Singular two-parameter eigenvalue problems and bivariate polynomial systems

Bor Plestenjak

Department of Mathematics

University of Ljubljana

Joint work with Andrej Muhič (IMFM Ljubljana)

Outline

- Systems of bivariate polynomials
- (Singular) two-parameter eigenvalue problem
- Extraction of the common finite regular part of two matrix pencils
- Application to systems of bivariate polynomials

Roots of a polynomial as an eigenvalue problem

Well-known: roots of a monic polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ are the eigenvalues of the companion matrix

$$C_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

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Thus, we can compute roots using an eigenvalue solver.

Can we generalize this to a system of bivariate polynomials?

$$\begin{aligned} p(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{ij}x^i y^j &= 0 \\ q(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} b_{ij}x^i y^j &= 0 \end{aligned}$$

Determinantal representations

$$p(\lambda,\mu)=\sum_{i=0}^{n}\sum_{j=0}^{n-i}a_{ij}\lambda^i\mu^j=0, \qquad q(\lambda,\mu)=\sum_{i=0}^{n}\sum_{j=0}^{n-i}b_{ij}\lambda^i\mu^j=0$$

Determinantal representation (Dixon (1902)): There exist matrices A_i, B_i, and C_i of dimension $n \times n$ such that

$$\begin{aligned} p(\lambda,\mu) &= \det(A_1+\lambda B_1+\mu C_1) \\ q(\lambda,\mu) &= \det(A_2+\lambda B_2+\mu C_2). \end{aligned}$$

Determinantal representations

$$p(\lambda, \mu) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_{ij} \lambda^i \mu^j = 0, \qquad q(\lambda, \mu) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} b_{ij} \lambda^i \mu^j = 0$$

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This leads to a two-parameter eigenvalue problem

$$\begin{aligned} (A_1 + \lambda B_1 + \mu C_1)x &= 0 \\ (A_2 + \lambda B_2 + \mu C_2)y &= 0. \end{aligned}$$

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Unfortunately, there isn't any construction of such $n\times n$ matrices.

Quadratic bivariate polynomials

$$\begin{aligned} a_{00} + a_{10}\lambda + a_{01}\mu + a_{20}\lambda^2 + a_{11}\lambda\mu + a_{02}\mu^2 &= 0 \\ b_{00} + b_{10}\lambda + b_{01}\mu + b_{20}\lambda^2 + b_{11}\lambda\mu + b_{02}\mu^2 &= 0 \end{aligned}$$

We write the above as:

$$\left(\begin{bmatrix} a_{00} & a_{10} & a_{01} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & a_{20} & a_{11} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & a_{02} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ \lambda \\ \mu \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} b_{00} & b_{10} & b_{01} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & b_{20} & b_{11} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & b_{02} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ \lambda \\ \mu \end{bmatrix} = 0.$$

This is a two-parameter eigenvalue problem with 3×3 matrices

$$\begin{aligned} (A_1 + \lambda B_1 + \mu C_1)x &= 0 \\ (A_2 + \lambda B_2 + \mu C_2)y &= 0. \end{aligned}$$

Cubic bivariate polynomials

$$\begin{aligned} a_{00} + a_{10}\lambda + a_{01}\mu + \cdots + a_{30}\lambda^3 + a_{21}\lambda^2\mu + a_{12}\lambda\mu^2 + a_{03}\mu^3 &= 0 \\ b_{00} + b_{10}\lambda + b_{01}\mu + \cdots + b_{30}\lambda^3 + b_{21}\lambda^2\mu + b_{12}\lambda\mu^2 + b_{03}\mu^3 &= 0 \end{aligned}$$

We can write this as a two-parameter eigenvalue problem with 6×6 matrices:

$$\left(\begin{bmatrix} a_{00} & a_{10} & a_{01} & a_{20} & a_{11} & a_{02} \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & a_{30} & a_{21} & a_{12} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ \lambda \\ \mu \\ \lambda^2 \\ \lambda\mu \\ \mu^2 \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} b_{00} & b_{10} & b_{01} & b_{20} & b_{11} & b_{02} \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -! \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & b_{30} & b_{21} & b_{12} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & b_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ \lambda \\ \mu \\ \lambda^2 \\ \lambda\mu \\ \mu^2 \end{bmatrix} = 0.$$

