On the quadratic two-parameter eigenvalue problem and its linearization ⋆

Andrej Muhič [a,*], Bor Plestenjak [b]

[a] *Institute of mathematics, physics and mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia.*

[b] *Department of Mathematics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia.*

Abstract

We introduce the quadratic two-parameter eigenvalue problem and linearize it as a singular two-parameter eigenvalue problem. This, together with an example from model updating, shows the need for numerical methods for singular two-parameter eigenvalue problems and for a better understanding of such problems.

There are various numerical methods for two-parameter eigenvalue problems, but only few for nonsingular ones. We present a method that can be applied to singular two-parameter eigenvalue problems including the linearization of the quadratic two-parameter eigenvalue problem. It is based on the staircase algorithm for the extraction of the common regular part of two singular matrix pencils.

AMS classification: 65F15, 15A18, 15A69, 15A22

Key words: two-parameter quadratic eigenvalue problem, singular two-parameter eigenvalue problem, model updating, linearization, Kronecker canonical form

1 Introduction

We consider the *quadratic two-parameter eigenvalue problem*

⋆ Supported in part by the Ministry of Higher Education, Science and Technology of Slovenia and by the Research Agency of the Republic of Slovenia.

* Corresponding author.

Email addresses: `andrej.muhic@fmf.uni-lj.si` (Andrej Muhič), `bor.plestenjak@fmf.uni-lj.si` (Bor Plestenjak).

$$
\begin{aligned}
Q_1(\lambda,\mu)x_1 &:= (A_1+\lambda B_1+\mu C_1+\lambda^2 D_1+\lambda\mu E_1+\mu^2 F_1)x_1 = 0\\
Q_2(\lambda,\mu)x_2 &:= (A_2+\lambda B_2+\mu C_2+\lambda^2 D_2+\lambda\mu E_2+\mu^2 F_2)x_2 = 0,
\end{aligned}
\tag{1}
$$

where $A_i, B_i, \ldots, F_i$ are given $n_i \times n_i$ complex matrices, $x_i \in \mathbb{C}^{n_i}$ is a nonzero vector for $i = 1, 2$ and $\lambda, \mu \in \mathbb{C}$. We say that (λ, μ) is an eigenvalue of (1) and the tensor product $x_1 \otimes x_2$ is the corresponding eigenvector. In the generic case the problem (1) has $4n_1n_2$ eigenvalues that are roots of the system of the bivariate polynomials $q_i(\lambda, \mu) = \det(Q_i(\lambda, \mu)) = 0$ for $i = 1, 2$.

Recently, a simpler quadratic two-parameter eigenvalue problem, where some of the quadratic terms $\lambda^2, \lambda\mu, \mu^2$ are missing, appeared in the study of linear time-delay systems for the single delay case [9]. Due to the missing terms the problem in [9] has $2n_1n_2$ eigenvalues and is easier to solve. Here we study the general case (1) where all quadratic terms are present in both equations.

Similarly to the quadratic eigenvalue problem (see, e.g., [11]), where we can linearize the problem to a generalized eigenvalue problem with matrices of double dimension, we can write (1) as a two-parameter eigenvalue problem

$$
\begin{aligned}
L_1(\lambda,\mu)w_1 &:= \left(A^{(1)}+\lambda B^{(1)}+\mu C^{(1)}\right) w_1 = 0\\
L_2(\lambda,\mu)w_2 &:= \left(A^{(2)}+\lambda B^{(2)}+\mu C^{(2)}\right) w_2 = 0,
\end{aligned}
\tag{2}
$$

with matrices of larger dimension. We take

$$
L_i(\lambda,\mu)w_i := \left(\overbrace{\begin{bmatrix} A_i & B_i & C_i \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}}^{A^{(i)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_i & E_i \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{B^{(i)}} + \mu \overbrace{\begin{bmatrix} 0 & 0 & F_i \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}}^{C^{(i)}} \right) \overbrace{\begin{bmatrix} x_i \\ \lambda x_i \\ \mu x_i \end{bmatrix}}^{w_i} = 0, \tag{3}
$$

where the matrices $A^{(i)}, B^{(i)}$, and $C^{(i)}$ are of size $3n_i \times 3n_i$ for $i = 1, 2$. In Section 3 we show that $\det(L_i(\lambda, \mu)) = q_i(\lambda, \mu)$ for $i = 1, 2$ and therefore (2) is a linearization of (1). One can observe that although the matrices in $L_i(\lambda, \mu)$, $i = 1, 2$, are of size $3n_i \times 3n_i$, the order of $\det(L_i(\lambda, \mu))$ is only $2n_i$. This is due to the structure of the matrices $B^{(i)}$ and $C^{(i)}$ that are not of full rank.

The eigenvalues of (2) are defined in a similar way as the eigenvalues of (1). A pair (λ, μ) is an *eigenvalue* if $L_i(\lambda, \mu)w_i = 0$ for a nonzero vector w_i for $i = 1, 2$, and the tensor product $w_1 \otimes w_2$ is the corresponding (right) *eigenvector*. Similarly, $v_1 \otimes v_2$ is a *left eigenvector* if $v_i \neq 0$ and $v_i^* L_i(\lambda, \mu) = 0$ for $i = 1, 2$.

The usual approach for a two-parameter eigenvalue problem of the form (2) is to define the *operator determinants*

$$
\begin{aligned}
\Delta_0 &= B^{(1)} \otimes C^{(2)} - C^{(1)} \otimes B^{(2)}, \\
\Delta_1 &= C^{(1)} \otimes A^{(2)} - A^{(1)} \otimes C^{(2)}, \\
\Delta_2 &= A^{(1)} \otimes B^{(2)} - B^{(1)} \otimes A^{(2)}
\end{aligned} \tag{4}
$$

on the tensor product space $\mathbb{C}^{3n_1} \otimes \mathbb{C}^{3n_2}$ (see, e.q., [2]) and consider the *coupled generalized eigenvalue problem*

$$\Delta_1 z = \lambda \Delta_0 z \quad \text{and} \quad \Delta_2 z = \mu \Delta_0 z, \tag{5}$$

where $z = w_1 \otimes w_2$. In the generic case, Δ_0 is nonsingular and we say that (2) is a nonsingular two-parameter eigenvalue problem. In this case it follows (see, e.g., [2]) that the matrices $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute, and (2) has $9n_1n_2$ eigenvalues (λ, μ), which can be computed from eigenvalues of $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ using standards tools for the generalized eigenvalue problem. For some numerical algorithms see, e.g., [7,8].

In our case, where the matrices $A^{(i)}, B^{(i)}$, and $C^{(i)}$ arise from the linearization (3), Δ_0 is singular and (2) is a *singular two-parameter eigenvalue problem.* The singularity is an obstacle for the available numerical methods for two-parameter eigenvalue problems, but we present a method than can overcome this problem and thus enables us to solve the quadratic two-parameter eigenvalue problem (1) via the linearization (3).

