Right Definite Multiparameter Sturm-Liouville Problems with Eigenparameter Dependent Boundary Conditions *

Tirthankar Bhattacharyya[†], Tomaž Košir[‡], and Bor Plestenjak[‡]

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Abstract

We study a system of ordinary differential equations that are linked by parameters and are subject to boundary conditions that depend on parameters. We assume certain definiteness conditions on the coefficient functions and on the boundary conditions that yield, in the corresponding abstract setting, a right definite case. Then the existence of the eigenvalues and completeness of eigenfunctions are well understood. We give results on location of the eigenvalues and oscillation of the eigenfunctions.

1 Introduction

In the paper we consider systems of multiparameter Sturm-Liouville problems with eigenparameter dependent boundary conditions. They were first studied by Bhattacharyya, Binding, and Seddighi in [2, 3]. Our results are on location of eigenvalues and oscillation of the corresponding eigenfunctions.

In one-parameter case one studies differential equation

$$-(py')' + qy = \lambda ry \tag{1}$$

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† Department of Mathematics, Indian Institute of Science, Bangalore 560012, India. E-MAIL: *tirtha@math.iisc.ernet.in*

‡ Department of Mathematics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia. E-MAIL: *tomaz.kosir@fmf.uni-lj.si* and *bor.plestenjak@fmf.uni-lj.si*

subject to boundary conditions

$$b_0 y(0) = d_0 \left(py'\right)(0) \tag{2}$$

and

$$b_1 y(1) = d_1 \left(py'\right)(1), \tag{3}$$

where p, p', q, and r are continuous functions on $[0,1]$, p and r are positive, and $(b_s, d_s) \in \mathbb{R}^2 \backslash \{0\}$, $s = 0, 1$. Then there are countably many real eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, accumulating at infinity, each with (up to a scalar multiple) unique eigenfunction y_n. The eigenfunctions $\{y_n\}_{i=0}^{\infty}$ are complete. One of the central topics in the Sturm-Liouville theory is the oscillation theory. Namely, the eigenfunction y_n possesses exactly n roots on $(0,1)$, i.e. y_n has the *oscillation count* equal to n (see [10, Ch. 8] for all of these). When the boundary conditions (2) and/or (3) are replaced by eigenparameter dependent boundary conditions

$$\left(a_0\lambda + b_0\right) y(0) = \left(c_0\lambda + d_0\right) \left(py'\right)(0), \tag{4}$$

and

$$\left(a_1\lambda + b_1\right) y(1) = \left(c_1\lambda + d_1\right) \left(py'\right)(1), \tag{5}$$

where

$$a_0 d_0 - b_0 c_0 < 0 \text{ and } a_1 d_1 - b_1 c_1 > 0 \tag{6}$$

there still are countably many eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, accumulating at infinity, each with (up to a scalar multiple) unique eigenfunction y_n. The eigenfunctions $\{y_n\}_{i=0}^{\infty}$ are complete and all the oscillation counts appear. However, there is a repeated oscillation count if either boundary conditions (2) and (5) or boundary conditions (3) and (4) are assumed, and there are two double oscillations counts or a triple oscillation count if (4) and (5) are assumed (see [8] for details). We remark that these results may fail if the sign conditions (6) are omitted. Then nonreal and nonsemisimple eigenvalues may occur [6, 7].

In a multiparameter generalization we consider a system of ordinary differential equations

$$-\left(p_j y_j'\right)' + q_j y_j = \left(\sum_{k=1}^{n} \lambda_k r_{jk}\right) y_j, \quad j = 1, 2, \cdots, n, \tag{7}$$

where p_j, p_j', q_j, and r_{jk} are real and continuous functions on $[0,1]$ and p_j are positive on $[0,1]$, subject to boundary conditions

$$\left(a_{j0}\lambda_j + b_{j0}\right) y_j(0) = \left(c_{j0}\lambda_j + d_{j0}\right) \left(p_j y_j'\right)(0) \tag{8}$$

and

$$(a_{j1}\lambda_j + b_{j1})\, y_j(1) = (c_{j1}\lambda_j + d_{j1}) \left(p_j y_j'\right)(1). \tag{9}$$

We assume the so-called Minkowski definiteness conditions on the functions r_{jk}, $j,k = 1,2,\ldots, n$, together with certain sign conditions on numbers $\omega_{js} = a_{js}d_{js} - b_{js}c_{js}$, $s = 0,1$. These conditions yield in the corresponding abstract setup a right definite case [3]. Then all the eigenvalues are real and the corresponding eigenfunctions are complete (see [1, 15, 16]). We generalize the oscillation theory to the general n-parameter case. We also discuss the behaviour of the eigenvalue hypersurfaces. These results for $n = 2$ and $a_{j0} = c_{j0} = 0$, $j = 1,2$, are given in [2].

2 Preliminaries

By a transformation of the independent variable, we can assume without loss of generality that p_j, $j = 1,2,\ldots,n$, are identically equal to 1 (see [8, Appendix]). Then differential equations (7) become

$$-y_j'' + q_j y_j = \left(\sum_{k=1}^{n} \lambda_k r_{jk}\right) y_j, \quad j = 1,2,\cdots,n, \tag{10}$$

and the boundary conditions (8) and (9) become

$$(a_{j0}\lambda_j + b_{j0})\, y_j(0) = (c_{j0}\lambda_j + d_{j0})\, y_j'(0) \tag{11}$$

and

$$(a_{j1}\lambda_j + b_{j1})\, y_j(1) = (c_{j1}\lambda_j + d_{j1})\, y_j'(1), \tag{12}$$

respectively.

