Multiparameter eigenvalue problems

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Outline

- Multiparameter eigenvalue problem
  - origin
  - applications
  - tensor product approach
  - right definite problem
- Numerical methods
- Related material

Two-parameter eigenvalue problem

- Two-parameter eigenvalue problem:

$$\begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y, \end{aligned} \tag{W}$$

  where A_i, B_i, C_i are $n \times n$ matrices, $\lambda, \mu \in \mathbb{C}$, and $x, y \in \mathbb{C}^n$.

- Eigenvalue: a pair (λ, μ) that satisfies (W) for nonzero x and y.

- Eigenvector: the tensor product $x \otimes y$.

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  where A_i, B_i, C_i are $n \times n$ matrices, $\lambda, \mu \in \mathbb{C}$, and $x, y \in \mathbb{C}^n$.

- Eigenvalue: a pair (λ, μ) that satisfies (W) for nonzero x and y.

- Eigenvector: the tensor product $x \otimes y$.

- **Problem: compute some (all) eigenvalues (λ, μ) and eigenvectors $x \otimes y$.**

Separation of variables: $\Delta u + \nu u = 0$ on Ω, $\quad u|_{\delta\Omega} = 0$

Rectangle: $\Omega = [0,a] \times [0,b] \Longrightarrow$ two S–L equations ($\nu = \lambda + \mu$)

$$\begin{aligned} x'' + \lambda x &= 0, & \qquad x(0) = x(a) &= 0, \\ y'' + \mu y &= 0, & \qquad y(0) = y(b) &= 0. \end{aligned}$$

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Circle: $\Omega = \{x^2 + y^2 \le a^2\}$, polar coordinates $\Longrightarrow$ a triangular situation

$$\begin{aligned} \Phi'' + \lambda \Phi &= 0, & \Phi(0) = \Phi(2\pi) &= 0, \\ r^{-1}(rR')' + (\nu - \lambda r^{-2})R &= 0, & R(0) < \infty, R(a) &= 0. \end{aligned}$$

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Ellipse: $\Omega = \{(x/a)^2 + (y/b)^2 \leq 1\}$, elliptic coordinates ($c$ focus)

$\Longrightarrow$ modified Mathieu's and Mathieu's DE ($4\lambda = c^2 \nu$)

$$\begin{aligned} &v_1'' + (2\lambda \cosh(2y_1) - \mu)v_1 = 0, & &v_1(0) = v_1(d) = 0, \\ &v_2'' - (2\lambda \cos(2y_1) - \mu)v_2 = 0, & &v_2(0) = v_2(\pi/2) = 0. \end{aligned}$$

Tensor product approach

$$
\begin{aligned}
A_1 x &= \lambda B_1 x + \mu C_1 x \\
A_2 y &= \lambda B_2 y + \mu C_2 y
\end{aligned}
\qquad \text{(W)}
$$

- On $\mathbb{C}^n \otimes \mathbb{C}^n$ of the dimension n^2 we define

$$
\begin{aligned}
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\
\Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2.
\end{aligned}
$$

- Two-parameter problem (W) is equivalent to a coupled GEP

$$
\begin{aligned}
\Delta_1 z &= \lambda \Delta_0 z \\
\Delta_2 z &= \mu \Delta_0 z,
\end{aligned}
\qquad (\Delta)
$$

  where $z = x \otimes y$.

- (W) is nonsingular $\Longleftrightarrow$ Δ_0 is nonsingular.
- $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute.

Right definite problem

$$(W) \quad \begin{array}{l} A_1x = \lambda B_1x + \mu C_1x \\ A_2y = \lambda B_2y + \mu C_2y \end{array} \qquad \begin{array}{l} \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \end{array} \qquad \begin{array}{l} \Delta_1 z = \lambda \Delta_0 z \\ \Delta_2 z = \mu \Delta_0 z \end{array} \quad (\Delta)$$

(W) is right definite when

- A_i, B_i, C_i real symmetric
- $\begin{vmatrix} x^TB_1x & x^TC_1x \\ y^TB_2y & y^TC_2y \end{vmatrix} = (x \otimes y)^T \Delta_0 (x \otimes y) > 0$ for $x, y \neq 0$ (equiv. Δ_0 s.p.d.)