In Section 2 we present some properties of singular two-parameter eigenvalue problems. For the particular case (3) we show in Section 3 that, under very mild conditions, the eigenvalues of (1) are exactly the regular eigenvalues of (5). In order to solve the quadratic two-parameter eigenvalue problem (1) using the linearization (3) we derive an algorithm for the extraction of the common regular part of two matrix pencils in Section 4. The algorithm, which is based on the staircase algorithm from [14], returns the $4n_1n_2 \times 4n_1n_2$ matrices $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ such that $\widetilde{\Delta}_0$ is nonsingular, the matrices $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ commute, and the eigenvalues of (1) are the eigenvalues of the matrix pencils $\widetilde{\Delta}_1 - \lambda\widetilde{\Delta}_0$ and $\widetilde{\Delta}_2 - \mu\widetilde{\Delta}_0$. In Section 5 we give some numerical examples. We show that the algorithm can be successfully applied to some other singular two-parameter eigenvalue problems, for example to the polynomial two-parameter eigenvalue problem and to the problems that appear in model updating [3]. Up to our knowledge, next to a very special case in [3], this is one of the first numerical methods for singular multiparameter eigenvalue problems.

2 Singular two-parameter eigenvalue problem

Let us consider a general two-parameter eigenvalue problem of the form (2). Multiparameter eigenvalue problems of this kind arise in a variety of applications [1], particularly in mathematical physics when the method of separation of variables is used to solve boundary value problems [16]. The theory for singular problems is scarce and there are no general results linking the eigenvalues of (2) to the eigenvalues of (5).

If Δ_0 is singular then there might still exist a nonsingular linear combination $\Delta = \alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2$. In such case (see [2]) the matrices $\Delta^{-1}\Delta_0$, $\Delta^{-1}\Delta_1$, and $\Delta^{-1}\Delta_2$ commute. If we consider the homogeneous problem

$$\begin{aligned}(\eta_0 A^{(1)} + \eta_1 B^{(1)} + \eta_2 C^{(1)})w_1 &= 0,\\ (\eta_0 A^{(2)} + \eta_1 B^{(2)} + \eta_2 C^{(2)})w_2 &= 0,\end{aligned} \tag{6}$$

where $(\eta_0, \eta_1, \eta_2) \neq (0,0,0)$, instead of (2), then we get η_0, η_1, and η_2 from the following three joined generalized eigenvalue problems $\Delta_0 z = \eta_0\Delta z$, $\Delta_1 z = \eta_1\Delta z$, and $\Delta_2 z = \eta_2\Delta z$. An eigenvalue of (6) with $\eta_0 \neq 0$ gives a finite eigenvalue $(\lambda, \mu) = (\eta_1/\eta_0, \eta_2/\eta_0)$ of (2), while the eigenvalues with $\eta_0 = 0$ are infinite eigenvalues of (2). If $\alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2$ is singular for all values of α_0, α_1, and α_2, i.e., $\det(\alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2) \equiv 0$, then also the homogeneous version of (2) is singular.

Theorem 1 ([2, Theorem 8.7.1]) *The following two statements for the homogeneous problem (6) are equivalent:*

(1) The matrix $\Delta = \sum_{i=0}^{2} \alpha_i\Delta_i$ is singular.
(2) There exist an eigenvalue (η_0, η_1, η_2) of (6) such that $\sum_{i=0}^{2} \eta_i\alpha_i = 0$.

It follows from Theorem 1 that when Δ_0 is singular and the polynomials $\det(L_1(\lambda, \mu))$ and $\det(L_2(\lambda, \mu))$ do not have a common factor, then the two-parameter eigenvalue problem (2) has less than $9n_1n_2$ finite eigenvalues and at least one infinite eigenvalue.

Another example of a singular two-parameter eigenvalue problem that appears in model updating is presented in the following example.

Example 2 *In model updating [3] one wants to adjust the matrices obtained from the finite element model so that some of the eigenfrequencies of the model match the measured eigenfrequencies. In a matrix formulation we can write the problem for the two frequencies as follows.*

Given $n \times n$ matrices A, B, C and two prescribed eigenvalues $\xi_1 \neq \xi_2$, find values of λ and μ such that two of the eigenvalues of the matrix $A + \lambda B + \mu C$ are equal to ξ_1 and ξ_2. The problem can be expressed as the following two-parameter eigenvalue problem

$$\begin{aligned}(A-\xi_1 I)x_1+\lambda Bx_2+\mu Cx_1&=0,\\(A-\xi_2 I)x_2+\lambda Bx_2+\mu Cx_2&=0,\end{aligned}\tag{7}$$

which can be shown to be singular.

3 Quadratic two-parameter eigenvalue problem

Let us take a closer look at the general quadratic two-parameter eigenvalue problem (1). From now on we will assume that $n_1 = n_2 = n$. By inspecting the Kronecker canonical structure of the matrix pencils (5) we will show that we get exactly $4n^2$ regular eigenvalues in the generic case.

Definition 3 *An $ln \times ln$ linear matrix pencil $L(\lambda,\mu) = A + \lambda B + \mu C$ is a linearization (see, e.g., [10]) (of order ln) of a matrix polynomial $Q(\lambda,\mu)$ if there exist matrix polynomials $P(\lambda,\mu)$ and $R(\lambda,\mu)$, whose determinant is a constant independent of λ and μ, such that*

$$\begin{bmatrix} Q(\lambda,\mu) & 0 \\ 0 & I_{l(n-1)} \end{bmatrix} = P(\lambda,\mu)L(\lambda,\mu)R(\lambda,\mu).$$

In our case,

$$P_i(\lambda,\mu)L_i(\lambda,\mu)R_i(\lambda,\mu) = \begin{bmatrix} Q_i(\lambda,\mu) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

where

$$P_i(\lambda,\mu) = \begin{bmatrix} I & B_i+\lambda D_i & C_i+\lambda E_i+\mu F_i \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad R_i(\lambda,\mu) = \begin{bmatrix} I & 0 & 0 \\ \lambda I & I & 0 \\ \mu I & 0 & I \end{bmatrix}.$$

This shows that (3) is a linearization of (1). In Appendix we show that we can linearize an arbitrary polynomial two-parameter eigenvalue problem into a two-parameter eigenvalue problem with matrices of higher dimension.

The linearization (3) is not optimal. Namely, it follows from the theory on determinantal representations [13] that there do exist matrices $A^{(i)}, B^{(i)}$, and $C^{(i)}$ of dimension $2n \times 2n$ such that $\det(L_i(\lambda,\mu)) = \det(Q_i(\lambda,\mu))$ for $i = 1, 2$. An appropriate pair of determinantal representations would result in a smaller and, more important, nonsingular two-parameter eigenvalue problem, but, unfortunately, there are no algorithms for the construction of such matrices.

In order to simplify the proofs of the next two lemmas, we introduce the Tracy–Singh product of partitioned matrices [12].

Definition 4 *Let an $m \times n$ matrix A be partitioned into the $m_i \times n_j$ blocks A_{ij} and a $p \times q$ matrix B into the $p_k \times q_l$ blocks B_{kl} such that $m = \sum_{i=1}^{r} m_i$, $n = \sum_{j=1}^{s} n_j$, $p = \sum_{k=1}^{t} p_k$, $q = \sum_{l=1}^{u} q_l$. The Tracy–Singh product $A \circledast B$ is a $mp \times nq$ matrix, defined as*

$$A \circledast B = (A_{ij} \circledast B)_{ij} = ((A_{ij} \otimes B_{kl})_{kl})_{ij},$$

where the (ij)th block of the product is the $m_i p \times n_j q$ matrix $A_{ij} \circledast B$, of which the (kl)th subblock equals the $m_i p_k \times n_j q_l$ matrix $A_{ij} \otimes B_{kl}$.

