Singular two-parameter eigenvalue problems

Bor Plestenjak
Department of Mathematics
University of Ljubljana

Joint work with Michiel Hochstenbach (TU Eindhoven) and Andrej Muhič (IMFM Ljubljana)

Outline

- Two-parameter eigenvalue problem (2EP)
- Nonsingular two-parameter eigenvalue problem
  - Harmonic Jacobi–Davidson
- Singular two-parameter eigenvalue problem
  - Extraction of the common finite regular part of two matrix pencils
  - Quadratic two-parameter eigenvalue problem (Q2EP)
  - Applications

Two-parameter eigenvalue problem

- Two-parameter eigenvalue problem:

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1)x &= 0 \\ (A_2 + \lambda B_2 + \mu C_2)y &= 0,\end{aligned} \qquad \text{(2EP)}$$

where A_i, B_i, C_i are $n \times n$ matrices, $\lambda, \mu \in \mathbb{C}$, $x, y \in \mathbb{C}^n$.

- Eigenvalue: a pair (λ, μ) that satisfies (2EP) for nonzero x and y.

- Eigenvector: the tensor product $x \otimes y$.

- There are n^2 eigenvalues, which are solutions of

$$\begin{aligned}\det(A_1 + \lambda B_1 + \mu C_1) &= 0 \\ \det(A_2 + \lambda B_2 + \mu C_2) &= 0.\end{aligned}$$

Atkinson (1972): Tensor product approach

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1)x &= 0 \\ (A_2 + \lambda B_2 + \mu C_2)y &= 0\end{aligned} \qquad \text{(2EP)}$$

- On $\mathbb{C}^n \otimes \mathbb{C}^n$ we define $n^2 \times n^2$ matrices

$$\begin{aligned}\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2 \\ \Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2.\end{aligned}$$

- nonsingular 2EP is equivalent to a coupled GEP

$$\begin{aligned}\Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z,\end{aligned} \qquad (\Delta)$$

where $z = x \otimes y$.

- 2EP is nonsingular $\Longleftrightarrow$ Δ_0 is nonsingular

- $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute

Numerical methods - Δ matrices

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned} \qquad \text{(2EP)}$$

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned}$$

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \qquad (\Delta)$$

Numerical methods - Δ matrices

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned} \qquad \text{(2EP)}$$

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned}$$

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \qquad (\Delta)$$

1. $Q^* \Delta_0 Z = R$ and $Q^* \Delta_1 Z = S$, where Q, Z are unitary, R, S upper triangular, the multiple values of $\lambda_i := s_{ii}/r_{ii}$ are clustered along the diagonal of $R^{-1}S$.

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ 0 & R_{22} & \cdots & R_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{pp} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ 0 & S_{22} & \cdots & S_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{pp} \end{bmatrix}.$$

2. Compute diagonal blocks $T_{11}, \ldots, T_{pp}$ of $T = Q^* \Delta_2 Z$.
3. Compute the eigenvalues $\mu_{i1}, \ldots, \mu_{im_i}$ of $T_{ii} w = \mu R_{ii} w$ for $i = 1, \ldots, p$.
4. Reindex the eigenvalues and compute the eigenvectors.

Time complexity: $\mathcal{O}(n^6)$

Hochstenbach, Košir, P. (2005) SIMAX

Jacobi–Davidson method

Subspace methods (Arnoldi, Lanczos, J-D, ...) compute eigenpairs from low dimensional subspaces. The main ingredients are:

- Extraction: We compute an approximation to an eigenpair from a given search subspace (Rayleigh-Ritz, harmonic Rayleigh-Ritz, ...).
- Expansion: After each step we expand the subspace by a new direction.

Jacobi–Davidson method

Subspace methods (Arnoldi, Lanczos, J-D, ...) compute eigenpairs from low dimensional subspaces. The main ingredients are:

- Extraction: We compute an approximation to an eigenpair from a given search subspace (Rayleigh-Ritz, harmonic Rayleigh-Ritz, ...).
- Expansion: After each step we expand the subspace by a new direction.

