

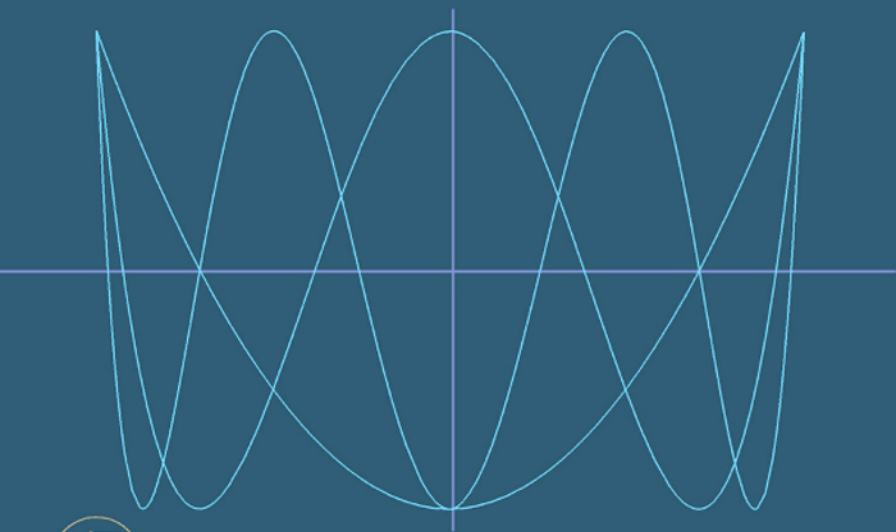
I. S. GRADSHTEYN
I. M. RYZHIK



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TABLE OF INTEGRALS, SERIES, AND PRODUCTS

SEVENTH EDITION



Edited by Alan Jeffrey and Daniel Zwillinger

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Seventh Edition

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Seventh Edition

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Preface to the Seventh Edition

Since the publication in 2000 of the completely reset sixth edition of Gradshteyn and Ryzhik, users of the reference work have continued to submit corrections, new results that extend the work, and suggestions for changes that improve the presentation of existing entries. It is a matter of regret to us that the structure of the book makes it impossible to acknowledge these individual contributions, so, as usual, the names of the many new contributors have been added to the acknowledgment list at the front of the book.

This seventh edition contains the corrections received since the publication of the sixth edition in 2000, together with a considerable amount of new material acquired from isolated sources. Following our previous conventions, an amended entry has a superscript “11” added to its entry reference number, where the equivalent superscript number for the sixth edition was “10.” Similarly, an asterisk on an entry’s reference number indicates a new result. When, for technical reasons, an entry in a previous edition has been removed, to preserve the continuity of numbering between the new and older editions the subsequent entries have not been renumbered, so the numbering will jump.

We wish to express our gratitude to all who have been in contact with us with the object of improving and extending the book, and we want to give special thanks to Dr. Victor H. Moll for his interest in the book and for the many contributions he has made over an extended period of time. We also wish to acknowledge the contributions made by Dr. Francis J. O’Brien Jr. of the Naval Station in Newport, in particular for results involving integrands where exponentials are combined with algebraic functions.

Experience over many years has shown that each new edition of Gradshteyn and Ryzhik generates a fresh supply of suggestions for new entries, and for the improvement of the presentation of existing entries and errata. In view of this, we do not expect this new edition to be free from errors, so all users of this reference work who identify errors, or who wish to propose new entries, are invited to contact the authors, whose email addresses are listed below. Corrections will be posted on the web site www.az-tec.com/gr/errata.

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Dr. N. I. Robinson	Dr. G. Solt	Dr. S. Wanzura
Dr. P. A. Robinson	Dr. J. Sondow	Dr. J. Ward
Dr. D. M. Rosenblum	Dr. A. Sørenssen	Dr. S. I. Warshaw
Dr. R. A. Rosthal	Dr. Marcus Spradlin	Dr. R. Weber
Dr. J. R. Roth	Dr. Andrzej Staruszkiewicz	Dr. Wei Qian
Dr. Klaus Rottbrand	Dr. Philip C. L. Stephenson	Dr. D. H. Werner
Dr. D. Roy	Dr. Edgardo Stockmeyer	Dr. E. Wetzel
Dr. E. Royer	Dr. J. C. Straton	Dr. Robert Whittaker
Dr. D. Rudermann	Mr. H. Suraweera	Dr. D. T. Wilton
Dr. Sanjib Sabhapandit	Dr. N. F. Svaiter	Dr. C. Wiuf
Dr. C. T. Sachradja	Dr. V. Svaiter	Dr. K. T. Wong
Dr. J. Sadiku	Dr. R. Szmytkowski	Mr. J. N. Wright
Dr. A. Sadiq	Dr. S. Tabachnik	Dr. J. D. Wright
Dr. Motohiko Saitoh	Dr. Erik Talvila	Dr. D. Wright
Dr. Naoki Saito	Dr. G. Tanaka	Dr. D. Wu
Dr. A. Salim	Dr. C. Tanguy	Dr. Michel Daoud Yacoub
Dr. J. H. Samson	Dr. G. K. Tannahill	Dr. Yu S. Yakovlev
Dr. Miguel A. Sanchis-Lozano	Dr. B. T. Tan	Dr. H.-C. Yang
Dr. J. A. Sanders	Dr. C. Tavarad	Dr. J. J. Yang
Dr. M. A. F. Sanjun	Dr. Gonçalo Tavares	Dr. Z. J. Yang
Dr. P. Sarquiz	Dr. Aba Teleki	Dr. J. J. Wang
Dr. Avadh Saxena	Dr. Arash Dahi Taleghani	Dr. Peter Widerin
Dr. Vito Scarola	Dr. D. Temperley	Mr. Chun Kin Au Yeung
Dr. O. Schärpf	Dr. A. J. Tervoort	Dr. Kazuya Yuasa
Dr. A. Scherzinger	Dr. Theodoros Theodoulidis	Dr. S. P. Yukon
Dr. B. Schizer	Dr. D. J. Thomas	Dr. B. Zhang
Dr. Martin Schmid	Dr. Michael Thorwart	Dr. Y. C. Zhang
Dr. J. Scholes	Dr. S. T. Thynell	Dr. Y. Zhao
Dr. Mel Schopper	Dr. D. C. Torney	Dr. Ralf Zimmer
Dr. H. J. Schulz	Dr. R. Tough	
Dr. G. J. Sears	Dr. B. F. Treadway	

The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x) dx = A$$

one may make a large number of substitutions of the form $x = \varphi(t)$ and thus obtain a number of “synonyms” of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such “synonyms” and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the “synonym” formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the “outer” functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single, simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of Chapter Two and the Newton–Leibniz formula, or to an integral of the form

$$\int_{-a}^a f(x) dx,$$

where $f(x)$ is an odd function. In such cases, the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\pi/4} \frac{(\cot x - 1)^{p-1}}{\sin^2 x} \ln \tan x dx = -\frac{\pi}{p} \operatorname{cosec} p\pi. \quad (0.1)$$

By making the natural substitution $u = \cot x - 1$, we obtain

$$\int_0^\infty u^{p-1} \ln(1+u) du = \frac{\pi}{p} \operatorname{cosec} p\pi. \quad (0.2)$$

Integrals similar to formula (0.1) are omitted in this new edition. Instead, we have formula (0.2).

As a second example, let us take

$$I = \int_0^{\pi/2} \ln(\tan^p x + \cot^p x) \ln \tan x \, dx = 0.$$

The substitution $u = \tan x$ yields

$$I = \int_0^\infty \frac{\ln(u^p + u^{-p}) \ln u}{1 + u^2} \, du.$$

If we now set $v = \ln u$, we obtain

$$I = \int_{-\infty}^\infty \frac{v e^v}{1 + e^{2v}} \ln(e^{pv} + e^{-pv}) \, dv = \int_{-\infty}^\infty v \frac{\ln(2 \cosh pv)}{2 \cosh v} \, dv.$$

The integrand is odd, and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the “inner” functions) of the functions in the integrand.

The functions are ordered as follows: First we have the elementary functions:

1. The function $f(x) = x$.
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
9. Elliptic functions.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel’s integrals.
12. The gamma function and related functions.
13. Bessel functions.
14. Mathieu functions.
15. Legendre functions.
16. Orthogonal polynomials.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Parabolic cylinder functions.
20. Meijer’s and MacRobert’s functions.
21. Riemann’s zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear

in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine function in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* We shall distinguish between all these functions and those listed above, and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x + \cos x}{\sin x - \cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form $A^{p/q}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions [whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$] are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \tan x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear [in the order determined by the position of $\varphi_2(x)$ in the list] after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions are the trigonometric and power functions [that is, $\varphi_2(x) = x$]. Then come

- combinations of trigonometric and exponential functions,
- combinations of trigonometric functions, exponential functions, and powers, etc.,
- combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function $e^{\frac{1}{x}}$ comes after e^x as regards complexity, but $\ln x$ and $\ln \frac{1}{x}$ are equally complex since $\ln \frac{1}{x} = -\ln x$. In the section on “powers and algebraic functions,” polynomials, rational functions, and powers of powers are formed from power functions of the form $(a + bx)^n$ and $(\alpha + \beta x)^\nu$.

*For any natural number n , the involution $(a + bx)^n$ of the binomial $a + bx$ is a polynomial. If n is a negative integer, $(a + bx)^n$ is a rational function. If n is irrational, the function $(a + bx)^n$ is not even an algebraic function.

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Use of the Tables*

For the effective use of the tables contained in this book, it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled *The Order of Presentation of the Formulas* (see page xxvii) and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function, and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln \sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Gradshteyn and Ryzhik's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals, and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are α , β , γ , δ , t , u , z , z_k , and Δ . The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k)$, $E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second, and third kinds.

The four inverse hyperbolic functions $\operatorname{arcsinh} z$, $\operatorname{arccosh} z$, $\operatorname{arctanh} z$, and $\operatorname{arccoth} z$ are introduced through the definitions

$$\begin{aligned}\operatorname{arcsin} z &= \frac{1}{i} \operatorname{arcsinh}(iz) \\ \operatorname{arccos} z &= \frac{1}{i} \operatorname{arccosh}(z) \\ \operatorname{arctan} z &= \frac{1}{i} \operatorname{arctanh}(iz) \\ \operatorname{arccot} z &= i \operatorname{arccoth}(iz)\end{aligned}$$

*Prepared by Alan Jeffrey for the English language edition.

or

$$\begin{aligned}\operatorname{arcsinh} z &= \frac{1}{i} \arcsin(iz) \\ \operatorname{arccosh} z &= i \arccos z \\ \operatorname{arctanh} z &= \frac{1}{i} \arctan(iz) \\ \operatorname{arccoth} z &= \frac{1}{i} \operatorname{arccot}(-iz)\end{aligned}$$

The numerical constants $\mathbf{C}$ and $\mathbf{G}$ which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant $\mathbf{C}$ is defined by the limit

$$\mathbf{C} = \lim_{s \rightarrow \infty} \left(\sum_{m=1}^s \frac{1}{m} - \ln s \right) = 0.577215 \dots$$

On occasion, other writers denote Euler's constant by the symbol γ , but this is also often used instead to denote the constant

$$\gamma = e^{\mathbf{C}} = 1.781072 \dots$$

Catalan's constant $\mathbf{G}$ is related to the complete elliptic integral

$$\mathbf{K} \equiv \mathbf{K}(k) \equiv \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

by the expression

$$\mathbf{G} = \frac{1}{2} \int_0^1 \mathbf{K} dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = 0.915965 \dots$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the *Index of Special Functions and Notation* on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n , $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{for } |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{for } |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \quad \text{for } n = 0, 1, \dots,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \dots$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n E_n \left(\frac{1}{2} \right) \quad \text{for } n = 0, 1, \dots$$

The E_n are all integral, and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $\dots$

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^* , uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \dots$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$\begin{aligned} B_1^* &= 1/6, & B_2^* &= 1/30, & B_3^* &= 1/42, & B_4^* &= 1/30, & B_5^* &= 5/66, \\ B_6^* &= 691/2730, & B_7^* &= 7/6, & B_8^* &= 3617/510, & \dots \end{aligned}$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$\begin{aligned} B_n(x) &= \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} B_{n-k} E_k(2x) \quad n = 0, 1, \dots \\ E_{n-1}(x) &= \frac{2^n}{n} \left\{ B_n \left(\frac{x+1}{2} \right) - B_n \left(\frac{x}{2} \right) \right\} \end{aligned}$$

or

$$E_{n-1}(x) = \frac{2}{n} \left\{ B_n(x) - 2^n B_n \left(\frac{x}{2} \right) \right\} \quad n = 1, 2, \dots$$

and

$$E_{n-2}(x) = 2 \binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k} - 1) B_{n-k} B_n(x) \quad n = 2, 3, \dots$$

There are also alternative definitions of the Euler polynomial of order n , and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1} \right) \left\{ B_n(x) - 2^n B_n \left(\frac{x}{2} \right) \right\}$$

the Euler polynomial of order n .

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$:

$$\begin{array}{lll} \operatorname{ns} u = \frac{1}{\operatorname{sn} u} & \operatorname{nc} u = \frac{1}{\operatorname{cn} u} & \operatorname{nd} u = \frac{1}{\operatorname{dn} u} \\ \operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} & \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} & \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u} \\ \operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} & \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} & \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u} \end{array}$$

The elliptic integral of the third kind is defined by Gradshteyn and Ryzhik to be

$$\begin{aligned}\Pi(\varphi, n^2, k) &= \int_0^\varphi \frac{da}{(1 - n^2 \sin^2 a) \sqrt{1 - k^2 \sin^2 a}} \\ &= \int_0^{\sin \varphi} \frac{dx}{(1 - n^2 x^2) \sqrt{(1 - x^2)(1 - k^2 x^2)}}\end{aligned}\quad (-\infty < n^2 < \infty)$$

The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function $\text{zn}(u, k)$, frequently written $Z(u)$, is defined by the relation

$$\text{zn}(u, k) = Z(u) = \int_0^u \left\{ \text{dn}^2 v - \frac{E}{K} \right\} dv = E(u) - \frac{E}{K} u.$$

This is related to the theta functions by the relationship

$$\text{zn}(u, k) = \frac{\partial}{\partial u} \ln \Theta(u)$$

giving

$$\begin{aligned}\text{(i).} \quad \text{zn}(u, k) &= \frac{\pi}{2K} \frac{\vartheta'_1\left(\frac{\pi u}{2K}\right)}{\vartheta_1\left(\frac{\pi u}{2K}\right)} - \frac{\text{cn } u \text{ dn } u}{\text{sn } u} \\ \text{(ii).} \quad \text{zn}(u, k) &= \frac{\pi}{2K} \frac{\vartheta'_2\left(\frac{\pi u}{2K}\right)}{\vartheta_2\left(\frac{\pi u}{2K}\right)} - \frac{\text{dn } u \text{ sn } u}{\text{cn } u} \\ \text{(iii).} \quad \text{zn}(u, k) &= \frac{\pi}{2K} \frac{\vartheta'_3\left(\frac{\pi u}{2K}\right)}{\vartheta_3\left(\frac{\pi u}{2K}\right)} - k^2 \frac{\text{sn } u \text{ cn } u}{\text{dn } u} \\ \text{(iv).} \quad \text{zn}(u, k) &= \frac{\pi}{2K} \frac{\vartheta'_4\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)}\end{aligned}$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ .

The Factorial (Gamma) Function

In older reference texts, the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions, the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d(\ln z!)}{dz} = \frac{(z!)' }{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -C + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$\Psi'(z+1) = \Psi'(z) - \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^{\infty} e^{-zt} t^{-n} dt \quad (n = 0, 1, \dots, \quad \operatorname{Re} z > 0).$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $\operatorname{Ei}(z)$ through the expressions

$$E_1(z) = -\operatorname{Ei}(-z) = \int_z^{\infty} e^{-t} t^{-1} dt$$

and

$$\operatorname{li}(z) = \int_0^z \frac{dt}{\ln t} = \operatorname{Ei}(\ln z) \quad [z > 1].$$

The functions $E_n(z)$ satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1} \{e^{-z} - z E_{n-1}(z)\} \quad [n > 1]$$

and

$$E'_n(z) = -E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function $E_n(z)$ has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \dots \right\} \quad \left[|\arg z| < \frac{3\pi}{2} \right]$$

while for large n ,

$$E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x+n)^6} + R(n, x) \right\},$$

where

$$-0.36n^{-4} \leq R(n, x) \leq \left(1 + \frac{1}{x+n-1} \right) n^{-4} \quad [x > 0].$$

The sine and cosine integrals $\operatorname{si}(x)$ and $\operatorname{ci}(x)$ are related to the functions $\operatorname{Si}(x)$ and $\operatorname{Ci}(x)$ by the integrals

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt = \operatorname{si}(x) + \frac{\pi}{2}$$

and

$$\text{Ci}(x) = \mathbf{C} + \ln x + \int_0^x \frac{(\cos t - 1)}{t} dt.$$

The hyperbolic sine and cosine integrals $\text{shi}(x)$ and $\text{chi}(x)$ are defined by the relations

$$\text{shi}(x) = \int_0^x \frac{\sinh t}{t} dt$$

and

$$\text{chi}(x) = \mathbf{C} + \ln x + \int_0^x \frac{(\cosh t - 1)}{t} dt.$$

Some authors write

$$\text{Cin}(x) = \int_0^x \frac{(1 - \cos t)}{t} dt$$

so that

$$\text{Cin}(x) = -\text{Ci}(x) + \ln x + \mathbf{C}.$$

The error function $\text{erf}(x)$ is defined by the relation

$$\text{erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

and the complementary error function $\text{erfc}(x)$ is related to the error function $\text{erfc}(x)$ and to $\Phi(x)$ by the expression

$$\text{erfc}(x) = 1 - \text{erf}(x).$$

The Fresnel integrals $S(x)$ and $C(x)$ are defined by Gradshteyn and Ryzhik as

$$S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 dt$$

and

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 dt.$$

Other definitions that are in use are

$$S_1(x) = \int_0^x \sin \frac{\pi t^2}{2} dt, \quad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2} dt,$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt, \quad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt.$$

These are related by the expressions

$$S(x) = S_1 \left(x \sqrt{\frac{2}{\pi}} \right) = S_2(x^2)$$

and

$$C(x) = C_1 \left(x \sqrt{\frac{2}{\pi}} \right) = C_2(x^2)$$

Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials $H_n(x)$ are related to the Hermite polynomials $He_n(x)$ by the relations

$$He_n(x) = 2^{-n/2} H_n \left(\frac{x}{\sqrt{2}} \right)$$

and

$$H_n(x) = 2^{n/2} He_n(x\sqrt{2}).$$

These functions satisfy the differential equations

$$\frac{d^2 H_n}{dx^2} - 2x \frac{d H_n}{dx} + 2n H_n = 0$$

and

$$\frac{d^2 He_n}{dx^2} - x \frac{d He_n}{dx} + n He_n = 0.$$

They obey the recurrence relations

$$H_{n+1} = 2x H_n - 2n H_{n-1}$$

and

$$He_{n+1} = x He_n - n He_{n-1}.$$

The first six orthogonal polynomials He_n are

$$He_0 = 1, \quad He_1 = x, \quad He_2 = x^2 - 1, \quad He_3 = x^3 - 3x, \quad He_4 = x^4 - 6x^2 + 3, \quad He_5 = x^5 - 10x^3 + 15x.$$

Sometimes the Chebyshev polynomial $U_n(x)$ of the second kind is defined as a solution of the equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + n(n+2)y = 0.$$

Bessel Functions

A variety of different notations for Bessel functions are in use. Some common ones involve the replacement of $Y_n(z)$ by $N_n(z)$ and the introduction of the symbol

$$\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n} \Gamma(n+1) J_n(z).$$

In the book by Gray, Mathews, and MacRobert, the symbol $Y_n(z)$ is used to denote $\frac{1}{2}\pi Y_n(z) + (\ln 2 - \mathbf{C}) J_n(z)$ while Neumann uses the symbol $Y^{(n)}(z)$ for the identical quantity.

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are sometimes denoted by $H_{s_\nu}(z)$ and $H_{i_\nu}(z)$, and some authors write $G_\nu(z) = \left(\frac{1}{2}\right) \pi i H_\nu^{(1)}(z)$.

The Neumann polynomial $O_n(t)$ is a polynomial of degree $n+1$ in $1/t$, with $O_0(t) = 1/t$. The polynomials $O_n(t)$ are defined by the generating function

$$\frac{1}{t-z} = J_0(z) O_0(t) + 2 \sum_{k=1}^{\infty} J_k(z) O_k(t),$$

giving

$$O_n(t) = \frac{1}{4} \sum_{k=0}^{[n/2]} \frac{n(n-k-1)!}{k!} \left(\frac{2}{t}\right)^{n-2k+1} \quad \text{for } n = 1, 2, \dots,$$

where $[\frac{1}{2}n]$ signifies the integral part of $\frac{1}{2}n$. The following relationship holds between three successive polynomials:

$$(n-1) O_{n+1}(t) + (n+1) O_{n-1}(t) - \frac{2(n^2-1)}{t} O_n(t) = \frac{2n}{t} \sin^2 \frac{n\pi}{2}.$$

The Airy functions $\text{Ai}(z)$ and $\text{Bi}(z)$ are independent solutions of the equation

$$\frac{d^2 u}{dz^2} - zu = 0.$$

The solutions can be represented in terms of Bessel functions by the expressions

$$\begin{aligned}\text{Ai}(z) &= \frac{1}{3}\sqrt{z} \left\{ I_{-1/3} \left(\frac{2}{3}z^{3/2} \right) - I_{1/3} \left(\frac{2}{3}z^{3/2} \right) \right\} = \frac{1}{\pi}\sqrt{\frac{z}{3}} K_{1/3} \left(\frac{2}{3}z^{3/2} \right) \\ \text{Ai}(-z) &= \frac{1}{3}\sqrt{z} \left\{ J_{1/3} \left(\frac{2}{3}z^{3/2} \right) + J_{-1/3} \left(\frac{2}{3}z^{3/2} \right) \right\}\end{aligned}$$

and by

$$\begin{aligned}\text{Bi}(z) &= \sqrt{\frac{z}{3}} \left\{ I_{-1/3} \left(\frac{2}{3}z^{3/2} \right) + I_{1/3} \left(\frac{2}{3}z^{3/2} \right) \right\}, \\ \text{Bi}(-z) &= \sqrt{\frac{z}{3}} \left\{ J_{-1/3} \left(\frac{2}{3}z^{3/2} \right) - J_{1/3} \left(\frac{2}{3}z^{3/2} \right) \right\}.\end{aligned}$$

Parabolic Cylinder Functions and Whittaker Functions

The differential equation

$$\frac{d^2 y}{dz^2} + (az^2 + bz + c)y = 0$$

has associated with it the two equations

$$\frac{d^2 y}{dz^2} + \left(\frac{1}{4}z^2 + a \right) y = 0 \quad \text{and} \quad \frac{d^2 y}{dz^2} - \left(\frac{1}{4}z^2 + a \right) y = 0,$$

the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing z by $ze^{i\pi/4}$ and a by $-ia$.

The solutions of the equation

$$\frac{d^2 y}{dz^2} - \left(\frac{1}{4}z^2 + a \right) y = 0$$

are sometimes written $U(a, z)$ and $V(a, z)$. These solutions are related to Whittaker's function $D_p(z)$ by the expressions

$$U(a, z) = D_{-a-\frac{1}{2}}(z)$$

and

$$V(a, z) = \frac{1}{\pi} \Gamma \left(\frac{1}{2} + a \right) \left\{ D_{-a-\frac{1}{2}}(-z) + (\sin \pi a) D_{-a-\frac{1}{2}}(z) \right\}.$$

Mathieu Functions

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Gradshteyn and Ryzhik is

$$\frac{d^2 y}{dz^2} + (a - 2k^2 \cos 2z)y = 0 \quad \text{with } k^2 = q.$$

Different notations involve the replacement of a and q in this equation by h and θ , λ and h^2 , and b and $c = 2\sqrt{q}$, respectively. The periodic solutions $\text{se}_n(z, q)$ and $\text{ce}_n(z, q)$ and the modified periodic solutions $\text{Se}_n(z, q)$ and $\text{Ce}_n(z, q)$ are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: *Tables Relating to Mathieu Functions*. National Bureau of Standards, Columbia University Press, New York, 1951.

Index of Special Functions

Notation	Name of the function and the number of the formula containing its definition	
$\beta(x)$		8.37
$\Gamma(z)$	Gamma function	8.31–8.33
$\gamma(a, x), \quad \Gamma(a, x)$	Incomplete gamma functions	8.35
$\Delta(n - k)$	Unit integer pulse function	18.1
$\xi(s)$		9.56
$\lambda(x, y)$		9.640
$\mu(x, \beta), \quad \mu(x, \beta, \alpha)$		9.640
$\nu(x), \quad \nu(x, \alpha)$		9.640
$\Pi(x)$	Lobachevskiy's angle of parallelism	1.48
$\Pi(\varphi, n, k)$	Elliptic integral of the third kind	8.11
$\zeta(u)$	Weierstrass zeta function	8.17
$\zeta(z, q), \quad \zeta(z)$	Riemann's zeta functions	9.51–9.54
$\Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}\right), \quad \Theta_1(u) = \vartheta_3\left(\frac{\pi u}{2K}\right)$	Jacobian theta function	8.191–8.196
$\left\{ \begin{array}{l} \vartheta_0(v \tau) = \vartheta_4(v \tau), \\ \vartheta_1(v \tau), \quad \vartheta_2(v \tau), \\ \vartheta_3(v \tau) \end{array} \right\}$	Elliptic theta functions	8.18, 8.19
$\sigma(u)$	Weierstrass sigma function	8.17
$\Phi(x)$	See probability integral	8.25
$\Phi(z, s, v)$	Lerch function	9.55
$\Phi(a, c; x) = {}_1F_1(\alpha; \gamma; x)$	Confluent hypergeometric function	9.21
$\left\{ \begin{array}{l} \Phi_1(\alpha, \beta, \gamma, x, y) \\ \Phi_2(\beta, \beta', \gamma, x, y) \\ \Phi_3(\beta, \gamma, x, y) \end{array} \right\}$	Degenerate hypergeometric series in two variables	9.26
$\psi(x)$	Euler psi function	8.36
$\wp(u)$	Weierstrass elliptic function	8.16
$\text{am}(u, k)$	Amplitude (of an elliptic function)	8.141
B_n	Bernoulli numbers	9.61, 9.71
$B_n(x)$	Bernoulli polynomials	9.620
$B(x, y)$	Beta functions	8.38
$B_x(p, q)$	Incomplete beta functions	8.39
$\text{bei}(z), \quad \text{ber}(z)$	Thomson functions	8.56

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<i>continued from previous page</i>		
Notation	Name of the function and the number of the formula containing its definition	
$\mathbf{C}$	Euler constant	9.73, 8.367
$C(x)$	Fresnel cosine integral	8.25
$C_\nu(a)$	Young functions	3.76
$C_n^\lambda(t)$	Gegenbauer polynomials	8.93
$C_n^\lambda(x)$	Gegenbauer functions	8.932 1
$\text{ce}_{2n}(z, q), \quad \text{ce}_{2n+1}(z, q)$	Periodic Mathieu functions (Mathieu functions of the first kind)	8.61
$\text{Ce}_{2n}(z, q), \quad \text{Ce}_{2n+1}(z, q)$	Associated (modified) Mathieu functions of the first kind	8.63
$\text{chi}(x)$	Hyperbolic cosine integral function	8.22
$\text{ci}(x)$	Cosine integral	8.23
$\text{cn}(u)$	Cosine amplitude	8.14
$D(k) \equiv \mathbf{D}$	Elliptic integral	8.112
$D(\varphi, k)$	Elliptic integral	8.111
$D_n(z), \quad D_p(z)$	Parabolic cylinder functions	9.24–9.25
$\text{dn } u$	Delta amplitude	8.14
e_1, e_2, e_3	(used with the Weierstrass function)	8.162
E_n	Euler numbers	9.63, 9.72
$E(\varphi, k)$	Elliptic integral of the second kind	8.11–8.12
$\left\{ \begin{array}{l} \mathbf{E}(k) = \mathbf{E} \\ \mathbf{E}(k') = \mathbf{E}' \end{array} \right\}$	Complete elliptic integral of the second kind	8.11–8.12
$E(p; a_r : q; \varrho_s : x)$	MacRobert's function	9.4
$\mathbf{E}_\nu(z)$	Weber function	8.58
$\text{Ei}(z)$	Exponential integral function	8.21
$\text{erf}(x)$	Error function	8.25
$\text{erfc}(x) = 1 - \text{erf}(x)$	Complementary error function	8.25
$F(\varphi, k)$	Elliptic integral of the first kind	8.11–8.12
${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$	Generalized hypergeometric series	9.14
${}_2F_1(\alpha, \beta; \gamma; z) = F(\alpha, \beta; \gamma; z)$	Gauss hypergeometric function	9.10–9.13
${}_1F_1(\alpha; \gamma; z) = \Phi(\alpha, \gamma; z)$	Degenerate hypergeometric function	9.21
$F_\Lambda(\alpha : \beta_1, \dots, \beta_n;$	Hypergeometric function of several variables	9.19
$\gamma_1, \dots, \gamma_n : z_1, \dots, z_n)$		
F_1, F_2, F_3, F_4	Hypergeometric functions of two variables	9.18
$\left\{ \begin{array}{l} \text{fe}_n(z, q), \text{Fe}_n(z, q) \dots \\ \text{Fey}_n(z, q), \text{Fek}_n(z, q) \dots \end{array} \right\}$	Other nonperiodic solutions of Mathieu's equation	8.64, 8.663
$\mathbf{G}$	Catalan constant	9.73
g_2, g_3	Invariants of the $\wp(u)$ -function	8.161
$\text{gd } x$	Gudermannian	1.49
$\left\{ \begin{array}{l} \text{ge}_n(z, q), \text{Ge}_n(z, q) \\ \text{Gey}_n(z, q), \text{Gek}_n(z, q) \end{array} \right\}$	Other nonperiodic solutions of Mathieu's equation	8.64, 8.663
$G_{p,q}^{m,n} \left(x \left \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right)$	Meijer's functions	9.3
<i>continued on next page</i>		

<i>continued from previous page</i>		
Notation	Name of the function and the number of the formula containing its definition	
$h(n)$	Unit integer function	18.1
$\text{hei}_\nu(z), \quad \text{her}_\nu(z)$	Thomson functions	8.56
$H_\nu^{(1)}(z), \quad H_\nu^{(2)}(z)$	Hankel functions of the first and second kinds	8.405, 8.42
$H(u) = \vartheta_1\left(\frac{\pi u}{2K}\right)$	Theta function	8.192
$H_1(u) = \vartheta_2\left(\frac{\pi u}{2K}\right)$	Theta function	8.192
$H_n(z)$	Hermite polynomials	8.95
$\mathbf{H}_\nu(z)$	Struve functions	8.55
$I_\nu(z)$	Bessel functions of an imaginary argument	8.406, 8.43
$I_x(p, q)$	Normalized incomplete beta function	8.39
$J_\nu(z)$	Bessel function	8.402, 8.41
$\mathbf{J}_\nu(z)$	Anger function	8.58
$k_\nu(x)$	Bateman's function	9.210 3
$\mathbf{K}(\mathbf{k}) = \mathbf{K}, \quad \mathbf{K}(\mathbf{k}') = \mathbf{K}'$	Complete elliptic integral of the first kind	8.11–8.12
$K_\nu(z)$	Bessel functions of imaginary argument	8.407, 8.43
$\text{kei}(z), \quad \text{ker}(z)$	Thomson functions	8.56
$L(x)$	Lobachevskiy's function	8.26
$\mathbf{L}_\nu(z)$	Modified Struve function	8.55
$L_n^\alpha(z)$	Laguerre polynomials	8.97
$\text{li}(x)$	Logarithm integral	8.24
$M_{\lambda, \mu}(z)$	Whittaker functions	9.22, 9.23
$O_n(x)$	Neumann's polynomials	8.59
$P_\nu^\mu(z), \quad P_\nu^\mu(x)$	Associated Legendre functions of the first kind	8.7, 8.8
$P_\nu(z), \quad P_\nu(x)$	Legendre functions and polynomials	8.82, 8.83, 8.91
$P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' \end{Bmatrix}$	Riemann's differential equation	9.160
$P_n^{(\alpha, \beta)}(x)$	Jacobi's polynomials	8.96
$Q_\nu^\mu(z), \quad Q_\nu^\mu(x)$	Associated Legendre functions of the second kind	8.7, 8.8
$Q_\nu(z), \quad Q_\nu(x)$	Legendre functions of the second kind	8.82, 8.83
$S(x)$	Fresnel sine integral	8.25
$S_n(x)$	Schläfli's polynomials	8.59
$s_{\mu, \nu}(z), \quad S_{\mu, \nu}(z)$	Lommel functions	8.57
$\text{se}_{2n+1}(z, q), \quad \text{se}_{2n+2}(z, q)$	Periodic Mathieu functions	8.61
$\text{Se}_{2n+1}(z, q), \quad \text{Se}_{2n+2}(z, q)$	Mathieu functions of an imaginary argument	8.63
$\text{shi}(x)$	Hyperbolic sine integral	8.22
$\text{si}(x)$	Sine integral	8.23
$\text{sn } u$	Sine amplitude	8.14
$T_n(x)$	Chebyshev polynomial of the 1 st kind	8.94
$U_n(x)$	Chebyshev polynomials of the 2 nd kind	8.94

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<i>continued from previous page</i>		
Notation	Name of the function and the number of the formula containing its definition	
$U_\nu(w, z), \quad V_\nu(w, z)$	Lommel functions of two variables	8.578
$W_{\lambda, \mu}(z)$	Whittaker functions	9.22, 9.23
$Y_\nu(z)$	Neumann functions	8.403, 8.41
$Z_\nu(z)$	Bessel functions	8.401
$\mathfrak{Z}_\nu(z)$	Bessel functions	

Notation

Symbol	Meaning
$\lfloor x \rfloor$	The integral part of the real number x (also denoted by $[x]$)
$\int_a^{(b+)} \int_a^{(b-)}$	Contour integrals; the path of integration starting at the point a extends to the point b (along a straight line unless there is an indication to the contrary), encircles the point b along a small circle in the positive (negative) direction, and returns to the point a , proceeding along the original path in the opposite direction.
$\int_C$	Line integral along the curve C
PV $\int$	Principal value integral
$\bar{z} = x - iy$	The complex conjugate of $z = x + iy$
$n!$	$= 1 \cdot 2 \cdot 3 \dots n$, $0! = 1$
$(2n + 1)!!$	$= 1 \cdot 3 \dots (2n + 1)$. (double factorial notation)
$(2n)!!$	$= 2 \cdot 4 \dots (2n)$. (double factorial notation)
$0!! = 1$ and $(-1)!! = 1$	(cf. 3.372 for $n = 0$)
$0^0 = 1$	(cf. 0.112 and 0.113 for $q = 0$)
$\binom{p}{n}$	$= \frac{p(p-1)\dots(p-n+1)}{1 \cdot 2 \dots n} = \frac{p!}{n!(p-n)!}$, $\binom{p}{0} = 1$, $\binom{p}{n} = \frac{p!}{n!(p-n)!}$ [$n = 1, 2, \dots, p \geq n$]
$\binom{x}{n}$	$= x(x-1)\dots(x-n+1)/n!$ [$n = 0, 1, \dots$]
$(a)_n$	$= a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ (Pochhammer symbol)
$\sum_{k=m}^n u_k$	$= u_m + u_{m+1} + \dots + u_n$. If $n < m$, we define $\sum_{k=m}^n u_k = 0$
$\sum'_n, \sum'_{m,n}$	Summation over all integral values of n excluding $n = 0$, and summation over all integral values of n and m excluding $m = n = 0$, respectively.
$\sum, \prod$	An empty $\sum$ has value 0, and an empty $\prod$ has value 1

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<i>continued from previous page</i>	
Symbol	Meaning
$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$	Kronecker delta
τ	Theta function parameter (cf. 8.18)
$\times$ and $\wedge$	Vector product (cf. 10.11)
$\cdot$	Scalar product (cf. 10.11)
∇ or “del”	Vector operator (cf. 10.21)
∇^2	Laplacian (cf. 10.31)
$\sim$	Asymptotically equal to
$\arg z$	The argument of the complex number $z = x + iy$
curl or rot	Vector operator (cf. 10.21)
div	Vector operator (divergence) (cf. 10.21)
$\mathcal{F}$	Fourier transform (cf. 17.21)
$\mathcal{F}_c$	Fourier cosine transform (cf. 17.31)
$\mathcal{F}_s$	Fourier sine transform (cf. 17.31)
grad	Vector operator (gradient) (cf. 10.21)
h_i and g_{ij}	Metric coefficients (cf. 10.51)
H	Hermitian transpose of a vector or matrix (cf. 13.123)
$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$	Heaviside step function
$\operatorname{Im} z \equiv y$	The imaginary part of the complex number $z = x + iy$
k	The letter k (when not used as an index of summation) denotes a number in the interval $[0, 1]$. This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1 - k^2}$ is denoted by k' .
$\mathcal{L}$	Laplace transform (cf. 17.11)
$\mathcal{M}$	Mellin transform (cf. 17.41)
$\mathbb{N}$	The natural numbers $(0, 1, 2, \dots)$
$O(f(z))$	The order of the function $f(z)$. Suppose that the point z approaches z_0 . If there exists an $M > 0$ such that $ g(z) \leq M f(z) $ in some sufficiently small neighborhood of the point z_0 , we write $g(z) = O(f(z))$.
<i>continued on next page</i>	

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Symbol	Meaning
q	The nome, a theta function parameter (cf. 8.18)
$\mathbb{R}$	The real numbers
$R(x)$	A rational function
$\operatorname{Re} z \equiv x$	The real part of the complex number $z = x + iy$
S_n^m	Stirling number of the first kind (cd. 9.74)
$\mathfrak{S}_n^m$	Stirling number of the second kind (cd. 9.74)
$\operatorname{sign} x = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$	The sign (signum) of the real number x
T	Transpose of a vector or matrix (cf. 13.115)
$\mathbb{Z}$	The integers $(0, \pm 1, \pm 2, \dots)$
Z_b	Bilateral z transform (cf. 18.1)
Z_u	Unilateral z transform (cf. 18.1)

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Note on the Bibliographic References

The letters and numbers following equations refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography beginning on page 1141. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources.

Some formulas were changed from their form in the source material. In such cases, the letter *a* appears at the end of the bibliographic references.

As an example, we may use the reference to equation 3.354–5:

ET I 118 (1) *a*

The key on page 1141 indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms*.

The Roman numeral denotes volume one of the work; 118 is the page on which the formula will be found; (1) refers to the number of the formula in this source; and the *a* indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases, the editors have used Russian editions of works published in other languages. Under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.

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0 Introduction

0.1 Finite Sums

0.11 Progressions

0.111 Arithmetic progression.

$$\sum_{k=0}^{n-1} (a + kr) = \frac{n}{2} [2a + (n-1)r] = \frac{n}{2} (a + l) \quad [l = a + (n-1)r \text{ is the last term}]$$

0.112 Geometric progression.

$$\sum_{k=1}^n aq^{k-1} = \frac{a(q^n - 1)}{q - 1} \quad [q \neq 1]$$

0.113 Arithmetic-geometric progression.

$$\sum_{k=0}^{n-1} (a + kr)q^k = \frac{a - [a + (n-1)r]q^n}{1 - q} + \frac{rq(1 - q^{n-1})}{(1 - q)^2}$$

$$[q \neq 1, \quad n > 1]$$

JO (5)

$$\mathbf{0.114^8} \sum_{k=1}^{n-1} k^2 x^k = \frac{(-n^2 + 2n - 1)x^{n+2} + (2n^2 - 2n - 1)x^{n+1} - n^2 x^n + x^2 + x}{(1 - x)^3}$$

0.12 Sums of powers of natural numbers

$$\begin{aligned} \mathbf{0.121} \quad \sum_{k=1}^n k^q &= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2} \binom{q}{1} B_2 n^{q-1} + \frac{1}{4} \binom{q}{3} B_4 n^{q-3} + \frac{1}{6} \binom{q}{5} B_6 n^{q-5} + \dots \\ &= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{qn^{q-1}}{12} - \frac{q(q-1)(q-2)}{720} n^{q-3} + \frac{q(q-1)(q-2)(q-3)(q-4)}{30,240} n^{q-5} - \dots \end{aligned}$$

[last term contains either n or n^2] CE 332

$$1. \quad \sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{CE 333}$$

$$2. \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{CE 333}$$

$$3. \quad \sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2 \quad \text{CE 333}$$

$$4. \quad \sum_{k=1}^n k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1) \quad \text{CE 333}$$

$$5. \quad \sum_{k=1}^n k^5 = \frac{1}{12} n^2(n+1)^2(2n^2+2n-1) \quad \text{CE 333}$$

$$6. \quad \sum_{k=1}^n k^6 = \frac{1}{42} n(n+1)(2n+1)(3n^4+6n^3-3n+1) \quad \text{CE 333}$$

$$7. \quad \sum_{k=1}^n k^7 = \frac{1}{24} n^2(n+1)^2(3n^4+6n^3-n^2-4n+2) \quad \text{CE 333}$$

$$\mathbf{0.122} \quad \sum_{k=1}^n (2k-1)^q = \frac{2^q}{q+1} n^{q+1} - \frac{1}{2} \binom{q}{1} 2^{q-1} B_2 n^{q-1} - \frac{1}{4} \binom{q}{3} 2^{q-3} (2^3-1) B_4 n^{q-3} - \dots$$

[last term contains either n or n^2 .]

$$1. \quad \sum_{k=1}^n (2k-1) = n^2$$

$$2. \quad \sum_{k=1}^n (2k-1)^2 = \frac{1}{3} n(4n^2-1) \quad \text{JO (32a)}$$

$$3. \quad \sum_{k=1}^n (2k-1)^3 = n^2(2n^2-1) \quad \text{JO (32b)}$$

$$4.^{11} \quad \sum_{k=1}^n (mk-1) = \frac{n}{2} [m(n+1)-2]$$

$$5.^{10} \quad \sum_{k=1}^n (mk-1)^2 = \frac{1}{6} n [m^2(n+1)(2n+1) - 6m(n+1) + 6]$$

$$6.^{10} \quad \sum_{k=1}^n (mk-1)^3 = \frac{1}{4} n [m^3 n(n+1)^2 - 2m^2(n+1)(2n+1) + 6m(n+1) - 4]$$

$$\mathbf{0.123} \quad \sum_{k=1}^n k(k+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$$

0.124

$$1. \quad \sum_{k=1}^q k(n^2 - k^2) = \frac{1}{4} q(q+1)(2n^2 - q^2 - q) \quad [q = 1, 2, \dots]$$

$$2.^{10} \quad \sum_{k=1}^n k(k+1)^3 = \frac{1}{60} n(n+1)(12n^3 + 63n^2 + 107n + 58)$$

$$\mathbf{0.125} \quad \sum_{k=1}^n k! \cdot k = (n+1)! - 1 \quad \text{AD (188.1)}$$

$$\mathbf{0.126} \quad \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} = \sqrt{\frac{e}{\pi}} K_{n+\frac{1}{2}} \left(\frac{1}{2} \right) \quad \text{WA 94}$$

0.13 Sums of reciprocals of natural numbers

$$0.131^{11} \quad \sum_{k=1}^n \frac{1}{k} = C + \ln n + \frac{1}{2n} - \sum_{k=2}^{\infty} \frac{A_k}{n(n+1) \dots (n+k-1)}, \quad \text{JO (59), AD (1876)}$$

where

$$A_k = \frac{1}{k} \int_0^1 x(1-x)(2-x)(3-x) \dots (k-1-x) dx$$

$$A_2 = \frac{1}{12}, \quad A_3 = \frac{1}{12}$$

$$A_4 = \frac{19}{120}, \quad A_5 = \frac{9}{20},$$

$$0.132^7 \quad \sum_{k=1}^n \frac{1}{2k-1} = \frac{1}{2} (C + \ln n) + \ln 2 + \frac{B_2}{8n^2} + \frac{(2^3-1)B_4}{64n^4} + \dots \quad \text{JO (71a)a}$$

$$0.133 \quad \sum_{k=2}^n \frac{1}{k^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)} \quad \text{JO (184f)}$$

0.14 Sums of products of reciprocals of natural numbers

$$1. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q][p+kq]} = \frac{n}{p(p+nq)} \quad \text{GI III (64)a}$$

$$2. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q][p+kq][p+(k+1)q]} = \frac{n(2p+nq+q)}{2p(p+q)(p+nq)[p+(n+1)q]} \quad \text{GI III (65)a}$$

$$3. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q][p+kq] \dots [p+(k+l)q]}$$

$$= \frac{1}{(l+1)q} \left\{ \frac{1}{p(p+q) \dots (p+lq)} - \frac{1}{(p+nq)[p+(n+1)q] \dots [p+(n+l)q]} \right\} \quad \text{AD (1856)a}$$

$$4. \quad \sum_{k=1}^n \frac{1}{[1+(k-1)q][1+(k-l)q+p]} = \frac{1}{p} \left[\sum_{k=1}^n \frac{1}{1+(k-1)q} - \sum_{k=1}^n \frac{1}{1+(k-1)q+p} \right] \quad \text{GI III (66)a}$$

$$0.142 \quad \sum_{k=1}^n \frac{k^2+k-1}{(k+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!} \quad \text{JO (157)}$$

0.15 Sums of the binomial coefficients

Notation: n is a natural number.

$$1. \quad \sum_{k=0}^m \binom{n+k}{n} = \binom{n+m+1}{n+1} \quad \text{KR 64 (70.1)}$$

$$2. \quad 1 + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1} \quad \text{KR 62 (58.1)}$$

$$3. \quad \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1} \quad \text{KR 62 (58.1)}$$

$$4. \quad \sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \quad [n \geq 1] \quad \text{KR 64 (70.2)}$$

0.152

$$1. \quad \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3} \right) \quad \text{KR 62 (59.1)}$$

$$2. \quad \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{(n-2)\pi}{3} \right) \quad \text{KR 62 (59.2)}$$

$$3. \quad \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{(n-4)\pi}{3} \right) \quad \text{KR 62 (59.3)}$$

0.153

$$1. \quad \binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right) \quad \text{KR 63 (60.1)}$$

$$2. \quad \binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right) \quad \text{KR 63 (60.2)}$$

$$3. \quad \binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \dots = \frac{1}{2} \left(2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right) \quad \text{KR 63 (60.3)}$$

$$4. \quad \binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \dots = \frac{1}{2} \left(2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right) \quad \text{KR 63 (60.4)}$$

0.154

$$1. \quad \sum_{k=0}^n (k+1) \binom{n}{k} = 2^{n-1} (n+2) \quad [n \geq 0] \quad \text{KR 63 (66.1)}$$

$$2. \quad \sum_{k=1}^n (-1)^{k+1} k \binom{n}{k} = 0 \quad [n \geq 2] \quad \text{KR 63 (66.2)}$$

$$3. \quad \sum_{k=0}^N (-1)^k \binom{N}{k} k^{n-1} = 0 \quad [N \geq n \geq 1; \quad 0^0 \equiv 1]$$

$$4. \quad \sum_{k=0}^n (-1)^k \binom{n}{k} k^n = (-1)^n n! \quad [n \geq 0; \quad 0^0 \equiv 1]$$

$$5. \quad \sum_{k=0}^n (-1)^k \binom{n}{k} (\alpha + k)^n = (-1)^n n! \quad [n \geq 0; \quad 0^0 \equiv 1]$$

$$6. \quad \sum_{k=0}^N (-1)^k \binom{N}{k} (\alpha + k)^{n-1} = 0 \quad [N \geq n \geq 1, \quad 0^0 \equiv 1 \quad N, n \in \mathbb{N}^+]$$

0.155

$$1. \quad \sum_{k=1}^n \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = \frac{n}{n+1} \quad \text{KR 63 (67)}$$

$$2. \quad \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1} \quad \text{KR 63 (68.1)}$$

$$3. \quad \sum_{k=0}^n \frac{\alpha^{k+1}}{k+1} \binom{n}{k} = \frac{(\alpha+1)^{n+1} - 1}{n+1} \quad \text{KR 63 (68.2)}$$

$$4. \quad \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{m=1}^n \frac{1}{m} \quad \text{KR 64 (69)}$$

0.156

$$1. \quad \sum_{k=0}^p \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p} \quad [m \text{ is a natural number}] \quad \text{KR 64 (71.1)}$$

$$2. \quad \sum_{k=0}^{n-p} \binom{n}{k} \binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!} \quad \text{KR 64 (71.2)}$$

0.157

$$1. \quad \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad \text{KR 64 (72.1)}$$

$$2. \quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \quad \text{KR 64 (72.2)}$$

$$3. \quad \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0 \quad \text{KR 64 (72.3)}$$

$$4. \quad \sum_{k=1}^n k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2} \quad \text{KR 64 (72.4)}$$

0.158¹⁰

$$1. \quad \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k = 4^n - \binom{2n}{n}$$

$$2. \quad \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^2 = 4^n - \binom{2n}{n} 3 \cdot 4^n$$

$$3. \quad \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^3 = (6n+13)4^n - 18n \binom{2n}{n}$$

$$4. \quad \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^4 = (32n^2 + 104n) \binom{2n}{n} - (60n+75)4^n$$

0.159¹⁰

$$1. \quad \sum_{k=0}^n \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k = \frac{1}{2} \left[4^n - \binom{2n}{n} \right]$$

$$\begin{aligned}
2. \quad & \sum_{k=0}^n \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k^2 = \frac{1}{2} \left[(2n+1) \binom{2n}{n} - 4^n \right] \\
3. \quad & \sum_{k=0}^n \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k^3 = \frac{(3n+2)}{4} \cdot 4^n - \frac{1}{2} \binom{2n}{n} (3n+1)
\end{aligned}$$

0.160¹⁰

$$\begin{aligned}
1. \quad & \sum_{k=n+1}^{2n} \binom{2n}{k} \alpha^k + \frac{1}{2} \binom{2n}{n} \alpha^n + \frac{(1+\alpha)^{2n-1}(1-\alpha)}{2} \sum_{k=0}^{n-1} \binom{2k}{k} \left[\frac{\alpha}{(1+\alpha)^2} \right]^k = \frac{1}{2} (1+\alpha)^{2n} \\
2. \quad & \sum_{r=0}^n (-1)^r \binom{n}{r} \frac{\Gamma(r+b)}{\Gamma(r+a)} = \frac{B(n+a-b, b)}{\Gamma(a-b)}
\end{aligned}$$

0.2 Numerical Series and Infinite Products

0.21 The convergence of numerical series

The series

$$\mathbf{0.211} \quad \sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \dots$$

is said to *converge absolutely* if the series

$$\mathbf{0.212} \quad \sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \dots,$$

composed of the absolute values of its terms converges. If the series **0.211** converges and the series **0.212** diverges, the series **0.211** is said to *converge conditionally*. Every absolutely convergent series converges.

