A continuation method for a weakly elliptic two-parameter eigenvalue problem

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Abstract

We show that the continuation method can be used to solve a weakly elliptic two-parameter eigenvalue problem. We generalize the continuation method for a nonsymmetric eigenvalue problem $Ax = \lambda x$ by T.Y. Li, Z. Zeng, and L. Cong (1992 *SIAM J. Numer. Anal.* **29**, 229–248) to two-parameter problems.

1 Introduction

We consider a system of two-parameter pencils

$$W_i(\boldsymbol{\lambda}) = \lambda_1 V_{i1} + \lambda_2 V_{i2} - V_{i0}, \quad i = 1, 2, \tag{1}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and V_{ij} are symmetric $n_i \times n_i$ matrices over $\mathbb{R}$ viewed as linear transformations on $\mathbb{C}^{n_i}$.

For a weakly elliptic problem [10] we require that one of the matrices $V_{11}, V_{12}, V_{21}, V_{22}$ is definite. Without loss of generality we may assume that V_{11} is a positive definite matrix. There exists a constant $c \in \mathbb{R}$ such that the linear substitution $\lambda_1 = \lambda_1' + c\lambda_2'$, $\lambda_2 = \lambda_2'$ transforms (1) into a similar problem where V_{12} is a definite matrix. Hence we may further assume that V_{12} is definite. Moreover, the following lemma shows that we may assume that V_{11} is an identity matrix and that V_{12} is a diagonal matrix.

Lemma 1 *We may assume that V_{11} is an identity matrix and that V_{12} is a diagonal matrix.*

Proof. Since V_{11} is a positive definite matrix, there exists Cholesky decomposition $V_{11} = LL^T$. For the symmetric matrix $L^{-1}V_{12}L^{-T}$ there exist an orthogonal matrix Q and a diagonal matrix D such that $L^{-1}V_{12}L^{-T} = QDQ^T$. The equation $V_{10}x_1 = \lambda_1 V_{11}x_1 + \lambda_2 V_{12}x_1$ is equivalent to $V_{10}'x_1' = \lambda_1 x_1' + \lambda_2 D x_1'$, where $V_{10}' = Q^T L^{-1} V_{10} L^{-T} Q$ and $x_1' = Q^T L^T x_1$. ■

We say that $\boldsymbol{\lambda}$ is an eigenvalue of the two-parameter problem (1) if

$$\operatorname{Ker} W_i(\boldsymbol{\lambda}) \neq \{0\}, \quad i = 1, 2.$$

On the tensor product space $S := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ of the dimension $N := n_1 n_2$ we define operator determinants

$$\Delta_0 = \begin{vmatrix} V_{11}^\dagger & V_{12}^\dagger \\ V_{21}^\dagger & V_{22}^\dagger \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} V_{10}^\dagger & V_{12}^\dagger \\ V_{20}^\dagger & V_{22}^\dagger \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} V_{11}^\dagger & V_{10}^\dagger \\ V_{21}^\dagger & V_{20}^\dagger \end{vmatrix},$$

where $V_{ij}^\dagger$ is the linear transformation on S induced by V_{ij}. So, $V_{1j}^\dagger = V_{1j} \otimes I$ and $V_{2j}^\dagger = I \otimes V_{2j}$ (see Atkinson [3] for details). We assume that the two-parameter problem (1) is nonsingular, that is the corresponding operator determinant Δ_0 is invertible. In this case the operators $\Gamma_i := \Delta_0^{-1}\Delta_i$ commute [3] and the problem (1) is equivalent to the simultaneous problem

$$\Gamma_i z = \lambda_i z, \quad i = 1, 2, \tag{2}$$

for decomposable tensors $z \in S$.

If $\boldsymbol{\lambda}$ is an eigenvalue of the problem (1), then

$$d_a := \dim \left(\bigcap_{j=0}^{N} \operatorname{Ker} \left[(\Gamma_1 - \lambda_1 I)^j (\Gamma_2 - \lambda_2 I)^{N-j} \right] \right)$$

is the algebraic multiplicity and

$$d_g := \dim \left(\operatorname{Ker}(\Gamma_1 - \lambda_1 I) \cap \operatorname{Ker}(\Gamma_2 - \lambda_2 I) \right) = \dim \operatorname{Ker} W_1(\boldsymbol{\lambda}) \cdot \dim \operatorname{Ker} W_2(\boldsymbol{\lambda})$$

is the geometric multiplicity of the eigenvalue (see [3]). We say that an eigenvalue $\boldsymbol{\lambda}$ is geometrically or algebraically simple when $d_g = 1$ or $d_a = 1$, respectively.

Multiparameter eigenvalue problems arise in a variety of applications [2], particularly in mathematical physics when the method of separation of variables is used to solve boundary value problems [8, 21]. One possible way to solve a multiparameter differential equation is to transform it into a matrix problem using discretization and solve this matrix problem numerically.

Instead of the original problem (1) we can treat the simultaneous problem (2) that can be solved by standard numerical methods. This approach looks interesting but it is completely unsuitable for large matrices. Namely, the matrices in (2) are of size $n_1 n_2 \times n_1 n_2$ and the time complexity is $\mathcal{O}(n_1^3 n_2^3)$. Therefore faster algorithms which use the structure of the two-parameter problem are needed.

Most numerical algorithms that exist for two–parameter eigenvalue problems [6, 4, 9, 19, 18] require a definite problem. Since our weakly elliptic problem is not definite in general, these methods cannot be applied.

Bohte [5] used Newton's method to find an eigenvalue pair. This method can be applied to a weakly elliptic problem, but its disadvantage is that Newton's method requires a very good initial approximation in order to converge. Such an approximation is hard to find if nothing else is known besides the initial matrices.

In order to fix the problem with the initial approximations we use the continuation method (see; for example, [1] for details about the continuation method). In this method we construct a homotopy and we have no problems with good initial approximations as we follow the solution curve.

The continuation method was first applied to the multiparameter eigenvalue problems by Müller [16]. It was used for right definite two-parameter problems by Shimasaki [19] and Plestenjak [18]. In this paper we show that the continuation method similar to the one that Li, Zeng, and Cong used in [12] for the nonsymmetric eigenvalue problem $Ax = \lambda x$ can be applied to a weakly elliptic two-parameter problem (1).

In Section 2 we state some auxiliary results by Košir [10] about the root subspace of the nonderogatory eigenvalue of the two-parameter problem. In Section 3 we construct the homotopy and show that its Jacobian is of full rank in general. In Section 4 we show that all eigenvalues are nonderogatory and that all bifurcations are turning points. In Section 5 we give more details about the continuation method and in Section 6 we present some numerical results that confirm that the continuation method is competitive with the QR algorithm for the simultaneous problem.

2 Auxiliary results

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ be an eigenvalue of the two-parameter problem (1) and let unit vectors $x_i, y_i \in \mathbb{C}^{n_i}$ be such that $W_i(\boldsymbol{\lambda})x_i = 0$ and $y_i^* W_i(\boldsymbol{\lambda}) = 0$ for $i = 1, 2$. We define the matrix

$$B_0 := \begin{pmatrix} y_1^* V_{11} x_1 & y_1^* V_{12} x_1 \\ y_2^* V_{21} x_2 & y_2^* V_{22} x_2 \end{pmatrix}. \tag{3}$$

We say that $\boldsymbol{\lambda}$ is a nonderogatory eigenvalue if $\dim \operatorname{Ker} W_i(\boldsymbol{\lambda}) = 1$ for $i = 1, 2$, and $\dim \operatorname{Ker} B_0 \leq 1$.