Linearization of a bivariate polynomial

$p(\lambda,\mu)=\sum_{i=0}^{n}\sum_{j=0}^{n-i}a_{ij}\lambda^i\mu^j=\det(L(\lambda,\mu))$, where

$$L(\lambda,\mu)=\begin{array}{c} \\ 1\\ 2\\ \vdots\\ k\end{array}\begin{array}{c}\begin{array}{cccc}1 & 2 & \dots & n\end{array}\\ \left[\begin{array}{cccc} S_1 & S_2 & \cdots & S_n\\ T_1 & -I_2 & & \\ & \ddots & \ddots & \\ & & T_n & -I_n\end{array}\right]\end{array},\qquad T_r=\left[\begin{array}{ccc}\lambda & & \\ & \ddots & \\ & & \lambda\\ & & \mu\end{array}\right],$$

$$\begin{aligned} S_1 &= [a_{00}]\\ S_2 &= [\,a_{10}\quad a_{01}\,]\\ &\vdots\\ S_{n-1} &= [\,a_{n-2,0}\quad a_{n-3,1}\quad\cdots\quad a_{0,n-2}\,]\\ S_n &= [\,a_{n-1,0}\quad\cdots\quad a_{0,n-1}\,]+\lambda\,[\,a_{n0}\quad\cdots\quad a_{1,n-1}\,]+\mu\,[\,0\quad\cdots\quad 0\quad a_{0n}\,].\end{aligned}$$

Matrices in $L(\lambda,\mu)$ are of dimension $\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)$.

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There exist linear matrix polynomials $G(\lambda,\mu)$ and $H(\lambda,\mu)$ with triangular matrices and a constant diagonal, such that

$$G(\lambda,\mu)L(\lambda,\mu)H(\lambda,\mu)=\left[\begin{array}{cc}p(\lambda,\mu) & 0\\ 0 & I\end{array}\right]$$

and $L(\lambda,\mu)$ is a linearization of $p(\lambda,\mu)$.

Two-parameter eigenvalue problem

- Two-parameter eigenvalue problem:

$$
\begin{aligned}
(A_1 + \lambda B_1 + \mu C_1)x &= 0 \\
(A_2 + \lambda B_2 + \mu C_2)y &= 0,
\end{aligned}
\qquad \text{(2EP)}
$$

  where A_i, B_i, C_i are $m \times m$ matrices, $\lambda, \mu \in \mathbb{C}$, $x, y \in \mathbb{C}^m$.

- Eigenvalue: a pair (λ, μ) that satisfies (2EP) for nonzero x and y.

- Eigenvector: the tensor product $x \otimes y$.

- Generically, there are m^2 eigenvalues, which are solutions of

$$
\begin{aligned}
\det(A_1 + \lambda B_1 + \mu C_1) &= 0 \\
\det(A_2 + \lambda B_2 + \mu C_2) &= 0.
\end{aligned}
$$

Tensor product approach

$$\begin{aligned}(A_1+\lambda B_1+\mu C_1)x&=0\\(A_2+\lambda B_2+\mu C_2)y&=0\end{aligned}\qquad \text{(2EP)}$$

- Atkinson (1972): on $\mathbb{C}^m\otimes\mathbb{C}^m$ we define $m^2\times m^2$ matrices

$$\begin{aligned}\Delta_0 &= B_1\otimes C_2-C_1\otimes B_2\\ \Delta_1 &= C_1\otimes A_2-A_1\otimes C_2\\ \Delta_2 &= A_1\otimes B_2-B_1\otimes A_2.\end{aligned}$$

- 2EP is nonsingular $\Longleftrightarrow$ Δ_0 is nonsingular
- nonsingular 2EP is equivalent to a coupled GEP

$$\begin{aligned}\Delta_1 z&=\lambda\Delta_0 z\\ \Delta_2 z&=\mu\Delta_0 z,\end{aligned}\qquad (\Delta)$$

  where $z=x\otimes y$.

- $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute
- Hochstenbach, Košir, P. (2005): numerical method of complexity $\mathcal{O}(m^6)$

Singular 2EP

Our linearization of bivariate polynomials leads to a singular 2EP.

$$\begin{array}{ccc} A_1x = \lambda B_1x + \mu C_1x & \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 & \Delta_1 z = \lambda \Delta_0 z \\ A_2y = \lambda B_2y + \mu C_2y & \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 & \Delta_2 z = \mu \Delta_0 z \\ \text{(2EP)} & \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 & (\Delta) \end{array}$$

2EP is singular iff Δ_0 is singular

We assume that all linear combinations of Δ_0, Δ_1, and Δ_2 are singular.

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2EP is singular iff Δ_0 is singular

We assume that all linear combinations of Δ_0, Δ_1, and Δ_2 are singular.

There are no general results linking the eigenvalues of singular (2EP) and (Δ).

Singular pencils

$$A - \lambda B \text{ is a singular pencil} \Leftrightarrow \det(A - \lambda B) \equiv 0$$

λ is a finite regular eigenvalue if

$$\text{rank}(A - \lambda B) < \max_{\mu \in \mathbb{C}} \text{rank}(A - \mu B).$$

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Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix pencil $A - \lambda B$ is singular, the only finite eigenvalue is $\lambda = 1$.