0.22 Convergence tests

Suppose that

$$\lim_{k \rightarrow \infty} |u_k|^{1/k} = q$$

If $q < 1$, the series **0.211** converges absolutely. On the other hand, if $q > 1$, the series **0.211** diverges. (Cauchy)

0.222 Suppose that

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = q$$

Here, if $q < 1$, the series **0.211** converges absolutely. If $q > 1$, the series **0.211** diverges. If $\left| \frac{u_{k+1}}{u_k} \right|$ approaches 1 but remains greater than unity, then the series **0.211** diverges. (d'Alembert)

0.223 Suppose that

$$\lim_{k \rightarrow \infty} k \left\{ \left| \frac{u_k}{u_{k+1}} \right| - 1 \right\} = q$$

Here, if $q > 1$, the series **0.211** converges absolutely. If $q < 1$, the series **0.211** diverges. (Raabe)

0.224 Suppose that $f(x)$ is a positive decreasing function and that

$$\lim_{k \rightarrow \infty} \frac{e^k f(e^k)}{f(k)} = q$$

for natural k . If $q < 1$, the series $\sum_{k=1}^{\infty} f(k)$ converges. If $q > 1$, this series diverges. (Ermakov)

0.225 Suppose that

$$\left| \frac{u_k}{u_{k+1}} \right| = 1 + \frac{q}{k} + \frac{|v_k|}{k^p},$$

where $p > 1$ and the $|v_k|$ are bounded, that is, the $|v_k|$ are all less than some M , which is independent of k . Here, if $q > 1$, the series **0.211** converges absolutely. If $q \leq 1$, this series diverges. (Gauss)

0.226 Suppose that a function $f(x)$ defined for $x \geq q \geq 1$ is continuous, positive, and decreasing. Under these conditions, the series

$$\sum_{k=1}^{\infty} f(k)$$

converges or diverges accordingly as the integral

$$\int_q^{\infty} f(x) dx$$

converges or diverges (the Cauchy integral test).

0.227 Suppose that all terms of a sequence $u_1, u_2, \dots, u_n$ are positive. In such a case, the series

$$1. \quad \sum_{k=1}^{\infty} (-1)^{k+1} u_k = u_1 - u_2 + u_3 - \dots$$

is called an *alternating series*.

If the terms of an alternating series decrease monotonically in absolute value and approach zero, that is, if

$$2. \quad u_{k+1} < u_k \text{ and } \lim_{k \rightarrow \infty} u_k = 0,$$

the series **0.227** 1 converges. Here, the remainder of the series is

$$3. \quad \sum_{k=n+1}^{\infty} (-1)^{k-n+1} u_k = \left| \sum_{k=1}^{\infty} (-1)^{k+1} u_k - \sum_{k=1}^n (-1)^{k+1} u_k \right| < u_{n+1} \quad (\text{Leibniz})$$

0.228 If the series

$$1. \quad \sum_{k=1}^{\infty} v_k = v_1 + v_2 + \dots + v_k + \dots$$

converges and the numbers u_k form a monotonic bounded sequence, that is, if $|u_k| < M$ for some number M and for all k , the series

$$2. \quad \sum_{k=1}^{\infty} u_k v_k = u_1 v_1 + u_2 v_2 + \dots + u_k v_k + \dots$$

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converges. (Abel)

0.229 If the partial sums of the series **0.228** 1 are bounded and if the numbers u_k constitute a monotonic sequence that approaches zero, that is, if

$$\left| \sum_{k=1}^n v_k \right| < M \quad [n = 1, 2, \dots] \quad \text{and} \quad \lim_{k \rightarrow \infty} u_k = 0,$$

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then the series **0.228** 2 converges (Dirichlet).

0.23–0.24 Examples of numerical series

0.231 Progressions

$$\begin{aligned} 1. \quad & \sum_{k=0}^{\infty} aq^k = \frac{a}{1-q} & [|q| < 1] \\ 2. \quad & \sum_{k=0}^{\infty} (a + kr)q^k = \frac{a}{1-q} + \frac{rq}{(1-q)^2} & [|q| < 1] \quad (\text{cf. } \mathbf{0.113}) \end{aligned}$$

0.232

$$\begin{aligned} 1. \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2 & (\text{cf. } \mathbf{1.511}) \\ 2. \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{\pi}{4} \\ & & (\text{cf. } \mathbf{1.643}) \\ 3.^* \quad & \sum_{k=1}^{\infty} \frac{k^a}{b^k} = \frac{1}{(b-1)^{a+1}} \sum_{i=1}^a \left[\frac{1}{b^{a-i}} \sum_{j=0}^i \frac{(-1)^j (a+1)! (i-j)^a}{j! (a+1-j)!} \right] \\ & & [a = 1, 2, 3, \dots, \quad b \neq 1] \end{aligned}$$

0.233

$$\begin{aligned} 1. \quad & \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots = \zeta(p) & [\operatorname{Re} p > 1] & \text{WH} \\ 2. \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^p} = (1 - 2^{1-p}) \zeta(p) & [\operatorname{Re} p > 0] & \text{WH} \\ 3.^{10} \quad & \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}|, \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} & & \text{FI II 721} \\ 4. \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2n}} = \frac{(2^{2n-1} - 1) \pi^{2n}}{(2n)!} |B_{2n}| & & \text{JO (165)} \\ 5. \quad & \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n} - 1) \pi^{2n}}{2 \cdot (2n)!} |B_{2n}| & & \text{JO (184b)} \\ 6. \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2} (2n)!} |E_{2n}| & & \text{JO (184d)} \end{aligned}$$

0.234

$$1. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \frac{\pi^2}{12} \quad \text{EU}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \quad \text{EU}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \mathbf{G} \quad \text{FI II 482}$$

$$4. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32} \quad \text{EU}$$

$$5. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96} \quad \text{EU}$$

$$6. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^5} = \frac{5\pi^5}{1536} \quad \text{EU}$$

$$7. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)^2} = \frac{\pi^2}{12} - \ln 2$$

$$8.^6 \quad \sum_{k=1}^{\infty} \frac{1}{k(2k+1)} = 2 - 2 \ln 2$$

$$9.* \quad \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n^2 \Gamma(n)} = \sqrt{\pi} \ln 4$$

$$\mathbf{0.235} \quad S_n = \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^n}$$

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{\pi^2 - 8}{16}, \quad S_3 = \frac{32 - 3\pi^2}{64}, \quad S_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}$$

JO (186)

0.236

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{k(4k^2-1)} = 2 \ln 2 - 1 \quad \text{BR 51a}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{1}{k(9k^2-1)} = \frac{3}{2} (\ln 3 - 1) \quad \text{BR 51a}$$

$$3. \quad \sum_{k=1}^{\infty} \frac{1}{k(36k^2-1)} = -3 + \frac{3}{2} \ln 3 + 2 \ln 2 \quad \text{BR 52, AD (6913.3)}$$

$$4. \quad \sum_{k=1}^{\infty} \frac{k}{(4k^2-1)^2} = \frac{1}{8} \quad \text{BR 52}$$

$$5. \quad \sum_{k=1}^{\infty} \frac{1}{k(4k^2-1)^2} = \frac{3}{2} - 2 \ln 2 \quad \text{BR 52}$$

$$\begin{aligned}
6. \quad & \sum_{k=1}^{\infty} \frac{12k^2 - 1}{k(4k^2 - 1)^2} = 2 \ln 2 && \text{AD (6917.3), BR 52} \\
7.^6 \quad & \sum_{k=1}^{\infty} \frac{1}{k(2k+1)^2} = 4 - \frac{\pi^2}{4} - 2 \ln 2
\end{aligned}$$

0.237

$$\begin{aligned}
1. \quad & \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2} && \text{AD (6917.2), BR 52} \\
2. \quad & \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8} \\
3. \quad & \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} = \frac{3}{4} && [\text{cf. } \mathbf{0.133}], \\
4. \quad & \sum'_{k=1, k \neq m}^{\infty} \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2} && [m \text{ is an integer}] \quad \text{AD (6916.1)} \\
5. \quad & \sum'_{k=1, k \neq m}^{\infty} \frac{(-1)^{k-1}}{(m-k)(m+k)} = \frac{3}{4m^2} && [m \text{ is an even number}] \quad \text{AD (6916.2)}
\end{aligned}$$

0.238

$$\begin{aligned}
1. \quad & \sum_{k=1}^{\infty} \frac{1}{(2k-1)2k(2k+1)} = \ln 2 - \frac{1}{2} && \text{GI III (93)} \\
2. \quad & \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)2k(2k+1)} = \frac{1}{2} (1 - \ln 2) && \text{GI III (94)a} \\
3. \quad & \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+3)(3k+4)} = \frac{1}{6} - \frac{1}{4} \ln 3 + \frac{\pi}{12\sqrt{3}} && \text{GI III (95)}
\end{aligned}$$

0.239

$$\begin{aligned}
1.^{11} \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-2} = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} + \ln 2 \right) && \text{GI III (85), BR* 161 (1)} \\
2.^7 \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-1} = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} - \ln 2 \right) && \text{BR* 161 (1)} \\
3. \quad & \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{4k-3} = \frac{1}{4\sqrt{2}} \left[\pi + 2 \ln (\sqrt{2} + 1) \right] && \text{BR* 161 (1)} \\
4. \quad & \sum_{k=1}^{\infty} (-1)^{\left[\frac{k+3}{2} \right]} \frac{1}{k} = \frac{\pi}{4} + \frac{1}{2} \ln 2 && \text{GI III (87)}
\end{aligned}$$

$$5. \quad \sum_{k=1}^{\infty} (-1)^{\left[\frac{k+3}{2}\right]} \frac{1}{2k-1} = \frac{\pi}{2\sqrt{2}}$$

$$6. \quad \sum_{k=1}^{\infty} (-1)^{\left[\frac{k+5}{3}\right]} \frac{1}{2k-1} = \frac{5\pi}{12} \quad \text{GI III (88)}$$

$$7. \quad \sum_{k=1}^{\infty} \frac{1}{(8k-1)(8k+1)} = \frac{1}{2} - \frac{\pi}{16} (\sqrt{2} + 1)$$

0.241

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{2^k k} = \ln 2 \quad \text{JO (172g)}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{1}{2^k k^2} = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2 \quad \text{JO (174)}$$

$$3.^{11} \quad \sum_{n=0}^{\infty} \binom{2n}{n} p^n = \frac{1}{\sqrt{1-4p}} \quad [0 \leq p < \tfrac{1}{4}]$$

$$4.^{10} \quad \sum_{n=1}^{\infty} \frac{p^n}{n^2} = \frac{\pi^2}{6} - \int_1^p \frac{\ln(1-x)}{x} dx \quad [0 \leq p \leq 1]$$

$$5.^{10} \quad \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j = 4^i - \binom{2i}{i} \\ \left[\binom{n}{m} = 0, \quad m < 0 \right]$$

$$6.^{10} \quad \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^2 = 4i \binom{2i}{i} - 3 \cdot 4^i \\ \left[\binom{n}{m} = 0, \quad m < 0 \right]$$

$$7.^{10} \quad \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^3 = (6i+13)4^i - 18i \binom{2i}{i} \\ \left[\binom{n}{m} = 0, \quad m < 0 \right]$$

$$8.^{10} \quad \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^4 = (32i^2 + 104i) \binom{2i}{i} - (60i + 75)4^i$$

$$9.^{10} \quad \sum_{j=n+1}^{2n} \binom{2n}{j} k^j + \frac{1}{2} \binom{2n}{n} k^n + \frac{(1+k)^{2n-1}(1-k)}{2} \sum_{i=0}^{n-1} \binom{2i}{i} \left[\frac{k}{(1+k)^2} \right]^i = \frac{1}{2}(1+k)^{2n}$$

$$10.^{10} \quad \sum_{k=0}^i \binom{i+k}{k} 2^{i-k} = 4^i$$

$$11.^{10} \sum_{k=0}^i \binom{i+k}{h} i^{-k} k = (i+1)4^i - (2i+1) \binom{2i}{i}$$

$$12.^{10} \sum_{k=0}^i \binom{2i}{k} = \frac{1}{2} \left[4^i + \binom{2i}{i} \right]$$

$$13.^{10} \sum_{k=0}^i \binom{2i}{k} k = \frac{i}{2} 4^i$$

$$14.^{10} \sum_{k=0}^i \binom{2i}{k} k^2 = (2i+1)i4^{i-1} - \frac{i^2}{2} \binom{2i}{i}$$

$$0.242 \sum_{k=0}^{\infty} (-1)^k \frac{1}{n^{2k}} = \frac{n^2}{n^2 + 1} \quad [n > 1]$$

0.243

$$1. \sum_{k=1}^{\infty} \frac{1}{[p + (k-1)q][p + kq] \dots [p + (k+l)q]} = \frac{1}{(l+1)q} \frac{1}{p(p+q) \dots (p+lq)}$$

(see also 0.141 3)

$$2.^7 \sum_{k=1}^{\infty} \frac{x^{k-1}}{[p + (k-1)q][p + (k-1)q + 1][p + (k-1)q + 2] \dots [p + (k-1)q + l]} = \frac{1}{l!} \int_0^i \frac{t^{p-1}(1-t)^t}{1-xt^q} dt$$

$[p > 0, \quad x^2 < 1] \quad \text{BR}^* 161 (2), \text{AD} (6.704)$

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left(\frac{1}{x} \tanh \left[\frac{(2k+1)\pi x}{2} \right] + x \tanh \left[\frac{(2k+1)\pi}{2x} \right] \right) = \frac{\pi^3}{16}$$

0.244

$$1. \sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p} \int_0^1 \frac{x^p - x^q}{1-x} dx \quad [p > -1, \quad q > -1, \quad p \neq q] \quad \text{GI III (90)}$$

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{p + (k-1)q} = \int_0^1 \frac{t^{p-1}}{1+t^q} dt \quad [p > 0, \quad q > 0] \quad \text{BR}^* 161 (1)$$

$$3.^{10} \sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p} \sum_{m=p+1}^q \frac{1}{m} \quad [q > p > -1, \quad p \text{ and } q \text{ integers}]$$

Summations of reciprocals of factorials

0.245

$$1. \sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828 \dots$$

$$2.^{11} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{2e} \approx 0.1839397 \dots$$

$$3. \quad \sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = \frac{1}{e} = 0.36787 \dots$$

$$4. \quad \sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1$$

$$5. \quad \sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2} \left(e + \frac{1}{e} \right) = 1.54308 \dots$$

$$6. \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2} \left(e - \frac{1}{e} \right) = 1.17520 \dots$$

$$7. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = \cos 1 = 0.54030 \dots$$

$$8. \quad \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} = \sin 1 = 0.84147 \dots$$

0.246

$$1. \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) = 2.27958530 \dots$$

$$2. \quad \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) = 1.590636855 \dots$$

$$3. \quad \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} = I_n(2)$$

$$4. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = J_0(2) = 0.22389078 \dots$$

$$5. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} = J_1(2) = 0.57672481 \dots$$

$$6. \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} = J_n(2)$$

$$\mathbf{0.247} \quad \sum_{k=1}^{\infty} \frac{k!}{(n+k-1)!} = \frac{1}{(n-2) \cdot (n-1)!}$$

$$\mathbf{0.248} \quad \sum_{k=1}^{\infty} \frac{k^n}{k!} = S_n,$$

$$S_1 = e,$$

$$S_2 = 2e,$$

$$S_3 = 5e,$$

$$S_4 = 15e$$

$$S_5 = 52e,$$

$$S_6 = 203e,$$

$$S_7 = 877e,$$

$$S_8 = 4140e$$

$$\mathbf{0.249^7} \quad \sum_{k=0}^{\infty} \frac{(k+1)^3}{k!} = 15e$$

0.25 Infinite products

0.250 Suppose that a sequence of numbers $a_1, a_2, \dots, a_k, \dots$ is given. If the limit $\lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k)$ exists, whether finite or infinite (but of definite sign), this limit is called the value of the *infinite product* $\prod_{k=1}^{\infty} (1 + a_k)$, and we write

$$1. \quad \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k) = \prod_{k=1}^{\infty} (1 + a_k)$$

If an infinite product has a finite *nonzero* value, it is said to converge. Otherwise, the infinite product is said to diverge. We assume that no a_k is equal to -1 . FI II 400

0.251 For the infinite product **0.250** 1. to converge, it is necessary that $\lim_{k \rightarrow \infty} a_k = 0$. FI II 403

0.252 If $a_k > 0$ or $a_k < 0$ for all values of the index k starting with some particular value, then, for the product **0.250** 1 to converge, it is necessary and sufficient that the series $\sum_{k=1}^{\infty} a_k$ converge.

0.253 The product $\prod_{k=1}^{\infty} (1 + a_k)$ is said to converge absolutely if the product $\prod_{k=1}^{\infty} (1 + |a_k|)$ converges. FI II 403

0.254 Absolute convergence of an infinite product implies its convergence.

0.255 The product $\prod_{k=1}^{\infty} (1 + a_k)$ converges absolutely if, and only if, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely. FI II 406

0.26 Examples of infinite products

$$\mathbf{0.261} \quad \prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1} \right) = \sqrt{2} \quad \text{EU}$$

0.262

$$1. \quad \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2} \right) = \frac{1}{2} \quad \text{FI II 401}$$

$$2. \quad \prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2} \right) = \frac{2}{\pi} \quad \text{FI II 401}$$

$$3. \quad \prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2} \right) = \frac{\pi}{4} \quad \text{FI II 401}$$

0.263

$$1. \quad e = \frac{2}{1} \cdot \left(\frac{4}{3} \right)^{1/2} \left(\frac{6 \cdot 8}{5 \cdot 7} \right)^{1/4} \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15} \right)^{1/8} \dots$$

$$2.^* \quad e = \left(\frac{2}{1} \right)^{1/2} \left(\frac{2^2}{1 \cdot 3} \right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3} \right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} \right)^{1/5} \dots$$

$$3.^* \quad \frac{\pi}{2} = \left(\frac{1}{2}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/4} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/8} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/16} \dots$$

where the n^{th} factor is the $(n+1)^{\text{th}}$ root of the product $\prod_{k=0}^n (k+1)^{(-1)^{k+1} \binom{n}{k}}$.

0.264

$$1. \quad e^C = \prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{1 + \frac{1}{k}} \quad \text{FI II 402}$$

$$2.^* \quad e^C = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \dots$$

where the n^{th} factor is the $(n+1)^{\text{th}}$ root of the product $\prod_{k=0}^n (k+1)^{(-1)^{k+1} \binom{n}{k}}$. Here C is the Euler constant, denoted in other works by γ .

$$\mathbf{0.265} \quad \frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \dots \quad \text{FI II 402}$$

$$\mathbf{0.266}^8 \quad \prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x} \quad [0 < x < 1] \quad \text{FI II 401}$$

0.3 Functional Series

0.30 Definitions and theorems

0.301 The series

$$1. \quad \sum_{k=1}^{\infty} f_k(x),$$

the terms of which are functions, is called a *functional series*. The set of values of the independent variable x for which the series **0.301** 1 converges constitutes what is called the *region of convergence* of that series.

0.302 A series that converges for all values of x in a region M is said to *converge uniformly* in that region if, for every $\varepsilon \geq 0$, there exists a number N such that, for $n > N$, the inequality

$$\left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \varepsilon$$

holds for *all* x in M .

0.303 If the terms of the functional series **0.301** 1 satisfy the inequalities:

$$|f_k(x)| < u_k \quad (k = 1, 2, 3, \dots),$$

throughout the region M , where the u_k are the terms of some *convergent* numerical series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots,$$

the series **0.301** 1 converges uniformly in M . (Weierstrass)

0.304 Suppose that the series **0.301** 1 converges uniformly in a region M and that a set of functions $g_k(x)$ constitutes (for each x) a monotonic sequence, and that these functions are uniformly bounded; that is, suppose that a number L exists such that the inequalities

$$1. \quad |g_n(x)| \leq L$$

hold for all n and x . Then, the series

$$2. \quad \sum_{k=1}^{\infty} f_k(x) g_k(x)$$

converges uniformly in the region M . (Abel)

FI II 451

0.305 Suppose that the partial sums of the series **0.301** 1 are uniformly bounded; that is, suppose that, for some L and for all n and x in M , the inequalities

$$\left| \sum_{k=1}^n f_k(x) \right| < L$$

hold. Suppose also that for each x the functions $g_n(x)$ constitute a monotonic sequence that approaches zero uniformly in the region M . Then, the series **0.304** 2 converges uniformly in the region M . (Dirichlet)

FI II 451

0.306⁶ If the functions $f_k(x)$ (for $k = 1, 2, 3, \dots$) are integrable on the interval $[a, b]$ and if the series **0.301** 1 made up of these functions converges uniformly on that interval, this series may be integrated *termwise*; that is,

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x) dx \quad [a \leq x \leq b] \quad \text{FI II 459}$$

0.307 Suppose that the functions $f_k(x)$ (for $k = 1, 2, 3, \dots$) have continuous derivatives $f'_k(x)$ on the interval $[a, b]$. If the series **0.301** 1 converges on this interval and if the series $\sum_{k=1}^{\infty} f'_k(x)$ of these derivatives converges uniformly, the series **0.301** 1 may be differentiated termwise; that is,

$$\left\{ \sum_{k=1}^{\infty} f_k(x) \right\}' = \sum_{k=1}^{\infty} f'_k(x) \quad \text{FI II 460}$$

0.31 Power series

0.311 A functional series of the form

$$1. \quad \sum_{k=0}^{\infty} a_k(x - \xi)^k = a_0 + a_1(x - \xi) + a_2(x - \xi)^2 + \dots$$

is called a *power series*. The following is true of any power series: if it is not everywhere convergent, the region of convergence is a circle with its center at the point ξ and a radius equal to R ; at every interior point of this circle, the power series **0.311** 1 converges absolutely, and outside this circle, it diverges. This circle is called the *circle of convergence*, and its radius is called the *radius of convergence*. If the series converges at all points of the complex plane, we say that the radius of convergence is infinite ($R = +\infty$).

0.312 Power series may be integrated and differentiated termwise inside the circle of convergence; that is,

$$\int_{\xi}^x \left\{ \sum_{k=0}^{\infty} a_k (x - \xi)^k \right\} dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x - \xi)^{k+1},$$

$$\frac{d}{dx} \left\{ \sum_{k=0}^{\infty} a_k (x - \xi)^k \right\} = \sum_{k=1}^{\infty} k a_k (x - \xi)^{k-1}.$$

The radius of convergence of a series that is obtained from termwise integration or differentiation of another power series coincides with the radius of convergence of the original series.

Operations on power series

0.313 Division of power series.

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_n + \frac{1}{a_0} \sum_{k=1}^n c_{n-k} a_k - b_n = 0,$$

or

$$c_n = \frac{(-1)^n}{a_0^n} \begin{bmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \cdots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n-1} b_0 - a_0 b_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix} \quad \text{AD (6360)}$$

0.314 Power series raised to powers.

$$\left(\sum_{k=0}^{\infty} a_k x^k \right)^n = \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_0 = a_0^n, \quad c_m = \frac{1}{m a_0} \sum_{k=1}^m (k n - m + k) a_k c_{m-k} \quad \text{for } m \geq 1 \quad [n \text{ is a natural number}] \quad \text{AD (6361)}$$

0.315 The substitution of one series into another.

$$\sum_{k=1}^{\infty} b_k y^k = \sum_{k=1}^{\infty} c_k x^k \quad y = \sum_{k=1}^{\infty} a_k x^k;$$

$$c_1 = a_1 b_1, \quad c_2 = a_2 b_1 + a_1^2 b_2, \quad c_3 = a_3 b_1 + 2 a_1 a_2 b_2 + a_1^3 b_3,$$

$$c_4 = a_4 b_1 + a_2^2 b_2 + 2 a_1 a_3 b_2 + 3 a_1^2 a_2 b_3 + a_1^4 b_4, \quad \dots$$

AD (6362)

0.316 Multiplication of power series

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k \quad c_n = \sum_{k=0}^n a_k b_{n-k} \quad \text{FI II 372}$$

Taylor series

0.317 If a function $f(x)$ has derivatives of all orders throughout a neighborhood of a point ξ , then we may write the series

$$1. \quad f(\xi) + \frac{(x-\xi)}{1!} f'(\xi) + \frac{(x-\xi)^2}{2!} f''(\xi) + \frac{(x-\xi)^3}{3!} f'''(\xi) + \dots,$$

which is known as the *Taylor series* of the function $f(x)$.

The Taylor series converges to the function $f(x)$ if the remainder

$$2. \quad R_n(x) = f(x) - f(\xi) - \sum_{k=1}^n \frac{(x-\xi)^k}{k!} f^{(k)}(\xi)$$

approaches zero as $n \rightarrow \infty$.

The following are different forms for the remainder of a Taylor series:

$$3. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{(n+1)!} f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < \theta < 1] \quad (\text{Lagrange})$$

$$4. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{n!} (1-\theta)^n f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < \theta < 1] \quad (\text{Cauchy})$$

$$5. \quad R_n(x) = \frac{\psi(x-\xi) - \psi(0)}{\psi'[(x-\xi)(1-\theta)]} \frac{(x-\xi)^n (1-\theta)^n}{n!} f^{(n+1)}(\xi + \theta(x-\xi))$$

$$[0 < \theta < 1], \quad (\text{Schl\"{o}milch})$$

where $\psi(x)$ is an arbitrary function satisfying the following two conditions: (1) It and its derivative $\psi'(x)$ are continuous in the interval $(0, x-\xi)$; and (2) the derivative $\psi'(x)$ does not change sign in that interval. If we set $\psi(x) = x^{p+1}$, we obtain the following form for the remainder:

$$R_n(x) = \frac{(x-\xi)^{n+1} (1-\theta)^{n-p-1}}{(p+1)n!} f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < p \leq n; \quad 0 < \theta < 1] \quad (\text{Rouch\'{e}})$$

$$6. \quad R_n(x) = \frac{1}{n!} \int_{\xi}^x f^{(n+1)}(t) (x-t)^n dt$$

0.318 Other forms in which a Taylor series may be written:

$$1.^{11} \quad f(a+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \dots$$

$$2. \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \quad (\text{Maclaurin series})$$

0.319 The Taylor series of functions of several variables:

$$f(x, y) = f(\xi, \eta) + (x - \xi) \frac{\partial f(\xi, \eta)}{\partial x} + (y - \eta) \frac{\partial f(\xi, \eta)}{\partial y} \\ + \frac{1}{2!} \left\{ (x - \xi)^2 \frac{\partial^2 f(\xi, \eta)}{\partial x^2} + 2(x - \xi)(y - \eta) \frac{\partial^2 f(\xi, \eta)}{\partial x \partial y} + (y - \eta)^2 \frac{\partial^2 f(\xi, \eta)}{\partial y^2} \right\} + \dots$$

0.32 Fourier series

0.320 Suppose that $f(x)$ is a *periodic* function of period $2l$ and that it is absolutely integrable (possibly improperly) over the interval $(-l, l)$. The following trigonometric series is called the *Fourier series* of $f(x)$:

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right),$$

the coefficients of which (the Fourier coefficients) are given by the formulas

$$2. \quad a_k = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{k\pi t}{l} dt = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(t) \cos \frac{k\pi t}{l} dt \quad (k = 0, 1, 2, \dots)$$

$$3.^{11} \quad b_k = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{k\pi t}{l} dt = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(t) \sin \frac{k\pi t}{l} dt \quad (k = 1, 2, \dots)$$

Convergence tests

0.321 The Fourier series of a function $f(x)$ at a point x_0 converges to the number

$$\frac{f(x_0 + 0) + f(x_0 - 0)}{2},$$

if, for some $h > 0$, the integral

$$\int_0^h \frac{|f(x_0 + t) + f(x_0 - t) - f(x_0 + 0) - f(x_0 - 0)|}{t} dt$$

exists. Here, it is assumed that the function $f(x)$ either is continuous at the point x_0 or has a discontinuity of the first kind (a *saltus*) at that point and that both one-sided limits $f(x_0 + 0)$ and $f(x_0 - 0)$ exist. (Dini) FI III 524

0.322 The Fourier series of a periodic function $f(x)$ that satisfies the Dirichlet conditions on the interval $[a, b]$ converges at every point x_0 to the value $\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]$. (Dirichlet)

We say that a function $f(x)$ satisfies the Dirichlet conditions on the interval $[a, b]$ if it is bounded on that interval and if the interval $[a, b]$ can be partitioned into a finite number of subintervals inside each of which the function $f(x)$ is continuous and monotonic.

0.323 The Fourier series of a function $f(x)$ at a point x_0 converges to $\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]$ if $f(x)$ is of bounded variation in some interval $(x_0 - h, x_0 + h)$ with center at x_0 . (Jordan–Dirichlet) FI III 528

The definition of a function of bounded variation. Suppose that a function $f(x)$ is defined on some interval $[a, b]$, where $a < b$. Let us partition this interval in an arbitrary manner into subintervals with the dividing points

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

and let us form the sum

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})|$$

Different partitions of the interval $[a, b]$ (that is, different choices of points of division x_i) yield, generally speaking, different sums. If the set of these sums is bounded above, we say that the function $f(x)$ is of *bounded variation* on the interval $[a, b]$. The least upper bound of these sums is called the *total variation* of the function $f(x)$ on the interval $[a, b]$.

0.324 Suppose that a function $f(x)$ is piecewise-continuous on the interval $[a, b]$ and that in each interval of continuity it has a piecewise-continuous derivative. Then, at every point x_0 of the interval $[a, b]$, the Fourier series of the function $f(x)$ converges to $\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]$.

0.325 A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a cosine series of the form

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

where

$$2. \quad a_k = \frac{2}{l} \int_0^l f(t) \cos \frac{k\pi t}{l} dt$$

0.326 A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a sine series of the form

$$1. \quad \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l},$$

where

$$2. \quad b_k = \frac{2}{l} \int_0^l f(t) \sin \frac{k\pi t}{l} dt$$

The convergence tests for the series **0.325** 1 and **0.326** 1 are analogous to the convergence tests for the series **0.320** 1 (see **0.321–0.324**).

0.327 The Fourier coefficients a_k and b_k (given by formulas **0.320** 2 and **0.320** 3) of an absolutely integrable function approach zero as $k \rightarrow \infty$.

If a function $f(x)$ is square-integrable on the interval $(-l, l)$, the equation of closure is satisfied:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{l} \int_{-l}^l f^2(x) dx \quad (\text{A. M. Lyapunov}) \quad \text{FI III 705}$$

0.328 Suppose that $f(x)$ and $\varphi(x)$ are two functions that are square-integrable on the interval $(-l, l)$ and that a_k, b_k and α_k, β_k are their Fourier coefficients. For such functions, the generalized equation of closure (Parseval's equation) holds:

$$\frac{a_0 \alpha_0}{2} + \sum_{k=1}^{\infty} (a_k \alpha_k + b_k \beta_k) = \frac{1}{l} \int_{-l}^l f(x) \varphi(x) dx \quad \text{FI III 709}$$

For examples of Fourier series, see **1.44** and **1.45**.

0.33 Asymptotic series

0.330 Included in the collection of all divergent series is the broad class of series known as *asymptotic* or *semiconvergent* series. *Despite the fact that these series diverge*, the values of the functions that they represent can be calculated with a high degree of accuracy if we take the sum of a suitable number of terms of such series. In the case of alternating asymptotic series, we obtain greatest accuracy if we break off the series in question at whatever term is of lowest absolute value. In this case, the error (in absolute value) does not exceed the absolute value of the first of the discarded terms (cf. **0.227 3**).

Asymptotic series have many properties that are analogous to the properties of convergent series, and, for that reason, they play a significant role in analysis.

The asymptotic expansion of a function is denoted as follows:

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}$$

This is the definition of an asymptotic expansion. The divergent series $\sum_{n=0}^{\infty} \frac{A_n}{z^n}$ is called the *asymptotic expansion* of a function $f(z)$ in a given region of values of $\arg z$ if the expression $R_n(z) = z^n [f(z) - S_n(z)]$, where $S_n(z) = \sum_{k=0}^n \frac{A_k}{z^k}$, satisfies the condition $\lim_{|z| \rightarrow \infty} R_n(z) = 0$ for fixed n . FI II 820

A divergent series that represents the asymptotic expansion of some function is called an *asymptotic series*.

0.331 Properties of asymptotic series

1. The operations of addition, subtraction, multiplication, and raising to a power can be performed on asymptotic series just as on absolutely convergent series. The series obtained as a result of these operations will also be asymptotic.
2. One asymptotic series can be divided by another, provided that the first term A_0 of the divisor is not equal to zero. The series obtained as a result of division will also be asymptotic. FI II 823-825
3. An asymptotic series can be integrated termwise, and the resultant series will also be asymptotic. In contrast, differentiation of an asymptotic series is, in general, not permissible. FI II 824
4. A single asymptotic expansion can represent different functions. On the other hand, a given function can be expanded in an asymptotic series in only one manner.

0.4 Certain Formulas from Differential Calculus

0.41 Differentiation of a definite integral with respect to a parameter

$$\mathbf{0.410} \quad \frac{d}{da} \int_{\psi(a)}^{\varphi(a)} f(x, a) dx = f(\varphi(a), a) \frac{d\varphi(a)}{da} - f(\psi(a), a) \frac{d\psi(a)}{da} + \int_{\psi(a)}^{\varphi(a)} \frac{d}{da} f(x, a) dx \quad \text{FI II 680}$$

0.411 In particular,

1. $\frac{d}{da} \int_b^a f(x) dx = f(a)$
2. $\frac{d}{db} \int_b^a f(x) dx = -f(b)$

0.42 The n^{th} derivative of a product (Leibniz's rule)

Suppose that u and v are n -times-differentiable functions of x . Then,

$$\frac{d^n(uv)}{dx^n} = u \frac{d^n v}{dx^n} + \binom{n}{1} \frac{du}{dx} \frac{d^{n-1} v}{dx^{n-1}} + \binom{n}{2} \frac{d^2 u}{dx^2} \frac{d^{n-2} v}{dx^{n-2}} + \binom{n}{3} \frac{d^3 u}{dx^3} \frac{d^{n-3} v}{dx^{n-3}} + \cdots + v \frac{d^n u}{dx^n}$$

or, symbolically,

$$\frac{d^n(uv)}{dx^n} = (u + v)^{(n)}$$

FI I 272

0.43 The n^{th} derivative of a composite function

0.430 If $f(x) = F(y)$ and $y = \varphi(x)$, then

$$1. \quad \frac{d^n}{dx^n} f(x) = \frac{U_1}{1!} F'(y) + \frac{U_2}{2!} F''(y) + \frac{U_3}{3!} F'''(y) + \cdots + \frac{U_n}{n!} F^{(n)}(y),$$

where

$$U_k = \frac{d^n}{dx^n} y^k - \frac{k}{1!} y \frac{d^n}{dx^n} y^{k-1} + \frac{k(k-1)}{2!} y^2 \frac{d^n}{dx^n} y^{k-2} - \cdots + (-1)^{k-1} k y^{k-1} \frac{d^n y}{dx^n} \quad \text{AD (7361) GO}$$

$$2. \quad \frac{d^n}{dx^n} f(x) = \sum \frac{n!}{i!j!h!\cdots k!} \frac{d^m F}{dy^m} \left(\frac{y'}{1!}\right)^i \left(\frac{y''}{2!}\right)^j \left(\frac{y'''}{3!}\right)^h \cdots \left(\frac{y^{(l)}}{l!}\right)^k,$$

Here, the symbol $\sum$ indicates summation over all solutions in non-negative integers of the equation $i + 2j + 3h + \cdots + lk = n$ and $m = i + j + h + \cdots + k$.

0.431

$$1. \quad (-1)^n \frac{d^n}{dx^n} F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}} F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}} \frac{n}{1!} F^{(n-1)}\left(\frac{1}{x}\right) \\ + \frac{(n-1)(n-2)}{x^{2n-2}} \frac{n(n-1)}{2!} F^{(n-2)}\left(\frac{1}{x}\right) + \cdots$$

AD (7362.1)

$$2. \quad (-1)^n \frac{d^n}{dx^n} e^{\frac{a}{x}} = \frac{1}{x^n} e^{\frac{a}{x}} \left[\left(\frac{a}{x}\right)^n + (n-1) \binom{n}{1} \left(\frac{a}{x}\right)^{n-1} + (n-1)(n-2) \binom{n}{2} \left(\frac{a}{x}\right)^{n-2} \right. \\ \left. + (n-1)(n-2)(n-3) \binom{n}{3} \left(\frac{a}{x}\right)^{n-3} + \cdots \right]$$

AD (7362.2)

0.432

$$1. \quad \frac{d^n}{dx^n} F(x^2) = (2x)^n F^{(n)}(x^2) + \frac{n(n-1)}{1!} (2x)^{n-2} F^{(n-1)}(x^2) \\ + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4} F^{(n-2)}(x^2) + \\ + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!} (2x)^{n-6} F^{(n-3)}(x^2) + \cdots$$

AD (7363.1)

$$2. \quad \frac{d^n}{dx^n} e^{ax^2} = (2ax)^n e^{ax^2} \left[1 + \frac{n(n-1)}{1! (4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2! (4ax^2)^2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3! (4ax^2)^3} + \dots \right]$$

AD (7363.2)

$$3. \quad \frac{d^n}{dx^n} (1 + ax^2)^p = \frac{p(p-1)(p-2)\dots(p-n+1)(2ax)^n}{(1+ax^2)^{n-p}} \times \left\{ 1 + \frac{n(n-1)}{1!(p-n+1)} \frac{1+ax^2}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)} \left(\frac{1+ax^2}{4ax^2} \right)^2 + \dots \right\},$$

AD (7363.3)

$$4. \quad \frac{d^{m-1}}{dx^{m-1}} (1-x^2)^{m-\frac{1}{2}} = (-1)^{m-1} \frac{(2m-1)!!}{m} \sin(m \arccos x)$$

AD (7363.4)

$$5. \quad (-1)^n \frac{\partial^n}{\partial a^n} \left(\frac{a}{a^2 + b^2} \right) = n! \left(\frac{a}{a^2 + b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n+1} (-1)^k \binom{n+1}{2k} \left(\frac{b}{a} \right)^{2k} \quad (3.944.12)$$

$$6. \quad (-1)^n \frac{\partial^n}{\partial a^n} \left(\frac{b}{a^2 + b^2} \right) = n! \left(\frac{a}{a^2 + b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n} (-1)^k \binom{n+1}{2k+1} \left(\frac{b}{a} \right)^{2k+1} \quad (3.944.11)$$

0.433

$$1. \quad \frac{d^n}{dx^n} F(\sqrt{x}) = \frac{F^{(n)}(\sqrt{x})}{(2\sqrt{x})^n} - \frac{n(n-1)}{1!} \frac{F^{(n-1)}(\sqrt{x})}{(2\sqrt{x})^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!} \frac{F^{(n-2)}(\sqrt{x})}{(2\sqrt{x})^{n+2}} - \dots$$

AD (7364.1)

$$2. \quad \frac{d^n}{dx^n} (1 + a\sqrt{x})^{2n-1} = \frac{(2n-1)!!}{2^n} \frac{a}{\sqrt{x}} \left(a^2 - \frac{1}{x} \right)^{n-1}$$

AD (7364.2)

$$\mathbf{0.434} \quad \frac{d^n}{dx^n} y^p = p \binom{n-p}{n} \left\{ -\binom{n}{1} \frac{1}{p-1} y^{p-1} \frac{d^n y}{dx^n} + \binom{n}{2} \frac{1}{p-2} y^{p-2} \frac{d^n (y^2)}{dx^n} - \dots \right\} \quad \text{AD (737.1)}$$

$$\mathbf{0.435} \quad \frac{d^n}{dx^n} \ln y = \left\{ \binom{n}{1} \frac{1}{1 \cdot y} \frac{d^n y}{dx^n} - \binom{n}{2} \frac{1}{2 \cdot y^2} \frac{d^n (y^2)}{dx^n} + \frac{d^n (y^3)}{dx^n} x^n - \dots \right\} \quad \text{AD (737.2)}$$

0.44 Integration by substitution

0.440¹¹ Let $f(g(x))$ and $g(x)$ be continuous in $[a, b]$. Further, let $g'(x)$ exist and be continuous there.

Then $\int_a^b f[g(x)]g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$

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1 Elementary Functions

1.1 Power of Binomials

1.11 Power series

$$\mathbf{1.110} \quad (1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \cdots + \frac{q(q-1)\cdots(q-k+1)}{k!}x^k + \cdots = \sum_{k=0}^{\infty} \binom{q}{k} x^k$$

If q is neither a natural number nor zero, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$, the series converges for $q > -1$ and diverges for $q \leq -1$. For $x = -1$, the series converges absolutely for $q > 0$ and diverges for $q < 0$. If $q = n$ is a natural number, the series **1.110** is reduced to the finite sum **1.111**. FI II 425

$$\mathbf{1.111} \quad (a+x)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k}$$

1.112

$$1. \quad (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$$

(see also **1.121** 2)

$$2. \quad (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}$$

$$3.^{11} \quad (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \cdots$$

$$4. \quad (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots$$

$$\mathbf{1.113} \quad \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k \quad [x^2 < 1]$$

1.114

$$1. \quad (1 + \sqrt{1+x})^q = 2^q \left[1 + \frac{q}{1!} \left(\frac{x}{4} \right) + \frac{q(q-3)}{2!} \left(\frac{x}{4} \right)^2 + \frac{q(q-4)(q-5)}{3!} \left(\frac{x}{4} \right)^3 + \cdots \right]$$

$[x^2 < 1, \quad q \text{ is a real number}]$

AD (6351.1)

$$\begin{aligned}
2. \quad \left(x + \sqrt{1+x^2}\right)^q &= 1 + \sum_{k=0}^{\infty} \frac{q^2 (q^2 - 2^2) (q^2 - 4^2) \dots [q^2 - (2k)^2] x^{2k+2}}{(2k+2)!} \\
&\quad + qx + q \sum_{k=1}^{\infty} \frac{(q^2 - 1^2) (q^2 - 3^2) \dots [q^2 - (2k-1)^2]}{(2k+1)!} x^{2k+1} \\
&\quad [x^2 < 1, \quad q \text{ is a real number}] \quad \text{AD(6351.2)}
\end{aligned}$$

1.12 Series of rational fractions

1.121

$$\begin{aligned}
1. \quad \frac{x}{1-x} &= \sum_{k=1}^{\infty} \frac{2^{k-1} x^{2^{k-1}}}{1+x^{2^{k-1}}} = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{1-x^{2^k}} \quad [x^2 < 1] \quad \text{AD (6350.3)} \\
2. \quad \frac{1}{x-1} &= \sum_{k=1}^{\infty} \frac{2^{k-1}}{x^{2^{k-1}} + 1} \quad [x^2 > 1] \quad \text{AD (6350.3)}
\end{aligned}$$

1.2 The Exponential Function

1.21 Series representation

1.211

$$\begin{aligned}
1.^{11} \quad e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
2. \quad a^x &= \sum_{k=0}^{\infty} \frac{(x \ln a)^k}{k!} \\
3. \quad e^{-x^2} &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!} \\
4.^* \quad e^x &= \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n
\end{aligned}$$

$$1.212 \quad e^x(1+x) = \sum_{k=0}^{\infty} \frac{x^k(k+1)}{k!}$$

$$1.213 \quad \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k)!} \quad [x < 2\pi] \quad \text{FI II 520}$$

$$1.214 \quad e^{e^x} = e \left(1 + x + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \dots\right) \quad \text{AD (6460.3)}$$

1.215

$$1. \quad e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \dots \quad \text{AD (6460.4)}$$

$$2. \quad e^{\cos x} = e \left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \dots\right) \quad \text{AD (6460.5)}$$

$$3. \quad e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots \quad \text{AD (6460.6)}$$

1.216

$$1. \quad e^{\arcsin x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \dots \quad \text{AD (6460.7)}$$

$$2. \quad e^{\arctan x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \dots \quad \text{AD (6460.8)}$$

1.217

$$1. \quad \pi \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{1}{x^2 + k^2} \quad (\text{cf. 1.421 3}) \quad \text{AD (6707.1)}$$

$$2. \quad \frac{2\pi}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x^2 + k^2} \quad (\text{cf. 1.422 3}) \quad \text{AD (6707.2)}$$

1.22 Functional relations**1.221**

$$1. \quad a^x = e^{x \ln a}$$

$$2. \quad a^{\log_a x} = a^{\frac{1}{\log_x a}} = x$$

1.222

$$1. \quad e^x = \cosh x + \sinh x$$

$$2. \quad e^{ix} = \cos x + i \sin x$$

$$1.223 \quad e^{ax} - e^{bx} = (a - b)x \exp \left[\frac{1}{2}(a + b)x \right] \prod_{k=1}^{\infty} \left[1 + \frac{(a - b)^2 x^2}{2k^2 \pi^2} \right] \quad \text{MO 216}$$

1.23 Series of exponentials

$$1.231 \quad \sum_{k=0}^{\infty} a^{kx} = \frac{1}{1 - a^x} \quad [a > 1 \text{ and } x < 0 \text{ or } 0 < a < 1 \text{ and } x > 0]$$

1.232

$$1. \quad \tanh x = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx} \quad [x > 0]$$

$$2. \quad \operatorname{sech} x = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x} \quad [x > 0]$$

$$3. \quad \operatorname{cosech} x = 2 \sum_{k=0}^{\infty} e^{-(2k+1)x} \quad [x > 0]$$

$$4.* \quad \sin x = \exp \left[- \sum_{n=1}^{\infty} \frac{\cos^{2n} x}{2n} \right] \quad [0 \leq x \leq \pi]$$

1.3–1.4 Trigonometric and Hyperbolic Functions

1.30 Introduction

The trigonometric and hyperbolic sines are related by the identities

$$\sinh x = \frac{1}{i} \sin(ix), \quad \sin x = \frac{1}{i} \sinh(ix).$$

The trigonometric and hyperbolic cosines are related by the identities

$$\cosh x = \cos(ix), \quad \cos x = \cosh(ix).$$

Because of this duality, every relation involving trigonometric functions has its formal counterpart involving the corresponding hyperbolic functions, and vice versa. In many (though not all) cases, both pairs of relationships are meaningful.

The idea of matching the relationships is carried out in the list of formulas given below. However, not all the meaningful “pairs” are included in the list.

1.31 The basic functional relations

1.311

1. $\sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$
 $= -i \sinh(ix)$
2. $\sinh x = \frac{1}{2} (e^x - e^{-x})$
 $= -i \sin(ix)$
3. $\cos x = \frac{1}{2} (e^{ix} + e^{-ix})$
 $= \cosh(ix)$
4. $\cosh x = \frac{1}{2} (e^x + e^{-x})$
 $= \cos(ix)$
5. $\tan x = \frac{\sin x}{\cos x} = \frac{1}{i} \tanh(ix)$
6. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{1}{i} \tan(ix)$
7. $\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = i \coth(ix)$
8. $\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = i \cot(ix)$

1.312

1. $\cos^2 x + \sin^2 x = 1$

$$2. \quad \cosh^2 x - \sinh^2 x = 1$$

1.313

1. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
2. $\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$
3. $\sin(x \pm iy) = \sin x \cosh y \pm i \sinh y \cos x$
4. $\sinh(x \pm iy) = \sinh x \cos y \pm i \sin y \cosh x$
5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
6. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
7. $\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y$
8. $\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$
9. $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
10. $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
11. $\tan(x \pm iy) = \frac{\tan x \pm i \tanh y}{1 \mp i \tan x \tanh y}$
12. $\tanh(x \pm iy) = \frac{\tanh x \pm i \tan y}{1 \pm i \tanh x \tan y}$

1.314

1. $\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$
2. $\sinh x \pm \sinh y = 2 \sinh \frac{1}{2}(x \pm y) \cosh \frac{1}{2}(x \mp y)$
3. $\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$
4. $\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$
5. $\cos x - \cos y = 2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(y - x)$
6. $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$
7. $\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}$
8. $\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$
- 9.*
$$\begin{aligned} \sin x \pm \cos y &= \pm 2 \sin \left[\frac{1}{2}(x + y) \pm \frac{\pi}{4} \right] \sin \left[\frac{1}{2}(x - y) \pm \frac{\pi}{4} \right] \\ &= \pm 2 \cos \left[\frac{1}{2}(x + y) \mp \frac{\pi}{4} \right] \cos \left[\frac{1}{2}(x - y) \mp \frac{\pi}{4} \right] \\ &= 2 \sin \left[\frac{1}{2}(x \pm y) \pm \frac{\pi}{4} \right] \cos \left[\frac{1}{2}(x \mp y) \mp \frac{\pi}{4} \right] \end{aligned}$$

$$10.^* \quad a \sin x \pm b \cos x = a \sqrt{1 + \left(\frac{b}{a}\right)^2} \sin \left[x \pm \arctan \left(\frac{b}{a} \right) \right] \quad [a \neq 0]$$

$$11.^* \quad \pm a \sin x + b \cos x = b \sqrt{1 + \left(\frac{a}{b}\right)^2} \cos \left[x \mp \arctan \left(\frac{a}{b} \right) \right] \quad [b \neq 0]$$

$$12.^* \quad a \sin x \pm b \cos y = q \sqrt{1 + \left(\frac{r}{q}\right)^2} \sin \left[\frac{1}{2}(x \pm y) + \arctan \left(\frac{r}{q} \right) \right] \\ q = (a + b) \cos \left[\frac{1}{2}(x \mp y) \right], \quad r = (a - b) \sin \left[\frac{1}{2}(x \mp y) \right] \quad [q \neq 0]$$

$$13.^* \quad a \cos x + b \cos y = t \sqrt{1 + \left(\frac{s}{t}\right)^2} \cos \left[\frac{1}{2}(x \mp y) + \arctan \left(\frac{s}{t} \right) \right] \quad [t \neq 0] \\ = -s \sqrt{1 + \left(\frac{t}{s}\right)^2} \cos \left[\frac{1}{2}(x \mp y) - \arctan \left(\frac{t}{s} \right) \right] \quad [s \neq 0] \\ s = (a - b) \sin \left[\frac{1}{2}(x \pm y) \right], \quad t = (a + b) \cos \left[\frac{1}{2}(x \pm y) \right]$$

1.315

1. $\sin^2 x - \sin^2 y = \sin(x + y) \sin(x - y) = \cos^2 y - \cos^2 x$
2. $\sinh^2 x - \sinh^2 y = \sinh(x + y) \sinh(x - y) = \cosh^2 x - \cosh^2 y$
3. $\cos^2 x - \sin^2 y = \cos(x + y) \cos(x - y) = \cos^2 y - \sin^2 x$
4. $\sinh^2 x + \cosh^2 y = \cosh(x + y) \cosh(x - y) = \cosh^2 x + \sinh^2 y$

1.316

1. $(\cos x + i \sin x)^n = \cos nx + i \sin nx \quad [n \text{ is an integer}]$
2. $(\cosh x + \sinh x)^n = \sinh nx + \cosh nx \quad [n \text{ is an integer}]$

1.317

1. $\sin \frac{x}{2} = \pm \sqrt{\frac{1}{2}(1 - \cos x)}$
2. $\sinh \frac{x}{2} = \pm \sqrt{\frac{1}{2}(\cosh x - 1)}$
3. $\cos \frac{x}{2} = \pm \sqrt{\frac{1}{2}(1 + \cos x)}$
4. $\cosh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x + 1)}$
5. $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$

$$6. \quad \tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

The signs in front of the radical in formulas **1.317** 1, **1.317** 2, and **1.317** 3 are taken so as to agree with the signs of the left-hand members. The sign of the left hand members depends in turn on the value of x .

