Numerical methods for two-parameter eigenvalue problems

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Outline

- Two-parameter eigenvalue problem (2EP)
- Jacobi–Davidson type methods for 2EP
- Harmonic Rayleigh–Ritz for 2EP
- Singular 2EP

Two-parameter eigenvalue problem

- Two-parameter eigenvalue problem:

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y, \end{aligned} \qquad \text{(2EP)}$$

  where A_i, B_i, C_i are $n \times n$ matrices, $\lambda, \mu \in \mathbb{C}$, $x, y \in \mathbb{C}^n$.

- Eigenvalue: a pair (λ, μ) that satisfies (2EP) for nonzero x and y.

- Eigenvector: the tensor product $x \otimes y$.

- There are n^2 eigenvalues.

- Goal: eigenvalues (λ, μ) close to a target (σ, τ) and eigenvectors $x \otimes y$.

Separation of variables: $\Delta \mathbf{u} + \nu \mathbf{u} = 0$ on Ω, $\mathbf{u}|_{\delta\Omega} = 0$

Rectangle: $\Omega = [0, a] \times [0, b] \Longrightarrow$ two S–L equations ($\nu = \lambda + \mu$)

$$\begin{aligned} x'' + \lambda x &= 0, \qquad & x(0) = x(a) &= 0, \\ y'' + \mu y &= 0, \qquad & y(0) = y(b) &= 0. \end{aligned}$$

Separation of variables: $\Delta u + \nu u = 0$ on Ω, $u|_{\delta\Omega} = 0$

Rectangle: $\Omega = [0, a] \times [0, b] \Longrightarrow$ two S–L equations ($\nu = \lambda + \mu$)

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\begin{aligned}
x'' + \lambda x &= 0, \qquad & x(0) = x(a) &= 0, \\
y'' + \mu y &= 0, \qquad & y(0) = y(b) &= 0.
\end{aligned}
$$

Circle: $\Omega = \{x^2 + y^2 \le a^2\}$, polar coordinates $\Longrightarrow$ a triangular situation

$$
\begin{aligned}
\Phi'' + \lambda \Phi &= 0, \qquad & \Phi(0) &= \Phi(2\pi) = 0, \\
r^{-1}(rR')' + (\nu - \lambda r^{-2})R &= 0, \qquad & R(0) &< \infty, R(a) = 0.
\end{aligned}
$$

Separation of variables: $\boldsymbol{\Delta} \mathbf{u} + \boldsymbol{\nu} \mathbf{u} = 0$ on $\boldsymbol{\Omega}$, $\mathbf{u}|_{\boldsymbol{\delta\Omega}} = 0$

Rectangle: $\Omega = [0, a] \times [0, b] \Longrightarrow$ two S–L equations ($\nu = \lambda + \mu$)

$$\begin{aligned} x'' + \lambda x = 0, &\qquad x(0) = x(a) = 0, \\ y'' + \mu y = 0, &\qquad y(0) = y(b) = 0. \end{aligned}$$

Circle: $\Omega = \{x^2 + y^2 \leq a^2\}$, polar coordinates $\Longrightarrow$ a triangular situation

$$\begin{aligned} \Phi'' + \lambda \Phi = 0, &\qquad \Phi(0) = \Phi(2\pi) = 0, \\ r^{-1}(rR')' + (\nu - \lambda r^{-2})R = 0, &\qquad R(0) < \infty, R(a) = 0. \end{aligned}$$

Ellipse: $\Omega = \{(x/a)^2 + (y/b)^2 \leq 1\}$, elliptic coordinates ($c$ focus)

$\Longrightarrow$ modified Mathieu's and Mathieu's DE ($4\lambda = c^2\nu$)

$$\begin{aligned} v_1'' + (2\lambda \cosh(2y_1) - \mu)v_1 = 0, &\quad v_1(0) = v_1(d) = 0, \\ v_2'' - (2\lambda \cos(2y_1) - \mu)v_2 = 0, &\quad v_2(0) = v_2(\pi/2) = 0. \end{aligned}$$

Tensor product approach

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned} \tag{2EP}$$

- On $\mathbb{C}^n \otimes \mathbb{C}^n$ we define $n^2 \times n^2$ matrices

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned}$$

- 2EP is equivalent to a coupled GEP

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z, \end{aligned} \tag{Δ}$$

  where $z = x \otimes y$.