Let us introduce some notation. For $s = 0,1$ and $j = 1,2,\ldots,n$, we write

$$\omega_{js} = a_{js}d_{js} - b_{js}c_{js}.$$

If a function y is in $L^2[0,1]$ then we denote by $\bar{r}_{jk}(y)$ the integral $\int_0^1 r_{jk}\,|y|^2$, and if $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is an n-tuple of functions in $L^2[0,1]$ then we denote by $\rho_0(\mathbf{y})$ the determinant $\det[\bar{r}_{jk}(y_j)]_{j,k=1}^{n}$. We write B_1 for the unit ball of $L^2[0,1]$.

In what follows we use the following assumptions:

(C) q_j and r_{jk}, $j,k = 1,2,\ldots,n$, are real and continuous functions on $[0,1]$,

(I) (a) $a_{j0} = c_{j0} = 0$, $(b_{j0}, d_{j0}) \neq (0,0)$, $j = 1, 2, \ldots, n$,
or

(b) $\omega_{j0} < 0$ for $j = 1, 2, \ldots, n$,

(II) $\omega_{j1} > 0$ for $j = 1, 2, \ldots, n$,

(III) $\bar{r}_{jk}(y) \leq 0$ for $j, k = 1, 2, \ldots, n$, $j \neq k$, and for all $y \in L^2[0,1]$, $y \neq 0$,

(IV) $\sum_{k=1}^{n} \bar{r}_{jk}(y) > 0$ for $j = 1, 2, \ldots, n$ and for all $y \in L^2[0,1]$, $y \neq 0$.

By scaling the constants a_{js}, b_{js}, c_{js} and d_{js} we can replace the inequalities in assumptions (Ib) and (II) by $\omega_{j0} = -1$ and $\omega_{j1} = 1$, respectively. We assume that these simplifications are done.

Following [5] we call the assumptions (III) and (IV) the *Minkowski conditions*. Since we assume (C), i.e., r_{jk} are continuous functions, it follows that the Minkowski condition (IV) is *uniform*, i.e. there exists a constant $\gamma > 0$ such that for all $y \in B_1$ and $j = 1, 2, \ldots, n$,

(IV') $\sum_{k=1}^{n} \bar{r}_{jk}(y) > \gamma$.

After an invertible transformation of parameters is performed, the uniform Minkowski conditions follow from uniform right definiteness and uniform ellipticity conditions [5, p. 19 and p. 23]. The latter conditions are more familiar in the literature on multiparameter spectral theory. A system of equations (10) (or more generally a system of equations (7)) is called *uniformly right definite* if there exists a constant $\gamma > 0$ such that $\rho_0(\mathbf{y}) > \gamma$ for all $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in B_1^n$, and it is called *uniformly elliptic* if there exist $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\gamma' > 0$ such that $\sum_{k=1}^{n} \alpha_k \rho_{0jk}(\mathbf{y}) > \gamma'$ for all $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in B_1^n$ and all j. Here $\rho_{0jk}(\mathbf{y})$ is the cofactor of $\rho_0(\mathbf{y})$ corresponding to $\bar{r}_{jk}(y_j)$. Note that if we assumed uniform right definiteness and uniform ellipticity it would not be possible, in general, to obtain the Minkowski conditions by an invertible linear transformation of parameters without losing the form of boundary conditions (11) and (12) and assumptions (Ib) and (II). After such a transformation of parameters, more general boundary conditions are obtained from (11) and (12); namely, each λ_j is replaced by a linear combination of all the eigenparameters λ_k, $k = 1, 2, \ldots, n$. (Multiparameter Sturm-Liouville problems with these general boundary conditions are studied in [3]). However, before eigenvalue and oscillation theory for such multiparameter problems can be discussed, some further analysis of one-parameter Sturm-Liouville differential equations with eigenparameter dependent boundary condition would be required. Here we follow the path of [2] and assume the stronger conditions and apply the available one-parameter analysis of Binding, Browne, and Seddighi [8].

At the end of this section we introduce a notion of Minkowski matrix and give a bound for its minimal singular value.

A real matrix $A = [a_{jk}]_{j,k=1}^n$ is called a *Minkowski matrix* if the following conditions hold:

1. $a_{jj} > 0$ for $j = 1, 2, \ldots, n$,
2. $a_{jk} \leq 0$ for $j, k = 1, 2, \ldots, n$, $j \neq k$,
3. $\sum_{k=1}^n a_{jk} \geq \gamma > 0$ for $j = 1, 2, \ldots, n$.

Constant γ above is called *a bound* of the Minkowski matrix A.