1.32 The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)

1.320

$$1. \quad \sin^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\} \quad \text{KR 56 (10, 2)}$$

$$2. \quad \sinh^{2n} x = \frac{(-1)^n}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cosh 2(n-k)x + \binom{2n}{n} \right\}$$

$$3. \quad \sin^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sin(2n-2k-1)x \quad \text{KR 56 (10, 4)}$$

$$4. \quad \sinh^{2n-1} x = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sinh(2n-2k-1)x$$

$$5. \quad \cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\} \quad \text{KR 56 (10, 1)}$$

$$6. \quad \cosh^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cosh 2(n-k)x + \binom{2n}{n} \right\}$$

$$7. \quad \cos^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cos(2n-2k-1)x \quad \text{KR 56 (10, 3)}$$

$$8. \quad \cosh^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cosh(2n-2k-1)x$$

Special cases

1.321

$$1. \quad \sin^2 x = \frac{1}{2} (-\cos 2x + 1)$$

$$2. \quad \sin^3 x = \frac{1}{4} (-\sin 3x + 3 \sin x)$$

$$3. \quad \sin^4 x = \frac{1}{8} (\cos 4x - 4 \cos 2x + 3)$$

$$4. \quad \sin^5 x = \frac{1}{16} (\sin 5x - 5 \sin 3x + 10 \sin x)$$

$$5. \quad \sin^6 x = \frac{1}{32} (-\cos 6x + 6 \cos 4x - 15 \cos 2x + 10)$$

$$6. \quad \sin^7 x = \frac{1}{64} (-\sin 7x + 7 \sin 5x - 21 \sin 3x + 35 \sin x)$$

1.322

$$1. \quad \sinh^2 x = \frac{1}{2} (\cosh 2x - 1)$$

$$2. \quad \sinh^3 x = \frac{1}{4} (\sinh 3x - 3 \sinh x)$$

$$3. \quad \sinh^4 x = \frac{1}{8} (\cosh 4x - 4 \cosh 2x + 3)$$

$$4. \quad \sinh^5 x = \frac{1}{16} (\sinh 5x - 5 \sinh 3x + 10 \sinh x)$$

$$5. \quad \sinh^6 x = \frac{1}{32} (\cosh 6x - 6 \cosh 4x + 15 \cosh 2x + 10)$$

$$6. \quad \sinh^7 x = \frac{1}{64} (\sinh 7x - 7 \sinh 5x + 21 \sinh 3x + 35 \sinh x)$$

1.323

$$1. \quad \cos^2 x = \frac{1}{2} (\cos 2x + 1)$$

$$2. \quad \cos^3 x = \frac{1}{4} (\cos 3x + 3 \cos x)$$

$$3. \quad \cos^4 x = \frac{1}{8} (\cos 4x + 4 \cos 2x + 3)$$

$$4. \quad \cos^5 x = \frac{1}{16} (\cos 5x + 5 \cos 3x + 10 \cos x)$$

$$5. \quad \cos^6 x = \frac{1}{32} (\cos 6x + 6 \cos 4x + 15 \cos 2x + 10)$$

$$6. \quad \cos^7 x = \frac{1}{64} (\cos 7x + 7 \cos 5x + 21 \cos 3x + 35 \cos x)$$

1.324

$$1. \quad \cosh^2 x = \frac{1}{2} (\cosh 2x + 1)$$

$$2. \quad \cosh^3 x = \frac{1}{4} (\cosh 3x + 3 \cosh x)$$

$$3. \quad \cosh^4 x = \frac{1}{8} (\cosh 4x + 4 \cosh 2x + 3)$$

$$4. \quad \cosh^5 x = \frac{1}{16} (\cosh 5x + 5 \cosh 3x + 10 \cosh x)$$

$$5. \quad \cosh^6 x = \frac{1}{32} (\cosh 6x + 6 \cosh 4x + 15 \cosh 2x + 10)$$

$$6. \quad \cosh^7 x = \frac{1}{64} (\cosh 7x + 7 \cosh 5x + 21 \cosh 3x + 35 \cosh x)$$

1.33 The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions

1.331

$$\begin{aligned}
 1.^7 \quad \sin nx &= n \cos^{n-1} x \sin x - \binom{n}{3} \cos^{n-3} x \sin^3 x + \binom{n}{5} \cos^{n-5} x \sin^5 x - \dots; \\
 &= \sin x \left\{ 2^{n-1} \cos^{n-1} x - \binom{n-2}{1} 2^{n-3} \cos^{n-3} x \right. \\
 &\quad \left. + \binom{n-3}{2} 2^{n-5} \cos^{n-5} x - \binom{n-4}{3} 2^{n-7} \cos^{n-7} x + \dots \right\}
 \end{aligned}$$

AD (3.175)

$$\begin{aligned}
 2. \quad \sinh nx &= x \sum_{k=1}^{[(n+1)/2]} \binom{n}{2k-1} \sinh^{2k-2} x \cosh^{n-2k+1} x \\
 &= \sinh x \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} 2^{n-2k-1} \cosh^{n-2k-1} x
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \cos nx &= \cos^n x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \binom{n}{4} \cos^{n-4} x \sin^4 x - \dots; \\
 &= 2^{n-1} \cos^n x - \frac{n}{1} 2^{n-3} \cos^{n-2} x + \frac{n}{2} \binom{n-3}{1} 2^{n-5} \cos^{n-4} x \\
 &\quad - \frac{n}{3} \binom{n-4}{2} 2^{n-7} \cos^{n-6} x + \dots
 \end{aligned}$$

AD (3.175)

$$\begin{aligned}
 4.^3 \quad \cosh nx &= \sum_{k=0}^{[n/2]} \binom{n}{2k} \sinh^{2k} x \cosh^{n-2k} x \\
 &= 2^{n-1} \cosh^n x + n \sum_{k=1}^{[n/2]} (-1)^k \frac{1}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \cosh^{n-2k} x
 \end{aligned}$$

1.332

$$1. \quad \sin 2nx = 2n \cos x \left\{ \sin x - \frac{4n^2 - 2^2}{3!} \sin^3 x + \frac{(4n^2 - 2^2)(4n^2 - 4^2)}{5!} \sin^5 x - \dots \right\} \quad \text{AD (3.171)}$$

$$\begin{aligned}
 &= (-1)^{n-1} \cos x \left\{ 2^{2n-1} \sin^{2n-1} x - \frac{2n-2}{1!} 2^{2n-3} \sin^{2n-3} x \right. \\
 &\quad + \frac{(2n-3)(2n-4)}{2!} 2^{2n-5} \sin^{2n-5} x \\
 &\quad \left. - \frac{(2n-4)(2n-5)(2n-6)}{3!} 2^{2n-7} \sin^{2n-7} x + \dots \right\} \quad \text{AD (3.173)}
 \end{aligned}$$

$$\begin{aligned}
2. \quad \sin(2n-1)x &= (2n-1) \left\{ \sin x - \frac{(2n-1)^2 - 1^2}{3!} \sin^3 x \right. \\
&\quad \left. + \frac{[(2n-1)^2 - 1^2][(2n-1)^2 - 3^2]}{5!} \sin^5 x - \dots \right\} \quad \text{AD (3.172)}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{n-1} \left\{ 2^{2n-2} \sin^{2n-1} x - \frac{2n-1}{1!} 2^{2n-4} \sin^{2n-3} x \right. \\
&\quad + \frac{(2n-1)(2n-4)}{2!} 2^{2n-6} \sin^{2n-5} x \\
&\quad \left. - \frac{(2n-1)(2n-5)(2n-6)}{3!} 2^{2n-8} \sin^{2n-7} x + \dots \right\} \quad \text{AD (3.174)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \cos 2nx &= 1 - \frac{4n^2}{2!} \sin^2 x + \frac{4n^2(4n^2 - 2^2)}{4!} \sin^4 x - \frac{4n^2(4n^2 - 2)(4n^2 - 4^2)}{6!} \sin^6 x + \dots \\
&\quad \text{AD (3.171)}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^n \left\{ 2^{2n-1} \sin^{2n} x - \frac{2n}{1!} 2^{2n-3} \sin^{2n-2} x \right. \\
&\quad \left. + \frac{2n(2n-3)}{2!} 2^{2n-5} \sin^{2n-4} x - \frac{2n(2n-4)(2n-5)}{3!} 2^{2n-7} \sin^{2n-6} x + \dots \right\} \\
&\quad \text{AD (3.173)a}
\end{aligned}$$

$$\begin{aligned}
4. \quad \cos(2n-1)x &= \cos x \left\{ 1 - \frac{(2n-1)^2 - 1^2}{2!} \sin^2 x \right. \\
&\quad \left. + \frac{[(2n-1)^2 - 1^2][(2n-1)^2 - 3^2]}{4!} \sin^4 x - \dots \right\} \quad \text{AD (3.172)}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{n-1} \cos x \left\{ 2^{2n-2} \sin^{2n-2} x - \frac{2n-3}{1!} 2^{2n-4} \sin^{2n-4} x \right. \\
&\quad + \frac{(2n-4)(2n-5)}{2!} 2^{2n-6} \sin^{2n-6} x \\
&\quad \left. - \frac{(2n-5)(2n-6)(2n-7)}{3!} 2^{2n-8} \sin^{2n-8} x + \dots \right\} \quad \text{AD (3.174)}
\end{aligned}$$

By using the formulas and values of **1.30**, we can write formulas for $\sinh 2nx$, $\sinh(2n-1)x$, $\cosh 2nx$, and $\cosh(2n-1)x$ that are analogous to those of **1.332**, just as was done in the formulas in **1.331**.

Special cases

1.333

1. $\sin 2x = 2 \sin x \cos x$
2. $\sin 3x = 3 \sin x - 4 \sin^3 x$
3. $\sin 4x = \cos x (4 \sin x - 8 \sin^3 x)$
4. $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$
5. $\sin 6x = \cos x (6 \sin x - 32 \sin^3 x + 32 \sin^5 x)$

$$6. \quad \sin 7x = 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x$$

1.334

$$1. \quad \sinh 2x = 2 \sinh x \cosh x$$

$$2. \quad \sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

$$3.^{11} \quad \sinh 4x = \cosh x (4 \sinh x + 8 \sinh^3 x)$$

$$4. \quad \sinh 5x = 5 \sinh x + 20 \sinh^3 x + 16 \sinh^5 x$$

$$5.^{11} \quad \sinh 6x = \cosh x (6 \sinh x + 32 \sinh^3 x + 32 \sinh^5 x)$$

$$6. \quad \sinh 7x = 7 \sinh x + 56 \sinh^3 x + 112 \sinh^5 x + 64 \sinh^7 x$$

1.335

$$1. \quad \cos 2x = 2 \cos^2 x - 1$$

$$2. \quad \cos 3x = 4 \cos^3 x - 3 \cos x$$

$$3. \quad \cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$$

$$4. \quad \cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x$$

$$5. \quad \cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$$

$$6. \quad \cos 7x = 64 \cos^7 x - 112 \cos^5 x + 56 \cos^3 x - 7 \cos x$$

1.336

$$1. \quad \cosh 2x = 2 \cosh^2 x - 1$$

$$2. \quad \cosh 3x = 4 \cosh^3 x - 3 \cosh x$$

$$3. \quad \cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1$$

$$4. \quad \cosh 5x = 16 \cosh^5 x - 20 \cosh^3 x + 5 \cosh x$$

$$5. \quad \cosh 6x = 32 \cosh^6 x - 48 \cosh^4 x + 18 \cosh^2 x - 1$$

$$6. \quad \cosh 7x = 64 \cosh^7 x - 112 \cosh^5 x + 56 \cosh^3 x - 7 \cosh x$$

1.337

$$1.* \quad \frac{\cos 3x}{\cos^3 x} = 1 - 3 \tan^2 x$$

$$2.* \quad \frac{\cos 4x}{\cos^4 x} = 1 - 6 \tan^2 x + \tan^4 x$$

$$3.* \quad \frac{\cos 5x}{\cos^5 x} = 1 - 10 \tan^2 x + 5 \tan^4 x$$

$$4.* \quad \frac{\cos 6x}{\cos^6 x} = 1 - 15 \tan^2 x + 15 \tan^4 x - \tan^6 x$$

$$5.* \quad \frac{\sin 3x}{\cos^3 x} = 3 \tan x - \tan^3 x$$

$$6.* \quad \frac{\sin 4x}{\cos^4 x} = 4 \tan x - 4 \tan^3 x$$

$$7.^* \quad \frac{\sin 5x}{\cos^5 x} = 5 \tan x - 10 \tan^3 x + \tan^5 x$$

$$8.^* \quad \frac{\sin 6x}{\cos^6 x} = 6 \tan x - 20 \tan^3 x + 6 \tan^5 x$$

$$9.^* \quad \frac{\cos 3x}{\sin^3 x} = \cot^3 x - 3 \cot x$$

$$10.^* \quad \frac{\cos 4x}{\sin^4 x} = \cot^4 x - 6 \cot^2 x + 1$$

$$11.^* \quad \frac{\cos 5x}{\sin^5 x} = \cot^5 x - 10 \cot^3 x + 5 \cot x$$

$$12.^* \quad \frac{\cos 6x}{\sin^6 x} = \cot^6 x - 15 \cot^4 x + 15 \cot^2 x - 1$$

$$13.^* \quad \frac{\sin 3x}{\sin^3 x} = 3 \cot^2 x - 1$$

$$14.^* \quad \frac{\sin 4x}{\sin^4 x} = 4 \cot^3 x - 4 \cot x$$

$$15.^* \quad \frac{\sin 5x}{\sin^5 x} = 5 \cot^4 x - 10 \cot^2 x + 1$$

$$16.^* \quad \frac{\sin 6x}{\sin^6 x} = 6 \cot^5 x - 20 \cot^3 x + 6 \cot x$$

1.34 Certain sums of trigonometric and hyperbolic functions

1.341

$$1. \quad \sum_{k=0}^{n-1} \sin(x + ky) = \sin \left(x + \frac{n-1}{2}y \right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2} \quad \text{AD (361.8)}$$

$$2. \quad \sum_{k=0}^{n-1} \sinh(x + ky) = \sinh \left(x + \frac{n-1}{2}y \right) \sinh \frac{ny}{2} \frac{1}{\sinh \frac{y}{2}}$$

$$3. \quad \sum_{k=0}^{n-1} \cos(x + ky) = \cos \left(x + \frac{n-1}{2}y \right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2} \quad \text{AD (361.9)}$$

$$4. \quad \sum_{k=0}^{n-1} \cosh(x + ky) = \cosh \left(x + \frac{n-1}{2}y \right) \sinh \frac{ny}{2} \frac{1}{\sinh \frac{y}{2}}$$

$$5. \quad \sum_{k=0}^{2n-1} (-1)^k \cos(x + ky) = \sin \left(x + \frac{2n-1}{2}y \right) \sin ny \sec \frac{y}{2} \quad \text{JO (202)}$$

$$6. \quad \sum_{k=0}^{n-1} (-1)^k \sin(x + ky) = \sin \left(x + \frac{n-1}{2}(y + \pi) \right) \sin \frac{n(y + \pi)}{2} \sec \frac{y}{2} \quad \text{AD (202a)}$$

Special cases**1.342**

$$1. \quad \sum_{k=1}^n \sin kx = \sin \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} \quad \text{AD (361.1)}$$

$$\begin{aligned} 2.^{10} \quad \sum_{k=0}^n \cos kx &= \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1 \\ &= \cos \frac{nx}{2} \sin \frac{n+1}{2}x \operatorname{cosec} \frac{x}{2} = \frac{1}{2} \left(1 + \frac{\sin \left(n + \frac{1}{2} \right) x}{\sin \frac{x}{2}} \right) \end{aligned} \quad \text{AD (361.2)}$$

$$3. \quad \sum_{k=1}^n \sin(2k-1)x = \sin^2 nx \operatorname{cosec} x \quad \text{AD (361.7)}$$

$$4. \quad \sum_{k=1}^n \cos(2k-1)x = \frac{1}{2} \sin 2nx \operatorname{cosec} x \quad \text{JO (207)}$$

1.343

$$1. \quad \sum_{k=1}^n (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos \left(\frac{2n+1}{2}x \right)}{2 \cos \frac{x}{2}} \quad \text{AD (361.11)}$$

$$2. \quad \sum_{k=1}^n (-1)^{k+1} \sin(2k-1)x = (-1)^{n+1} \frac{\sin 2nx}{2 \cos x} \quad \text{AD (361.10)}$$

$$3. \quad \sum_{k=1}^n \cos(4k-3)x + \sum_{k=1}^n \sin(4k-1)x = \sin 2nx (\cos 2nx + \sin 2nx) (\cos x + \sin x) \operatorname{cosec} 2x \quad \text{JO (208)}$$

1.344

$$1. \quad \sum_{k=1}^{n-1} \sin \frac{\pi k}{n} = \cot \frac{\pi}{2n} \quad \text{AD (361.19)}$$

$$2. \quad \sum_{k=1}^{n-1} \sin \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \right) \quad \text{AD (361.18)}$$

$$3. \quad \sum_{k=0}^{n-1} \cos \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right) \quad \text{AD (361.17)}$$

1.35 Sums of powers of trigonometric functions of multiple angles**1.351**

$$\begin{aligned} 1. \quad \sum_{k=1}^n \sin^2 kx &= \frac{1}{4} [(2n+1) \sin x - \sin(2n+1)x] \operatorname{cosec} x \\ &= \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2 \sin x} \end{aligned} \quad \text{AD (361.3)}$$

$$2. \quad \sum_{k=1}^n \cos^2 kx = \frac{n-1}{2} + \frac{1}{2} \cos nx \sin(n+1)x \operatorname{cosec} x$$

$$= \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2 \sin x}$$

AD (361.4)a

$$3. \quad \sum_{k=1}^n \sin^3 kx = \frac{3}{4} \sin \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} - \frac{1}{4} \sin \frac{3(n+1)x}{2} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2}$$

JO (210)

$$4. \quad \sum_{k=1}^n \cos^3 kx = \frac{3}{4} \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + \frac{1}{4} \cos \frac{3(n+1)x}{2} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2}$$

JO (211)a

$$5. \quad \sum_{k=1}^n \sin^4 kx = \frac{1}{8} [3n - 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x]$$

JO (212)

$$6. \quad \sum_{k=1}^n \cos^4 kx = \frac{1}{8} [3n + 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x]$$

JO (213)

1.352

$$1.^{11} \quad \sum_{k=1}^{n-1} k \sin kx = \frac{\sin nx}{4 \sin^2 \frac{x}{2}} - \frac{n \cos \left(\frac{2n-1}{2}x \right)}{2 \sin \frac{x}{2}}$$

AD (361.5)

$$2.^{11} \quad \sum_{k=1}^{n-1} k \cos kx = \frac{n \sin \left(\frac{2n-1}{2}x \right)}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}}$$

AD (361.6)

1.353

$$1. \quad \sum_{k=1}^{n-1} p^k \sin kx = \frac{p \sin x - p^n \sin nx + p^{n+1} \sin(n-1)x}{1 - 2p \cos x + p^2}$$

AD (361.12)a

$$2. \quad \sum_{k=1}^{n-1} p^k \sinh kx = \frac{p \sinh x - p^n \sinh nx + p^{n+1} \sinh(n-1)x}{1 - 2p \cosh x + p^2}$$

$$3. \quad \sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p \cos x - p^n \cos nx + p^{n+1} \cos(n-1)x}{1 - 2p \cos x + p^2}$$

AD (361.13)aj

$$4. \quad \sum_{k=0}^{n-1} p^k \cosh kx = \frac{1 - p \cosh x - p^n \cosh nx + p^{n+1} \cosh(n-1)x}{1 - 2p \cosh x + p^2}$$

JO (396)

1.36 Sums of products of trigonometric functions of multiple angles**1.361**

$$1. \quad \sum_{k=1}^n \sin kx \sin(k+1)x = \frac{1}{4} [(n+1) \sin 2x - \sin 2(n+1)x] \operatorname{cosec} x$$

JO (214)

$$2. \quad \sum_{k=1}^n \sin kx \sin(k+2)x = \frac{n}{2} \cos 2x - \frac{1}{2} \cos(n+3)x \sin nx \operatorname{cosec} x$$

JO (216)

$$\begin{aligned}
3. \quad 2 \sum_{k=1}^n \sin kx \cos(2k-1)y &= \sin \left(ny + \frac{n+1}{2}x \right) \sin \frac{n(x+2y)}{2} \operatorname{cosec} \frac{x+2y}{2} \\
&\quad - \sin \left(ny - \frac{n+1}{2}x \right) \sin \frac{n(2y-x)}{2} \operatorname{cosec} \frac{2y-x}{2}
\end{aligned}$$

JO (217)

1.362

$$1. \quad \sum_{k=1}^n \left(2^k \sin^2 \frac{x}{2^k} \right)^2 = \left(2^n \sin^2 \frac{x}{2^n} \right)^2 - \sin^2 x \quad \text{AD (361.15)}$$

$$2. \quad \sum_{k=1}^n \left(\frac{1}{2^k} \sec \frac{x}{2^k} \right)^2 = \operatorname{cosec}^2 x - \left(\frac{1}{2^n} \operatorname{cosec} \frac{x}{2^n} \right)^2 \quad \text{AD (361.14)}$$

1.37 Sums of tangents of multiple angles**1.371**

$$1. \quad \sum_{k=0}^n \frac{1}{2^k} \tan \frac{x}{2^k} = \frac{1}{2^n} \cot \frac{x}{2^n} - 2 \cot 2x \quad \text{AD (361.16)}$$

$$2. \quad \sum_{k=0}^n \frac{1}{2^{2k}} \tan^2 \frac{x}{2^k} = \frac{2^{2n+2} - 1}{3 \cdot 2^{2n-1}} + 4 \cot^2 2x - \frac{1}{2^{2n}} \cot^2 \frac{x}{2^n} \quad \text{AD (361.20)}$$

1.38 Sums leading to hyperbolic tangents and cotangents**1.381**

$$1. \quad \sum_{k=0}^{n-1} \frac{\tanh \left(x \frac{1}{n \sin^2 \left(\frac{2k+1}{4n} \pi \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left(\frac{2k+1}{4n} \pi \right)}} = \tanh(2nx) \quad \text{JO (402)a}$$

$$2. \quad \sum_{k=1}^{n-1} \frac{\tanh \left(x \frac{1}{n \sin^2 \left(\frac{k\pi}{2n} \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left(\frac{k\pi}{2n} \right)}} = \coth(2nx) - \frac{1}{2n} (\tanh x + \coth x) \quad \text{JO (403)}$$

$$3. \quad \sum_{k=0}^{n-1} \frac{\tanh \left(x \frac{2}{(2n+1) \sin^2 \left(\frac{2k+1}{2(2n+1)} \pi \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left(\frac{2k+1}{2(2n+1)} \pi \right)}} = \tanh (2n+1) x - \frac{\tanh x}{2n+1} \quad \text{JO (404)}$$

$$4. \quad \sum_{k=1}^n \frac{\tanh \left(x \frac{2}{(2n+1) \sin^2 \left(\frac{k\pi}{2(2n+1)} \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left(\frac{k\pi}{(2n+1)} \right)}} = \coth (2n+1) x - \frac{\coth x}{2n+1} \quad \text{JO (405)}$$

1.382

$$1. \quad \sum_{k=0}^{n-1} \frac{1}{\left(\frac{\sin^2 \left(\frac{2k+1}{4n} \pi \right)}{\sinh x} + \frac{1}{2} \tanh \left(\frac{x}{2} \right) \right)} = 2n \tanh (nx) \quad \text{JO (406)}$$

$$2. \quad \sum_{k=1}^{n-1} \frac{1}{\left(\frac{\sin^2 \left(\frac{k\pi}{2n} \right)}{\sinh x} + \frac{1}{2} \tanh \left(\frac{x}{2} \right) \right)} = 2n \coth (nx) - 2 \coth x \quad \text{JO (407)}$$

$$3. \quad \sum_{k=0}^{n-1} \frac{1}{\left(\frac{\sin^2 \left(\frac{2k+1}{2(2n+1)} \pi \right)}{\sinh x} + \frac{1}{2} \tanh \left(\frac{x}{2} \right) \right)} = (2n+1) \tanh \left(\frac{(2n+1)x}{2} \right) - \tanh \frac{x}{2} \quad \text{JO (408)}$$

$$4. \quad \sum_{k=1}^n \frac{1}{\left(\frac{\sin^2 \left(\frac{k\pi}{2n+1} \right)}{\sinh x} + \frac{1}{2} \tanh \left(\frac{x}{2} \right) \right)} = (2n+1) \coth \left(\frac{(2n+1)x}{2} \right) - \coth \frac{x}{2} \quad \text{JO (409)}$$

1.39 The representation of cosines and sines of multiples of the angle as finite products

1.391

$$1. \quad \sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n \text{ is even}] \quad \text{JO (568)}$$

$$2. \quad \cos nx = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n \text{ is even}] \quad \text{JO (569)}$$

$$3. \quad \sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n \text{ is odd}] \quad \text{JO (570)}$$

$$4. \quad \cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n \text{ is odd}] \quad \text{JO (571)a}$$

1.392

$$1. \quad \sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin \left(x + \frac{k\pi}{n} \right) \quad \text{JO (548)}$$

$$2. \quad \cos nx = 2^{n-1} \prod_{k=1}^n \sin \left(x + \frac{2k-1}{2n} \pi \right) \quad \text{JO (549)}$$

1.393

$$1. \quad \prod_{k=0}^{n-1} \cos \left(x + \frac{2k}{n} \pi \right) = \frac{1}{2^{n-1}} \cos nx \quad [n \text{ odd}]$$

$$= \frac{1}{2^{n-1}} [(-1)^{\frac{n}{2}} - \cos nx] \quad [n \text{ even}] \quad \text{JO (543)}$$

$$2.^{11} \quad \prod_{k=0}^{n-1} \sin \left(x + \frac{2k}{n} \pi \right) = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sin nx \quad [n \text{ odd}]$$

$$= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} (1 - \cos nx) \quad [n \text{ even}] \quad \text{JO (544)}$$

$$1.394 \quad \prod_{k=0}^{n-1} \left\{ x^2 - 2xy \cos \left(\alpha + \frac{2k\pi}{n} \right) + y^2 \right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n} \quad \text{JO (573)}$$

1.395

$$1. \quad \cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cos x - \cos \left(y + \frac{2k\pi}{n} \right) \right\} \quad \text{JO (573)}$$

$$2. \quad \cosh nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cosh x - \cos \left(y + \frac{2k\pi}{n} \right) \right\} \quad \text{JO (538)}$$

1.396

$$1. \quad \prod_{k=1}^{n-1} \left(x^2 - 2x \cos \frac{k\pi}{n} + 1 \right) = \frac{x^{2n} - 1}{x^2 - 1} \quad \text{KR 58 (28.1)}$$

$$2. \quad \prod_{k=1}^n \left(x^2 - 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x - 1} \quad \text{KR 58 (28.2)}$$

$$3. \quad \prod_{k=1}^n \left(x^2 + 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x + 1} \quad \text{KR 58 (28.3)}$$

$$4. \quad \prod_{k=0}^{n-1} \left(x^2 - 2x \cos \frac{(2k+1)\pi}{2n} + 1 \right) = x^{2n} + 1 \quad \text{KR 58 (28.4)}$$

1.41 The expansion of trigonometric and hyperbolic functions in power series**1.411**

$$1. \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$2. \quad \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$3. \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$4. \quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$5. \quad \tan x = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1)}{(2k)!} |B_{2k}| x^{2k-1} \quad \left[x^2 < \frac{\pi^2}{4} \right] \quad \text{FI II 523}$$

$$6.^{11} \quad \tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315} x^7 + \cdots = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1)}{(2k)!} B_{2k} x^{2k-1} \quad \left[x^2 < \frac{\pi^2}{4} \right]$$

$$7. \quad \cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} \quad [x^2 < \pi^2] \quad \text{FI II 523a}$$

$$8. \quad \coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1} \quad [x^2 < \pi^2] \quad \text{FI II 522a}$$

$$9. \quad \sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k} \quad \left[x^2 < \frac{\pi^2}{4} \right] \quad \text{CE 330a}$$

$$10. \quad \operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \quad \left[x^2 < \frac{\pi^2}{4} \right] \quad \text{CE 330}$$

$$11. \quad \operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) |B_{2k}| x^{2k-1}}{(2k)!} \quad [x^2 < \pi^2] \quad \text{CE 329a}$$

$$12. \quad \operatorname{cosech} x = \frac{1}{x} - \frac{1}{6}x + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) B_{2k}}{(2k)!} x^{2k-1} \quad [x^2 < \pi^2] \quad \text{JO (418)}$$

1.412

$$1. \quad \sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!} \quad \text{JO (452)a}$$

$$2. \quad \cos^2 x = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!} \quad \text{JO (443)}$$

$$3. \quad \sin^3 x = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k+1} - 3}{(2k+1)!} x^{2k+1} \quad \text{JO (452a)a}$$

$$4. \quad \cos^3 x = \frac{1}{4} \sum_{k=0}^{\infty} (-1)^k \frac{(3^{2k} + 3) x^{2k}}{(2k)!} \quad \text{JO (443a)}$$

1.413

$$1. \quad \sinh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{4k-2}}{(4k-1)!} \quad \text{JO (508)}$$

$$2. \quad \cosh x = \sec x + \sec x \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{4k}}{(4k)!} \quad \text{JO (507)}$$

$$3. \quad \sinh x = \sec x \sum_{k=1}^{\infty} (-1)^{[k/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!} \quad \text{JO (510)}$$

$$4. \quad \cosh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{[(k-1)/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!} \quad \text{JO (509)}$$

1.414

$$1. \quad \cos \left[n \ln \left(x + \sqrt{1+x^2} \right) \right] = 1 - \sum_{k=0}^{\infty} (-1)^k \frac{(n^2 + 0^2)(n^2 + 2^2) \cdots [n^2 + (2k)^2]}{(2k+2)!} x^{2k+2} \quad [x^2 < 1] \quad \text{AD (6456.1)}$$

$$2. \quad \sin \left[n \ln \left(x + \sqrt{1+x^2} \right) \right] = nx - n \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(n^2+1^2)(n^2+3^2)\dots[n^2+(2k-1)^2] x^{2k+1}}{(2k+1)!} \\ [x^2 < 1] \quad \text{AD (6456.2)}$$

Power series for $\ln \sin x$, $\ln \cos x$, and $\ln \tan x$ see **1.518**.

1.42 Expansion in series of simple fractions

1.421

$$1. \quad \tan \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 - x^2} \quad \text{BR* (191), AD (6495.1)}$$

$$2.^{10} \quad \tanh \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2}$$

$$3. \quad \cot \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2} = \frac{1}{\pi x} + \frac{x}{\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k(x-k)} \quad \text{AD (6495.2), JO (450a)}$$

$$4. \quad \coth \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} \quad (\text{cf. } \mathbf{1.217} \ 1)$$

$$5. \quad \tan^2 \frac{\pi x}{2} = x^2 \sum_{k=1}^{\infty} \frac{2(2k-1)^2 - x^2}{(1^2 - x^2)^2 (3^2 - x^2)^2 \dots [(2k-1)^2 - x^2]^2} \quad \text{JO (450)}$$

1.422

$$1. \quad \sec \frac{\pi x}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k-1}{(2k-1)^2 - x^2} \quad \text{AD (6495.3)a}$$

$$2. \quad \sec^2 \frac{\pi x}{2} = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k-1-x)^2} + \frac{1}{(2k-1+x)^2} \right\} \quad \text{JO (451)a}$$

$$3. \quad \operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2} \quad (\text{see also } \mathbf{1.217} \ 2) \quad \text{AD (6495.4)a}$$

$$4. \quad \operatorname{cosec}^2 \pi x = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^2} = \frac{1}{\pi^2 x^2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{x^2 + k^2}{(x^2 - k^2)^2} \quad \text{JO (446)}$$

$$5. \quad \frac{1+x \operatorname{cosec} x}{2x^2} = \frac{1}{x^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(x^2 - k^2 \pi^2)} \quad \text{JO (449)}$$

$$6. \quad \operatorname{cosec} \pi x = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x^2 - k^2} \quad \text{JO (450b)}$$

$$\mathbf{1.423} \quad \frac{\pi^2}{4m^2} \operatorname{cosec}^2 \frac{\pi}{m} + \frac{\pi}{4m} \cot \frac{\pi}{m} - \frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{(1 - k^2 m^2)^2} \quad \text{JO (477)}$$

1.43 Representation in the form of an infinite product

1.431

$$1. \quad \sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2} \right) \quad \text{EU}$$

$$2. \quad \sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2 \pi^2} \right) \quad \text{EU}$$

$$3. \quad \cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2 \pi^2} \right) \quad \text{EU}$$

$$4. \quad \cosh x = \prod_{k=0}^{\infty} \left(1 + \frac{4x^2}{(2k+1)^2 \pi^2} \right) \quad \text{EU}$$

1.432

$$1.^{11} \quad \cos x - \cos y = 2 \left(1 - \frac{x^2}{y^2} \right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(2k\pi + y)^2} \right) \left(1 - \frac{x^2}{(2k\pi - y)^2} \right) \quad \text{AD (653.2)}$$

$$2. \quad \cosh x - \cos y = 2 \left(1 + \frac{x^2}{y^2} \right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{(2k\pi + y)^2} \right) \left(1 + \frac{x^2}{(2k\pi - y)^2} \right) \quad \text{AD (653.1)}$$

$$1.433 \quad \cos \frac{\pi x}{4} - \sin \frac{\pi x}{4} = \prod_{k=1}^{\infty} \left[1 + \frac{(-1)^k x}{2k-1} \right] \quad \text{BR* 189}$$

$$1.434 \quad \cos^2 x = \frac{1}{4} (\pi + 2x)^2 \prod_{k=1}^{\infty} \left[1 - \left(\frac{\pi + 2x}{2k\pi} \right)^2 \right]^2 \quad \text{MO 216}$$

$$1.435 \quad \frac{\sin \pi(x+a)}{\sin \pi a} = \frac{x+a}{a} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k-a} \right) \left(1 + \frac{x}{k+a} \right) \quad \text{MO 216}$$

$$1.436 \quad 1 - \frac{\sin^2 \pi x}{\sin^2 \pi a} = \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{x}{k-a} \right)^2 \right] \quad \text{MO 216}$$

$$1.437 \quad \frac{\sin 3x}{\sin x} = - \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{2x}{x+k\pi} \right)^2 \right] \quad \text{MO 216}$$

$$1.438 \quad \frac{\cosh x - \cos a}{1 - \cos a} = \prod_{k=-\infty}^{\infty} \left[1 + \left(\frac{x}{2k\pi + a} \right)^2 \right] \quad \text{MO 216}$$

1.439

$$1. \quad \sin x = x \prod_{k=1}^{\infty} \cos \frac{x}{2^k} \quad [|x| < 1] \quad \text{AD (615), MO 216}$$

$$2. \quad \frac{\sin x}{x} = \prod_{k=1}^{\infty} \left[1 - \frac{4}{3} \sin^2 \left(\frac{x}{3^k} \right) \right] \quad \text{MO 216}$$

1.44–1.45 Trigonometric (Fourier) series

1.441

1. $\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \quad [0 < x < 2\pi] \quad \text{FI III 539}$
2. $\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\frac{1}{2} \ln [2(1 - \cos x)] \quad [0 < x < 2\pi] \quad \text{FI III 530a, AD (6814)}$
3. $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2} \quad [-\pi < x < \pi] \quad \text{FI III 542}$
4. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln \left(2 \cos \frac{x}{2} \right) \quad [-\pi < x < \pi] \quad \text{FI III 550}$

1.442

- 1.¹¹ $\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\pi}{4} \operatorname{sign} x \quad [-\pi < x < \pi] \quad \text{FI III 541}$
2. $\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} = \frac{1}{2} \ln \cot \frac{x}{2} \quad [0 < x < \pi]$
BR* 168, JO (266), GI III(195)
3. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \quad \left[-\frac{\pi}{2} < x < \frac{\pi}{2} \right] \quad \text{BR* 168, JO (268)a}$
- 4.¹⁰ $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k-1)x}{2k-1} = \frac{\pi}{4} \quad \left[-\frac{\pi}{2} < x < \frac{\pi}{2} \right]$
 $= -\frac{\pi}{4} \quad \left[\frac{\pi}{2} < x < \frac{3\pi}{2} \right]$
BR* 168, JO (269)

1.443

- 1.⁸ $\sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k} \rho^k$
 $= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \left(\frac{x}{2} \right)$
 $\left[0 \leq x \leq 2, \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor \right] \quad \text{CE 340, GE 71}$
2. $\sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1} \rho^k$
 $= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1} \left(\frac{x}{2} \right)$
 $\left[0 < x < 1; \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor \right] \quad \text{CE 340}$

$$3. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad [0 \leq x \leq 2\pi] \quad \text{FI III 547}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \quad [-\pi \leq x \leq \pi] \quad \text{FI III 544}$$

$$5. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} \quad [0 \leq x \leq 2\pi]$$

$$6. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48} \quad [0 \leq x \leq 2\pi] \quad \text{AD (6617)}$$

$$7. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^5} = \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240} \quad [0 \leq x \leq 2\pi] \quad \text{AD (6818)}$$

1.444

$$1. \quad \sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} = \sin 2x - (\pi - 2x) \sin^2 x - \sin x \cos x \ln(4 \sin^2 x) \quad [0 \leq x \leq \pi] \quad \text{BR* 168, GI III (190)}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} = \cos 2x - \left(\frac{\pi}{2} - x\right) \sin 2x + \sin^2 x \ln(4 \sin^2 x) \quad [0 \leq x \leq \pi] \quad \text{BR* 168}$$

$$3. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} = \sin x - \frac{x}{2} (1 + \cos x) - \sin x \ln \left| 2 \cos \frac{x}{2} \right| \quad \text{MO 213}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} = \cos x - \frac{x}{2} \sin x - (1 + \cos x) \ln \left| 2 \cos \frac{x}{2} \right| \quad \text{MO 213}$$

$$5. \quad \sum_{k=0}^{\infty} (-1)^k \frac{\sin(2k+1)x}{(2k+1)^2} = \frac{\pi}{4} x \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right] \\ = \frac{\pi}{4} (\pi - x) \quad \left[\frac{\pi}{2} \leq x \leq \frac{3}{2}\pi \right] \quad \text{MO 213}$$

$$6.^6 \quad \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\pi}{4} \left(\frac{\pi}{2} - |x| \right) \quad [-\pi \leq x \leq \pi] \quad \text{FI III 546}$$

$$7. \quad \sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} = \frac{1}{2} - \frac{\pi}{4} \sin x \quad \left[0 \leq x \leq \frac{\pi}{2} \right] \quad \text{JO (591)}$$

1.445

$$1. \quad \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi}{2} \frac{\sinh \alpha(\pi - x)}{\sinh \alpha\pi} \quad [0 < x < 2\pi] \quad \text{BR* 157, JO (411)}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh \alpha(\pi - x)}{\sinh \alpha\pi} - \frac{1}{2\alpha^2} \quad [0 \leq x \leq 2\pi] \quad \text{BR* 257, JO (410)}$$

3.
$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh \alpha x}{\sinh \alpha \pi} - \frac{1}{2\alpha^2} \quad [-\pi \leq x \leq \pi] \quad \text{FI III 546}$$
4.
$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi}{2} \frac{\sinh \alpha x}{\sinh \alpha \pi} \quad [-\pi < x < \pi] \quad \text{FI III, 546}$$
5.
$$\sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin \{ \alpha [(2m+1)\pi - x] \}}{2 \sin \alpha \pi} \quad \left[\text{if } x = 2m\pi, \text{ then } \sum \cdots = 0 \right]$$

$$[2m\pi < x < (2m+2)\pi, \quad \alpha \text{ not an integer}] \quad \text{MO 213}$$
6.
$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos [\alpha \{ (2m+1)\pi - x \}]}{\alpha \sin \alpha \pi}$$

$$[2m\pi \leq x \leq (2m+2)\pi, \quad \alpha \text{ not an integer}] \quad \text{MO 213}$$
7.
$$\sum_{k=1}^{\infty} (-1)^k \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin [\alpha (2m\pi - x)]}{2 \sin \alpha \pi} \quad \left[\text{if } x = (2m+1)\pi, \text{ then } \sum \cdots = 0 \right],$$

$$[(2m-1)\pi < x < (2m+1)\pi, \alpha \text{ not an integer}] \quad \text{FI III 545a}$$
8.
$$\sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos [\alpha (2m\pi - x)]}{\alpha \sin \alpha \pi}$$

$$[(2m-1)\pi \leq x \leq (2m+1)\pi, \alpha \text{ not an integer}] \quad \text{FI III 545a}$$
- 9.*
$$\sum_{n=-\infty}^{\infty} \frac{e^{in\alpha}}{(n-\beta)^2 + \gamma^2} = \frac{\pi}{\gamma} \frac{e^{i\beta(\alpha-2\pi)} \sinh(\gamma\alpha) + e^{i\beta\alpha} \sinh[\gamma(2\pi-\alpha)]}{\cosh(2\pi\gamma) - \cos(2\pi\beta)}$$

$$[0 \leq \alpha \leq 2\pi]$$
- 1.446
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8} \cos^2 x - \frac{1}{3} \cos x$$

$$\left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right] \quad \text{BR* 256, GI III (189)}$$
- 1.447
1.
$$\sum_{k=1}^{\infty} p^k \sin kx = \frac{p \sin x}{1 - 2p \cos x + p^2}$$

$$[|p| < 1] \quad \text{FI II 559}$$
2.
$$\sum_{k=0}^{\infty} p^k \cos kx = \frac{1 - p \cos x}{1 - 2p \cos x + p^2}$$

$$[|p| < 1] \quad \text{FI II 559}$$
3.
$$1 + 2 \sum_{k=1}^{\infty} p^k \cos kx = \frac{1 - p^2}{1 - 2p \cos x + p^2}$$

$$[|p| < 1] \quad \text{FI II 559a, MO 213}$$

1.448

1.
$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \arctan \frac{p \sin x}{1 - p \cos x}$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{FI II 559}$$
2.
$$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln(1 - 2p \cos x + p^2)$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{FI II 559}$$
3.
$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \sin x}{1 - p^2}$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{JO (594)}$$
4.
$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2}$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{JO (259)}$$
5.
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \sin x + p^2}{1 - 2p \sin x + p^2}$$

$$[0 < x < \pi, \quad p^2 \leq 1] \quad \text{JO (261)}$$
6.
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \cos x}{1 - p^2}$$

$$[0 < x < \pi, \quad p^2 \leq 1] \quad \text{JO (597)}$$

1.449

1.
$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p \cos x} \sin(p \sin x)$$

$$[p^2 \leq 1] \quad \text{JO (486)}$$
 2.
$$\sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p \cos x} \cos(p \sin x)$$

$$[p^2 \leq 1] \quad \text{JO (485)}$$
- Let $S(x) = -\frac{1}{x} \cos x + \frac{1}{x}$ and $C(x) = \frac{1}{x} \sin x$.
- 3.*
$$\sum_{n=1}^{\infty} \frac{n}{n^2 - a^2} S(nx) = \frac{\pi}{2} [C(ax) - \cot(\pi a) S(ax)]$$

$$[0 < x < 2\pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$
 - 4.*
$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} C(nx) = \frac{1}{2a^2} - \frac{\pi}{2a} [S(ax) - \cot(\pi a) C(ax)]$$

$$[0 \leq x \leq 2\pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$
 - 5.*
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \operatorname{cosec}(\pi a) S(ax)$$

$$[-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$6.^* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - a^2} C(nx) = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cosec}(\pi a) C(ax) \quad [-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$7.^* \quad \sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)^2 - a^2} S(nx) = \frac{\pi}{4} \left[C(ax) + \tan\left(\frac{\pi a}{2}\right) S(ax) \right] \\ [0 < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$8.^* \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - a^2} C(nx) = -\frac{\pi}{4a} \left[S(ax) - \tan\left(\frac{\pi a}{2}\right) C(ax) \right] \\ [0 \leq x \leq \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$9.^* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2 - a^2} S(nx) = \frac{\pi}{4a} \sec\left(\frac{\pi a}{2}\right) S(ax) \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \dots\right]$$

$$10.^* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2 - a^2} C(nx) = \frac{\pi}{4} \sec\left(\frac{\pi a}{2}\right) C(ax) \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \dots\right]$$

Fourier expansions of hyperbolic functions

1.451

$$1. \quad \sinh x = \cos x \sum_{k=0}^{\infty} \frac{(1^2 + 0^2)(1^2 + 2^2) \dots [1^2 + (2k)^2]}{(2k+1)!} \sin^{2k+1} x \quad \text{JO (504)}$$

$$2. \quad \cosh x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{(1^2 + 1^2)(1^2 + 3^2) \dots [1^2 + (2k-1)^2]}{(2k)!} \sin^{2k} x \quad \text{JO (503)}$$

1.452

$$1. \quad \sinh(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \cos(2k+1)\theta}{(2k+1)!} \\ [x^2 < 1] \quad \text{JO (391)}$$

$$2. \quad \cosh(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k} \cos 2k\theta}{(2k)!} \\ [x^2 < 1] \quad \text{JO (390)}$$

$$3. \quad \sinh(x \cos \theta) = \operatorname{cosec}(x \sin \theta) \sum_{k=1}^{\infty} \frac{x^{2k} \sin 2k\theta}{(2k)!} \\ [x^2 < 1, \quad x \sin \theta \neq 0] \quad \text{JO (393)}$$

$$4. \quad \cosh(x \cos \theta) = \operatorname{cosec}(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \sin(2k+1)\theta}{(2k+1)!} \\ [x^2 < 1, \quad x \sin \theta \neq 0] \quad \text{JO (392)}$$

1.46 Series of products of exponential and trigonometric functions

1.461

$$1. \quad \sum_{k=0}^{\infty} e^{-kt} \sin kx = \frac{1}{2} \frac{\sin x}{\cosh t - \cos x} \quad [t > 0] \quad \text{MO 213}$$

$$2. \quad 1 + 2 \sum_{k=1}^{\infty} e^{-kt} \cos kx = \frac{\sinh t}{\cosh t - \cos x} \quad [t > 0] \quad \text{MO 213}$$

$$1.462^9 \quad \sum_{k=1}^{\infty} \frac{\sin kx \sin ky}{k} e^{-2k|t|} = \frac{1}{4} \ln \left[\frac{\sin^2 \frac{x+y}{2} + \sinh^2 t}{\sin^2 \frac{x-y}{2} + \sinh^2 t} \right] \quad \text{MO 214}$$

1.463

$$1. \quad e^{x \cos \varphi} \cos (x \sin \varphi) = \sum_{n=0}^{\infty} \frac{x^n \cos n\varphi}{n!} \quad [x^2 < 1] \quad \text{AD (6476.1)}$$

$$2. \quad e^{x \cos \varphi} \sin (x \sin \varphi) = \sum_{n=1}^{\infty} \frac{x^n \sin n\varphi}{n!} \quad [x^2 < 1] \quad \text{AD (6476.2)}$$

1.47 Series of hyperbolic functions

1.471

$$1. \quad \sum_{k=1}^{\infty} \frac{\sinh kx}{k!} = e^{\cosh x} \sinh (\sinh x). \quad \text{JO (395)}$$

$$2. \quad \sum_{k=0}^{\infty} \frac{\cosh kx}{k!} = e^{\cosh x} \cosh (\sinh x). \quad \text{JO (394)}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left[\frac{1}{x} \tanh \frac{(2m+1)\pi x}{2} + x \tanh \frac{(2m+1)\pi}{2x} \right] = \frac{\pi^3}{16}$$

1.472

$$1. \quad \sum_{k=1}^{\infty} p^k \sinh kx = \frac{p \sinh x}{1 - 2p \cosh x + p^2} \quad [p^2 < 1] \quad \text{JO (396)}$$

$$2. \quad \sum_{k=0}^{\infty} p^k \cosh kx = \frac{1 - p \cosh x}{1 - 2p \cosh x + p^2} \quad [p^2 < 1] \quad \text{JO (397)a}$$

1.48 Lobachevskiy's "Angle of Parallelism" $\Pi(x)$

1.480 Definition.

$$1. \quad \Pi(x) = 2 \operatorname{arccot} e^x = 2 \arctan e^{-x} \quad [x \geq 0] \quad \text{LO III 297, LOI 120}$$

$$2. \quad \Pi(x) = \pi - \Pi(-x) \quad [x < 0] \quad \text{LO III 183, LOI 193}$$

1.481 Functional relations

$$1. \quad \sin \Pi(x) = \frac{1}{\cosh x} \quad \text{LO III 297}$$

$$2. \quad \cos \Pi(x) = \tanh x \quad \text{LO III 297}$$

$$3. \quad \tan \Pi(x) = \frac{1}{\sinh x} \quad \text{LO III 297}$$

$$4. \quad \cot \Pi(x) = \sinh x \quad \text{LO III 297}$$

$$5. \quad \sin \Pi(x + y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)} \quad \text{LO III 297}$$

$$6. \quad \cos \Pi(x + y) = \frac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)} \quad \text{LO III 183}$$

1.482 Connection with the Gudermannian.

$$\text{gd}(-x) = \Pi(x) - \frac{\pi}{2}$$

(Definite) integral of the angle of parallelism: cf. **4.581** and **4.561**.

