

On linearizations of the quadratic two-parameter eigenvalue problems

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Abstract

We present several transformations that can be used to solve the quadratic two-parameter eigenvalue problem (QMEP), by formulating an associated linear multiparameter eigenvalue problem. Two of these transformations are generalizations of the well-known linearization of the quadratic eigenvalue problem and linearize the QMEP as a singular two-parameter eigenvalue problem. The third one substitutes all nonlinear terms with new variables and adds new equations for their relations. The QMEP is thus transformed into a nonsingular five-parameter eigenvalue problem. The advantage of these transformations is that they enable one to solve the QMEP using the existing numerical methods for the multiparameter eigenvalue problems. We also consider several special cases of the QMEP, where some of the quadratic terms are missing.

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Dedicated to the 65th birthday of Dan Sorensen.

1. Introduction

The linear multiparameter eigenvalue problem [1] and in particular the two-parameter case, has been studied for several decades. For an overview of the recent work on numerical solutions see, e.g., [3, 4, 8, 9] and references therein.

Currently, there is an increasing interest in the quadratic two-parameter eigenvalue problem (QMEP) [5, 9], which has a general form

$$\begin{aligned} Q_1(\lambda, \mu) x_1 &:= (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \lambda \mu A_{11} + \mu^2 A_{02}) x_1 = 0, \\ Q_2(\lambda, \mu) x_2 &:= (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \lambda \mu B_{11} + \mu^2 B_{02}) x_2 = 0. \end{aligned} \tag{1}$$

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where A_{ij}, B_{ij} are given $n_i \times n_i$ complex matrices, $x_i \in \mathbb{C}^{n_i}$ is a nonzero vector for $i = 1, 2$, and $\lambda, \mu \in \mathbb{C}$. We say that (λ, μ) is an eigenvalue of (1) and the tensor product $x_1 \otimes x_2$ is the corresponding eigenvector. We note that the QMEP is a recently recognized new type of eigenvalue problem. See [10] for a nice overview of standard and generalized eigenvalue problems.

In the generic case the QMEP (1) has $4n_1n_2$ eigenvalues that are the roots of the system of the bivariate characteristic polynomials $\det(Q_i(\lambda, \mu)) = 0$ of order $2n_i$ for $i = 1, 2$. This follows from Bézout's theorem (see, e.g., [2]), which states that two projective curves of orders n and m with no common component have precisely nm points of intersection counting multiplicities. To simplify the notation, we will assume from now on that $n_1 = n_2 = n$.

It is well known that one can solve the quadratic eigenvalue problem by linearizing it as a generalized eigenvalue problem with matrices of double dimension (see, e.g., [11]). This approach was generalized to the QMEP in [9], where (1) is linearized as a singular two-parameter eigenvalue problem

$$\begin{aligned} L_1(\lambda, \mu)w_1 &:= \left(A^{(1)} + \lambda B^{(1)} + \mu C^{(1)} \right) w_1 = 0 \\ L_2(\lambda, \mu)w_2 &:= \left(A^{(2)} + \lambda B^{(2)} + \mu C^{(2)} \right) w_2 = 0, \end{aligned} \quad (2)$$

where

$$L_i(\lambda, \mu)w_i = \left(\overbrace{\begin{bmatrix} A_{00} & A_{10} & A_{01} \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}}^{A^{(i)}} + \lambda \overbrace{\begin{bmatrix} 0 & A_{20} & A_{11} \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{B^{(i)}} + \mu \overbrace{\begin{bmatrix} 0 & 0 & A_{02} \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}}^{C^{(i)}} \right) \overbrace{\begin{bmatrix} x_i \\ \lambda x_i \\ \mu x_i \end{bmatrix}}^{w_i} \quad (3)$$

and the matrices $A^{(i)}, B^{(i)}$, and $C^{(i)}$ are of size $3n \times 3n$ for $i = 1, 2$. The numerical method for singular two-parameter eigenvalue problems presented in [9] can then be used to solve the problem (2) and retrieve the eigenpairs of (1).

In this paper we present new relations between the QMEP and the linear multiparameter eigenvalue problem that lead to new numerical methods for solving the QMEP. For some special cases of (1), where some of the terms are missing, we provide linearizations that are more efficient than for the general case. For example, a simplified QMEP, where all of the terms λ^2 and μ^2 are missing, appears in the study of linear time-delay systems for the single delay case [5]. In subsection 5.3 we show that such problem can be studied as a nonsingular three-parameter eigenvalue problem.

In Section 2 we give a short overview of the linear multiparameter eigenvalue problems. In Section 3 we give two linearizations of the QMEP as a singular two-parameter eigenvalue problem while in Section 4 we show that one may also treat the QMEP as a five-parameter eigenvalue problem. Some special cases of the QMEP are considered in Section 5, and in Section 6 we extend the methods to polynomial two-parameter eigenvalue problems.

2. The linear multiparameter eigenvalue problem

The homogeneous multiparameter eigenvalue problem (MEP) has the form

$$W_i^h(\boldsymbol{\eta})x_i = \sum_{j=0}^k \eta_j V_{ij}x_i = 0, \quad i = 1, \dots, k, \quad (4)$$

where V_{ij} is an $n_i \times n_i$ complex matrix for $j = 0, \dots, k$. A nonzero $(k+1)$ -tuple $\boldsymbol{\eta} = (\eta_0, \eta_1, \dots, \eta_k)$ that satisfies (4) for a nonzero $x_i \in \mathbb{C}^{n_i}$ is called an *eigenvalue* while the tensor product $\mathbf{x} = x_1 \otimes \dots \otimes x_k$ is the corresponding *eigenvector*.