Jacobi–Davidson method:

- a new subspace direction is orthogonal or oblique to the last Ritz vector,
- approximate solutions of certain correction equations are used for expansion.

J-D method can be efficiently applied to two-parameter eigenvalue problems. This is not clear for subspace methods based on Krylov subspaces.

J-D for the right definite 2EP

Right definite 2EP: Matrices $A_1, B_1, C_1, A_2, B_2, C_2$ are symmetric, Δ_0 is s.p.d.

Extraction: Ritz–Galerkin: search spaces $=$ test spaces: $u \in \mathcal{U}_k$, $v \in \mathcal{V}_k$

$$
\begin{aligned}
(A_1 - \sigma B_1 - \tau C_1)u &\perp \mathcal{U}_k \\
(A_2 - \sigma B_2 - \tau C_2)v &\perp \mathcal{V}_k
\end{aligned}
$$

$\Rightarrow$ projected right definite two-parameter eigenvalue problem

$$
\begin{aligned}
U_k^T A_1 U_k c &= \sigma U_k^T B_1 U_k c + \tau U_k^T C_1 U_k c \\
V_k^T A_2 V_k d &= \sigma V_k^T B_2 V_k d + \tau V_k^T C_2 V_k d
\end{aligned}
$$

Ritz value: (σ, τ), Ritz vectors: $u = U_k c$, $v = V_k d$, where $c, d \in \mathbb{R}^k$

Expansion: Correction equation for new directions s, t:

$$
\begin{aligned}
(I - uu^T)(A_1 - \sigma B_1 - \tau C_1)(I - uu^T)s &= -(A_1 - \sigma B_1 - \tau C_1)u \\
(I - vv^T)(A_2 - \sigma B_2 - \tau C_2)(I - vv^T)t &= -(A_2 - \sigma B_2 - \tau C_2)v.
\end{aligned}
$$

Works well for exterior eigenvalues. Hochstenbach, P. (2002) SIMAX

Two-sided J-D for a general nonsingular 2EP

Petrov–Galerkin conditions: search spaces $u_i \in \mathcal{U}_{ik}$, test spaces $v_i \in \mathcal{V}_{ik}$

$$\begin{aligned}(A_1 - \sigma B_1 - \tau C_1)u_1 &\perp \mathcal{V}_{1k} \\ (A_2 - \sigma B_2 - \tau C_2)u_2 &\perp \mathcal{V}_{2k},\end{aligned}$$

$\Rightarrow$ projected two-parameter eigenvalue problem

$$\begin{aligned}V_{1k}^* A_1 U_{1k} c_1 &= \sigma V_{1k}^* B_1 U_{1k} c_1 + \tau V_{1k}^* C_1 U_{1k} c_1 \\ V_{2k}^* A_2 U_{2k} c_2 &= \sigma V_{2k}^* B_2 U_{2k} c_2 + \tau V_{2k}^* C_2 U_{2k} c_2,\end{aligned}$$

where $u_i = U_{ik} c_i \neq 0$ for $i = 1, 2$ and $\sigma, \tau \in \mathbb{C}$.

Petrov value: (σ, τ), Petrov vectors: $u_i = U_{ik} c_i$, $v_i = V_{ik} d_i$, where $c_i, d_i \in \mathbb{C}^k$

Usually performs better than the one-sided method.

Works well for exterior eigenvalues. Hochstenbach, Košir, P. (2005) SIMAX

Harmonic Rayleigh–Ritz for 2EP

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

Spectral transformation: $(A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x$

Harmonic Rayleigh–Ritz: $Au - \theta Bu \perp (A - \tau B)\,\mathcal{U}_k$

2EP:

$$\begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y \end{aligned}$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:

$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\,u &\perp \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\,v &\perp \mathcal{V}_k \end{aligned}$$

Harmonic Rayleigh–Ritz for 2EP

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

Spectral transformation: $(A - \tau B)^{-1} Bx = (\lambda - \tau)^{-1} x$

Harmonic Rayleigh–Ritz: $Au - \theta Bu \perp (A - \tau B)\,\mathcal{U}_k$

2EP:
$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned}$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:
$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\, u &\perp \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\, v &\perp \mathcal{V}_k \end{aligned}$$

Spectral transformation: ? ? ?