Theorem 5 ([12, Theorem 5]) *In the case of balanced partitioning, where all blocks in matrix A and B are of the same size, respectively, the Tracy–Singh product $A \circledast B$ and the Kronecker product $A \otimes B$ are permutation equivalent.*

All our block matrices have balanced partition and some properties are easier to be obtained when we work with the Tracy-Singh product instead of the Kronecker product. Since this is just a reordering of columns and rows, we will denote by TS the map that reorders the elements of $A \otimes B$ so that $TS(A \otimes B) = A \circledast B$.

Lemma 6 *In the generic case, the $9n^2 \times 9n^2$ matrix Δ_0 in (5) has rank $6n^2$.*

Proof. If we apply the Tracy–Singh reordering to Δ_0, then we obtain

$$TS(\Delta_0) = \begin{array}{c c} & \begin{array}{cc} \scriptstyle 3n^2 & \scriptstyle 6n^2 \end{array} \\ \begin{array}{c} \scriptstyle 3n^2 \\ \scriptstyle 6n^2 \end{array} & \begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix} \end{array},$$

where

$$T = \left[\begin{array}{c|c|c} 0 & 0 & I \otimes F_2 \\ \hline 0 & 0 & 0 \\ \hline I \otimes I & 0 & 0 \\ \hline 0 & -I \otimes D_2 & -I \otimes E_2 \\ \hline -I \otimes I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

and

$$S = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & D_1 \otimes F_2 & 0 & -F_1 \otimes D_2 & E_1 \otimes F_2 - F_1 \otimes E_2 \\ \hline 0 & 0 & 0 & -F_1 \otimes I & 0 & 0 \\ \hline D_1 \otimes I & 0 & 0 & E_1 \otimes I & 0 & 0 \end{array}\right]$$

From the above block representations of S and T it is easy to see, under the general assumption that matrices D_1, F_1, D_2, and F_2 are all nonsingular, that each of the matrices S and T is of rank $3n^2$. It follows that in the generic case the rank of Δ_0 is indeed $6n^2$. $\square$

Lemma 7 *In the generic case, the* $9n^2 \times 9n^2$ *matrices* Δ_1 *and* Δ_2 *in (5) are of rank* $8n^2$.

Proof. Let us consider a related problem, where $W_1'(\lambda, \mu) = A_1 + \mu C_1 + \mu^2 F_1$ and $W_2'(\lambda, \mu) = W_2(\lambda, \mu)$. We linearize W_1' by the $2n \times 2n$ matrix pencil

$$L_1'(\lambda, \mu) = \overbrace{\begin{bmatrix} A_1 & C_1 \\ 0 & -I \end{bmatrix}}^{A'^{(1)}} + \mu \overbrace{\begin{bmatrix} 0 & F_1 \\ I & 0 \end{bmatrix}}^{C'^{(1)}}$$

and W_2' as in (3). The related problem has an eigenvalue of the form $(0, \mu)$ if and only if this is true for the original problem (1). Let us show that the $6n^2 \times 6n^2$ matrix

$$\Delta_1' = C'^{(1)} \otimes A^{(2)} - A'^{(1)} \otimes C^{(2)}$$

from the coupled generalized eigenvalue problem (of the form (5)) of the related problem is nonsingular.

Suppose that Δ_1' is singular. Then, by Theorem 1, the homogeneous version of the linearization of the related problem has an eigenvalue $(\eta_0, 0, \eta_2)$ such that $(\eta_0, \eta_2) \neq (0, 0)$. In the generic case, the matrix $C'^{(1)}$ is nonsingular, so $\eta_0 \neq 0$. This implies that the original problem (1) has an eigenvalue of the form $(0, \mu)$. Since this is not true in the generic case, Δ_1' has to be nonsingular.

The block structure of $TS(\Delta_1)$ is

$$TS(\Delta_1) = \begin{array}{c} \\ 3n^2 \\ 3n^2 \\ 3n^2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 3n^2 & 3n^2 & 3n^2 \end{array} \\ \begin{bmatrix} \times & \times & \times \\ 0 & Z & 0 \\ \times & 0 & \times \end{bmatrix} \end{array}, \quad \text{where} \quad Z = \begin{array}{c} \\ n^2 \\ n^2 \\ n^2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} n^2 & n^2 & n^2 \end{array} \\ \begin{bmatrix} 0 & 0 & I \otimes F_2 \\ 0 & 0 & 0 \\ I \otimes I & 0 & 0 \end{bmatrix} \end{array}.$$

The four corner blocks of $TS(\Delta_1)$ represent the nonsingular matrix $TS(\Delta_1')$ of the related problem. The central $3n^2 \times 3n^2$ block Z of $TS(\Delta_1)$ is of maximal rank $2n^2$ in the generic case, where we assume that matrix F_2 is nonsingular. It follows that the matrix Δ_1 is of rank $8n^2$.

Similarly we can show that if the problem (1) does not have an eigenvalue with $\mu = 0$ and if the matrix $[D_2 \; E_2]$ is of full rank, which is true in the generic case, then the matrix Δ_2 has rank $8n^2$. $\square$

Lemma 8 *In the generic case, where we assume that the matrices D_1, D_2, F_1, and F_2 are nonsingular, we can construct bases for the kernels of Δ_0, Δ_1, and Δ_2 in (5) as follows:*

(1) A basis for $\mathrm{Ker}(\Delta_1)$ *consists of the vectors*

$$\begin{bmatrix} 0 & e_i^T & 0 \end{bmatrix}^T \otimes \begin{bmatrix} 0 & e_j^T & 0 \end{bmatrix}^T, \quad i, j = 1, \dots, n.$$

(2) A basis for $\mathrm{Ker}(\Delta_2)$ *consists of the vectors*

$$\begin{bmatrix} 0 \\ D_1^{-1} E_1 e_i \\ -e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ D_2^{-1} E_2 e_j \\ -e_j \end{bmatrix}, \quad i, j = 1, \dots, n.$$

(3) The kernels of Δ_1 and Δ_2 are included in the kernel of Δ_0. A basis for $\mathrm{Ker}(\Delta_0)$ *consists of the vectors in (1) and (2), and the vectors*

$$\begin{bmatrix} 0 \\ D_1^{-1}(E_1 - F_1) e_i \\ -e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ D_2^{-1}(E_2 - F_2) e_j \\ -e_j \end{bmatrix}, \quad i, j = 1, \dots, n.$$

Proof. One can confirm the lemma by a direct computation. $\square$

In a similar way we can find bases for the kernels of Δ_0^*, Δ_1^*, and Δ_2^*.