If (W) is right definite then

- eigenpairs are real
- there exist n^2 linearly independent eigenvectors
- eigenvectors of distinct eigenvalues are Δ_0-orthogonal, i.e. $(x_1 \otimes y_1)^T \Delta_0 (x_2 \otimes y_2) = 0$

Outline

- Multiparameter eigenvalue problem
- Numerical methods
  - using Δ matrices
  - Jacobi–Davidson type method
    - right definite case
    - general case
- Related material

Algorithm with Δ matrices

$$\Delta_1 z = \lambda \Delta_0 z$$
$$\Delta_2 z = \mu \Delta_0 z$$

1. Generalized Schur decomposition $Q^*\Delta_0 Z = R$ and $Q^*\Delta_1 Z = S$, where Q and Z are unitary, R and S are upper triangular, and the multiple values of $\lambda_i := s_{ii}/r_{ii}$ are clustered along the diagonal of $R^{-1}S$.

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ 0 & R_{22} & \cdots & R_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{pp} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ 0 & S_{22} & \cdots & S_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{pp} \end{bmatrix},$$

   where the size of R_{ii} and S_{ii} is m_i and $m_1 + \cdots + m_p = n^2$.

2. Compute diagonal blocks $T_{11}, \ldots, T_{pp}$ of $T = Q^*\Delta_2 Z$, partitioned conformally with R and S.
3. Compute the eigenvalues $\mu_{i1}, \ldots, \mu_{im_i}$ of the GEP $T_{ii}w = \mu R_{ii} w$ for $i = 1, \ldots, p$.
4. Reindex $(\lambda_1, \mu_{11}), \ldots, (\lambda_1, \mu_{1m_1}); \ldots; (\lambda_p, \mu_{p1}), \ldots, (\lambda_p, \mu_{pm_p})$ into $(\lambda_1, \mu_1), \ldots, (\lambda_{n^2}, \mu_{n^2})$.
5. For each eigenvalue (λ_j, μ_j) compute the eigenvectors $x_j \otimes y_j$ for $j = 1, \ldots, n^2$.

The time complexity is $\mathcal{O}(n^6)$.

Some available numerical methods

- Blum, Curtis, Geltner (1978), and Browne, Sleeman (1982): gradient method,
- Bohte (1980): Newton's method for eigenvalues,
- Slivnik, Tomšič (1986): solving (Δ) with standard numerical methods for RD problem,
- Ji, Jiang, Lee (1992): Generalized Rayleigh Quotient Iteration.
- Shimasaki (1995): continuation method for a special class of RD problems.
- P. (1999): continuation method for RD problem, Tensor Rayleigh Quotient Iteration
- P. (2000): continuation method for weakly elliptic problem.
- Hochstenbach, P. (2002): Jacobi-Davidson type method for RD problem.
- Hochstenbach, Košir, P. (2005): Two-sided Jacobi-Davidson type method.

Subspace methods and Jacobi–Davidson method

Subspace methods compute eigenpairs from low dimensional subspaces. They work as follows:

- Extraction: We start with a given search subspace from which approximations to eigenpairs are computed. In the extraction we usually have to solve the same type of eigenvalue problem as the original one, but of a smaller dimension.
- Expansion: After each step we expand the subspace by a new direction.

As the search subspace grows the eigenpair approximations should converge to an eigenpair of the original problem.

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Jacobi–Davidson method is a subspace method where:

- a new direction to the subspace is orthogonal or oblique to the last chosen Ritz vector,
- approximate solutions of certain correction equations are used for expansion.