- 2EP is nonsingular $\Longleftrightarrow$ Δ_0 is nonsingular
- $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute.

Right definite problem

$$(2EP) \quad \begin{array}{l} A_1x = \lambda B_1x + \mu C_1x \\ A_2y = \lambda B_2y + \mu C_2y \end{array} \qquad \begin{array}{l} \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \end{array} \qquad \begin{array}{l} \Delta_1 z = \lambda \Delta_0 z \\ \Delta_2 z = \mu \Delta_0 z \end{array} \quad (\Delta)$$

2EP is right definite when A_i, B_i, C_i are Hermitian and Δ_0 is positive definite.

If 2EP is right definite then

- eigenpairs are real
- there exist n^2 linearly independent eigenvectors
- eigenvectors of distinct eigenvalues are Δ_0-orthogonal, i.e., $(x_1 \otimes y_1)^T \Delta_0 (x_2 \otimes y_2) = 0$

Numerical methods - Δ matrices

$$\text{(2EP)} \quad \begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned} \qquad \begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned} \qquad \begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \quad (\Delta)$$

Hochstenbach, Košir, P. (2005):

1. $Q^* \Delta_0 Z = R$ and $Q^* \Delta_1 Z = S$, where Q, Z are unitary, R, S upper triangular, and the multiple values of $\lambda_i := s_{ii}/r_{ii}$ are clustered along the diagonal of $R^{-1}S$.

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ 0 & R_{22} & \cdots & R_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{pp} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ 0 & S_{22} & \cdots & S_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{pp} \end{bmatrix}.$$

2. Compute diagonal blocks $T_{11}, \ldots, T_{pp}$ of $T = Q^* \Delta_2 Z$.
3. Compute the eigenvalues $\mu_{i1}, \ldots, \mu_{im_i}$ of the GEP $T_{ii} w = \mu R_{ii} w$ for $i = 1, \ldots, p$.
4. Reindex $(\lambda_1, \mu_{11}), \ldots, (\lambda_1, \mu_{1m_1}); \ldots; (\lambda_p, \mu_{p1}), \ldots, (\lambda_p, \mu_{pm_p})$ into $(\lambda_1, \mu_1), \ldots, (\lambda_{n^2}, \mu_{n^2})$.
5. For each eigenvalue (λ_j, μ_j) compute its eigenvector $x_j \otimes y_j$.

Time complexity: $\mathcal{O}(n^6)$

Algorithms that work with matrices A_i, B_i, C_i

$$
\text{(2EP)} \quad \begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y \end{aligned} \qquad \begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned} \qquad \begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \quad (\Delta)
$$

- Gradient method: Blum, Curtis, Geltner (1978), Browne, Sleeman (1982)
- Newton's method for eigenvalues: Bohte (1980)
- Generalized Rayleigh Quotient Iteration: Ji, Jiang, Lee (1992)
- Continuation method: Shimasaki (1995) for a special class of RD 2EP, P. (1999) for RD 2EP, P. (2000) for weakly elliptic 2EP,
- Jacobi-Davidson: Hochstenbach, P. (2002) for RD 2EP, Hochstenbach, Košir, P. (2005) for nonsingular 2EP, Hochstenbach, P. (2008) - harmonic extraction

Jacobi–Davidson method

Subspace methods (Arnoldi, Lanczos, J-D, ...) compute eigenpairs from low dimensional subspaces. The main ingredients are:

- Extraction: We compute an approximation to an eigenpair from a given search subspace (Rayleigh-Ritz, harmonic Rayleigh-Ritz, ...).
- Expansion: After each step we expand the subspace by a new direction.

Jacobi–Davidson method:

- a new subspace direction is orthogonal or oblique to the last chosen Ritz vector,
- approximate solutions of certain correction equations are used for expansion.

J-D method can be efficiently applied to two-parameter eigenvalue problems.
This is not clear for subspace methods based on Krylov subspaces.