Lemma 9 *A basis for* $\mathrm{Ker}(\Delta_0^*)$ *in (5) is*

$$\overbrace{\begin{bmatrix} 0 \\ e_i \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ e_j \\ 0 \end{bmatrix}}^{\mathrm{Ker}(\Delta_1^*)}, \quad \overbrace{\begin{bmatrix} 0 \\ 0 \\ e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ e_j \end{bmatrix}}^{\mathrm{Ker}(\Delta_2^*)}, \quad \begin{bmatrix} 0 \\ e_i \\ e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ e_j \\ e_j \end{bmatrix}, \quad i, j = 1, \dots, n.$$

Proof. It is easy to see that the above vectors are indeed in the subspaces $\mathrm{Ker}(\Delta_1^*)$, $\mathrm{Ker}(\Delta_2^*)$, and $\mathrm{Ker}(\Delta_0^*)$, respectively. From Lemmas 6 and 7 it follows that these vectors form bases for the mentioned kernels. $\square$

Lemma 10 *The matrices Δ_1^* and Δ_2^* in (5) act on $\mathrm{Ker}(\Delta_0^*)$ as*

$$\Delta_1^* \begin{bmatrix}0\\x\\x\end{bmatrix} \otimes \begin{bmatrix}0\\y\\y\end{bmatrix} = -\Delta_2^* \begin{bmatrix}0\\x\\x\end{bmatrix} \otimes \begin{bmatrix}0\\y\\y\end{bmatrix} = \begin{bmatrix}0\\x\\x\end{bmatrix} \otimes \begin{bmatrix}y\\0\\0\end{bmatrix} - \begin{bmatrix}x\\0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\y\\y\end{bmatrix},$$

$$\Delta_1^* \begin{bmatrix}0\\0\\x\end{bmatrix} \otimes \begin{bmatrix}0\\0\\y\end{bmatrix} = \begin{bmatrix}0\\0\\x\end{bmatrix} \otimes \begin{bmatrix}y\\0\\0\end{bmatrix} - \begin{bmatrix}x\\0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\\y\end{bmatrix},$$

$$\Delta_2^* \begin{bmatrix}0\\x\\0\end{bmatrix} \otimes \begin{bmatrix}0\\y\\0\end{bmatrix} = \begin{bmatrix}x\\0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\y\\0\end{bmatrix} - \begin{bmatrix}0\\x\\0\end{bmatrix} \otimes \begin{bmatrix}y\\0\\0\end{bmatrix}.$$

The images of Δ_1^* and Δ_2^* restricted to $\mathrm{Ker}(\Delta_0^*)$ coincide.

Using the above straightforward lemma one can easily check that for each triple $(\alpha_0, \alpha_1, \alpha_2) \neq (0,0,0)$ there exist a triple $(a,b,c) \neq (0,0,0)$ such that

$$(\alpha_0\Delta_0^* + \alpha_1\Delta_1^* + \alpha_2\Delta_2^*) \left(a \begin{bmatrix}0\\x\\x\end{bmatrix} \otimes \begin{bmatrix}0\\y\\y\end{bmatrix} + b \begin{bmatrix}0\\x\\0\end{bmatrix} \otimes \begin{bmatrix}0\\y\\0\end{bmatrix} + c \begin{bmatrix}0\\0\\x\end{bmatrix} \otimes \begin{bmatrix}0\\0\\y\end{bmatrix} \right) = 0$$

for arbitrary nonzero vectors x and y. One solution is $a = \alpha_1\alpha_2$, $b = \alpha_1^2 - \alpha_1\alpha_2$, and $c = \alpha_2^2 - \alpha_1\alpha_2$. The problem is thus singular even if we study it in the homogeneous form (6).

In order to show that the eigenvalues of (1) agree with the finite regular eigenvalues of (5), we introduce the Kronecker canonical form, for more details see, e.g., [6,14].

Definition 11 *Let $A - \lambda B \in \mathbb{C}^{m\times n}$ be a matrix pencil. There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}(A - \lambda B)Q = \widetilde{A} - \lambda\widetilde{B} = \mathrm{diag}(A_1 - \lambda B_1, \ldots, A_k - \lambda B_k)$$

is the Kronecker canonical form. Each block $A_i - \lambda B_i$, $i = 1, \dots, k$, must be of one of the following forms:

$$J_j(\alpha) = \begin{bmatrix} \alpha-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha-\lambda \end{bmatrix} \in \mathbb{C}^{j\times j}, \quad N_j = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{j\times j},$$

$$L_j = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix} \in \mathbb{C}^{j\times(j+1)}, \quad L_j^T = \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix} \in \mathbb{C}^{(j+1)\times j},$$

that represent finite regular, infinite regular, right singular, and left singular blocks, respectively.

Definition 12 *The normal rank of the square matrix pencil $A - \lambda B$ is $n_r = \max_{s\in\mathbb{C}} \operatorname{rank}(A - sB)$. We say that $\lambda \in \mathbb{C}$ is a finite regular eigenvalue of the matrix pencil if $\operatorname{rank}(A - \lambda B) < n_r$.*

Definition 13 *A pair $(\lambda, \mu) \in \mathbb{C}^2$ is a finite regular eigenvalue of the matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if all of the following statements are true:*

(1) λ is a finite regular eigenvalue of $\Delta_1 - \lambda\Delta_0$,
(2) μ is a finite regular eigenvalue of $\Delta_2 - \mu\Delta_0$,
(3) there exists a common eigenvector z in the intersection of the regular parts of the pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ such that

$$(\Delta_1 - \lambda\Delta_0)z = 0 \quad \text{and} \quad (\Delta_2 - \mu\Delta_0)z = 0.$$

It follows from the linearization that all eigenvalues of the initial quadratic two-parameter eigenvalue problem (1) are finite eigenvalues of the linearized two-parameter eigenvalue problem (3). Next we show that all eigenvalues of (1) are finite regular eigenvalues of (5). The equivalence of both sets of eigenvalues is established in Theorem 17 below.

Lemma 14 *The eigenvalues of the quadratic two-parameter eigenvalue problem (1) are finite regular eigenvalues of the matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ in (5).*

Proof. It follows from Lemmas 6, 7, and 8 that the normal rank of the $9n^2 \times 9n^2$ pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ is exactly $8n^2$.

Let a vector of the form $[x_1^T\ \lambda x_1^T\ \mu x_1^T]^T \otimes [x_2^T\ \lambda x_2^T\ \mu x_2^T]^T$ be an eigenvector for the eigenvalue (λ, μ) that we get from the linearization. The first block components x_1 and x_2 of such vector are both nonzero. All vectors in the kernels of Δ_1 and Δ_2 have their first block component zero, so we have $\mathrm{rank}(\Delta_1 - \lambda\Delta_0) < 8n^2$ and $\mathrm{rank}(\Delta_2 - \mu\Delta_0) < 8n^2$. $\square$

Now we have enough information to determine the Kronecker canonical structure of the matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$.

Lemma 15 *The $9n^2 \times 9n^2$ pencil $\Delta_1^* - \lambda\Delta_0^*$ in (5) has at least $2n^2$ first root vectors for the infinite eigenvalues. The same is true for the pencil $\Delta_2^* - \mu\Delta_0^*$.*

Proof. The first root vector for an infinite eigenvalue is vector z_1 in the chain $\Delta_0^* z_0 = 0$, $\Delta_1^* z_0 = \Delta_0^* z_1$ such that $\Delta_1^* z_1 \neq 0$. We have to show that we can find $2n^2$ such linearly independent vectors.