1.49 The hyperbolic amplitude (the Gudermannian) $\text{gd } x$ **1.490** Definition.

$$1. \quad \text{gd } x = \int_0^x \frac{dt}{\cosh t} = 2 \arctan e^x - \frac{\pi}{2} \quad \text{JA}$$

$$2. \quad x = \int_0^{\text{gd } x} \frac{dt}{\cos t} = \ln \tan \left(\frac{\text{gd } x}{2} + \frac{\pi}{4} \right) \quad \text{JA}$$

1.491 Functional relations.

$$1. \quad \cosh x = \sec(\text{gd } x) \quad \text{AD (343.1), JA}$$

$$2. \quad \sinh x = \tan(\text{gd } x) \quad \text{AD (343.2), JA}$$

$$3. \quad e^x = \sec(\text{gd } x) + \tan(\text{gd } x) = \tan \left(\frac{\pi}{4} + \frac{\text{gd } x}{2} \right) = \frac{1 + \sin(\text{gd } x)}{\cos(\text{gd } x)} \quad \text{AD (343.5), JA}$$

$$4. \quad \tanh x = \sin(\text{gd } x) \quad \text{AD (343.3), JA}$$

$$5. \quad \tanh \frac{x}{2} = \tan \left(\frac{1}{2} \text{gd } x \right) \quad \text{AD (343.4), JA}$$

$$6. \quad \arctan(\tanh x) = \frac{1}{2} \text{gd } 2x \quad \text{AD (343.6a)}$$

1.492 If $\gamma = \text{gd } x$, then $ix = \text{gd } i\gamma$ JA**1.493** Series expansion.

$$1. \quad \frac{\text{gd } x}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tanh^{2k+1} \frac{x}{2} \quad \text{JA}$$

2. $\frac{x}{2} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \tan^{2k+1} \left(\frac{1}{2} \operatorname{gd} x \right)$ JA
3. $\operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \dots$ JA
4. $x = \operatorname{gd} x + \frac{(\operatorname{gd} x)^3}{6} + \frac{(\operatorname{gd} x)^5}{24} + \frac{61(\operatorname{gd} x)^7}{5040} + \dots$ $\left[\operatorname{gd} x < \frac{\pi}{2} \right]$ JA

1.5 The Logarithm

1.51 Series representation

1.511 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$
 $[-1 < x \leq 1]$

1.512

1. $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$
 $[0 < x \leq 2]$
2. $\ln x = 2 \left[\frac{x-1}{x+1} + \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 + \dots \right] = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{x-1}{x+1} \right)^{2k-1}$
 $[0 < x]$
3. $\ln x = \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x-1}{x} \right)^k$
 $[x \geq \frac{1}{2}]$ AD (644.6)
- 4.* $\ln x = \lim_{\epsilon \rightarrow 0} \left(\frac{x^\epsilon - 1}{\epsilon} \right)$

1.513

1. $\ln \frac{1+x}{1-x} = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k-1}$ $[x^2 < 1]$ FI II 421
2. $\ln \frac{x+1}{x-1} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)x^{2k-1}}$ $[x^2 > 1]$ AD (644.9)
3. $\ln \frac{x}{x-1} = \sum_{k=1}^{\infty} \frac{1}{kx^k}$ $[x \leq -1 \text{ or } x > 1]$ JO (88a)
4. $\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k}$ $[-1 \leq x < 1]$ JO (88b)
5. $\frac{1-x}{x} \ln \frac{1}{1-x} = 1 - \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$ $[-1 \leq x < 1]$ JO (102)

$$6. \quad \frac{1}{1-x} \ln \frac{1}{1-x} = \sum_{k=1}^{\infty} x^k \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1] \quad \text{JO (88e)}$$

$$7. \quad \frac{(1-x)^2}{2x^3} \ln \frac{1}{1-x} = \frac{1}{2x^2} - \frac{3}{4x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k(k+1)(k+2)} \quad [-1 \leq x < 1] \quad \text{AD (6445.1)}$$

$$\mathbf{1.514} \quad \ln(1 - 2x \cos \varphi + x^2) = -2 \sum_{k=1}^{\infty} \frac{\cos k\varphi}{k} x^k; \quad \ln(x + \sqrt{1+x^2}) = \operatorname{arcsinh} x$$

(see **1.631**, **1.641**, **1.642**, **1.646**) $[x^2 \leq 1, \quad x \cos \varphi \neq 1]$ MO 98, FI II 485

1.515

$$1.^{11} \quad \ln(1 + \sqrt{1+x^2}) = \ln 2 + \frac{1 \cdot 1}{2 \cdot 2} x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} x^4 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6} x^6 - \dots$$

$$= \ln 2 - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k} (k!)^2} x^{2k} \quad [x^2 \leq 1] \quad \text{JO (91)}$$

$$2. \quad \ln(1 + \sqrt{1+x^2}) = \ln x + \frac{1}{x} - \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \dots$$

$$= \ln x + \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k-1} \cdot k! (k-1)! (2k+1) x^{2k+1}} \quad [x^2 \geq 1] \quad \text{AD (644.4)}$$

$$3. \quad \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) = x - \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1} (k-1)! k!}{(2k+1)!} x^{2k+1}$$

$$[x^2 \leq 1] \quad \text{JO (93)}$$

$$4. \quad \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} (k!)^2}{(2k+1)!} x^{2k+1} \quad [x^2 \leq 1] \quad \text{JO (94)}$$

1.516

$$1. \quad \frac{1}{2} \{\ln(1 \pm x)\}^2 = \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1} x^{k+1}}{k+1} \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1] \quad \text{JO (86), JO (85)}$$

$$2. \quad \frac{1}{6} \{\ln(1+x)\}^3 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k+2}}{k+2} \sum_{n=1}^k \frac{1}{n+1} \sum_{m=1}^n \frac{1}{m} \quad [x^2 < 1] \quad \text{AD (644.14)}$$

$$3. \quad -\ln(1+x) \cdot \ln(1-x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \sum_{n=1}^{2k-1} \frac{(-1)^{n+1}}{n} \quad [x^2 < 1] \quad \text{JO (87)}$$

$$4. \quad \frac{1}{4x} \left\{ \frac{1+x}{\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + 2 \ln(1-x) \right\} = \frac{1}{2x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{(2k-1)2k(2k+1)}$$

$$[0 < x < 1] \quad \text{AD (6445.2)}$$

1.517

$$1.^6 \quad \frac{1}{2x} \left\{ 1 - \ln(1+x) - \frac{1-x}{\sqrt{x}} \arctan \sqrt{x} \right\} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k-1}}{(2k-1)2k(2k+1)} \quad [0 < x \leq 1] \quad \text{AD (6445.3)}$$

$$2. \quad \frac{1}{2} \arctan x \ln \frac{1+x}{1-x} = \sum_{k=1}^{\infty} \frac{x^{4k-2}}{2k-1} \sum_{n=1}^{2k-1} \frac{(-1)^{n-1}}{2n-1} \quad [x^2 < 1] \quad \text{BR* 163}$$

$$3. \quad \frac{1}{2} \arctan x \ln(1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2k+1} \sum_{n=1}^{2k} \frac{1}{n} \quad [x^2 \geq 1] \quad \text{AD (6455.3)}$$

1.518

$$1. \quad \begin{aligned} \ln \sin x &= \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \dots \\ &= \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} B_{2k} x^{2k}}{k(2k)!} \end{aligned} \quad [0 < x < \pi] \quad \text{AD (643.1)a}$$

$$2.^3 \quad \begin{aligned} \ln \cos x &= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \dots \\ &= -\sum_{k=1}^{\infty} \frac{2^{2k-1} (2^{2k} - 1) |B_{2k}|}{k(2k)!} x^{2k} = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin^{2k} x}{k} \end{aligned} \quad \left[x^2 < \frac{\pi^2}{4} \right] \quad \text{FI II 524}$$

$$3. \quad \begin{aligned} \ln \tan x &= \ln x + \frac{x^2}{3} + \frac{7}{90} x^4 + \frac{62}{2835} x^6 + \frac{127}{18,900} x^8 + \dots \\ &= \ln x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k-1} - 1) 2^{2k} B_{2k} x^{2k}}{k(2k)!} \end{aligned} \quad \left[0 < x < \frac{\pi}{2} \right] \quad \text{AD (643.3)a}$$

1.52 Series of logarithms (cf. 1.431)**1.521**

$$1. \quad \sum_{k=1}^{\infty} \ln \left(1 - \frac{4x^2}{(2k-1)^2 \pi^2} \right) = \ln \cos x \quad \left[-\frac{\pi}{2} < x < \frac{\pi}{2} \right]$$

$$2. \quad \sum_{k=1}^{\infty} \ln \left(1 - \frac{x^2}{k^2 \pi^2} \right) = \ln \sin x - \ln x \quad [0 < x < \pi]$$

1.6 The Inverse Trigonometric and Hyperbolic Functions

1.61 The domain of definition

The principal values of the inverse trigonometric functions are defined by the inequalities:

1. $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}; \quad 0 \leq \arccos x \leq \pi \quad [-1 \leq x \leq 1] \quad \text{FI II 553}$
2. $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}; \quad 0 < \operatorname{arccot} x < \pi \quad [-\infty < x < +\infty] \quad \text{FI II 552}$

1.62–1.63 Functional relations

1.621 The relationship between the inverse and the direct trigonometric functions.

1. $\arcsin(\sin x) = x - 2n\pi \quad \left[2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi + \frac{\pi}{2}\right]$
 $\quad \quad \quad = -x + (2n+1)\pi \quad \left[(2n+1)\pi - \frac{\pi}{2} \leq x \leq (2n+1)\pi + \frac{\pi}{2}\right]$
2. $\arccos(\cos x) = x - 2n\pi \quad [2n\pi \leq x \leq (2n+1)\pi]$
 $\quad \quad \quad = -x + 2(n+1)\pi \quad [(2n+1)\pi \leq x \leq 2(n+1)\pi]$
3. $\arctan(\tan x) = x - n\pi \quad \left[n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}\right]$
4. $\operatorname{arccot}(\cot x) = x - n\pi \quad [n\pi < x < (n+1)\pi]$

1.622 The relationship between the inverse trigonometric functions, the inverse hyperbolic functions, and the logarithm.

1. $\arcsin z = \frac{1}{i} \ln \left(iz + \sqrt{1 - z^2} \right) = \frac{1}{i} \operatorname{arcsinh}(iz)$
2. $\arccos z = \frac{1}{i} \ln \left(z + \sqrt{z^2 - 1} \right) = \frac{1}{i} \operatorname{arccosh} z$
3. $\arctan z = \frac{1}{2i} \ln \frac{1 + iz}{1 - iz} = \frac{1}{i} \operatorname{arctanh}(iz)$
4. $\operatorname{arccot} z = \frac{1}{2i} \ln \frac{iz - 1}{iz + 1} = i \operatorname{arccoth}(iz)$
5. $\operatorname{arcsinh} z = \ln \left(z + \sqrt{z^2 + 1} \right) = \frac{1}{i} \arcsin(iz)$
6. $\operatorname{arccosh} z = \ln \left(z + \sqrt{z^2 - 1} \right) = i \arccos z$
7. $\operatorname{arctanh} z = \frac{1}{2} \ln \frac{1 + z}{1 - z} = \frac{1}{i} \arctan(iz)$
8. $\operatorname{arccoth} z = \frac{1}{2} \ln \frac{z + 1}{z - 1} = \frac{1}{i} \operatorname{arccot}(-iz)$

Relations between different inverse trigonometric functions

1.623

1. $\arcsin x + \arccos x = \frac{\pi}{2}$ NV 43
2. $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$ NV 43

1.624

1. $\arcsin x = \arccos \sqrt{1-x^2} \quad [0 \leq x \leq 1]$ NV 47 (5)
 $= -\arccos \sqrt{1-x^2} \quad [-1 \leq x \leq 0]$ NV 46 (2)
2. $\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}} \quad [x^2 < 1]$
3. $\arcsin x = \operatorname{arccot} \frac{\sqrt{1-x^2}}{x} \quad [0 < x \leq 1]$
 $= \operatorname{arccot} \frac{x}{\sqrt{1-x^2}} - \pi \quad [-1 \leq x < 0]$ NV 49 (10)
4. $\arccos x = \arcsin \sqrt{1-x^2} \quad [0 \leq x \leq 1]$
 $= \pi - \arcsin \sqrt{1-x^2} \quad [-1 \leq x \leq 0]$ NV 48 (6)
5. $\arccos x = \arctan \frac{\sqrt{1-x^2}}{x} \quad [0 < x \leq 1]$
 $= \pi + \arctan \frac{\sqrt{1-x^2}}{x} \quad [-1 \leq x < 0]$ NV 48 (8)
6. $\arccos x = \operatorname{arccot} \frac{x}{\sqrt{1-x^2}} \quad [-1 \leq x < 1]$ NV 46 (4)
7. $\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}$ NV 6 (3)
8. $\arctan x = \arccos \frac{1}{\sqrt{1+x^2}} \quad [x \geq 0]$
 $= -\arccos \frac{1}{\sqrt{1+x^2}} \quad [x \leq 0]$ NV 48 (7)
9. $\arctan x = \operatorname{arccot} \frac{1}{x} \quad [x > 0]$
 $= -\operatorname{arccot} \frac{1}{x} - \pi \quad [x < 0]$ NV 49 (9)
- 10.¹¹ $\operatorname{arccot} x = \arcsin \frac{1}{\sqrt{1+x^2}} \quad [x > 0]$
 $= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} \quad [x < 0]$ NV 49 (11)
11. $\operatorname{arccot} x = \arccos \frac{x}{\sqrt{1+x^2}}$ NV 46 (4)

$$\begin{aligned}
 12. \quad \operatorname{arccot} x &= \arctan \frac{1}{x} & [x > 0] \\
 &= \pi + \arctan \frac{1}{x} & [x < 0]
 \end{aligned}
 \quad \text{NV 49 (12)}$$

1.625

$$\begin{aligned}
 1. \quad \arcsin x + \arcsin y &= \arcsin \left(x\sqrt{1-y^2} + y\sqrt{1-x^2} \right) & [xy \leq 0 \text{ or } x^2 + y^2 \leq 1] \\
 &= \pi - \arcsin \left(x\sqrt{1-y^2} + y\sqrt{1-x^2} \right) & [x > 0, \quad y > 0 \text{ and } x^2 + y^2 > 1] \\
 &= -\pi - \arcsin \left(x\sqrt{1-y^2} + y\sqrt{1-x^2} \right) & [x < 0, \quad y < 0 \text{ and } x^2 + y^2 > 1]
 \end{aligned}
 \quad \text{NV 54(1), GI I (880)}$$

$$\begin{aligned}
 2. \quad \arcsin x + \arcsin y &= \arccos \left(\sqrt{1-x^2}\sqrt{1-y^2} - xy \right) & [x \geq 0, \quad y \geq 0] \\
 &= -\arccos \left(\sqrt{1-x^2}\sqrt{1-y^2} - xy \right) & [x < 0, \quad y < 0]
 \end{aligned}
 \quad \text{NV 55}$$

$$\begin{aligned}
 3. \quad \arcsin x + \arcsin y &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} & [xy \leq 0 \text{ or } x^2 + y^2 < 1] \\
 &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} + \pi & [x > 0, \quad y > 0 \text{ and } x^2 + y^2 > 1] \\
 &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} - \pi & [x < 0, \quad y < 0 \text{ and } x^2 + y^2 > 1]
 \end{aligned}
 \quad \text{NV 56}$$

$$\begin{aligned}
 4. \quad \arcsin x - \arcsin y &= \arcsin \left(x\sqrt{1-y^2} - y\sqrt{1-x^2} \right) & [xy \geq 0 \text{ or } x^2 + y^2 \leq 1] \\
 &= \pi - \arcsin \left(x\sqrt{1-y^2} - y\sqrt{1-x^2} \right) & [x > 0, \quad y < 0 \text{ and } x^2 + y^2 > 1] \\
 &= -\pi - \arcsin \left(x\sqrt{1-y^2} - y\sqrt{1-x^2} \right) & [x < 0, \quad y > 0 \text{ and } x^2 + y^2 > 1]
 \end{aligned}
 \quad \text{NV 55(2)}$$

$$\begin{aligned}
 5. \quad \arcsin x - \arcsin y &= \arccos \left(x\sqrt{1-x^2}\sqrt{1-y^2} + xy \right) & [xy > y] \\
 &= -\arccos \left(\sqrt{1-x^2}\sqrt{1-y^2} + xy \right) & [x < y]
 \end{aligned}
 \quad \text{NV 56}$$

$$\begin{aligned}
 6. \quad \arccos x + \arccos y &= \arccos \left(xy - \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x + y \geq 0] \\
 &= 2\pi - \arccos \left(xy - \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x + y < 0]
 \end{aligned}
 \quad \text{NV 57 (3)}$$

$$\begin{aligned}
 7.^{11} \quad \arccos x - \arccos y &= -\arccos \left(xy + \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x \geq y] \\
 &= \arccos \left(xy + \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x < y]
 \end{aligned}
 \quad \text{NV 57 (4)}$$

$$\begin{aligned}
8. \quad \arctan x + \arctan y &= \arctan \frac{x+y}{1-xy} & [xy < 1] \\
&= \pi + \arctan \frac{x+y}{1-xy} & [x > 0, \quad xy > 1] \\
&= -\pi + \arctan \frac{x+y}{1-xy} & [x < 0, \quad xy > 1]
\end{aligned}$$

NV 59(5), GI I (879)

$$\begin{aligned}
9. \quad \arctan x - \arctan y &= \arctan \frac{x-y}{1+xy} & [xy > -1] \\
&= \pi + \arctan \frac{x-y}{1+xy} & [x > 0, \quad xy < -1] \\
&= -\pi + \arctan \frac{x-y}{1+xy} & [x < 0, \quad xy < -1]
\end{aligned}$$

NV 59(6)

1.626

$$\begin{aligned}
1. \quad 2 \arcsin x &= \arcsin (2x\sqrt{1-x^2}) & \left[|x| \leq \frac{1}{\sqrt{2}} \right] \\
&= \pi - \arcsin (2x\sqrt{1-x^2}) & \left[\frac{1}{\sqrt{2}} < x \leq 1 \right] \\
&= -\pi - \arcsin (2x\sqrt{1-x^2}) & \left[-1 \leq x < -\frac{1}{\sqrt{2}} \right]
\end{aligned}$$

NV 61 (7)

$$\begin{aligned}
2. \quad 2 \arccos x &= \arccos (2x^2 - 1) & [0 \leq x \leq 1] \\
&= 2\pi - \arccos (2x^2 - 1) & [-1 \leq x < 0]
\end{aligned}$$

NV 61 (8)

$$\begin{aligned}
3. \quad 2 \arctan x &= \arctan \frac{2x}{1-x^2} & [|x| < 1] \\
&= \arctan \frac{2x}{1-x^2} + \pi & [x > 1] \\
&= \arctan \frac{2x}{1-x^2} - \pi & [x < -1]
\end{aligned}$$

NV 61 (9)

1.627

$$\begin{aligned}
1. \quad \arctan x + \arctan \frac{1}{x} &= \frac{\pi}{2} & [x > 0] \\
&= -\frac{\pi}{2} & [x < 0]
\end{aligned}$$

GI I (878)

$$\begin{aligned}
2. \quad \arctan x + \arctan \frac{1-x}{1+x} &= \frac{\pi}{4} & [x > -1] \\
&= -\frac{3}{4}\pi & [x < -1]
\end{aligned}$$

NV 62, GI I (881)

1.628

$$\begin{aligned}
 1. \quad \arcsin \frac{2x}{1+x^2} &= -\pi - 2 \arctan x & [x \leq -1] \\
 &= 2 \arctan x & [-1 \leq x \leq 1] \\
 &= \pi - 2 \arctan x & [x \geq 1]
 \end{aligned}$$

NV 65

$$\begin{aligned}
 2. \quad \arccos \frac{1-x^2}{1+x^2} &= 2 \arctan x & [x \geq 0] \\
 &= -2 \arctan x & [x \leq 0]
 \end{aligned}$$

NV 66

$$1.629 \quad \frac{2x-1}{2} - \frac{1}{\pi} \arctan \left(\tan \frac{2x-1}{2} \pi \right) = E(x) \quad \text{GI (886)}$$

1.631 Relations between the inverse hyperbolic functions.

$$\begin{aligned}
 1. \quad \operatorname{arsinh} x &= \operatorname{arcosh} \sqrt{x^2+1} = \operatorname{arctanh} \frac{x}{\sqrt{x^2+1}} & \text{JA} \\
 2. \quad \operatorname{arcosh} x &= \operatorname{arsinh} \sqrt{x^2-1} = \operatorname{arctanh} \frac{\sqrt{x^2-1}}{x} & \text{JA} \\
 3. \quad \operatorname{arctanh} x &= \operatorname{arsinh} \frac{x}{\sqrt{1-x^2}} = \operatorname{arcosh} \frac{1}{\sqrt{1-x^2}} = \operatorname{arcoth} \frac{1}{x} & \text{JA} \\
 4. \quad \operatorname{arsinh} x \pm \operatorname{arsinh} y &= \operatorname{arsinh} \left(x\sqrt{1+y^2} \pm y\sqrt{1+x^2} \right) & \text{JA} \\
 5. \quad \operatorname{arcosh} x \pm \operatorname{arcosh} y &= \operatorname{arcosh} \left(xy \pm \sqrt{(x^2-1)(y^2-1)} \right) & \text{JA} \\
 6. \quad \operatorname{arctanh} x \pm \operatorname{arctanh} y &= \operatorname{arctanh} \frac{x \pm y}{1 \pm xy} & \text{JA}
 \end{aligned}$$

1.64 Series representations**1.641**

$$\begin{aligned}
 1. \quad \arcsin x &= \frac{\pi}{2} - \arccos x = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots \\
 &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1} = x F \left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2 \right) \\
 & \quad [x^2 \leq 1] \quad \text{FI II 479}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \operatorname{arsinh} x &= x - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 - \dots; \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1} \\
 &= x F \left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -x^2 \right) \\
 & \quad [x^2 \leq 1] \quad \text{FI II 480}
 \end{aligned}$$

1.642

1.
$$\begin{aligned} \operatorname{arcsinh} x &= \ln 2x + \frac{1}{2} \frac{1}{2x^2} - \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{4x^4} + \dots \\ &= \ln 2x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \end{aligned} \quad [x \geq 1]$$
 AD (6480.2)a
2.
$$\operatorname{arccosh} x = \ln 2x - \sum_{k=1}^{\infty} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \quad [x \geq 1]$$
 AD (6480.3)a

1.643

1.
$$\begin{aligned} \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \end{aligned} \quad [x^2 \leq 1]$$
 FI II 479
2.
$$\operatorname{arctanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \quad [x^2 < 1]$$
 AD (6480.4)

1.644

1.
$$\begin{aligned} \arctan x &= \frac{x}{\sqrt{1+x^2}} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} \left(\frac{x^2}{1+x^2} \right)^k \\ &= \frac{x}{\sqrt{1+x^2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{1+x^2}\right) \end{aligned} \quad [x^2 < \infty]$$
 AD (641.3)
2.
$$\arctan x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)x^{2k+1}}$$
 AD (641.4)

1.645

1.
$$\begin{aligned} \operatorname{arcsec} x &= \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2 \cdot 3x^3} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)! x^{-(2k+1)}}{(k!)^2 2^{2k} (2k+1)} \\ &= \frac{\pi}{2} - \frac{1}{x} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{x^2}\right) \end{aligned} \quad [x^2 > 1]$$
 AD (641.5)
2.
$$(\arcsin x)^2 = \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2 x^{2k+2}}{(2k+1)!(k+1)} \quad [x^2 \leq 1]$$
 AD (642.2), GI III (152)a
3.
$$(\arcsin x)^3 = x^3 + \frac{3!}{5!} 3^2 \left(1 + \frac{1}{3^2}\right) x^5 + \frac{3!}{7!} 3^2 \cdot 5^2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right) x^7 + \dots \quad [x^2 \leq 1]$$
 BR* 188, AD (642.2), GI III (153)a

1.646

1. $\operatorname{arcsinh} \frac{1}{x} = \operatorname{arcosech} x = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k} (k!)^2 (2k+1)} x^{-2k-1}$
 $[x^2 \geq 1]$ AD (6480.5)
2. $\operatorname{arccosh} \frac{1}{x} = \operatorname{arcsech} x = \ln \frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 2k} x^{2k}$ $[0 < x \leq 1]$ AD (6480.6)
3. $\operatorname{arcsinh} \frac{1}{x} = \operatorname{arcosech} x = \ln \frac{2}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k)!}{2^{2k} (k!)^2 2k} x^{2k}$
 $[0 < x \leq 1]$ AD (6480.7)a
4. $\operatorname{artanh} \frac{1}{x} = \operatorname{arcoth} x = \sum_{k=0}^{\infty} \frac{x^{-(2k+1)}}{2k+1}$ $[x^2 > 1]$ AD (6480.8)

1.647

1.
$$\sum_{k=1}^{\infty} \frac{\tanh(2k-1)(\pi/2)}{(2k-1)^{4n+3}} = \frac{\pi^{4n+3}}{2} \left(2 \sum_{j=1}^n \frac{(-1)^{j-1} (2^{2j}-1) (2^{4n-2j+4}-1) B_{2j-1}^* B_{4n-2j+3}^*}{(2j)!(4n-2j+4)!} \right. \\ \left. + \frac{(-1)^n (2^{2n+2}-1)^2 B_{2n+1}^{*2}}{[(2n+2)!]^2} \right)$$

 $n = 0, 1, 2, \dots,$
2.
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \operatorname{sech}(2k-1)(\pi/2)}{(2k-1)^{4n+1}} = \frac{\pi^{4n+1}}{2^{4n+3}} \left(2 \sum_{j=1}^{n-1} \frac{(-1)^j B_{2j}^* B_{4n-2j}^*}{(2j)!(4n-2j)!} + \frac{2B_{4n}^*}{(4n)!} + \frac{(-1)^n B_{2n}^{*2}}{[(2n)!]^2} \right),$$

 $n = 1, 2, \dots$

(The summation term on the right is to be omitted for $n = 1$.) (See page xxxiii for the definition of B_r^* .)

2 Indefinite Integrals of Elementary Functions

2.0 Introduction

2.00 General remarks

We omit the constant of integration in all the formulas of this chapter. Therefore, the equality sign (=) means that the functions on the left and right of this symbol differ by a constant. For example (see 201 15), we write

$$\int \frac{dx}{1+x^2} = \arctan x = -\arctan x$$

although

$$\arctan x = -\arctan x + \frac{\pi}{2}.$$

When we integrate certain functions, we obtain the logarithm of the absolute value (for example, $\int \frac{dx}{\sqrt{1+x^2}} = \ln |x + \sqrt{1+x^2}|$). In such formulas, the absolute-value bars in the argument of the logarithm are omitted for simplicity in writing.

In certain cases, it is important to give the complete form of the primitive function. Such primitive functions, written in the form of definite integrals, are given in Chapter 2 and in other chapters.

Closely related to these formulas are formulas in which the limits of integration and the integrand depend on the same parameter.

A number of formulas lose their meaning for certain values of the constants (parameters) or for certain relationships between these constants (for example, formula 2.02 8 for $n = -1$ or formula 2.02 15 for $a = b$). These values of the constants and the relationships between them are for the most part completely clear from the very structure of the right-hand member of the formula (the one not containing an integral sign). Therefore, throughout the chapter, we omit remarks to this effect. However, if the value of the integral is given by means of some other formula for those values of the parameters for which the formula in question loses meaning, we accompany this second formula with the appropriate explanation.

The letters $x, y, t, \dots$ denote independent variables; $f, g, \varphi, \dots$ denote functions of $x, y, t, \dots$; $f', g', \varphi', \dots, f'', g'', \varphi'', \dots$ denote their first, second, etc., derivatives; $a, b, m, p, \dots$ denote constants, by which we generally mean arbitrary real numbers. If a particular formula is valid only for certain values of the constants (for example, only for positive numbers or only for integers), an appropriate remark is made, provided the restriction that we make does not follow from the form of the formula itself. Thus, in formulas 2.148 4 and 2.424 6, we make no remark since it is clear from the form of these formulas themselves that n must be a natural number (that is, a positive integer).

2.01 The basic integrals

$$1. \quad \int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$2. \quad \int \frac{dx}{x} = \ln x$$

$$3. \quad \int e^x dx = e^x$$

$$4. \quad \int a^x dx = \frac{a^x}{\ln a}$$

$$5. \quad \int \sin x dx = -\cos x$$

$$6.^{11} \quad \int \cos x dx = \sin x$$

$$7. \quad \int \frac{dx}{\sin^2 x} = -\cot x$$

$$8.^{11} \quad \int \frac{dx}{\cos^2 x} = \tan x$$

$$16. \quad \int \frac{dx}{1-x^2} = \operatorname{arctanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$$

$$17. \quad \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x = -\arccos x$$

$$18. \quad \int \frac{dx}{\sqrt{x^2+1}} = \operatorname{arcsinh} x = \ln \left(x + \sqrt{x^2+1} \right)$$

$$19. \quad \int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x = \ln \left(x + \sqrt{x^2-1} \right)$$

$$20. \quad \int \sinh x dx = \cosh x$$

$$21. \quad \int \cosh x dx = \sinh x$$

$$22.^{11} \quad \int \frac{dx}{\sinh^2 x} = -\coth x$$

$$23. \quad \int \frac{dx}{\cosh^2 x} = \tanh x$$

$$24. \quad \int \tanh x dx = \ln \cosh x$$

$$25. \quad \int \coth x dx = \ln \sinh x$$

$$26. \quad \int \frac{dx}{\sinh x} = \ln \tanh \frac{x}{2}$$

$$9. \quad \int \frac{\sin x}{\cos^2 x} dx = \sec x$$

$$10. \quad \int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x$$

$$11. \quad \int \tan x dx = -\ln \cos x$$

$$12. \quad \int \cot x dx = \ln \sin x$$

$$13. \quad \int \frac{dx}{\sin x} = \ln \tan \frac{x}{2}$$

$$14. \quad \int \frac{dx}{\cos x} = \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) = \ln (\sec x + \tan x)$$

$$15. \quad \int \frac{dx}{1+x^2} = \arctan x = \frac{\pi}{2} - \operatorname{arccot} x$$

2.02 General formulas

1. $\int a f dx = a \int f dx$
2. $\int [a f \pm b \varphi \pm c \psi \pm \dots] dx = a \int f dx \pm b \int \varphi dx \pm c \int \psi dx \pm \dots$
3. $\frac{d}{dx} \int f dx = f$
4. $\int f' dx = f$
5. $\int f' \varphi dx = f \varphi - \int f \varphi' dx$ [integration by parts]
6. $\int f^{(n+1)} \varphi dx = \varphi f^{(n)} - \varphi' f^{(n-1)} + \varphi'' f^{(n-2)} - \dots + (-1)^n \varphi^{(n)} f + (-1)^{n+1} \int \varphi^{(n+1)} f dx$
7. $\int f(x) dx = \int f[\varphi(y)] \varphi'(y) dy$ $[x = \varphi(y)]$ [change of variable]
- 8.¹¹ $\int (f)^n f' dx = \frac{(f)^{n+1}}{n+1}$ $[n \neq -1]$
 For $n = -1$
 $\int \frac{f' dx}{f} = \ln f$
9. $\int (a f + b)^n f' dx = \frac{(a f + b)^{n+1}}{a(n+1)}$
10. $\int \frac{f' dx}{\sqrt{a f + b}} = \frac{2\sqrt{a f + b}}{a}$
11. $\int \frac{f' \varphi - \varphi' f}{\varphi^2} dx = \frac{f}{\varphi}$
12. $\int \frac{f' \varphi - \varphi' f}{f \varphi} dx = \ln \frac{f}{\varphi}$
13. $\int \frac{dx}{f(f \pm \varphi)} = \pm \int \frac{dx}{f \varphi} \mp \int \frac{dx}{\varphi(f \pm \varphi)}$
14. $\int \frac{f' dx}{\sqrt{f^2 + a}} = \ln(f + \sqrt{f^2 + a})$
15. $\int \frac{f dx}{(f+a)(f+b)} = \frac{a}{a-b} \int \frac{dx}{(f+a)} - \frac{b}{a-b} \int \frac{dx}{(f+b)}$
 For $a = b$
 $\int \frac{f dx}{(f+a)^2} = \int \frac{dx}{f+a} - a \int \frac{dx}{(f+a)^2}$
16. $\int \frac{f dx}{(f+\varphi)^n} = \int \frac{dx}{(f+\varphi)^{n-1}} - \int \frac{\varphi dx}{(f+\varphi)^n}$
17. $\int \frac{f' dx}{p^2 + q^2 f^2} = \frac{1}{pq} \arctan \frac{q f}{p}$

18. $\int \frac{f' dx}{q^2 f^2 - p^2} = \frac{1}{2pq} \ln \frac{qf - p}{qf + p}$
19. $\int \frac{f dx}{1 - f} = -x + \int \frac{dx}{1 - f}$
20. $\int \frac{f^2 dx}{f^2 - a^2} = \frac{1}{2} \int \frac{f dx}{f - a} + \frac{1}{2} \int \frac{f dx}{f + a}$
21. $\int \frac{f' dx}{\sqrt{a^2 - f^2}} = \arcsin \frac{f}{a}$
22. $\int \frac{f' dx}{af^2 + bf} = \frac{1}{b} \ln \frac{f}{af + b}$
23. $\int \frac{f' dx}{f\sqrt{f^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{f}{a}$
24. $\int \frac{(f'\varphi - f\varphi') dx}{f^2 + \varphi^2} = \arctan \frac{f}{\varphi}$
25. $\int \frac{(f'\varphi - f\varphi') dx}{f^2 - \varphi^2} = \frac{1}{2} \ln \frac{f - \varphi}{f + \varphi}$

2.1 Rational Functions

2.10 General integration rules

2.101 To integrate an arbitrary rational function $\frac{F(x)}{f(x)}$, where $F(x)$ and $f(x)$ are polynomials with no common factors, we first need to separate out the integral part $E(x)$ [where $E(x)$ is a polynomial], if there is an integral part, and then to integrate separately the integral part and the remainder; thus:

$$\int \frac{F(x) dx}{f(x)} = \int E(x) dx + \int \frac{\varphi(x)}{f(x)} dx.$$

Integration of the remainder, which is then a proper rational function (that is, one in which the degree of the numerator is less than the degree of the denominator) is based on the decomposition of the fraction into elementary fractions, the so-called *partial fractions*.

2.102 If $a, b, c, \dots, m$ are roots of the equation $f(x) = 0$ and if $\alpha, \beta, \gamma, \dots, \mu$ are their corresponding multiplicities, so that $f(x) = (x-a)^\alpha(x-b)^\beta \dots (x-m)^\mu$, then $\frac{\varphi(x)}{f(x)}$ can be decomposed into the following partial fractions:

$$\begin{aligned} \frac{\varphi(x)}{f(x)} = & \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \dots + \frac{A_1}{x-a} + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \dots + \frac{B_1}{x-b} + \dots \\ & + \frac{M_\mu}{(x-m)^\mu} + \frac{M_{\mu-1}}{(x-m)^{\mu-1}} + \dots + \frac{M_1}{x-m}, \end{aligned}$$

where the numerators of the individual fractions are determined by the following formulas:

$$\begin{aligned} A_{\alpha-k+1} &= \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, & B_{\beta-k+1} &= \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, & \dots, & & M_{\mu-k+1} &= \frac{\psi_m^{(k-1)}(m)}{(k-1)!}, \\ \psi_1(x) &= \frac{\varphi(x)(x-a)^\alpha}{f(x)}, & \psi_2(x) &= \frac{\varphi(x)(x-b)^\beta}{f(x)}, & \dots, & & \psi_m(x) &= \frac{\varphi(x)(x-m)^\mu}{f(x)} \end{aligned}$$

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If $a, b, \dots, m$ are simple roots, that is, if $\alpha = \beta = \dots = \mu = 1$, then

$$\frac{\varphi(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \dots + \frac{M}{x-m},$$

where

$$A = \frac{\varphi(a)}{f'(a)}, \quad B = \frac{\varphi(b)}{f'(b)}, \quad \dots, \quad M = \frac{\varphi(m)}{f'(m)}.$$

If some of the roots of the equation $f(x) = 0$ are imaginary, we group together the fractions that represent conjugate roots of the equation. Then, after certain manipulations, we represent the corresponding pairs of fractions in the form of real fractions of the form

$$\frac{M_1x + N_1}{x^2 + 2Bx + C} + \frac{M_2x + N_2}{(x^2 + 2Bx + C)^2} + \dots + \frac{M_px + N_p}{(x^2 + 2Bx + C)^p}.$$

2.103 Thus, the integration of a proper rational fraction $\frac{\varphi(x)}{f(x)}$ reduces to integrals of the form $\int \frac{g dx}{(x-a)^\alpha}$ or $\int \frac{Mx + N}{(A + 2Bx + Cx^2)^p} dx$. Fractions of the first form yield rational functions for $\alpha > 1$ and logarithms for $\alpha = 1$. Fractions of the second form yield rational functions and logarithms or arctangents:

1. $\int \frac{g dx}{(x-a)^\alpha} = g \int \frac{d(x-a)}{(x-a)^\alpha} = -\frac{g}{(\alpha-1)(x-a)^{\alpha-1}}$
2. $\int \frac{g dx}{x-a} = g \int \frac{d(x-a)}{x-a} = g \ln |x-a|$
3. $\int \frac{Mx + N}{(A + 2Bx + Cx^2)^p} dx = \frac{NB - MA + (NC - MB)x}{2(p-1)(AC - B^2)(A + 2Bx + Cx^2)^{p-1}} + \frac{(2p-3)(NC - MB)}{2(p-1)(AC - B^2)} \int \frac{dx}{(A + 2Bx + Cx^2)^{p-1}}$
4. $\int \frac{dx}{A + 2Bx + Cx^2} = \frac{1}{\sqrt{AC - B^2}} \arctan \frac{Cx + B}{\sqrt{AC - B^2}} \quad \text{for } [AC > B^2]$
 $= \frac{1}{2\sqrt{B^2 - AC}} \ln \left| \frac{Cx + B - \sqrt{B^2 - AC}}{Cx + B + \sqrt{B^2 - AC}} \right| \quad \text{for } [AC < B^2]$
5. $\int \frac{(Mx + N) dx}{A + 2Bx + Cx^2} = \frac{M}{2C} \ln |A + 2Bx + Cx^2| + \frac{NC - MB}{C\sqrt{AC - B^2}} \arctan \frac{Cx + B}{\sqrt{AC - B^2}} \quad \text{for } [AC > B^2]$
 $= \frac{M}{2C} \ln |A + 2Bx + Cx^2| + \frac{NC - MB}{2C\sqrt{B^2 - AC}} \ln \left| \frac{Cx + B - \sqrt{B^2 - AC}}{Cx + B + \sqrt{B^2 - AC}} \right| \quad \text{for } [AC < B^2]$

The Ostrogradskiy–Hermite method

2.104 By means of the Ostrogradskiy–Hermite method, we can find the rational part of $\int \frac{\varphi(x)}{f(x)} dx$ without finding the roots of the equation $f(x) = 0$ and without decomposing the integrand into partial fractions:

$$\int \frac{\varphi(x)}{f(x)} dx = \frac{M}{D} + \int \frac{N}{Q} dx$$

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Here, M , N , D , and Q are rational functions of x . Specifically, D is the greatest common divisor of the function $f(x)$ and its derivative $f'(x)$; $Q = \frac{f(x)}{D}$; M is a polynomial of degree no higher than $m-1$, where m is the degree of the polynomial D ; N is a polynomial of degree no higher than $n-1$, where n is the degree of the polynomial Q . The coefficients of the polynomials M and N are determined by equating the coefficients of like powers of x in the following identity:

$$\varphi(x) = M'Q - M(T - Q') + ND$$

where $T = \frac{f'(x)}{D}$ and M' and Q' are the derivatives of the polynomials M and Q .

2.11–2.13 Forms containing the binomial $a + bx^k$

2.110 Reduction formulas for $z_k = a + bx^k$ and an explicit expression for the general case.

$$\begin{aligned} 1. \quad \int x^n z_k^m dx &= \frac{x^{n+1} z_k^m}{km + n + 1} + \frac{amk}{km + n + 1} \int x^n z_k^{m-1} dx \\ &= \frac{x^{n+1}}{m+1} \sum_{s=0}^p \frac{(ak)^s (m+1)m(m-1)\dots(m-s+1)z_k^{m-s}}{[mk+n+1][(m-1)k+n+1]\dots[(m-s)k+n+1]} \\ &\quad + \frac{(ak)^{p+1}m(m-1)\dots(m-p+1)(m-p)}{[mk+n+1][(m-1)k+n+1]\dots[(m-p)k+n+1]} \int x^n z_k^{m-p-1} dx \end{aligned}$$

LA 126(4)

$$2. \quad \int x^n z_k^m dx = \frac{-x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)} \int x^n z_k^{m+1} dx$$

LA 126 (6)

$$3. \quad \int x^n z_k^m dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1} \int x^{n+k} z_k^{m-1} dx$$

$$4. \quad \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)} \int x^{n-k} z_k^{m+1} dx$$

LA 125 (2)

$$5. \quad \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)} \int x^{n-k} z_k^m dx$$

LA 126 (3)

$$6. \quad \int x^n z_k^m dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)} \int x^{n+k} z_k^m dx$$

LA 126 (5)

$$7.* \quad \int x^n (nx^b + c)^k dx = \frac{n^k}{b} \sum_{i=0}^k \frac{(-1)^i k! \Gamma\left(\frac{a+1}{b}\right) \left(n^b + \frac{c}{n}\right)^{k-i}}{(k-i)! \Gamma\left(\frac{a+1}{b} + i + 1\right)} x^{a+1+ib}$$

[$a, b, k \geq 0$ are all integers]

$$8.* \quad \int x^n z_k^m dx = \frac{b^m}{k} \sum_{i=0}^m \frac{(-1)^i m! J! \left(x^k + \frac{a}{b}\right)^{m-i} x^{k(J+i+1)}}{(m-i)!(J+i+1)!}$$

$$J = \frac{n+1}{k} - 1 \quad [a, b, k, m, n \text{ real}, \quad k \neq 0, \quad m \geq 0 \text{ an integer}]$$

Forms containing the binomial $z_1 = a + bx$ **2.111**

$$1. \quad \int z_1^m dx = \frac{z_1^{m+1}}{b(m+1)}$$

For $m = -1$

$$\int \frac{dx}{z_1} = \frac{1}{b} \ln z_1$$

$$2. \quad \int \frac{x^n dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b} \int \frac{x^{n-1} dx}{z_1^m}$$

For $n = m - 1$, we may use the formula

$$3.^8 \quad \int \frac{x^{m-1} dx}{z_1^m} = -\frac{x^{m-1}}{z_1^{m-1}(m-1)b} + \frac{1}{b} \int \frac{x^{m-2} dx}{z_1^{m-1}}$$

For $m = 1$

$$\int \frac{x^n dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2x^{n-2}}{(n-2)b^3} - \dots + (-1)^{n-1} \frac{a^{n-1}x}{1 \cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}} \ln z_1$$

$$4. \quad \int \frac{x^n dx}{z_1^2} = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{ka^{k-1}x^{n-k}}{(n-k)b^{k+1}} + (-1)^{n-1} \frac{a^n}{b^{n+1}z_1} + (-1)^{n+1} \frac{na^{n-1}}{b^{n+1}} \ln z_1$$

$$5. \quad \int \frac{x dx}{z_1} = \frac{x}{b} - \frac{a}{b^2} \ln z_1$$

$$6. \quad \int \frac{x^2 dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3} \ln z_1$$

2.113

$$1. \quad \int \frac{dx}{z_1^2} = -\frac{1}{bz_1}$$

$$2. \quad \int \frac{x dx}{z_1^2} = -\frac{x}{bz_1} + \frac{1}{b^2} \ln z_1 = \frac{a}{b^2 z_1} + \frac{1}{b^2} \ln z_1$$

$$3. \quad \int \frac{x^2 dx}{z_1^2} = \frac{x}{b^2} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3} \ln z_1$$

2.114

$$1. \quad \int \frac{dx}{z_1^3} = -\frac{1}{2bz_1^2}$$

$$2. \quad \int \frac{x dx}{z_1^3} = -\left[\frac{x}{b} + \frac{a}{2b^2}\right] \frac{1}{z_1^2}$$

$$3. \quad \int \frac{x^2 dx}{z_1^3} = \left[\frac{2ax}{b^2} + \frac{3a^2}{2b^3}\right] \frac{1}{z_1^2} + \frac{1}{b^3} \ln z_1$$

$$4.^6 \quad \int \frac{x^3 dx}{z_1^3} = \left[\frac{x^3}{b} + 2\frac{a}{b^2}x^2 - 2\frac{a^2}{b^3}x - \frac{5a^3}{2b^4}\right] \frac{1}{z_1^2} - 3\frac{a}{b^4} \ln z_1$$

2.115

1. $\int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3}$
2. $\int \frac{x dx}{z_1^4} = -\left[\frac{x}{2b} + \frac{a}{6b^2}\right] \frac{1}{z_1^3}$
3. $\int \frac{x^2 dx}{z_1^4} = -\left[\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right] \frac{1}{z_1^3}$
4. $\int \frac{x^3 dx}{z_1^4} = \left[\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^2} + \frac{11a^3}{6b^4}\right] \frac{1}{z_1^3} + \frac{1}{b^4} \ln z_1$

2.116

1. $\int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4}$
2. $\int \frac{x dx}{z_1^5} = -\left[\frac{x}{3b} + \frac{a}{12b^2}\right] \frac{1}{z_1^4}$
3. $\int \frac{x^2 dx}{z_1^5} = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right] \frac{1}{z_1^4}$
4. $\int \frac{x^3 dx}{z_1^5} = -\left[\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4}\right] \frac{1}{z_1^4}$

2.117

1. $\int \frac{dx}{x^n z_1^m} = \frac{-1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)} \int \frac{dx}{x^{n-1}z_1^m}$
2. $\int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}}$
3. $\int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a} \int \frac{dx}{xz_1^{m-1}}$
4. $\int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n} \ln \frac{z_1}{x}$

2.118

1. $\int \frac{dx}{xz_1} = -\frac{1}{a} \ln \frac{z_1}{x},$
2. $\int \frac{dx}{x^2 z_1} = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{z_1}{x}$
3. $\int \frac{dx}{x^3 z_1} = -\frac{1}{2ax^2} + \frac{b}{a^2 x} - \frac{b^2}{a^3} \ln \frac{z_1}{x}$

2.119

1. $\int \frac{dx}{xz_1^2} = \frac{1}{az_1} - \frac{1}{a^2} \ln \frac{z_1}{x}$

$$\begin{aligned}
2. \quad & \int \frac{dx}{x^2 z_1^2} = - \left[\frac{1}{ax} + \frac{2b}{a^2} \right] \frac{1}{z_1} + \frac{2b}{a^3} \ln \frac{z_1}{x} \\
3. \quad & \int \frac{dx}{x^3 z_1^2} = \left[-\frac{1}{2ax^2} + \frac{3b}{2a^2x} + \frac{3b^2}{a^3} \right] \frac{1}{z_1} - \frac{3b^2}{a^4} \ln \frac{z_1}{x}
\end{aligned}$$

2.121

$$\begin{aligned}
1. \quad & \int \frac{dx}{x z_1^3} = \left[\frac{3}{2a} + \frac{bx}{a^2} \right] \frac{1}{z_1^2} - \frac{1}{a^3} \ln \frac{z_1}{x} \\
2. \quad & \int \frac{dx}{x^2 z_1^3} = - \left[\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2x}{a^3} \right] \frac{1}{z_1^2} + \frac{3b}{a^4} \ln \frac{z_1}{x} \\
3. \quad & \int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{2ax^2} + \frac{2b}{a^2x} + \frac{9b^2}{a^3} + \frac{6b^3x}{a^4} \right] \frac{1}{z_1^2} - \frac{6b^2}{a^5} \ln \frac{z_1}{x}
\end{aligned}$$

2.122

$$\begin{aligned}
1. \quad & \int \frac{dx}{x z_1^4} = \left[\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3} \right] \frac{1}{z_1^3} - \frac{1}{a^4} \ln \frac{z_1}{x} \\
2. \quad & \int \frac{dx}{x^2 z_1^4} = - \left[\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2x}{a^3} + \frac{4b^3x^2}{a^4} \right] \frac{1}{z_1^3} + \frac{4b}{a^5} \ln \frac{z_1}{x} \\
3. \quad & \int \frac{dx}{x^3 z_1^4} = \left[-\frac{1}{2ax^2} + \frac{5b}{2a^2x} + \frac{55b^2}{3a^3} + \frac{25b^3x}{a^4} + \frac{10b^4x^2}{a^5} \right] \frac{1}{z_1^3} - \frac{10b^2}{a^6} \ln \frac{z_1}{x}
\end{aligned}$$

2.123

$$\begin{aligned}
1.^{11} \quad & \int \frac{dx}{x z_1^5} = \left[\frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2x^2}{2a^3} + \frac{b^3x^3}{a^4} \right] \frac{1}{z_1^4} - \frac{1}{a^5} \ln \frac{z_1}{x} \\
2. \quad & \int \frac{dx}{x^2 z_1^5} = \left[-\frac{1}{ax} - \frac{125b}{12a^2} - \frac{65b^2x}{3a^3} - \frac{35b^3x^2}{2a^4} - \frac{5b^4x^3}{a^5} \right] \frac{1}{z_1^4} + \frac{5b}{a^6} \ln \frac{z_1}{x} \\
3. \quad & \int \frac{dx}{x^3 z_1^5} = \left[-\frac{1}{2ax^2} + \frac{3b}{a^2x} + \frac{125b^2}{4a^3} + \frac{65b^3x}{a^4} + \frac{105b^4x^2}{2a^5} + \frac{15b^5x^3}{a^6} \right] \frac{1}{z_1^4} - \frac{15b^2}{a^7} \ln \frac{z_1}{x}
\end{aligned}$$

2.124 Forms containing the binomial $z_2 = a + bx^2$.

$$\begin{aligned}
1. \quad & \int \frac{dx}{z_2} = \frac{1}{\sqrt{ab}} \arctan x \sqrt{\frac{b}{a}} \quad \text{if } [ab > 0] \quad (\text{see also } \mathbf{2.141} \text{ 2}) \\
& = \frac{1}{2i\sqrt{ab}} \ln \frac{a + xi\sqrt{ab}}{a - xi\sqrt{ab}} \quad \text{if } [ab < 0] \quad (\text{see also } \mathbf{2.143} \text{ 2 and } \mathbf{2.1433}) \\
2. \quad & \int \frac{x dx}{z_2^m} = -\frac{1}{2b(m-1)z_2^{m-1}} \quad (\text{see also } \mathbf{2.145} \text{ 2, } \mathbf{2.145} \text{ 6, and } \mathbf{2.18})
\end{aligned}$$