Lemma 2 *If $\boldsymbol{\lambda}$ is an algebraically simple eigenvalue of the two-parameter problem (1), then matrix B_0 is nonsingular.*

Proof. The lemma follows from Lemma 3 in [10]. ■

Lemma 3 *If $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is a geometrically simple eigenvalue of the two-parameter problem (1), then the following is true:*

a)

$$\dim \left(\bigcap_{j=0}^{2} \operatorname{Ker} \left[(\Gamma_1 - \lambda_1 I)^j (\Gamma_2 - \lambda_2 I)^{2-j} \right] \right) = 1 + \dim \operatorname{Ker} B_0.$$

b) A nonzero vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{C}^{\mathbf{2}}$ such that $B_0 \boldsymbol{a} = 0$ exists if and only if there exists a vector $z_i \in \mathbb{C}^{n_i}$ such that

$$W_i(\boldsymbol{\lambda}) z_i + a_1 V_{i1} x_i + a_2 V_{i2} x_i = 0$$

for $i = 1, 2$.

Proof. The lemma follows from Theorem 5 and Theorem 15 in [10]. ■

3 A continuation method

Let $A_{10} \in \mathbb{R}^{n_1 \times n_1}$ be a diagonal matrix, $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ a symmetric matrix with zeros on its diagonal and $A_{20} \in \mathbb{R}^{n_2 \times n_2}$ a symmetric matrix. Let us denote

$$\begin{aligned} W_1(\boldsymbol{\lambda}, t) &:= \lambda_1 V_{11} + \lambda_2 V_{12} - (1-t)A_{10} - t(1-t)A_{11} - tV_{10}, \\ W_2(\boldsymbol{\lambda}, t) &:= \lambda_2 V_{21} + \lambda_2 V_{22} - (1-t)A_{20} - tV_{20}, \end{aligned}$$

where $t \in [0,1]$. Suppose that matrices A_{10} and A_{20} are such that all the eigenvalues of the two-parameter problem

$$\begin{aligned} (A_{10} - \lambda_1 V_{11} - \lambda_2 V_{12})x_1 &= 0, \\ (A_{20} - \lambda_1 V_{21} - \lambda_2 V_{22})x_2 &= 0 \end{aligned} \tag{4}$$

are algebraically simple. We introduce the homotopy

$$H : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C} \times [0,1] \longrightarrow \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C},$$

$$H(x_1, x_2, \lambda_1, \lambda_2, t) := \begin{pmatrix} W_1(\boldsymbol{\lambda}, t)x_1 \\ W_2(\boldsymbol{\lambda}, t)x_2 \\ \frac{1}{2}(c_{11}x_{11}^2 + \ldots + c_{1n_1}x_{1n_1}^2 - 1) \\ \frac{1}{2}(c_{21}x_{21}^2 + \ldots + c_{2n_2}x_{2n_2}^2 - 1) \end{pmatrix}, \tag{5}$$

where c_{ij} are randomly chosen real positive numbers.

A solution of the equation $H(x_1, x_2, \lambda_1, \lambda_2, t) = 0$ is a solution of the two-parameter problem

$$W_i(\boldsymbol{\lambda}, t)x_i = 0, \quad i = 1, 2, \tag{6}$$

which is equal to the problem (1) at $t = 1$ and equal to the problem (4) at $t = 0$. Clearly, the problem (6) is weakly elliptic for every $t \in [0,1]$.

In the continuation method we start from the known eigenpair of (4) at $t = 0$ and follow the solution curve to a solution of (1) at $t = 1$. This approach is reasonable if the solution of the problem (4) is considerably easier to calculate than a solution of the original problem (1). To have N independent solution curves at $t = 0$, we further require that all eigenvalues of the problem (4) are algebraically simple.

The following lemma shows that it is possible to choose matrices A_{10} and A_{20} which satisfy the above demands. From the constructive proof of the lemma a method for obtaining the initial problem (4) and its solution can be easily derived.

Lemma 4 *There exist a diagonal matrix $A_{10} \in \mathbb{R}^{n_1 \times n_1}$ and a symmetric matrix $A_{20} \in \mathbb{R}^{n_2 \times n_2}$ such that all eigenvalues of the two-parameter problem (4) are algebraically simple.*

Proof. Let us denote $A_{10} = \mathrm{diag}(e_1, e_2, \ldots, e_{n_1})$ and $V_{12} = \mathrm{diag}(d_1, d_2, \ldots, d_{n_1})$. Eigenpairs of the first equation of (4) are points (λ_1, λ_2) on the lines

$$e_j - \lambda_1 - \lambda_2 d_j = 0, \quad j = 1, \ldots, n_1. \tag{7}$$

When we insert λ_1 from equation (7) into the second equation of (4), we obtain n_1 generalized eigenvalue problems

$$(A_{20} - e_j V_{21})x_2 = \lambda_2(V_{22} - d_j V_{21})x_2, \quad j = 1, \ldots, n_1. \tag{8}$$

It is easy to see that nonsingularity of Δ_0 implies nonsingularity of $V_{22} - d_j V_{21}$ for $j = 1, \ldots, n_1$.

The lines (7) intersect in finitely many points

$$(\lambda_{11}, \lambda_{21}), \ldots, (\lambda_{1k}, \lambda_{2k}). \tag{9}$$

It remains to show that we can choose A_{20} such that all eigenvalues λ_2 of the generalized problems (8) are algebraically simple and that they all differ from the λ_2-coordinates of intersections (9).

Let us show that there exists an open dense subset $\mathcal{M}$ of all $n_2 \times n_2$ symmetric matrices such that all eigenvalues of (4) are simple for $A_{20} \in \mathcal{M}$. For each $j = 1, \ldots, n_1$, there exist an orthogonal matrix Q_j and a diagonal matrix $G_j = \operatorname{diag}(g_{j1}, g_{j2}, \ldots, g_{jn_2})$ such that

$$V_{22} - d_j V_{21} = Q_j G_j Q_j^T.$$

For a fixed $j \in \{1, \ldots, n_1\}$ we set

$$A_{20} = e_j V_{21} + Q_j F_j Q_j^T,$$

where $F_j = \operatorname{diag}(f_{j1}, f_{j2}, \ldots, f_{jn_2})$, and rewrite (8) as

$$F_j x_2' = \lambda_2 G_j x_2', \quad j = 1, \ldots, n_1,$$

where $x_2' = Q_j^T x_2$. If diagonal elements f_{jk} satisfy the equation

$$\frac{f_{jk}}{g_{jk}} \neq \frac{f_{jl}}{g_{jl}} \quad \text{for} \quad k \neq l,$$

then all eigenvalues of the generalized eigenvalue problems (8) are algebraically simple. It is easy to see that it is possible to choose f_{jk} such that all eigenvalues λ_2 of the generalized problems (8) differ from the λ_2-coordinates of intersections (9).