Lemma 2.1 *If A is a Minkowski matrix with a bound γ and $\sigma_n(A)$ is its minimal singular value then*

$$\sigma_n(A) \geq \frac{\gamma}{\sqrt{n}}.$$

Proof. The minimal singular value satisfies a relation $\sigma_n(A) = \min_{\|x\|_2=1} \|Ax\|_2$ (see e.g. [12, p. 428]). We choose a vector $x = [x_j]_{j=1}^n$ with a norm $\|x\|_2 = 1$. Suppose that k is such that $|x_k| \geq |x_j|$ for $j = 1, 2, \ldots, n$. Then we have

$$\left|\sum_{j=1}^n a_{kj} x_j\right| \geq |a_{kk} x_k| - \left|\sum_{j=1, j\neq k}^n a_{kj} x_j\right| \geq a_{kk}|x_k| + \sum_{j=1, j\neq k}^n a_{kj}|x_j| \geq \left(\sum_{j=1}^n a_{kj}\right)|x_k| \geq \gamma |x_k|.$$

Because we assume that $\|x\|_2 = 1$ it follows that $|x_k| \geq \frac{1}{\sqrt{n}}$. The above inequality implies that $\|Ax\|_2 \geq \frac{\gamma}{\sqrt{n}}$. $\square$

3 Eigenvalues in the case that boundary conditions at one end depend on eigenparameter

We first consider in detail the problem (10), (11) and (12) under assumptions (C), (Ia) and (II)-(IV) and study the properties of the corresponding eigenvalue hypersurfaces. This is a generalization of two-parameter results proved in [2]. The proofs here are similar and depend on results in [8]. A crucial new step is an application of Hadamard's Inverse Function Theorem [13, Thm. A].

Let us now fix j and consider Sturm-Liouville problem (10), (11) and (12) under assumptions (Ia) and (II)-(IV). We write $\boldsymbol{\lambda}_j$ for the set of parameters λ_l, $l \neq j$.

Lemma 3.1 *There exists an infinite sequence $\lambda_j = \lambda_j^{(m)}(\boldsymbol{\lambda}_j), m = 0, 1, 2, \ldots$, of real eigenvalue hypersurfaces. Each of the functions $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ depends continuously on all $\lambda_l \in \boldsymbol{\lambda}_j$ and for each value $\boldsymbol{\lambda}_j \in \mathbb{R}^{n-1}$ the sequence of eigenvalues $\left\{\lambda_j^{(m)}(\boldsymbol{\lambda}_j)\right\}_{m=0}^{\infty}$ is strictly increasing.*

Proof. We fix $j = 1$ for simplicity. We view the boundary value problem

$$-y_1'' + \left(q_1 - \sum_{k=2}^{n} \lambda_k r_{1k}\right) y_1 = \lambda_j r_{11} y_1$$

together with (11) and (12) as a parameterized one parameter Sturm-Liouville boundary value problem with eigenparameter dependent boundary condition. The existence of $\lambda_1^{(m)}(\boldsymbol{\lambda}_1)$ with required properties follows by [8, Thms 3.1 and 3.2]. □

Lemma 3.2 *To each eigenvalue $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ there exists a real eigenfunction $y_j^{(m)} = y_j^{(m)}\left(x, \boldsymbol{\lambda}_j^{(m)}\right)$ with $\left\|y_j^{(m)}\right\| = 1$ for all $\boldsymbol{\lambda}_j$ and such that for each $x \in [0,1]$ and each compact set $K_j \subset \mathbb{R}^{n-1}$ the eigenfunction $y_j^{(m)}$ and its derivative with respect to x depend continuously on $\boldsymbol{\lambda}_j \in K_j$. Furthermore, there exists a sequence of natural numbers $N_j^{(m)} = N_j^{(m)}(\boldsymbol{\lambda}_j)$, $m = 0, 1, 2, \ldots$, such that $y_j^{(m)}$ has m zeros on the interval $(0,1)$ for $m \leq N_j^{(m)}$ and $m-1$ zeros on $(0,1)$ for $m > N_j^{(m)}$.*

Proof. The proof is similar to the proof of [2, Lemma 2.2]. For simplicity we fix $j = 1$ and suppress it. Let

$$\mathbf{y} = \begin{pmatrix} y \\ \frac{d}{dx} y \end{pmatrix} \text{ and } A(x, \boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 \\ q - \lambda\left(\boldsymbol{\lambda}^{(m)}\right) r_1 - \sum_{l=2}^{n} \lambda_l r_l & 0 \end{pmatrix}.$$

Then $\mathbf{y}$ is a solution of

$$\mathbf{y}' = A(x, \boldsymbol{\lambda})\mathbf{y}.$$

Observe that A is a continuous function of x and $\boldsymbol{\lambda}$. Then for $\boldsymbol{\lambda}$ lying in a compact subset K the operator norm $\|A(x, \boldsymbol{\lambda})\|$ on $L^2[0,1] \oplus L^2[0,1]$ has an upper bound which may depend on x. Then the function $f_{\boldsymbol{\lambda}} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f_{\boldsymbol{\lambda}}(x, \alpha) = A(x, \boldsymbol{\lambda})\alpha$, for $x \in [0,1]$ and $\alpha \in \mathbb{R}^2$ is Lipschitz. The continuity of $y^{(m)}(x, \boldsymbol{\lambda})$ and $\frac{d}{dx} y^{(m)}(x, \boldsymbol{\lambda})$ then follows by [14, Thm. 3.2] using the same arguments as in the proof of [2, Lemma 2.2]. The existence of $\lambda_1^{(m)}(\boldsymbol{\lambda})$ with required properties follows by [8, Thm. 3.1]. □

Theorem 3.3 *Partial derivative of $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ with respect to $\lambda_l \in \boldsymbol{\lambda}_j$ exists and is equal to*

$$\frac{\partial \lambda_j^{(m)}}{\partial \lambda_l} = -\left(\bar{r}_{jj}(y_j^{(m)}) + \frac{y_j^{(m)}(1)^2}{(c_{j1}\lambda_j + d_{j1})^2} \right)^{-1} \cdot \bar{r}_{jl}(y_j^{(m)}), \tag{13}$$

where $y_j^{(m)}(1) = y_j^{(m)}(1, \boldsymbol{\lambda}_j)$. Moreover, the derivative $\frac{\partial \lambda_j^{(m)}}{\partial \lambda_l}(\boldsymbol{\lambda}_j)$ is continuous, positive, and bounded on the entire $\mathbb{R}^{n-1}$.

