 3) - A$. For $\lambda_0 = 1$ the matrix $A + \lambda_0 B = \mathrm{diag}(2, 2, 3)$ clearly has a semisimple double eigenvalue.

As before, if we take the singular two-parameter eigenvalue problem (7), then the corresponding Δ-matrices are of size 27×27. The KCF of the pencil $\Delta_1 - \lambda\Delta_0$ now consists of 1 block L_1, 2 blocks L_2, 1 block L_0^T, 2 blocks L_1^T, 9 blocks N_1, and a regular part of size 8. We can extract the regular part using the staircase algorithm. The following table shows the intermediate Kronecker structures in the staircase algorithm that extracts the regular part from the singular pencil.

action	size	KCF
initial pencil	27×27	$1L_1,\ 2L_2,\ 1L_0^T,\ 2L_1^T,\ 9N_1,\ R(8)$
R-C compression	15×16	$1L_1,\ 2L_2,\ 2L_0^T,\ R(8)$
R-C compression	13×16	$1L_1,\ 2L_2,\ R(8)$
C-R compression	10×13	$1L_0,\ 2L_1,\ R(8)$
C-R compression	8×10	$2L_0,\ R(8)$
C-R compression	8×8	$R(8)$

We obtain an 8×8 regular matrix pencil. Its eigenvalues are the solutions of the initial problem, where the solution $\lambda = 1$, which gives a semisimple double eigenvalue, appears with multiplicity four. The values λ with the corresponding eigenvalues μ are presented in the following table.

λ	μ
$0.9292 \pm 0.1987i$	$2.2723 \pm 0.6372i$
$0.6324 \pm 0.0558i$	$2.1585 \pm 0.0228i$
1.0000	2.0000

Example 17. We take the same pair of matrices as in the previous example, but this time we apply the singular system (5) from Section 6, which can detect only semisimple solutions. The corresponding Δ-matrices are of size 9×9. The staircase algorithm, applied to the pencil $\Delta_1 - \lambda\Delta_0$, extracts the regular part 2×2 in the following way.

action	size	KCF
initial pencil	9×9	$1L_0,\ 2L_1,\ 1L_0^T,\ 2L_1^T,\ R(2)$
R-C compression	6×7	$1L_0,\ 2L_1,\ 2L_0^T,\ R(2)$
R-C compression	4×7	$1L_0,\ 2L_1,\ R(2)$
C-R compression	2×4	$2L_0,\ R(2)$
C-R compression	2×2	$R(2)$

The final regular part has double eigenvalue 1. As explained in Section 6, this approach detects only values λ, such that $A + \lambda B$ has a semisimple double eigenvalue.

Example 18. Steps of the staircase algorithm rely on the right decision about the numerical rank. This is the reason why the algorithm is very sensitive and can in double precision be used in practice only for matrices not larger then 10×10. For larger matrices the gap between the significant singular values and those that should be zero virtually disappears and the obtained results have no meaning.

The following table shows the gaps between the smallest nonzero and the next singular value for random matrices of size 10×10 in the staircase algorithm applied to the matrix pencil $\Delta_1 - \lambda\Delta_0$ from Section 7. We repeated the computation in higher precision where the gaps are clearly visible. This shows that the algorithm should work in exact arithmetic, whereas in practice it is limited to small examples only.

matrix size	k	σ_k	σ_{k+1}	$\widetilde{\sigma}_k$	$\widetilde{\sigma}_{k+1}$
300×300	190	1.8e-01	1.4e-14	1.8e-01	2.0e-110
190×190	180	8.3e-03	2.0e-15	8.3e-03	9.8e-88
180×180	170	8.3e-03	3.2e-14	8.3e-03	1.6e-86
170×170	160	8.3e-03	5.7e-13	8.3e-03	5.2e-85
160×160	150	8.3e-03	1.3e-11	8.3e-03	1.1e-83
10×160	5	2.0e-01	2.6e-06	2.0e-01	3.8e-79
150×155	145	8.3e-01	4.9e-11	8.3e-01	3.0e-83
95×10	5	2.4e-01	1.0e-02	2.4e-01	4.0e-76

In the above table, k denotes the rank of the submatrix, σ_k and σ_{k+1} are singular values computed in double precision in Matlab, while $\widetilde{\sigma}_k$ and $\widetilde{\sigma}_{k+1}$ are singular values computed in virtual precision arithmetic in Matlab from initial matrices `vpa(A,80)` and `vpa(B,80)`.

10. Conclusions

We present a method that can find all values λ such that the matrix $A + \lambda B$ has a multiple eigenvalue and uses only standard numerical linear algebra tools. The method returns a nonsingular generalized eigenvalue problem such that its eigenvalues are the values λ that we are looking for. This is an elegant way to solve the double eigenvalue problem, but unfortunately, due to its sensitivity, it is numerically limited to matrices of small size only.

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