From Lemma 10 we see that all vectors in $\mathrm{Ker}(\Delta_0)$, which are of the form $[0\ \times\ \times]^T \otimes [0\ \times\ \times]^T$ by Lemma 8, are obviously orthogonal to $\Delta_1^* \mathrm{Ker}(\Delta_0^*)$. As the whole space is an orthogonal sum of $\mathrm{Im}(\Delta_0^*)$ and $\mathrm{Ker}(\Delta_0)$, it follows that $\Delta_1^* \mathrm{Ker}(\Delta_0^*)$ is a subspace of $\mathrm{Im}(\Delta_0^*)$. So, there exist $2n^2$ linearly independent vectors z_1 such that $\Delta_0^* z_1$ is in $\Delta_1^* \mathrm{Ker}(\Delta_0^*)$. $\square$

Lemma 16 *The Kronecker canonical form of the $9n^2 \times 9n^2$ pencil $\Delta_1 - \lambda\Delta_0$ from (5) has n^2 L_0, n^2 L_0^T, $2n^2$ N_2 blocks, and the finite regular part of size $4n^2$. The same is true for the pencil $\Delta_2 - \mu\Delta_0$.*

Proof. The regular Kronecker canonical structure of the transposed pencil $\Delta_1^* - \lambda\Delta_0^*$ is the same as of $\Delta_1 - \lambda\Delta_0$. The right (left) singular structure of $\Delta_1^* - \lambda\Delta_0^*$ is the left (right) singular structure of $\Delta_1 - \lambda\Delta_0$. The pencil $\Delta_1 - \lambda\Delta_0$ has a regular part of size at least $4n^2$ by Lemma 14. The number of L_0 and L_0^T blocks is n^2 by Lemmas 7, 8, and 9. In addition, it follows from Lemma 15 that the pencil has $2n^2$ N_2 blocks. Thus we have completely determined the Kronecker canonical structure of $\Delta_1 - \lambda\Delta_0$. $\square$

Theorem 17 *The eigenvalues of the quadratic two-parameter eigenvalue problem (1) are exactly the finite regular eigenvalues of the coupled generalized eigenvalue problem (5).*

Proof. We know that (1) has $4n^2$ eigenvalues, which are also finite regular eigenvalues of the linearized two-parameter eigenvalue problem (2), and we have proved in Lemma 14 that all eigenvalues of (2) are finite regular eigenvalues of (5). By Lemma 16, it follows that (5) can not have more than $4n^2$ finite regular eigenvalues, and thus, the sets of eigenvalues must be equal. □

In the next section we describe the algorithm that computes the common regular part of two matrix pencils. It follows from Theorem 17 that such an algorithm can solve the quadratic two-parameter eigenvalue problem linearized as a singular two-parameter eigenvalue problem.

4 Extraction of the common regular subspace of two singular matrix pencils

We would like to recover the finite regular eigenvalues of the matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$. Here we are not interested in the infinite part.

Instead of the Kronecker canonical form we will use the generalized upper-triangular form, where the transformation matrices P and Q are unitary, see, e.g., [4,14]. For the matrix pencil $A - \lambda B$ there exist unitary matrices P and Q, partitioned as $P = [P_1\ P_2]$ and $Q = [Q_1\ Q_2]$, such that

$$P^*(A - \lambda B)Q = \left[\begin{array}{cc|cc} A_\mu - \lambda B_\mu & & & \\ \times & A_\infty - \lambda B_\infty & & \\ \hline \times & \times & A_f - \lambda B_f & \\ \times & \times & \times & A_\epsilon - \lambda B_\epsilon \end{array}\right]. \tag{8}$$

The pencils $A_\mu - \lambda B_\mu$, $A_\infty - \lambda B_\infty$, $A_f - \lambda B_f$, and $A_\epsilon - \lambda B_\epsilon$ contain the left singular, the infinite regular, the finite regular, and the right singular structure, respectively. The most simple case of a right singular structure is when $\mathrm{Ker}(A) \cap \mathrm{Ker}(B)$ is nontrivial. The eigenvectors of such matrix pencil are then not well defined.

We are particularly interested in the lower right block of (8), where we find the finite regular and the right singular structure. We say that $\mathcal{P}$ and $\mathcal{Q}$ form a pair of left and right reducing subspaces [15] of $A - \lambda B$, respectively, if $\mathcal{P} = A\mathcal{Q} + B\mathcal{Q}$ and $\dim(\mathcal{P}) = \dim(\mathcal{Q}) - n_s$, where n_s is the number of right singular blocks in the Kronecker canonical form of $A - \lambda B$.

Below we provide a sketch of the algorithm that computes the lower right block of (8) and the matrices P_2 and Q_2.

Algorithm 1 *Given an $m \times n$ matrix pencil $A - \lambda B$, the algorithm returns matrices A_1, B_1, P, and Q, where P and Q have orthonormal columns, such that the columns of P and Q form a basis for a pair of left and right reducing subspaces of $A - \lambda B$ and $A_1 - \lambda B_1 = P^*(A - \lambda B)Q$ is equivalent to the lower right block of (8), which contains the finite regular and the right singular structure of the matrix pencil $A - \lambda B$.*

$A_1 = A$, $B_1 = B$, $P = I_m$, $Q = I_n$, $j = 1$.
Repeat,

(1) (a) Compute the singular value decomposition $U_0\Sigma_0V_0^*$ of the $m_j \times n_j$ matrix B_1. Let $r_j = \operatorname{rank}(B_1)$ and partition $U_0 = \begin{array}{cc} & \begin{array}{cc} r_j & m_j - r_j \end{array} \\ m_j & \left[\begin{array}{cc} U_{0a} & U_{0b} \end{array}\right] \end{array}$.

(b) If $r_j = m_j$ then exit and return A_1, B_1, P, and Q.

(2) (a) Compute the $(m_j - r_j) \times n_j$ matrix $H = U_{0b}^*A_1$.

(b) Compute the singular value decomposition $H = U_1\Sigma_1V_1^*$. Let $c_j = \operatorname{rank}(H)$ and partition $V_1 = \begin{array}{cc} & \begin{array}{cc} c_j & n_j - c_j \end{array} \\ n_j & \left[\begin{array}{cc} V_{1a} & V_{1b} \end{array}\right] \end{array}$.

(3) Now we have

$$U_0^*(A_1 - \lambda B_1)V_1 = \begin{array}{cc} & \begin{array}{cc} c_j & n_j - c_j \end{array} \\ \begin{array}{c} r_j \\ m_j - r_j \end{array} & \left[\begin{array}{cc} \times & \widehat{A}_1 \\ \times & 0 \end{array}\right] \end{array} - \lambda \begin{array}{cc} & \begin{array}{cc} c_j & n_j - c_j \end{array} \\ \begin{array}{c} r_j \\ m_j - r_j \end{array} & \left[\begin{array}{cc} \times & \widehat{B}_1 \\ 0 & 0 \end{array}\right] \end{array}.$$

(4) Set $A_1 = \widehat{A}_1$, $B_1 = \widehat{B}_1$, $P = PU_{0a}$, $Q = QV_{1b}$, $j = j + 1$, and go to (1).