The determinant of the polynomial

$$p_j(\lambda_2) = \det(A_{20} - e_j V_{21} - \lambda_2(V_{22} - d_j V_{21}))$$

as a polynomial in λ_2 can be considered as a polynomial P_j in all elements of the symmetric matrix A_{20}. This polynomial is not identically zero because there exists A_{20} such that all eigenvalues of the generalized eigenvalue problem (8) are simple. Hence, the zeros of P_j is a variety of codimension 1 in $\mathbb{R}^{n_2(n_2+1)/2}$ and let $\mathcal{M}_j'$ be its complement. Since $p_j(\lambda_2)$ is not identically equal to any λ_2-coordinate of intersections (9) there exists an open dense subset $\mathcal{M}_j''$ of all $n_2 \times n_2$ symmetric matrices such that if $A_{20} \in \mathcal{M}_j''$ then all eigenvalues λ_2 of the generalized problem (8) differ from all λ_2-coordinates of intersections (9).

Let $\mathcal{M} := \bigcup_{j=1}^{n_1} (\mathcal{M}_j' \cap \mathcal{M}_j'')$. If $A_{20} \in \mathcal{M}$ then all eigenvalues of (4) are algebraically simple. ■

Remark. If $A_{20} \in \mathcal{M}$ and the matrix $A_{20} - e_j V_{21}$ is positive definite for all j, then all eigenvalues are real. Hence, if we randomly choose a symmetric matrix A'_{20} and take $A_{20} = A'_{20} + kI$ for large $k \in \mathbb{R}$, then all eigenvalues of (4) are real and algebraically simple.

Eigenvalues of a weakly elliptic two-parameter problem have a very important property. That is, either λ_1 and λ_2 are both real or they are both strictly complex. Based on this property we later use one version of the algorithm for the real and the second one for the complex solution curve.

Lemma 5 *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ be an eigenvalue of the weakly elliptic two-parameter problem (1). Then either $\lambda_1, \lambda_2 \in \mathbb{R}$ or $\lambda_1, \lambda_2 \notin \mathbb{R}$.*

Proof. Take a nonzero vector x_1 such that

$$\lambda_1 V_{11} x_1 + \lambda_2 V_{12} x_1 - V_{10} x_1 = 0.$$

Then

$$\lambda_1 x_1^* V_{11} x_1 + \lambda_2 x_1^* V_{12} x_1 - x_1^* V_{10} x_1 = 0.$$

Since all matrices V_{ij} are symmetric, $x_1^* V_{1j} x_1 \in \mathbb{R}$ for $j = 0, 1, 2$. Furthermore, V_{11} and V_{12} are both definite, therefore $x_1^* V_{1j} x_1 \neq 0$ for $j = 1, 2$. It follows that either $\lambda_1, \lambda_2 \in \mathbb{R}$ or $\lambda_1, \lambda_2 \notin \mathbb{R}$. ■

Let

$$G := \{(x_1, x_2, \lambda_1, \lambda_2, t) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C} \times [0,1] \mid H(x_1, x_2, \lambda_1, \lambda_2, t) = 0\}$$

denote a solution set for the homotopy (5). We say that $t \in [0,1]$ is a singular point if the two-parameter problem (6) has a multiple eigenvalue. It is easy to see that there are only finitely many singular points $t \in [0,1]$. On each interval $(a,b) \subset [0,1]$ which does not contain any singular point the solution set consists of N solution curves.

The Jacobian of H is a $(n_1 + n_2 + 3) \times (n_1 + n_2 + 2)$ matrix equal to

$$H' = (\, H_{x_1} \quad H_{x_2} \quad H_{\lambda_1} \quad H_{\lambda_2} \quad H_t \,),$$

where

$$H_{x_1} = \begin{pmatrix} W_1(\boldsymbol{\lambda}, t) & & \\ 0 & & \\ c_{11}x_{11} & \cdots & c_{1n_1}x_{1n_1} \\ 0 & & \end{pmatrix}, \quad H_{x_2} = \begin{pmatrix} 0 & & \\ W_2(\boldsymbol{\lambda}, t) & & \\ 0 & & \\ c_{21}x_{21} & \cdots & c_{2n_2}x_{2n_2} \end{pmatrix},$$

$$H_{\lambda_1} = \begin{pmatrix} V_{11}x_1 \\ V_{21}x_2 \\ 0 \\ 0 \end{pmatrix}, \quad H_{\lambda_2} = \begin{pmatrix} V_{12}x_1 \\ V_{22}x_2 \\ 0 \\ 0 \end{pmatrix}, \quad H_t = \begin{pmatrix} (A_{10} + (1-2t)A_{11} - V_{10})x_1 \\ (A_{20} - V_{20})x_2 \\ 0 \\ 0 \end{pmatrix}.$$

We numerically follow the eigenpath by a prediction-correction scheme using the arclength as the parameter. Let $\boldsymbol{p} = (x_1, x_2, \lambda_1, \lambda_2, t) \in G$. We calculate the tangent vector and

predict the next point on the solution curve using Euler's method. The unit tangent vector $\dot{\boldsymbol{p}} = (\dot{x_1}, \dot{x_2}, \dot{\lambda_1}, \dot{\lambda_2}, \dot{t})$ satisfies the equations

$$H_{x_1}\dot{x}_1 + H_{x_2}\dot{x}_2 + H_{\lambda_1}\dot{\lambda}_1 + H_{\lambda_2}\dot{\lambda}_2 + H_t\dot{t} = 0 \tag{10}$$

and

$$\|\dot{x}_1\|^2 + \|\dot{x}_2\|^2 + |\dot{\lambda}_1|^2 + |\dot{\lambda}_1|^2 + |\dot{t}|^2 = 1.$$

A simple way to calculate the unit tangent vector is to set $\dot{t} = 1$, solve the linear system

$$H_{(\boldsymbol{x},\boldsymbol{\lambda})} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = -H_t\dot{t}, \tag{11}$$

where $H_{(\boldsymbol{x},\boldsymbol{\lambda})} = (\, H_{x_1} \quad H_{x_2} \quad H_{\lambda_1} \quad H_{\lambda_2} \,)$, and normalize the tangent vector. This method works well as long as matrix $H_{(\boldsymbol{x},\boldsymbol{\lambda})}$ is nonsingular. We show that this is true for an algebraically simple eigenvalue.

