A continuation method for a weakly elliptic two-parameter eigenvalue problem

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Weakly elliptic two-parameter eigenvalue problem

- We consider two-parameter eigenvalue problem

$$
\begin{aligned}
A_1 x_1 &= \lambda B_1 x + \mu C_1 x, \\
A_2 x_2 &= \lambda B_2 y + \mu C_2 y.
\end{aligned} \tag{W}
$$

  where A_i, B_i, C_i are $n \times n$ real symmetric matrices, $\lambda, \mu \in \mathbb{C}$, and $x, y \in \mathbb{C}^n$.

- Eigenvalue: A pair (λ, μ) if $\operatorname{Ker}(A_i - \lambda B_i - \mu C_i) \neq \{0\}, \quad i = 1, 2.$

- Weakly elliptic: One of the matrices B_1, C_1, B_2, C_2 is definite.

- If we assume that both matrices B_1 and C_1 are positive definite then for an eigenvalue (λ, μ) we have either $\lambda, \mu \in \mathbb{R}$ or $\lambda, \mu \notin \mathbb{R}$.

Equivalent problem in a tensor product space

- On $S := \mathbb{C}^n \otimes \mathbb{C}^n$ of the dimension n^2 we define

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned}$$

- Two-parameter problem (W) is equivalent to

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z, \end{aligned}$$

  where $z = x \otimes y$. If (W) is nonsingular then Δ_0 is invertible.

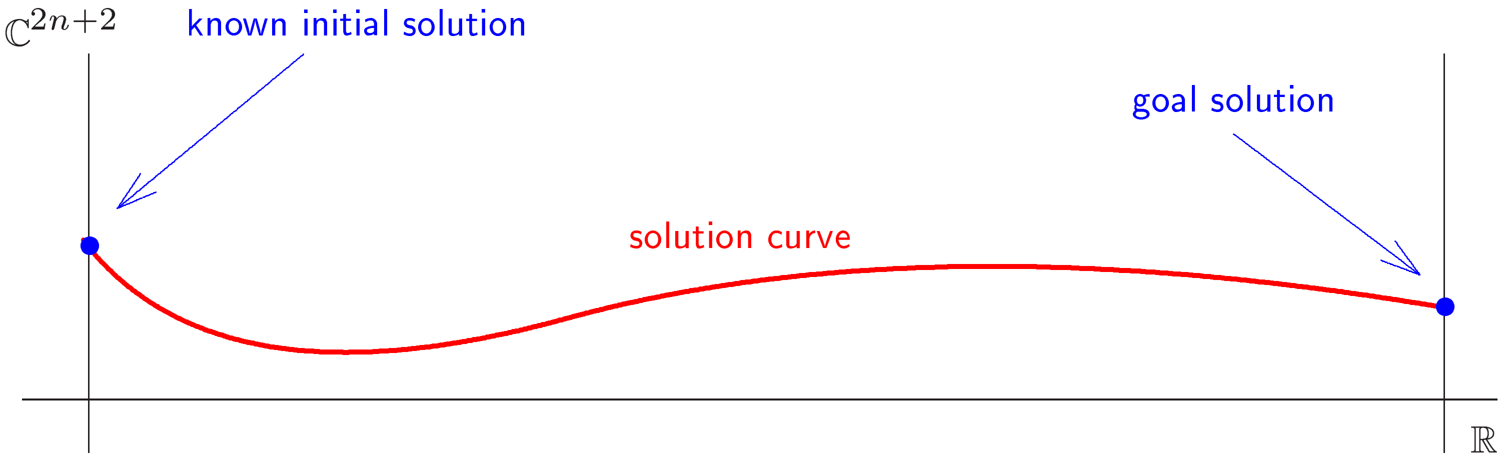
- $\Gamma_i := \Delta_0^{-1}\Delta_i$ commute and the problem (W) is equivalent to the simultaneous problem in S:

$$\begin{aligned} \Gamma_1 z &= \lambda z, \\ \Gamma_2 z &= \mu z. \end{aligned} \qquad (\Gamma)$$

Available numerical methods

- We can treat the simultaneous problem (Γ) but matrices are of size $n^2 \times n^2$.
- Most numerical algorithms for two-parameter problems require a definite problem.
- We can use Newton's method, but we need good approximations.
- This problem can be fixed if we use the continuation method (for a definite two-parameter problem: Müller (1982), Shimasaki (1995), P. (2000)).
- The new continuation method for a weakly elliptic two-parameter problem is similar to the one that Li, Zeng, and Cong (1992) used for the nonsymmetric eigenvalue problem $Ax = \lambda x$.