Harmonic Rayleigh–Ritz for 2EP

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

Spectral transformation: $(A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x$

Harmonic Rayleigh–Ritz: $Au - \theta Bu \perp (A - \tau B)\,\mathcal{U}_k$

2EP:
$$\begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y \end{aligned}$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:
$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\,u &\perp \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\,v &\perp \mathcal{V}_k \end{aligned}$$

Spectral transformation: ? ? ?

Harmonic Rayleigh–Ritz:
$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\,u &\perp (A_1 - \sigma B_1 - \tau C_1)\,\mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\,v &\perp (A_2 - \sigma B_2 - \tau C_2)\,\mathcal{V}_k \end{aligned}$$

J-D with harmonic Rayleigh–Ritz for 2EP

1. $s = u_1$ and $t = v_1$ (starting vectors), $U_0 = V_0 = [\ ]$

   for $k = 1, 2, \ldots$

2. $(U_{k-1}, s) \rightarrow U_k, \quad (V_{k-1}, t) \rightarrow V_k$

3. Extract appropriate harmonic Ritz pair $((\xi_1, \xi_2), U_k c \otimes V_k d)$

4. Take $u = U_k c$, $v = V_k d$ and compute tensor Rayleigh quotient

$$\theta = \frac{(u \otimes v)^* \Delta_1 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* A_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* A_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}$$

$$\eta = \frac{(u \otimes v)^* \Delta_2 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* B_1 u)(v^* A_2 v) - (u^* A_1 u)(v^* B_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}$$

5. $$\begin{aligned} r_1 &= (A_1 - \theta B_1 - \eta C_1) u \\ r_2 &= (A_2 - \theta B_2 - \eta C_2) v \end{aligned}$$

6. Stop if $(\|r_1\|^2 + \|r_2\|^2)^{1/2} \le \varepsilon$

7. Solve (approximately) an $s \perp u, t \perp v$ from corr. equation(s)

Hochstenbach, P. (2008) ETNA

Singular 2EP

$$\begin{array}{c} A_1 x = \lambda B_1 x + \mu C_1 x \\ A_2 y = \lambda B_2 y + \mu C_2 y \\ \text{(2EP)} \end{array} \qquad \begin{array}{c} \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \end{array} \qquad \begin{array}{c} \Delta_1 z = \lambda \Delta_0 z \\ \Delta_2 z = \mu \Delta_0 z \\ (\Delta) \end{array}$$

2EP is singular iff Δ_0 is singular

We assume that all linear combinations of Δ_0, Δ_1, and Δ_2 are singular.

Singular 2EP

$$\begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y \end{aligned}$$

(2EP)

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned}$$

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned}$$

(Δ)

2EP is singular iff Δ_0 is singular

We assume that all linear combinations of Δ_0, Δ_1, and Δ_2 are singular.

There are no general results linking the eigenvalues of singular (2EP) and (Δ).

We know:

$$\begin{aligned} (A_1 + \lambda B_1 + \mu C_1)x &= 0 \\ (A_2 + \lambda B_2 + \mu C_2)y &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \Delta_1(x \otimes y) &= \lambda \Delta_0(x \otimes y) \\ \Delta_2(x \otimes y) &= \mu \Delta_0(x \otimes y) \end{aligned}$$

Singular pencils

$$A - \lambda B \text{ is a singular pencil} \Leftrightarrow \det(A - \lambda B) \equiv 0$$

λ is a finite regular eigenvalue if

$$\operatorname{rank}(A - \lambda B) < \max_{\mu \in \mathbb{C}} \operatorname{rank}(A - \mu B).$$

Singular pencils

$$A - \lambda B \text{ is a singular pencil} \Leftrightarrow \det(A - \lambda B) \equiv 0$$

λ is a finite regular eigenvalue if

$$\operatorname{rank}(A - \lambda B) < \max_{\mu \in \mathbb{C}} \operatorname{rank}(A - \mu B).$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix pencil $A - \lambda B$ is singular, the only finite eigenvalue is $\lambda = 1$.