Algorithm 1, which is based on Algorithm 4.1 from [14], starts with the $m \times n$ matrices A and B. It reduces them using consequent row and column compressions, until B_1 has full row rank. For the reduction we use the singular value decomposition. For additional details, see [14].

Algorithm 1 has a dual form, which is based on Algorithm 4.5 from [14], where column and row compressions are interchanged. The dual algorithm, presented in Algorithm 2, computes a pencil representing the finite regular structure together with the left singular structure of the matrix pencil $A - \lambda B$.

Algorithm 2 *Given an $m \times n$ matrix pencil $A - \lambda B$, the algorithm returns matrices A_1, B_1, P, and Q, where P and Q have orthonormal columns, such that the columns of Q and P form a basis for a pair of left and right reducing subspaces of the matrix pencil $A^* - \lambda B^*$, and $A_1 - \lambda B_1 = P^*(A - \lambda B)Q$ contains the finite regular and the left singular structure of the matrix pencil $A - \lambda B$.*

$A_1 = A,\ B_1 = B,\ P = I_m,\ Q = I_n,\ j = 1.$
Repeat,

(1) (a) *Compute the singular value decomposition* $U_0 \Sigma_0 V_0^*$ *of the* $m_j \times n_j$ *matrix* B_1. *Let* $c_j = \operatorname{rank}(B_1)$ *and partition* $V_0 = \begin{array}{cc} & \begin{array}{cc} c_j & n_j - c_j \end{array} \\ n_j & \begin{bmatrix} V_{0a} & V_{0b} \end{bmatrix} \end{array}$.
(b) *If* $c_j = n_j$ *then exit and return* $A_1,\ B_1,\ P,$ *and* Q.

(2) (a) *Compute the* $m_j \times (n_j - c_j)$ *matrix* $H = A_1 V_{0b}$.
(b) *Compute the singular value decomposition* $H = U_1 \Sigma_1 V_1^*$. *Let* $r_j = \operatorname{rank}(H)$ *and partition* $U_1 = \begin{array}{cc} & \begin{array}{cc} r_j & m_j - r_j \end{array} \\ m_j & \begin{bmatrix} U_{1a} & U_{1b} \end{bmatrix} \end{array}$.

(3) *Now we have*

$$U_1^*(A_1 - \lambda B_1) V_0 = \begin{array}{cc} & \begin{array}{cc} c_j & n_j - c_j \end{array} \\ \begin{array}{c} r_j \\ m_j - r_j \end{array} & \begin{bmatrix} \times & \times \\ \widehat{A}_1 & 0 \end{bmatrix} \end{array} - \lambda \begin{array}{cc} & \begin{array}{cc} c_j & n_j - c_j \end{array} \\ \begin{array}{c} r_j \\ m_j - r_j \end{array} & \begin{bmatrix} \times & 0 \\ \widehat{B}_1 & 0 \end{bmatrix} \end{array}.$$

(4) *Set* $A_1 = \widehat{A}_1,\ B_1 = \widehat{B}_1,\ P = PU_{1a},\ Q = QV_{0b},\ j = j + 1,$ *and go to* (1).

We apply these two algorithms to compute the common regular structure of the matrix pencils $\Delta_1 - \lambda \Delta_0$ and $\Delta_2 - \mu \Delta_0$. From now on we will use the notation that the vector space spanned by the columns of a matrix A is denoted by $\mathcal{A}$.

Algorithm 3 *Given* $m \times n$ *matrix pencils* $\Delta_1 - \lambda \Delta_0$ *and* $\Delta_2 - \mu \Delta_0$, *the algorithm returns matrices* P *and* Q *with orthonormal columns, such that the matrix pencils* $P^* \Delta_1 Q - \lambda P^* \Delta_0 Q$ *and* $P^* \Delta_2 Q - \mu P^* \Delta_0 Q$ *contain the common regular part of the initial pencils. The columns of* Q *form a basis for the common finite regular subspace.*

$P = I_m,\ Q = I_n.$

(1) *Separation of the finite part from the infinite part.*
(a) *Apply Algorithm 1 to the pencils* $P^* \Delta_1 Q - \lambda P^* \Delta_0 Q$ *and* $P^* \Delta_2 Q - \mu P^* \Delta_0 Q$. *We get* P_1, Q_1 *and* P_2, Q_2, *respectively.*
(b) *Compute matrices* Q_3 *and* P_3 *with orthonormal columns such that* $\mathcal{Q}_3 = \mathcal{Q}_1 \cap \mathcal{Q}_2$ *and* $\mathcal{P}_3 = \mathcal{P}_1 + \mathcal{P}_2$. *Update* $Q = QQ_3,\ P = PP_3$.
(c) *If* Q_3 *is a square matrix, then go to (2.a). Otherwise, go to (1.a).*

(2) *Separation of the finite regular part from the right singular part.*
(a) *Apply Algorithm 2 to the pencils* $P^* \Delta_1 Q - \lambda P^* \Delta_0 Q$ *and* $P^* \Delta_2 Q - \mu P^* \Delta_0 Q$. *We get* P_1, Q_1 *and* P_2, Q_2, *respectively.*
(b) *Compute matrices* Q_3 *and* P_3 *with orthonormal columns such that*

$\mathcal{Q}_3 = \mathcal{Q}_1 + \mathcal{Q}_2$ and $\mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_2$. Update $Q = QQ_3$, $P = PP_3$.

(c) If Q_3 is a square matrix, then return P, Q and exit. Otherwise, go to (2.a).

In the first phase of Algorithm 3 we compute the common finite regular and right singular structure of $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ using Algorithm 1. We start with $P = I_m$ and $Q = I_n$. In step (1a) we separately compute the basis for the common finite regular and right singular subspace for each of the deflated pencils $P^*\Delta_1 Q - \lambda P^*\Delta_0 Q$ and $P^*\Delta_2 Q - \mu P^*\Delta_0 Q$. If the subspaces do not agree we compute their intersection in step (1b) and repeat the process with the updated P and Q. At the end of the first phase the matrix $P^*\Delta_0 Q$ has full row rank. In a similar way, in the second phase of the algorithm we separate the finite regular and the right singular structure using Algorithm 2.

Algorithm 3 terminates in a finite number of steps. In the first phase the row rank of $P^*\Delta_0 Q$ and the number of columns in Q decrease until $P^*\Delta_0 Q$ has full row rank. In the second phase the column rank of $P^*\Delta_0 Q$ and the number of columns in P decrease until $P^*\Delta_0 Q$ has full column rank. In the end we get square matrices $P^*\Delta_i Q$ for $i = 0, 1, 2$, where $P^*\Delta_0 Q$ is nonsingular.

The above algorithm has a dual form. We can start with Algorithm 2 in the first phase and use Algorithm 1 in the second phase, but then we have to compute Q_3 as an orthogonal basis for $\mathcal{Q}_1 + \mathcal{Q}_2$ and P_3 as an orthogonal basis for $\mathcal{P}_1 \cap \mathcal{P}_2$ in the first step. In the second step we compute Q_3 as an orthogonal basis for $\mathcal{Q}_1 \cap \mathcal{Q}_2$ and P_3 as an orthogonal basis for $\mathcal{P}_1 + \mathcal{P}_2$.