We can study the MEP (4) in the tensor product space $\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$, which is isomorphic to \mathbb{C}^N , where $N = n_1 \dots n_k$, as follows. The linear transformations V_{ij} induce linear transformations V_{ij}^\dagger on \mathbb{C}^N . For a decomposable tensor,

$$V_{ij}^\dagger(x_1 \otimes \dots \otimes x_k) = x_1 \otimes \dots \otimes V_{ij}^\dagger x_i \otimes \dots \otimes x_k.$$

V_{ij}^\dagger is then extended to all of \mathbb{C}^N by linearity. On \mathbb{C}^N we define operator determinants

$$\Delta_0 = \begin{vmatrix} V_{11}^\dagger & V_{12}^\dagger & \dots & V_{1k}^\dagger \\ V_{21}^\dagger & V_{22}^\dagger & \dots & V_{2k}^\dagger \\ \vdots & \vdots & & \vdots \\ V_{k1}^\dagger & V_{k2}^\dagger & \dots & V_{kk}^\dagger \end{vmatrix}$$

and

$$\Delta_i = \begin{vmatrix} V_{11}^\dagger & \dots & V_{1,i-1}^\dagger & V_{10}^\dagger & V_{1,i+1}^\dagger & \dots & V_{1k}^\dagger \\ V_{21}^\dagger & \dots & V_{2,i-1}^\dagger & V_{20}^\dagger & V_{2,i+1}^\dagger & \dots & V_{2k}^\dagger \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{k1}^\dagger & \dots & V_{k,i-1}^\dagger & V_{k0}^\dagger & V_{k,i+1}^\dagger & \dots & V_{kk}^\dagger \end{vmatrix}$$

for $i = 1, \dots, k$.

A homogeneous MEP is called *nonsingular* if there exists a nonsingular linear combination

$$\Delta = \sum_{i=0}^k \alpha_i \Delta_i$$

of operator determinants $\Delta_0, \dots, \Delta_k$. A nonsingular homogeneous MEP is equivalent to the joint generalized eigenvalue problems

$$\Delta_i \mathbf{x} = \eta_i \Delta \mathbf{x}, \quad i = 0, \dots, k,$$

for decomposable tensors $\mathbf{x} = x_1 \otimes \dots \otimes x_k \in \mathbb{C}^N$. It turns out that the matrices $\Gamma_i := \Delta^{-1} \Delta_i$ commute for $i = 0, \dots, k$ (see [1]).

Theorem 1 ([1, Theorem 8.7.1]). *The following two statements for the homogeneous multiparameter eigenvalue problem (4) are equivalent:*

1. *The matrix $\Delta = \sum_{i=0}^k \alpha_i \Delta_i$ is nonsingular.*
2. *If $\boldsymbol{\eta} = (\eta_0, \eta_1, \dots, \eta_k)$ is an eigenvalue of (4) then $\sum_{i=0}^k \eta_i \alpha_i \neq 0$.*

Let us remark that we usually study the nonhomogeneous multiparameter eigenvalue problem

$$W_i(\boldsymbol{\lambda}) x_i = V_{i0} x_i + \sum_{j=1}^k \lambda_j V_{ij} x_i = 0, \quad i = 1, \dots, k, \quad (5)$$

where λ is a k -tuple $\lambda = (\lambda_1, \dots, \lambda_k)$. Such a problem is called nonsingular when Δ_0 is nonsingular. One can see that $W_i^h((1, \lambda_1, \dots, \lambda_k)) = W_i(\lambda)$ and instead of (5) we can study the homogeneous problem (4).

If η is an eigenvalue of (4), such that η_0 is nonzero, then $\lambda = (\eta_1/\eta_0, \dots, \eta_k/\eta_0)$ is an eigenvalue of (5). If (5) is nonsingular, then we can take $\Delta = \Delta_0$ and it follows from Theorem 1 that all eigenvalues of (4) are such that $\eta_0 \neq 0$.

If Δ_0 is singular, then there exists at least one eigenvalue η of (4) having $\eta_0 = 0$. In this case we say that (5) has an infinite eigenvalue. The finite eigenvalues of (5) can be numerically computed from the joint generalized eigenvalue problems

$$\Delta_i x = \lambda_i \Delta_0 x, \quad i = 1, \dots, k,$$

where $x = x_1 \otimes \dots \otimes x_k$, using the generalized staircase algorithm for the extraction of the common regular part of singular pencils from [9].

3. Two different linearizations by MEP

The following straightforward generalization of the linearization of a standard univariate matrix polynomial (see, e.g., [7]) is given in [9].

Definition 2. An $ln \times ln$ linear matrix pencil $L(\lambda, \mu) = A + \lambda B + \mu C$ is a linearization of order ln of an $n \times n$ matrix polynomial $Q(\lambda, \mu)$ if there exist matrix polynomials $P(\lambda, \mu)$ and $R(\lambda, \mu)$, whose determinant is a nonzero constant independent of λ and μ , such that

$$\begin{bmatrix} Q(\lambda, \mu) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix} = P(\lambda, \mu)L(\lambda, \mu)R(\lambda, \mu).$$

It follows from [9, Theorem 22] that the two-parameter eigenvalue problem (2) is indeed a linearization of the QMEP (1). As shown in [9], (2) is singular even in the homogeneous setting (4) and in the general case the QMEP (1) has $4n^2$ eigenvalues which are (see [9, Theorem 17]) exactly the finite eigenvalues of (2).

Another linearization of the two-parameter matrix polynomial was presented even earlier by Khazanov [6]. In his approach we first write $Q_1(\lambda, \mu)x_1 = 0$ as a polynomial in λ :

$$(A_{00} + \mu A_{01} + \mu^2 A_{02} + \lambda(A_{10} + \mu A_{11}) + \lambda^2 A_{20})x_1 = 0. \quad (6)$$

Then we use the standard first companion form (see, e.g., [11]) and linearize (6) as

$$\left(\begin{bmatrix} A_{00} + \mu A_{01} + \mu^2 A_{02} & A_{10} + \mu A_{11} \\ 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & A_{20} \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ \lambda x_1 \end{bmatrix} = 0. \quad (7)$$

We rewrite (7) as a quadratic polynomial in μ

$$\left(\begin{bmatrix} A_{00} & A_{10} + \lambda A_{20} \\ \lambda I & -I \end{bmatrix} + \mu \begin{bmatrix} A_{01} & A_{11} \\ 0 & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} A_{02} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ \lambda x_1 \end{bmatrix} = 0$$

and linearize it using the first companion form as

$$\left(\left(\begin{array}{cc|cc} A_{00} & A_{10} + \lambda A_{20} & A_{01} & A_{11} \\ \lambda I & -I & 0 & 0 \\ \hline 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{array} \right) + \mu \left(\begin{array}{cc|cc} 0 & 0 & A_{02} & 0 \\ 0 & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array} \right) \right) \begin{bmatrix} x_1 \\ \lambda x_1 \\ \mu x_1 \\ \lambda \mu x_1 \end{bmatrix} = 0,$$

which is equivalent to

$$\left(\begin{bmatrix} A_{00} & A_{10} & A_{01} & A_{11} \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & A_{20} & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & A_{02} & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ \lambda x_1 \\ \mu x_1 \\ \lambda \mu x_1 \end{bmatrix} = 0. \quad (8)$$

It is obvious from the construction itself that (8) is really a linearization of $Q_1(\lambda, \mu)$. We can repeat this for the second polynomial $Q_2(\lambda, \mu)$ and obtain a linear two-parameter eigenvalue problem. The matrices in (8) are of size $4n \times 4n$, which makes the Khazanov linearization less efficient than the linearization (3), where matrices are of size $3n \times 3n$. In fact, we now show that the linearization (3) is a reduction of the linearization (8).