JD-like method for the right definite case: extraction

Ritz–Galerkin conditions: search spaces = test spaces: $u \in \mathcal{U}_k$, $v \in \mathcal{V}_k$

$$\begin{aligned}(A_1 - \sigma B_1 - \tau C_1)u &\perp \mathcal{U}_k \\ (A_2 - \sigma B_2 - \tau C_2)v &\perp \mathcal{V}_k\end{aligned}$$

$\Rightarrow$ projected right definite two-parameter problem

$$\begin{aligned}U_k^T A_1 U_k c &= \sigma U_k^T B_1 U_k c + \tau U_k^T C_1 U_k c \\ V_k^T A_2 V_k d &= \sigma V_k^T B_2 V_k d + \tau V_k^T C_2 V_k d\end{aligned}$$

Ritz vectors: $u = U_k c$, $v = V_k d$ for $c, d \in \mathbb{R}^k$

Ritz value: (σ, τ), Ritz pair: $((\sigma, \tau), u \otimes v)$

JD-like method for the two-parameter RD eigenvalue problem

1. $s = u_1$ and $t = v_1$ (starting vectors), $U_0 = V_0 = [\ ]$

   for $k = 1, 2, \ldots$

2. $(U_{k-1}, s) \rightarrow U_k$

   $(V_{k-1}, t) \rightarrow V_k$

3. Extract appropriate Ritz pair $((\sigma, \tau), c \otimes d)$ of

$$\begin{aligned} U_k^T A_1 U_k c &= \sigma U_k^T B_1 U_k c + \tau U_k^T C_1 U_k c \\ V_k^T A_2 V_k d &= \sigma V_k^T B_2 V_k d + \tau V_k^T C_2 V_k d \end{aligned}$$

4. $$\begin{aligned} r_1 &= (A_1 - \sigma B_1 - \tau C_1) u \\ r_2 &= (A_2 - \sigma B_2 - \tau C_2) v \end{aligned}$$

5. Stop if $(\|r_1\|^2 + \|r_2\|^2)^{1/2} \le \varepsilon$

6. Solve (approximately) an $s \perp u, t \perp v$ from corr. equation(s)

JD-like method: expansion: $s \perp u$, $t \perp v$

$$A_1(u+s) = \lambda B_1(u+s) + \mu C_1(u+s)$$
$$A_2(v+t) = \lambda B_2(v+t) + \mu C_2(v+t)$$

Two correction equations

$$(I-uu^T)(A_1-\sigma B_1-\tau C_1)(I-uu^T)s = -r_1$$
$$(I-vv^T)(A_2-\sigma B_2-\tau C_2)(I-vv^T)t = -r_2$$

- We solve the equations only approximately with a Krylov subspace method with initial guess 0 (e.g., few steps of MINRES or GMRES).
- We suggest the preconditioner $M_i = A_i - \lambda_{\mathrm{T}} B_i - \mu_{\mathrm{T}} C_i$, where $(\lambda_{\mathrm{T}}, \mu_{\mathrm{T}})$ is the target.

Theorem: Ritz pair $((\sigma,\tau), u \otimes v)$ is an approximation to $((\lambda,\mu),(u+s)\otimes(v+t))$

$$\Rightarrow \qquad \sqrt{(\lambda-\sigma)^2+(\mu-\tau)^2} = \mathcal{O}(\|s\|^2+\|t\|^2)$$

JD-like method: computing more eigenpairs

Standard deflation techniques can not be applied:

- $(x \otimes y)^{\perp_{\Delta_0}}$ can not be written as $\mathcal{U} \otimes \mathcal{V}$, where $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$.
- there can exist eigenvalues $(\lambda, \mu) \neq (\lambda', \mu')$ with eigenvectors $x \otimes y_1$ and $x \otimes y_2$, respectively.

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- there can exist eigenvalues $(\lambda, \mu) \neq (\lambda', \mu')$ with eigenvectors $x \otimes y_1$ and $x \otimes y_2$, respectively.

Eigenvectors are Δ_0-orthogonal:

$$(x_1 \otimes y_1)^T \Delta_0 (x_2 \otimes y_2) = 0$$

Our approach: In extraction we consider only Ritz vectors that are Δ_0-orthogonal to the already computed eigenvectors.