Proposition 1 *Let $\boldsymbol{p} = (x_1, x_2, \lambda_1, \lambda_2, t) \in G$. Matrix $H_{(\boldsymbol{x},\boldsymbol{\lambda})}$ is nonsingular if and only if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is an algebraically simple eigenvalue.*

Proof. Suppose that $(z_1, z_2, \mu_1, \mu_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C}$ satisfies the equation

$$H_{(\boldsymbol{x},\boldsymbol{\lambda})} \begin{pmatrix} z_1 \\ z_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} = 0.$$

It follows that

$$\begin{aligned}
W_1(\boldsymbol{\lambda}, t)z_1 + \mu_1 V_{11}x_1 + \mu_2 V_{12}x_1 &= 0, && (12)\\
W_2(\boldsymbol{\lambda}, t)z_2 + \mu_1 V_{21}x_2 + \mu_2 V_{22}x_2 &= 0, && (13)\\
c_{11}x_{11}z_{11} + \cdots + c_{1n_1}x_{1n_1}z_{1n_1} &= 0, && (14)\\
c_{21}x_{21}z_{21} + \cdots + c_{2n_2}x_{2n_2}z_{2n_2} &= 0. && (15)
\end{aligned}$$

On the other hand $\boldsymbol{p} \in G$ and therefore

$$\begin{aligned}
W_1(\boldsymbol{\lambda}, t)x_1 &= 0, && (16)\\
W_2(\boldsymbol{\lambda}, t)x_2 &= 0, && (17)\\
c_{11}x_{11}^2 + \cdots + c_{1n_1}x_{1n_1}^2 &= 1, && (18)\\
c_{21}x_{21}^2 + \cdots + c_{2n_2}x_{2n_2}^2 &= 1. && (19)
\end{aligned}$$

For the first part we show that if $\boldsymbol{\lambda}$ is an algebraically simple eigenvalue, then $H_{(\boldsymbol{x},\boldsymbol{\lambda})}$ is nonsingular. It follows from $W_i(\boldsymbol{\lambda})^* = W_i(\overline{\boldsymbol{\lambda}})$, where $\overline{\boldsymbol{\lambda}} = (\overline{\lambda_1}, \overline{\lambda_2})$, that $x_i^T W_i(\boldsymbol{\lambda}) = 0$ for

$i = 1, 2$. We multiply (12) and (13) by x_1^T and x_2^T, respectively, and obtain a homogeneous system

$$\begin{pmatrix} x_1^T V_{11} x_1 & x_1^T V_{12} x_1 \\ x_2^T V_{21} x_2 & x_2^T V_{22} x_2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0. \tag{20}$$

We recognize the matrix in equation (20) as matrix B_0 in (3) that is nonsingular by Lemma 2, therefore $\mu_1 = \mu_2 = 0$.

What remains of the equations (12) and (13) is $W_1(\boldsymbol{\lambda}, t)z_1 = 0$ and $W_2(\boldsymbol{\lambda}, t)z_2 = 0$. Since $\dim \operatorname{Ker} W_i(\boldsymbol{\lambda}, t) = 1$ for $i = 1, 2$, there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $z_1 = \alpha_1 x_1$ and $z_2 = \alpha_2 x_2$. It follows from (14) and (18) that

$$0 = \alpha_1(c_{11}x_{11}^2 + \cdots + c_{1n_1}x_{1n_1}^2) = \alpha_1.$$

In a similar manner (15) and (19) yield $\alpha_2 = 0$.

For the second part we assume that $\boldsymbol{\lambda}$ is a multiple eigenvalue. To show that $H_{(\boldsymbol{x},\boldsymbol{\lambda})}$ is singular, we consider two possibilities:

a) $\dim \operatorname{Ker} W_i(\boldsymbol{\lambda}, t) = 1$ for $i = 1, 2$. We take a left eigenvector y_i of $W_i(\boldsymbol{\lambda}, t)$ for $i = 1, 2$. By Lemma 3 there exist a nonzero pair (μ_1, μ_2) and vectors z_1, z_2, such that the equations (12) and (13) hold. By adding appropriate multiples of x_1 and x_2 to z_1 and z_2, respectively, it is easy to see that z_1 and z_2 can be chosen in such a way that all the equations (12)–(15) are satisfied. Then (z_1, z_2, μ_1, μ_2) lies in the kernel of $H_{(\boldsymbol{x},\boldsymbol{\lambda})}$.

b) We have at least two linearly independent eigenvectors for the eigenvalue $\boldsymbol{\lambda}$. Without loss of generality we may assume that $\dim \operatorname{Ker} W_1(\boldsymbol{\lambda}, t) > 1$. Then there exists a nonzero vector $z_1 \in \operatorname{Ker} W_1(\boldsymbol{\lambda}, t)$ which satisfies the equation (14). ■

The unit tangent vector is uniquely determined if the Jacobian H' is of full rank. We show that for randomly chosen real matrices A_{10}, A_{11}, and A_{20}, H' is of full rank in every point $\boldsymbol{p} \in G$, such that $t \in [0, 1)$. We need the following result which follows from Corollary 4.4 in [14].

Lemma 6 *Let $\mathcal{A} \subset \mathbb{C}^m$ and $\mathcal{B} \subset \mathbb{C}^n$. If 0 is a regular value of the polynomial map $P : \mathcal{A} \times \mathcal{B} \to \mathbb{C}^q$, then there exists real algebraic subset $\mathcal{B}' \subset \mathbb{R}^n$ with zero measure such that 0 is a regular value of $P(\cdot, y) : \mathcal{A} \to \mathbb{C}^q$ for each $y \in \mathcal{B} \cap \mathbb{R}^n - \mathcal{B}'$.*

Proposition 2 *For all diagonal matrices $A_{10} \in \mathbb{R}^{n_1 \times n_1}$, symmetric matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ with zero diagonal entries, and symmetric matrices $A_{20} \in \mathbb{R}^{n_2 \times n_2}$ in a dense open set the Jacobian H' is of full rank at every point $(x_1, x_2, \lambda_1, \lambda_2, t) \in G$ such that $t \in [0, 1)$.*

Proof. We treat nonzero elements of matrices A_{10}, A_{11}, and A_{20} as variables of H. First we consider homotopy H as a complex transformation

$$\widetilde{H} : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C} \times [0, 1] \times \mathbb{C}^{n_1(n_1+1)/2} \times \mathbb{C}^{n_2(n_2+1)/2} \to \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C},$$

where we assume that matrices A_{10}, A_{11}, A_{20} are complex. The Jacobian of $\widetilde{H}$ has the form

$$\widetilde{H}' = \begin{bmatrix} H_{(\boldsymbol{x},\boldsymbol{\lambda})} & H_t & H_{A_1} & H_{A_2} \end{bmatrix},$$

where

$$H_{A_1} = \begin{pmatrix} S_1 \\ 0 \\ 0 \end{pmatrix}, \quad H_{A_2} = \begin{pmatrix} 0 \\ S_2 \\ 0 \end{pmatrix},$$

$S_i \in \mathbb{C}^{n_i \times n_i(n_i+1)/2}$, and

$$S_1 = t(1-t) \begin{pmatrix} \frac{1}{t}x_{11} & x_{12} & \cdots & x_{1n_1} & 0 & & 0 & \cdots & 0 \\ 0 & x_{11} & & 0 & \frac{1}{t}x_{12} & \cdots & x_{1n_1} & & \vdots \\ \vdots & & \ddots & & & \ddots & & & 0 \\ 0 & & & x_{11} & 0 & & x_{12} & \cdots & \frac{1}{t}x_{1n_1} \end{pmatrix},$$

$$S_2 = (1-t) \begin{pmatrix} x_{21} & x_{22} & \cdots & x_{2n_2} & 0 & & 0 & \cdots & 0 \\ 0 & x_{21} & & 0 & x_{22} & \cdots & x_{2n_2} & & \vdots \\ \vdots & & \ddots & & & \ddots & & & 0 \\ 0 & & & x_{21} & 0 & & x_{22} & \cdots & x_{2n_2} \end{pmatrix}.$$

Since vectors x_1, x_2 are nonzero, matrices S_1, S_2 are of full rank and therefore $\widetilde{H}'$ is of full rank for $t \in (0,1)$. At $t = 0$ all eigenvalues are algebraically simple, hence $\widetilde{H}'$ is of full rank by Proposition 11. The proof now follows from the generalized Sard's theorem [13] and from Lemma 6. ■

From now on we assume that the Jacobian H' is of full rank at every point $\boldsymbol{p} \in G$ and $t \in [0,1)$. If follows from Proposition 2 that this is the case in general.