Theorem 3. *The Khazanov linearization (8) of the $n \times n$ quadratic matrix polynomial $Q_1(\lambda, \mu)$ can be reduced to the linearization (3) proposed in [9].*

PROOF. If we multiply the matrices in (8) by the nonsingular matrices with a constant determinant

$$E(\lambda, \mu) = \begin{bmatrix} I & \mu A_{11} & -\lambda A_{11} & A_{11} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} \quad \text{and} \quad F(\lambda, \mu) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & \mu I & 0 & I \end{bmatrix}$$

from the left and the right side, respectively, then we obtain

$$\begin{bmatrix} A_{00} & A_{10} & A_{01} & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} + \lambda \begin{bmatrix} 0 & A_{20} & A_{11} & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & A_{02} & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This clearly shows, in view of the leading 3×3 block, that the linearization (3) is a reduction of the linearization proposed by Khazanov. \square

Not surprisingly, the two-parameter eigenvalue problem that we obtain when we linearize Q_1 and Q_2 by the Khazanov linearization, is singular as well. We omit the details, but using similar technique as in [9] one can show that all linear combinations of the corresponding operator determinants Δ_0, Δ_1 , and Δ_2 are singular.

Because it produces smaller matrices, the linearization proposed by Muhič and Plestenjak in [9] is more suitable for the general QMEP than the Khazanov linearization. But, as we will see later, the approach by Khazanov may be more efficient for some special QMEPs, where some of the terms are missing.

Finally, we note that in fact both linearizations are not optimal in view of the following observations. The bivariate polynomial $\det(Q_1(\lambda, \mu))$ is of order $2n$. In theory (see [12]), for a given bivariate polynomial $p(\lambda, \mu)$ of order $2n$, there should exist a so-called determinantal representation with matrices A, B , and C of size $2n \times 2n$, such that $\det(A + \lambda B + \mu C) = p(\lambda, \mu)$. However, it is not known how to construct the matrices A, B , and C . Even if this could be done, it is unlikely that A, B , and C could be constructed as 2×2 block matrices, which would make the construction useful for the QMEP.

4. Linearization like method

The approach proposed in the previous section is to linearize the QMEP as a two-parameter eigenvalue problem, which we can later solve using the operator determinants and the algorithm for the extraction of the common regular part of singular pencils from [9]. In the final step of this procedure we have to compute the finite eigenvalues of the coupled singular pencils

$$\begin{aligned} (\Delta_1 - \lambda\Delta_0)z &= 0 \\ (\Delta_2 - \mu\Delta_0)z &= 0. \end{aligned} \tag{9}$$

The matrices Δ_0, Δ_1 , and Δ_2 in (9) are of size $9n^2 \times 9n^2$ if we use linearization (3) or $16n^2 \times 16n^2$ if we use the Khazanov linearization (8). In both cases the common regular part that contains all the finite eigenvalues of (1) has dimension $4n^2$.

A new approach that we present in this section, is not a linearization in the sense of Definition 2. Yet, it involves multiparameter eigenvalue problems and in the end we obtain the eigenvalues of (1) from a pair of generalized eigenvalue problems of the kind (9). The advantage is that the matrices are of size $8n^2 \times 8n^2$, which is smaller, and, even more important, the obtained pencils are not singular.

We start with the QMEP (1) and introduce new variables $\alpha = \lambda^2, \beta = \lambda\mu$, and $\gamma = \mu^2$. Then we can write (1) as a linear five-parameter eigenvalue problem

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \alpha A_{20} + \beta A_{11} + \gamma A_{02})x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \alpha B_{20} + \beta B_{11} + \gamma B_{02})x_2 &= 0 \\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) y_1 &= 0 \\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) y_2 &= 0 \\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \mu \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) y_3 &= 0. \end{aligned} \tag{10}$$

It is easy to see that each eigenpair of the QMEP (1) gives an eigenpair of (10). Namely, if $((\lambda, \mu), x_1 \otimes x_2)$ is an eigenpair of (1) then

$$\left((\lambda, \mu, \lambda^2, \lambda\mu, \mu^2), x_1 \otimes x_2 \otimes \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \mu \end{bmatrix} \right)$$

is an eigenpair of (10).

The next lemma shows that, in contrast to the singular two-parameter eigenvalue problems of the linearizations from Section 3, the five-parameter problem (10) is nonsingular.

Lemma 4. *In the general case, the homogeneous version of the obtained five-parameter eigenvalue problem (10) is nonsingular. In particular, the related operator determinants Δ_3, Δ_4 , and Δ_5 are all nonsingular.*

PROOF. The homogeneous version of (10), where we write $\lambda = \tilde{\lambda}/\tilde{\eta}, \mu = \tilde{\mu}/\tilde{\eta}, \alpha = \tilde{\alpha}/\tilde{\eta}, \beta = \tilde{\beta}/\tilde{\eta}, \gamma = \tilde{\gamma}/\tilde{\eta}$, and multiply all equations by $\tilde{\eta}$, results in the following system (it suffices to look at the

determinants only):

$$\begin{aligned}
\det(\tilde{\eta}A_{00} + \tilde{\lambda}A_{10} + \tilde{\mu}A_{01} + \tilde{\alpha}A_{20} + \tilde{\beta}A_{11} + \tilde{\gamma}A_{02}) &= 0 \\
\det(\tilde{\eta}B_{00} + \tilde{\lambda}B_{10} + \tilde{\mu}B_{01} + \tilde{\alpha}B_{20} + \tilde{\beta}B_{11} + \tilde{\gamma}B_{02}) &= 0 \\
\tilde{\alpha}\tilde{\eta} - \tilde{\lambda}^2 &= 0 \\
\tilde{\beta}\tilde{\eta} - \tilde{\lambda}\tilde{\mu} &= 0 \\
\tilde{\gamma}\tilde{\eta} - \tilde{\mu}^2 &= 0.
\end{aligned} \tag{11}$$