Proof. For simplicity we assume $j = 1$ and $l = 2$ and we write $\boldsymbol{\lambda}'$ for the set of remaining parameters λ_r, $r = 3, 4, \ldots, n$. Furthermore, we fix $\boldsymbol{\lambda}' \in \mathbb{R}^{n-2}$ and a nonnegative integer m, and we suppress m.

Assume that $y_1 = y_1(x, \lambda_2, \boldsymbol{\lambda}')$ is the eigenfunction corresponding to $\lambda_1(\lambda_2, \boldsymbol{\lambda}')$ and that $z_1 = z_1(x, \lambda_2 + \epsilon, \boldsymbol{\lambda}')$ is the eigenfunction corresponding to $\lambda_1(\lambda_2 + \epsilon, \boldsymbol{\lambda}')$ for some $\epsilon > 0$. So we have

$$-y_1'' + q_1 y_1 = \left(\lambda_1(\lambda_2, \boldsymbol{\lambda}') r_{11} + \lambda_2 r_{12} + \sum_{t=2}^{n} \lambda_t r_{1t} \right) y_1 \tag{14}$$

and

$$-z_1'' + q_1 z_1 = \left(\lambda_1(\lambda_2 + \epsilon, \boldsymbol{\lambda}') r_{11} + (\lambda_2 + \epsilon) r_{12} + \sum_{t=2}^{n} \lambda_t r_{1t} \right) z_1. \tag{15}$$

Multiplying the first equation by z_1 and the second by y_1, subtracting and integrating, we obtain

$$(y_1' z_1 - y_1 z_1')\big|_0^1 = (\lambda_1(\lambda_2 + \epsilon, \boldsymbol{\lambda}') - \lambda_1(\lambda_2, \boldsymbol{\lambda}')) \int_0^1 r_{11} y_1 z_1 + \epsilon \int_0^1 r_{12} y_1 z_1. \tag{16}$$

Dividing by ϵ and using the continuity established in Lemmas 3.1 and 3.2, we have

$$-\left(\frac{y_1(1)^2 \omega_{11}}{(c_{11}\lambda_1 + d_{11})^2} \right) \frac{\partial \lambda_1}{\partial \lambda_2} = \bar{r}_{11}(y_1) \frac{\partial \lambda_1}{\partial \lambda_2} + \bar{r}_{12}(y_1).$$

Then

$$\frac{\partial\lambda_1}{\partial\lambda_2} = -\left(\bar{r}_{11}(y_1) + \frac{y_1(1)^2}{(c_{11}\lambda_1 + d_{11})^2}\right)^{-1} \bar{r}_{12}(y_1). \tag{17}$$

Since y_1 and $\bar{r}_{jk}$ are continuous it follows that $\frac{\partial\lambda_1}{\partial\lambda_2}$ is continuous. Note that $\|y_1\| = 1$ by Lemma 3.2. Then the Minkowski condition (III) and identity (17) imply that $\frac{\partial\lambda_1}{\partial\lambda_2} > 0$ for all $(\lambda_2, \boldsymbol{\lambda}') \in \mathbb{R}^{n-1}$. By the continuity of r_{12} it follows that $M_{12} = \max\{r_{12}(x); 0 \leq x \leq 1\}$ is finite. The uniform Minkowski conditions imply that $r_{11}(y_1) > n\gamma$. Using these and identity (17) it follows that $\frac{\partial\lambda_1}{\partial\lambda_2}(\boldsymbol{\lambda}') < \frac{M_{12}}{n\gamma}$ for all $\boldsymbol{\lambda}' \in \mathbb{R}^{n-1}$.

A straightforward calculation shows that (12) with $j = 1$ implies that

$$-f_{11}(y_1) = \frac{y_1(1)}{c_{11}\lambda_1 + d_{11}}.$$

For other derivatives, one carries out the same calculation with the roles of 1 and 2 replaced by j and l, respectively. □

For each n-tuple $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ of nonnegative integers we consider the set of eigenvalue hypersurfaces $\lambda_j = \lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$, $j = 1, 2, \ldots, n$. We fix $\mathbf{m}$ and, for simplicity of notation, we suppress it. Consider next the function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\boldsymbol{\lambda}) = (\lambda_j - \lambda_j(\boldsymbol{\lambda}_j))_{j=1}^n$. Assume that $y_j = y_j(x, \boldsymbol{\lambda}_j)$ is the eigenfunction corresponding to $\lambda_j(\boldsymbol{\lambda}_j)$ and write $f_{j1}(y_j) = -\frac{y_j(1)}{c_{j1}\lambda_j + d_{j1}}$. By Theorem 3.3 it follows that function F is a C^1-function. Its Jacobian matrix is equal to

$$J(F) = \begin{pmatrix} 1 & -\frac{\partial\lambda_1}{\partial\lambda_2} & \cdots & -\frac{\partial\lambda_1}{\partial\lambda_n} \\ -\frac{\partial\lambda_2}{\partial\lambda_1} & 1 & \cdots & -\frac{\partial\lambda_2}{\partial\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial\lambda_n}{\partial\lambda_1} & -\frac{\partial\lambda_n}{\partial\lambda_2} & \cdots & 1 \end{pmatrix}.$$