Finite regular eigenvalues

(λ, μ) is a finite regular eigenvalue of (2EP) if for $i = 1, 2$:

$$\text{rank}(A_i + \lambda B_i + \mu C_i) < \max_{(s,t)\in\mathbb{C}^2} \text{rank}(A_i + sB_i + tC_i).$$

(λ, μ) is a finite regular eigenvalue of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if:

1. $\text{rank}(\Delta_1 - \lambda\Delta_0) < \max_{s\in\mathbb{C}} \text{rank}(\Delta_1 - s\Delta_0)$,

2. $\text{rank}(\Delta_2 - \mu\Delta_0) < \max_{t\in\mathbb{C}} \text{rank}(\Delta_2 - t\Delta_0)$,

3. a common regular eigenvector z of $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ exists, such that

$$\begin{aligned}(\Delta_1 - \lambda\Delta_0)z &= 0,\\ (\Delta_2 - \mu\Delta_0)z &= 0.\end{aligned}$$

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$$\begin{aligned}(\Delta_1 - \lambda\Delta_0)z &= 0,\\ (\Delta_2 - \mu\Delta_0)z &= 0.\end{aligned}$$

Conjecture: Finite regular eigenvalues of (2EP) $=$ finite regular eig's of (Δ).

Simple finite regular eigenvalues

$$\begin{array}{ccc} A_1x = \lambda B_1x + \mu C_1x & \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 & \Delta_1 z = \lambda \Delta_0 z \\ A_2y = \lambda B_2y + \mu C_2y & \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 & \Delta_2 z = \mu \Delta_0 z \\ \text{(2EP)} & \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 & (\Delta) \end{array}$$

Muhič, P. (2009): Suppose that all finite eigenvalues of a regular 2EP are algebraically simple. Then (λ, μ) is a finite regular eigenvalue of 2EP iff (λ, μ) is a finite regular eigenvalue of Δ.

Numerical method for singular 2EP

$$\begin{array}{ccc} A_1 x = \lambda B_1 x + \mu C_1 x & \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 & \Delta_1 z = \lambda \Delta_0 z \\ A_2 y = \lambda B_2 y + \mu C_2 y & \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 & \Delta_2 z = \mu \Delta_0 z \\ \text{(2EP)} & \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 & (\Delta) \end{array}$$

We extract the common finite regular part of matrix pencils (Δ).

We get matrices $\widetilde{\Delta}_0, \widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ such that $\widetilde{\Delta}_0$ is nonsingular and eigenvalues of

$$\begin{aligned} \widetilde{\Delta}_1 \widetilde{z} &= \lambda \widetilde{\Delta}_0 \widetilde{z} \\ \widetilde{\Delta}_2 \widetilde{z} &= \mu \widetilde{\Delta}_0 \widetilde{z} \end{aligned} \qquad (\widetilde{\Delta})$$

are finite regular eigenvalues of (Δ).

Generalized upper-triangular form

For a matrix pencil $A - \lambda B$ there exist unitary matrices P and Q such that:

$$P^H(A-\lambda B)Q = \left[\begin{array}{cc|cc} A_\mu - \lambda B_\mu & & & \\ \times & A_\infty - \lambda B_\infty & & \\ \hline \times & \times & A_f - \lambda B_f & \\ \times & \times & \times & A_\epsilon - \lambda B_\epsilon \end{array}\right].$$

Pencils $A_\mu - \lambda B_\mu$, $A_\infty - \lambda B_\infty$, $A_f - \lambda B_f$, and $A_\epsilon - \lambda B_\epsilon$ contain the left singular, the infinite regular, the finite regular, and the right singular structure, respectively.

Van Dooren (1979), Demmel and Kågström (1993), software package GUPTRI.

Our extraction algorithm is based on Van Dooren's staircase algorithm (1979).

Algorithms RRS and RLS

RRS: Extraction of the regular and the right singular part. We get matrices P, Q with orthonormal columns such that

- $P^H(A - \lambda B)Q = \begin{bmatrix} A_f - \lambda B_f & \\ \times & A_\epsilon - \lambda B_\epsilon \end{bmatrix}$
  is a finite regular and right singular structure of pencil $A - \lambda B$,
- the columns of Q are a basis for the eigenspace of the finite regular and the right singular part.

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  is a finite regular and right singular structure of pencil $A - \lambda B$,
- the columns of Q are a basis for the eigenspace of the finite regular and the right singular part.

RLS: Extraction of the regular and the left singular part. We get matrices P, Q with orthonormal columns such that

- $P^H(A - \lambda B)Q = \begin{bmatrix} A_\mu - \lambda B_\mu & \\ \times & A_f - \lambda B_f \end{bmatrix}$
  is a left singular and a finite regular structure of pencil $A - \lambda B$,
- the columns of P are a basis for the left eigenspace of the left singular and the finite regular part.

Algorithm for the common finite regular part

We start with matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$, $P = I$ and $Q = I$.