Suppose that $(\tilde{\eta}, \tilde{\lambda}, \tilde{\mu}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is an eigenvalue of (11) such that $\tilde{\alpha} = 0$. Then the equations (11) transform into

$$\begin{aligned}
\det(\tilde{\eta}A_{00} + \tilde{\lambda}A_{10} + \tilde{\mu}A_{01} + \tilde{\beta}A_{11} + \tilde{\gamma}A_{02}) &= 0 \\
\det(\tilde{\eta}B_{00} + \tilde{\lambda}B_{10} + \tilde{\mu}B_{01} + \tilde{\beta}B_{11} + \tilde{\gamma}B_{02}) &= 0 \\
-\tilde{\lambda}^2 &= 0 \\
\tilde{\beta}\tilde{\eta} - \tilde{\lambda}\tilde{\mu} &= 0 \\
\tilde{\gamma}\tilde{\eta} - \tilde{\mu}^2 &= 0.
\end{aligned} \tag{12}$$

From the third equation we get $\tilde{\lambda} = 0$, by substituting this in the fourth equation we get $\tilde{\eta}\tilde{\beta} = 0$. We consider two options:

- a) $\tilde{\eta} = 0$. In this case it follows from the last row of (12) that $\tilde{\mu} = 0$. What remains from the first two rows of (12) is the system

$$\begin{aligned}
\det(\tilde{\beta}A_{11} + \tilde{\gamma}A_{02}) &= 0 \\
\det(\tilde{\beta}B_{11} + \tilde{\gamma}B_{02}) &= 0,
\end{aligned}$$

which has no solutions in the generic case.

- b) $\tilde{\eta} \neq 0$. Then $\tilde{\beta} = 0$ and it follows from the last row of (12) that $\tilde{\gamma} = \tilde{\mu}^2/\tilde{\eta}$. From the first two rows of (12) we obtain the system

$$\begin{aligned}
\det\left(A_{00} + \frac{\tilde{\mu}}{\tilde{\eta}}A_{01} + \frac{\tilde{\mu}^2}{\tilde{\eta}^2}A_{02}\right) &= 0 \\
\det\left(B_{00} + \frac{\tilde{\mu}}{\tilde{\eta}}B_{01} + \frac{\tilde{\mu}^2}{\tilde{\eta}^2}B_{02}\right) &= 0,
\end{aligned}$$

which again has no solutions in the generic case.

Therefore, in the generic case the problem (10) does not have an eigenvalue with $\tilde{\alpha} = 0$. It follows from Theorem 1 that Δ_3 is nonsingular. Similarly we can obtain that Δ_4 and Δ_5 are nonsingular. \square

In the generic case we can assume that the QMEP (1) does not have an eigenvalue (λ, μ) such that $\lambda = 0$. If we take $\Delta = \Delta_3$ then the appropriate system of coupled matrix pencils is

$$\begin{aligned}
(\Delta_0 - \tilde{\eta}\Delta)z &= 0, & (\Delta_1 - \tilde{\lambda}\Delta)z &= 0, & (\Delta_2 - \tilde{\mu}\Delta)z &= 0, \\
(\Delta_3 - \tilde{\alpha}\Delta)z &= 0, & (\Delta_4 - \tilde{\beta}\Delta)z &= 0, & (\Delta_5 - \tilde{\gamma}\Delta)z &= 0,
\end{aligned}$$

where $z = x_1 \otimes x_2 \otimes y_1 \otimes y_2 \otimes y_3$. Clearly, $\tilde{\alpha} \equiv 1$. As we are only interested in the solution of the QMEP (1), it is enough to consider just two of the above matrix pencils.

Theorem 5. *In the generic case, the pair of matrix pencils*

$$\begin{aligned}(\Delta_1 - \tilde{\lambda}\Delta_3)z &= 0 \\ (\Delta_2 - \tilde{\mu}\Delta_3)z &= 0,\end{aligned}$$

associated to the five-parameter eigenvalue problem (10), has $8n^2$ eigenvalues $(\tilde{\lambda}, \tilde{\mu})$, of which

- a) $4n^2$ eigenvalues are such that $\tilde{\lambda} \neq 0$. Each such eigenvalue corresponds to a finite eigenvalue (λ, μ) of the QMEP (1), where

$$\lambda = 1/\tilde{\lambda}, \quad \mu = \tilde{\mu}\lambda^2; \tag{13}$$

- b) the remaining $4n^2$ eigenvalues are such that $\tilde{\lambda} = 0$. These spurious eigenvalues are a result of the transformation and are not related to the eigenvalues of (1).

PROOF. a) We know from the construction that to each eigenvalue (λ, μ) of (2) there corresponds the eigenvalue $(\lambda, \mu, \lambda^2, \lambda\mu, \mu^2)$ of (10) and the eigenvalue $(1/\lambda^2, 1/\lambda, \mu/\lambda^2, 1, \mu/\lambda, \mu^2/\lambda^2)$ in the homogeneous setting (11). In the generic case (1) has $4n^2$ eigenvalues that can be extracted from (2) using the equations (13).

b) Suppose that $(0, \tilde{\lambda}, \tilde{\mu}, 1, \tilde{\beta}, \tilde{\gamma})$ is an eigenvalue of (11). It follows from the last three rows of (11) that

$$\begin{aligned}\tilde{\lambda}^2 &= 0 \\ \tilde{\lambda}\tilde{\mu} &= 0 \\ \tilde{\mu}^2 &= 0,\end{aligned} \tag{14}$$

therefore $\tilde{\lambda} = \tilde{\mu} = 0$. From the first two equations of (11) we get a two-parameter eigenvalue problem

$$\begin{aligned}\det(A_{20} + \tilde{\beta}A_{11} + \tilde{\gamma}A_{02}) &= 0 \\ \det(B_{20} + \tilde{\beta}B_{11} + \tilde{\gamma}B_{02}) &= 0,\end{aligned}$$

which has n^2 eigenvalues $(\tilde{\beta}, \tilde{\gamma})$ in the generic case. Together with (14) we can now count that (10) has $4n^2$ eigenvalues with $\tilde{\lambda} = 0$. \square

The transformation of the QMEP to a five-parameter eigenvalue problem has an advantage that in the end we work with nonsingular pencils and therefore we can apply more efficient numerical methods. A disadvantage is that the 5×5 operator determinants Δ_i are not as sparse and thus more expensive to compute than for the two-parameter eigenvalue problems from Section 3.