J-D for the right definite case

Extraction: Ritz–Galerkin conditions: search spaces = test spaces: $u \in \mathcal{U}_k$, $v \in \mathcal{V}_k$

$$\begin{aligned}(A_1 - \sigma B_1 - \tau C_1)u &\perp \mathcal{U}_k\\ (A_2 - \sigma B_2 - \tau C_2)v &\perp \mathcal{V}_k\end{aligned}$$

$\Rightarrow$ projected right definite two-parameter eigenvalue problem

$$\begin{aligned}U_k^T A_1 U_k c &= \sigma U_k^T B_1 U_k c + \tau U_k^T C_1 U_k c\\ V_k^T A_2 V_k d &= \sigma V_k^T B_2 V_k d + \tau V_k^T C_2 V_k d\end{aligned}$$

Ritz value: (σ, τ), Ritz vectors: $u = U_k c$, $v = V_k d$, where $c, d \in \mathbb{R}^k$

Expansion: Correction equation for new directions s, t:

$$\begin{aligned}(I - uu^T)(A_1 - \sigma B_1 - \tau C_1)(I - uu^T)s &= -(A_1 - \sigma B_1 - \tau C_1)u\\ (I - vv^T)(A_2 - \sigma B_2 - \tau C_2)(I - vv^T)t &= -(A_2 - \sigma B_2 - \tau C_2)v.\end{aligned}$$

Works well for exterior eigenvalues.

Two-sided J-D for a general problem

Petrov–Galerkin conditions: search spaces $u_i \in \mathcal{U}_{ik}$, test spaces $v_i \in \mathcal{V}_{ik}$

$$\begin{aligned}(A_1 - \sigma B_1 - \tau C_1)u_1 &\perp \mathcal{V}_{1k}\\(A_2 - \sigma B_2 - \tau C_2)u_2 &\perp \mathcal{V}_{2k},\end{aligned}$$

$\Rightarrow$ projected two-parameter eigenvalue problem

$$\begin{aligned}V_{1k}^* A_1 U_{1k} c_1 &= \sigma V_{1k}^* B_1 U_{1k} c_1 + \tau V_{1k}^* C_1 U_{1k} c_1\\V_{2k}^* A_2 U_{2k} c_2 &= \sigma V_{2k}^* B_2 U_{2k} c_2 + \tau V_{2k}^* C_2 U_{2k} c_2,\end{aligned}$$

where $u_i = U_{ik} c_i \neq 0$ for $i = 1, 2$ and $\sigma, \tau \in \mathbb{C}$.

Petrov value: (σ, τ), Petrov vectors: $u_i = U_{ik} c_i$, $v_i = V_{ik} d_i$, where $c_i, d_i \in \mathbb{C}^k$

Usually performs better than the one-sided method.

Works well for exterior eigenvalues.

Harmonic Rayleigh–Ritz for 2EP

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

Spectral transformation: $(A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x$

Harmonic Rayleigh–Ritz: $Au - \theta Bu \perp (A - \tau B)\,\mathcal{U}_k$

2EP:

$$
\begin{aligned}
A_1x &= \lambda B_1x + \mu C_1x \\
A_2y &= \lambda B_2y + \mu C_2y
\end{aligned}
$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:

$$
\begin{aligned}
(A_1 - \theta B_1 - \eta C_1)\,u \quad &\perp \quad \mathcal{U}_k \\
(A_2 - \theta B_2 - \eta C_2)\,v \quad &\perp \quad \mathcal{V}_k
\end{aligned}
$$

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$$(A_1 - \theta B_1 - \eta C_1)\, u \quad \perp \quad \mathcal{U}_k$$
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Spectral transformation: ? ? ?

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Rayleigh–Ritz:

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Spectral transformation: ? ? ?

Harmonic Rayleigh–Ritz:

$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\, u &\perp (A_1 - \sigma B_1 - \tau C_1)\,\mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\, v &\perp (A_2 - \sigma B_2 - \tau C_2)\,\mathcal{V}_k \end{aligned}$$

J-D with harmonic Rayleigh–Ritz for 2EP

1. $s = u_1$ and $t = v_1$ (starting vectors), $U_0 = V_0 = [\,]$

   for $k = 1, 2, \ldots$

2. $(U_{k-1}, s) \to U_k, \quad (V_{k-1}, t) \to V_k$

3. Extract appropriate harmonic Ritz pair $((\xi_1, \xi_2), c \otimes d)$

4. Take $u = U_k c$, $v = V_k d$ and compute tensor Rayleigh quotient

$$\theta = \frac{(u \otimes v)^* \Delta_1 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* A_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* A_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}$$

$$\eta = \frac{(u \otimes v)^* \Delta_2 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* B_1 u)(v^* A_2 v) - (u^* A_1 u)(v^* B_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}$$