Forms containing the binomial $z_3 = a + bx^3$

Notation: $\alpha = \sqrt[3]{\frac{a}{b}}$

2.125

$$\begin{aligned} 1. \quad \int \frac{x^n dx}{z_3^m} &= \frac{x^{n-2}}{z_3^{m-1}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)} \int \frac{x^{n-3} dx}{z_3^m} \\ 2. \quad \int \frac{x^n dx}{z_3^m} &= \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-3m}{3a(m-1)} \int \frac{x^n dx}{z_3^{m-1}} \end{aligned} \quad \text{LA 133 (1)}$$

2.126

$$\begin{aligned} 1. \quad \int \frac{dx}{z_3} &= \frac{\alpha}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} + \sqrt{3} \arctan \frac{x\sqrt{3}}{2\alpha - x} \right\} \\ &= \frac{\alpha}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} + \sqrt{3} \arctan \frac{2x - \alpha}{\alpha\sqrt{3}} \right\} \quad (\text{see also } \mathbf{2.141} \text{ 3 and } \mathbf{2.143}) \\ 2. \quad \int \frac{x dx}{z_3} &= -\frac{1}{3b\alpha} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} - \sqrt{3} \arctan \frac{2x - \alpha}{\alpha\sqrt{3}} \right\} \end{aligned}$$

(see also **2.145** 3. and **2.145** 7)

$$\begin{aligned} 3. \quad \int \frac{x^2 dx}{z_3} &= \frac{1}{3b} \ln(1 + x^3 \alpha^{-3}) = \frac{1}{3b} \ln z_3 \\ 4. \quad \int \frac{x^3 dx}{z_3} &= \frac{x}{b} - \frac{a}{b} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1}) \\ 5. \quad \int \frac{x^4 dx}{z_3} &= \frac{x^2}{2b} - \frac{a}{b} \int \frac{x dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 2}) \end{aligned}$$

2.127

$$\begin{aligned} 1. \quad \int \frac{dx}{z_3^2} &= \frac{x}{3az_3} + \frac{2}{3a} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1}) \\ 2. \quad \int \frac{x dx}{z_3^2} &= \frac{x^2}{3az_3} + \frac{1}{3a} \int \frac{x dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 2}) \\ 3. \quad \int \frac{x^2 dx}{z_3^2} &= -\frac{1}{3bz_3} \\ 4. \quad \int \frac{x^3 dx}{z_3^2} &= -\frac{x}{3bz_3} + \frac{1}{3b} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1}) \end{aligned}$$

2.128

$$\begin{aligned} 1. \quad \int \frac{dx}{x^n z_3^m} &= -\frac{1}{(n-1)ax^{n-1}z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)} \int \frac{dx}{x^{n-3}z_3^m} \\ 2. \quad \int \frac{dx}{x^n z_3^m} &= \frac{1}{3a(m-1)x^{n-1}z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)} \int \frac{dx}{x^n z_3^{m-1}} \end{aligned} \quad \text{LA 133 (2)}$$

2.129

1. $\int \frac{dx}{xz_3} = \frac{1}{3a} \ln \frac{x^3}{z_3}$
2. $\int \frac{dx}{x^2 z_3} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x dx}{z_3}$ (see **2.126** 2)
3. $\int \frac{dx}{x^3 z_3} = -\frac{1}{2ax^2} - \frac{b}{a} \int \frac{dx}{z_3}$ (see **2.126** 1)

2.131

1. $\int \frac{dx}{x z_3^2} = \frac{1}{3a z_3} + \frac{1}{3a^2} \ln \frac{x^3}{z_3}$
2. $\int \frac{dx}{x^2 z_3^2} = -\left[\frac{1}{ax} + \frac{4bx^2}{3a^2} \right] \frac{1}{z_3} - \frac{4b}{3a^2} \int \frac{x dx}{z_3}$ (see **2.126** 2)
3. $\int \frac{dx}{x^3 z_3^2} = -\left[\frac{1}{2ax^2} + \frac{5bx}{6a^2} \right] \frac{1}{z_3} - \frac{5b}{3a^2} \int \frac{dx}{z_3}$ (see **2.126** 1)

Forms containing the binomial $z_4 = a + bx^4$

Notation: $\alpha = \sqrt[4]{\frac{a}{b}}$ $\alpha' = \sqrt[4]{-\frac{a}{b}}$

2.132

- 1.⁸ $\int \frac{dx}{z_4} = \frac{\alpha}{4a\sqrt{2}} \left\{ \ln \frac{x^2 + \alpha x \sqrt{2} + \alpha^2}{x^2 - \alpha x \sqrt{2} + \alpha^2} + 2 \arctan \frac{\alpha x \sqrt{2}}{\alpha^2 - x^2} \right\}$ for $ab > 0$ (see also **2.141** 4)
 $= \frac{\alpha'}{4a} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} + 2 \arctan \frac{x}{\alpha'} \right\}$ for $ab < 0$ (see also **2.143** 5)
2. $\int \frac{x dx}{z_4} = \frac{1}{2\sqrt{ab}} \arctan x^2 \sqrt{\frac{b}{a}}$ for $ab > 0$ (see also **2.145** 4)
 $= \frac{1}{4i\sqrt{ab}} \ln \frac{a + x^2 i \sqrt{ab}}{a - x^2 i \sqrt{ab}}$ for $ab < 0$ (see also **2.145** 8)
3. $\int \frac{x^2 dx}{z_4} = \frac{1}{4b\alpha\sqrt{2}} \left\{ \ln \frac{x^2 - \alpha x \sqrt{2} + \alpha^2}{x^2 + \alpha x \sqrt{2} + \alpha^2} + 2 \arctan \frac{\alpha x \sqrt{2}}{\alpha^2 - x^2} \right\}$ for $ab > 0$
 $= -\frac{1}{4b\alpha'} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} - 2 \arctan \frac{x}{\alpha'} \right\}$ for $ab < 0$
4. $\int \frac{x^3 dx}{z_4} = \frac{1}{4b} \ln z_4$

2.133

1. $\int \frac{x^n dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m-n-5}{4a(m-1)} \int \frac{x^n dx}{z_4^{m-1}}$ LA 134 (1)
2. $\int \frac{x^n dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b} - \frac{(n-3)a}{b(n+1-4m)} \int \frac{x^{n-4} dx}{z_4^m}$

2.134

$$1. \quad \int \frac{dx}{z_4^2} = \frac{x}{4az_4} + \frac{3}{4a} \int \frac{dx}{z_4} \quad (\text{see } \mathbf{2.132} \text{ 1})$$

$$2. \quad \int \frac{x dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a} \int \frac{x dx}{z_4} \quad (\text{see } \mathbf{2.132} \text{ 2})$$

$$3. \quad \int \frac{x^2 dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a} \int \frac{x^2 dx}{z_4} \quad (\text{see } \mathbf{2.132} \text{ 3})$$

$$4. \quad \int \frac{x^3 dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4}$$

$$\mathbf{2.135} \quad \int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1}z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a} \int \frac{dx}{x^{n-4}z_4^m}$$

For $n=1$

$$\int \frac{dx}{xz_4^m} = \frac{1}{a} \int \frac{dx}{xz_4^{m-1}} - \frac{b}{a} \int \frac{dx}{x^{-3}z_4^m}$$

2.136

$$1. \quad \int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a} \ln \frac{x^4}{z_4}$$

$$2. \quad \int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x^2 dx}{z_4} \quad (\text{see } \mathbf{2.132} \text{ 3})$$

2.14 Forms containing the binomial $1 \pm x^n$ **2.141**

$$1. \quad \int \frac{dx}{1+x} = \ln(1+x)$$

$$2.^{11} \quad \int \frac{dx}{1+x^2} = \arctan x = -\arctan \left(\frac{1}{x} \right) \quad (\text{see also } \mathbf{2.124} \text{ 1})$$

$$3. \quad \int \frac{dx}{1+x^3} = \frac{1}{3} \ln \frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}} \arctan \frac{x\sqrt{3}}{2-x} \quad (\text{see also } \mathbf{2.126} \text{ 1})$$

$$4. \quad \int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}} \arctan \frac{x\sqrt{2}}{1-x^2}$$

(see also **2.132** 1)

$$\mathbf{2.142} \quad \int \frac{dx}{1+x^n} = -\frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} P_k \cos \left(\frac{2k+1}{n} \pi \right) + \frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} Q_k \sin \left(\frac{2k+1}{n} \pi \right)$$

for n a positive even number

TI (43)a

$$= \frac{1}{n} \ln(1+x) - \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \left(\frac{2k+1}{n} \pi \right) + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \left(\frac{2k+1}{n} \pi \right)$$

for n a positive odd number

TI (45)

where

$$P_k = \frac{1}{2} \ln \left(x^2 - 2x \cos \left(\frac{2k+1}{n} \pi \right) + 1 \right)$$

$$Q_k = \arctan \frac{x \sin \left(\frac{2k+1}{n} \pi \right)}{1 - x \cos \left(\frac{2k+1}{n} \pi \right)} = \arctan \frac{x - \cos \left(\frac{2k+1}{n} \pi \right)}{\sin \left(\frac{2k+1}{n} \pi \right)}$$

2.143

1. $\int \frac{dx}{1-x} = -\ln(1-x)$
 2. $\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \frac{1+x}{1-x} = \operatorname{arctanh} x \quad [-1 < x < 1] \quad (\text{see also } \mathbf{2.141} \text{ 1})$
 3. $\int \frac{dx}{x^2-1} = \frac{1}{2} \ln \frac{x-1}{x+1} = -\operatorname{arccoth} x \quad [x > 1, \quad x < -1]$
 4. $\int \frac{dx}{1-x^3} = \frac{1}{3} \ln \frac{\sqrt{1+x+x^2}}{1-x} + \frac{1}{\sqrt{3}} \arctan \frac{x\sqrt{3}}{2+x} \quad (\text{see also } \mathbf{2.126} \text{ 1})$
 5. $\int \frac{dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \arctan x = \frac{1}{2} (\operatorname{arctanh} x + \arctan x)$
- (see also **2.132** 1)

2.144

1. $\int \frac{dx}{1-x^n} = \frac{1}{n} \ln \frac{1+x}{1-x} - \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} P_k \cos \frac{2k}{n} \pi + \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} Q_k \sin \frac{2k}{n} \pi$
for n a positive even number TI (47)
- where $P_k = \frac{1}{2} \ln \left(x^2 + 2x \cos \frac{2k+1}{n} \pi + 1 \right), \quad Q_k = \arctan \frac{x + \cos \frac{2k+1}{n} \pi}{\sin \frac{2k+1}{n} \pi}$
2. $\int \frac{dx}{1-x^n} = -\frac{1}{n} \ln(1-x) + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \frac{2k+1}{n} \pi + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \frac{2k+1}{n} \pi$
for n a positive odd number TI (49)
- where $P_k = \frac{1}{2} \ln \left(x^2 - 2x \cos \frac{2k}{n} \pi + 1 \right), \quad Q_k = \arctan \frac{x - \cos \frac{2k}{n} \pi}{\sin \frac{2k}{n} \pi}$

2.145

1. $\int \frac{x dx}{1+x} = x - \ln(1+x)$
2. $\int \frac{x dx}{1+x^2} = \frac{1}{2} \ln(1+x^2)$
3. $\int \frac{x dx}{1+x^3} = -\frac{1}{6} \ln \frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} \quad (\text{see also } \mathbf{2.126} \text{ 2})$

4. $\int \frac{x dx}{1+x^4} = \frac{1}{2} \arctan x^2$
5. $\int \frac{x dx}{1-x} = -\ln(1-x) - x$
6. $\int \frac{x dx}{1-x^2} = -\frac{1}{2} \ln(1-x^2)$
7. $\int \frac{x dx}{1-x^3} = -\frac{1}{6} \ln \frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}}$ (see also **2.126** 2)
8. $\int \frac{x dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x^2}{1-x^2}$ (see also **2.132** 2)

2.146 For m and n natural numbers.

$$1. \quad \int \frac{x^{m-1} dx}{1+x^{2n}} = -\frac{1}{2n} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n} \pi + x^2 \right\} \\ + \frac{1}{n} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n} \arctan \frac{x - \cos \frac{2k-1}{2n} \pi}{\sin \frac{2k-1}{2n} \pi} \\ [m < 2n] \quad \text{TI (44)a}$$

$$2. \quad \int \frac{x^{m-1} dx}{1+x^{2n+1}} = (-1)^{m+1} \frac{\ln(1+x)}{2n+1} - \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n+1} \pi + x^2 \right\} \\ + \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \arctan \frac{x - \cos \frac{2k-1}{2n+1} \pi}{\sin \frac{2k-1}{2n+1} \pi} \\ [m \leq 2n] \quad \text{TI (46)a}$$

$$3.^{11} \quad \int \frac{x^{m-1} dx}{1-x^{2n}} = \frac{1}{2n} \{ (-1)^{m+1} \ln(1+x) - \ln(1-x) \} - \frac{1}{2n} \sum_{k=1}^{n-1} \cos \frac{km\pi}{n} \ln \left(1 - 2x \cos \frac{k\pi}{n} + x^2 \right) \\ + \frac{1}{n} \sum_{k=1}^{n-1} \sin \frac{km\pi}{n} \arctan \left(\frac{x - \cos \frac{k\pi}{n}}{\sin \frac{k\pi}{n}} \right) \\ [m < 2n] \quad \text{TI (48)}$$

$$4. \quad \int \frac{x^{m-1} dx}{1-x^{2n+1}} = -\frac{1}{2n+1} \ln(1-x) \\ + (-1)^{m+1} \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left(1 + 2x \cos \frac{2k-1}{2n+1} \pi + x^2 \right) \\ + (-1)^{m+1} \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \arctan \frac{x + \cos \frac{2k-1}{2n+1} \pi}{\sin \frac{2k-1}{2n+1} \pi} \\ [m \leq 2n] \quad \text{TI (50)}$$

2.147

$$1. \quad \int \frac{x^m dx}{1-x^{2n}} = \frac{1}{2} \int \frac{x^m dx}{1-x^n} + \frac{1}{2} \int \frac{x^m dx}{1+x^n}$$

$$2. \quad \int \frac{x^m dx}{(1+x^2)^n} = -\frac{1}{2n-m-1} \cdot \frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1} \int \frac{x^{m-2} dx}{(1+x^2)^n} \quad \text{LA 139 (28)}$$

$$3. \quad \int \frac{x^m}{1+x^2} dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{1+x^2} dx$$

$$4. \quad \int \frac{x^m dx}{(1-x^2)^n} = \frac{1}{2n-m-1} \frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-m-1} \int \frac{x^{m-2} dx}{(1-x^2)^n} \\ = \frac{1}{2n-2} \frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-2} \int \frac{x^{m-2} dx}{(1-x^2)^{n-1}}$$

LA 139 (33)

$$5. \quad \int \frac{x^m dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int \frac{x^{m-2} dx}{1-x^2}$$

2.148

$$1. \quad \int \frac{dx}{x^m (1+x^2)^n} = -\frac{1}{m-1} \frac{1}{x^{m-1} (1+x^2)^{n-1}} - \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2} (1+x^2)^n} \quad \text{LA 139 (29)}$$

For $m = 1$

$$\int \frac{dx}{x (1+x^2)^n} = \frac{1}{2n-2} \frac{1}{(1+x^2)^{n-1}} + \int \frac{dx}{x (1+x^2)^{n-1}} \quad \text{LA 139 (31)}$$

For $m = 1$ and $n = 1$

$$\int \frac{dx}{x (1+x^2)} = \ln \frac{x}{\sqrt{1+x^2}}$$

$$2. \quad \int \frac{dx}{x^m (1+x^2)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2} (1+x^2)}$$

$$3. \quad \int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2} \frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}} \quad \text{FI II 40}$$

$$4. \quad \int \frac{dx}{(1+x^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)(2n-5) \cdots (2n-2k+1)}{2^k (n-1)(n-2) \cdots (n-k) (1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1} (n-1)!} \arctan x \quad \text{TI (91)}$$

2.149

$$1. \quad \int \frac{dx}{x^m (1-x^2)^n} = -\frac{1}{(m-1)x^{m-1} (1-x^2)^{n-1}} + \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2} (1-x^2)^n} \quad \text{LA 139 (34)}$$

For $m = 1$

$$\int \frac{dx}{x (1-x^2)^n} = \frac{1}{2(n-1) (1-x^2)^{n-1}} + \int \frac{dx}{x (1-x^2)^{n-1}} \quad \text{LA 139 (36)}$$

For $m = 1$ and $n = 1$

$$\int \frac{dx}{x (1-x^2)} = \ln \frac{x}{\sqrt{1-x^2}}$$

$$2. \quad \int \frac{dx}{(1-x^2)^n} = \frac{1}{2n-2} \frac{x}{(1-x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1-x^2)^{n-1}} \quad \text{LA 139 (35)}$$

$$3. \quad \int \frac{dx}{(1-x^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)(2n-5) \cdots (2n-2k+1)}{2^k (n-1)(n-2) \cdots (n-k) (1-x^2)^{n-k}} + \frac{(2n-3)!!}{2^n \cdot (n-1)!} \ln \frac{1+x}{1-x} \quad \text{TI (91)}$$

2.15 Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$

Notation: $z = a + bx$; $t = \alpha + \beta x$; $\Delta = a\beta - \alpha b$

$$2.151 \quad \int z^n t^m dx = \frac{z^{n+1} t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b} \int z^n t^{m-1} dx$$

2.152

$$1. \quad \int \frac{z}{t} dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2} \ln t$$

$$2. \quad \int \frac{t}{z} dx = \frac{\beta x}{b} - \frac{\Delta}{b^2} \ln z$$

$$2.153 \quad \begin{aligned} \int \frac{t^m dx}{z^n} &= \frac{1}{(m-n+1)b} \frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b} \int \frac{t^{m-1} dx}{z^n} \\ &= \frac{1}{(n-1)\Delta} \frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta} \int \frac{t^m dx}{z^{n-1}} \\ &= -\frac{1}{(n-1)b} \frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b} \int \frac{t^{m-1} dx}{z^{n-1}} \end{aligned}$$

$$2.154 \quad \int \frac{dx}{zt} = \frac{1}{\Delta} \ln \frac{t}{z}$$

$$2.155 \quad \begin{aligned} \int \frac{dx}{z^n t^m} &= -\frac{1}{(m-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta} \int \frac{dx}{t^{m-1} z^n} \\ &= \frac{1}{(n-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta} \int \frac{dx}{t^m z^{n-1}} \end{aligned}$$

$$2.156 \quad \int \frac{x dx}{zt} = \frac{1}{\Delta} \left(\frac{a}{b} \ln z - \frac{\alpha}{\beta} \ln t \right)$$

2.16 Forms containing the trinomial $a + bx^k + cx^{2k}$

2.160 Reduction formulas for $R_k = a + bx^k + cx^{2k}$.

$$1. \quad \int x^{m-1} R_k^n dx = \frac{x^m R_k^{n+1}}{ma} - \frac{(m+k+nk)b}{ma} \int x^{m+k-1} R_k^n dx - \frac{(m+2k+2kn)c}{ma} \int x^{m+2k-1} R_k^n dx$$

$$2. \quad \int x^{m-1} R_k^n dx = \frac{x^m R_k^n}{m} - \frac{bkn}{m} \int x^{m+k-1} R_k^{n-1} dx - \frac{2ckn}{m} \int x^{m+2k-1} R_k^{n-1} dx$$

$$3. \quad \begin{aligned} \int x^{m-1} R_k^n dx &= \frac{x^{m-2k} R_k^{n+1}}{(m+2kn)c} - \frac{(m-2k)a}{(m+2kn)c} \int x^{m-2k-1} R_k^n dx - \frac{(m-k+kn)b}{(m+2kn)c} \int x^{m-k-1} R_k^n dx \\ &= \frac{x^m R_k^n}{m+2kn} + \frac{2kna}{m+2kn} \int x^{m-1} R_k^{n-1} dx + \frac{bkn}{m+2kn} \int x^{m+k-1} R_k^{n-1} dx \end{aligned}$$

2.161 Forms containing the trinomial $R_2 = a + bx^2 + cx^4$.

Notation: $f = \frac{b}{2} - \frac{1}{2} \sqrt{b^2 - 4ac}$, $g = \frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 4ac}$,

$$h = \sqrt{b^2 - 4ac}, \quad q = \sqrt[4]{\frac{a}{c}}, \quad l = 2a(n-1)(b^2 - 4ac), \quad \cos \alpha = -\frac{b}{2\sqrt{ac}}$$

1.
$$\int \frac{dx}{R_2} = \frac{c}{h} \left\{ \int \frac{dx}{cx^2 + f} - \int \frac{dx}{cx^2 + g} \right\} \quad [h^2 > 0] \quad \text{LA 146 (5)}$$

$$= \frac{1}{4cq^3 \sin \alpha} \left\{ \sin \frac{\alpha}{2} \ln \frac{x^2 + 2qx \cos \frac{\alpha}{2} + q^2}{x^2 - 2qx \cos \frac{\alpha}{2} + q^2} + 2 \cos \frac{\alpha}{2} \arctan \frac{x^2 - q^2}{2qx \sin \frac{\alpha}{2}} \right\} \quad [h^2 < 0] \quad \text{LA 146 (8)a}$$
2.
$$\int \frac{x dx}{R_2} = \frac{1}{2h} \ln \frac{cx^2 + f}{cx^2 + g} \quad [h^2 > 0] \quad \text{LA 146 (6)}$$

$$= \frac{1}{2cq^2 \sin \alpha} \arctan \frac{x^2 - q^2 \cos \alpha}{q^2 \sin \alpha} \quad [h^2 < 0] \quad \text{LA 146 (9)a}$$
3.
$$\int \frac{x^2 dx}{R_2} = \frac{g}{h} \int \frac{dx}{cx^2 + g} - \frac{f}{h} \int \frac{dx}{cx^2 + f} \quad [h^2 > 0] \quad \text{LA 146 (7)}$$
4.
$$\int \frac{dx}{R_2^2} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_2} + \frac{b^2 - 6ac}{l} \int \frac{dx}{R_2} + \frac{bc}{l} \int \frac{x^2 dx}{R_2}$$
5.
$$\int \frac{dx}{R_2^n} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_{n-1}^2} + \frac{(4n-7)bc}{l} \int \frac{x^2 dx}{R_2^{n-1}} + \frac{2(n-1)h^2 + 2ac - b^2}{l} \int \frac{dx}{R_2^{n-1}} \quad [n > 1] \quad \text{LA 146}$$
- 6.⁹
$$\int \frac{dx}{x^m R_2^n} = -\frac{1}{(m-1)ax^{m-1}R_2^{n-1}} - \frac{(m+2n-3)b}{(m-1)a} \int \frac{dx}{x^{m-2}R_2^n} - \frac{(m+4n-5)bc}{(m-1)a} \int \frac{dx}{x^{m-4}R_2^n} \quad \text{LA 147 (12)a}$$

2.17 Forms containing the quadratic trinomial $a + bx + cx^2$ and powers of x

Notation: $R = a + bx + cx^2$; $\Delta = 4ac - b^2$

2.171

1.
$$\int x^{m+1} R^n dx = \frac{x^m R^{n+1}}{c(m+2n+2)} - \frac{am}{c(m+2n+2)} \int x^{m-1} R^n dx - \frac{b(m+n+1)}{c(m+2n+2)} \int x^m R^n dx \quad \text{TI (97)}$$
 2.
$$\int \frac{R^n dx}{x^{m+1}} = -\frac{R^{n+1}}{amx^m} + \frac{b(n-m+1)}{am} \int \frac{R^n dx}{x^m} + \frac{c(2n-m+2)}{am} \int \frac{R^n dx}{x^{m-1}} \quad \text{LA 142(3), TI (96)a}$$
 3.
$$\int \frac{dx}{R^{n+1}} = \frac{b+2cx}{n\Delta R^n} + \frac{(4n-2)c}{n\Delta} \int \frac{dx}{R^n} \quad \text{TI (94)a}$$
 4.
$$\int \frac{dx}{R^{n+1}} = \frac{(2cx+b)}{2n+1} \sum_{k=0}^{n-1} \frac{2k(2n+1)(2n-1)(2n-3)\dots(2n-2k+1)c^k}{n(n-1)\dots(n-k)\Delta^{k+1}R^{n-k}} + 2^n \frac{(2n-1)!!c^n}{n!\Delta^n} \int \frac{dx}{R} \quad \text{TI (96)a}$$
- 2.172**¹¹
$$\int \frac{dx}{R} = \frac{1}{\sqrt{-\Delta}} \ln \frac{\sqrt{-\Delta} - (b+2cx)}{(b+2cx) + \sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}} \operatorname{arctanh} \frac{b+2cx}{\sqrt{-\Delta}} \quad \text{for } [\Delta < 0]$$

$$= \frac{-2}{b+2cx} \quad \text{for } [\Delta = 0, b \text{ and } c \text{ non-zero}]$$

$$= \frac{2}{\sqrt{\Delta}} \arctan \frac{b+2cx}{\sqrt{\Delta}} \quad \text{for } [\Delta > 0]$$

2.173

$$1. \quad \int \frac{dx}{R^2} = \frac{b+2cx}{\Delta R} + \frac{2c}{\Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$2. \quad \int \frac{dx}{R^3} = \frac{b+2cx}{\Delta} \left\{ \frac{1}{2R^2} + \frac{3c}{\Delta R} \right\} + \frac{6c^2}{\Delta^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

2.174

$$1. \quad \int \frac{x^m dx}{R^n} = -\frac{x^{m-1}}{(2n-m-1)cR^{n-1}} - \frac{(n-m)b}{(2n-m-1)c} \int \frac{x^{m-1} dx}{R^n} + \frac{(m-1)a}{(2n-m-1)c} \int \frac{x^{m-2} dx}{R^n}$$

For $m = 2n - 1$, this formula is inapplicable. Instead, we may use

$$2. \quad \int \frac{x^{2n-1} dx}{R^n} = \frac{1}{c} \int \frac{x^{2n-3} dx}{R^{n-1}} - \frac{a}{c} \int \frac{x^{2n-3} dx}{R^n} - \frac{b}{c} \int \frac{x^{2n-2} dx}{R^n}$$

2.175

$$1. \quad \int \frac{x dx}{R} = \frac{1}{2c} \ln R - \frac{b}{2c} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$2. \quad \int \frac{x dx}{R^2} = -\frac{2a+bx}{\Delta R} - \frac{b}{\Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$3. \quad \int \frac{x dx}{R^3} = -\frac{2a+bx}{2\Delta R^2} - \frac{3b(b+2cx)}{2\Delta^2 R} - \frac{3bc}{\Delta^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$4. \quad \int \frac{x^2 dx}{R} = \frac{x}{c} - \frac{b}{2c^2} \ln R + \frac{b^2-2ac}{2c^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$5. \quad \int \frac{x^2 dx}{R^2} = \frac{ab+(b^2-2ac)x}{c\Delta R} + \frac{2a}{\Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$6. \quad \int \frac{x^2 dx}{R^3} = \frac{ab+(b^2-2ac)x}{2c\Delta R^2} + \frac{(2ac+b^2)(b+2cx)}{2c\Delta^2 R} + \frac{2ac+b^2}{\Delta^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$7. \quad \int \frac{x^3 dx}{R} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2-ac}{2c^3} \ln R - \frac{b(b^2-3ac)}{2c^3} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$8. \quad \int \frac{x^3 dx}{R^2} = \frac{1}{2c^2} \ln R + \frac{a(2ac-b^2)+b(3ac-b^2)x}{c^2\Delta R} - \frac{b(6ac-b^2)}{2c^2\Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$9. \quad \int \frac{x^3 dx}{R^3} = -\left(\frac{x^2}{c} + \frac{abx}{c\Delta} + \frac{2a^2}{c\Delta}\right) \frac{1}{2R^2} - \frac{3ab}{2c\Delta} \int \frac{dx}{R^2} \quad (\text{see } \mathbf{2.173 } 1)$$

$$\mathbf{2.176} \quad \int \frac{dx}{x^m R^n} = \frac{-1}{(m-1)ax^{m-1}R^{n-1}} - \frac{b(m+n-2)}{a(m-1)} \int \frac{dx}{x^{m-1}R^n} - \frac{c(m+2n-3)}{a(m-1)} \int \frac{dx}{x^{m-2}R^n}$$

2.177

$$1. \quad \int \frac{dx}{xR} = \frac{1}{2a} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172})$$

$$2. \quad \int \frac{dx}{xR^2} = \frac{1}{2a^2} \ln \frac{x^2}{R} + \frac{1}{2aR} \left\{ 1 - \frac{b(b+2cx)}{\Delta} \right\} - \frac{b}{2a^2} \left(1 + \frac{2ac}{\Delta} \right) \int \frac{dx}{R}$$

(see **2.172**)

$$3. \quad \int \frac{dx}{xR^3} = \frac{1}{4aR^2} + \frac{1}{2a^2R} + \frac{1}{2a^3} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R^3} - \frac{b}{2a^2} \int \frac{dx}{R^2} - \frac{b}{2a^3} \int \frac{dx}{R}$$

(see **2.172**, **2.173**)

$$4. \quad \int \frac{dx}{x^2R} = -\frac{b}{2a^2} \ln \frac{x^2}{R} - \frac{1}{ax} + \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{R}$$

(see **2.172**)

$$5. \quad \int \frac{dx}{x^2R^2} = -\frac{b}{a^3} \ln \frac{x^2}{R} - \frac{a+bx}{a^2xR} + \frac{(b^2 - 3ac)(b+2cx)}{a^2\Delta R} - \frac{1}{\Delta} \left(\frac{b^4}{a^3} - \frac{6b^2c}{a^2} + \frac{6c^2}{a} \right) \int \frac{dx}{R}$$

(see **2.172**)

$$6. \quad \int \frac{dx}{x^2R^3} = -\frac{1}{axR^2} - \frac{3b}{a} \int \frac{dx}{xR^3} - \frac{5c}{a} \int \frac{dx}{R^3}$$

(see **2.173** and **2.177** 3)

$$7. \quad \int \frac{dx}{x^3R} = -\frac{ac - b^2}{2a^3} \ln \frac{x^2}{R} + \frac{b}{a^2x} - \frac{1}{2ax^2} + \frac{b(3ac - b^2)}{2a^3} \int \frac{dx}{R}$$

(see **2.172**)

$$8. \quad \int \frac{dx}{x^3R^2} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2x} \right) \frac{1}{R} + \left(\frac{3b^2}{a^2} - \frac{2c}{a} \right) \int \frac{dx}{xR^2} + \frac{9bc}{2a^2} \int \frac{dx}{R^2}$$

(see **2.173** 1 and **2.177** 2)

$$9. \quad \int \frac{dx}{x^3R^3} = \left(\frac{-1}{2ax^2} + \frac{2b}{a^2x} \right) \frac{1}{R^2} + \left(\frac{6b^2}{a^2} - \frac{3c}{a} \right) \int \frac{dx}{xR^3} + \frac{10bc}{a^2} \int \frac{dx}{R^3}$$

(see **2.173** 2 and **2.177** 3)

2.18 Forms containing the quadratic trinomial $a+bx+cx^2$ and the binomial $\alpha+\beta x$

Notation: $R = a+bx+cx^2$; $z = \alpha+\beta x$; $A = a\beta^2 - \alpha b\beta + c\alpha^2$;

$$B = b\beta - 2c\alpha; \quad \Delta = 4ac - b^2$$

$$1. \quad \int z^m R^n dx = \frac{\beta z^{m-1} R^{n+1}}{(m+2n+1)c} - \frac{(m+n)B}{(m+2n+1)c} \int z^{m-1} R^n dx - \frac{(m-1)A}{(m+2n+1)c} \int z^{m-2} R^n dx$$

$$2. \quad \int \frac{R^n dx}{z^m} = -\frac{1}{(m-2n-1)\beta} \frac{R^n}{z^{m-1}} - \frac{2nA}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^m} \\ - \frac{nB}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}}; \quad \text{LA 184 (4)a}$$

$$= \frac{-\beta}{(m-1)A} \frac{R^{n+1}}{z^{m-1}} - \frac{(m-n-2)B}{(m-1)A} \int \frac{R^n dx}{z^{m-1}} - \frac{(m-2n-3)c}{(m-1)A} \int \frac{R^n dx}{z^{m-2}} \quad \text{LA 148 (5)}$$

$$= -\frac{1}{(m-1)\beta} \frac{R^n}{z^{m-1}} + \frac{nB}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}} + \frac{2nc}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-2}} \quad \text{LA 418 (6)}$$

$$3. \quad \int \frac{z^m dx}{R^n} = \frac{\beta}{(m-2n+1)c} \frac{z^{m-1}}{R^{n-1}} - \frac{(m-n)B}{(m-2n+1)c} \int \frac{z^{m-1} dx}{R^n} - \frac{(m-1)A}{(m-2n+1)c} \int \frac{z^{m-2} dx}{R^n}$$

LA 147 (1)

$$= \frac{b+2cx}{(n-1)\Delta} \frac{z^m}{R^{n-1}} - \frac{2(m-2n+3)c}{(n-1)\Delta} \int \frac{z^m dx}{R^{n-1}} - \frac{Bm}{(n-1)\Delta} \int \frac{z^{m-1} dx}{R^{n-1}}$$

LA 148 (3)

$$4.^3 \quad \int \frac{dx}{z^m R^n} = -\frac{\beta}{(m-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{(m+n-2)B}{(m-1)A} \int \frac{dx}{z^{m-1} R^n} - \frac{(m+2n-3)c}{(m-1)A} \int \frac{dx}{z^{m-2} R^n}$$

LA 148 (7)

$$= \frac{\beta}{2(n-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{B}{2A} \int \frac{dx}{z^{m-1} R^n} + \frac{(m+2n-3)\beta^2}{2(n-1)A} \int \frac{dx}{z^m R^{n-1}}$$

LA 148 (8)

For $m = 1$ and $n = 1$

$$\int \frac{dx}{zR} = \frac{\beta}{2A} \ln \frac{z^2}{R} - \frac{B}{2A} \int \frac{dx}{R}$$

For $A = 0$

$$\int \frac{dx}{z^m R^n} = -\frac{\beta}{(m+n-1)B} \frac{1}{z^m R^{n-1}} - \frac{(m+2n-2)c}{(m+n-1)B} \int \frac{dx}{z^{m-1} R^n}$$

LA 148 (9)

2.2 Algebraic Functions

2.20 Introduction

2.201 The integrals $\int R \left(x, \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right)^r, \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right)^s, \dots \right) dx$, where $r, s, \dots$ are rational numbers, can be reduced to integrals of rational functions by means of the substitution

$$\frac{\alpha x + \beta}{\gamma x + \delta} = t^m, \quad \text{FI II 57}$$

where m is the common denominator of the fractions $r, s, \dots$

2.202 Integrals of the form $\int x^m (a + bx^n)^p dx$,* where m, n , and p are rational numbers, can be expressed in terms of elementary functions only in the following cases:

- (a) When p is an integer; then, this integral takes the form of a sum of the integrals shown in **2.201**;
- (b) When $\frac{m+1}{n}$ is an integer: by means of the substitution $x^n = z$, this integral can be transformed to the form $\frac{1}{n} \int (a + bz)^p z^{\frac{m+1}{n}-1} dz$, which we considered in **2.201**;
- (c) When $\frac{m+1}{n} + p$ is an integer; by means of the same substitution $x^n = z$, this integral can be reduced to an integral of the form $\frac{1}{n} \int \left(\frac{a+bz}{z} \right)^p z^{\frac{m+1}{n}+p-1} dz$, considered in **2.201**;

For reduction formulas for integrals of binomial differentials, see **2.110**.

*Translator: The authors term such integrals “integrals of binomial differentials.”

2.21 Forms containing the binomial $a + bx^k$ and $\sqrt{x}$

Notation: $z_1 = a + bx$.

$$\mathbf{2.211} \quad \int \frac{dx}{z_1 \sqrt{x}} = \frac{2}{\sqrt{ab}} \arctan \sqrt{\frac{bx}{a}} \quad [ab > 0]$$

$$= \frac{1}{i\sqrt{ab}} \ln \frac{a - bx + 2i\sqrt{xab}}{z_1} \quad [ab < 0]$$

$$\mathbf{2.212} \quad \int \frac{x^m \sqrt{x}}{z_1} dx = 2\sqrt{x} \sum_{k=0}^m \frac{(-1)^k a^k x^{m-k}}{(2m-2k+1)b^{k+1}} + (-1)^{m+1} \frac{a^{m+1}}{b^{m+1}} \int \frac{dx}{z_1 \sqrt{x}}$$

(see **2.211**)

2.213

$$1. \quad \int \frac{\sqrt{x} dx}{z_1} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211})$$

$$2. \quad \int \frac{x\sqrt{x} dx}{z_1} = \left(\frac{x}{3b} - \frac{a}{b^2}\right) 2\sqrt{x} + \frac{a^2}{b^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211})$$

$$3. \quad \int \frac{x^2 \sqrt{x} dx}{z_1} = \left(\frac{x^2}{5b} - \frac{xa}{3b^2} + \frac{a^2}{b^3}\right) 2\sqrt{x} - \frac{a^3}{b^3} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211})$$

$$4. \quad \int \frac{dx}{z_1^2 \sqrt{x}} = \frac{\sqrt{x}}{az_1} + \frac{1}{2a} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211})$$

$$5. \quad \int \frac{\sqrt{x} dx}{z_1^2} = -\frac{\sqrt{x}}{bz_1} + \frac{1}{2b} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211})$$

$$6. \quad \int \frac{x\sqrt{x} dx}{z_1^2} = \frac{2x\sqrt{x}}{bz_1} - \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{see } \mathbf{2.213} \ 5)$$

$$7. \quad \int \frac{x^2 \sqrt{x} dx}{z_1^2} = \left(\frac{x^2}{3b} - \frac{5ax}{3b^2}\right) \frac{2\sqrt{x}}{z_1} + \frac{5a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{see } \mathbf{2.213} \ 5)$$

$$8. \quad \int \frac{dx}{z_1^3 \sqrt{x}} = \left(\frac{1}{2az_1^2} + \frac{3}{4a^2 z_1}\right) \sqrt{x} + \frac{3}{8a^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211})$$

$$9. \quad \int \frac{\sqrt{x} dx}{z_1^3} = \left(-\frac{1}{2bz_1^2} + \frac{1}{4abz_1}\right) \sqrt{x} + \frac{1}{8ab} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211})$$

$$10. \quad \int \frac{x\sqrt{x} dx}{z_1^3} = -\frac{2x\sqrt{x}}{bz_1^2} + \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^3} \quad (\text{see } \mathbf{2.213} \ 9)$$

$$11. \quad \int \frac{x^2 \sqrt{x} dx}{z_1^3} = \left(\frac{x^2}{b} + \frac{5ax}{b^2}\right) \frac{2\sqrt{x}}{z_1^2} - \frac{15a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^3} \quad (\text{see } \mathbf{2.213} \ 9)$$

Notation: $z_2 = a + bx^2$, $\alpha = \sqrt[4]{\frac{a}{b}}$, $\alpha' = \sqrt[4]{-\frac{a}{b}}$.

$$\mathbf{2.214} \quad \int \frac{dx}{z_2 \sqrt{x}} = \frac{1}{b\alpha^3 \sqrt{2}} \left[\ln \frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \arctan \frac{\alpha\sqrt{2x}}{\alpha^2 - x} \right] \quad \left[\frac{a}{b} > 0\right]$$

$$= \frac{1}{2b\alpha'^3} \left(\ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} - 2 \arctan \frac{\sqrt{x}}{\alpha'} \right) \quad \left[\frac{a}{b} < 0\right]$$

$$\begin{aligned}
 2.215 \quad \int \frac{\sqrt{x} dx}{z_2} &= \frac{1}{b\alpha\sqrt{2}} \left[-\ln \frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \arctan \frac{\alpha\sqrt{2x}}{\alpha^2 - x} \right] & \left[\frac{a}{b} > 0 \right] \\
 &= \frac{1}{2b\alpha'} \left[\ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} + 2 \arctan \frac{\sqrt{x}}{\alpha'} \right] & \left[\frac{a}{b} < 0 \right]
 \end{aligned}$$

2.216

1. $\int \frac{x\sqrt{x} dx}{z_2} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_2\sqrt{x}}$ (see 2.214)
2. $\int \frac{x^2\sqrt{x} dx}{z_2} = \frac{2x\sqrt{x}}{3b} - \frac{a}{b} \int \frac{\sqrt{x} dx}{z_2}$ (see 2.215)
3. $\int \frac{dx}{z_2^2\sqrt{x}} = \frac{\sqrt{x}}{2az_2} + \frac{3}{4a} \int \frac{dx}{z_2\sqrt{x}}$ (see 2.214)
4. $\int \frac{\sqrt{x} dx}{z_2^2} = \frac{x\sqrt{x}}{2az_2} + \frac{1}{4a} \int \frac{\sqrt{x} dx}{z_2}$ (see 2.215)
5. $\int \frac{x\sqrt{x} dx}{z_2^2} = -\frac{\sqrt{x}}{2bz_2} + \frac{1}{4b} \int \frac{dx}{z_2\sqrt{x}}$ (see 2.214)
6. $\int \frac{x^2\sqrt{x} dx}{z_2^2} = -\frac{x\sqrt{x}}{2bz_2} + \frac{3}{4b} \int \frac{\sqrt{x} dx}{z_2}$ (see 2.215)
7. $\int \frac{dx}{z_2^3\sqrt{x}} = \left(\frac{1}{4az_2^2} + \frac{7}{16a^2z_2} \right) \sqrt{x} + \frac{21}{32a^2} \int \frac{dx}{z_2\sqrt{x}}$ (see 2.214)
8. $\int \frac{\sqrt{x} dx}{z_2^3} = \left(\frac{1}{4az_2^2} + \frac{5}{16a^2z_2} \right) x\sqrt{x} + \frac{5}{32a^2} \int \frac{\sqrt{x} dx}{z_2}$ (see 2.215)
9. $\int \frac{x\sqrt{x} dx}{z_2^3} = \frac{(bx^2 - 3a)\sqrt{x}}{16abz_2^2} + \frac{3}{32ab} \int \frac{dx}{z_2\sqrt{x}}$ (see 2.214)
10. $\int \frac{x^2\sqrt{x} dx}{z_2^3} = -\frac{2x\sqrt{x}}{5bz_2^2} + \frac{3a}{5b} \int \frac{\sqrt{x} dx}{z_2^3}$ (see 2.216 8)

2.22–2.23 Forms containing $\sqrt[n]{(a + bx)^k}$

Notation: $z = a + bx$.

$$2.220 \quad \int x^n \sqrt[l]{z^{lm+f}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{ln - lk + l(m+1) + f} \right\} \frac{l \sqrt[l]{z^{l(m+1)+f}}}{b^{n+1}}$$

The square root

$$2.221 \quad \int x^n \sqrt{z^{2m+1}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{2n - 2k + 2m + 1} \right\} \frac{2\sqrt{z^{2m+1}}}{b^{n+1}}$$

2.222

$$1. \quad \int \frac{dx}{\sqrt{z}} = \frac{2}{b} \sqrt{z}$$

$$2. \quad \int \frac{x \, dx}{\sqrt{z}} = \left(\frac{1}{3}z - a \right) \frac{2\sqrt{z}}{b^2}$$

$$3. \quad \int \frac{x^2 \, dx}{\sqrt{z}} = \left(\frac{1}{5}z^2 - \frac{2}{3}az + a^2 \right) \frac{2\sqrt{z}}{b^3}$$

2.223

$$1. \quad \int \frac{dx}{\sqrt{z^3}} = -\frac{2}{b\sqrt{z}}$$

$$2. \quad \int \frac{x \, dx}{\sqrt{z^3}} = (z + a) \frac{2}{b^2\sqrt{z}}$$

$$3. \quad \int \frac{x^2 \, dx}{\sqrt{z^3}} = \left(\frac{z^2}{3} - 2az - a^2 \right) \frac{2}{b^3\sqrt{z}}$$

2.224

$$1. \quad \int \frac{z^m \, dx}{x^n \sqrt{z}} = -\frac{z^m \sqrt{z}}{(n-1)ax^{n-1}} + \frac{2m-2n+3}{2(n-1)} \frac{b}{a} \int \frac{z^m \, dx}{x^{n-1} \sqrt{z}}$$

$$2. \quad \int \frac{z^m \, dx}{x^n \sqrt{z}} = -z^m \sqrt{z} \left\{ \frac{1}{(n-1)ax^{n-1}} + \sum_{k=1}^{n-2} \frac{(2m-2n+3)(2m-2n+5) \dots (2m-2n+2k+1)}{2^k(n-1)(n-2) \dots (n-k-1)x^{n-k-1}} \frac{b^k}{a^{k+1}} \right\} \\ + \frac{(2m-2n+3)(2m-2n+5) \dots (2m-3)(2m-1)}{2^{n-1}(n-1)!x} \frac{b^{n-1}}{a^{n-1}} \int \frac{z^m \, dx}{x \sqrt{z}}$$

For $n = 1$

$$3. \quad \int \frac{z^m}{x \sqrt{z}} \, dx = \frac{2z^m}{(2m-1)\sqrt{z}} + a \int \frac{z^{m-1}}{x \sqrt{z}} \, dx$$

$$4. \quad \int \frac{z^m}{x \sqrt{z}} \, dx = \sum_{k=1}^m \frac{2a^{m-k} z^k}{(2k-1)\sqrt{z}} + a^m \int \frac{dx}{x \sqrt{z}}$$

$$5.^6 \quad \int \frac{dx}{x \sqrt{z}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{z} - \sqrt{a}}{\sqrt{z} + \sqrt{a}} \right| \quad [a > 0] \\ = \frac{2}{\sqrt{-a}} \arctan \frac{\sqrt{z}}{\sqrt{-a}} \quad [a < 0]$$

2.225

$$1. \quad \int \frac{\sqrt{z} \, dx}{x} = 2\sqrt{z} + a \int \frac{dx}{x \sqrt{z}} \quad (\text{see 2.224 4})$$

$$2. \quad \int \frac{\sqrt{z} \, dx}{x^2} = -\frac{\sqrt{z}}{x} + \frac{b}{2} \int \frac{dx}{x \sqrt{z}} \quad (\text{see 2.224 4})$$

$$3. \quad \int \frac{\sqrt{z} \, dx}{x^3} = -\frac{\sqrt{z^3}}{2ax^2} + \frac{b\sqrt{z}}{4ax} - \frac{b^2}{8a} \int \frac{dx}{x \sqrt{z}} \quad (\text{see 2.224 4})$$

2.226

$$1. \quad \int \frac{\sqrt{z^3} dx}{x} = \left(\frac{z}{3} + a\right) 2\sqrt{z} + a^2 \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \ 4)$$

$$2. \quad \int \frac{\sqrt{z^3} dx}{x^2} = -\frac{\sqrt{z^5}}{ax} + \frac{3b}{2a} \int \frac{\sqrt{z^3} dx}{x} \quad (\text{see } \mathbf{2.226} \ 1)$$

$$3. \quad \int \frac{\sqrt{z^3} dx}{x^3} = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right) \sqrt{z^5} + \frac{3b^2}{8a^2} \int \frac{\sqrt{z^3} dx}{x} \quad (\text{see } \mathbf{2.226} \ 1)$$

$$\mathbf{2.227} \quad \int \frac{dx}{xz^m\sqrt{z}} = \sum_{k=0}^{m-1} \frac{2}{(2k+1)a^{m-k}z^k\sqrt{z}} + \frac{1}{a^m} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \ 4)$$

2.228

$$1. \quad \int \frac{dx}{x^2\sqrt{z}} = -\frac{\sqrt{z}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \ 4)$$

$$2. \quad \int \frac{dx}{x^3\sqrt{z}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right) \sqrt{z} + \frac{3b^2}{8a^2} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \ 4)$$

2.229

$$1. \quad \int \frac{dx}{x\sqrt{z^3}} = \frac{2}{a\sqrt{z}} + \frac{1}{a} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \ 4)$$

$$2. \quad \int \frac{dx}{x^2\sqrt{z^3}} = \left(-\frac{1}{ax} - \frac{3b}{a^2}\right) \frac{1}{\sqrt{z}} - \frac{3b}{2a^2} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \ 4)$$

$$3. \quad \int \frac{dx}{x^3\sqrt{z^3}} = \left(-\frac{1}{2ax^2} + \frac{5b}{4a^2x} + \frac{15b^2}{4a^3}\right) \frac{1}{\sqrt{z}} + \frac{15b^2}{8a^3} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \ 4)$$

Cube root**2.231**

$$1. \quad \int \sqrt[3]{z^{3m+1}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3(m+1) + 1} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+1}}}{b^{n+1}}$$

$$2. \quad \int \frac{x^n dx}{\sqrt[3]{z^{3m+2}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3(m-1) - 2} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+2}}}$$

$$3. \quad \int \sqrt[3]{z^{3m+2}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3(m+1) + 2} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+2}}}{b^{n+1}}$$

$$4. \quad \int \frac{x^n dx}{\sqrt[3]{z^{3m+1}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3(m-1) - 1} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+1}}}$$

$$5. \quad \int \frac{z^n dx}{x^m \sqrt[3]{x^2}} = -\frac{z^{n+\frac{1}{3}}}{(m-1)ax^{m-1}} + \frac{3n-3m+4}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[3]{z^2}}$$

For $m = 1$

$$\int \frac{z^n dx}{x \sqrt[3]{z^2}} = \frac{3z^n}{(3n-2)\sqrt[3]{z^2}} + a \int \frac{z^{n-1} dx}{x \sqrt[3]{z^2}}$$

$$6. \quad \int \frac{dx}{xz^n \sqrt[3]{z^2}} = \frac{3\sqrt[3]{z}}{(3n-1)az^n} + \frac{1}{a} \int \frac{\sqrt[3]{z} dx}{xz^n}$$

$$2.232 \quad \int \frac{dx}{x \sqrt[3]{z^2}} = \frac{1}{\sqrt[3]{a^2}} \left\{ \frac{3}{2} \ln \frac{\sqrt[3]{z} - \sqrt[3]{a}}{\sqrt[3]{x}} - \sqrt{3} \arctan \frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z} + 2\sqrt[3]{a}} \right\}$$

2.233

$$1. \quad \int \frac{\sqrt[3]{z} dx}{x} = 3\sqrt[3]{z} + a \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see } 2.232)$$

$$2. \quad \int \frac{\sqrt[3]{z} dx}{x^2} = -\frac{z\sqrt[3]{z}}{ax} + \frac{b}{a}\sqrt[3]{z} + \frac{b}{3} \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see } 2.232)$$

$$3. \quad \int \frac{\sqrt[3]{z} dx}{x^3} = \left(-\frac{1}{2ax^2} + \frac{b}{3a^2x} \right) z\sqrt[3]{z} - \frac{b^2}{3a^2}\sqrt[3]{z} - \frac{b^2}{9a} \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see } 2.232)$$

$$4. \quad \int \frac{dx}{x^2 \sqrt[3]{z^2}} = -\frac{\sqrt[3]{z}}{ax} - \frac{2b}{3a} \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see } 2.232)$$

$$5. \quad \int \frac{dx}{x^3 \sqrt[3]{z^2}} = \left[-\frac{1}{2ax^2} + \frac{5b}{6a^2x} \right] \sqrt[3]{z} + \frac{5b^2}{9a^2} \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see } 2.232)$$

2.234

$$1. \quad \int \frac{z^n dx}{x^m \sqrt[3]{z^2}} = -\frac{z^n \sqrt[3]{z^2}}{(m-1)ax^{m-1}} + \frac{3n-3m+5}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[3]{z}}$$

For $m = 1$:

$$2. \quad \int \frac{z^n dx}{x \sqrt[3]{z}} = \frac{3z^n}{(3n-1)\sqrt[3]{z}} + a \int \frac{z^{n-1} dx}{x \sqrt[3]{z}}$$

$$3. \quad \int \frac{dx}{xz^n \sqrt[3]{z}} = \frac{3\sqrt[3]{z^2}}{(3n-2)az^n} + \frac{1}{a} \int \frac{\sqrt[3]{z^2} dx}{xz^n}$$

$$2.235 \quad \int \frac{dx}{x \sqrt[3]{z}} = \frac{1}{\sqrt[3]{a^2}} \left\{ \frac{3}{2} \ln \frac{\sqrt[3]{z} - \sqrt[3]{a}}{\sqrt[3]{x}} + \sqrt{3} \arctan \frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z} + 2\sqrt[3]{a}} \right\}$$

2.236

$$1. \quad \int \frac{\sqrt[3]{z^2} dx}{x} = \frac{3}{2} \sqrt[3]{z^2} + a \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see } 2.235)$$

$$2. \quad \int \frac{\sqrt[3]{z^2} dx}{x^2} = -\frac{\sqrt[3]{z^5}}{ax} + \frac{b}{a} \sqrt[3]{z^2} + \frac{2b}{3} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see } 2.235)$$

$$3. \quad \int \frac{\sqrt[3]{z^2} dx}{x^3} = \left[-\frac{1}{2ax^2} + \frac{b}{6a^2x} \right] z^{5/3} - \frac{b^2}{6a^2} \sqrt[3]{z^2} - \frac{b^2}{9a} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

$$4. \quad \int \frac{dx}{x^2 \sqrt[3]{z}} = -\frac{\sqrt[3]{z^2}}{ax} - \frac{b}{3a} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

$$5. \quad \int \frac{dx}{x^3 \sqrt[3]{z}} = \left[-\frac{1}{2ax^2} + \frac{2b}{3a^2x} \right] \sqrt[3]{z} + \frac{2b^2}{9a^2} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

2.24 Forms containing $\sqrt{a+bx}$ and the binomial $\alpha + \beta x$

Notation: $z = a + bx$, $t = \alpha + \beta x$, $\Delta = a\beta - b\alpha$.