5. Special cases of the quadratic two-parameter eigenvalue problem

In this section we study special cases of the QMEP, where some of the quadratic terms $\lambda^2, \lambda\mu, \mu^2$ are missing. There are two reasons to do so. First, applications may lead to these special types instead of the general form (1); an example are linear time-delay systems for the single delay case [5]. Second, we can use the special structure to develop special tailored methods that are more efficient and simpler in nature than the approaches for the general QMEP (1).

5.1. Both equations missing the $\lambda\mu$ term

If both $\lambda\mu$ terms in (1) are missing (i.e., $A_{11} = B_{11} = 0$), then the QMEP has the form

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \mu^2 A_{02}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \mu^2 B_{02}) x_2 &= 0. \end{aligned} \tag{15}$$

Lemma 6. *In the generic case, the QMEP (15) has $4n^2$ finite solutions.*

PROOF. The bivariate polynomials $\det(A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \mu^2 A_{02})$ and $\det(B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \mu^2 B_{02})$ are of order $2n$. By Bézout's theorem, in the generic case such polynomial system has $4n^2$ solutions.

To see that in the general case all $4n^2$ solutions are finite, we study the homogeneous version of (15). We set $\lambda = \tilde{\lambda}/\tilde{\eta}$, $\mu = \tilde{\mu}/\tilde{\eta}$, and multiply both equations by $\tilde{\eta}$. If the homogeneous system has a projective solution $(\tilde{\eta}, \tilde{\lambda}, \tilde{\mu})$ such that $\tilde{\eta} = 0$, then $(\tilde{\lambda}, \tilde{\mu})$ is a nonzero solution of

$$\begin{aligned} \det(\tilde{\lambda}^2 A_{20} + \tilde{\mu}^2 A_{02}) &= 0 \\ \det(\tilde{\lambda}^2 B_{20} + \tilde{\mu}^2 B_{02}) &= 0. \end{aligned}$$

Since the above system does not have a nonzero solution in the general case, it follows that $\tilde{\eta} \neq 0$ and all eigenvalues of (15) are finite. \square

Denoting $\alpha = \lambda^2$ and $\gamma = \mu^2$, we propose the following transformation to a linear four-parameter eigenvalue problem:

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \alpha A_{20} + \gamma A_{02}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \alpha B_{20} + \gamma B_{02}) x_2 &= 0 \\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) y_1 &= 0 \\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \mu \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) y_3 &= 0. \end{aligned} \tag{16}$$

Note that (16) is the five-parameter eigenvalue problem (10) without the parameter β and without the fourth equation, which is unnecessary due to the missing $\lambda\mu$ terms.

Theorem 7. *In the generic case, the four-parameter eigenvalue problem (16) is nonsingular and there is one-to-one relationship between the eigenpairs of (15) and (16): $((\lambda, \mu), x_1 \otimes x_2)$ is an eigenpair of (15) if and only if*

$$\left((\lambda, \mu, \lambda^2, \mu^2), \left(x_1 \otimes x_2 \otimes \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \mu \end{bmatrix} \right) \right)$$

(up to scaling of the eigenvector) is an eigenpair of (16).

PROOF. It is easy to see that an eigenpair of (15) gives an eigenpair of (16). This gives $4n^2$ finite eigenvalues of (16). As we know that the four-parameter eigenvalue problem (16) has exactly $4n^2$ eigenvalues, they must all be finite and related to the eigenvalues of (15). Since all eigenvalues of (16) are finite, the corresponding operator determinant Δ_0 is nonsingular. \square

Although not being a true linearization in the sense of Definition 2, we call (16) a *minimal-order linearization*, because of the following properties:

- the eigenvalues of (15) correspond exactly to those of (16);
- the operator determinant Δ_0 is nonsingular in general.

In addition, (16) is a symmetric linearization: if all of A_{ij}, B_{ij} , $i, j \in \{0, 1, 2\}$ are symmetric (or Hermitian), then all matrices in the linearization are also symmetric (or Hermitian). This implies that the operator determinants are also symmetric (or Hermitian).

5.2. Both equations missing the μ^2 (or λ^2) terms

If both μ^2 terms in (1) are missing (i.e., $A_{02} = B_{02} = 0$), then the QMEP has the form

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \lambda \mu A_{11}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \lambda \mu B_{11}) x_2 &= 0. \end{aligned} \tag{17}$$

Lemma 8. *In the generic case, the QMEP (17) has $3n^2$ finite solutions.*

PROOF. The homogeneous system of the characteristic polynomials of (17) is given by

$$\begin{aligned} \det(\tilde{\eta}^2 A_{00} + \tilde{\lambda} \tilde{\eta} A_{10} + \tilde{\mu} \tilde{\eta} A_{01} + \tilde{\mu}^2 A_{20} + \tilde{\lambda} \tilde{\mu} A_{11}) &= 0 \\ \det(\tilde{\eta}^2 B_{00} + \tilde{\lambda} \tilde{\eta} B_{10} + \tilde{\mu} \tilde{\eta} B_{01} + \tilde{\mu}^2 B_{20} + \tilde{\lambda} \tilde{\mu} B_{11}) &= 0. \end{aligned}$$

We get infinite solutions of (17) if we put $\tilde{\eta} = 0$. Then we are looking for nonzero $(\tilde{\lambda}, \tilde{\mu})$ such that

$$\begin{aligned} \tilde{\mu}^n \det(\tilde{\mu} A_{20} + \tilde{\lambda} A_{11}) &= 0 \\ \tilde{\mu}^n \det(\tilde{\mu} B_{20} + \tilde{\lambda} B_{11}) &= 0. \end{aligned} \tag{18}$$