4 Bifurcations

In a bifurcation point at least two solution curves intersect. It is vital for our method to be able to detect the bifurcation point and to switch onto a bifurcating branch. It is easy to see that bifurcation points are inevitable. Namely, we start following the eigenpaths at $t = 0$ where all eigenvalues are algebraically simple. Since the number of real eigenvalues at $t = 0$ in general differs from the number of real eigenvalues at $t = 1$, there exist transition points where real eigenvalues change into the complex ones or vice versa. Each transition point is clearly also a bifurcation point.

Definition 1 *Let $\boldsymbol{p} \in G$, $t \in [0,1)$, and let $\dot{\boldsymbol{p}} = (\dot{x_1}, \dot{x_2}, \dot{\lambda_1}, \dot{\lambda_2}, \dot{t})$ be its tangent vector. We say that $\boldsymbol{p}$*

a) is a singular point if $\dot{t} = 0$;

b) is a turning point of order k if $\dot{t} = 0$ and exactly k solution curves intersect in $\boldsymbol{p}$;

c) is a bifurcation point if $\boldsymbol{p}$ *lies in the intersection of at least two solution curves.*

Singular points present numerical problems as we can not apply the equation (11) for the tangent vector. In this section we establish the equality between the set of singular points, the set of turning points, and the set of bifurcation points in our situation. First of all, the following lemma shows that singular points are identical to points with multiple eigenvalues.

Lemma 7 *Let* $\boldsymbol{p} = (x_1, x_2, \lambda_1, \lambda_2, t) \in G$, $t \in [0, 1)$, *be a point on the solution curve and let* $\dot{\boldsymbol{p}} = (\dot{x_1}, \dot{x_2}, \dot{\lambda_1}, \dot{\lambda_2}, \dot{t})$ *be the corresponding tangent vector. Then* $\dot{t} = 0$ *if and only if* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ *is a multiple eigenvalue of* (6).

Proof. By Proposition 1, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is a multiple eigenvalue if and only if $H_{(\boldsymbol{x},\boldsymbol{\lambda})}$ is singular. The result now follows from the equation (10). ■

We have already verified that each transition point is a bifurcation point. It follows from Lemma 7 that each bifurcation point is a singular point. What remains to show is that each singular point is a transition point. This will follow from the result that all eigenvalues of (6) are nonderogatory for $t \in [0, 1)$.

Lemma 8 *Let* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ *be an eigenvalue of* (6) *for* $t \in [0, 1)$. *If algebraic multiplicity of* $\boldsymbol{\lambda}$ *is greater than two, then* $\boldsymbol{\lambda}$ *is a nonderogatory eigenvalue.*

Proof. Suppose that there exists a derogatory eigenvalue with algebraic multiplicity greater than two. We show that in such a case $\dim \operatorname{Ker} H' \geq 2$, which is a contradiction to Proposition 2. Since $\dim \operatorname{Ker} H_{(\boldsymbol{x},\boldsymbol{\lambda})} \leq \dim \operatorname{Ker} H'$, it is enough to find two linearly independent vectors in $\operatorname{Ker} H_{(\boldsymbol{x},\boldsymbol{\lambda})}$. We have to consider the following four situations.

a) $\dim \operatorname{Ker} W_i(\boldsymbol{\lambda}, t) = 2$ for $i = 1, 2$. By the same arguments as in the proof of Proposition 1 it follows that there exist vectors $z_1 \in \operatorname{Ker} W_1(\boldsymbol{\lambda}, t)$ and $z_2 \in \operatorname{Ker} W_2(\boldsymbol{\lambda}, t)$ such that $(z_1, 0, 0, 0), (0, z_2, 0, 0, 0) \in \operatorname{Ker} H_{(\boldsymbol{x},\boldsymbol{\lambda})}$.

b) Geometric multiplicity of $\boldsymbol{\lambda}$ is greater or equal to 3 and there exists $i \in \{1, 2\}$ such that $\dim \operatorname{Ker} W_i(\boldsymbol{\lambda}, t) \geq 3$. Without loss of generality we may assume that $\dim \operatorname{Ker} W_1(\boldsymbol{\lambda}, t) = 3$. It follows that there exist linearly independent vectors $z_1, z_2 \in \operatorname{Ker} W_1(\boldsymbol{\lambda}, t)$ such that $(z_i, 0, 0, 0) \in \operatorname{Ker} H_{(\boldsymbol{x},\boldsymbol{\lambda})}$ for $i = 1, 2$.

c) Geometric multiplicity of $\boldsymbol{\lambda}$ is 2. Suppose that $\dim \operatorname{Ker} W_1(\boldsymbol{\lambda}, t) = 2$ and let $x_2 \in W_2(\boldsymbol{\lambda}, t)$. Then there exist two linearly independent vectors $x_1, \widetilde{x}_1 \in \operatorname{Ker} W_1(\boldsymbol{\lambda}, t)$ and there exist a nonzero pair $(a_1, a_2) \in \mathbb{C}^2$ and vectors $z_i \in \mathbb{C}^{n_i}$ such that

$$W_i(\boldsymbol{\lambda}) z_i + a_1 V_{i1} x_i + a_2 V_{i2} x_i = 0$$

for $i = 1, 2$ (see [11, Section 6]). We consider $H_{(\boldsymbol{x},\boldsymbol{\lambda})}$ in the point $(x_1, x_2, \lambda_1, \lambda_2, t)$. It is easy to see that $\boldsymbol{\varphi_1} := (z_1, z_2, a_1, a_2) \in \operatorname{Ker} H_{(\boldsymbol{x},\boldsymbol{\lambda})}$. On the other hand, there exists a linear combination z of vectors $x_1, \widetilde{x}_1$ such that $\boldsymbol{\varphi_2} := (z, 0, 0, 0) \in \operatorname{Ker} H_{(\boldsymbol{x},\boldsymbol{\lambda})}$. The vectors $\boldsymbol{\varphi_1}$ and $\boldsymbol{\varphi_2}$ are clearly linearly independent.

d) Geometric multiplicity of $\boldsymbol{\lambda}$ is 1. By Lemma 3, $\dim \operatorname{Ker} B_0 = 2$ and therefore $\dim \operatorname{Ker} H_{(\boldsymbol{x},\boldsymbol{\lambda})} = 2$. ■

It follows from Lemma 8 that all eigenvalues with algebraic multiplicity at least three are nonderogatory. We handle the remaining case of a double eigenvalue in the following lemma.