Theorem 18 *Let all eigenvalues of the quadratic two-parameter eigenvalue problem (1) be semisimple. If we linearize (1) as the two-parameter eigenvalue problem (3) and apply Algorithm 3 to the coupled generalized eigenvalue problem (5), then we get matrices P and Q with orthonormal columns that define the $4n^2 \times 4n^2$ matrices $\widetilde{\Delta}_i = P^*\Delta_i Q$ for $i = 0, 1, 2$ such that $\widetilde{\Delta}_0$ is nonsingular and the matrices $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ commute.*

Proof. Since all eigenvalues are semisimple, the problem (1) has $4n^2$ linearly independent eigenvectors $x_{1k} \otimes x_{2k}$ with the corresponding eigenvalues (λ_k, μ_k) for $k = 1, \ldots, 4n^2$. Then $w_k := [x_{1k}^T \ \lambda_k x_{1k}^T \ \mu_k x_{1k}^T]^T \otimes [x_{2k}^T \ \lambda_k x_{2k}^T \ \mu_k x_{2k}^T]^T$ for $k = 1, \ldots, 4n^2$ are the eigenvectors of (2).

From Lemmas 8, 9, and 16 we can deduce that Algorithm 3 returns the matrix Q such that $\mathrm{Im}(Q) = \mathrm{Lin}(w_1, \ldots, w_{4n^2})$. From Theorem 17 we know that for each $k = 1, \ldots, 4n^2$ there exists a nonzero vector $z_k \in \mathbb{C}^{4n^2}$ such that $\widetilde{\Delta}_1 z_k = \lambda_k \widetilde{\Delta}_0 z_k$ and $\widetilde{\Delta}_2 z_k = \mu_k \widetilde{\Delta}_0 z_k$. The linearly independent vectors $z_1, \ldots, z_{4n^2}$ form a complete common set of eigenvectors for the matrices $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$, which therefore commute. □

It follows from Theorem 18 that one can numerically solve the quadratic two-parameter eigenvalue problem (1) by the linearization (3) and Algorithm 3. The algorithm from [7] can be applied to compute the eigenvalues of the projected coupled generalized eigenvalue problem of the form (5).

5 Numerical examples

We present some small numerical examples to show that singular two-parameter eigenvalue problems can be solved with Algorithm 3. The numerical examples were computed in Matlab 7.4, while the exact eigenvalues were obtained in Mathematica 6.0 using variable precision. In each example we computed the maximum relative error of the computed eigenvalues as

$$\max_{i=1,\ldots,k} \frac{\|[\widetilde{\lambda}_i\ \widetilde{\mu}_i] - [\lambda_i\ \mu_i]\|_2}{\|[\lambda_i\ \mu_i]\|_2},$$

where $(\widetilde{\lambda}_1, \widetilde{\mu}_1), \ldots, (\widetilde{\lambda}_k, \widetilde{\mu}_k)$ and $(\lambda_1, \mu_1), \ldots, (\lambda_k, \mu_k)$ are the computed and the exact eigenvalues, respectively.

Example 19 *We consider the quadratic two-parameter eigenvalue problem*

$$\left(\begin{bmatrix}3 & 4\\ 6 & 1\end{bmatrix} + \lambda \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} + \mu \begin{bmatrix}4 & 1\\ 2 & 4\end{bmatrix} + \lambda^2 \begin{bmatrix}6 & 7\\ 5 & 2\end{bmatrix} + \lambda\mu \begin{bmatrix}1 & 3\\ 7 & 1\end{bmatrix} + \mu^2 \begin{bmatrix}4 & 1\\ 6 & 3\end{bmatrix}\right) x_1 = 0,$$

$$\left(\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix} + \lambda \begin{bmatrix}1 & 4\\ 8 & 2\end{bmatrix} + \mu \begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix} + \lambda^2 \begin{bmatrix}2 & 6\\ 1 & 3\end{bmatrix} + \lambda\mu \begin{bmatrix}7 & 2\\ 3 & 7\end{bmatrix} + \mu^2 \begin{bmatrix}3 & 5\\ 5 & 2\end{bmatrix}\right) x_2 = 0,$$

which has 16 *eigenvalues. The largest and the smallest (by absolute value) eigenvalue are* $(-7.5130, 3.8978)$ *and* $(-0.2658 \pm 0.8007i, 0.3141 \mp 0.1077i)$*, respectively.*

The matrices Δ_0*,* Δ_1*, and* Δ_2 *from the linearization (3) are of size* 36×36*. Algorithm 3 returns the* 16×16 *matrices* $\widetilde{\Delta}_0$*,* $\widetilde{\Delta}_1$*, and* $\widetilde{\Delta}_2$ *such that* $\widetilde{\Delta}_0$ *is nonsingular and that* $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ *and* $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ *commute. From* $\widetilde{\Delta}_0$*,* $\widetilde{\Delta}_1$*, and* $\widetilde{\Delta}_2$ *we get all* 16 *eigenvalues of the quadratic two-parameter eigenvalue problem. The maximum relative error of the computed eigenvalues is* $1.8 \cdot 10^{-14}$*.*

Example 20 *A cubic two-parameter eigenvalue problem has the form*

$$\begin{aligned}(A_{00}^{(1)} + \cdots + \lambda^3 A_{30}^{(1)} + \lambda^2\mu A_{21}^{(1)} + \lambda\mu^2 A_{12}^{(1)} + \mu^3 A_{03}^{(1)})x_1 &= 0\\ (A_{00}^{(2)} + \cdots + \lambda^3 A_{30}^{(2)} + \lambda^2\mu A_{21}^{(2)} + \lambda\mu^2 A_{12}^{(2)} + \mu^3 A_{03}^{(2)})x_2 &= 0.\end{aligned} \tag{9}$$

If $A_{jk}^{(i)}$ are general $n\times n$ matrices, then the problem has $9n^2$ eigenvalues. Similarly to the quadratic two-parameter eigenvalue problem, we linearize (9) as a two-parameter eigenvalue problem, a possible linearization (see Appendix) is

$$L_i(\lambda,\mu)=\begin{bmatrix} A_{00}^{(i)} & A_{10}^{(i)} & A_{01}^{(i)} & A_{20}^{(i)}+\lambda A_{30}^{(i)} & A_{11}^{(i)}+\lambda A_{21}^{(i)} & A_{02}^{(i)}+\lambda A_{12}^{(i)}+\mu A_{03}^{(i)} \\ \lambda I & -I & 0 & 0 & 0 & 0 \\ \mu I & 0 & -I & 0 & 0 & 0 \\ 0 & \lambda I & 0 & -I & 0 & 0 \\ 0 & 0 & \lambda I & 0 & -I & 0 \\ 0 & 0 & \mu I & 0 & 0 & -I \end{bmatrix}$$

for $i=1,2$. The corresponding operator determinant Δ_0 is of rank $20n^2$ and thus singular.