In the generic case the polynomials $\det(\tilde{\mu} A_{20} + \tilde{\lambda} A_{11})$ and $\det(\tilde{\mu} B_{20} + \tilde{\lambda} B_{11})$ do not have a nonzero solution. Therefore, the only option for (18) is $\tilde{\mu} = 0$ and $\tilde{\lambda} \neq 0$. So, in the projective coordinates, $(\tilde{\eta}, \tilde{\lambda}, \tilde{\mu}) = (0, 1, 0)$ is a solution of multiplicity n^2 , and there are n^2 infinite and $3n^2$ finite eigenvalues of the QMEP (17). \square

If we apply the approach by Khazanov from Section 3 (see (6) and (7)), and linearize polynomials in (17) as quadratic polynomials in λ using the standard first companion form, we obtain the following linearization of (17):

$$\begin{aligned} \left(\begin{bmatrix} A_{00} & A_{10} \\ 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & A_{20} \\ I & 0 \end{bmatrix} + \mu \begin{bmatrix} A_{01} & A_{11} \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ \lambda x_1 \end{bmatrix} &= 0 \\ \left(\begin{bmatrix} B_{00} & B_{10} \\ 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & B_{20} \\ I & 0 \end{bmatrix} + \mu \begin{bmatrix} B_{01} & B_{11} \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_2 \\ \lambda x_2 \end{bmatrix} &= 0. \end{aligned} \tag{19}$$

Clearly, if $((\lambda, \mu), x_1 \otimes x_2)$ is an eigenpair of (17) then $\left((\lambda, \mu), \begin{bmatrix} x_1 \\ \lambda x_1 \end{bmatrix} \otimes \begin{bmatrix} x_2 \\ \lambda x_2 \end{bmatrix} \right)$ is an eigenpair of (19).

Proposition 9. *In the generic case, the two-parameter eigenvalue problem (19) is nonsingular in the homogeneous setting. In particular, the related operator determinant Δ_2 is nonsingular.*

PROOF. Suppose that the homogeneous version of (19) has an eigenvalue $(\tilde{\eta}, \tilde{\lambda}, \tilde{\mu})$ such that $\tilde{\mu} = 0$. Then $(\tilde{\eta}, \tilde{\lambda})$ is a nonzero solution of

$$\begin{aligned} \det \left(\tilde{\eta} \begin{bmatrix} A_{00} & A_{10} \\ 0 & -I \end{bmatrix} + \tilde{\lambda} \begin{bmatrix} 0 & A_{20} \\ I & 0 \end{bmatrix} \right) &= 0 \\ \det \left(\tilde{\eta} \begin{bmatrix} B_{00} & B_{10} \\ 0 & -I \end{bmatrix} + \tilde{\lambda} \begin{bmatrix} 0 & B_{20} \\ I & 0 \end{bmatrix} \right) &= 0. \end{aligned} \tag{20}$$

But, since (20) has no nonzero solutions in the general case, it follows that $\tilde{\mu} \neq 0$ and Δ_2 is nonsingular by Theorem 1. \square

Theorem 10. *In the generic case, the pair of generalized eigenvalue problems*

$$\begin{aligned} (\Delta_0 - \tilde{\eta}\Delta_2)z &= 0 \\ (\Delta_1 - \tilde{\lambda}\Delta_2)z &= 0, \end{aligned}$$

associated to the two-parameter eigenvalue problem (19), has $4n^2$ eigenvalues $(\tilde{\eta}, \tilde{\lambda})$, where

- a) $3n^2$ eigenvalues are such that $\tilde{\eta} \neq 0$. Each such eigenvalue corresponds to a finite eigenvalue (λ, μ) of the QMEP (17), where

$$\lambda = \tilde{\lambda}/\tilde{\eta}, \quad \mu = 1/\tilde{\eta}.$$

- b) The remaining n^2 eigenvalues are such that $\tilde{\eta} = 0$.

PROOF. a) We know that each of the $3n^2$ eigenvalues (λ, μ) of (17) is an eigenvalue of (19) and thus corresponds to the eigenvalue $(1/\mu, \lambda/\mu, 1)$ of the homogeneous version of (19).

- b) Let $(0, \tilde{\lambda}, 1)$ be an eigenvalue of the homogeneous version of (19). Then

$$\begin{aligned} \det \left(\tilde{\lambda} \begin{bmatrix} 0 & A_{20} \\ I & 0 \end{bmatrix} + \begin{bmatrix} A_{01} & A_{11} \\ 0 & 0 \end{bmatrix} \right) &= 0 \\ \det \left(\tilde{\lambda} \begin{bmatrix} 0 & B_{20} \\ I & 0 \end{bmatrix} + \begin{bmatrix} B_{01} & B_{11} \\ 0 & 0 \end{bmatrix} \right) &= 0, \end{aligned}$$

which has n^2 solutions in the generic case. \square

The transformation to (19) does introduce n^2 spurious eigenvalues, but we believe that a transformation to a multiparameter eigenvalue problem of a smaller size is not possible, i.e., the Δ_i matrices corresponding to (19) are of the smallest possible size.

Let us mention that we could also write (17) as a four-parameter eigenvalue problem by applying (10) without the fourth equation. This would again lead to matrices Δ_i of the size $4n^2 \times 4n^2$. An advantage of this transformation is that it preserves symmetry, while, on the other hand, (19) has fewer parameters.

5.3. Both equations missing both the λ^2 and μ^2 terms

If both λ^2 and μ^2 terms in (1) are missing (i.e., $A_{20} = A_{02} = B_{20} = B_{02} = 0$), then the QMEP has the form

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda\mu A_{11}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda\mu B_{11}) x_2 &= 0. \end{aligned} \tag{21}$$

Lemma 11. *In the generic case, the QMEP (21) has $2n^2$ finite solutions.*

PROOF. The homogeneous system of the characteristic polynomials of (17) is given by

$$\begin{aligned} \det(\tilde{\eta}^2 A_{00} + \tilde{\lambda}\tilde{\eta}A_{10} + \tilde{\mu}\tilde{\eta}A_{01} + \tilde{\lambda}\tilde{\mu}A_{11}) &= 0 \\ \det(\tilde{\eta}^2 B_{00} + \tilde{\lambda}\tilde{\eta}B_{10} + \tilde{\mu}\tilde{\eta}B_{01} + \tilde{\lambda}\tilde{\mu}B_{11}) &= 0. \end{aligned}$$

To count the infinite solutions, we insert $\tilde{\eta} = 0$ and look for nonzero $(\tilde{\lambda}, \tilde{\mu})$ such that

$$\det(\tilde{\lambda}\tilde{\mu}A_{11}) = \det(\tilde{\lambda}\tilde{\mu}B_{11}) = 0.$$

This system has roots $(1, 0)$ and $(0, 1)$, each of multiplicity n^2 . Together we have $2n^2$ infinite eigenvalues in the generic case, while the remaining $2n^2$ eigenvalues are finite. \square

The above case appears in the study of linear time-delay systems for the single delay case [5], where it is solved by a transformation to a coupled pair of quadratic eigenvalue problems (QEP).