Lemma 9 *Solution curve of H that starts at $t = 0$ does not include a point $\boldsymbol{p}_0 = (x_1^{(0)}, x_2^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, t^{(0)}) \in G$, $t^{(0)} \in [0, 1)$, with the algebraic and the geometric multiplicity of the eigenvalue $\boldsymbol{\lambda}^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)})$ both equal to 2.*

Proof. Since at $t = 0$ all eigenvalues are algebraically simple, $t^{(0)} > 0$. Without loss of generality we may assume that $\dim \operatorname{Ker} W_1(\boldsymbol{\lambda}^{(0)}, t^{(0)}) = 2$. There exists a vector $z_1 \in \operatorname{Ker} W_1(\boldsymbol{\lambda}^{(0)}, t^{(0)})$ such that

$$c_{11} x_{11}^{(0)} z_{11} + \cdots + c_{1n_1} x_{1n_1}^{(0)} z_{1n_1} = 0$$

and

$$c_{11} z_{11}^2 + \cdots + c_{1n_1} z_{1n_1}^2 = 1.$$

It is easy to see that each point on the curve

$$\vartheta(\alpha) := (\cos\alpha\ x_1^{(0)} + \sin\alpha\ z_1, x_2^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, t^{(0)}), \quad \alpha \in \mathbb{R},$$

is a solution of H.

Let us consider H as an analytic function

$$H(x_1, x_2, \lambda_1, \lambda_2, t) : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C} \times \mathbb{C}.$$

The Jacobian $H'(\boldsymbol{p}_0)$ is of full rank and by the Implicit Function Theorem there are unique analytic functions

$$x_1(u), x_2(u), \lambda_1(u), \lambda_2(u), t(u), \tag{21}$$

defined in a neighbourhood $\mathcal{A} \subset \mathbb{C}$ of 0 such that

$$H(x_1(u), x_2(u), \lambda_1(u), \lambda_2(u), t(u)) = 0 \tag{22}$$

for each $u \in \mathcal{A}$ and $t(0) = t^{(0)}$, $x_i(0) = x_i^{(0)}$, $\lambda_i(0) = \lambda_i^{(0)}$ for $i = 1, 2$. Differentiating (22) with respect to u yields

$$H_{x_1} \frac{dx_1}{du} + H_{x_2} \frac{dx_2}{du} + H_{\lambda_1} \frac{d\lambda_1}{du} + H_{\lambda_2} \frac{d\lambda_2}{du} + H_t \frac{dt}{du} = 0. \tag{23}$$

The matrix $H_{(\boldsymbol{x},\boldsymbol{\lambda})} = [\, H_{x_1} \quad H_{x_2} \quad H_{\lambda_1} \quad H_{\lambda_2} \,]$ is singular at $\boldsymbol{p}_0$ due to the multiple eigenvalue $\boldsymbol{\lambda}$. Since the Jacobian $H' = \left[\, H_{(\boldsymbol{x},\boldsymbol{\lambda})} \quad H_t \,\right]$ is of full rank, at $u = 0$, $H_t \neq 0$ and (23) yields $\frac{dt}{du}(0) = 0$. One of the solution curves is ϑ where $t \equiv t^{(0)}$ and consequently t is constant for all solution curves which pass $\boldsymbol{p}$. This is a contradiction to the assumption that $\boldsymbol{p}$ lies on the solution curve which starts at $t = 0$. ■

Proposition 3 *Let $\boldsymbol{p} = (x_1, x_2, \lambda_1, \lambda_2, t)$ be a singular point of H. Then $\boldsymbol{p}$ is a bifurcation point.*

Proof. If follows from Lemma 7, Lemma 8, and Lemma 9 that $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is a nonderogatory eigenvalue. Therefore, $\boldsymbol{\lambda}$ is a geometrically simple eigenvalue and k solution curves intersect at $\boldsymbol{p}$, where k is the algebraic multiplicity of $\boldsymbol{\lambda}$. ■

The order of a turning point $\boldsymbol{p} = (x_1, x_2, \lambda_1, \lambda_2, t)$ equals the algebraic multiplicity of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Therefore, it is possible to determine the order of the turning point using the algorithm for computing a basis for the root subspace at a nonderogatory eigenvalue, which was treated in [17]. The following proposition by Li and Wang describes the behaviour of solution curves in a turning point.

Proposition 4 [15, Proposition 2.4] *Let $\boldsymbol{p}$ be a turning point of order k and let $\gamma_j \subset G$, $j = 1, \ldots, k$, be the k solution curves passing through $\boldsymbol{p}$. Then the tangent vectors of γ_j at $\boldsymbol{p}$ differ only by complex scalar factors of the form $e^{l\pi i/k}$ for certain $1 \leq |l| < k$.*

In practice, turning points of orders higher than 2 do occur very rarely. From now on we therefore assume that all turning points are quadratic. If necessary, a slight modification of our algorithm could treat the general case.

The situation in a quadratic turning point is depicted in Figure 1. The left solution curve γ is real and the right solution curve γ' is complex. They join in a turning point p_T which is a transition point from real to complex space. It follows from Proposition 4 that if $\boldsymbol{\phi}$ is the tangent vector for γ at p_T then $i\boldsymbol{\phi}$ is the tangent vector for γ' at the same point. This allows us to switch to another branch and continue to follow the solution curve.

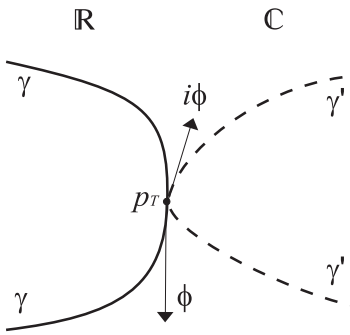

Figure 1: Change from real to complex space in a turning point.

5 The method

First of all, we find a diagonal matrix A_{10} and a symmetric matrix A_{20} such that at $t = 0$ all eigenvalues are algebraically simple and that the initial problem (4) is easily solved. We can follow the proof of Lemma 4 to construct matrices A_{10} and A_{20} and to

calculate the eigenpairs of the initial problem (4). We take a random matrix for A_{11} that is required only in order that the Jacobian H' is of full rank. Based on the discussion from the previous section we assume that all turning points are quadratic.

The method is a generalization of the method by Li, Zeng, and Cong in [12].

5.1 Following the solution curve

Let $\boldsymbol{p}_0 := (x_1^{(0)}, x_2^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, t^{(0)}) \in G$. We set $\dot{t} = 1$, solve the linear system (11), normalize the solution, and obtain the unit tangent vector $\dot{\boldsymbol{p}}_0 := (\dot{x}_1, \dot{x}_2, \dot{\lambda}_1, \dot{\lambda}_2, \dot{t})$. We have to choose the correct sign of $\dot{t}$ in $\dot{\boldsymbol{p}}_0$. In the initial point at $t = 0$ we always choose $\dot{t} > 0$. For $t > 0$ we choose a sign such that the real part of the inner product of the tangent vector $\dot{\boldsymbol{p}}_0$ and the tangent vector from the previous step is positive. This strategy guarantees that we follow the curve in the same direction.

We consider two situations.

1. $\dot{t} > 0$. In this case we use the following prediction-correction scheme. First, we calculate Euler's prediction point

$$\boldsymbol{p}_P := (x_1^{(P)}, x_2^{(P)}, \lambda_1^{(P)}, \lambda_2^{(P)}, t^{(P)}) = \boldsymbol{p}_0 + h\dot{\boldsymbol{p}}_0,$$

   where h is a stepsize (see Subsection 5.3 for details). Next, we calculate new point $\boldsymbol{p}_1 := (x_1^{(1)}, x_2^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)}, t^{(1)})$ on the solution set G using $\boldsymbol{p}_P$ as an initial approximation. We search for $\boldsymbol{p}_1$ in the plane which contains $\boldsymbol{p}_P$ and is perpendicular to the tangent vector $\dot{\boldsymbol{p}}_0$. A different approach is used whether we follow a real or a complex curve.