Two-sided JD-like method for a general problem: extraction

Petrov–Galerkin conditions: search spaces $u_i \in \mathcal{U}_{ik}$, test spaces $v_i \in \mathcal{V}_{ik}$

$$\begin{aligned} (A_1 - \sigma B_1 - \tau C_1)u_1 &\perp \mathcal{V}_{1k}, \\ (A_2 - \sigma B_2 - \tau C_2)u_2 &\perp \mathcal{V}_{2k}, \end{aligned}$$

$\Rightarrow$ projected two-parameter problem

$$\begin{aligned} V_{1k}^* A_1 U_{1k} c_1 &= \sigma V_{1k}^* B_1 U_{1k} c_1 + \tau V_{1k}^* C_1 U_{1k} c_1, \\ V_{2k}^* A_2 U_{2k} c_2 &= \sigma V_{2k}^* B_2 U_{2k} c_2 + \tau V_{2k}^* C_2 U_{2k} c_2, \end{aligned}$$

where $u_i = U_{ik} c_i \neq 0$ for $i = 1, 2$ and $\sigma, \tau \in \mathbb{C}$.

Petrov vectors: $u_i = U_{ik} c_i$, $v_i = V_{ik} d_i$, $c_i, d_i \in \mathbb{C}^k$

Petrov value: (σ, τ), Petrov triple: $((\sigma, \tau), u_1 \otimes u_2, v_1 \otimes v_2)$

Residuals:

$$\begin{aligned} r_i^R &= (A_i - \sigma B_i - \tau C_i) u_i, \\ r_i^L &= (A_i - \sigma B_i - \tau C_i)^* v_i \end{aligned}$$

Two-sided JD-like algorithm

1. $s_i = u_i$ and $t_i = v_i$ (starting vectors), $U_{i0} = V_{i0} = [\ ]$

   for $k = 1, 2, \ldots$

2. $(U_{i,k-1}, s_i) \rightarrow U_{ik}$
   $(V_{i,k-1}, t_i) \rightarrow V_{ik}$

3. Extract appropriate Petrov triple $((\sigma, \tau), c_1 \otimes c_2, d_1 \otimes d_2)$ of

$$\begin{aligned} V_{1k}^* A_1 U_{1k} c_1 &= \sigma V_{1k}^* B_1 U_{1k} c_1 + \tau V_{1k}^* C_1 U_{1k} c_1, \\ V_{2k}^* A_2 U_{2k} c_2 &= \sigma V_{2k}^* B_2 U_{2k} c_2 + \tau V_{2k}^* C_2 U_{2k} c_2, \end{aligned}$$

4. $$\begin{aligned} r_i^R &= (A_i - \sigma B_i - \tau C_i) u_i \\ r_i^L &= (A_i - \sigma B_i - \tau C_i)^* v_i \end{aligned}$$

5. Stop if $\left(\|r_1^R\|^2 + \|r_2^R\|^2 + \|r_1^L\|^2 + \|r_2^L\|^2 \right)^{1/2} < \varepsilon$

6. Solve (approximately) s_i and t_i from correction equations

Outline

- Multiparameter eigenvalue problem
- Numerical methods
- Related material
  - Inverse eigenvalue problems
  - Determinantal representations
  - Structured matrices
- Conclusions

Inverse eigenvalue problems

We have $n \times n$ matrices $A_0, A_1, \ldots, A_k$, $k \leq n$, and we are looking for a linear combination

$$M = A_0 - \lambda_1 A_1 - \cdots - \lambda_k A_k,$$

such that M has eigenvalues $\alpha_1, \ldots, \alpha_k$.