Theorem 12 ([5, Theorem 3]). *If $((\lambda, \mu), x_1 \otimes x_2)$ is an eigenpair of (21) then*

a) λ is an eigenvalue with corresponding eigenvector $x_1 \otimes x_2$ of the QEP

$$\begin{aligned} &[\lambda^2(A_{11} \otimes B_{10} - A_{10} \otimes B_{11}) + \lambda(A_{11} \otimes B_{00} - A_{00} \otimes B_{11} \\ &\quad - A_{10} \otimes B_{01} + A_{01} \otimes B_{10}) + A_{01} \otimes B_{00} - A_{00} \otimes B_{01}]z = 0. \end{aligned}$$

b) μ is an eigenvalue with corresponding eigenvector $x_1 \otimes x_2$ of the QEP

$$\begin{aligned} &[\mu^2(A_{11} \otimes B_{01} - A_{01} \otimes B_{11}) + \mu(A_{11} \otimes B_{00} - A_{00} \otimes B_{11} \\ &\quad + A_{10} \otimes B_{01} - A_{01} \otimes B_{10}) + A_{10} \otimes B_{00} - A_{00} \otimes B_{10}]z = 0. \end{aligned}$$

We propose an alternative solution using a linearization like method. We can write (21) as a three-parameter eigenvalue problem

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \beta A_{11}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \beta B_{11}) x_2 &= 0 \\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right) y &= 0, \end{aligned} \tag{22}$$

which is in fact the five-parameter eigenvalue problem (10) without the third and the fifth equation.

Theorem 13. *In the generic case, the three-parameter eigenvalue problem (22) is nonsingular and there is one-to-one relationship between the eigenpairs of (21) and (22): $((\lambda, \mu), x_1 \otimes x_2)$ is an eigenpair of (21) if and only if*

$$\left((\lambda, \mu, \lambda\mu), \left(x_1 \otimes x_2 \otimes \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \right) \right)$$

(up to scaling of the eigenvector) is an eigenpair of (22).

PROOF. The proof is similar to that of Theorem 7. □

It follows from Theorem 13 that (22) is a minimal-order linearization of (21), which holds also for the pair of QEP from Theorem 12. The matrices are not identical, but, if we linearize the QEP from Theorem 12, then in both cases one has to solve a generalized eigenvalue problem of size $2n^2 \times 2n^2$ and the methods have the same complexity.

5.4. Each equation contains exactly one of the λ^2 and μ^2 terms

Without going into details we study two additional special cases where both equations miss the $\lambda\mu$ term and have exactly one of the remaining λ^2 and μ^2 terms. The first QMEP has the form

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20}) x_2 &= 0. \end{aligned} \tag{23}$$

Using a similar approach as in the previous special cases one may show that in the generic case the QMEP (23) has $2n^2$ finite eigenvalues. We can write (23) as a three-parameter eigenvalue problem

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \gamma A_{20}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \gamma B_{20}) x_2 &= 0 \\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) y &= 0. \end{aligned} \tag{24}$$

which is in fact the five-parameter eigenvalue problem (10) without the fourth and the fifth equation. In the generic case, the three-parameter eigenvalue problem (24) is nonsingular and there is one-to-one relationship between the eigenpairs of (24) and (23); in fact, (24) is a symmetry preserving minimal-order linearization in the same sense as before.

The second QMEP has the form

$$\begin{aligned} (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20}) x_1 &= 0 \\ (B_{00} + \lambda B_{10} + \mu B_{01} + \mu^2 B_{02}) x_2 &= 0. \end{aligned} \tag{25}$$

In the generic case the QMEP (25) has $4n^2$ finite eigenvalues, which is same as for the general QMEP (1). One option is to write (25) as a four-parameter eigenvalue problem, that we obtain if we take (10) without the third equation.

Another option is to linearize (25) as a two-parameter eigenvalue problem with matrices of size $2n \times 2n$ using the Khazanov linearization. We obtain

$$\begin{aligned} \left(\begin{bmatrix} A_{00} & A_{10} \\ 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & A_{20} \\ I & 0 \end{bmatrix} + \mu \begin{bmatrix} A_{01} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ \lambda x_1 \end{bmatrix} &= 0 \\ \left(\begin{bmatrix} B_{00} & B_{01} \\ 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} B_{10} & 0 \\ 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & B_{02} \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} x_2 \\ \mu x_2 \end{bmatrix} &= 0. \end{aligned} \quad (26)$$

In the generic case, the two-parameter eigenvalue problem (26) is nonsingular and there is one-to-one relationship between the eigenpairs of (26) and (25), which makes (26) a minimal-order linearization.

5.5. Symmetric quadratic two-parameter eigenvalue problems

We now focus on the general QMEP (1), where all matrices are symmetric (or Hermitian). We would like to linearize the QMEP so that the symmetry is preserved. For this situation we propose the following symmetric linearization (it is sufficient to write it down for the first of the two polynomials only)

$$\left(\begin{bmatrix} A_{00} & 0 & 0 \\ 0 & -A_{20} & -\frac{1}{2}A_{11} \\ 0 & -\frac{1}{2}A_{11} & -A_{02} \end{bmatrix} + \lambda \begin{bmatrix} A_{10} & A_{20} & \frac{1}{2}A_{11} \\ A_{20} & 0 & 0 \\ \frac{1}{2}A_{11} & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} A_{01} & \frac{1}{2}A_{11} & A_{02} \\ \frac{1}{2}A_{11} & 0 & 0 \\ A_{02} & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ \lambda x_1 \\ \mu x_1 \end{bmatrix} = 0. \quad (27)$$

We will now show that if an additional condition holds then this really is a linearization.