   If $\boldsymbol{\lambda} \in \mathbb{R}^2$, we add the equation

$$\dot{\boldsymbol{p}}_0^*(\boldsymbol{p}_1 - \boldsymbol{p}_P) = 0 \tag{24}$$

   to the equation $H(\boldsymbol{p}_1) = 0$ and solve the obtained system with Newton's method using $\boldsymbol{p}_P$ as an initial approximation.

   If $\boldsymbol{\lambda} \notin \mathbb{R}^2$, then we add the real part of the equation (24) to the equation $H(\boldsymbol{p}_1) = 0$. We consider the obtained system as a real system of $2n_1 + 2n_2 + 5$ equations in $2n_1 + 2n_2 + 5$ unknowns and use Newton's method as in the previous real case.

2. $\dot{t} < 0$. In this case we passed a turning point. We use a special algorithm to detect a turning point $\boldsymbol{p}_T$ (see Subsection 5.2 for details). We continue the calculation from $\boldsymbol{p}_T$ using $i\dot{\boldsymbol{p}}_T$ as the tangent vector.

We end the algorithm when t is close enough to 1.

5.2 Calculation of the turning point

Suppose that after one step of the prediction-correction scheme we obtain a point $\boldsymbol{p}_1$ with the unit tangent vector $\dot{\boldsymbol{p}}_1 := (\dot{x}_1^{(1)}, \dot{x}_2^{(1)}, \dot{\lambda}_1^{(1)}, \dot{\lambda}_2^{(1)}, \dot{t}^{(1)})$ and $\dot{t}^{(1)} < 0$. We passed a turning

point $\boldsymbol{p}_T = (x_1^{(T)}, x_2^{(T)}, \lambda_1^{(T)}, \lambda_2^{(T)}, t^{(T)})$ where the tangent vector $\dot{\boldsymbol{p}}_T$ has the component $\dot{t}^{(T)} = 0$.

If we are following a real curve then for the part of the curve between points $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ we can use λ_1 or λ_2 as a parameter. If we choose λ_1 then the following algorithm calculates the turning point $\boldsymbol{p}_T$.

1. Calculate
$$\lambda_1^{(2)} = \frac{\dot{t}^{(0)}\lambda_1^{(1)} - \dot{t}^{(1)}\lambda_1^{(0)}}{\dot{t}^{(1)} - \dot{t}^{(0)}} \tag{25}$$
as an approximation for the value of λ_1 at the turning point $\boldsymbol{p}_T$. Compute the stepsize
$$h = \frac{\lambda_1^{(2)} - \lambda_1^{(0)}}{\dot{\lambda}_1^{(0)}}$$
that returns Euler's prediction point $\boldsymbol{p}_P = \boldsymbol{p}_0 + h\dot{\boldsymbol{p}}_0$ such that $\lambda_1^{(P)} = \lambda_1^{(2)}$.
Solve the system $H(x_1^{(2)}, x_2^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)}, t^{(2)}) = 0$, where $\lambda_1^{(2)}$ is fixed, with Newton's method using the initial approximation $(x_1^{(P)}, x_2^{(P)}, \lambda_2^{(P)}, t^{(P)})$.

2. If $\dot{t}(\boldsymbol{p}_2) > 0$, then replace $\boldsymbol{p}_0$ with $\boldsymbol{p}_2$, otherwise replace $\boldsymbol{p}_1$ with $\boldsymbol{p}_2$.

3. If $|\dot{t}(\boldsymbol{p}_2)|$ is small enough, then set $\boldsymbol{p}_T = \boldsymbol{p}_2$, otherwise repeat step 1.

A similar strategy is used for a complex curve. In this case we use the imaginary part of λ_1 as a parameter. The last two steps of the algorithm are the same as in the real case. The modified first step is:

1. Calculate $\mathrm{Im}(\lambda_1^{(2)})$ from (25) as an approximation for $\mathrm{Im}(\lambda_1^{(T)})$ at the turning point. Compute the stepsize
$$h = \frac{\mathrm{Im}(\lambda_1^{(2)} - \lambda_1^{(0)})}{\mathrm{Im}(\dot{\lambda}_1^{(0)})}$$
that returns Euler's prediction point $\boldsymbol{p}_P$ such that that $\mathrm{Im}(\lambda_1^{(P)}) = \mathrm{Im}(\lambda_1^{(2)})$.
Consider the system $H(x_1^{(2)}, x_2^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)}, t^{(2)}) = 0$, where $\mathrm{Im}(\lambda_1^{(2)})$ is fixed, as a real system of $2n_1 + 2n_2 + 4$ equations in $2n_1 + 2n_2 + 4$ unknowns. Solve it with Newton's method using the initial approximation $(x_1^{(P)}, x_2^{(P)}, \mathrm{Re}(\lambda_1^{(P)}), \lambda_2^{(P)}, t^{(P)})$.

5.3 Adjusting the step size

In the beginning we choose the initial stepsize h in accordance with the size of the tangent vector $\dot{t}$. In the algorithm we decrease or increase the stepsize corresponding to the behaviour of the solution curve.

- If any of the following situations appear, then we decrease the stepsize to $h/2$ and return to the last well-calculated point:

  a) Newton's method fails to converge from a given prediction point to a nearby point on the solution curve;

b) search for a bifurcation point fails to converge;

c) we calculate $t_1 < t_0$;

d) the angle between the two consecutive tangent vectors (respectively, eigenvectors) is larger than the maximal allowed angle.

- If the curve is flat, then we increase the stepsize to $2h$. The criteria for this is a small angle between the two consecutive tangent vectors and a small angle between the two consecutive eigenvectors.
- If the prediction $t^P = t^{(0)} + h\dot{t}$ is greater than 1, then we decrease h so that $t^P = 1$.

5.4 Avoiding the curve switching

In contrast to a right definite two-parameter problem, where the minimal angle between eigenvectors exists [18], eigenvectors can be arbitrary close in our case. Because of this it may happen that in a numerical process we switch from one solution curve to another. Curve switching may appear for many reasons, for instance:

- the stepsize is too large;
- the criteria for convergence of Newton's method is not strict enough;
- the maximal allowed angle between the two consecutive eigenvectors is too large.

To avoid curve switching we use more strict criteria for stepsize control and convergence of Newton's method and we restrict the allowed angle between the two consecutive eigenvectors.

From our numerical experiments it follows that the phenomenon of curve switching is not so frequent that it would be advisable to follow all curves with high precision. Instead of this we follow all curves with moderate precision. In the end we compare all eigenvalues and if we find two close eigenvalues we compare their eigenvectors. If the eigenvectors are also close to each other then it is very likely that a curve switching did appear when we followed one of these two curves. We repeat the computation for both curves using higher precision and stricter criteria.