Continuation method



$t = 0$ $\longrightarrow$ $t = 1$

$$S_1 x = \lambda B_1 x + \mu C_1 x \qquad A_1 x = \lambda B_1 x + \mu C_1 x$$
$$S_2 y = \lambda B_2 y + \mu C_2 y \qquad A_2 y = \lambda B_2 y + \mu C_2 y$$

initial problem W_0 with known solutions — homotopy — original problem W

Simple initial problem

Lemma

It is possible to construct symmetric matrices S_1 and S_2 in a way that

a) all eigenvalues of the two-parameter problem

$$\begin{aligned} S_1 x &= \lambda B_1 x + \mu C_1 x, \\ S_2 y &= \lambda B_2 y + \mu C_2 y. \end{aligned} \qquad (W_0)$$

are algebraically simple,

b) the construction reveals the solutions of (W_0).

Our homotopy $H : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times [0,1] \longrightarrow \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ is

$$H(x, y, \lambda, \mu, t) := \begin{pmatrix} (\lambda B_1 + \mu C_2 - (1-t)S_1 - tA_1)x \\ (\lambda B_2 + \mu C_2 - (1-t)S_2 - tA_2)y \\ (c_{11}x_1^2 + \ldots + c_{1n}x_n^2 - 1)/2 \\ (c_{21}y_1^2 + \ldots + c_{2n}y_n^2 - 1)/2 \end{pmatrix},$$

where c_{ij} are randomly chosen real positive numbers.

- Solution of $H(x, y, \lambda, \mu, t) = 0$ is a solution of

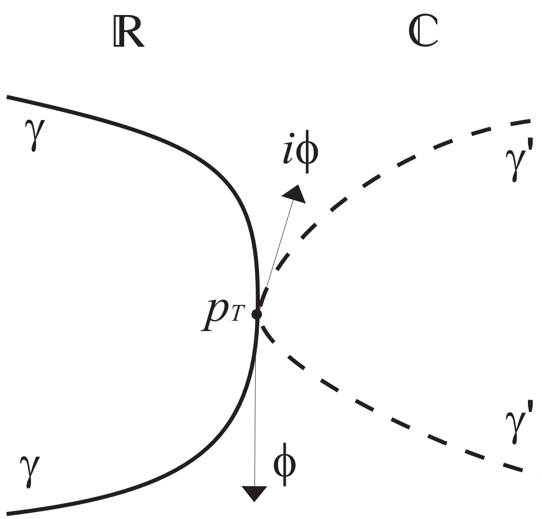
$$\begin{aligned} ((1-t)S_1 + tA_1)x &= \lambda B_1 x + \mu C_1 x, \\ ((1-t)S_2 + tA_2)y &= \lambda B_2 y + \mu C_2 y. \end{aligned} \qquad (W_t)$$

  which is equal to (W) at $t = 1$ and is equal to (W_0) at $t = 0$.

- (W_t) is weakly elliptic for $t \in [0, 1]$.

Some properties of the homotopy

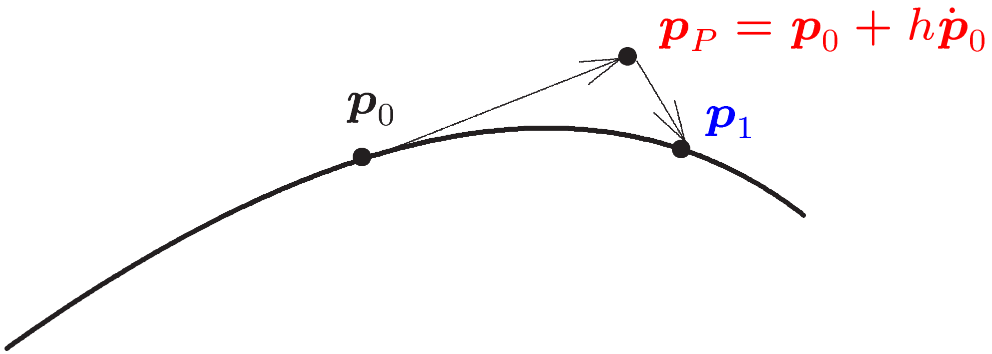
- There are only finitely many values $t \in [0,1]$ where (W_t) has a multiple eigenvalue.
- All bifurcations are turning points.
- We can not avoid the turning points.