2.241

$$1. \quad \int \frac{z^m t^n dx}{\sqrt{z}} = \frac{2}{(2n+2m+1)\beta} t^{n+1} z^{m-1} \sqrt{z} + \frac{(2m-1)\Delta}{(2n+2m+1)\beta} \int \frac{z^{m-1} t^n dx}{\sqrt{z}} \quad \text{LA 176 (1)}$$

$$2. \quad \int \frac{t^n z^m dx}{\sqrt{z}} = 2\sqrt{z^{2m+1}} \sum_{k=0}^n \binom{n}{k} \frac{\alpha^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p} a^p}{2k-2p+2m+1}$$

2.242

$$1.^{11} \quad \int \frac{t dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z}}{b} + \beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2}$$

$$2. \quad \int \frac{t^2 dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z}}{b} + 2\alpha\beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + \beta^2 \left(\frac{z^2}{5} - \frac{2}{3}za + a^2 \right) \frac{2\sqrt{z}}{b^3}$$

$$3. \quad \int \frac{t^3 dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z}}{b} + 3\alpha^2\beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + 3\alpha\beta^2 \left(\frac{z^2}{5} - \frac{2}{3}za + a^2 \right) \frac{2\sqrt{z}}{b^3} \\ + \beta^3 \left(\frac{z^3}{7} - \frac{3z^2a}{5} + za^2 - a^3 \right) \frac{2\sqrt{z}}{b^4}$$

$$4. \quad \int \frac{tz dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^3}}{3b} + \beta \left(\frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2}$$

$$5. \quad \int \frac{t^2 z dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^3}}{3b} + 2\alpha\beta \left(\frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2} + \beta^2 \left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3} \right) \frac{2\sqrt{z^3}}{b^3}$$

$$6. \quad \int \frac{t^3 z dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^3}}{3b} + 3\alpha^2\beta \left(\frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2} + 3\alpha\beta^2 \left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3} \right) \frac{2\sqrt{z^3}}{b^3} \\ + \beta^3 \left(\frac{z^3}{9} - \frac{3z^2a}{7} + \frac{3za^2}{5} - \frac{a^3}{3} \right) \frac{2\sqrt{z^3}}{b^4}$$

$$7. \quad \int \frac{tz^2 dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^5}}{5b} + \beta \left(\frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2}$$

$$8. \quad \int \frac{t^2 z^2 dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^5}}{5b} + 2\alpha\beta \left(\frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2} + \beta^2 \left(\frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5} \right) \frac{2\sqrt{z^5}}{b^3}$$

$$9. \quad \int \frac{t^3 z^2 dx}{\sqrt{z}} = \frac{2\alpha^3 \sqrt{z^5}}{5b} + 3\alpha^2 \beta \left(\frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2} + 3\alpha \beta^2 \left(\frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5} \right) \frac{2\sqrt{z^5}}{b^3} \\ + \beta^3 \left(\frac{z^3}{11} - \frac{3z^2 a}{9} + \frac{3za^2}{7} - \frac{a^3}{5} \right) \frac{2\sqrt{z^5}}{b^4}$$

$$10. \quad \int \frac{tz^3 dx}{\sqrt{z}} = \frac{2\alpha \sqrt{z^7}}{7b} + \beta \left(\frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2}$$

$$11. \quad \int \frac{t^2 z^3 dx}{\sqrt{z}} = \frac{2\alpha^2 \sqrt{z^7}}{7b} + 2\alpha \beta \left(\frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2} + \beta^2 \left(\frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7} \right) \frac{2\sqrt{z^7}}{b^3}$$

$$12. \quad \int \frac{t^3 z^3 dx}{\sqrt{z}} = \frac{2\alpha^3 \sqrt{z^7}}{7b} + 3\alpha^2 \beta \left(\frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2} + 3\alpha \beta^2 \left(\frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7} \right) \frac{2\sqrt{z^7}}{b^3} \\ + \beta^3 \left(\frac{z^3}{13} - \frac{3z^2 a}{11} + \frac{3za^2}{9} - \frac{a^3}{7} \right) \frac{2\sqrt{z^7}}{b^4}$$

2.243

$$1. \quad \int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{t^{n+1}}{z^m} \sqrt{z} - \frac{(2n-2m+3)\beta}{(2m-1)\Delta} \int \frac{t^n dx}{z^{m-1} \sqrt{z}} \\ = -\frac{2}{(2m-1)b} \frac{t^n}{z^m} \sqrt{z} + \frac{2n\beta}{(2m-1)b} \int \frac{t^{n-1} dx}{z^{m-1} \sqrt{z}}$$

LA 176 (2)

$$2. \quad \int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{\sqrt{z^{2m-1}}} \sum_{k=0}^n \binom{n}{k} \frac{a^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p} a^p}{2k-2p-2m+1}$$

2.244

$$1. \quad \int \frac{t dx}{z \sqrt{z}} = -\frac{2a}{b \sqrt{z}} + \frac{2\beta(z+a)}{b^2 \sqrt{z}}$$

$$2. \quad \int \frac{t^2 dx}{z \sqrt{z}} = -\frac{2\alpha^2}{b \sqrt{z}} + \frac{4\alpha\beta(z+a)}{b^2 \sqrt{z}} + \frac{2\beta^2 \left(\frac{z^2}{3} - 2za - a^2 \right)}{b^3 \sqrt{z}}$$

$$3. \quad \int \frac{t^3 dx}{z \sqrt{z}} = -\frac{2\alpha^3}{b \sqrt{z}} + \frac{6\alpha^2 \beta(z+a)}{b^2 \sqrt{z}} + \frac{6\alpha \beta^2 \left(\frac{z^2}{3} - 2za - a^2 \right)}{b^3 \sqrt{z}} + \frac{2\beta^3 \left(\frac{z^3}{5} - z^2 a + 3za^2 + a^3 \right)}{b^4 \sqrt{z}}$$

$$4. \quad \int \frac{t dx}{z^2 \sqrt{z}} = -\frac{2a}{3b \sqrt{z^3}} - \frac{2\beta \left(z - \frac{a}{3} \right)}{b^2 \sqrt{z^3}}$$

$$5. \quad \int \frac{t^2 dx}{z^2 \sqrt{z}} = -\frac{2a^2}{3b \sqrt{z^3}} - \frac{4\alpha\beta \left(z - \frac{a}{3} \right)}{b^2 \sqrt{z^3}} + \frac{2\beta^2 \left(z^2 + 2az - \frac{a^2}{3} \right)}{b^3 \sqrt{z^3}}$$

$$6. \quad \int \frac{t^3 dx}{z^2 \sqrt{z}} = -\frac{2\alpha^3}{3b \sqrt{z^3}} - \frac{6\alpha^2 \beta \left(z - \frac{a}{3} \right)}{b^2 \sqrt{z^3}} + \frac{6\alpha \beta^2 \left(z^2 + 2za - \frac{a^2}{3} \right)}{b^3 \sqrt{z^3}} + \frac{2\beta^3 \left(\frac{z^3}{3} - 3z^2 a - 3za^2 + \frac{a^3}{3} \right)}{b^4 \sqrt{z^3}}$$

$$7. \quad \int \frac{t dx}{z^3 \sqrt{z}} = -\frac{2\alpha}{5b \sqrt{z^5}} - \frac{2\beta \left(\frac{z}{3} - \frac{a}{5} \right)}{b^2 \sqrt{z^5}}$$

$$\begin{aligned}
 8. \quad \int \frac{t^2 dx}{z^3 \sqrt{z}} &= -\frac{2\alpha^2}{5b\sqrt{z^5}} - \frac{4\alpha\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{2\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3\sqrt{z^5}} \\
 9. \quad \int \frac{t^3 dx}{z^3 \sqrt{z}} &= -\frac{2\alpha^3}{5b\sqrt{z^5}} - \frac{6\alpha^2\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{6\alpha\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3\sqrt{z^5}} \\
 &\quad + \frac{2\beta^3\left(z^3 + 3z^2a - za^2 + \frac{a^3}{5}\right)}{b^4\sqrt{z^5}}
 \end{aligned}$$

2.245

$$\begin{aligned}
 1. \quad \int \frac{z^m dx}{t^n \sqrt{z}} &= -\frac{2}{(2n-2m-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta} \int \frac{z^{m-1} dx}{t^n \sqrt{z}} \\
 &= -\frac{1}{(n-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta} \int \frac{z^{m-1}}{t^{n-1} \sqrt{z}} dx \\
 &= -\frac{1}{(n-1)\Delta} \frac{z^m}{t^{n-1}} \sqrt{z} - \frac{(2n-2m-3)b}{2(n-1)\Delta} \int \frac{z^m dx}{t^{n-1} \sqrt{z}} \\
 2. \quad \int \frac{z^m dz}{t^n \sqrt{z}} &= -z^m \sqrt{z} \left[\frac{1}{(n-1)\Delta} \frac{1}{t^{n-1}} \right. \\
 &\quad \left. + \sum_{k=2}^{n-1} \frac{(2n-2m-3)(2n-2m-5)\dots(2n-2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k} \frac{1}{t^{n-k}} \right] \\
 &\quad - \frac{(2n-2m-3)(2n-2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1} \cdot (n-1)!\Delta^n} \int \frac{z^m dx}{t \sqrt{z}}
 \end{aligned}$$

LA 176 (3)

For $n = 1$

$$\begin{aligned}
 3. \quad \int \frac{z^m dx}{t \sqrt{z}} &= \frac{2}{(2m-1)\beta} \frac{z^m}{\sqrt{z}} + \frac{\Delta}{\beta} \int \frac{z^{m-1} dx}{t \sqrt{z}} \\
 4. \quad \int \frac{z^m dx}{t \sqrt{z}} &= 2 \sum_{k=0}^{m-1} \frac{\Delta^k}{(2m-2k-1)\beta^{k+1}} \frac{z^{m-k}}{\sqrt{z}} + \frac{\Delta^m}{\beta^m} \int \frac{dx}{t \sqrt{z}}
 \end{aligned}$$

$$\begin{aligned}
 2.246 \quad \int \frac{dx}{t \sqrt{z}} \frac{1}{\sqrt{\beta\Delta}} \ln \frac{\beta\sqrt{z} - \sqrt{\beta\Delta}}{\beta\sqrt{z} + \sqrt{\beta\Delta}} & \quad [\beta\Delta > 0] \\
 &= \frac{2}{\sqrt{-\beta\Delta}} \arctan \frac{\beta\sqrt{z}}{\sqrt{-\beta\Delta}} \quad [\beta\Delta < 0] \\
 &= -\frac{2\sqrt{z}}{bt} \quad [\Delta = 0]
 \end{aligned}$$

$$2.247 \quad \int \frac{dx}{tz^m \sqrt{z}} = \frac{2}{z^{m-1} \sqrt{z}} + \sum_{k=1}^m \frac{\beta^{k-1} z^k}{\Delta^k (2m-2k+1)} + \frac{\beta^m}{\Delta^m} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

2.248

$$1. \quad \int \frac{dx}{tz \sqrt{z}} = \frac{2}{\Delta \sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{t \sqrt{z}} \quad (\text{see 2.246})$$

$$2. \quad \int \frac{dx}{tz^2\sqrt{z}} = \frac{2}{3\Delta z\sqrt{z}} + \frac{2\beta}{\Delta^2\sqrt{z}} + \frac{\beta^2}{\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$3. \quad \int \frac{dx}{tz^3\sqrt{z}} = \frac{2}{5\Delta z^2\sqrt{z}} + \frac{2\beta}{3\Delta^2 z\sqrt{z}} + \frac{2\beta^2}{\Delta^3\sqrt{z}} + \frac{\beta^3}{\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$4. \quad \int \frac{dx}{t^2\sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$5. \quad \int \frac{dx}{t^2 z\sqrt{z}} = -\frac{1}{\Delta t\sqrt{z}} - \frac{3b}{\Delta^2\sqrt{z}} - \frac{3b\beta}{2\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$6. \quad \int \frac{dx}{t^2 z^2\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{5b}{3\Delta^2 z\sqrt{z}} - \frac{5b\beta}{\Delta^3\sqrt{z}} - \frac{5b\beta^2}{2\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$7. \quad \int \frac{dx}{t^2 z^3\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{7b}{5\Delta^2 z^2\sqrt{z}} - \frac{7b\beta}{3\Delta^3 z\sqrt{z}} - \frac{7b\beta^2}{\Delta^4\sqrt{z}} - \frac{7b\beta^3}{2\Delta^4} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$8. \quad \int \frac{dx}{t^3\sqrt{z}} = -\frac{\sqrt{z}}{2\Delta t^2} + \frac{3b\sqrt{z}}{4\Delta^2 t} + \frac{3b^2}{8\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$9. \quad \int \frac{dx}{t^3 z\sqrt{z}} = -\frac{1}{2\Delta t^2\sqrt{z}} + \frac{5b}{4\Delta^2 t\sqrt{z}} + \frac{15b^2}{4\Delta^3\sqrt{z}} + \frac{15b^2\beta}{8\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$10. \quad \int \frac{dx}{t^3 z^2\sqrt{z}} = -\frac{1}{2\Delta t^2 z\sqrt{z}} + \frac{7b\sqrt{z}}{4\Delta^2 t z\sqrt{z}} + \frac{35b^2}{12\Delta^2 z\sqrt{z}} + \frac{35b^2\beta}{4\Delta^4\sqrt{z}} + \frac{35b^2\beta^2}{8\Delta^4} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$11. \quad \int \frac{dx}{t^3 z^3\sqrt{z}} = -\frac{1}{2\Delta t^2 z^2\sqrt{z}} + \frac{9b}{4\Delta^2 t z^2\sqrt{z}} + \frac{63b^2}{20\Delta^3 z^2\sqrt{z}} + \frac{21b^2\beta}{4\Delta^4 z\sqrt{z}} + \frac{63b^2\beta^2}{4\Delta^5\sqrt{z}} + \frac{63b^2\beta^3}{8\Delta^5} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$12. \quad \int \frac{z dx}{t\sqrt{z}} = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$13. \quad \int \frac{z^2 dx}{t\sqrt{z}} = \frac{2z\sqrt{z}}{3\beta} + \frac{2\Delta\sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$14. \quad \int \frac{z^3 dx}{t\sqrt{z}} = \frac{2z^2\sqrt{z}}{5\beta} + \frac{2\Delta z\sqrt{z}}{3\beta^2} + \frac{2\Delta^2\sqrt{z}}{\beta^3} + \frac{\Delta^3}{\beta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$15. \quad \int \frac{z dx}{t^2\sqrt{z}} = -\frac{z\sqrt{z}}{\Delta t} + \frac{b\sqrt{z}}{\beta\Delta} + \frac{b}{2\beta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$16. \quad \int \frac{z^2 dx}{t^2\sqrt{z}} = -\frac{z^2\sqrt{z}}{\Delta t} + \frac{bz\sqrt{z}}{\beta\Delta} + \frac{3b\sqrt{z}}{\beta^2} + \frac{3b\Delta}{2\beta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$17. \quad \int \frac{z^3 dx}{t^2 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{\Delta t} + \frac{bz^2 \sqrt{z}}{\beta \Delta} + \frac{5bz \sqrt{z}}{3\beta^2} + \frac{5b\Delta \sqrt{z}}{\beta^3} + \frac{5\Delta^2 b}{2\beta^3} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

$$18.^3 \quad \int \frac{z dx}{t^3 \sqrt{z}} = -\frac{z \sqrt{z}}{2\Delta t^2} + \frac{bz \sqrt{z}}{4\Delta^2 t} - \frac{b^2 \sqrt{z}}{4\beta \Delta^2} + \frac{b^2}{8\beta \Delta} \int \frac{dx}{t \sqrt{z}} \quad (\text{see 2.246})$$

$$19. \quad \int \frac{z^2 dx}{t^3 \sqrt{z}} = -\frac{z^2 \sqrt{z}}{2\Delta t^2} + \frac{bz^2 \sqrt{z}}{4\Delta^2 t} + \frac{b^2 z \sqrt{z}}{4\beta \Delta^2} + \frac{3b^2 \sqrt{z}}{4\beta^2 \Delta} + \frac{3b^2}{8\beta^2} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

$$20. \quad \int \frac{z^3 dx}{t^3 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{2\Delta t^2} + \frac{3bz^3 \sqrt{z}}{\Delta^2 t} + \frac{3b^2 z^2 \sqrt{z}}{4\beta \Delta^2} + \frac{5b^2 z \sqrt{z}}{4\beta^2 \Delta} + \frac{15b^2 \sqrt{z}}{4\beta^3} + \frac{15b^2 \Delta}{8\beta^3} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

2.249

$$1. \quad \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{\sqrt{z}}{t^{n-1} z^m} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta} \int \frac{dx}{t^n z^{m-1} \sqrt{z}}$$

LA 177 (4)

$$= -\frac{1}{(n-1)\Delta} \frac{\sqrt{z}}{z^m t^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta} \int \frac{dx}{t^{n-1} z^m \sqrt{z}}$$

$$2. \quad \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{\sqrt{z}}{z^m} \left[\frac{-1}{(n-1)\Delta} \frac{1}{t^{n-1}} \right. \\ \left. + \sum_{k=2}^{n-1} (-1)^k \frac{(2n+2m-3)(2n+2m-5) \dots (2n+2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2) \dots (n-k)\Delta^k} \cdot \frac{1}{t^{n-k}} \right] \\ + (-1)^{n-1} \frac{(2n+2m-3)(2n+2m-5) \dots (-2m+3)(-2m+1)b^{n-1}}{2^{n-1}(n-1)!\Delta^{n-1}} \int \frac{dx}{t z^m \sqrt{z}}$$

For $n = 1$

$$\int \frac{dx}{z^m t \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{1}{z^{m-1} \sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{t z^{m-1} \sqrt{z}}$$

2.25 Forms containing $\sqrt{a + bx + cx^2}$

Integration techniques

2.251 It is possible to rationalize the integrand in integrals of the form $\int R\left(x, \sqrt{a + bx + cx^2}\right) dx$ by using one or more of the following three substitutions, known as the “Euler substitutions”:

1. $\sqrt{a + bx + cx^2} = xt \pm \sqrt{a} \quad \text{for } a > 0;$
2. $\sqrt{a + bx + cx^2} = t \pm x\sqrt{c} \quad \text{for } c > 0;$
3. $\sqrt{c(x-x_1)(x-x_2)} = t(x-x_1) \quad \text{when } x_1 \text{ and } x_2 \text{ are real roots of the equation } a + bx + cx^2 = 0.$

2.252 Besides the Euler substitutions, there is also the following method of calculating integrals of the form $\int R\left(x, \sqrt{a+bx+cx^2}\right) dx$. By removing the irrational expressions in the denominator and performing simple algebraic operations, we can reduce the integrand to the sum of some rational function of x and an expression of the form $\frac{P_1(x)}{P_2(x)\sqrt{a+bx+cx^2}}$, where $P_1(x)$ and $P_2(x)$ are both polynomials.

By separating the integral portion of the rational function $\frac{P_1(x)}{P_2(x)}$ from the remainder and decomposing the latter into partial fractions, we can reduce the integral of these partial fractions to the sum of integrals, each of which is in one of the following three forms:

1. $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}}$, where $P(x)$ is a polynomial of some degree r ;
2. $\int \frac{dx}{(x+p)^k \sqrt{a+bx+cx^2}}$;
3. $\int \frac{(Mx+N) dx}{(a+\beta x+x^2)^m \sqrt{c(a_1+b_1x+x^2)}}, \quad \left(a_1 = \frac{a}{c}, \quad b_1 = \frac{b}{c}\right).$

In more detail:

1. $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}} = Q(x)\sqrt{a+bx+cx^2} + \lambda \int \frac{dx}{\sqrt{a+bx+cx^2}}$, where $Q(x)$ is a polynomial of degree $(r-1)$. Its coefficients, and also the number λ , can be calculated by the method of undetermined coefficients from the identity

$$P(x) = Q'(x)(a+bx+cx^2) + \frac{1}{2}Q(x)(b+2cx) + \lambda \quad \text{LI II 77}$$

Integrals of the form $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}}$ (where $r \leq 3$) can also be calculated by use of formulas **2.26**.

2. Integrals of the form $\int \frac{P(x) dx}{(x+p)^k \sqrt{a+bx+cx^2}}$, where the degree n of the polynomial $P(x)$ is lower than k can, by means of the substitution $t = \frac{1}{x+p}$, be reduced to an integral of the form $\int \frac{P(t) dt}{\sqrt{a+\beta t+\gamma t^2}}$. (See also **2.281**).
3. Integrals of the form $\int \frac{(Mx+N) dx}{(\alpha+\beta x+x^2)^m \sqrt{c(a_1+b_1x+x^2)}}$ can be calculated by the following procedure:

- If $b_1 \neq \beta$, by using the substitution

$$x = \frac{a_1 - \alpha}{\beta b_1} + \frac{t-1}{t+1} \frac{\sqrt{(a_1 - \alpha)^2 - (\alpha b_1 - a_1 \beta)(\beta - b_1)}}{\beta - b_1}$$

we can reduce this integral to an integral of the form $\int \frac{P(t) dt}{(t^2 + p)^m \sqrt{c(t^2 + q)}}$, where $P(t)$ is a polynomial of degree no higher than $2m - 1$. The integral $\int \frac{P(t) dt}{(t^2 + p)^m \sqrt{t^2 + q}}$ can be reduced to the sum of integrals of the forms $\int \frac{t dt}{(t^2 + p)^k \sqrt{t^2 + q}}$ and $\int \frac{dt}{(t^2 + p)^k \sqrt{t^2 + q}}$.

- If $b_1 = \beta$, we can reduce it to integrals of the form $\int \frac{P(t) dt}{(t^2 + p)^m \sqrt{c(t^2 + q)}}$ by means of the substitution $t = x + \frac{b_1}{2}$.

The integral $\int \frac{t dt}{(t^2 + p)^k \sqrt{c(t^2 + q)}}$ can be evaluated by means of the substitution $t^2 + q = u^2$.

The integral $\int \frac{dt}{(t^2 + p)^k \sqrt{c(t^2 + q)}}$ can be evaluated by means of the substitution $\frac{t}{\sqrt{t^2 + q}} = v$ (see also **2.283**). FI II 78-82

2.26 Forms containing $\sqrt{a + bx + cx^2}$ and integral powers of x

Notation: $R = a + bx + cx^2$, $\Delta = 4ac - b^2$

For simplified formulas for the case $b = 0$, see **2.27**.

2.260

$$1. \quad \int x^m \sqrt{R^{2n+1}} dx = \frac{x^{m-1} \sqrt{R^{2n+3}}}{(m+2n+2)c} - \frac{(2m+2n+1)b}{2(m+2n+2)c} \int x^{m-1} \sqrt{R^{2n+1}} dx \\ - \frac{(m-1)a}{(m+2n+2)c} \int x^{m-2} \sqrt{R^{2n+1}} dx$$

TI (192)a

$$2. \quad \int \sqrt{R^{2n+1}} dx = \frac{2cx + b}{4(n+1)c} \sqrt{R^{2n+1}} + \frac{2n+1}{8(n+1)} \frac{\Delta}{c} \int \sqrt{R^{2n-1}} dx$$

TI (188)

$$3. \quad \int \sqrt{R^{2n+1}} dx = \frac{(2cx + b)\sqrt{R}}{4(n+1)c} \left\{ R^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)} \left(\frac{\Delta}{c}\right)^{k+1} R^{n-k-1} \right\} \\ + \frac{(2n+1)!!}{8^{n+1}(n+1)!} \left(\frac{\Delta}{c}\right)^{n+1} \int \frac{dx}{\sqrt{R}}$$

TI (190)

2.261¹¹ For $n = -1$

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln \left(\frac{2\sqrt{cR} + 2cx + b}{\sqrt{\Delta}} \right) \quad [c > 0]$$

TI (127)

$$= \frac{1}{\sqrt{c}} \operatorname{arcsinh} \left(\frac{2cx + b}{\sqrt{\Delta}} \right) \quad [c > 0, \quad \Delta > 0]$$

DW

$$= \frac{1}{\sqrt{c}} \ln(2cx + b) \quad [c > 0, \quad \Delta = 0]$$

DW

$$= \frac{-1}{\sqrt{-c}} \arcsin \left(\frac{2cx + b}{\sqrt{-\Delta}} \right) \quad [c < 0, \quad \Delta < 0]$$

TI (128)

2.262

1. $\int \sqrt{R} dx = \frac{(2cx+b)\sqrt{R}}{4c} + \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}}$ (see **2.261**)
2. $\int x\sqrt{R} dx = \frac{\sqrt{R^3}}{3c} - \frac{(2cx+b)b}{8c^2} \sqrt{R} - \frac{b\Delta}{16c^2} \int \frac{dx}{\sqrt{R}}$ (see **2.261**)
3. $\int x^2\sqrt{R} dx = \left(\frac{x}{4c} - \frac{5b}{24c^2}\right) \sqrt{R^3} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right) \frac{(2cx+b)\sqrt{R}}{4c} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}}$
(see **2.261**)
4. $\int x^3\sqrt{R} dx = \left(\frac{x^2}{5c} - \frac{7bx}{40c^2} + \frac{7b^2}{48c^3} - \frac{2a}{15c^2}\right) \sqrt{R^3} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right) \frac{(2cx+b)\sqrt{R}}{4c}$
 $- \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}}$
(see **2.261**)
5. $\int \sqrt{R^3} dx = \left(\frac{R}{8c} + \frac{3\Delta}{64c^2}\right) (2cx+b)\sqrt{R} + \frac{3\Delta^2}{128c^2} \int \frac{dx}{\sqrt{R}}$
(see **2.261**)
6. $\int x\sqrt{R^3} dx = \frac{\sqrt{R^5}}{5c} - (2cx+b) \left(\frac{b}{16c^2} \sqrt{R^3} + \frac{3\Delta b}{128c^3} \sqrt{R}\right) - \frac{3\Delta^2 b}{256c^3} \int \frac{dx}{\sqrt{R}}$
(see **2.261**)
7. $\int x^2\sqrt{R^3} dx = \left(\frac{x}{6c} - \frac{7b}{60c^2}\right) \sqrt{R^5} + \left(\frac{7b^2}{24c^2} - \frac{a}{6c}\right) \left(2x + \frac{b}{c}\right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c} \sqrt{R}\right)$
 $+ \left(\frac{7b^2}{4c} - a\right) \frac{\Delta^2}{256c^3} \int \frac{dx}{\sqrt{R}}$
(see **2.261**)
8. $\int x^3\sqrt{R^3} dx = \left(\frac{x^2}{7c} - \frac{3bx}{28c^2} + \frac{3b^2}{40c^3} - \frac{2a}{35c^2}\right) \sqrt{R^5}$
 $- \left(\frac{3b^3}{16c^3} - \frac{ab}{4c^2}\right) \left(2x + \frac{b}{c}\right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c} \sqrt{R}\right)$
 $- \left(\frac{3b^2}{4c} - a\right) \frac{3\Delta^2 b}{512c^4} \int \frac{dx}{\sqrt{R}}$
(see **2.261**)

2.263

1. $\int \frac{x^m dx}{\sqrt{R^{2n+1}}} = \frac{x^{m-1}}{(m-2n)c\sqrt{R^{2n-1}}} - \frac{(2m-2n-1)b}{2(m-2n)c} \int \frac{x^{m-1} dx}{\sqrt{R^{2n+1}}} - \frac{(m-1)a}{(m-2n)c} \int \frac{x^{m-2} dx}{\sqrt{R^{2n+1}}}$

TI (193)a

For $m = 2n$

2. $\int \frac{x^{2n} dx}{\sqrt{R^{2n+1}}} = -\frac{x^{2n-1}}{(2n-1)c\sqrt{R^{2n-1}}} - \frac{b}{2c} \int \frac{x^{2n-1}}{\sqrt{R^{2n+1}}} dx + \frac{1}{c} \int \frac{x^{2n-2}}{\sqrt{R^{2n-1}}} dx$

TI (194)a

$$3. \quad \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} + \frac{8(n-1)c}{(2n-1)\Delta} \int \frac{dx}{\sqrt{R^{2n-1}}} \quad \text{TI (189)}$$

$$4. \quad \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} \left\{ 1 + \sum_{k=1}^{n-1} \frac{8^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \frac{c^k}{\Delta^k} R^k \right\} \\ [n \geq 1]. \quad \text{TI (191)}$$

2.264

$$1. \quad \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$2. \quad \int \frac{x dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$3. \quad \int \frac{x^2 dx}{\sqrt{R}} = \left(\frac{x}{2c} - \frac{3b}{4c^2} \right) \sqrt{R} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c} \right) \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$4. \quad \int \frac{x^3 dx}{\sqrt{R}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2} \right) \sqrt{R} - \left(\frac{5b^3}{16c^3} - \frac{3ab}{4c^2} \right) \int \frac{dx}{\sqrt{R}} \\ (\text{see } \mathbf{2.261})$$

$$5. \quad \int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta\sqrt{R}}$$

$$6. \quad \int \frac{x dx}{\sqrt{R^3}} = -\frac{2(2a+bx)}{\Delta\sqrt{R}}$$

$$7. \quad \int \frac{x^2 dx}{\sqrt{R^3}} = -\frac{(\Delta-b^2)x-2ab}{c\Delta\sqrt{R}} + \frac{1}{c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$8. \quad \int \frac{x^3 dx}{\sqrt{R^3}} = \frac{c\Delta x^2 + b(10ac-3b^2)x + a(8ac-3b^2)}{c^2\Delta\sqrt{R}} - \frac{3b}{2c^2} \int \frac{dx}{\sqrt{R}} \\ (\text{see } \mathbf{2.261})$$

$$\mathbf{2.265} \quad \int \frac{\sqrt{R^{2n+1}}}{x^m} dx = -\frac{\sqrt{R^{2n+3}}}{(m-1)ax^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-1}} dx \\ + \frac{(2n-m+4)c}{(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-2}} dx \quad \text{TI (195)}$$

$$\text{For } m=1 \quad \int \frac{\sqrt{R^{2n+1}}}{x} dx = \frac{\sqrt{R^{2n+1}}}{2n+1} + \frac{b}{2} \int \sqrt{R^{2n-1}} dx + a \int \frac{\sqrt{R^{2n-1}}}{x} dx \quad \text{TI (198)}$$

$$\text{For } a=0 \quad \int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^m} dx = \frac{2\sqrt{(bx+cx^2)^{2n+3}}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)c}{(2n-2m+3)b} \int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^{m-1}} dx \quad \text{LA 169 (3)}$$

For $m=0$ see **2.260** 2 and **2.260** 3.

For $n=-1$ and $m=1$:

$$\mathbf{2.266}^8 \int \frac{dx}{x\sqrt{R}} = -\frac{1}{\sqrt{a}} \ln \frac{2a+bx+2\sqrt{aR}}{x} \quad [a > 0] \quad \text{TI (137)}$$

$$= \frac{1}{\sqrt{-a}} \arcsin \frac{2a+bx}{x\sqrt{b^2-4ac}} \quad [a < 0, \quad \Delta < 0] \quad \text{TI (138)}$$

$$= \frac{1}{\sqrt{-a}} \arctan \frac{2a+bx}{2\sqrt{-a}\sqrt{R}} \quad [a < 0] \quad \text{LA 178 (6)a}$$

$$= -\frac{1}{\sqrt{a}} \operatorname{arcsinh} \frac{2a+bx}{x\sqrt{\Delta}} \quad [a > 0, \quad \Delta > 0] \quad \text{DW}$$

$$= -\frac{1}{\sqrt{a}} \operatorname{arctanh} \frac{2a+bx}{2\sqrt{a}\sqrt{R}} \quad [a > 0]$$

$$= \frac{1}{\sqrt{a}} \ln \frac{x}{2a+bx} \quad [a > 0, \quad \Delta = 0]$$

$$= -\frac{2\sqrt{bx+cx^2}}{bx} \quad [a = 0, \quad b \neq 0] \quad \text{LA 170 (16)}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arccosh} \left(\frac{2a+bx}{x\sqrt{-\Delta}} \right) \quad [a > 0, \Delta < 0]$$

2.267

$$1. \quad \int \frac{\sqrt{R} dx}{x} = \sqrt{R} + a \int \frac{dx}{x\sqrt{R}} + \frac{b}{2} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266})$$

$$2. \quad \int \frac{\sqrt{R} dx}{x^2} = -\frac{\sqrt{R}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{R}} + c \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{\sqrt{bx+cx^2}}{x^2} dx = -\frac{2\sqrt{bx+cx^2}}{x} + c \int \frac{dx}{\sqrt{bx+cx^2}} \quad (\text{see } \mathbf{2.261})$$

$$3. \quad \int \frac{\sqrt{R} dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4ax}\right) \sqrt{R} - \left(\frac{b^2}{8a} - \frac{c}{2}\right) \int \frac{dx}{x\sqrt{R}} \quad (\text{see } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{\sqrt{bx+cx^2}}{x^3} dx = -\frac{2\sqrt{(bx+cx^2)^3}}{3bx^3}$$

$$4. \quad \int \frac{\sqrt{R^3}}{x} dx = \frac{\sqrt{R^3}}{3} + \frac{2bcx+b^2+8ac}{8c} \sqrt{R} + a^2 \int \frac{dx}{x\sqrt{R}} + \frac{b(12ac-b^2)}{16c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266})$$

$$5. \quad \int \frac{\sqrt{R^3}}{x^2} dx = -\frac{\sqrt{R^5}}{ax} + \frac{cx+b}{a} \sqrt{R^3} + \frac{3}{4}(2cx+3b)\sqrt{R} + \frac{3}{2}ab \int \frac{dx}{x\sqrt{R}} + \frac{3(4ac+b^2)}{8} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{\sqrt{(bx+cx^2)^3}}{x^2} = \frac{\sqrt{(bx+cx^2)^3}}{2x} + \frac{3b}{4} \sqrt{bx+cx^2} + \frac{3b^2}{8} \int \frac{dx}{\sqrt{bx+cx^2}}$$

(see **2.261**)

$$6. \quad \int \frac{\sqrt{R^3}}{x^3} dx = - \left(\frac{1}{2ax^2} + \frac{b}{4a^2x} \right) \sqrt{R^5} + \frac{bcx + 2ac + b^2}{4a^2} \sqrt{R^3} + \frac{3(bcx + 2ac + b^2)}{4a} \sqrt{R} \\ + \frac{3}{8} (4ac + b^2) \int \frac{dx}{x\sqrt{R}} + \frac{3}{2} bc \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{\sqrt{(bx + cx^2)^3}}{x^3} dx = \left(c - \frac{2b}{x} \right) \sqrt{bx + cx^2} + \frac{3bc}{2} \int \frac{dx}{\sqrt{bx + cx^2}} \quad (\text{see } \mathbf{2.261})$$

$$\mathbf{2.268} \quad \int \frac{dx}{x^m \sqrt{R^{2n+1}}} = - \frac{1}{(m-1)a x^{m-1} \sqrt{R^{2n-1}}} - \frac{(2n+2m-3)b}{2(m-1)a} \int \frac{dx}{x^{m-1} \sqrt{R^{2n+1}}} - \frac{(2n+m-2)c}{(m-1)a} \int \frac{dx}{x^{m-2} \sqrt{R^{2n+1}}} \quad \text{TI (196)}$$

For $m = 1$

$$\int \frac{dx}{x \sqrt{R^{2n+1}}} = \frac{1}{(2n-1)a \sqrt{R^{2n-1}}} - \frac{b}{2a} \int \frac{dx}{\sqrt{R^{2n+1}}} + \frac{1}{a} \int \frac{dx}{x \sqrt{R^{2n-1}}} \quad \text{TI (199)}$$

For $a = 0$

$$\int \frac{dx}{x^m \sqrt{(bx + cx^2)^{2n+1}}} = - \frac{2}{(2n+2m-1)bx^m \sqrt{(bx + cx^2)^{2n-1}}} - \frac{(4n+2m-2)c}{(2n+2m-1)b} \int \frac{dx}{x^{m-1} \sqrt{(bx + cx^2)^{2n+1}}} \quad (\text{cf. } \mathbf{2.265})$$

2.269

$$1. \quad \int \frac{dx}{x\sqrt{R}} \quad (\text{see } \mathbf{2.266})$$

$$2. \quad \int \frac{dx}{x^2 \sqrt{R}} = - \frac{\sqrt{R}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{R}} \quad (\text{see } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{dx}{x^2 \sqrt{bx + cx^2}} = \frac{2}{3} \left(-\frac{1}{bx^2} + \frac{2c}{b^2x} \right) \sqrt{bx + cx^2}$$

$$3. \quad \int \frac{dx}{x^3 \sqrt{R}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x} \right) \sqrt{R} + \left(\frac{3b^2}{8a^2} - \frac{c}{2a} \right) \int \frac{dx}{x\sqrt{R}} \quad (\text{see } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{dx}{x^3 \sqrt{bx + cx^2}} = \frac{2}{5} \left(-\frac{1}{bx^3} + \frac{4c}{3b^2x^2} - \frac{8c^2}{3b^3x} \right) \sqrt{bx + cx^2}$$

$$4. \quad \int \frac{dx}{x\sqrt{R^3}} = - \frac{2(bcx - 2ac + b^2)}{a\Delta\sqrt{R}} + \frac{1}{a} \int \frac{dx}{x\sqrt{R}} \quad (\text{see } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{dx}{x\sqrt{(bx+cx^2)^3}} = \frac{2}{3} \left(-\frac{1}{bx} + \frac{4c}{b^2} + \frac{8c^2x}{b^3} \right) \frac{1}{\sqrt{bx+cx^2}}$$

$$5.11 \quad \int \frac{dx}{x^2\sqrt{R^3}} = -\frac{A}{\sqrt{R}} - \frac{3b}{2a^2} \int \frac{dx}{x\sqrt{R}}$$

where $A = \left(-\frac{1}{ax} - \frac{b(10ac-3b^2)}{a^2\Delta} - \frac{c(8ac-3b^2)x}{a^2\Delta} \right)$ (see **2.266**)

For $a = 0$

$$\int \frac{dx}{x^2\sqrt{(bx+cx^2)^3}} = \frac{2}{5} \left(-\frac{1}{bx^2} + \frac{2c}{b^2x} - \frac{8c^2}{b^3} - \frac{16c^3x}{b^4} \right) \frac{1}{\sqrt{bx+cx^2}}$$

$$6. \quad \int \frac{dx}{x^3\sqrt{R^3}} = \left(-\frac{1}{ax^2} + \frac{5b}{2a^2x} - \frac{15b^4-62acb^2+24a^2c^2}{2a^3\Delta} - \frac{bc(15b^2-52ac)x}{2a^3\Delta} \right) \frac{1}{2\sqrt{R}} + \frac{15b^2-12ac}{8a^3} \int \frac{dx}{x\sqrt{R}}$$

(see **2.266**)

For $a = 0$

$$\int \frac{dx}{x^3\sqrt{(bx+cx^2)^3}} = \frac{2}{7} \left(-\frac{1}{bx^3} + \frac{8c}{5b^2x^2} - \frac{16c^2}{5b^3x} + \frac{64c^3}{5b^4} + \frac{128c^4x}{5b^5} \right) \frac{1}{\sqrt{bx+cx^2}}$$

2.27 Forms containing $\sqrt{a+cx^2}$ and integral powers of x

Notation: $u = \sqrt{a+cx^2}$.

$$I_1 = \frac{1}{\sqrt{c}} \ln(x\sqrt{c}+u) \quad [c > 0]$$

$$= \frac{1}{\sqrt{-c}} \arcsin x\sqrt{-\frac{c}{a}} \quad [c < 0 \text{ and } a > 0]$$

$$I_2 = \frac{1}{2\sqrt{a}} \ln \frac{u-\sqrt{a}}{u+\sqrt{a}} \quad [a > 0 \text{ and } c > 0]$$

$$= \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a}-u}{\sqrt{a}+u} \quad [a > 0 \text{ and } c > 0]$$

$$= \frac{1}{\sqrt{-a}} \operatorname{arcsec} x\sqrt{-\frac{c}{a}} = \frac{1}{\sqrt{-a}} \arccos \frac{1}{x}\sqrt{-\frac{a}{c}} \quad [a < 0 \text{ and } c > 0]$$

2.271

1.	$\int u^5 dx = \frac{1}{6}xu^5 + \frac{5}{24}axu^3 + \frac{5}{16}a^2xu + \frac{5}{16}a^3I_1$	DW
2.	$\int u^3 dx = \frac{1}{4}xu^3 + \frac{3}{8}axu + \frac{3}{8}a^2I_1$	DW
3.	$\int u dx = \frac{1}{2}xu + \frac{1}{2}aI_1$	DW
4.	$\int \frac{dx}{u} = I_1$	DW

$$5. \quad \int \frac{dx}{u^3} = \frac{1}{a} \frac{x}{u} \quad \text{DW}$$

$$6. \quad \int \frac{dx}{u^{2n+1}} = \frac{1}{a^n} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \frac{c^k x^{2k+1}}{u^{2k+1}}$$

$$7. \quad \int \frac{x dx}{u^{2n+1}} = -\frac{1}{(2n-1)cu^{2n-1}} \quad \text{DW}$$

2.272

$$1. \quad \int x^2 u^3 dx = \frac{1}{6} \frac{xu^5}{c} - \frac{1}{24} \frac{axu^3}{c} - \frac{1}{16} \frac{a^2 xu}{c} - \frac{1}{16} \frac{a^3}{c} I_1 \quad \text{DW}$$

$$2. \quad \int x^2 u dx = \frac{1}{4} \frac{xu^3}{c} - \frac{1}{8} \frac{axu}{c} - \frac{1}{8} \frac{a^2}{c} I_1 \quad \text{DW}$$

$$3. \quad \int \frac{x^2}{u} dx = \frac{1}{2} \frac{xu}{c} - \frac{1}{2} \frac{a}{c} I_1 \quad \text{DW}$$

$$4. \quad \int \frac{x^2}{u^3} dx = -\frac{x}{cu} + \frac{1}{c} I_1 \quad \text{DW}$$

$$5. \quad \int \frac{x^2}{u^5} dx = \frac{1}{3} \frac{x^3}{au^3} \quad \text{DW}$$

$$6. \quad \int \frac{x^2 dx}{u^{2n+1}} = \frac{1}{a^{n-1}} \sum_{k=0}^{n-2} \frac{(-1)^k}{2k+3} \binom{n-2}{k} \frac{c^k x^{2k+3}}{u^{2k+3}}$$

$$7. \quad \int \frac{x^3 dx}{u^{2n+1}} = -\frac{1}{(2n-3)c^2 u^{2n-3}} + \frac{a}{(2n-1)c^2 u^{2n-1}} \quad \text{DW}$$

2.273

$$1. \quad \int x^4 u^3 dx = \frac{1}{8} \frac{x^3 u^5}{c} - \frac{axu^5}{16c^2} + \frac{a^2 xu^3}{64c^2} + \frac{3a^3 xu}{128c^2} + \frac{3a^4}{128c^2} I_1 \quad \text{DW}$$

$$2. \quad \int x^4 u dx = \frac{1}{6} \frac{x^3 u^3}{c} - \frac{axu^3}{8c^2} + \frac{a^2 xu}{16c^2} + \frac{a^3}{16c^2} I_1 \quad \text{DW}$$

$$3. \quad \int \frac{x^4}{u} dx = \frac{1}{4} \frac{x^3 u}{c} - \frac{3axu}{8c^2} + \frac{3a^2}{8c^2} I_1 \quad \text{DW}$$

$$4. \quad \int \frac{x^4}{u^3} dx = \frac{1}{2} \frac{xu}{c^2} + \frac{ax}{c^2 u} - \frac{3a}{2c^2} I_1 \quad \text{DW}$$

$$5. \quad \int \frac{x^4}{u^5} dx = -\frac{x}{c^2 u} - \frac{1}{3} \frac{x^3}{cu^3} + \frac{1}{c^2} I_1 \quad \text{DW}$$

$$6. \quad \int \frac{x^4}{u^7} dx = \frac{1}{5} \frac{x^5}{au^5} \quad \text{DW}$$

$$7. \quad \int \frac{x^4 dx}{u^{2n+1}} = \frac{1}{a^{n-2}} \sum_{k=0}^{n-3} \frac{(-1)^k}{2k+5} \binom{n-3}{k} \frac{c^k x^{2k+5}}{u^{2k+5}}$$

$$8. \quad \int \frac{x^5 dx}{u^{2n+1}} = -\frac{1}{(2n-5)c^3 u^{2n-5}} + \frac{2a}{(2n-3)c^2 u^{2n-3}} - \frac{a^2}{(2n-1)c^3 u^{2n-1}} \quad \text{DW}$$

2.274

$$1. \quad \int x^6 u^3 dx = \frac{1}{10} \frac{x^5 u^5}{c} - \frac{ax^3 u^5}{16c^2} + \frac{a^2 x u^5}{32c^3} - \frac{a^3 x u^3}{128c^3} - \frac{3a^4 x u}{256c^3} - \frac{3}{256} \frac{a^5}{c^3} I_1$$

$$2. \quad \int x^6 u dx = \frac{1}{8} \frac{x^5 u^3}{c} - \frac{5}{48} \frac{ax^3 u^3}{c^2} + \frac{5a^2 x u^3}{64c^3} - \frac{5a^3 x u}{128c^3} - \frac{5}{128} \frac{a^4}{c^3} I_1$$

$$3. \quad \int \frac{x^6}{u} dx = \frac{1}{6} \frac{x^5 u}{c} - \frac{5}{24} \frac{ax^3 u}{c^2} + \frac{5}{16} \frac{a^2 x u}{c^3} - \frac{5}{16} \frac{a^3}{c^3} I_1 \quad \text{DW}$$

$$4. \quad \int \frac{x^6}{u^3} dx = \frac{1}{4} \frac{x^5}{cu} - \frac{5}{8} \frac{ax^3}{c^2 u} - \frac{15}{8} \frac{a^2 x}{c^3 u} + \frac{15}{8} \frac{a^2}{c^3} I_1 \quad \text{DW}$$

$$5. \quad \int \frac{x^6}{u^5} dx = \frac{1}{2} \frac{x^5}{cu^3} + \frac{10}{3} \frac{ax^3}{c^2 u^3} + \frac{5}{2} \frac{a^2 x}{c^3 u^3} - \frac{5}{2} \frac{a}{c^3} I_1 \quad \text{DW}$$

$$6. \quad \int \frac{x^6}{u^7} dx = -\frac{23}{15} \frac{x^5}{cu^5} - \frac{7}{3} \frac{ax^3}{c^2 u^5} - \frac{a^2 x}{c^3 u^5} + \frac{1}{c^3} I_1 \quad \text{DW}$$

$$7. \quad \int \frac{x^6}{u^9} dx = \frac{1}{7} \frac{x^7}{au^7} \quad \text{DW}$$

$$8. \quad \int \frac{x^6 dx}{u^{2n+1}} = \frac{1}{a^{n-3}} \sum_{k=0}^{n-4} \frac{(-1)^k}{2k+7} \binom{n-4}{k} \frac{c^k x^{2k+7}}{u^{2k+7}}$$

$$9. \quad \int \frac{x^7 dx}{u^{2n+1}} = -\frac{1}{(2n-7)c^4 u^{2n-7}} + \frac{3a}{(2n-5)c^4 u^{2n-5}} - \frac{3a^2}{(2n-3)c^4 u^{2n-3}} + \frac{a^3}{(2n-1)c^4 u^{2n-1}} \quad \text{DW}$$