Finite regular eigenvalues

(λ, μ) is a finite regular eigenvalue of (2EP) if for $i = 1, 2$:

$$\text{rank}(A_i + \lambda B_i + \mu C_i) < \max_{(s,t)\in\mathbb{C}^2} \text{rank}(A_i + sB_i + tC_i).$$

(λ, μ) is a finite regular eigenvalue of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if:

1. $\text{rank}(\Delta_1 - \lambda\Delta_0) < \max_{s\in\mathbb{C}} \text{rank}(\Delta_1 - s\Delta_0)$,

2. $\text{rank}(\Delta_2 - \mu\Delta_0) < \max_{t\in\mathbb{C}} \text{rank}(\Delta_2 - t\Delta_0)$,

3. a common regular eigenvector z of $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ exists, such that

$$\begin{aligned}(\Delta_1 - \lambda\Delta_0)z &= 0,\\ (\Delta_2 - \mu\Delta_0)z &= 0.\end{aligned}$$

Finite regular eigenvalues

(λ, μ) is a finite regular eigenvalue of (2EP) if for $i = 1, 2$:

$$\text{rank}(A_i + \lambda B_i + \mu C_i) < \max_{(s,t)\in\mathbb{C}^2} \text{rank}(A_i + sB_i + tC_i).$$

(λ, μ) is a finite regular eigenvalue of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if:

1. $\text{rank}(\Delta_1 - \lambda\Delta_0) < \max_{s\in\mathbb{C}} \text{rank}(\Delta_1 - s\Delta_0)$,
2. $\text{rank}(\Delta_2 - \mu\Delta_0) < \max_{t\in\mathbb{C}} \text{rank}(\Delta_2 - t\Delta_0)$,
3. a common regular eigenvector z of $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ exists, such that

$$\begin{aligned}(\Delta_1 - \lambda\Delta_0)z &= 0,\\ (\Delta_2 - \mu\Delta_0)z &= 0.\end{aligned}$$

Conjecture: Finite regular eigenvalues of (2EP) = finite regular eig's of (Δ).

Simple finite regular eigenvalues agree

$$\begin{array}{ccc} A_1x = \lambda B_1x + \mu C_1x & \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 & \Delta_1 z = \lambda \Delta_0 z \\ A_2y = \lambda B_2y + \mu C_2y & \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 & \Delta_2 z = \mu \Delta_0 z \\ \text{(2EP)} & \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 & (\Delta) \end{array}$$

Theorem (Muhič, P. (2009)). Suppose that all finite eigenvalues of a regular 2EP are algebraically simple. Then (λ, μ) is a finite regular eigenvalue of 2EP iff (λ, μ) is a finite regular eigenvalue of Δ.

Simple finite regular eigenvalues agree

$$\begin{array}{ccc} A_1x = \lambda B_1x + \mu C_1x & \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 & \Delta_1 z = \lambda \Delta_0 z \\ A_2y = \lambda B_2y + \mu C_2y & \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 & \Delta_2 z = \mu \Delta_0 z \\ \text{(2EP)} & \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 & (\Delta) \end{array}$$

Theorem (Muhič, P. (2009)). Suppose that all finite eigenvalues of a regular 2EP are algebraically simple. Then (λ, μ) is a finite regular eigenvalue of 2EP iff (λ, μ) is a finite regular eigenvalue of Δ.

More than two parameters?

Simple eigenvalues of a singular multiparameter eigenvalue problem are common regular eigenvalues of the associated system of singular generalized eigenvalue problems.

Numerical method for singular 2EP

$$A_1x = \lambda B_1x + \mu C_1x \qquad \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \qquad \Delta_1 z = \lambda\Delta_0 z$$
$$A_2y = \lambda B_2y + \mu C_2y \qquad \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \qquad \Delta_2 z = \mu\Delta_0 z$$
$$(\text{2EP}) \qquad \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \qquad (\Delta)$$

We extract the common finite regular part of matrix pencils (Δ).