Change from real to complex space in a quadratic turning point

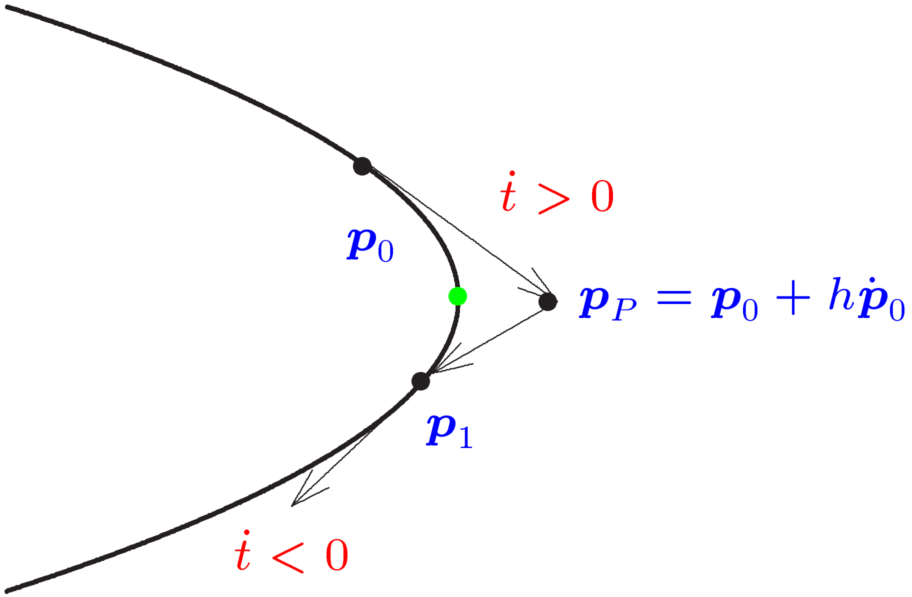
Following the homotopy curve

- We numerically follow the homotopy curve by a prediction-correction scheme using arc length as the parameter.
- Euler's method is used for the predictor.
- Newton's method is used for the corrector.