2.275

$$1. \quad \int \frac{u^5}{x} dx = \frac{u^5}{5} + \frac{1}{3} au^3 + a^2 u + a^3 I_2 \quad \text{DW}$$

$$2. \quad \int \frac{u^3}{x} dx = \frac{u^3}{3} + au + a^2 I_2 \quad \text{DW}$$

$$3. \quad \int \frac{u}{x} dx = u + a I_2 \quad \text{DW}$$

$$4. \quad \int \frac{dx}{xu} = I_2 \quad \text{DW}$$

$$5. \quad \int \frac{dx}{xu^{2n+1}} = \frac{1}{a^n} I_2 + \sum_{k=0}^{n-1} \frac{1}{(2k+1)a^{n-k} u^{2k+1}}$$

$$6. \quad \int \frac{u^5}{x^2} dx = -\frac{u^5}{x} + \frac{5}{4} cxu^3 + \frac{15}{8} acxu + \frac{15}{8} a^2 I_1 \quad \text{DW}$$

$$7. \quad \int \frac{u^3}{x^2} dx = -\frac{u^3}{x} + \frac{3}{2} cxu + \frac{3}{2} a I_1 \quad \text{DW}$$

$$8. \quad \int \frac{u}{x^2} dx = -\frac{u}{x} + c I_1 \quad \text{DW}$$

$$9. \quad \int \frac{dx}{x^2 u^{2n+1}} = -\frac{1}{a^{n+1}} \left\{ \frac{u}{x} + \sum_{k=1}^n \frac{(-1)^{k+1}}{2k-1} \binom{n}{k} c^k \left(\frac{x}{u} \right)^{2k-1} \right\}$$

2.276

$$1. \quad \int \frac{u^5}{x^3} dx = -\frac{u^5}{2x^2} + \frac{5}{6}cu^3 + \frac{5}{2}acu + \frac{5}{2}a^2cI_2 \quad \text{DW}$$

$$2. \quad \int \frac{u^3}{x^3} dx = -\frac{u^3}{2x^2} + \frac{3}{2}cu + \frac{3}{2}acI_2 \quad \text{DW}$$

$$3. \quad \int \frac{u}{x^3} dx = -\frac{u}{2x^2} + \frac{c}{2}I_2 \quad \text{DW}$$

$$4. \quad \int \frac{dx}{x^3 u} = -\frac{u}{2ax^2} - \frac{c}{2a}I_2 \quad \text{DW}$$

$$5. \quad \int \frac{dx}{x^3 u^3} = -\frac{1}{2ax^2 u} - \frac{3c}{2a^2 u} - \frac{3c}{2a^2}I_2 \quad \text{DW}$$

$$6. \quad \int \frac{dx}{x^3 u^5} = -\frac{1}{2ax^2 u^3} - \frac{5}{6} \frac{c}{a^2 u^3} - \frac{5}{2} \frac{c}{a^3 u} - \frac{5}{2} \frac{c}{a^3}I_2 \quad \text{DW}$$

$$7. \quad \int \frac{u^5}{x^4} dx = -\frac{au^3}{3x^3} - \frac{2acu}{x} + \frac{c^2 xu}{2} + \frac{5}{2}acI_1 \quad \text{DW}$$

$$8. \quad \int \frac{u^3}{x^4} dx = -\frac{u^3}{3x^3} - \frac{cu}{x} + cI_1 \quad \text{DW}$$

$$9. \quad \int \frac{u}{x^4} dx = -\frac{u^3}{3ax^3} \quad \text{DW}$$

$$10. \quad \int \frac{dx}{x^4 u^{2n+1}} = \frac{1}{a^{n+2}} \left\{ -\frac{u^3}{3x^3} + (n+1) \frac{cu}{x} + \sum_{k=2}^{n+1} \frac{(-1)^k}{2k-3} \binom{n+1}{k} c^k \left(\frac{x}{u} \right)^{2k-3} \right\}$$

2.277

$$1. \quad \int \frac{u^3}{x^5} dx = -\frac{u^3}{4x^4} - \frac{3}{8} \frac{cu^3}{ax^2} + \frac{3}{8} \frac{c^2 u}{a} + \frac{3}{8} c^2 I_2 \quad \text{DW}$$

$$2. \quad \int \frac{u}{x^5} dx = -\frac{u}{4x^4} - \frac{1}{8} \frac{cu}{ax^2} - \frac{1}{8} \frac{c^2}{a} I_2 \quad \text{DW}$$

$$3. \quad \int \frac{dx}{x^5 u} = -\frac{u}{4ax^4} + \frac{3}{8} \frac{cu}{a^2 x^2} + \frac{3}{8} \frac{c^2}{a^2} I_2 \quad \text{DW}$$

$$4. \quad \int \frac{dx}{x^5 u^3} = -\frac{1}{4ax^4 u} + \frac{5}{8} \frac{c}{a^2 x^2 u} + \frac{15}{8} \frac{c^2}{a^3 u} + \frac{15}{8} \frac{c^2}{a^3} I_2 \quad \text{DW}$$

2.278

$$1. \quad \int \frac{u^3}{x^6} dx = -\frac{u^5}{5ax^5} \quad \text{DW}$$

$$2. \quad \int \frac{u}{x^6} dx = -\frac{u^3}{5ax^5} + \frac{2}{15} \frac{cu^3}{a^2 x^3} \quad \text{DW}$$

$$3. \quad \int \frac{dx}{x^6 u} = \frac{1}{a^3} \left(-\frac{u^5}{5x^5} + \frac{2}{3} \frac{cu^3}{x^3} - \frac{c^2 u}{x} \right) \quad \text{DW}$$

$$4. \quad \int \frac{dx}{x^6 u^{2n+1}} = \frac{1}{a^{n+3}} \left\{ -\frac{u^5}{5x^5} + \frac{1}{3} \binom{n+2}{1} \frac{cu^3}{x^3} - \binom{n+2}{2} \frac{c^2 u}{x} + \sum_{k=3}^{n+2} \frac{(-1)^k}{2k-5} \binom{n+2}{k} c^k \left(\frac{x}{u} \right)^{2k-5} \right\}$$

2.28 Forms containing $\sqrt{a + bx + cx^2}$ and first- and second-degree polynomials

Notation: $R = a + bx + cx^2$

See also 2.252

$$2.281^3 \quad \int \frac{dx}{(x+p)^n \sqrt{R}} = -\int \frac{t^{n-1} dt}{\sqrt{c + (b-2pc)t + (a-bp+cp^2)t^2}} \quad \left[t = \frac{1}{x+p} > 0 \right]$$

2.282

$$1.^3 \quad \int \frac{\sqrt{R} dx}{x+p} = c \int \frac{x dx}{\sqrt{R}} + (b-cp) \int \frac{dx}{\sqrt{R}} + (a-bp+cp^2) \int \frac{dx}{(x+p)\sqrt{R}} \quad [x+p > 0]$$

$$2. \quad \int \frac{dx}{(x+p)(x+q)\sqrt{R}} = \frac{1}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{1}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}$$

$$3. \quad \int \frac{\sqrt{R} dx}{(x+p)(x+q)} = \frac{1}{q-p} \int \frac{\sqrt{R} dx}{x+p} + \frac{1}{p-q} \int \frac{\sqrt{R} dx}{x+q}$$

$$4. \quad \int \frac{(x+p)\sqrt{R} dx}{x+q} = \int \sqrt{R} dx + (p-q) \int \frac{\sqrt{R} dx}{x+q}$$

$$5. \quad \int \frac{(rx+s) dx}{(x+p)(x+q)\sqrt{R}} = \frac{s-pr}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{s-qr}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}$$

$$2.283 \quad \int \frac{(Ax+B) dx}{(p+R)^n \sqrt{R}} = \frac{A}{c} \int \frac{du}{(p+u^2)^n} + \frac{2Bc-Ab}{2c} \int \frac{(1-cv^2)^{n-1} dv}{\left[p+a-\frac{b^2}{4c}-cpv^2 \right]^n},$$

where $u = \sqrt{R}$ and $v = \frac{b+2cx}{2c\sqrt{R}}$.

$$2.284 \quad \int \frac{Ax+B}{(p+R)\sqrt{R}} dx = \frac{A}{c} I_1 + \frac{2Bc-Ab}{\sqrt{c^2 p [b^2 - 4(a+p)c]}} I_2,$$

where

$$I_1 = \frac{1}{\sqrt{p}} \arctan \sqrt{\frac{R}{p}} \quad [p > 0]$$

$$= \frac{1}{2\sqrt{-p}} \ln \frac{\sqrt{-p} - \sqrt{R}}{\sqrt{-p} + \sqrt{R}} \quad [p < 0]$$

$$\begin{aligned}
I_2 &= \arctan \sqrt{\frac{p}{b^2 - 4(a+p)c}} \frac{b + 2cx}{\sqrt{R}} & [p \{b^2 - 4(a+p)c\} > 0, \quad p < 0] \\
&= -\arctan \sqrt{\frac{p}{b^2 - 4(a+p)c}} \frac{b + 2cx}{\sqrt{R}} & [p \{b^2 - 4(a+p)c\} > 0, \quad p > 0] \\
&= \frac{1}{2i} \ln \frac{\sqrt{4(a+p)c - b^2} \sqrt{R} + \sqrt{p}(b + 2cx)}{\sqrt{4(a+p)c - b^2} \sqrt{R} - \sqrt{p}(b + 2cx)} & [p \{b^2 - 4(a+p)c\} < 0, \quad p > 0] \\
&= \frac{1}{2i} \ln \frac{\sqrt{b^2 - 4(a+p)c} \sqrt{R} - \sqrt{-p}(b + 2cx)}{\sqrt{b^2 - 4(a+p)c} \sqrt{R} + \sqrt{-p}(b + 2cx)} & [p \{b^2 - 4(a+p)c\} < 0, \quad p < 0]
\end{aligned}$$

2.29 Integrals that can be reduced to elliptic or pseudo-elliptic integrals

2.290 Integrals of the form $\int R(x, \sqrt{P(x)}) dx$, where $P(x)$ is a third- or fourth-degree polynomial, can, by means of algebraic transformations, be reduced to a sum of integrals expressed in terms of elementary functions and elliptic integrals (see **8.11**). Since the substitutions that transform the given integral into an elliptic integral in the normal Legendre form are different for different intervals of integration, the corresponding formulas are given in the chapter on definite integrals (see **3.13**, **3.17**).

2.291 Certain integrals of the form $\int R(x, \sqrt{P(x)}) dx$, where $P_n(x)$ is a polynomial of not more than fourth degree, can be reduced to integrals of the form $\int R(x, \sqrt[k]{P_n(x)}) dx$ with $k \geq 2$. Below are examples of this procedure.

$$\begin{aligned}
1. \quad & \int \frac{dx}{\sqrt{1-x^6}} = -\int \frac{dz}{\sqrt{3+3z^2+z^4}} & [x^2 = \frac{1}{1+z^2}] \\
2. \quad & \int \frac{dx}{\sqrt{a+bx^2+cx^4+dx^6}} = \frac{1}{2} \int \frac{dz}{\sqrt{az+bz^2+cz^3+dz^4}} & [x^2 = z] \\
3. \quad & \int (a+2bx+cx^2+gx^3)^{\pm 1/3} dx = \frac{3}{2} \int \frac{z^2 A^{\pm \frac{1}{3}} dz}{B} \\
& \left[a+2bx+cx^2 = z^3, \quad A = g \left(\frac{-b + \sqrt{b^2 + (z^3-a)c}}{c} \right)^3 + z^3, \quad B = \sqrt{b^2 + (z^3-a)c} \right] \\
4. \quad & \int \frac{dx}{\sqrt{a+bx+cx^2+dx^3+cx^4+bx^5+ax^6}} \\
& = -\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(z+1)p}} - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} & [x = z + \sqrt{z^2-1}] \\
& = -\frac{1}{\sqrt{2}} \int \frac{d}{\sqrt{(z+1)p}} + \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} & [x = z - \sqrt{z^2-1}]
\end{aligned}$$

where $p = 2a(4z^3 - 3z) + 2b(2z^2 - 1) + 2cz + d$.

$$\begin{aligned}
5. \quad \int \frac{dx}{\sqrt{a+bx^2+cx^4+bx^6+ax^8}} &= \frac{1}{2} \int \frac{dy}{\sqrt{y}\sqrt{a+by+cy^2+by^3+ay^4}} & [x = \sqrt{y}] \\
&= -\frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} + \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} & [y = z + \sqrt{z^2-1}] \\
&= \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} - \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} & [y = z - \sqrt{z^2-1}]
\end{aligned}$$

where $p = 2a(2z^2 - 1) + 2bz + c$.

$$\begin{aligned}
6. \quad \int \frac{dx}{\sqrt{a+bx^4+cx^8}} &= \frac{1}{2} \sqrt{\frac{a}{c}} \int \frac{dt}{\sqrt{t}\sqrt{ab_1t^2+at^4}} & \left[x = \sqrt[8]{\frac{a}{c}} \sqrt{t} \right]; \\
&= -\frac{1}{2\sqrt{2}} \sqrt[8]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} - \int \frac{dz}{\sqrt{(z-1)p}} \right\} & [t = z + \sqrt{z^2-1}] \\
&= -\frac{1}{2\sqrt{2}} \sqrt[8]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} + \int \frac{dz}{\sqrt{(z-1)p}} \right\} & [t = z - \sqrt{z^2-1}]
\end{aligned}$$

where $p = 2a(2z^2 - 1) + b_1$; $b_1 = b\sqrt{\frac{a}{c}}$.

$$7. \quad \int \frac{x dx}{\sqrt[4]{a+bx^2+cx^4}} = 2 \int \frac{z^2 dz}{\sqrt{A+Bz^4}} \quad [a+bx^2+cx^4 = z^4, \quad A = b^2 - 4ac, \quad B = 4c]$$

$$8. \quad \int \frac{dx}{\sqrt[4]{a+2bx^2+cx^4}} = \int \frac{\sqrt{b^2-a(c-z^4)}+b}{(c-z^4)\sqrt{b^2-a(c-z^4)}} z^2 dz = \int R_1(z^4) z^2 dz + \int \frac{R_2(z^4) z^2 dz}{\sqrt{b^2-a(c-z^4)}},$$

where $R_1(z^4)$ and $R_2(z^4)$ are rational functions of z^4 and $a+2bx^2+cx^4 = x^4z^4$.

2.292 In certain cases, integrals of the form $\int R(x, \sqrt{P(x)}) dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be expressed in terms of elementary functions. Such integrals are called *pseudo-elliptic* integrals.

Thus, if the relations

$$f_1(x) = f_1\left(\frac{1}{k^2x}\right), \quad f_2(x) = f_2\left(\frac{1-k^2x}{k^2(1-x)}\right), \quad f_3(x) = f_3\left(\frac{1-x}{1-k^2x}\right),$$

hold, then

$$\begin{aligned}
1. \quad \int \frac{f_1(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_1(z) dz & [z = \sqrt{x(1-x)(1-k^2x)}] \\
2. \quad \int \frac{f_2(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_2(z) dz & \left[z = \frac{\sqrt{x(1-k^2x)}}{\sqrt{1-x}} \right] \\
3. \quad \int \frac{f_3(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_3(z) dz & \left[z = \frac{\sqrt{x(1-x)}}{\sqrt{1-k^2x}} \right]
\end{aligned}$$

where $R_1(z)$, $R_2(z)$, and $R_3(z)$ are rational functions of z .

2.3 The Exponential Function

2.31 Forms containing e^{ax}

$$2.311 \quad \int e^{ax} dx = \frac{e^{ax}}{a}$$

2.312 a^x in the integrands should be replaced with $e^{x \ln a} = a^x$

2.313

$$1. \quad \int \frac{dx}{a + be^{mx}} = \frac{1}{am} [mx - \ln(a + be^{mx})] \quad \text{PE (410)}$$

$$2. \quad \int \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} = x - \ln(1 + e^x) \quad \text{PE (409)}$$

$$2.314 \quad \int \frac{dx}{ae^{mx} + be^{-mx}} = \frac{1}{m\sqrt{ab}} \arctan \left(e^{mx} \sqrt{\frac{a}{b}} \right) \quad [ab > 0] \\ = \frac{1}{2m\sqrt{-ab}} \ln \left| \frac{b + e^{mx}\sqrt{-ab}}{b - e^{mx}\sqrt{-ab}} \right| \quad [ab < 0] \quad \text{PE (411)}$$

$$2.315 \quad \int \frac{dx}{\sqrt{a + be^{mx}}} = \frac{1}{m\sqrt{a}} \ln \frac{\sqrt{a + be^{mx}} - \sqrt{a}}{\sqrt{a + be^{mx}} + \sqrt{a}} \quad [a > 0] \\ = \frac{2}{m\sqrt{-a}} \arctan \frac{\sqrt{a + be^{mx}}}{\sqrt{-a}} \quad [a < 0]$$

2.32 The exponential combined with rational functions of x

2.321

$$1. \quad \int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$2.^{11} \quad \int x^n e^{ax} dx = e^{ax} \left(\sum_{k=0}^n \frac{(-1)^k k!}{a^{k+1}} \binom{n}{k} x^{n-k} \right)$$

2.322

$$1. \quad \int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right)$$

$$2. \quad \int x^2 e^{ax} dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right)$$

$$3. \quad \int x^3 e^{ax} dx = e^{ax} \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4} \right)$$

$$4.^{10} \quad \int x^4 e^{ax} dx = e^{ax} \left(\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{24}{a^5} \right)$$

$$2.323 \quad \int P_m(x) e^{ax} dx = \frac{e^{ax}}{a} \sum_{k=0}^m (-1)^k \frac{P^{(k)}(x)}{a^k},$$

where $P_m(x)$ is a polynomial in x of degree m and $P^{(k)}(x)$ is the k^{th} derivative of $P_m(x)$ with respect to x .

2.324

1. $\int \frac{e^{ax} dx}{x^m} = \frac{1}{m-1} \left[-\frac{e^{ax}}{x^{m-1}} + a \int \frac{e^{ax} dx}{x^{m-1}} \right]$
2. $\int \frac{e^{ax}}{x^n} dx = -e^{ax} \sum_{k=1}^{n-1} \frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!} \text{Ei}(ax)$

2.325

1. $\int \frac{e^{ax}}{x} dx = \text{Ei}(ax)$
2. $\int \frac{e^{ax}}{x^2} dx = -\frac{e^{ax}}{x} + a \text{Ei}(ax)$
3. $\int \frac{e^{ax}}{x^3} dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2} \text{Ei}(ax)$
- 4.* $\int \frac{e^{ax}}{x^4} dx = -\frac{e^{ax}}{3x^3} - \frac{ae^{ax}}{6x^2} - \frac{a^2e^{ax}}{6x} + \frac{a^3}{6} \text{Ei}(ax)$
- 5.* $\int \frac{e^{\pm ax^n}}{x^m} dx = \frac{1}{m-1} \left[-\frac{e^{\pm ax^n}}{x^{m-1}} \pm na \int \frac{e^{\pm ax^n}}{x^{m-n}} dx \right] \quad [m \neq 1]$
- 6.* $\int \frac{e^{ax^n}}{x^m} dx = \frac{(-1)^{z+1} a^z \Gamma(-z, -ax^n)}{n}$
 $= \frac{(-1)^{z+1} a^z}{n} \int_{-ax^n}^{\infty} \frac{e^{-t}}{t^{z+1}} dt$
 $z = \frac{m-1}{n}, \quad \text{for } \Gamma(\alpha, x) \text{ see 8.350.2} \quad [n \neq 0]$
- 7.* $\int \frac{e^{ax^n}}{x} dx = \frac{\text{Ei}(ax^n)}{n} \quad [a \neq 0, \quad n \neq 0]$
- 8.* $\int \frac{e^{ax^n}}{x^m} dx = -e^{ax^n} \frac{\sum_{k=0}^{z-1} k! \frac{a^{z-k-1}}{x^{n(k+1)}}}{nz!} + \frac{a^z \text{Ei}(ax^n)}{nz!}$
 $\left[a \neq 0, \quad z = \frac{m-1}{n} = 1, 2, \dots, \quad m = 2, 3, \dots \right]$
- 9.* $\int \frac{e^{ax^n}}{x^m} dx = -\frac{e^{ax^n}}{nx^n} + \frac{a \text{Ei}(ax^n)}{n} \quad \left[a \neq 0, \quad z = \frac{m-1}{n} = 1 \right]$
- 10.* $\int \frac{e^{ax^n}}{x^m} dx = -\frac{e^{ax^n}}{2nx^{2n}} - \frac{ae^{ax^n}}{2nx^n} + \frac{a^2 \text{Ei}(ax^n)}{2n} \quad \left[a \neq 0, \quad z = \frac{m-1}{n} = 2 \right]$
- 11.* $\int \frac{e^{ax^n}}{x^m} dx = -\frac{e^{ax^n}}{3nx^{3n}} - \frac{e^{ax^n}}{6nx^{2n}} - \frac{a^2e^{ax^n}}{6nx^n} + \frac{a^3 \text{Ei}(ax^n)}{6n}$
 $\left[a \neq 0, \quad z = \frac{m-1}{n} = 3 \right]$
- 12.* $\int \frac{e^{ax^2}}{x^2} dx = -\frac{e^{ax^2}}{x} + \sqrt{a\pi} \text{erfi}(\sqrt{ax})$ where $\text{erfi}(z) = \frac{\text{erf}(iz)}{i}$

$$13.* \quad \int e^{(ax^2+2bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{ac-b^2}{a}\right) \operatorname{erfi}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right) \quad [a \neq 0]$$

$$2.326 \quad \int \frac{xe^{ax} dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)} \quad [a \neq 0]$$

2.33

$$1.^8 \quad \int e^{-(ax^2+2bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-ac}{a}\right) \operatorname{erf}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right) \quad [a \neq 0]$$

$$2.* \quad \int e^{ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \operatorname{erfi}(\sqrt{a}x) \quad \text{where } \operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i} \quad [a \neq 0]$$

$$3.* \quad \int e^{ax^2+bx+c} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{ac-b^2}{a}\right) \operatorname{erfi}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right) \quad \text{where } \operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i} \quad [a \neq 0]$$

$$4.* \quad \int x^m e^{\pm ax^n} dx = \pm \frac{x^{m+1-n}}{na} \mp \frac{m+1-n}{na} \int x^{m-n} e^{\pm ax^n} dx \quad [a \neq 0, \quad n \neq 0]$$

$$5.* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left[(\gamma-1)! \sum_{k=0}^{\gamma-1} (-1)^{k+1-\gamma} \frac{x^{nk}}{k! a^{\gamma-k}} \right] \quad \left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 1, 2, \dots \right]$$

$$6.* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{na} \quad \left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 1 \right]$$

$$7.* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left(\frac{x^n}{a} - \frac{1}{a^2} \right) \quad \left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 2 \right]$$

$$8.* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left(\frac{x^{2n}}{a} - \frac{2x^n}{a^2} + \frac{2}{a^3} \right) \quad \left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 3 \right]$$

$$9.* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left(\frac{x^{3n}}{a} - \frac{3x^{2n}}{a^2} + \frac{6x^n}{a^3} - \frac{6}{a^4} \right) \quad \left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 4 \right]$$

$$10.* \quad \int x^m e^{-\beta x^n} dx = -\frac{\Gamma(\gamma, \beta x^n)}{n\beta^\gamma} \quad \text{for } \Gamma(\alpha, x) \text{ see 8.350.2}$$

$$= -\frac{1}{n\beta^\gamma} \int_{\beta x^n}^{\infty} t^{\gamma-1} e^{-t} dt \quad \left[\gamma = \frac{m+1}{n}, \quad \beta \neq 0, \quad n \neq 0 \right]$$

$$11.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{(\gamma-1)!}{n} \exp(-\beta x^n) \left[\sum_{k=0}^{\gamma-1} \frac{x^{nk}}{k! \beta^{\gamma-k}} \right] \quad \left[\gamma = \frac{m+1}{n} = 1, 2, \dots \right]$$

$$12.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n\beta} \quad \left[\gamma = \frac{m+1}{n} = 1 \right]$$

$$13.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n} \left(\frac{x^n}{\beta} + \frac{1}{\beta^2} \right) \quad \left[\gamma = \frac{m+1}{n} = 2 \right]$$

$$14.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n} \left(\frac{x^{2n}}{\beta} + \frac{2x^n}{\beta^2} + \frac{2}{\beta^3} \right) \quad \left[\gamma = \frac{m+1}{n} = 3 \right]$$

$$15.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n} \left(\frac{x^{3n}}{\beta} + \frac{3x^{2n}}{\beta^2} + \frac{6x^n}{\beta^3} + \frac{6}{\beta^4} \right) \quad \left[\gamma = \frac{m+1}{n} = 4 \right]$$

$$16.* \quad \int e^{-\beta x^n} dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \operatorname{erf}(\sqrt{\beta} x) \quad [\beta \neq 0]$$

$$17.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = -\frac{\beta^z \Gamma(-z, \beta x^n)}{n} \\ = -\frac{\beta^z}{n} \int_{\beta x^n}^{\infty} \frac{e^{-t}}{t^{z+a}} dt$$

$$z = \frac{m-1}{n}$$

$$18.* \quad \int \frac{\exp(-\beta x^n)}{x} dx = \frac{\operatorname{Ei}(-\beta x^n)}{n}$$

$$19.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = (-1)^z \frac{\exp(-\beta x^n)}{nz!} \sum_{k=0}^{z-1} (-1)^k! \frac{\beta^{z-k-1}}{x^{n(k+1)}} + (-1)^z \frac{\beta^z}{nz!} \operatorname{Ei}(-\beta x^n) \quad \left[z = \frac{m-1}{n} = 1, 2, \dots, \quad m = 2, 3, \dots \right]$$

$$20.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = -\frac{\exp(-\beta x^n)}{nx^n} - \frac{\beta \operatorname{Ei}(-\beta x^n)}{n} \quad \left[z = \frac{m-1}{n} = 1 \right]$$

$$21.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = -\frac{\exp(-\beta x^n)}{2nx^{2n}} + \frac{\beta \exp(-\beta x^n)}{2nx^n} + \frac{\beta^2 \operatorname{Ei}(-\beta x^n)}{2n} \quad \left[z = \frac{m-1}{n} = 2 \right]$$

$$22.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = -\frac{\exp(-\beta x^n)}{3nx^{3n}} + \frac{\beta \exp(-\beta x^n)}{6nx^{2n}} - \frac{\beta^2 \exp(-\beta x^n)}{6nx^n} - \frac{\beta^3 \operatorname{Ei}(-\beta x^n)}{6n} \quad \left[z = \frac{m-1}{n} = 3 \right]$$

$$23.* \quad \int \frac{\exp(-\beta x^2)}{x^2} dx = -\frac{\exp(-\beta x^2)}{x} - \sqrt{\beta\pi} \operatorname{erf}(\sqrt{\beta} x)$$

2.4 Hyperbolic Functions

2.41–2.43 Powers of $\sinh x$, $\cosh x$, $\tanh x$, and $\coth x$

$$\begin{aligned}
 2.411 \quad \int \sinh^p x \cosh^q x \, dx &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sinh^p x \cosh^{q-2} x \, dx \\
 &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{p+q} - \frac{p-1}{p+q} \int \sinh^{p-2} x \cosh^q x \, dx \\
 &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{q+1} - \frac{p-1}{q+1} \int \sinh^{p-2} x \cosh^{q+2} x \, dx \\
 &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+1} - \frac{q-1}{p+1} \int \sinh^{p+2} x \cosh^{q-2} x \, dx \\
 &= \frac{\sinh^{p+1} x \cosh^{q+1} x}{p+1} - \frac{p+q+2}{p+1} \int \sinh^{p+2} x \cosh^q x \, dx \\
 &= -\frac{\sinh^{p+1} x \cosh^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sinh^p x \cosh^{q+2} x \, dx
 \end{aligned}$$

2.412

$$\begin{aligned}
 1. \quad \int \sinh^p x \cosh^{2n} x \, dx &= \frac{\sinh^{p+1} x}{2n+p} \left[\cosh^{2n-1} x \right. \\
 &\quad \left. + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \cosh^{2n-2k-1} x \right] \\
 &\quad + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \sinh^p x \, dx
 \end{aligned}$$

This formula is applicable for arbitrary real p , except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

$$2. \quad \int \sinh^{2m} x \, dx = (-1)^m \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \frac{\sinh(2m-2k)x}{2m-2k} \quad \text{TI (543)}$$

$$\begin{aligned}
 3. \quad \int \sinh^{2m+1} x \, dx &= \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \frac{\cosh(2m-2k+1)x}{2m-2k+1}; \quad \text{TI (544)} \\
 &= (-1)^n \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\cosh^{2k+1} x}{2k+1} \quad \text{GU (351) (5)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int \sinh^p x \cosh^{2n+1} x \, dx \\
 &= \frac{\sinh^{p+1} x}{2n+p+1} \left\{ \cosh^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1)\dots(n-k+1) \cosh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right\}
 \end{aligned}$$

This formula is applicable for arbitrary real p , except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.413

$$\begin{aligned}
1. \quad \int \cosh^p x \sinh^{2n} x \, dx &= \frac{\cosh^{p+1} x}{2n+p} \left[\sinh^{2n-1} x \right. \\
&\quad \left. + \sum_{k=1}^{n-1} (-1)^k \frac{(2n-1)(2n-3)\dots(2n-2k+1) \sinh^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \right] \\
&\quad + (-1)^n \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \cosh^p x \, dx
\end{aligned}$$

This formula is applicable for arbitrary real p , except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

$$2. \quad \int \cosh^{2m} x \, dx = \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \frac{\sinh(2m-2k)x}{2m-2k} \quad \text{TI (541)}$$

$$\begin{aligned}
3. \quad \int \cosh^{2m+1} x \, dx &= \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \frac{\sinh(2m-2k+1)x}{2m-2k+1} \quad \text{TI (542)} \\
&= \sum_{k=0}^m \binom{m}{k} \frac{\sinh^{2k+1} x}{2k+1} \quad \text{GU (351) (8)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int \cosh^p x \sinh^{2n+1} x \, dx &= \frac{\cosh^{p+1} x}{2n+p+1} \left[\sinh^{2n} x \right. \\
&\quad \left. + \sum_{k=1}^n (-1)^k \frac{2^k n(n-1)\dots(n-k+1) \sinh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right]
\end{aligned}$$

This formula is applicable for arbitrary real p , except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.414

$$\begin{aligned}
1. \quad \int \sinh ax \, dx &= \frac{1}{a} \cosh ax \\
2. \quad \int \sinh^2 ax \, dx &= \frac{1}{4a} \sinh 2ax - \frac{x}{2} \\
3. \quad \int \sinh^3 x \, dx &= -\frac{3}{4} \cosh x + \frac{1}{12} \cosh 3x = \frac{1}{3} \cosh^3 x - \cosh x \\
4. \quad \int \sinh^4 x \, dx &= \frac{3}{8}x - \frac{1}{4} \sinh 2x + \frac{1}{32} \sinh 4x = \frac{3}{8}x - \frac{3}{8} \sinh x \cosh x + \frac{1}{4} \sinh^3 x \cosh x \\
5. \quad \int \sinh^5 x \, dx &= \frac{5}{8} \cosh x - \frac{5}{48} \cosh 3x + \frac{1}{80} \cosh 5x \\
&= \frac{4}{5} \cosh x + \frac{1}{5} \sinh^4 x \cosh x - \frac{4}{15} \cosh^3 x \\
6. \quad \int \sinh^6 x \, dx &= -\frac{5}{16}x + \frac{15}{64} \sinh 2x - \frac{3}{64} \sinh 4x + \frac{1}{192} \sinh 6x \\
&= -\frac{5}{16}x + \frac{1}{6} \sinh^5 x \cosh x - \frac{5}{24} \sinh^3 x \cosh x + \frac{5}{16} \sinh x \cosh x
\end{aligned}$$

7.
$$\int \sinh^7 x \, dx = -\frac{35}{64} \cosh x + \frac{7}{64} \cosh 3x - \frac{7}{320} \cosh 5x + \frac{1}{448} \cosh 7x$$
$$= -\frac{24}{35} \cosh x + \frac{8}{35} \cosh^3 x - \frac{6}{35} \cosh x \sinh^4 x + \frac{1}{7} \cosh x \sinh^6 x$$
8.
$$\int \cosh ax \, dx = \frac{1}{a} \sinh ax$$
9.
$$\int \cosh^2 ax \, dx = \frac{x}{2} + \frac{1}{4a} \sinh 2ax$$
10.
$$\int \cosh^3 x \, dx = \frac{3}{4} \sinh x + \frac{1}{12} \sinh 3x = \sinh x + \frac{1}{3} \sinh^3 x$$
11.
$$\int \cosh^4 x \, dx = \frac{3}{8} x + \frac{1}{4} \sinh 2x + \frac{1}{32} \sinh 4x = \frac{3}{8} x + \frac{3}{8} \sinh x \cosh x + \frac{1}{4} \sinh x \cosh^3 x$$
12.
$$\int \cosh^5 x \, dx = \frac{5}{8} \sinh x + \frac{5}{48} \sinh 3x + \frac{1}{80} \sinh 5x$$
$$= \frac{4}{5} \sinh x + \frac{1}{5} \cosh^4 x \sinh x + \frac{4}{15} \sinh^3 x$$
13.
$$\int \cosh^6 x \, dx = \frac{5}{16} x + \frac{15}{64} \sinh 2x + \frac{3}{64} \sinh 4x + \frac{1}{192} \sinh 6x$$
$$= \frac{5}{16} x + \frac{5}{16} \sinh x \cosh x + \frac{5}{24} \sinh x \cosh^3 x + \frac{1}{6} \sinh x \cosh^5 x$$
14.
$$\int \cosh^7 x \, dx = \frac{35}{64} \sinh x + \frac{7}{64} \sinh 3x + \frac{7}{320} \sinh 5x + \frac{1}{448} \sinh 7x$$
$$= \frac{24}{35} \sinh x + \frac{8}{35} \sinh^3 x + \frac{6}{35} \sinh x \cosh^4 x + \frac{1}{7} \sinh x \cosh^6 x$$

2.415

1.
$$\int \sinh ax \cosh bx \, dx = \frac{\cosh(a+b)x}{2(a+b)} + \frac{\cosh(a-b)x}{2(a-b)}$$
2.
$$\int \sinh ax \cosh ax \, dx = \frac{1}{4a} \cosh 2ax$$
3.
$$\int \sinh^2 x \cosh x \, dx = \frac{1}{3} \sinh^3 x$$
4.
$$\int \sinh^3 x \cosh x \, dx = \frac{1}{4} \sinh^4 x$$
5.
$$\int \sinh^4 x \cosh x \, dx = \frac{1}{5} \sinh^5 x$$
6.
$$\int \sinh x \cosh^2 x \, dx = \frac{1}{3} \cosh^3 x$$
7.
$$\int \sinh^2 x \cosh^2 x \, dx = -\frac{x}{8} + \frac{1}{32} \sinh 4x$$
8.
$$\int \sinh^3 x \cosh^2 x \, dx = \frac{1}{5} \left(\sinh^2 x - \frac{2}{3} \right) \cosh^3 x$$
9.
$$\int \sinh^4 x \cosh^2 x \, dx = \frac{x}{16} - \frac{1}{64} \sinh 2x - \frac{1}{64} \sinh 4x + \frac{1}{192} \sinh 6x$$

10. $\int \sinh x \cosh^3 x \, dx = \frac{1}{4} \cosh^4 x$
11. $\int \sinh^2 x \cosh^3 x \, dx = \frac{1}{5} \left(\cosh^2 x + \frac{2}{3} \right) \sinh^3 x$
12. $\int \sinh^3 x \cosh^3 x \, dx = -\frac{3}{64} \cosh 2x + \frac{1}{192} \cosh 6x = \frac{1}{48} \cosh^3 2x - \frac{1}{16} \cosh 2x$
 $= \frac{\sinh^6 x}{6} + \frac{\sinh^4 x}{4} = \frac{\cosh^6 x}{6} - \frac{\cosh^4 x}{4}$
13. $\int \sinh^4 x \cosh^3 x \, dx = \frac{1}{7} \sinh^3 x \left(\cosh^4 x - \frac{3}{5} \cosh^2 x - \frac{2}{5} \right) = \frac{1}{7} \left(\cosh^2 x + \frac{2}{5} \right) \sinh^5 x$
14. $\int \sinh x \cosh^4 x \, dx = \frac{1}{5} \cosh^5 x$
15. $\int \sinh^2 x \cosh^4 x \, dx = -\frac{x}{16} - \frac{1}{64} \sinh 2x + \frac{1}{64} \sinh 4x + \frac{1}{192} \sinh 6x$
16. $\int \sinh^3 x \cosh^4 x \, dx = \frac{1}{7} \cosh^3 x \left(\sinh^4 x + \frac{3}{5} \sinh^2 x - \frac{2}{5} \right) = \frac{1}{7} \left(\sinh^2 x - \frac{2}{5} \right) \cosh^5 x$
17. $\int \sinh^4 x \cosh^4 x \, dx = \frac{3x}{128} - \frac{1}{128} \sinh 4x + \frac{1}{1024} \sinh 8x$

2.416

$$1.^{10} \quad \int \frac{\sinh^p x}{\cosh^{2n} x} \, dx = \frac{\sinh^{p+1} x}{2n-1} \left[\operatorname{sech}^{2n-1} x \right. \\
+ \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{sech}^{2n-2k-1} x \left. \right] \\
+ \frac{(2n-p-2)(2n-p-4) \dots (-p+2)(-p)}{(2n-1)!!} \int \sinh^p x \, dx$$

This formula is applicable for arbitrary real p . For $\int \sinh^p x \, dx$, where p is a natural number, see **2.412 2** and **2.412 3**. For $n = 0$ and p a negative integer, we have for this integral:

$$2. \quad \int \frac{dx}{\sinh^{2m} x} = \frac{\cosh x}{2m-1} \left[-\operatorname{cosech}^{2m-1} x \right. \\
+ \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{2^k (m-1)(m-2) \dots (m-k)}{(2m-3)(2m-5) \dots (2m-2k-1)} \operatorname{cosec} h^{2m-2k-1} x \left. \right]$$

$$3. \quad \int \frac{dx}{\sinh^{2m+1} x} = \frac{\cosh x}{2m} \left[-\operatorname{cosech}^{2m} x \right. \\
+ \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m-1)(m-2) \dots (m-k)} \operatorname{cosec} h^{2m-2k} x \left. \right] \\
+ (-1)^m \frac{(2m-1)!!}{(2m)!!} \ln \tanh \frac{x}{2}$$

2.417

$$1. \quad \int \frac{\sinh^p x}{\cosh^{2n+1} x} dx = \frac{\sinh^{p+1} x}{2n} \left[\operatorname{sech}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \operatorname{sech}^{2n-2k} x \right] + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!} \int \frac{\sinh^p x}{\cosh x} dx$$

This formula is applicable for arbitrary real p . For $n = 0$ and p integral, we have

$$2. \quad \int \frac{\sinh^{2m+1} x}{\cosh x} dx = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k} \sinh^{2k} x + (-1)^m \ln \cosh x \\ = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k} \binom{m}{k} \cosh^{2k} x + (-1)^m \ln \cosh x \quad [m \geq 1]$$

$$3. \quad \int \frac{\sinh^{2m} x}{\cosh x} dx = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k-1} \sinh^{2k-1} x + (-1)^m \arctan(\sinh x) \\ [m \geq 1]$$

$$4. \quad \int \frac{dx}{\sinh^{2m+1} x \cosh x} = \sum_{k=1}^m \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+2} + (-1)^m \ln \tanh x$$

$$5. \quad \int \frac{dx}{\sinh^{2m} x \cosh x} = \sum_{k=1}^m \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+1} + (-1)^m \arctan \sinh x$$

2.418

$$1. \quad \int \frac{\cosh^p x}{\sinh^{2n} x} dx = -\frac{\cosh^{p+1} x}{2n-1} \left[\operatorname{cosech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x \right] + \frac{(-1)^n (2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \cosh^p x dx$$

This formula is applicable for arbitrary real p . For the integral $\int \cosh^p x dx$, where p is a natural number, see 2.413 2 and 2.413 3. If p is a negative integer, we have for this integral:

$$2. \quad \int \frac{dx}{\cosh^{2m} x} = \frac{\sinh x}{2m-1} \left\{ \operatorname{sech}^{2m-1} x + \sum_{k=1}^{m-1} \frac{2^k(m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{sech}^{2m-2k-1} x \right\}$$

$$3. \quad \int \frac{dx}{\cosh^{2m+1} x} = \frac{\sinh x}{2m} \left\{ \operatorname{sech}^{2m} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k(m-1)(m-2)\dots(m-k)} \operatorname{sech}^{2m-2k} x \right\} + \frac{(2m-1)!!}{(2m)!!} \arctan \sinh x$$

2.419

$$1. \quad \int \frac{\cosh^p x}{\sinh^{2n+1} x} dx = -\frac{\cosh^{p+1} x}{2n} \left[\operatorname{cosech}^{2n} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \operatorname{cosec} h^{2n-2k} x \right] + \frac{(-1)^n (2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\cosh^p x}{\sinh x} dx$$

This formula is applicable for arbitrary real p . For $n = 0$ and p an integer

$$2. \quad \int \frac{\cosh^{2m} x}{\sinh x} dx = \sum_{k=1}^m \frac{\cosh^{2k-1} x}{2k-1} + \ln \tanh \frac{x}{2}$$

$$3. \quad \int \frac{\cosh^{2m+1} x}{\sinh x} dx = \sum_{k=1}^m \frac{\cosh^{2k} x}{2k} + \ln \sinh x$$

$$= \sum_{k=1}^m \binom{m}{k} \frac{\sinh^{2k} x}{2k} + \ln \sinh x$$

$$4. \quad \int \frac{dx}{\sinh x \cosh^{2m} x} = \sum_{k=1}^m \frac{\operatorname{sech}^{2m-2k+1} x}{2m-2k+1} + \ln \tanh \frac{x}{2}$$

$$5. \quad \int \frac{dx}{\sinh x \cosh^{2m+1} x} = \sum_{k=1}^m \frac{\operatorname{sech}^{2m-2k+2} x}{2m-2k+2} + \ln \tanh x$$

2.421 In formulas **2.421** 1 and **2.421** 2, $s = 1$ for m odd and $m < 2n + 1$; in all other cases, $s = 0$.
GI (351)(11, 13)

$$1.^{10} \quad \int \frac{\sinh^{2n+1} x}{\cosh^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^{n+k} \binom{n}{k} \frac{\cosh^{2k-m+1} x}{2k-m+1} + s(-1)^{n+\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln \cosh x$$

$$2. \quad \int \frac{\cosh^{2n+1} x}{\sinh^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n \binom{n}{k} \frac{\sinh^{2k-m+1} x}{2k-m+1} + s \binom{n}{\frac{m-1}{2}} \ln \sinh x$$

2.422

$$1. \quad \int \frac{dx}{\sinh^{2m} x \cosh^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^{k+1}}{2m-2k-1} \binom{m+n-1}{k} \tanh^{2k-2m+1} x$$

$$2. \quad \int \frac{dx}{\sinh^{2m+1} x \cosh^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \tanh^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln \tanh x$$

GI (351)(15)

2.423

$$1. \quad \int \frac{dx}{\sinh x} = \ln \tanh \frac{x}{2} = \frac{1}{2} \ln \frac{\cosh x - 1}{\cosh x + 1}$$

2. $\int \frac{dx}{\sinh^2 x} = -\coth x$
3. $\int \frac{dx}{\sinh^3 x} = -\frac{\cosh x}{2 \sinh^2 x} - \frac{1}{2} \ln \tanh \frac{x}{2}$
4. $\int \frac{dx}{\sinh^4 x} = -\frac{\cosh x}{3 \sinh^3 x} + \frac{2}{3} \coth x = -\frac{1}{3} \coth^3 x + \coth x$
5. $\int \frac{dx}{\sinh^5 x} = -\frac{\cosh x}{4 \sinh^4 x} + \frac{3}{8} \frac{\cosh x}{\sinh^2 x} + \frac{3}{8} \ln \tanh \frac{x}{2}$
6. $\int \frac{dx}{\sinh^6 x} = -\frac{\cosh x}{5 \sinh^5 x} + \frac{4}{15} \coth^3 x - \frac{4}{5} \coth x$
 $= -\frac{1}{5} \coth^5 x + \frac{2}{3} \coth^3 x - \coth x$
7. $\int \frac{dx}{\sinh^7 x} = -\frac{\cosh x}{6 \sinh^2 x} \left(\frac{1}{\sinh^4 x} - \frac{5}{4 \sinh^2 x} + \frac{15}{8} \right) - \frac{5}{16} \ln \tanh \frac{x}{2}$
8. $\int \frac{dx}{\sinh^8 x} = \coth x - \coth^3 x + \frac{3}{5} \coth^5 x - \frac{1}{7} \coth^7 x$
9. $\int \frac{dx}{\cosh x} = \arctan (\sinh x)$
 $= \arcsin (\tanh x)$
 $= 2 \arctan (e^x)$
 $= \operatorname{gd} x$
10. $\int \frac{dx}{\cosh^2 x} = \tanh x$
11. $\int \frac{dx}{\cosh^3 x} = \frac{\sinh x}{2 \cosh^2 x} + \frac{1}{2} \arctan (\sinh x)$
12. $\int \frac{dx}{\cosh^4 x} = \frac{\sinh x}{3 \cosh^3 x} + \frac{2}{3} \tanh x$
 $= -\frac{1}{3} \tanh^3 x + \tanh x$
13. $\int \frac{dx}{\cosh^5 x} = \frac{\sinh x}{4 \cosh^4 x} + \frac{3}{8} \frac{\sinh x}{\cosh^2 x} + \frac{3}{8} \arctan (\sinh x)$
14. $\int \frac{dx}{\cosh^6 x} = \frac{\sinh x}{5 \cosh^5 x} - \frac{4}{15} \tanh^3 x + \frac{4}{5} \tanh x$
 $= \frac{1}{5} \tanh^5 x - \frac{2}{3} \tanh^3 x + \tanh x$
15. $\int \frac{dx}{\cosh^7 x} = \frac{\sinh x}{6 \cosh^2 x} \left(\frac{1}{\cosh^4 x} + \frac{5}{4 \cosh^2 x} + \frac{15}{8} \right) + \frac{5}{16} \arctan (\sinh x)$
16. $\int \frac{dx}{\cosh^8 x} = -\frac{1}{7} \tanh^7 x + \frac{3}{5} \tanh^5 x - \tanh^3 x + \tanh x$
17. $\int \frac{\sinh x}{\cosh x} dx = \ln \cosh x$

18. $\int \frac{\sinh^2 x}{\cosh x} dx = \sinh x - \arctan(\sinh x)$
19. $\int \frac{\sinh^3 x}{\cosh x} dx = \frac{1}{2} \sinh^2 x - \ln \cosh x$
 $= \frac{1}{2} \cosh^2 x - \ln \cosh x$
20. $\int \frac{\sinh^4 x}{\cosh x} dx = \frac{1}{3} \sinh^3 x - \sinh x + \arctan(\sinh x)$
21. $\int \frac{\sinh x}{\cosh^2 x} dx = -\frac{1}{\cosh x}$
22. $\int \frac{\sinh^2 x}{\cosh^2 x} dx = x - \tanh x$
23. $\int \frac{\sinh^3 x}{\cosh^2 x} dx = \cosh x + \frac{1}{\cosh x}$
24. $\int \frac{\sinh^4 x}{\cosh^2 x} dx = -\frac{3}{2}x + \frac{1}{4} \sinh 2x + \tanh x$
25. $\int \frac{\sinh x}{\cosh^3 x} dx = -\frac{1}{2 \cosh^2 x}$
 $= \frac{1}{2} \tanh^2 x$
26. $\int \frac{\sinh^2 x}{\cosh^3 x} dx = -\frac{\sinh x}{2 \cosh^2 x} + \frac{1}{2} \arctan(\sinh x)$
27. $\int \frac{\sinh^3 x}{\cosh^3 x} dx = -\frac{1}{2} \tanh^2 x + \ln \cosh x$
 $= \frac{1}{2 \cosh^2 x} + \ln \cosh x$
28. $\int \frac{\sinh^4 x}{\cosh^3 x} dx = \frac{\sinh x}{2 \cosh x} + \sinh x - \frac{3}{2} \arctan(\sinh x)$
29. $\int \frac{\sinh x}{\cosh^4 x} dx = -\frac{1}{3 \cosh^3 x}$
30. $\int \frac{\sinh^2 x}{\cosh^4 x} dx = \frac{1}{3} \tanh^3 x$
31. $\int \frac{\sinh^3 x}{\cosh^4 x} dx = -\frac{1}{\cosh x} + \frac{1}{3 \cosh^3 x}$
32. $\int \frac{\sinh^4 x}{\cosh^4 x} dx = -\frac{1}{3} \tanh^3 x - \tanh x + x$
33. $\int \frac{\cosh x}{\sinh x} dx = \ln \sinh x$
34. $\int \frac{\cosh^2 x}{\sinh x} dx = \cosh x + \ln \tanh \frac{x}{2}$

$$35. \quad \int \frac{\cosh^3 x}{\sinh x} dx = \frac{1}{2} \cosh^2 x + \ln \sinh x$$

$$36. \quad \int \frac{\cosh^4 x}{\sinh x} dx = \frac{1}{3} \cosh^3 x + \cosh x + \ln \tanh \frac{x}{2}$$

$$37. \quad \int \frac{\cosh x}{\sinh^2 x} dx = -\frac{1}{\sinh x}$$

$$38. \quad \int \frac{\cosh^2 x}{\sinh^2 x} dx = x - \coth x$$

$$39. \quad \int \frac{\cosh^3 x}{\sinh^2 x} dx = \sinh x - \frac{1}{\sinh x}$$

$$40. \quad \int \frac{\cosh^4 x}{\sinh^2 x} dx = \frac{3}{2}x + \frac{1}{4} \sinh 2x - \coth x$$

$$41. \quad \int \frac{\cosh x}{\sinh^3 x} dx = -\frac{1}{2 \sinh^2 x} \\ = -\frac{1}{2} \coth^2 x$$

$$42. \quad \int \frac{\cosh^2 x}{\sinh^3 x} dx = -\frac{\cosh x}{2 \sinh^2 x} + \ln \tanh \frac{x}{2}$$

$$43. \quad \int \frac{\cosh^3 x}{\sinh^3 x} dx = -\frac{1}{2 \sinh^2 x} + \ln \sinh x \\ = -\frac{1}{2} \coth^2 x + \ln \sinh x$$

$$44. \quad \int \frac{\cosh^4 x}{\sinh^3 x} dx = -\frac{\cosh x}{2 \sinh^2 x} + \cosh x + \frac{3}{2} \ln \tanh \frac{x}{2}$$

$$45. \quad \int \frac{\cosh x}{\sinh^4 x} dx = -\frac{1}{3 \sinh^3 x}$$

$$46. \quad \int \frac{\cosh^2 x}{\sinh^4 x} dx = -\frac{1}{3} \coth^3 x$$

$$47. \quad \int \frac{\cosh^3 x}{\sinh^4 x} dx = -\frac{1}{\sinh x} - \frac{1}{3 \sinh^3 x}$$

$$48. \quad \int \frac{\cosh^4 x}{\sinh^4 x} dx = -\frac{1}{3} \coth^3 x - \coth x + x$$

$$49. \quad \int \frac{dx}{\sinh x \cosh x} = \ln \tanh x$$

$$50. \quad \int \frac{dx}{\sinh x \cosh^2 x} = \frac{1}{\cosh x} + \ln \tanh \frac{x}{2}$$

$$51. \quad \int \frac{dx}{\sinh x \cosh^3 x} = \frac{1}{2 \cosh^2 x} + \ln \tanh x \\ = -\frac{1}{2} \tanh^2 x + \ln \tanh x$$

52. $\int \frac{dx}{\sinh x \cosh^4 x} = \frac{1}{\cosh x} + \frac{1}{3 \cosh^3 x} + \ln \tanh \frac{x}{2}$
53. $\int \frac{dx}{\sinh^2 x \cosh x} = -\frac{1}{\sinh x} - \arctan \sinh x$
54. $\int \frac{dx}{\sinh^2 x \cosh^2 x} = -2 \coth 2x$
55. $\int \frac{dx}{\sinh^2 x \cosh^3 x} = -\frac{\sinh x}{2 \cosh^2 x} - \frac{1}{\sinh x} - \frac{3}{2} \arctan \sinh x$
56. $\int \frac{dx}{\sinh^2 x \cosh^4 x} = \frac{1}{3 \sinh x \cosh^3 x} - \frac{8}{3} \coth 2x$
57. $\int \frac{dx}{\sinh^3 x \cosh x} = -\frac{1}{2 \sinh^2 x} - \ln \tanh x$
 $= -\frac{1}{2} \coth^2 x + \ln \coth x$
58. $\int \frac{dx}{\sinh^3 x \cosh^2 x} = -\frac{1}{\cosh x} - \frac{\cosh x}{2 \sinh^2 x} - \frac{3}{2} \ln \tanh \frac{x}{2}$
59. $\int \frac{dx}{\sinh^3 x \cosh^3 x} = -\frac{2 \cosh 2x}{\sinh^2 2x} - 2 \ln \tanh x$
 $= \frac{1}{2} \tanh^2 x - \frac{1}{2} \coth^2 x - 2 \ln \tanh x$
60. $\int \frac{dx}{\sinh^3 x \cosh^4 x} = -\frac{2}{\cosh x} - \frac{1}{3 \cosh^2 x} - \frac{\cosh x}{2 \sinh^2 x} - \frac{5}{2} \ln \tanh \frac{x}{2}$
61. $\int \frac{dx}{\sinh^4 x \cosh x} = \frac{1}{\sinh x} - \frac{1}{3 \sinh^3 x} + \arctan \sinh x$
62. $\int \frac{dx}{\sinh^4 x \cosh^2 x} = -\frac{1}{3 \cosh x \sinh^3 x} + \frac{8}{3} \coth 2x$
63. $\int \frac{dx}{\sinh^4 x \cosh^3 x} = \frac{2}{\sinh x} - \frac{1}{3 \sinh^3 x} + \frac{\sinh x}{2 \cosh^2 x} + \frac{5}{2} \arctan \sinh x$
64. $\int \frac{dx}{\sinh^4 x \cosh^4 x} = 8 \coth 2x - \frac{8}{3} \coth^3 2x$

2.424

1. $\int \tanh^p x \, dx = -\frac{\tanh^{p-1} x}{p-1} + \int \tanh^{p-2} x \, dx \quad [p \neq 1]$
2. $\int \tanh^{2n+1} x \, dx = \sum_{k=1}^n \frac{(-1)^{k-1}}{2k} \binom{n}{k} \frac{1}{\cosh^{2k} x} + \ln \cosh x$
 $= -\sum_{k=1}^n \frac{\tanh^{2n-2k+2} x}{2n-2k+2} + \ln \cosh x$
3. $\int \tanh^{2n} x \, dx = -\sum_{k=1}^n \frac{\tanh^{2n-2k+1} x}{2n-2k+1} + x$ GU (351)(12)
4. $\int \coth^p x \, dx = -\frac{\coth^{p-1} x}{p-1} + \int \coth^{p-2} x \, dx \quad [p \neq 1]$

$$\begin{aligned}
 5. \quad \int \coth^{2n+1} x \, dx &= - \sum_{k=1}^n \frac{1}{2n} \binom{n}{k} \frac{1}{\sinh^{2k} x} + \ln \sinh x \\
 &= - \sum_{k=1}^n \frac{\coth^{2n-2k+2} x}{2n-2k+2} + \ln \sinh x
 \end{aligned}$$

$$6. \quad \int \coth^{2n} x \, dx = - \sum_{k=1}^n \frac{\coth^{2n-2k+1} x}{2n-2k+1} + x \quad \text{GU (351)(14)}$$

For formulas containing powers of $\tanh x$ and $\coth x$ equal to $n = 1, 2, 3, 4$, see **2.423** 17, **2.423** 22, **2.423** 27, **2.423** 32, **2.423** 33, **2.423** 38, **2.423** 43, **2.423** 48.