If the problem (1) has a multiple eigenvalue such that the geometric multiplicity is smaller than the algebraic one, then at least two curves collide at $t = 1$ independently of the precision used. If such an eigenvalue is nonderogatory, then it is possible to calculate a basis for the root subspace using the algorithm in [17]. Košir proved in [10, Theorem 19] that if $\boldsymbol{\lambda}$ is a real geometrically simple eigenvalue of a weakly elliptic system (1), then $\boldsymbol{\lambda}$ is a nonderogatory eigenvalue.

5.5 Time complexity

If we treat the simultaneous problem (2) by standard numerical methods then the time complexity is $\mathcal{O}(n_1^3 n_2^3)$. Using this approach we are dealing with matrices of order $n_1 n_2$. On the other hand, matrices in the continuation method are of moderate size $\mathcal{O}(n_1 + n_2)$

and one step of Newton's method has time complexity $\mathcal{O}((n_1+n_2)^3)$. We have to multiply this quantity first with $n_1 n_2$ as we are following $n_1 n_2$ solution curves and second with the number of steps that Newton's method is used per curve. If we take $n = n_1 = n_2$ and denote the average number of times that Newton's method is evaluated per curve by $\phi(n)$, then the time complexity of the continuation method is $\mathcal{O}(n^5\phi(n))$.

Numerical experiments in the next section indicate that it is possible that $\phi(n)$ is of order $o(n)$ and that the continuation method requires less operations for large n than the simultaneous problem (2). In order to confirm or reject this hypothesis further properties of the algorithm have to be studied and experiments with larger matrices have to be performed.

The continuation method allows an elegant parallel implementation. Each processor can follow its own solution curve, therefore n^2 processors can decrease the time complexity to $\mathcal{O}(n^3\phi(n))$. Let us also remark that no sophisticated parallel computers are needed for the parallel computation of the solution curves. We simply divide n^2 curves between the available serial computers and join their results when they finish. For comparison, if we use n^2 processors then the time complexity of the parallel algorithm for the simultaneous problem (2) is $\mathcal{O}(n^4)$ [7, 20].

Another advantage of the continuation method is its low consumption of memory. The simultaneous problem requires $\mathcal{O}(n^4)$ data for the matrices, while the continuation method requires only $\mathcal{O}(n^2)$. As a result, the size of the main memory does not limit the continuation method as much as it does the simultaneous problem.

6 Numerical examples

The method was implemented in Matlab and tested numerically on randomly generated weakly elliptic two-parameter problems. The results show that the complexity of the continuation method is comparable with the complexity of solving the simultaneous problem (2) with standard methods.

We generate a random weakly elliptic problem with matrices of size n using the following sequence:

```
rand('state',0);
V11=eye(n); p=rand(n,1); V12=diag(p); p=rand(n); V10=p+p';
p=rand(n); V21=p+p'; p=rand(n); V22=p+p'; p=rand(n); V20=p+p';
c=rand(n,2)/2+0.5;
```

We tested the method for different values of n and compared the number of operations with the solution of the simultaneous problem (2). The continuation method successfully calculated all 2025 eigenpairs for the largest matrices on test of order $n = 45$.

In the algorithm we use 10^{-7} as the minimal allowed stepsize h. We end the algorithm when $|t-1| \leq 10^{-6}$. In each step we iterate the Newton's method until the residue

$$\left(\|W_1(\boldsymbol{\lambda},t)x_1\|^2 + \|W_2(\boldsymbol{\lambda},t)x_2\|^2\right)^{\frac{1}{2}}$$

is less than 10^{-7} for $t < 1-10^{-5}$ and less than 10^{-10} for $t \geq 1-10^{-5}$. To continue, the real part of the inner product between the last two consecutive eigenvectors has to be greater

than 0.85, otherwise we decrease the stepsize h. The same applies to the eigenvector part of the last two consecutive tangent vectors. On the other hand, if both inner products have the real part larger than 0.95, we increase the stepsize.

In the end we compare the eigenpairs and recalculate all the curves where

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\| \leq 10^{-3}(1 + \|\boldsymbol{\lambda}\|)$$

and the minimum singular value of the matrix

$$(x_1 \otimes x_2 \quad x_1' \otimes x_2')$$

is smaller than 10^{-3}.

For the matrix A_{10} we take $A_{10} = \mathrm{diag}(1, \ldots, n)$ and for A_{20} we take $A_{20} = 0$. It is easy to see that if we take $A_2 = \mathrm{diag}(m, m+1, \ldots, n+m)$, where $m = n\|V_{21}\|$, then all initial eigenpairs are algebraically simple and real.

Table 1 shows the statistics for different matrix orders. The total number of operations for the continuation method in the second column also includes operations for the recalculated curves. It is clearly seen from the fourth column that the ratio is decreasing. The fifth column contains the average number of evaluations of Newton's method per curve and the sixth column contains the average number of bifurcations per curve. Practically all solution curves contain bifurcation points and there are no easy paths as in the algorithm by Li, Zeng, and Cong (see [12]).

Matrix order	Number of operations		ratio	Aver. no. of	Aver. no. of
n	continuation	QR	(cont. : QR)	steps	bifurcations
5	$50.7 \cdot 10^6$	$82.4 \cdot 10^4$	61.5	210	1.4
10	$13.5 \cdot 10^8$	$49.3 \cdot 10^7$	27.5	317	1.2
15	$16.0 \cdot 10^9$	$51.9 \cdot 10^7$	30.7	556	2.7
20	$57.0 \cdot 10^9$	$29.5 \cdot 10^8$	19.3	594	2.2
25	$19.3 \cdot 10^{10}$	$11.3 \cdot 10^9$	17.0	720	3.0
30	$46.1 \cdot 10^{10}$	$32.8 \cdot 10^9$	14.1	778	3.3
35	$10.3 \cdot 10^{11}$	$81.2 \cdot 10^9$	12.7	868	3.2
40	$21.2 \cdot 10^{11}$	$18.0 \cdot 10^{10}$	11.8	975	3.2
45	$36.1 \cdot 10^{11}$	$36.6 \cdot 10^{10}$	9.9	956	2.8

Table 1: Statistics for $n = 5, 10, \ldots, 45$. Flops count for the continuation method, for the simultaneous problem (2), their ratio, the average number of evaluations of Newton's method per curve and the average number of bifurcations per curve.

Table 2 shows how repeated computation of some curves contribute the total number of operations. The percentage of operations for recalculated curves is greater from the percentage of recalculated curves itself. This is a consequence of higher precision and more strict criteria used for repeated computation.

Since higher precision and more strict criteria is used, repeated computation require more operations per curve than the initial computation. The number of recalculated curves can be decreased by more strict criteria but this increases the number of operations

Matrix order n	Number of recalculated curves	Percentage of recalculated curves	Percentage of flops for recalculated curves
15	12	5.3	14.4
20	10	2.5	4.1
25	2	0.3	0.5
30	31	3.4	5.0
35	26	2.1	4.4
40	40	2.5	2.6
45	52	2.6	4.7

Table 2: Statistics for recalculated curves for $n = 15, 20, \ldots, 45$.

per curve in the initial stage. On the other hand, more relaxed criteria increases the number of recalculated curves and decreases the number of operation in the initial stage. In order to minimize the total number of operations one should carefully calibrate the criteria.

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