We get matrices $\widetilde{\Delta}_0, \widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ such that $\widetilde{\Delta}_0$ is nonsingular and eigenvalues of

$$\begin{aligned} \widetilde{\Delta}_1\widetilde{z} &= \lambda\widetilde{\Delta}_0\widetilde{z} \\ \widetilde{\Delta}_2\widetilde{z} &= \mu\widetilde{\Delta}_0\widetilde{z} \end{aligned} \qquad (\widetilde{\Delta})$$

are finite regular eigenvalues of (Δ).

Kronecker canonical structure

For a matrix pencil $A - \lambda B$ there exist nonsingular matrices P and Q such that

$$P^{-1}(A - \lambda B)Q = \widetilde{A} - \lambda\widetilde{B} = \mathrm{diag}(A_1 - \lambda B_1, \ldots, A_b - \lambda B_b)$$

is the Kronecker canonical form (KCF). Regular blocks

$$J_j(\alpha) = \begin{bmatrix} \alpha - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha - \lambda \end{bmatrix}, \quad N_j = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix},$$

and singular blocks

$$L_j = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \quad L_j^T = \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix},$$

represent finite regular, infinite regular, right singular, and left singular blocks, respectively.

Generalized upper-triangular form

Instead of KCF, we use the generalized upper-triangular form (GUPTRI):

$$P^H(A-\lambda B)Q = \left[\begin{array}{cc|cc} A_\mu - \lambda B_\mu & & & \\ \times & A_\infty - \lambda B_\infty & & \\ \hline \times & \times & A_f - \lambda B_f & \\ \times & \times & \times & A_\epsilon - \lambda B_\epsilon \end{array}\right].$$

Pencils $A_\mu - \lambda B_\mu$, $A_\infty - \lambda B_\infty$, $A_f - \lambda B_f$, and $A_\epsilon - \lambda B_\epsilon$ contain the left singular, the infinite regular, the finite regular, and the right singular structure, respectively.

Van Dooren (1979), Demmel and Kågström (1993), software package GUPTRI.

Our extraction algorithm is based on Van Dooren's staircase algorithm (1979).

Algorithms RRS and RLS

RRS: Using SVD for the row and column compressions of the matrices A and B we find matrices P, Q with orthonormal columns such that

- $P^H(A-\lambda B)Q = \begin{bmatrix} A_f - \lambda B_f & \\ \times & A_\epsilon - \lambda B_\epsilon \end{bmatrix}$ is a finite regular and right singular structure of pencil $A - \lambda B$,
- the columns of Q are a basis for the eigenspace of the finite regular and the right singular part.

Algorithms RRS and RLS

RRS: Using SVD for the row and column compressions of the matrices A and B we find matrices P, Q with orthonormal columns such that

- $P^H(A-\lambda B)Q = \begin{bmatrix} A_f - \lambda B_f & \\ \times & A_\epsilon - \lambda B_\epsilon \end{bmatrix}$ is a finite regular and right singular structure of pencil $A - \lambda B$,
- the columns of Q are a basis for the eigenspace of the finite regular and the right singular part.

RLS: Using SVD for the column and row compressions of the matrices A and B we find matrices P, Q with orthonormal columns such that

- $P^H(A - \lambda B)Q = \begin{bmatrix} A_\mu - \lambda B_\mu & \\ \times & A_f - \lambda B_f \end{bmatrix}$ is a left singular and a finite regular structure of pencil $A - \lambda B$,
- the columns of P are a basis for the left eigenspace of the left singular and the finite regular part.

Algorithm for the common finite regular part

We start with matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$, $P = I$ and $Q = I$.

1. Separate the infinite and the finite part.
   (a) Apply RRS to $P^H\Delta_1 Q - \lambda P^H\Delta_0 Q$ and $P^H\Delta_2 Q - \mu P^H\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Compute Q and P such that $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.
   (c) If $\mathcal{Q} = \mathcal{Q}_1$ return P, Q and proceed to (2a). Otherwise, proceed to (1a).

2. Separate the finite regular part from the right singular part.
   (a) Apply RLS on $P^H\Delta_1 Q - \lambda P^H\Delta_0 Q$ and $P^H\Delta_2 Q - \mu P^H\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Compute Q and P such that $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$.
   (c) If $\mathcal{P} = \mathcal{P}_1$ return P, Q and exit. Otherwise, proceed to (2a).