Lemma 3.4 *The determinant of the Jacobian matrix $J(F)$ is positive for all $\boldsymbol{\lambda} \in \mathbb{R}^n$.*

Proof. Recall that the uniform Minkowski condition (IV') holds. Then $\bar{r}_{jk}(y_j) \leq 0$ for $j \neq k$ and $\sum_{k=1}^n \bar{r}_{jk}(y_j) \geq \gamma > 0$. Let s_j be the sum of the entries of the j-th row of the Jacobian matrix $J(F)$. Take $j = 1$ and apply Theorem 3.3 to show that

$$s_1 = 1 - \sum_{k=2}^n \frac{\partial\lambda_1}{\partial\lambda_k} = 1 + \sum_{k=2}^n \frac{\bar{r}_{1k}(y_1)}{\bar{r}_{11} + f_{11}(y_1)^2} \geq 1 + \sum_{k=2}^n \frac{\bar{r}_{1k}(y_1)}{\bar{r}_{11}} \geq \frac{\gamma}{R} > 0,$$

where $R = \max\{\bar{r}_{kk}(y_k);\ k = 1, 2, \ldots, n\}$. In a similar way we see that $s_j \geq \frac{\gamma}{R} > 0$ for $j = 2, 3, \ldots, n$. The Gershgorin Circle Theorem (see e.g. [12, p. 341]) implies then there is a constant $\beta > 0$ such that real parts of all the eigenvalues of $J(F)$ are greater than β. Since nonreal eigenvalues, if any, occur in conjugate pairs it follows that the determinant $\det J(F)$ is positive for all $\boldsymbol{\lambda} \in \mathbb{R}^n$. □

Lemma 3.5 *The function F is* proper [13], *i.e.* $\|\boldsymbol{\lambda}\|_2 \to \infty$ *implies* $\|F(\boldsymbol{\lambda})\|_2 \to \infty$.

Proof. We write $F = (F_j)_{j=1}^n$. The inner product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is denoted by $\langle \mathbf{a}, \mathbf{b} \rangle$ and the p-norm of a vector $\mathbf{a}\mathbb{R}^n$ is denoted by $\|\mathbf{a}\|_p$. By the Mean Value Theorem applied to $F_j : \mathbb{R}^n \to \mathbb{R}$ and vectors $\mathbf{a}, \boldsymbol{\lambda} \in \mathbb{R}^n$ there is a vector $\mathbf{b}_j$ in the convex hull of the set $\{\mathbf{a}, \boldsymbol{\lambda}\}$ such that

$$(F_j(\boldsymbol{\lambda}) - F_j(\mathbf{a}))^2 = \langle \operatorname{grad} F_j(\mathbf{b}_j), \boldsymbol{\lambda} - \mathbf{a} \rangle,\ j = 1, 2, \ldots, n, \tag{18}$$

where $\operatorname{grad} F_j(\mathbf{b}_j)$ is the gradient of F_j at $\mathbf{b}_j$. By the definition of F it follows that

$$\operatorname{grad} F_j(\mathbf{b}_j) = \begin{pmatrix} -\frac{\partial \lambda_j}{\partial \lambda_1}(\mathbf{b}_j) & -\frac{\partial \lambda_j}{\partial \lambda_2}(\mathbf{b}_j) & \cdots & -\frac{\partial \lambda_j}{\partial \lambda_n}(\mathbf{b}_j) \end{pmatrix}.$$

Next we consider the matrix

$$G = \begin{pmatrix} 1 & -\frac{\partial \lambda_1}{\partial \lambda_2}(\mathbf{b}_1) & \cdots & -\frac{\partial \lambda_1}{\partial \lambda_n}(\mathbf{b}_1) \\ -\frac{\partial \lambda_2}{\partial \lambda_1}(\mathbf{b}_2) & 1 & \cdots & -\frac{\partial \lambda_2}{\partial \lambda_n}(\mathbf{b}_2) \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial \lambda_n}{\partial \lambda_1}(\mathbf{b}_n) & -\frac{\partial \lambda_n}{\partial \lambda_2}(\mathbf{b}_n) & \cdots & 1 \end{pmatrix}.$$

We apply Theorem 3.3 and use the uniform Minkowski conditions to prove that G is a Minkowski matrix with bound γ. Calculations are similar to those in the proof of Lemma 3.4 and we omit them. Next it follows by relations (18) and Lemma 2.1 that

$$\|F(\boldsymbol{\lambda}) - F(\mathbf{a})\|_4^2 = \|G(\boldsymbol{\lambda} - \mathbf{a})\|_2 \geq \frac{\gamma}{\sqrt{n}} \|\boldsymbol{\lambda} - \mathbf{a}\|_2.$$

Finally, if $\|\boldsymbol{\lambda}\|_2 \to \infty$ then $\|F(\boldsymbol{\lambda})\|_2 \to \infty$ since the 2-norm and the 4-norm on $\mathbb{R}^n$ are equivalent. Hence F is a proper function. □

Theorem 3.6 *For each n-tuple $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ of nonnegative integers the set of eigenvalue hypersurfaces $\lambda_j = \lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$, $j = 1, 2, \ldots, n$, has exactly one point of intersection in $\mathbb{R}^n$.*

Proof. We fix $\mathbf{m}$ and suppress it. We consider the function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\boldsymbol{\lambda}) = (\lambda_j - \lambda_j(\boldsymbol{\lambda}_j))_{j=1}^n$. Lemmas 3.4 and 3.5 tell us that F is a proper function and that the determinant of its Jacobian is positive for all $\boldsymbol{\lambda} \in \mathbb{R}^n$. By Hadamard's Inverse Function Theorem [13, Thm. A] it follows that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Hence the inverse image $F^{-1}(0)$, which is the intersection of the eigenvalue hypersurfaces $\lambda_j = \lambda_j(\boldsymbol{\lambda}_j)$, $j = 1, 2, \ldots, n$, is a single point. □

Next we describe the limiting behaviour of the eigenvalue hypersurfaces.