This is a multiparameter eigenvalue problem

$$\begin{aligned} (A_0 - \alpha_1 I)x_1 &= \lambda_1 A_1 x_1 + \cdots + \lambda_k A_k x_1 \\ &\vdots \\ (A_0 - \alpha_k I)x_k &= \lambda_1 A_1 x_k + \cdots + \lambda_k A_k x_k. \end{aligned}$$

Inverse eigenvalue problems

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&\vdots \\
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\end{aligned}$$

Cottin (2001): Dynamic model updating. In a spring-mass model the mass matrix is known and the stiffness parameters of two springs have to be updated based on the outside measurements of the natural frequencies.

These problems are singular, so we can not apply the existing methods.

Determinantal representations

Zeroes of a polynomial $p(x) = a_0 x^n + \cdots + a_n$ are the eigenvalues of the companion matrix.

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Vinnikov (1989): For a given polynomial

$$p(x, y) = a_{00} x^n y^n + a_{01} x^n y^{n-1} + a_{10} x^{n-1} y^n + \cdots + a_{nn}$$

exists a determinantal representation A, B, C such that

$$p(x, y) = \det(A + xB + yC).$$

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$$p(x,y) = \det(A + xB + yC).$$

$$\begin{array}{l} p(\lambda,\mu) = 0 \\ q(\lambda,\mu) = 0 \end{array} \quad \Longrightarrow \quad \text{companion problem} \quad \begin{array}{l} A_1 u = \lambda B_1 u + \mu C_1 u \\ A_2 v = \lambda B_2 v + \mu C_2 v. \end{array}$$

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$$\begin{matrix} p(\lambda, \mu) = 0 \\ q(\lambda, \mu) = 0 \end{matrix} \quad \Longrightarrow \quad \text{companion problem} \quad \begin{matrix} A_1 u = \lambda B_1 u + \mu C_1 u \\ A_2 v = \lambda B_2 v + \mu C_2 v. \end{matrix}$$

Question: How to construct a determinantal representation?

Structured matrices

If we apply finite differences to

$$
\begin{aligned}
-\Big(p_1(x_1)y_1'(x_1)\Big)' + q_1(x_1)y_1(x_1) &= \Big(\lambda\, a_{11}(x_1) + \mu\, a_{12}(x_1)\Big)\, y_1(x_1)\\
-\Big(p_2(x_2)y_2'(x_2)\Big)' + q_2(x_2)y_2(x_2) &= \Big(\lambda\, a_{21}(x_2) + \mu\, a_{22}(x_2)\Big)\, y_2(x_2)
\end{aligned}
$$

then matrix A_i is tridiagonal and B_i, C_i are diagonal for $i = 1, 2$.

Can we exploit this structure for efficient methods for

$$
\begin{aligned}
\Delta_1 z &= \lambda \Delta_0 z\\
\Delta_2 z &= \mu \Delta_0 z,
\end{aligned}
\qquad (\Delta)
$$

where

$$
\begin{aligned}
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2\\
\Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2\\
\Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2?
\end{aligned}
$$

Conclusion

Twoparameter eigenvalue problems can be solved numerically...

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Twoparameter eigenvalue problems can be solved numerically...

Thank you for your attention.

Literature

1. F. V. Atkinson, *Multiparameter eigenvalue problems*, Academic Press, New York, 1972.
2. P., *A continuation method for a right definite two-parameter eigenvalue problem*, SIAM J. Matrix Anal. Appl., 21 (2000), 1163–1184.
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4. M. E. Hochstenbach, P., *A Jacobi–Davidson type method for a right definite two-parameter eigenvalue problem*, SIAM J. Matrix Anal. Appl., 24 (2002), 392–410.
5. M. E. Hochstenbach, T. Košir, and P., *A Jacobi–Davidson type method for a the two-parameter eigenvalue problem*, SIAM J. Matrix Anal. Appl., 26 (2005), 477–497.

Multiparameter eigenvalue problems

$$\begin{aligned}
V_{10}x_1 &= \lambda_1 V_{11}x_1 + \lambda_2 V_{12}x_1 + \cdots + \lambda_k V_{1k}x_1 \\
V_{20}x_2 &= \lambda_1 V_{21}x_2 + \lambda_2 V_{22}x_2 + \cdots + \lambda_k V_{2k}x_2 \\
&\vdots \\
V_{k0}x_k &= \lambda_1 V_{k1}x_k + \lambda_2 V_{k2}x_k + \cdots + \lambda_k V_{kk}x_k.
\end{aligned}$$

A k-tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ is an eigenvalue and $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_k$ is the corresponding eigenvector.