In the end $\widetilde{\Delta}_0 = P^H\Delta_0 Q$, $\widetilde{\Delta}_1 = P^H\Delta_1 Q$, $\widetilde{\Delta}_2 = P^H\Delta_2 Q$ and $\widetilde{\Delta}_0$ is nonsingular.

Quadratic 2EP

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x &= 0 \\ (A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y &= 0,\end{aligned} \quad \text{(Q2EP)}$$

where $A_i, B_i, \ldots, F_i$ are $n \times n$ matrices, (λ, μ) is an eigenvalue, and $x \otimes y$ is the corresponding eigenvector. In the generic case the problem has $4n^2$ eigenvalues that are solutions of

$$\begin{aligned}\det(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1) &= 0 \\ \det(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2) &= 0.\end{aligned}$$

Jarlebring (2008): Q2EP of a simpler form, with some of the terms $\lambda^2, \lambda\mu, \mu^2$ missing, appears in the study of linear time-delay systems for the single delay.

Linearization

$$(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x = 0$$
$$(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y = 0 \qquad \text{(Q2EP)}$$

Vinnikov (1989): It follows from the theory on determinantal representations that one could write Q2EP as a 2EP with $2n \times 2n$ matrices.

Since there is no construction this is just a theoretical result.

Linearization

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x &= 0\\ (A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y &= 0\end{aligned} \qquad \text{(Q2EP)}$$

Vinnikov (1989): It follows from the theory on determinantal representations that one could write Q2EP as a 2EP with $2n \times 2n$ matrices.

Since there is no construction this is just a theoretical result.

We can write Q2EP as a two-parameter eigenvalue problem with $3n \times 3n$ matrices:

$$\left(\begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_1 & E_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ \lambda x \\ \mu x \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} y \\ \lambda y \\ \mu y \end{bmatrix} = 0.$$

Weak linearization

If we multiply the matrix of the first equation

$$\begin{bmatrix} A_1 & B_1+\lambda D_1 & C_1+\lambda E_1+\mu F_1 \\ \lambda I & -I & 0 \\ \mu I & 0 & -I \end{bmatrix}$$

from left by the unimodular polynomial

$$E(\lambda,\mu)=\begin{bmatrix} I & B_1+\lambda D_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} I & 0 & C_1+\lambda E_1+\mu F_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and from right by the unimodular polynomial

$$F(\lambda,\mu)=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mu I & 0 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ \lambda I & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we obtain

$$\begin{bmatrix} A_1+\lambda B_1+\mu C_1+\lambda^2 D_1+\lambda\mu E_1+\mu^2 F_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Linearization is a singular 2EP

$$A^{(1)} + \lambda B^{(1)} + \mu C^{(1)} = \begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_1 & E_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}$$

$$A^{(2)} + \lambda B^{(2)} + \mu C^{(2)} = \begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}.$$

The matrices of the corresponding pair of generalized eigenvalue problems are

$$\begin{aligned} \Delta_0 &= B^{(1)} \otimes C^{(2)} - C^{(1)} \otimes B^{(2)}, \\ \Delta_1 &= C^{(1)} \otimes A^{(2)} - A^{(1)} \otimes C^{(2)}, \\ \Delta_2 &= A^{(1)} \otimes B^{(2)} - B^{(1)} \otimes A^{(2)}. \end{aligned}$$

Lemma. In the generic case (matrices D_1, D_2, F_1, F_2 are all nonsingular):

1. $\mathrm{rank}(\Delta_1) = \mathrm{rank}(\Delta_2) = 8n^2$,
2. $\mathrm{rank}(\Delta_0) = 6n^2$,
3. $\det(\alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2) = 0$ for all $\alpha_0, \alpha_1, \alpha_2$.