Proposition 3.7 *The eigenvalue hypersurfaces have the following properties :*

1. $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ *is an increasing function in each parameter* $\lambda_l \in \boldsymbol{\lambda}_j$,
2. $\lambda_j^{(0)}(\boldsymbol{\lambda}_j) < \min\left\{0, -\frac{d_{j1}}{c_{j1}}\right\}$ *for all* j,
3. $\lim_{\lambda_k \to \infty} \lambda_j^{(0)}(\boldsymbol{\lambda}_j) = \min\left\{0, -\frac{d_{j1}}{c_{j1}}\right\}$ *for all* j *and* $k \neq j$,
4. $\lim_{\lambda_k \to \infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = \infty$ *for* $m > 0$, $j, k = 1, 2, \ldots, n$, $j \neq k$,
5. $\lim_{\lambda_k \to -\infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = -\infty$ *for* $m \geq 0$, $j, k = 1, 2, \ldots, n$, $j \neq k$.

Proof. We shall prove only the property 2 in detail. The property 1 is obvious from positivity of all the partial derivatives, the proof of 3 depends on the fact that f is an increasing function in each $\lambda_k \in \boldsymbol{\lambda}_j$ which follows from [8], and the proofs of 4 and 5 follow by considering the corresponding asymptotic problems and are similar to the proof of [2, Lemma 3.4].

For 2, one has to go back to [8, pp. 60-64]. Consider the jth. equation as a one-parameter problem, but depending on $\boldsymbol{\lambda}_j$. Let θ be the Prüfer angle. Thus θ is a function of $x \in [0, 1]$, the eigenparameter λ_j and the $n-1$ constants $\boldsymbol{\lambda}_j$. The zeroth eigensurface $\lambda_j^{(0)}$ is the intersection point of $f(\lambda_j) = \cot\theta(1, \lambda_j, \boldsymbol{\lambda}_j)$ with the hyperbola $g_j(\lambda_j) = (a_j\lambda_j + b_j)/(c_j\lambda_j + d_j)$. Now because of the assumptions on a_j, b_j, c_j and d_j, the hyperbola is increasing. On the other hand, the graph of f has countably many branches. The hyperbola cuts the leftmost branch of f in the left half plane. Since the vertical asymptote for the hyperbola is $-d_j/c_j$, the point of intersection has to lie on the left of this vertical line also. Hence 2 is proved. ■

Suppose that $\boldsymbol{\lambda} \in \mathbb{R}^n$ is an eigenvalue of the problem (10), (11) and (12) under assumptions (C), (Ia) and (II)-(IV) and that $y_j(\boldsymbol{\lambda})$, $j = 1, 2, \ldots, n$, are the corresponding eigenfunctions. Let h_j be the number of zeros of $y_j(\boldsymbol{\lambda})$ on the interval $(0,1)$. The n-tuple of nonnegative integers $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ is called the *oscillation count* of $\boldsymbol{\lambda}$ and h_j is called the j-th *oscillation count* of $\boldsymbol{\lambda}$.

By [8, Thm. 3.1] it follows that on each hypersurface $\lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$ with $m_j > 0$ we have 2^n oscillation counts. The j-th oscillation count changes when we cross the hyperplane $\lambda_j = -\frac{d_{j1}}{c_{j1}}$. Then $N_j^{(m_j)}$ is determined so that

$$\lambda_j^{N_j^{(m_j)}-1}(\boldsymbol{\lambda}_j) < -\frac{d_{j1}}{c_{j1}} \leq \lambda_j^{N_j(m_j)}(\boldsymbol{\lambda}_j).$$

Hence

$$h_j = \begin{cases} m_j, & \text{if } \lambda_j^{(m_j)} < -\frac{d_{j1}}{c_{j1}} \\ m_j - 1, & \text{otherwise.} \end{cases} \tag{19}$$

The following result now follows by Proposition 3.7 and above relations (19).

Theorem 3.8 *If there are M eigenvalues with the same oscillation count then:*

1. $M \leq 2^n$,
2. *there is at most one oscillation count corresponding to $M = 2^n$ eigenvalues,*
3. *for $M \neq 2^k$, $k = 1, 2, \ldots, n-1$, there is only a finite number of oscillation counts that correspond to M eigenvalues,*
4. *for $M = 2^k$, $k = 1, 2, \ldots, n-1$, there is an infinite number of oscillation counts that correspond to M eigenvalues.*

4 Eigenvalue hypersurfaces in the case that boundary conditions at both ends are eigenparameter dependent

Now we consider the problem (10), (11) and (12) under assumptions (Ib) and (II)-(IV) and study the properties for the corresponding eigenvalue hypersurfaces. The arguments in the proofs are similar to those above under assumption (Ia). We specify which results are used in the proofs but do not give all details.