Multiparameter eigenvalue problems

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A k-tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ is an eigenvalue and $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_k$ is the corresponding eigenvector.

$$\Delta_0 = \begin{vmatrix} V_{11}^\dagger & V_{12}^\dagger & \cdots & V_{1k}^\dagger \\ V_{21}^\dagger & V_{22}^\dagger & \cdots & V_{2k}^\dagger \\ \vdots & \vdots & & \vdots \\ V_{k1}^\dagger & V_{k2}^\dagger & \cdots & V_{kk}^\dagger \end{vmatrix}, \quad \Delta_i = \begin{vmatrix} V_{11}^\dagger & \cdots & V_{1,i-1}^\dagger & V_{10}^\dagger & V_{1,i+1}^\dagger & \cdots & V_{1k}^\dagger \\ V_{21}^\dagger & \cdots & V_{2,i-1}^\dagger & V_{20}^\dagger & V_{2,i+1}^\dagger & \cdots & V_{2k}^\dagger \\ \vdots & & \vdots & \vdots & & \vdots & \\ V_{k1}^\dagger & \cdots & V_{k,i-1}^\dagger & V_{k0}^\dagger & V_{k,i+1}^\dagger & \cdots & V_{kk}^\dagger \end{vmatrix}, \quad i = 1, \ldots, k,$$

where $V_{ij}^\dagger$ is defined by $V_{ij}^\dagger(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes V_{ij}x_i \otimes \cdots \otimes x_k$ and linearity.

A nonsingular MEP (i.e., Δ_0 nonsingular) is equivalent to the associated joined problem

$$\Delta_i \boldsymbol{x} = \lambda_i \Delta_0 \boldsymbol{x}, \quad i = 1, \ldots, k,$$

for decomposable tensors $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_k$, where the matrices $\Delta_0^{-1}\Delta_i$ commute.

Two-sided vs. one-sided JD

Statistics of the Jacobi–Davidson type method using the same set of 10 random initial vectors for computing 10 closest eigenvalues to the origin, matrices are of size 100.

For each eigenvalue we select the closest Petrov value to the origin until the residual becomes smaller than $\varepsilon_{\text{change}}$; in the remaining steps we select Petrov triple with the minimum residual.

two-sided correction equation									
	$\varepsilon_{\text{change}} = 10^{-1}$			$\varepsilon_{\text{change}} = 10^{-1.5}$			$\varepsilon_{\text{change}} = 10^{-2}$		
GMRES	In 10	Conv.	Iter.	In 10	Conv.	Iter.	In 10	Conv.	Iter.
5	3.4	4.2	400.0	3.3	3.9	400.0	2.7	3.0	400.0
10	4.7	7.4	324.5	5.9	8.0	387.8	5.3	6.2	400.0
20	6.8	9.4	255.3	6.6	9.2	301.8	6.9	9.4	300.3
40	7.2	9.5	284.0	7.3	9.5	292.3	7.0	9.0	354.9

one-sided correction equation									
	$\varepsilon_{\text{change}} = 10^{-1}$			$\varepsilon_{\text{change}} = 10^{-1.5}$			$\varepsilon_{\text{change}} = 10^{-2}$		
GMRES	In 10	Conv.	Iter.	In 10	Conv.	Iter.	In 10	Conv.	Iter.
5	2.0	5.2	400.0	1.3	1.3	400.0	1.2	0.5	400.0
10	2.9	7.1	357.3	2.6	3.0	400.0	1.9	1.9	400.0
20	3.5	9.9	189.5	3.0	3.7	400.0	1.9	2.1	400.0
40	3.0	9.9	143.8	3.5	4.0	380.5	2.9	3.2	400.0