Regular eigenvalues

$$
\begin{aligned}
A^{(1)} + \lambda B^{(1)} + \mu C^{(1)} &= \begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_1 & E_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \\
A^{(2)} + \lambda B^{(2)} + \mu C^{(2)} &= \begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

The matrices of the corresponding pair of generalized eigenvalue problems are

$$
\begin{aligned}
\Delta_0 &= B^{(1)} \otimes C^{(2)} - C^{(1)} \otimes B^{(2)}, \\
\Delta_1 &= C^{(1)} \otimes A^{(2)} - A^{(1)} \otimes C^{(2)}, \\
\Delta_2 &= A^{(1)} \otimes B^{(2)} - B^{(1)} \otimes A^{(2)}.
\end{aligned}
$$

Theorem. Kronecker canonical form of pencil $\Delta_1 - \lambda\Delta_0$ (and $\Delta_2 - \mu\Delta_0$) has $n^2\ L_0$, $n^2\ L_0^T$, $2n^2\ N_2$ blocks, and the finite regular part of size $4n^2$.

Theorem. The eigenvalues of Q2EP are regular eigenvalues of the coupled matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ from the weak linearization.

Q2EP example

$$\left(\begin{bmatrix} -3 & 4 \\ 6 & -1 \end{bmatrix} + \lambda \begin{bmatrix} 7 & 2 \\ -2 & 1 \end{bmatrix} + \mu \begin{bmatrix} 4 & -1 \\ 9 & 4 \end{bmatrix} + \lambda^2 \begin{bmatrix} 6 & 7 \\ 5 & 2 \end{bmatrix} + \lambda\mu \begin{bmatrix} 10 & -3 \\ 7 & 1 \end{bmatrix} + \mu^2 \begin{bmatrix} 4 & 8 \\ 6 & -3 \end{bmatrix}\right) x = 0,$$

$$\left(\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix} + \lambda \begin{bmatrix} -1 & -4 \\ 8 & 2 \end{bmatrix} + \mu \begin{bmatrix} 2 & 3 \\ -4 & -1 \end{bmatrix} + \lambda^2 \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} + \lambda\mu \begin{bmatrix} 7 & -2 \\ 3 & 7 \end{bmatrix} + \mu^2 \begin{bmatrix} 3 & -5 \\ -5 & 2 \end{bmatrix}\right) y = 0.$$

Matrices Δ_0, Δ_1, Δ_2 obtained in the linearization are of size 36×36.

The algorithm for the extraction of the common finite regular part returns matrices $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$ of size 16×16, such that $\widetilde{\Delta}_0$ is nonsingular and eigenvalues of

$$\begin{aligned} \widetilde{\Delta}_1 \widetilde{z} &= \lambda \widetilde{\Delta}_0 \widetilde{z} \\ \widetilde{\Delta}_2 \widetilde{z} &= \mu \widetilde{\Delta}_0 \widetilde{z} \end{aligned} \qquad (\widetilde{\Delta})$$

are the eigenvalue of the Q2EP.

From $(\widetilde{\Delta})$ we compute all 16 eigenvalues of Q2EP. The largest and the smallest one (in absolute value) are $(1.799, -2.166)$ and $(0.007 \pm 0.167i, -0.507 \pm 0.1i)$.

Cubic two-parameter eigenvalue problem

$$\begin{aligned}(S_{00}+\lambda S_{10}+\mu S_{01}+\cdots+\lambda^3 S_{30}+\lambda^2\mu S_{21}+\lambda\mu^2 S_{12}+\mu^3 S_{03})x &= 0\\ (T_{00}+\lambda T_{10}+\mu T_{01}+\cdots+\lambda^3 T_{30}+\lambda^2\mu T_{21}+\lambda\mu^2 T_{12}+\mu^3 T_{03})y &= 0.\end{aligned}$$