5.
$$r_1 = (A_1 - \theta B_1 - \eta C_1) u$$
$$r_2 = (A_2 - \theta B_2 - \eta C_2) v$$

6. Stop if $(\|r_1\|^2 + \|r_2\|^2)^{1/2} \le \varepsilon$

7. Solve (approximately) an $s \perp u$, $t \perp v$ from corr. equation(s)

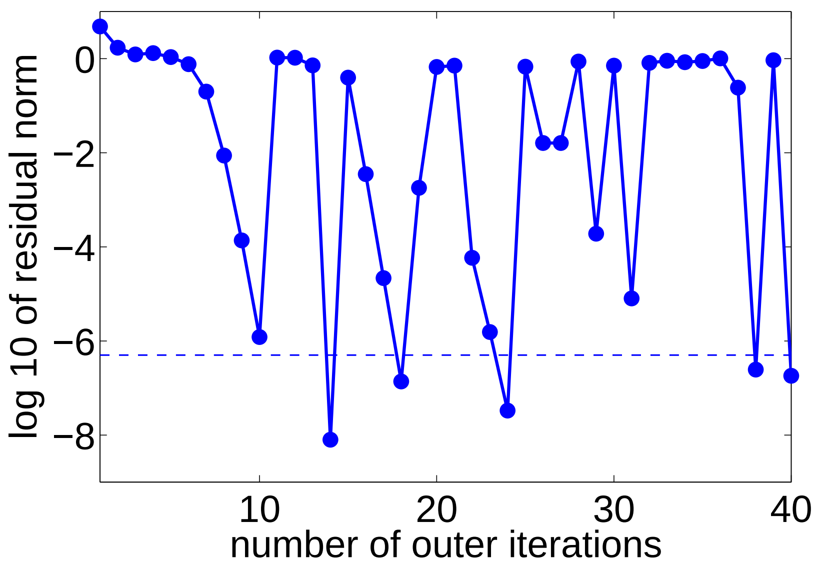
Hochstenbach, P. (2008) ETNA

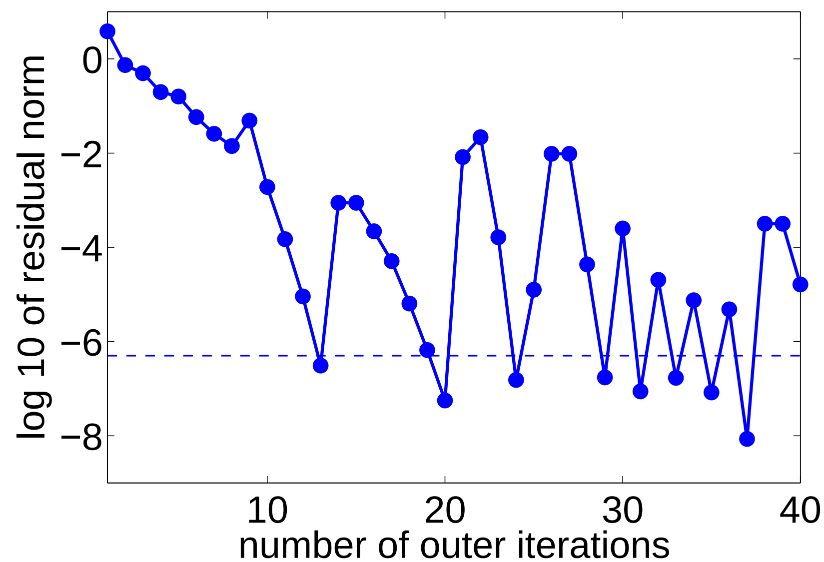
Numerical example, $n = 1000$

Goal: 50 eigenvalues closest to the origin using at most 2500 outer iterations.

	Rayleigh-Ritz			Harmonic Rayleigh-Ritz				
GMRES	eigs	in 10	in 30	iter	time	in 10	in 30	in 50
8	12	9	12	226	119	10	30	46
16	19	10	19	106	73	10	30	44
32	22	10	22	89	87	10	29	40
64	30	10	29	93	118	10	28	40

Convergence graphs for the Rayleigh-Ritz (left) and the harmonic R-R extraction (right) for the first 40 outer iterations (8 GMRES steps for the correction eq.).