Lemma 4.1 *There exists an infinite sequence* $\left\{\lambda_j^{(m)}(\boldsymbol{\lambda}_j)\right\}_{m=0}^{\infty}$ *of real eigenvalues. Each of* $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ *depend continuously on all* $\lambda_l \in \boldsymbol{\lambda}_j$ *and the sequence of eigenvalues* $\left\{\lambda_j^{(m)}(\boldsymbol{\lambda}_j)\right\}_{m=0}^{\infty}$ *is (strictly) increasing for each* $\boldsymbol{\lambda}_j \in \mathbb{R}^{n-1}$.

Proof. We fix $j=1$ for simplicity. We view boundary value problem

$$-y_1'' + \left(q_1 - \sum_{k=2}^{n} \lambda_k r_{1k}\right) y_1 = \lambda_j r_{11} y_1$$

together with (11) and (12) as a parameterized one parameter Sturm-Liouville boundary value problem with eigenparameter dependent boundary conditions. The existence of $\lambda_1^{(m)}(\boldsymbol{\lambda}_1)$ with required properties follows by [8, Thms 4.2 and 4.3]. □

Lemma 4.2 *To each eigenvalue* $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ *there exists a real eigenfunction* $y_j^{(m)}(x, \boldsymbol{\lambda}_j)$ *of norm 1 for all* $\boldsymbol{\lambda}_j$ *such that for each* $x \in [0,1]$ *and each compact set* $K_j \subset \mathbb{R}^{n-1}$ *the eigenfunction* $y_j^{(m)}(x, \boldsymbol{\lambda}_j)$ *and its derivative with respect to* x *depend continuously on* $\boldsymbol{\lambda}_j \in K_j$. *Furthermore, there exists a sequence of natural numbers* $\left\{N_{j1}^{(m)}(\boldsymbol{\lambda}_j)\right\}_{m=0}^{\infty}$ *and* $\left\{N_{j2}^{(m)}(\boldsymbol{\lambda}_j)\right\}_{m=0}^{\infty}$ *such that* $y_m(\boldsymbol{\lambda}_j)$ *has* m *zeros on* $(0,1)$ *for* $m \le N_{j1}^{(m)}(\boldsymbol{\lambda}_j)$, $m-1$ *zeros on* $(0,1)$ *for* $N_{j1}^{(m)}(\boldsymbol{\lambda}_j) < m < N_{j2}^{(m)}(\boldsymbol{\lambda}_j)$ *and* $m-2$ *zeros on* $(0,1)$ *for* $m \ge N_{j2}^{(m)}(\boldsymbol{\lambda}_j)$.

The proof is the same as the proof of Lemma 3.2. Only at the end of it the existence of $\lambda_1^{(m)}(\boldsymbol{\lambda}_1)$ with required properties follows by [8, Thm. 4.2].

Proposition 4.3 *Partial derivatives of* $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ *with respect to* $\lambda_l \in \boldsymbol{\lambda}_j$ *exist and are equal to*

$$\frac{\partial \lambda_j^{(m)}}{\partial \lambda_l} = -\left(\bar{r}_{jj}(y_j^{(m)}) + \sum_{s=0}^{1} \frac{y_j^{(m)}(s)^2}{(c_{js}\lambda_j + d_{js})^2}\right)^{-1} \cdot \bar{r}_{jl}(y_j^{(m)}). \tag{20}$$

Proof. For simplicity we assume $j=1$ and $l=2$. We use the notation of the proof of Proposition 3.3. Consider the identity (16). Dividing it by ϵ, using the boundary conditions (11) and (12), and the continuity established in Lemmas 4.1 and 4.2 we obtain

$$-\frac{\partial \lambda_1}{\partial \lambda_2}\left(\frac{y_1(1)^2 \omega_{11}}{(c_{11}\lambda_1 + d_{11})^2} - \frac{y_1(0)^2 \omega_{10}}{(c_{10}\lambda_1 + d_{10})^2}\right) = \frac{\partial \lambda_1}{\partial \lambda_2}\bar{r}_{11}(y_1) + \bar{r}_{12}(y_1).$$

A straightforward calculation shows that assumptions (Ib) and (II) together with the boundary conditions (11) and (12) imply that

$$-f_{1s}(y_1) = \frac{y_1(s)}{c_{1s}\lambda_1 + d_{1s}}, \; s = 0, 1.$$

For other derivatives, one carries out the same calculation with the roles of 1 and 2 replaced by j and l, respectively. $\square$

Theorem 4.4 *The set of eigenvalue hypersurfaces $\lambda_j = \lambda_j^{m_j}(\boldsymbol{\lambda}_j)$, $j = 1, 2, \ldots, n$, has exactly one intersection point in $\mathbb{R}^n$ for each n-tuple $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ of nonnegative integers.*

Proof. The proof is almost identical to the proof of Lemmas 3.4 and 3.5, and Theorem 3.6. We use Proposition 4.3 to show that function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\boldsymbol{\lambda}) = (\lambda_j - \lambda_j(\boldsymbol{\lambda}_j))_{j=1}^n$ is a C^1 function and to show that its Jacobian matrix has a positive determinant. Further we show that F is a proper function and hence it is a diffeomorphism by Hadamard's Inverse Function Theorem [13, Thm. A]. Then $F^{-1}(0)$ is the intersection point of the eigenvalue hypersurfaces. $\square$

The limiting behaviour of the eigenvalue hypersurfaces follows by [8, Thm. 4.4 and Cor. 4.5].