In the general case the problem has $9n^2$ eigenvalues. A possible linearization is

$$\left(\begin{bmatrix} S_{00} & S_{10} & S_{01} & S_{20} & S_{11} & S_{02}\\ & -I & & & & \\ & & -I & & & \\ & & & -I & & \\ & & & & -I & \\ & & & & & -I\end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & S_{30} & S_{21} & S_{12}\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & S_{03}\\ 0 & 0 & 0 & 0 & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\end{bmatrix}\right)\widetilde{x} = 0$$

$$\left(\begin{bmatrix} T_{00} & T_{10} & T_{01} & T_{20} & T_{11} & T_{02}\\ & -I & & & & \\ & & -I & & & \\ & & & -I & & \\ & & & & -I & \\ & & & & & -I\end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & T_{30} & T_{21} & T_{12}\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & T_{03}\\ 0 & 0 & 0 & 0 & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\end{bmatrix}\right)\widetilde{y} = 0,$$

$\widetilde{x} = [1 \quad \lambda \quad \mu \quad \lambda^2 \quad \lambda\mu \quad \mu^2]^T \otimes x$ and $\widetilde{y} = [1 \quad \lambda \quad \mu \quad \lambda^2 \quad \lambda\mu \quad \mu^2]^T \otimes y$.

Problem is singular, $\mathrm{rank}(\Delta_0) = 20n^2$.

Similarly we can linearize all bivariate matrix polynomials.

Model updating as a singular 2EP

Model updating (Cottin 2001, Cottin and Reetz 2006): Parameters of finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find a perturbation of matrix A by a linear combination of matrices B and C, such that $A + \lambda B + \mu C$ has the prescribed eigenvalues σ_1 and σ_2.

Model updating as a singular 2EP

Model updating (Cottin 2001, Cottin and Reetz 2006): Parameters of finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find a perturbation of matrix A by a linear combination of matrices B and C, such that $A+\lambda B+\mu C$ has the prescribed eigenvalues σ_1 and σ_2.

The problem, usually treated as an optimization problem, can be expressed as a 2EP

$$\begin{aligned}(A-\sigma_1 I+\lambda B+\mu C)x &= 0,\\ (A-\sigma_2 I+\lambda B+\mu C)y &= 0.\end{aligned}$$

$\det(B\otimes C-C\otimes B)=0$ and this is a singular 2EP.

Jacobi-Davidson method for singular 2EP

- Jacobi–Davidson method was successfully applied to nonsingular 2EPs.
- Now that we have a solver for singular 2EPs, we can use it in the outer step.
- This gives us a J-D for singular 2EP.

Zeros of bivariate polynomials

Suppose that we have a system of two bivariate polynomials

$$
\begin{aligned}
p(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{ij}x^i y^j &= 0 \\
q(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} b_{ij}x^i y^j &= 0.
\end{aligned}
$$

Such system has n^2 solutions.

Zeros of bivariate polynomials

Suppose that we have a system of two bivariate polynomials

$$
\begin{aligned}
p(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{ij}x^i y^j &= 0 \\
q(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} b_{ij}x^i y^j &= 0.
\end{aligned}
$$

Such system has n^2 solutions.

We can linearize it as a singular 2EP with matrices of size $\dfrac{n(n+1)}{2} \times \dfrac{n(n+1)}{2}$.

Matrices of 2EP are very large, but:

- We can apply Jacobi–Davidson method to compute solutions close to (x_0, y_0).
- Matrices are sparse, we need $\mathcal{O}(n^2)$ flops for one MV multiplication in J-D.
- This is the same order as for one evaluation of $p(x,y)$ and $q(x,y)$.

Bibliography

Thank you for your attention!

Bibliography:

- M. E. Hochstenbach, A. Muhič, BP: On linearizations of the quadratic two-parameter eigenvalue problems, March 2010, preprint.
- A. Muhič, BP: On the quadratic two-parameter eigenvalue problem and its linearization, Linear Algebra Appl. 432 (2010) 2529–2542.
- A. Muhič, BP: On the singular two-parameter eigenvalue problem, Electron. J. Linear Algebra 18 (2009) 420–437.
- M. E. Hochstenbach, BP: Harmonic Rayleigh-Ritz for the multiparameter eigenvalue problem, Electron. Trans. Numer. Anal. 29 (2008) 81–96.
- M. E. Hochstenbach, T. Košir, BP: A Jacobi-Davidson type method for the two-parameter eigenvalue problem, SIAM J. Matrix Anal. Appl. 26 (2005) 477–497.