Model updating as a singular 2EP

Model updating (Cottin 2001, Cottin and Reetz 2006): finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find the smallest perturbation of matrix A by a linear combination of matrices B and C, such that $A - \lambda B - \mu C$ has the prescribed eigenvalues σ_1 and σ_2.

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The problem can be expressed as a two-parameter eigenvalue problem

$$\begin{aligned} (A - \sigma_1 I)x &= \lambda Bx + \mu Cx, \\ (A - \sigma_2 I)y &= \lambda By + \mu Cy. \end{aligned}$$

$\det(B \otimes C - C \otimes B) = 0 \implies$ singular 2EP

Quadratic 2EP as a singular 2EP

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x &= 0\\(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y &= 0,\end{aligned} \tag{Q2EP}$$

where $A_i, B_i, \ldots, F_i$ are $n \times n$ matrices, (λ, μ) is an eigenvalue, and $x \otimes y$ is the corresponding eigenvector. In the generic case the problem has $4n^2$ eigenvalues.

Quadratic 2EP as a singular 2EP

$$
\begin{aligned}
(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x &= 0 \\
(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y &= 0,
\end{aligned}
\qquad \text{(Q2EP)}
$$

where $A_i, B_i, \ldots, F_i$ are $n \times n$ matrices, (λ, μ) is an eigenvalue, and $x \otimes y$ is the corresponding eigenvector. In the generic case the problem has $4n^2$ eigenvalues.

We can write Q2EP as a two-parameter eigenvalue problem, one option is

$$
\begin{aligned}
\left(\begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_1 & \frac{1}{2}E_1 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & \frac{1}{2}E_1 & F_1 \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ \lambda x \\ \mu x \end{bmatrix} &= 0 \\
\left(\begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_2 & \frac{1}{2}E_2 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & \frac{1}{2}E_2 & F_2 \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} y \\ \lambda y \\ \mu y \end{bmatrix} &= 0,
\end{aligned}
$$

where matrices are of size $3n \times 3n$. Singular 2EP

Numerical method for singular 2EP ($\det(\Delta_0) = 0$)

$$
\text{(2EP)} \quad \begin{array}{l} A_1 x = \lambda B_1 x + \mu C_1 x \\ A_2 y = \lambda B_2 y + \mu C_2 y \end{array} \qquad \begin{array}{l} \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \end{array} \qquad \begin{array}{l} \Delta_1 z = \lambda \Delta_0 z \\ \Delta_2 z = \mu \Delta_0 z \end{array} \quad (\Delta)
$$

For singular 2EP, there are no general results linking the eigenv. of (2EP) and (Δ).

Numerical method for singular 2EP ($\det(\Delta_0) = 0$)

$$
\text{(2EP)} \quad \begin{array}{l} A_1 x = \lambda B_1 x + \mu C_1 x \\ A_2 y = \lambda B_2 y + \mu C_2 y \end{array} \qquad \begin{array}{l} \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \end{array} \qquad \begin{array}{l} \Delta_1 z = \lambda \Delta_0 z \\ \Delta_2 z = \mu \Delta_0 z \end{array} \quad (\Delta)
$$

For singular 2EP, there are no general results linking the eigenv. of (2EP) and (Δ).

Numerical method: we extract the common regular part of matrix pencils (Δ). Thus we obtain matrices $\widetilde{\Delta}_0, \widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$, such that $\widetilde{\Delta}_0$ is nonsingular and eigenvalues of

$$
\begin{array}{l} \widetilde{\Delta}_1 \widetilde{z} = \lambda \widetilde{\Delta}_0 \widetilde{z} \\ \widetilde{\Delta}_2 \widetilde{z} = \mu \widetilde{\Delta}_0 \widetilde{z} \end{array} \quad (\widetilde{\Delta})
$$

are common regular eigenvalues of (Δ).

For Q2EP and model updating we can show that

regular eigenvalues of (2EP) = eigenvalues of ($\widetilde{\Delta}$) = regular eigenvalues of (Δ).

Conclusions

J-D works for nonsingular two-parameter eigenvalue problems.

The harmonic approach can be generalized to the 2EP.

Singular 2EP: work in progress ...