Proposition 4.5 *The eigenvalue hypersurfaces have the following properties :*

1. $\lambda_j^{(0)}(\boldsymbol{\lambda}_j) < \min\{0, -\frac{d_{j0}}{c_{j0}}, -\frac{d_{j1}}{c_{j1}}\}$ *for all* j,
2. $\lim_{\lambda_k \to \infty} \lambda_j^{(0)}(\boldsymbol{\lambda}_j) = \min\{0, -\frac{d_{j0}}{c_{j0}}, -\frac{d_{j1}}{c_{j1}}\}$ *for all* j *and* $k \neq j$,
3. $\lim_{\lambda_k \to -\infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = -\infty$ *for* $m \geq 0$, $j, k = 1, 2, \ldots, n$, $j \neq k$,
4. $\lim_{\lambda_k \to \infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = -\infty$ *for* $m \geq 0$, $j, k = 1, 2, \ldots, n$, $j \neq k$.

Suppose that $\boldsymbol{\lambda} \in \mathbb{R}^n$ is an eigenvalue of the problem (10), (11) and (12) under assumptions (C), (Ib) and (II)-(IV) and that $y_j(\boldsymbol{\lambda})$, $j = 1, 2, \ldots, n$, are the corresponding eigenfunctions. By [8, Thm. 4.2] it follows that on each hypersurface $\lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$ with $m_j > 0$ we have 3^n oscillation counts. That is, the j-th oscillation count changes when we cross the hyperplanes $\lambda_j = -\frac{d_{js}}{c_{js}}$,

$s = 0, 1$. Write $e_0 = \min\left\{-\frac{d_{js}}{c_{js}}, s = 0, 1\right\}$ and $e_1 = \max\left\{-\frac{d_{js}}{c_{js}}, s = 0, 1\right\}$. Then the numbers $N_{jk}^{(m_j)}$, $k = 1, 2$, are determined so that

$$\lambda_j^{N_{j1}^{(m_j)}-1}(\boldsymbol{\lambda}_j) < e_0 \leq \lambda_j^{N_{j2}^{(m_j)}-1}(\boldsymbol{\lambda}_j) < e_1 \leq \lambda_j^{N_j(m_j)}(\boldsymbol{\lambda}_j).$$

It further follows that

$$h_j = \begin{cases} m_j, & \text{if } \lambda_j^{(m_j)} < e_0 \\ m_j - 1, & \text{if } e_0 \leq \lambda_j^{(m_j)} < e_1 \\ m_j - 2, & \text{otherwise.} \end{cases} \tag{21}$$

Proposition 4.5 and above relations (21) are used to obtain the following result.

Theorem 4.6 *If there are M eigenvalues with the same oscillation count then:*

1. $M \leq 3^n$,
2. *there is at most one oscillation count corresponding to $M = 3^n$ eigenvalues,*
3. *for $M \neq 3^k$, $k = 1, 2, \ldots, n-1$, there is only a finite number of oscillation counts that correspond to M eigenvalues,*
4. *for $M = 3^k$, $k = 1, 2, \ldots, n-1$, there is an infinite number of oscillation counts that correspond to M eigenvalues.*

References

[1] F.V. Atkinson. *Multiparameter Eigenvalue Problems*, Academic Press, New York, London, 1972.

[2] T. Bhattacharyya, P.A. Binding, K. Seddighi. *Two parameter right definite Sturm-Liouville problems with eigenparameter dependent boundary conditions*, submitted.

[3] T. Bhattacharyya, P.A. Binding, K. Seddighi. *Multiparameter Sturm-Liouville problems with eigenparameter dependent boundary conditions*, submitted.

[4] P. A. Binding. Multiparameter definiteness conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **89** (1981), 319-332.

[5] P. A. Binding, P. J. Browne. A definiteness result for determinantal operators, in *Ordinary differential equations and operators (Dundee, 1982)*, 17–30, Lecture Notes in Math., **1032**, Springer-Verlag, Berlin, New York, 1983.

[6] P. A. Binding, P. J. Browne. Application of two parameter eigencurves to Sturm-Liouville problems with eigenparameter-dependent boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), 1205-1218.

[7] P. A. Binding, P. J. Browne. Oscillation theory for indefinite Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), 1123-1136.

[8] P. A. Binding, P. J. Browne, K. Seddighi. Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Proc. Edinburgh Math. Soc.* **37** (1993), 57-72.

[9] P. A. Binding, P. J. Browne, K. Seddighi. Two parameter asymptotic spectra in the uniformly elliptic case, *Result. Math.* **31** (1997), 1-13.

[10] E. A. Coddington, N. Levinson. *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[11] C. T. Fulton. Two point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh, Sect. A* **77** (1977), 293-308.

[12] G. H. Golub, C. F. van Loan, *Matrix Computations*, 2nd edition, The John Hopkins Univ. Press, Baltimore, London, 1989.

[13] W. B. Gordon. On the diffeomorphisms of Euclidean spaces, *Amer. Math. Monthly* **79** (1972), 755-759. (See also Addendum **80** (1973), 674-675; and Application **84** (1977), 28-29.)

[14] J. K. Hale, *Ordinary Differential Equations*, Wiley-InterScience, New York, 1969.

[15] B. D. Sleeman. *Multiparameter Spectral Theory*, Pitman, London, 1978.

[16] H. Volkmer. *Multiparameter Eigenvalue Problems and Expansion Theorems*, Lecture Notes in Math. **1356**, Springer-Verlag, Berlin, New York, 1988.

[17] J. Walter. Regular eigenvalue problems with eigenvalue parameter in the boundary conditions, *Math. Z.* **133** (1973), 301-312.