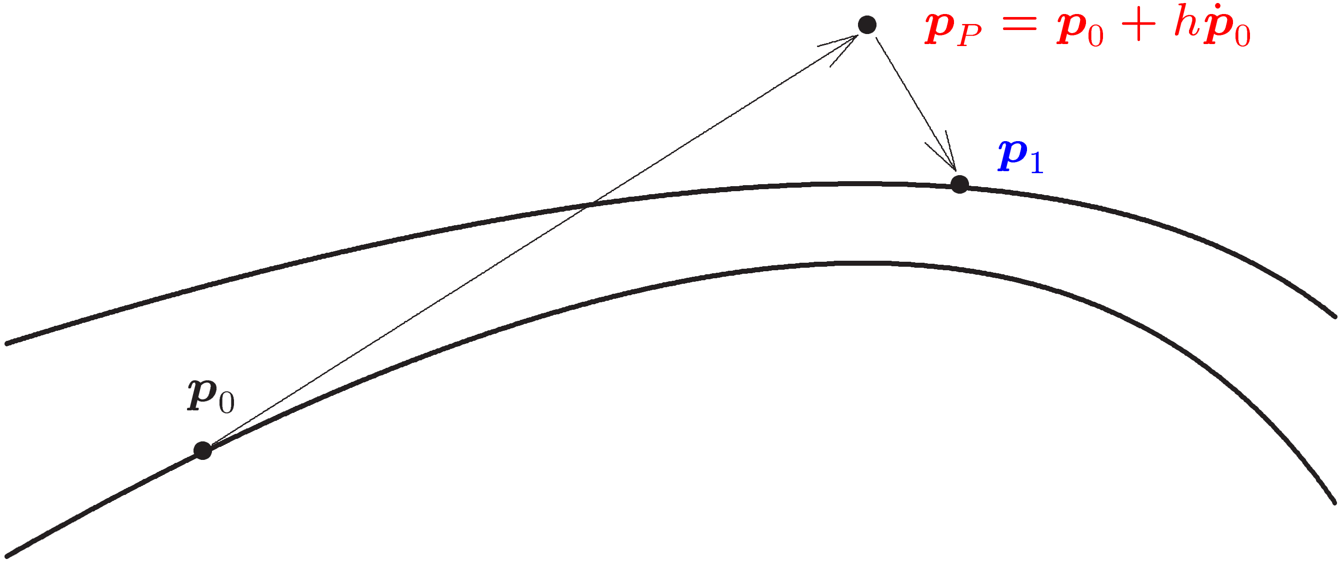


Miscellaneous

- We end the algorithm when t is close enough to 1.
- In the algorithm we decrease or increase the step size when the homotopy curve is very steep or flat, respectively.
- We detect turning points with special algorithm.



Curve switching



Curve switching may appear. In the end we compare all eigenvalues and if we find two close eigenvalues with close eigenvectors we repeat the computation for both curves using higher precision and stricter criteria.

Time complexity

- In tensor product space matrices Γ_i are of order n^2 and the time complexity is $\mathcal{O}(n^6)$.
- Matrices in the continuation method are of moderate size $\mathcal{O}(n)$.
  - One predictor-corrector step has time complexity $\mathcal{O}(n^3)$.
  - We have to multiply this number with n^2 as we are following n^2 curves.
  - We have to multiply it further with the number of P-C steps per curve.

  If $\phi(n)$ is the average number of P-C steps for one curve then the time complexity is

$$\mathcal{O}(n^5\phi(n)).$$

- Is $\phi(n)$ of order $o(n)$?
  - Based on numerical results, it is possible.
- The continuation method allows an elegant parallel implementation.

Numerical examples

n	Number of operations: continuation	Number of operations: QR	ratio (cont. : QR)	Aver. no. of P-C steps	Aver. no. of bifurcations
5	$50.7 \cdot 10^6$	$82.4 \cdot 10^4$	61.5	210	1.4
10	$13.5 \cdot 10^8$	$49.3 \cdot 10^7$	27.5	317	1.2
15	$16.0 \cdot 10^9$	$51.9 \cdot 10^7$	30.7	556	2.7
20	$57.0 \cdot 10^9$	$29.5 \cdot 10^8$	19.3	594	2.2
25	$19.3 \cdot 10^{10}$	$11.3 \cdot 10^9$	17.0	720	3.0
30	$46.1 \cdot 10^{10}$	$32.8 \cdot 10^9$	14.1	778	3.3
35	$10.3 \cdot 10^{11}$	$81.2 \cdot 10^9$	12.7	868	3.2
40	$21.2 \cdot 10^{11}$	$18.0 \cdot 10^{10}$	11.8	975	3.2
45	$36.1 \cdot 10^{11}$	$36.6 \cdot 10^{10}$	9.9	956	2.8

Table 1: Statistics for $n = 5, 10, \ldots, 45$. Flops count for the continuation method, for the simultaneous problem (Γ), their ratio, the average number of evaluations of Newton's method per curve and the average number of bifurcations per curve.

Conclusions

- Continuation method for a weakly elliptic two-parameter problem is a method that:
  - works without approximations for eigenpairs,
  - works with matrices of modest size $\mathcal{O}(n)$ instead of n^2,
  - allows parallel implementation.
- Some drawbacks:
  - Even if we are interested in a small portion of eigenvalues, we have to compute all the eigenvalues.
- Questions:
  - Is the time complexity below $\mathcal{O}(n^6)$?
  - Continuation method for a general two-parameter problem with symmetric matrices.