Using the software package GUPTRI [5] for the evaluation of the generalized upper-triangular form we observe the following interesting structure:

a) The Kronecker structure of $\Delta_1-\lambda\Delta_0$ (and same for $\Delta_2-\mu\Delta_0$) consists of $4n^2$ L_0, $4n^2$ L_0^T, n^2 L_1, n^2 L_1^T, $6n^2$ N_1, $2n^2$ N_2, $2n^2$ N_3, n^2 N_4, and the regular part of size $9n^2$.
b) $\dim(\mathrm{Ker}(\Delta_0))=16n^2$, $\dim(\mathrm{Ker}(\Delta_1))=5n^2$, *and* $\dim(\mathrm{Ker}(\Delta_2))=5n^2$.
c) $\dim(\mathrm{Ker}(\Delta_1)\cap\mathrm{Ker}(\Delta_0))=4n^2$, $\dim(\mathrm{Ker}(\Delta_2)\cap\mathrm{Ker}(\Delta_0))=4n^2$, *and* $\dim(\mathrm{Ker}(\Delta_0)\cap\mathrm{Ker}(\Delta_1)\cap\mathrm{Ker}(\Delta_2))=n^2$.

Due to the complex Kronecker canonical structure, we did not attempt to prove the structure in theory as we did for the quadratic case.

Using Algorithm 3 for the extraction of the common regular part, we are able to compute all eigenvalues of the cubic two-parameter eigenvalue problem. For the test case we reuse the matrices from Example 19 and add the matrices

$$A_{30}^{(1)}=\begin{bmatrix}3&5\\2&4\end{bmatrix},\ A_{21}^{(1)}=\begin{bmatrix}1&7\\2&8\end{bmatrix},\ A_{12}^{(1)}=\begin{bmatrix}4&9\\1&1\end{bmatrix},\ A_{03}^{(1)}=\begin{bmatrix}5&8\\6&3\end{bmatrix},$$

$$A_{30}^{(2)}=\begin{bmatrix}2&3\\2&7\end{bmatrix},\ A_{21}^{(2)}=\begin{bmatrix}6&5\\9&1\end{bmatrix},\ A_{12}^{(2)}=\begin{bmatrix}5&7\\8&8\end{bmatrix},\ A_{03}^{(2)}=\begin{bmatrix}3&1\\3&5\end{bmatrix}.$$

The matrices Δ_0, Δ_1 and Δ_2 from the linearization are of size 144×144. Algorithm 3 returns the matrices $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ of size 36×36. The matrices $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ commute. From $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ we get all 36 eigenvalues of the cubic two-parameter eigenvalue problem. The largest and the smallest (by absolute value) eigenvalue are $(18.8604, 9.9061)$ and $(0.0477, 0.7640)$, respectively. The maximum relative error of the computed eigenvalues is $1.6\cdot10^{-13}$.

Example 21 *We simulate a model updating problem with the matrices*

$$A = \begin{bmatrix} 9 & 5 & 2 & -1 & -8 \\ -5 & 0 & 5 & 8 & -2 \\ 2 & -9 & 8 & 8 & 6 \\ 0 & 6 & 4 & -1 & -9 \\ 7 & -1 & -6 & 7 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & -9 & -1 & 6 & 0 \\ -6 & 4 & 6 & -9 & 4 \\ 2 & -1 & 0 & 3 & -1 \\ -4 & 8 & -5 & -2 & -3 \\ -6 & 0 & 3 & 6 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} -6 & 3 & 0 & 3 & 4 \\ 3 & -2 & 7 & -3 & -3 \\ -3 & 7 & 6 & -4 & 6 \\ 0 & 7 & 2 & -3 & 1 \\ -6 & 1 & 6 & 0 & -2 \end{bmatrix}.$$

We are looking for parameters λ and μ such that two eigenvalues of the matrix $A+\lambda B+\mu C$ are $\sigma_1 = 2$ and $\sigma_2 = 3$. If we write this as a two-parameter eigenvalue problem (7) and apply Algorithm 3 we obtain 20 *suitable pairs* (λ, μ). *The closest solution to* $(0,0)$, *which corresponds to the smallest perturbation of A, is* $(0.2593, 0.0067)$. *The maximum relative error of the computed eigenvalues is* $2.5 \cdot 10^{-13}$.

Acknowledgements

The authors would like to thank Paul Binding and Tomaž Košir for fruitful discussions. We also thank the anonymous referees for the careful reading of the manuscript and for many valuable suggestions.

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A Linearization of two-parameter matrix polynomials

Theorem 22 *Let*

$$P(\lambda,\mu)=\sum_{i=0}^{k}\sum_{j=0}^{k-i}\lambda^i\mu^j A_{ij}$$

be a two-parameter matrix polynomial, where A_{ij} is an $n\times n$ matrix for each i,j. Let us define

$$\begin{aligned}
K_{ij}(\lambda,\mu)&=A_{ij},\quad i+j<k-1,\\
K_{ij}(\lambda,\mu)&=A_{ij}+\lambda A_{i+1,j},\quad i+j=k-1,\ i\neq 0,\\
K_{0,k-1}(\lambda,\mu)&=A_{0,k-1}+\lambda A_{1,k-1}+\mu A_{0,k}.
\end{aligned}$$

The linear matrix polynomial

$$L(\lambda,\mu) = \begin{array}{cc} & \begin{array}{cccc} n & 2n & \dots & kn \end{array} \\ \begin{array}{c} n \\ 2n \\ \vdots \\ kn \end{array} & \begin{bmatrix} K_0 & K_1 & \cdots & K_k \\ T_1 & -I_{2n} & & \\ & \ddots & \ddots & \\ & & T_k & -I_{kn} \end{bmatrix} \end{array}, \quad \textit{where} \quad T_r = \begin{bmatrix} \lambda I_n & & \\ & \ddots & \\ & & \lambda I_n \\ & & \mu I_n \end{bmatrix}$$

and $K_r = \begin{bmatrix} K_{r0} & K_{r-1,1} & \cdots & K_{0r} \end{bmatrix}$ *is an* $n \times (r+1)n$ *block matrix for* $r = 1, \ldots, k$, *is a linearization of* $P(\lambda,\mu)$.

Proof. If we take

$$F(\lambda,\mu) = \begin{array}{cc} & \begin{array}{cccc} n & 2n & \dots & kn \end{array} \\ \begin{array}{c} n \\ 2n \\ \vdots \\ kn \end{array} & \begin{bmatrix} I_n & & & \\ T_1 & I_{2n} & & \\ & \ddots & \ddots & \\ & & T_k & I_{kn} \end{bmatrix} \end{array},$$

then we obtain

$$L(\lambda,\mu)F(\lambda,\mu) = \begin{array}{cc} & \begin{array}{cccc} n & 2n & \dots & kn \end{array} \\ \begin{array}{c} n \\ 2n \\ \vdots \\ kn \end{array} & \begin{bmatrix} P(\lambda,\mu) & H_1 & \cdots & H_{k-1} \\ & -I_{2n} & & \\ & & \ddots & \\ & & & -I_{kn} \end{bmatrix} \end{array}$$

for some matrices $H_1, \ldots, H_{k-1}$. Now,

$$\begin{bmatrix} I_n & H_1 & \cdots & H_{k-1} \\ & -I_{2n} & & \\ & & \ddots & \\ & & & -I_{kn} \end{bmatrix} L(\lambda,\mu)F(\lambda,\mu) = \begin{bmatrix} P(\lambda,\mu) & \\ 0 & I_{(k+2)(k-1)n/2} \end{bmatrix}$$

and $L(\lambda,\mu)$ is a linearization of $P(\lambda,\mu)$. $\square$