Proposition 14. *The linear matrix pencil (27) is a linearization of the bivariate quadratic matrix polynomial $Q_1(\lambda, \mu)$ from (1) if the $2n \times 2n$ matrix*

$$\begin{bmatrix} A_{20} & \frac{1}{2}A_{11} \\ \frac{1}{2}A_{11} & A_{02} \end{bmatrix} \quad (28)$$

is nonsingular.

PROOF. Let $[z_1^T \ z_2^T \ z_3^T] \neq 0$ and (λ, μ) be such that

$$\left(\begin{bmatrix} A_{00} & 0 & 0 \\ 0 & -A_{20} & -\frac{1}{2}A_{11} \\ 0 & -\frac{1}{2}A_{11} & -A_{02} \end{bmatrix} + \lambda \begin{bmatrix} A_{10} & A_{20} & \frac{1}{2}A_{11} \\ A_{20} & 0 & 0 \\ \frac{1}{2}A_{11} & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} A_{01} & \frac{1}{2}A_{11} & A_{02} \\ \frac{1}{2}A_{11} & 0 & 0 \\ A_{02} & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0. \quad (29)$$

The last two rows of (29) can be rewritten as

$$\begin{bmatrix} A_{20} & \frac{1}{2}A_{11} \\ \frac{1}{2}A_{11} & A_{02} \end{bmatrix} \begin{bmatrix} z_2 - \lambda z_1 \\ z_3 - \mu z_1 \end{bmatrix} = 0.$$

Since the matrix (28) is nonsingular, it follows that $z_2 = \lambda z_1$ and $z_3 = \mu z_1$, which yields $z_1 \neq 0$. From the first row of (29) we then obtain $Q_1(\lambda, \mu)z_1 = 0$. \square

6. Bivariate matrix polynomials of higher order

The linearizations and transformations for the QMEP may be generalized to the polynomial two-parameter problems of higher order

$$\begin{aligned} P_1(\lambda, \mu)x_1 &= \sum_{i=0}^k \sum_{j=0}^{k-i} \lambda^i \mu^j A_{ij}x_1 = 0 \\ P_2(\lambda, \mu)x_2 &= \sum_{i=0}^k \sum_{j=0}^{k-i} \lambda^i \mu^j B_{ij}x_2 = 0, \end{aligned} \tag{30}$$

where A_{ij} and B_{ij} are $n \times n$ matrices. It follows from Bézout's theorem that in the generic case the problem (30) has k^2n^2 eigenvalues.

A generalization of the linearization (3) was given in [9], where (30) is linearized as a two-parameter eigenvalue problem with matrices of size $\frac{1}{2}k(k+1)n \times \frac{1}{2}k(k+1)n$. The obtained two-parameter eigenvalue problem is singular and has $\frac{1}{4}k^2(k+1)^2n^2$ eigenvalues, where the eigenvalues of (30) correspond to the finite ones. We now turn our attention to the other techniques.

The Khazanov linearization can also be generalized for polynomials of higher order; the procedure is similar to the quadratic case. First we linearize $P_1(\lambda, \mu)$ as a polynomial of λ , then we rearrange the obtained linearization as a polynomial of μ , and finally we linearize this as a polynomial of μ . We obtain a singular two-parameter eigenvalue problem with matrices of size $k^2n \times k^2n$ that has k^4n^2 eigenvalues, where, as before, the eigenvalues of (30) correspond to the finite ones.

In a similar way as in Section 4 we can transform (30) to a $((k+1)(k+2)/2 - 1)$ -parameter eigenvalue problem, where each term $\lambda^i \mu^j$ is substituted as a new parameter. Such multiparameter eigenvalue problem has $n^2 2^{((k+1)(k+2)/2 - 3)}$ eigenvalues.

For example, if we compare the dimensions of the final Δ_i matrices for the case of a generic cubic polynomial ($k = 3$), we obtain the following orders:

- a) linearization from [9]: $36n^2 \times 36n^2$,
- b) the Khazanov linearization: $81n^2 \times 81n^2$,
- c) transformation to a 9-parameter eigenvalue problem: $128n^2 \times 128n^2$.

Clearly, if k is greater than 2, then linearization a) is the most efficient. However, when some of the terms are missing, some other method may be more efficient, as the next example shows.

Example 15. *Suppose that we have a special system of cubic matrix polynomials of the form*

$$\begin{aligned} P_1(\lambda, \mu)x_1 &:= (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^3 A_{30} + \mu^3 A_{03})x_1 = 0, \\ P_2(\lambda, \mu)x_2 &:= (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^3 B_{30} + \mu^3 B_{03})x_2 = 0. \end{aligned} \tag{31}$$

In the generic case, the problem (31) has $9n^2$ eigenvalues. If we introduce new variables $\alpha = \lambda^3$ and $\beta = \mu^3$

then we can write (31) as a four-parameter eigenvalue problem

$$\begin{aligned}
(A_{00} + \lambda A_{10} + \mu A_{01} + \alpha A_{30} + \beta A_{03})x_1 &= 0 \\
(B_{00} + \lambda B_{10} + \mu B_{01} + \alpha B_{30} + \beta B_{03})x_2 &= 0
\end{aligned} \tag{32}$$

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) y_1 = 0$$

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) y_2 = 0.$$

If $((\lambda, \mu), x_1 \otimes x_2)$ is an eigenpair of (31) then

$$\left((\lambda, \mu, \lambda^3, \mu^3), x_1 \otimes x_2 \otimes \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \mu \\ \mu^2 \end{bmatrix} \right)$$

is an eigenpair of (32). The four-parameter eigenvalue problem (32), which has $9n^2$ eigenvalues, is thus nonsingular. As there are no spurious eigenvalues, (32) is a minimal-order linearization to solve (31).

7. Conclusions

We presented several transformations that can be applied to solve the QMEP via the multi-parameter eigenvalue problems. This enables one to apply the numerical methods that exist for multiparameter problems and solve the QMEP numerically. The approaches can also be extended to polynomial two-parameter eigenvalue problems of higher order.

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