1. Separate the infinite and the finite part.
   (a) Apply RRS to $P^H\Delta_1 Q - \lambda P^H\Delta_0 Q$ and $P^H\Delta_2 Q - \mu P^H\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Compute Q and P such that $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.
   (c) If $\mathcal{Q} = \mathcal{Q}_1$ return P, Q and proceed to (2a). Otherwise, proceed to (1a).

2. Separate the finite regular part from the right singular part.
   (a) Apply RLS on $P^H\Delta_1 Q - \lambda P^H\Delta_0 Q$ and $P^H\Delta_2 Q - \mu P^H\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Compute Q and P such that $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$.
   (c) If $\mathcal{P} = \mathcal{P}_1$ return P, Q and exit. Otherwise, proceed to (2a).

In the end $\widetilde{\Delta}_0 = P^H\Delta_0 Q$, $\widetilde{\Delta}_1 = P^H\Delta_1 Q$, $\widetilde{\Delta}_2 = P^H\Delta_2 Q$ and $\widetilde{\Delta}_0$ is nonsingular.

Simple example

$$\begin{aligned} 1.2\lambda^2 + 1.5\mu^2 + 3\lambda\mu + 2\lambda - 1.7 &= 0 \\ 2.1\lambda^2 - 1.3\mu^2 + 4\lambda\mu - 1.1\mu + 1.5 &= 0 \end{aligned}$$

1) linearization:

$$\left(\begin{bmatrix} -1.7 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1.2 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 1.5 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) x = 0$$

$$\left(\begin{bmatrix} 1.5 & 0 & -1.1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 2.1 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & -1.3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) y = 0.$$

2) reduction: from 9×9 matrices $\Delta_0, \Delta_1, \Delta_2$ we obtain 4×4 matrices

$$\widetilde{\Delta}_0 = \begin{bmatrix} 0.7029 & 0.0313 & -0.8942 & 1.1668 \\ -5.9907 & 3.1228 & -1.7334 & 2.5729 \\ -0.1487 & 0.8985 & 1.6088 & -1.4158 \\ 3.6521 & -1.9782 & -1.2353 & 0.8949 \end{bmatrix}, \quad \widetilde{\Delta}_1 = \begin{bmatrix} 0.0740 & 0.6112 & -0.5910 & -0.2663 \\ -1.0909 & 0.0322 & 0.8003 & 0.9691 \\ -0.1381 & 0.3022 & 0.2461 & -0.1749 \\ 0.6584 & 1.1501 & -0.0361 & 0.3001 \end{bmatrix}$$

$$\widetilde{\Delta}_2 = \begin{bmatrix} -1.6497 & 0.5854 & -0.0124 & 0.6501 \\ 3.7601 & -4.2970 & -5.0612 & 3.2551 \\ 1.4724 & -1.0724 & 0.7050 & -1.1030 \\ -4.4639 & 2.9827 & 1.3934 & -0.5497 \end{bmatrix}.$$

3) the eigenvalue pairs of the commuting matrices $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ are the roots

$$(-14.0565, 6.3499), \quad (0.9888, -0.8866), \quad (0.2551, -1.1530), \quad (0.1186, 0.8704)$$

Numerical example: polynomials of small order

Time in seconds for 100 random bivariate systems

n	Linearization to 2EP	PHClab	Mathematica NSolve
2	0.21	21.47	1.31
3	0.93	23.58	2.92
4	4.20	26.23	5.96
5	29.96	30.25	10.92
6	161.8	38.27	26.97
⋮	⋮	⋮	⋮
10	-	128.13	499.21

Environment: Windows 7, Intel Core Duo P8700 2.53GHz, 4GB RAM

- Matlab 7.9.0
- PHClab 1.0.1
- Mathematica 7

Polynomials of large order

We have a system of two bivariate polynomials with n^2 roots, where n is large

$$p(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n-i}a_{ij}x^iy^j=0,\qquad q(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n-i}b_{ij}x^iy^j=0.$$

We linearize it as a singular 2EP and apply Jacobi–Davidson method:

- J-D was successfully applied to nonsingular 2EPs.
- J-D is a subspace method, in each outer step we solve a smaller projected 2EP.
- Now that we have a solver for singular 2EPs, we can use it in the outer step.
- This gives us a J-D for singular 2EP.

Matrices of 2EP are of size $n(n+1)/2\times n(n+1)/2$, but:

- matrices are sparse, we need $\mathcal{O}(n^2)$ flops for one MV multiplication
- same order as for one evaluation of $p(x,y)$ and $q(x,y)$
- might be competitive to compute solutions close to a given target (x_0,y_0)

Thank you!

- M. E. Hochstenbach, A. Muhič, B.P.:
  On linearizations of the quadratic two-parameter eigenvalue problems,
  March 2010, preprint.
- A. Muhič, B.P.:
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