Powers of hyperbolic functions and hyperbolic functions of linear functions of the argument

2.425

$$\begin{aligned}
 1. \quad \int \sinh(ax+b) \sinh(cx+d) \, dx &= \frac{1}{2(a+c)} \sinh[(a+c)x+b+d] \\
 &\quad - \frac{1}{2(a-c)} \sinh[(a-c)x+b-d] \\
 &\quad [a^2 \neq c^2] \quad \text{GU (352)(2a)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int \sinh(ax+b) \cosh(cx+d) \, dx &= \frac{1}{2(a+c)} \cosh[(a+c)x+b+d] \\
 &\quad + \frac{1}{2(a-c)} \cosh[(a-c)x+b-d] \\
 &\quad [a^2 \neq c^2] \quad \text{GU (352)(2c)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int \cosh(ax+b) \cosh(cx+d) \, dx &= \frac{1}{2(a+c)} \sinh[(a+c)x+b+d] \\
 &\quad + \frac{1}{2(a-c)} \sinh[(a-c)x+b-d] \\
 &\quad [a^2 \neq c^2] \quad \text{GU (352)(2b)}
 \end{aligned}$$

When $a = c$:

$$4. \quad \int \sinh(ax+b) \sinh(ax+d) \, dx = -\frac{x}{2} \cosh(b-d) + \frac{1}{4a} \sinh(2ax+b+d) \quad \text{GU (352)(3a)}$$

$$5. \quad \int \sinh(ax+b) \cosh(ax+d) \, dx = \frac{x}{2} \sinh(b-d) + \frac{1}{4a} \cosh(2ax+b+d) \quad \text{GU (352)(3c)}$$

$$6. \quad \int \cosh(ax+b) \cosh(ax+d) \, dx = \frac{x}{2} \cosh(b-d) + \frac{1}{4a} \sinh(2ax+b+d) \quad \text{GU (352)(3b)}$$

2.426

$$\begin{aligned}
 1. \quad \int \sinh ax \sinh bx \sinh cx \, dx &= \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} \\
 &\quad - \frac{\cosh(a-b+c)x}{4(a-b+c)} - \frac{\cosh(a+b-c)x}{4(a+b-c)} \\
 &\quad \text{GU (352)(4a)}
 \end{aligned}$$

$$2. \quad \int \sinh ax \sinh bx \cosh cx \, dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} - \frac{\sinh(-a+b+c)x}{4(-a+b+c)} - \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4b)

$$3. \quad \int \sinh ax \cosh bx \cosh cx \, dx = \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} + \frac{\cosh(a-b+c)x}{4(a-b+c)} + \frac{\cosh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4c)

$$4. \quad \int \cosh ax \cosh bx \cosh cx \, dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} + \frac{\sinh(-a+b+c)x}{4(-a+b+c)} + \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4d)

2.427

$$1. \quad \int \sinh^p x \sinh ax \, dx = \frac{1}{p+a} \left\{ \sinh px \cosh ax - p \int \sinh^{p-1} x \cosh(a-1)x \, dx \right\}$$

$$2. \quad \int \sinh^p x \sinh(2n+1)x \, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \times \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k+1)x \right. \\ \left. - \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k)x \right] \\ + \frac{\Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p+1-2n)} \int \sinh^{p-2n} x \sinh x \, dx$$

[p is not a negative integer]

$$3. \quad \int \sinh^p x \sinh 2nx \, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \times \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k)x \right. \\ \left. - \frac{\Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k-1)x \right]$$

[p is not a negative integer] GU (352)(5)a

2.428

$$1. \quad \int \sinh^p x \cosh ax \, dx = \frac{1}{p+a} \left\{ \sinh^p x \sinh ax - p \int \sinh^{p-1} x \sinh(a-1)x \, dx \right\}$$

$$\begin{aligned}
2. \quad \int \sinh^p x \cosh(2n+1)x \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \\
&\times \left\{ \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k+1)x \right. \right. \\
&\quad \left. \left. - \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k)x \right] \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p+1-2n)} \int \sinh^{p-2n} x \cosh x \, dx \right\} \\
&\quad [p \text{ is not a negative integer}]
\end{aligned}$$

$$\begin{aligned}
3. \quad \int \sinh^p x \cosh 2nx \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \\
&\times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k)x \right. \right. \\
&\quad \left. \left. - \frac{\Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k-1)x \right] \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p}{2} - n + 1\right)}{2^{2n} \Gamma(p+1-2n)} \int \sinh^{p-2n} x \, dx \right\} \\
&\quad [p \text{ is not a negative integer}] \quad \text{GU (352)(6)a}
\end{aligned}$$

2.429

$$\begin{aligned}
1. \quad \int \cosh^p x \sinh ax \, dx &= \frac{1}{p+a} \left\{ \cosh^p x \cosh ax + p \int \cosh^{p-1} x \sinh(a-1)x \, dx \right\} \\
2. \quad \int \cosh^p x \sinh(2n+1)x \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k+1)x \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cosh^{p-n} x \sinh(n+1)x \, dx \right] \\
&\quad [p \text{ is not a negative integer}] \\
3. \quad \int \cosh^p x \sinh 2nx \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k)x \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \int \cosh^{p-n} x \sinh nx \, dx \right] \\
&\quad [p \text{ is not a negative integer}] \quad \text{GU (352)(7)a}
\end{aligned}$$

2.431

1.
$$\int \cosh^p x \cosh ax \, dx = \frac{1}{p+a} \left\{ \cosh^p x \sinh ax + p \int \cosh^{p-1} x \cosh(a-1)x \, dx \right\}$$
2.
$$\int \cosh^p x \cosh(2n+1)x \, dx = \frac{\Gamma(p+1)}{\Gamma(\frac{p+3}{2}+n)} \left[\sum_{k=0}^{n-1} \frac{\Gamma(\frac{p+1}{2}+n-k)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \sinh(2n-k+1)x \right. \\ \left. + \frac{\Gamma(\frac{p+3}{2})}{2^n \Gamma(p-n+1)} \int \cosh^{p-n} x \cosh(n+1)x \, dx \right] \\ [p \text{ is not a negative integer}]$$
3.
$$\int \cosh^p x \cosh 2nx \, dx = \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2}+n+1)} \left[\sum_{k=0}^{n-1} \frac{\Gamma(\frac{p}{2}+n-k)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \sinh(2n-k)x \right. \\ \left. + \frac{\Gamma(\frac{p}{2}+1)}{2^n \Gamma(p-n+1)} \cosh^{p-n} x \cosh nx \, dx \right] \\ [p \text{ is not a negative integer}] \quad \text{GU (352)(8)a}$$

2.432

1.
$$\int \sinh(n+1)x \sinh^{n-1} x \, dx = \frac{1}{n} \sinh^n x \sinh nx$$
2.
$$\int \sinh(n+1)x \cosh^{n-1} x \, dx = \frac{1}{n} \cosh^n x \cosh nx$$
3.
$$\int \cosh(n+1)x \sinh^{n-1} x \, dx = \frac{1}{n} \sinh^n x \cosh nx$$
4.
$$\int \cosh(n+1)x \cosh^{n-1} x \, dx = \frac{1}{n} \cosh^n x \sinh nx$$

2.433

1.
$$\int \frac{\sinh(2n+1)x}{\sinh x} \, dx = 2 \sum_{k=0}^{n-1} \frac{\sinh(2n-2k)x}{2n-2k} + x$$
2.
$$\int \frac{\sinh 2nx}{\sinh x} \, dx = 2 \sum_{k=0}^{n-1} \frac{\sinh(2n-2k-1)x}{2n-2k-1} \quad \text{GU (352)(5d)}$$
3.
$$\int \frac{\cosh(2n+1)x}{\sinh x} \, dx = 2 \sum_{k=0}^{n-1} \frac{\cosh(2n-2k)x}{2n-2k} + \ln \sinh x$$
4.
$$\int \frac{\cosh 2nx}{\sinh x} \, dx = 2 \sum_{k=0}^{n-1} \frac{\cosh(2n-2k-1)x}{2n-2k-1} + \ln \tanh \frac{x}{2} \quad \text{GU (352)(6d)}$$
5.
$$\int \frac{\sinh(2n+1)x}{\cosh x} \, dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\cosh(2n-2k)x}{2n-2k} + (-1)^n \ln \cosh x$$

$$6. \quad \int \frac{\sinh 2nx}{\cosh x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\cosh(2n-2k-1)x}{2n-2k-1} \quad \text{GU (352)(7d)}$$

$$7. \quad \int \frac{\cosh(2n+1)x}{\cosh x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\sinh(2n-2k)x}{2n-2k} + (-1)^n x$$

$$8. \quad \int \frac{\cosh 2nx}{\cosh x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\sinh(2n-2k-1)x}{2n-2k-1} + (-1)^n \arcsin(\tanh x) \quad \text{GU (352)(8d)}$$

$$9. \quad \int \frac{\sinh 2x}{\sinh^n x} dx = -\frac{2}{(n-2) \sinh^{n-2} x}$$

For $n = 2$:

$$10. \quad \int \frac{\sinh 2x}{\sinh^2 x} dx = 2 \ln \sinh x$$

$$11. \quad \int \frac{\sinh 2x}{\cosh^n x} dx = \frac{2}{(2-n) \cosh^{n-2} x}$$

For $n = 2$:

$$12. \quad \int \frac{\sinh 2x}{\cosh^2 x} dx = 2 \ln \cosh x$$

$$13. \quad \int \frac{\cosh 2x}{\sinh x} dx = 2 \cosh x + \ln \tanh \frac{x}{2}$$

$$14. \quad \int \frac{\cosh 2x}{\sinh^2 x} dx = -\coth x + 2x$$

$$15. \quad \int \frac{\cosh 2x}{\sinh^3 x} dx = -\frac{\cosh x}{2 \sinh^2 x} + \frac{3}{2} \ln \tanh \frac{x}{2}$$

$$16. \quad \int \frac{\cosh 2x}{\cosh x} dx = 2 \sinh x - \arcsin(\tanh x)$$

$$17. \quad \int \frac{\cosh 2x}{\cosh^2 x} dx = -\tanh x + 2x$$

$$18. \quad \int \frac{\cosh 2x}{\cosh^3 x} dx = -\frac{\sinh x}{2 \cosh^2 x} + \frac{3}{2} \arcsin(\tanh x)$$

$$19. \quad \int \frac{\sinh 3x}{\sinh x} dx = x + \sinh 2x$$

$$20. \quad \int \frac{\sinh 3x}{\sinh^2 x} dx = 3 \ln \tanh \frac{x}{2} + 4 \cosh x$$

$$21. \quad \int \frac{\sinh 3x}{\sinh^3 x} dx = -3 \coth x + 4x$$

$$22. \quad \int \frac{\sinh 3x}{\cosh^n x} dx = \frac{4}{(3-n) \cosh^{n-3} x} - \frac{1}{(1-n) \cosh^{n-1} x}$$

For $n = 1$ and $n = 3$:

$$23. \quad \int \frac{\sinh 3x}{\cosh x} dx = 2 \sinh^2 x - \ln \cosh x$$

$$24. \quad \int \frac{\sinh 3x}{\cosh^3 x} dx = \frac{1}{2 \cosh^2 x} + 4 \ln \cosh x$$

$$25. \quad \int \frac{\cosh 3x}{\sinh^n x} dx = \frac{4}{(3-n) \sinh^{n-3} x} + \frac{1}{(1-n) \sinh^{n-1} x}$$

For $n = 1$ and $n = 3$:

$$26. \quad \int \frac{\cosh 3x}{\sinh x} dx = 2 \sinh^2 x + \ln \sinh x$$

$$27. \quad \int \frac{\cosh 3x}{\sinh^3 x} dx = -\frac{1}{2 \sinh^2 x} + 4 \ln \sinh x$$

$$28. \quad \int \frac{\cosh 3x}{\cosh x} dx = \sinh 2x - x$$

$$29. \quad \int \frac{\cosh 3x}{\cosh^2 x} dx = 4 \sinh x - 3 \arcsin(\tanh x)$$

$$30. \quad \int \frac{\cosh 3x}{\cosh^3 x} dx = 4x - 3 \tanh x$$

2.44–2.45 Rational functions of hyperbolic functions

2.441

$$1. \quad \int \frac{A + B \sinh x}{(a + b \sinh x)^n} dx = \frac{aB - bA}{(n-1)(a^2 + b^2)} \cdot \frac{\cosh x}{(a + b \sinh x)^{n-1}} \\ + \frac{1}{(n-1)(a^2 + b^2)} \int \frac{(n-1)(aA + bB) + (n-2)(aB - bA) \sinh x}{(a + b \sinh x)^{n-1}} dx$$

For $n = 1$:

$$2. \quad \int \frac{A + B \sinh x}{a + b \sinh x} dx = \frac{B}{b} x - \frac{aB - bA}{b} \int \frac{dx}{a + b \sinh x} \quad (\text{see 2.441 3})$$

$$3. \quad \int \frac{dx}{a + b \sinh x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a \tanh \frac{x}{2} - b + \sqrt{a^2 + b^2}}{a \tanh \frac{x}{2} - b - \sqrt{a^2 + b^2}} \\ = \frac{2}{\sqrt{a^2 + b^2}} \operatorname{arctanh} \frac{a \tanh \frac{x}{2} - b}{\sqrt{a^2 + b^2}}$$

2.442

$$1. \quad \int \frac{A + B \cosh x}{(a + b \sinh x)^n} dx = -\frac{B}{(n-1)b(a + b \sinh x)^{n-1}} + A \int \frac{dx}{(a + b \sinh x)^n}$$

For $n = 1$:

$$2. \quad \int \frac{A + B \cosh x}{a + b \sinh x} dx = \frac{B}{b} \ln(a + b \sinh x) + A \int \frac{dx}{a + b \sinh x}$$

(see 2.441 3)

2.443

$$1. \quad \int \frac{A + B \cosh x}{(a + b \cosh x)^n} dx = \frac{aB - bA}{(n-1)(a^2 - b^2)} \cdot \frac{\sinh x}{(a + b \cosh x)^{n-1}} \\ + \frac{1}{(n-1)(a^2 - b^2)} \int \frac{(n-1)(aA - bB) + (n-2)(aB - bA) \cosh x}{(a + b \cosh x)^{n-1}} dx$$

For $n = 1$:

$$2. \quad \int \frac{A + B \cosh x}{a + b \cosh x} dx = \frac{B}{b}x - \frac{aB - bA}{b} \int \frac{dx}{a + b \cosh x} \quad (\text{see } \mathbf{2.443} \ 3)$$

$$3. \quad \int \frac{dx}{a + b \cosh x} = \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \cosh x}{a + b \cosh x} \quad [b^2 > a^2, \quad x < 0] \\ = -\frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \cosh x}{a + b \cosh x} \quad [b^2 > a^2, \quad x > 0] \\ = \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2} \tanh \frac{x}{2}}{a + b - \sqrt{a^2 - b^2} \tanh \frac{x}{2}} \quad [a^2 > b^2]$$

2.444

$$1. \quad \int \frac{dx}{\cosh a + \cosh x} = \operatorname{cosech} a \left[\ln \cosh \frac{x+a}{2} - \ln \cosh \frac{x-a}{2} \right] \\ = 2 \operatorname{cosech} a \operatorname{arctanh} \left(\tanh \frac{x}{2} \tanh \frac{a}{2} \right)$$

$$2.^{11} \quad \int \frac{dx}{\cos a + \cosh x} = 2 \operatorname{cosec} a \operatorname{arctan} \left(\tanh \frac{x}{2} \tan \frac{a}{2} \right)$$

2.445

$$1. \quad \int \frac{B \sinh x}{(a + b \cosh x)^n} dx = -\frac{B}{(n-1)b(a + b \cosh x)^{n-1}} \quad [n \neq 1]$$

For $n = 1$:

$$2. \quad \int \frac{B \sinh x}{a + b \cosh x} dx = \frac{B}{b} \ln(a + b \cosh x) \quad (\text{see } \mathbf{2.443} \ 3)$$

In evaluating definite integrals by use of formulas **2.441**–**2.443** and **2.445**, one may not take the integral over points at which the integrand becomes infinite, that is, over the points

$$x = \operatorname{arcsinh} \left(-\frac{a}{b} \right)$$

in formulas **2.441** or **2.442** or over the points

$$x = \operatorname{arccosh} \left(-\frac{a}{b} \right)$$

in formulas **2.443** or **2.445**. Formulas **2.443** are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

2.446

$$\begin{aligned}
1. \quad \int \frac{A + B \cosh x}{(\varepsilon + \cosh x)^n} dx &= \frac{B \sinh x}{(1-n)(\varepsilon + \cosh x)^n} + \left(\varepsilon A + \frac{n}{n-1} B \right) \frac{(n-1)!}{(2n-1)!!} \sinh x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!} \\
&\quad \times \frac{\varepsilon^h}{(\varepsilon + \cosh x)^{n-k}} \quad [\varepsilon = \pm 1, \quad n > 1]
\end{aligned}$$

For $n = 1$:

$$2. \quad \int \frac{A + B \cosh x}{\varepsilon + \cosh x} dx = Bx + (\varepsilon A - B) \frac{\cosh x - \varepsilon}{\sinh x} \quad [\varepsilon = \pm 1]$$

2.447

$$\begin{aligned}
1. \quad \int \frac{\sinh x \, dx}{a \cosh x + b \sinh x} &= \frac{a \ln \cosh \left(x + \operatorname{arctanh} \frac{b}{a} \right) bx}{a^2 - b^2} \quad [a > |b|] \\
&= \frac{bx - a \ln \sinh \left(x + \operatorname{arctanh} \frac{a}{b} \right)}{b^2 - a^2} \quad [b > |a|] \quad \text{MZ 215}
\end{aligned}$$

For $a = b = 1$:

$$2. \quad \int \frac{\sinh x \, dx}{\cosh x + \sinh x} = \frac{x}{2} + \frac{1}{4} e^{-2x}$$

For $a = -b = 1$:

$$3. \quad \int \frac{\sinh x \, dx}{\cosh x - \sinh x} = -\frac{x}{2} + \frac{1}{4} e^{2x} \quad \text{MZ 215}$$

2.448

$$\begin{aligned}
1. \quad \int \frac{\cosh x \, dx}{a \cosh x + b \sinh x} &= \frac{ax - b \ln \cosh \left(x + \operatorname{arctanh} \frac{b}{a} \right)}{a^2 - b^2} \quad [a > |b|] \\
&= \frac{-ax + b \ln \sinh \left(x + \operatorname{arctanh} \frac{a}{b} \right)}{b^2 - a^2} \quad [b > |a|]
\end{aligned}$$

For $a = b = 1$:

$$2. \quad \int \frac{\cosh x \, dx}{\cosh x + \sinh x} = \frac{x}{2} - \frac{1}{4} e^{-2x}$$

For $a = -b = 1$:

$$3. \quad \int \frac{\cosh x \, dx}{\cosh x - \sinh x} = \frac{x}{2} + \frac{1}{4} e^{2x} \quad \text{MZ 214, 215}$$

2.449

$$\begin{aligned}
1.^6 \quad \int \frac{dx}{(a \cosh x + b \sinh x)^n} &= \frac{1}{\sqrt{(a^2 - b^2)^n}} \int \frac{dx}{\sinh^n \left(x + \operatorname{arctanh} \frac{b}{a} \right)} \quad [a > |b|] \\
&= \frac{1}{\sqrt{(b^2 - a^2)^n}} \int \frac{dx}{\cosh^n \left(x + \operatorname{arctanh} \frac{a}{b} \right)} \quad [b > |a|]
\end{aligned}$$

For $n = 1$:

$$2. \quad \int \frac{dx}{a \cosh x + b \sinh x} = \frac{1}{\sqrt{a^2 - b^2}} \arctan \left| \sinh \left(x + \operatorname{arctanh} \frac{b}{a} \right) \right| \quad [a > |b|]$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \left| \tanh \frac{x + \operatorname{arctanh} \frac{a}{b}}{2} \right| \quad [b > |a|]$$

For $a = b = 1$:

$$3. \quad \int \frac{ax}{\cosh x + \sinh x} = -e^{-x} = \sinh x - \cosh x$$

For $a = -b = 1$:

$$4. \quad \int \frac{dx}{\cosh x - \sinh x} = e^x = \sinh x + \cosh x$$

MZ 214

2.451

$$1. \quad \int \frac{A + B \cosh x + C \sinh x}{(a + b \cosh x + c \sinh x)^n} dx$$

$$= \frac{Bc - Cb + (Ac - Ca) \cosh x + (Ab - Ba) \sinh x}{(1 - n)(a^2 - b^2 + c^2)(a + b \cosh x + c \sinh x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 + c^2)}$$

$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba) \cosh x - (n-2)(Ac - Ca) \sinh x}{(a + b \cosh x + c \sinh x)^{n-1}} dx$$

$$= \frac{Bc - Cb - Ca \cosh x - Ba \sinh x}{(n-1)a(a + b \cosh x + c \sinh x)^n} + \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2} \right] (c \cosh x + b \sinh x) \frac{(n-1)!}{(2n-1)!!}$$

$$\times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k} \frac{1}{(a + b \cosh x + c \sinh x)^{n-k}}$$

$[a^2 + c^2 \neq b^2]$

$[a^2 + c^2 = b^2]$

$$2. \quad \int \frac{A + B \cosh x + C \sinh x}{a + b \cosh x + c \sinh x} dx = \frac{Cb - Bc}{b^2 - c^2} \ln(a + b \cosh x + c \sinh x)$$

$$+ \frac{Bb - Cc}{b^2 - c^2} x + \left(A - a \frac{Bb - Cc}{b^2 - c^2} \right) \int \frac{dx}{a + b \cosh x + c \sinh x}$$

$[b^2 \neq c^2]$ (see **2.451** 4)

$$3. \quad \int \frac{A + B \cosh x + C \sinh x}{a + b \cosh x \pm b \sinh x} dx = \frac{C \mp B}{2a} (\cosh x \mp \sinh x) + \left[\frac{A}{a} - \frac{(B \mp C)b}{2a^2} \right] x$$

$$+ \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)b}{2a^2} \right] \ln(a + b \cosh x \pm b \sinh x)$$

$[ab \neq 0]$

$$\begin{aligned}
4. \quad & \int \frac{dx}{a + b \cosh x + c \sinh x} \\
&= \frac{2}{\sqrt{b^2 - a^2 - c^2}} \arctan \frac{(b-a) \tanh \frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}} \quad [b^2 > a^2 + c^2 \text{ and } a \neq b] \\
&= \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{(a-b) \tanh \frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b) \tanh \frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}} \quad [b^2 < a^2 + c^2 \text{ and } a \neq b] \\
&= \frac{1}{c} \ln \left(a + c \tanh \frac{x}{2} \right) \quad [a = b \text{ and } c \neq 0] \\
&= \frac{2}{(a-b) \tanh \frac{x}{2} + c} \quad [b^2 = a^2 + c^2]
\end{aligned}$$

GU (351)(18)

2.452

$$\begin{aligned}
1. \quad & \int \frac{A + B \cosh x + C \sinh x}{(a_1 + b_1 \cosh x + c_1 \sinh x)(a_2 + b_2 \cosh x + c_2 \sinh x)} dx \\
&= A_0 \ln \frac{a_1 + b_1 \cosh x + c_1 \sinh x}{a_2 + b_2 \cosh x + c_2 \sinh x} + A_1 \int \frac{dx}{a_1 + b_1 \cosh x + c_1 \sinh x} + A_2 \int \frac{dx}{a_2 + b_2 \cosh x + c_2 \sinh x}
\end{aligned}$$

where

GU (351)(19)

$$\begin{aligned}
A_0 &= \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, & A_1 &= \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ b_1 & c_1 & A \\ B & C & A \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & A \\ C & A & B \\ C & A & B \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ A & B & C \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \\
A_2 &= \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ C & B & A \\ c_2 & b_2 & a_2 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & A \\ C & A & B \\ c_2 & a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ b_2 & a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, & & \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \right].
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int \frac{A \cosh^2 x + 2B \sinh x \cosh x + C \sinh^2 x}{a \cosh^2 x + 2b \sinh x \cosh x + c \sinh^2 x} dx \\
&= \frac{1}{4b^2 - (a+c)^2} \left\{ [4Bb - (A+C)(a+c)]x \right. \\
&\quad + [(A+C)b - B(a+c)] \ln (a \cosh^2 x + 2b \sinh x \cosh x + c \sinh^2 x) \\
&\quad \left. + [2(A-C)b^2 - 2Bb(a-c) + (Ca - Ac)(a+c)] f(x) \right\}
\end{aligned}$$

where

GU (351)(24)

$$\begin{aligned}
 f(x) &= \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \tanh x + b - \sqrt{b^2 - ac}}{c \tanh x + b + \sqrt{b^2 - ac}} & [b^2 > ac] \\
 &= \frac{1}{\sqrt{ac - b^2}} \arctan \frac{c \tanh x + b}{\sqrt{ac - b^2}} & [b^2 < ac] \\
 &= -\frac{1}{c \tanh x + b} & [b^2 = ac]
 \end{aligned}$$

2.453

1.
$$\int \frac{(A + B \sinh x) dx}{\sinh x (a + b \sinh x)} = \frac{1}{a} \left[A \ln \left| \tanh \frac{x}{2} \right| + (aB - bA) \int \frac{dx}{a + b \sinh x} \right]$$

(see **2.441** 3)
2.
$$\int \frac{(A + B \sinh x) dx}{\sinh x (a + b \cosh x)} = \frac{A}{a^2 - b^2} \left(a \ln \left| \tanh \frac{x}{2} \right| + b \ln \left| \frac{a + b \cosh x}{\sinh x} \right| \right) + B \int \frac{dx}{a + b \cosh x}$$

(see **2.443** 3)

For $a^2 = b^2 = 1$:

3.
$$\int \frac{(A + B \sinh x) dx}{\sinh x (1 + \cosh x)} = \frac{A}{2} \left(\ln \left| \tanh \frac{x}{2} \right| - \frac{1}{2} \tanh^2 \frac{x}{2} \right) + B \tanh \frac{x}{2}$$
4.
$$\int \frac{(A + B \sinh x) dx}{\sinh x (1 - \cosh x)} = \frac{A}{2} \left(-\ln \left| \coth \frac{x}{2} \right| + \frac{1}{2} \coth^2 \frac{x}{2} \right) + B \coth \frac{x}{2}$$

2.454

1.
$$\int \frac{(A + B \sinh x) dx}{\cosh x (a + b \sinh x)} = \frac{1}{a^2 + b^2} \left[(Aa + Bb) \arctan (\sinh x) + (Ab - Ba) \ln \left| \frac{a + b \sinh x}{\cosh x} \right| \right]$$
2.
$$\int \frac{(A + B \cosh x) dx}{\sinh x (a + b \sinh x)} = \frac{1}{a} \left(A \ln \left| \tanh \frac{x}{2} \right| + B \ln \left| \frac{\sinh x}{a + b \sinh x} \right| - Ab \int \frac{dx}{a + b \sinh x} \right)$$

(see **2.441** 3)

2.455

1.
$$\int \frac{(A + B \cosh x) dx}{\sinh x (a + b \cosh x)} = \frac{1}{a^2 - b^2} \left[(Aa + Bb) \ln \left| \tanh \frac{x}{2} \right| + (Ab - Ba) \ln \left| \frac{a + b \cosh x}{\sinh x} \right| \right]$$

For $a^2 = b^2 = 1$:

2.
$$\int \frac{(A + B \cosh x) dx}{\sinh x (1 + \cosh x)} = \frac{A + B}{2} \ln \left| \tanh \frac{x}{2} \right| - \frac{A - B}{4} \tanh^2 \frac{x}{2}$$
3.
$$\int \frac{(A + B \cosh x) dx}{\sinh x (1 - \cosh x)} = \frac{A + B}{4} \coth^2 \frac{x}{2} - \frac{A - B}{2} \ln \coth \frac{x}{2}$$

- 2.456**
$$\int \frac{(A + B \cosh x) dx}{\cosh x (a + b \sinh x)} = \frac{A}{a^2 + b^2} \left[a \arctan (\sinh x) + b \ln \left| \frac{a + b \sinh x}{\cosh x} \right| \right] + B \int \frac{dx}{a + b \sinh x}$$

(see **2.441** 3)

2.457

$$1. \quad \int \frac{(A + B \cosh x) dx}{\cosh x (a + b \cosh x)} = \frac{1}{a} \left[A \arctan \sinh x - (Ab - Ba) \int \frac{dx}{a + b \cosh x} \right]$$

(see **2.443 3**)**2.458**

$$1. \quad \int \frac{dx}{a + b \sinh^2 x}$$

$$= \frac{1}{\sqrt{a(b-a)}} \arctan \left(\sqrt{\frac{b}{a}} - 1 \tanh x \right) \quad \left[\frac{b}{a} > 1 \right]$$

$$= \frac{1}{\sqrt{a(a-b)}} \operatorname{arctanh} \left(\sqrt{1 - \frac{b}{a}} \tanh x \right) \quad \left[0 < \frac{b}{a} < 1 \text{ or } \frac{b}{a} < 0 \text{ and } \sinh^2 x < -\frac{a}{b} \right]$$

$$= \frac{1}{\sqrt{a(a-b)}} \operatorname{arccoth} \left(\sqrt{1 - \frac{b}{a}} \tanh x \right) \quad \left[\frac{b}{a} < 0 \text{ and } \sinh^2 x > -\frac{a}{b} \right]$$

MZ 195

$$2. \quad \int \frac{dx}{a + b \cosh^2 x}$$

$$= \frac{1}{\sqrt{-a(a+b)}} \arctan \left(\sqrt{-\left(1 + \frac{b}{a}\right)} \coth x \right) \quad \left[\frac{b}{a} < -1 \right]$$

$$= \frac{1}{\sqrt{a(a+b)}} \operatorname{arctanh} \left(\sqrt{1 + \frac{b}{a}} \coth x \right) \quad \left[-1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x > -\frac{a}{b} \right]$$

$$= \frac{1}{\sqrt{a(a+b)}} \operatorname{arccoth} \left(\sqrt{1 + \frac{b}{a}} \coth x \right) \quad \left[\frac{b}{a} > 0 \text{ or } -1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x < -\frac{a}{b} \right]$$

MZ 202

For $a^2 = b^2 = 1$:

$$3. \quad \int \frac{dx}{1 + \sinh^2 x} = \tanh x$$

$$4. \quad \int \frac{dx}{1 - \sinh^2 x} = \frac{1}{\sqrt{2}} \operatorname{arctanh} (\sqrt{2} \tanh x) \quad [\sinh^2 x < 1]$$

$$= \frac{1}{\sqrt{2}} \operatorname{arccoth} (\sqrt{2} \tanh x) \quad [\sinh^2 x > 1]$$

$$5. \quad \int \frac{dx}{1 + \cosh^2 x} = \frac{1}{\sqrt{2}} \operatorname{arccoth} (\sqrt{2} \coth x)$$

$$6. \quad \int \frac{dx}{1 - \cosh^2 x} = \coth x$$

2.459

$$1. \quad \int \frac{dx}{(a + b \sinh^2 x)^2} = \frac{1}{2a(b-a)} \left[\frac{b \sinh x \cosh x}{a + b \sinh^2 x} + (b - 2a) \int \frac{dx}{a + b \sinh^2 x} \right]$$

(see **2.458 1**)

MZ 196

$$2. \quad \int \frac{dx}{(a + b \cosh^2 x)^2} = \frac{1}{2a(a+b)} \left[-\frac{b \sinh x \cosh x}{a + b \cosh^2 x} + (2a+b) \int \frac{dx}{a + b \cosh^2 x} \right]$$

(see **2.458 2**) MZ 203

$$3. \quad \int \frac{dx}{(a + b \sinh^2 x)^3} = \frac{1}{8pa^3} \left[\left(3 - \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \tanh x) + \left(3 - \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \tanh x}{1 + p^2 \tanh^2 x} \right. \\ \left. + \left(1 + \frac{2}{p^2} - \frac{1}{p^2} \tanh^2 x \right) \frac{2p \tanh x}{(1 + p^2 \tanh^2 x)^2} \right]$$

$\left[p^2 = \frac{b}{a} - 1 > 0 \right]$

$$= \frac{1}{8qa^3} \left[\left(3 + \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{arctanh}(q \tanh x) + \left(3 + \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \tanh x}{1 - q^2 \tanh^2 x} \right. \\ \left. + \left(1 - \frac{2}{q^2} + \frac{1}{q^2} \tanh^2 x \right) \frac{2q \tanh x}{(1 - q^2 \tanh^2 x)^2} \right]$$

$\left[q^2 = 1 - \frac{b}{a} > 0 \right]$ MZ 196

$$4. \quad \int \frac{dx}{(a + b \cosh^2 x)^3} = \frac{1}{8pa^3} \left[\left(3 - \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \coth x) + \left(3 - \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \coth x}{1 + p^2 \coth^2 x} \right. \\ \left. + \left(1 + \frac{2}{p^2} - \frac{1}{p^2} \coth^2 x \right) \frac{2p \coth x}{(1 + p^2 \coth^2 x)^2} \right]$$

$\left[p^2 = -1 - \frac{b}{a} > 0 \right]$

$$= \frac{1}{8qa^3} \left[\left(3 + \frac{2}{q^2} + \frac{3}{q^4} \right) \varphi(x)^* + \left(3 + \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \coth x}{1 - q^2 \coth^2 x} \right. \\ \left. + \left(1 - \frac{2}{q^2} + \frac{1}{q^2} \coth^2 x \right) \frac{2q \coth x}{(1 - q^2 \coth^2 x)^2} \right]$$

$\left[q^2 = 1 + \frac{b}{a} > 0 \right]$

2.46 Algebraic functions of hyperbolic functions

2.461

$$1. \quad \int \sqrt{\tanh x} dx = \operatorname{arctanh} \sqrt{\tanh x} - \arctan \sqrt{\tanh x}$$

MZ 221

*In 2.459.4, if $\frac{b}{a} < 0$ and $\cosh^2 x > -\frac{a}{b}$, then $\varphi(x) = \operatorname{arctanh}(q \coth x)$. If $\frac{b}{a} < 0$, but $\cosh^2 x < -\frac{a}{b}$, or if $\frac{b}{a} > 0$, then $\varphi(x) = \operatorname{arccoth}(q \coth x)$.

$$2. \quad \int \sqrt{\coth x} \, dx = \operatorname{arccoth} \sqrt{\coth x} - \arctan \sqrt{\coth x} \quad \text{MZ 222}$$

2.462

$$\begin{aligned} 1. \quad \int \frac{\sinh x \, dx}{\sqrt{a^2 + \sinh^2 x}} &= \operatorname{arcsinh} \frac{\cosh x}{\sqrt{a^2 - 1}} = \ln \left(\cosh x + \sqrt{a^2 + \sinh^2 x} \right) \quad [a^2 > 1] \\ &= \operatorname{arccosh} \frac{\cosh x}{\sqrt{1 - a^2}} = \ln \left(\cosh x + \sqrt{a^2 + \sinh^2 x} \right) \quad [a^2 < 1] \\ &= \ln \cosh x \quad [a^2 = 1] \end{aligned}$$

$$2. \quad \int \frac{\sinh x \, dx}{\sqrt{a^2 - \sinh^2 x}} = \operatorname{arcsin} \frac{\cosh x}{\sqrt{a^2 + 1}} \quad [\sinh^2 x < a^2]$$

$$\begin{aligned} 3. \quad \int \frac{\sinh x \, dx}{\sqrt{\sinh^2 x - a^2}} &= \operatorname{arccosh} \frac{\cosh x}{\sqrt{a^2 + 1}} = \ln \left(\cosh x + \sqrt{\sinh^2 x - a^2} \right) \\ & \quad [\sinh^2 x > a^2] \quad \text{MZ 199} \end{aligned}$$

$$4. \quad \int \frac{\cosh x \, dx}{\sqrt{a^2 + \sinh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{a} = \ln \left(\sinh x + \sqrt{a^2 + \sinh^2 x} \right)$$

$$5. \quad \int \frac{\cosh x \, dx}{\sqrt{a^2 - \sinh^2 x}} = \operatorname{arcsin} \frac{\sinh x}{a} \quad [\sinh^2 x < a^2]$$

$$\begin{aligned} 6. \quad \int \frac{\cosh x \, dx}{\sqrt{\sinh^2 x - a^2}} &= \operatorname{arccosh} \frac{\sinh x}{a} = \ln \left(\sinh x + \sqrt{\sinh^2 x - a^2} \right) \\ & \quad [\sinh^2 x > a^2] \end{aligned}$$

$$7. \quad \int \frac{\sinh x \, dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\cosh x}{a} = \ln \left(\cosh x + \sqrt{a^2 + \cosh^2 x} \right)$$

$$8. \quad \int \frac{\sinh x \, dx}{\sqrt{a^2 - \cosh^2 x}} = \operatorname{arcsin} \frac{\cosh x}{a} \quad [\cosh^2 x < a^2]$$

$$\begin{aligned} 9. \quad \int \frac{\sinh x \, dx}{\sqrt{\cosh^2 x - a^2}} &= \operatorname{arccosh} \frac{\cosh x}{a} = \ln \left(\cosh x + \sqrt{\cosh^2 x - a^2} \right) \\ & \quad [\cosh^2 x > a^2] \quad \text{MZ 215, 216} \end{aligned}$$

$$10. \quad \int \frac{\cosh x \, dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{\sqrt{a^2 + 1}} = \ln \left(\sinh x + \sqrt{a^2 + \cosh^2 x} \right)$$

$$11. \quad \int \frac{\cosh x \, dx}{\sqrt{a^2 - \cosh^2 x}} = \operatorname{arcsin} \frac{\sinh x}{\sqrt{a^2 - 1}} \quad [\cosh^2 x < a^2]$$

$$\begin{aligned} 12. \quad \int \frac{\cosh x \, dx}{\sqrt{\cosh^2 x - a^2}} &= \operatorname{arccosh} \frac{\sinh x}{\sqrt{a^2 - 1}} \quad [a^2 > 1] \\ &= \ln \sinh x \quad [a^2 = 1] \end{aligned}$$

$$\begin{aligned}
 13. \quad \int \frac{\coth x \, dx}{\sqrt{a + b \sinh x}} &= 2\sqrt{a} \operatorname{arccoth} \sqrt{1 + \frac{b}{a} \sinh x} & [b \sinh x > 0, \quad a > 0] \\
 &= 2\sqrt{a} \operatorname{arctanh} \sqrt{1 + \frac{b}{a} \sinh x} & [b \sinh x < 0, \quad a > 0] \\
 &= 2\sqrt{-a} \operatorname{arctanh} \sqrt{-\left(1 + \frac{b}{a} \sinh x\right)} & a < 0
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \int \frac{\tanh x \, dx}{\sqrt{a + b \cosh x}} &= 2\sqrt{a} \operatorname{arccoth} \sqrt{1 + \frac{b}{a} \cosh x} & [b \cosh x > 0, \quad a > 0] \\
 &= 2\sqrt{a} \operatorname{arctanh} \sqrt{1 + \frac{b}{a} \cosh x} & [b \cosh x < 0, \quad a > 0] \\
 &= 2\sqrt{-a} \operatorname{arctanh} \sqrt{-\left(1 + \frac{b}{a} \cosh x\right)} & [a < 0]
 \end{aligned}$$

MZ 220, 221

2.463

$$\begin{aligned}
 1. \quad \int \frac{\sinh x \sqrt{a + b \cosh x}}{p + q \cosh x} \, dx \\
 &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arccoth} \sqrt{\frac{q(a + b \cosh x)}{aq - bp}} & \left[b \cosh x > 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \cosh x)}{aq - bp}} & \left[b \cosh x < 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{bp - aq}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \cosh x)}{bp - aq}} & \left[\frac{aq - bp}{q} < 0 \right]
 \end{aligned}$$

MZ 220

$$\begin{aligned}
 2. \quad \int \frac{\cosh x \sqrt{a + b \sinh x}}{p + q \sinh x} \, dx \\
 &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arccoth} \sqrt{\frac{q(a + b \sinh x)}{aq - bp}} & \left[b \sinh x > 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \sinh x)}{aq - bp}} & \left[b \sinh x < 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{bp - aq}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \sinh x)}{bp - aq}} & \left[\frac{aq - bp}{q} < 0 \right]
 \end{aligned}$$

MZ 221

2.464

$$1. \quad \int \frac{dx}{\sqrt{k^2 + k'^2 \cosh^2 x}} = \int \frac{dx}{\sqrt{1 + k'^2 \sinh^2 x}} = F(\arcsin(\tanh x), k) \quad [x > 0] \quad \text{BY (295.00)(295.10)}$$

$$2. \quad \int \frac{dx}{\sqrt{\cosh^2 x - k^2}} = \int \frac{dx}{\sqrt{\sinh^2 x + k'^2}} = F\left(\arcsin\left(\frac{1}{\cosh x}\right), k\right) \quad [x > 0] \quad \text{BY (295.40)(295.30)}$$

$$3. \quad \int \frac{dx}{\sqrt{1 - k'^2 \cosh^2 x}} = F\left(\arcsin\left(\frac{\tanh x}{k}\right), k\right) \quad \left[0 < x < \operatorname{arccosh} \frac{1}{k'}\right] \quad \text{BY (295.20)}$$

Notation: In **2.464** 4–**2.464** 8, we set $\alpha = \operatorname{arccos} \frac{1 - \sinh 2ax}{1 + \sinh 2ax}$, $r = \frac{1}{\sqrt{2}}$ $[ax > 0]$

$$4. \quad \int \frac{dx}{\sqrt{\sinh 2ax}} = \frac{1}{2a} F(\alpha, r) \quad \text{BY (296.50)}$$

$$5. \quad \int \sqrt{\sinh 2ax} dx = \frac{1}{2a} [F(\alpha, r) - 2E(\alpha, r)] + \frac{1}{a} \frac{\sqrt{\sinh 2ax (1 + \sinh^2 2ax)}}{1 + \sinh 2ax} \quad \text{BY (296.53)}$$

$$6. \quad \int \frac{\cosh^2 2ax dx}{(1 + \sinh 2ax)^2 \sqrt{\sinh 2ax}} = \frac{1}{2a} E(\alpha, r) \quad \text{BY (296.51)}$$

$$7. \quad \int \frac{(1 - \sinh 2ax)^2 dx}{(1 + \sinh 2ax)^2 \sqrt{\sinh 2ax}} = \frac{1}{2a} [2E(\alpha, r) - F(\alpha, r)] \quad \text{BY (296.55)}$$

$$8. \quad \int \frac{\sqrt{\sinh 2ax} dx}{(1 + \sinh 2ax)^2} = \frac{1}{4a} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (296.54)}$$

Notation: In **2.464** 9–**2.464** 15, we set $\alpha = \arcsin \sqrt{\frac{\cosh 2ax - 1}{\cosh 2ax}}$, $r = \frac{1}{\sqrt{2}}$ $[x \neq 0]$:

$$9. \quad \int \frac{dx}{\sqrt{\cosh 2ax}} = \frac{1}{a\sqrt{2}} F(\alpha, r) \quad \text{BY (296.00)}$$

$$10. \quad \int \sqrt{\cosh 2ax} dx = \frac{1}{a\sqrt{2}} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sinh 2ax}{a\sqrt{\cosh 2ax}} \quad \text{BY (296.03)}$$

$$11. \quad \int \frac{dx}{\sqrt{\cosh^3 2ax}} = \frac{1}{a\sqrt{2}} [2E(\alpha, r) - F(\alpha, r)] \quad \text{BY (296.04)}$$

$$12. \quad \int \frac{dx}{\sqrt{\cosh^5 2ax}} = \frac{1}{3\sqrt{2}a} F(\alpha, r) + \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}} \quad \text{BY (296.04)}$$

$$13. \quad \int \frac{\sinh^2 2ax dx}{\sqrt{\cosh 2ax}} = -\frac{\sqrt{2}}{3a} F(\alpha, r) + \frac{1}{3a} \sinh 2ax \sqrt{\cosh 2ax} \quad \text{BY (296.07)}$$

$$14. \quad \int \frac{\tanh^2 2ax dx}{\sqrt{\cosh 2ax}} = \frac{\sqrt{2}}{3a} F(\alpha, r) - \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}} \quad \text{BY (296.05)}$$

$$15. \quad \int \frac{\sqrt{\cosh 2ax} dx}{p^2 + (1 - p^2) \cosh 2ax} = \frac{1}{a\sqrt{2}} \Pi(\alpha, p^2, r) \quad \text{BY (296.02)}$$

Notation: In **2.464** 16–**2.464** 20, we set:

$$\alpha = \operatorname{arccos} \frac{\sqrt{a^2 + b^2} - a - b \sinh x}{\sqrt{a^2 + b^2} + a + b \sinh x},$$

$$r = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2\sqrt{a^2 + b^2}}} \quad \left[a > 0, \quad b > 0, \quad x > -\operatorname{arcsinh} \frac{a}{b}\right]$$

$$16. \quad \int \frac{dx}{\sqrt{a + b \sinh x}} = \frac{1}{\sqrt[4]{a^2 + b^2}} F(\alpha, r) \quad \text{BY (298.00)}$$

$$17. \quad \int \sqrt{a + b \sinh x} \, dx = \sqrt[4]{a^2 + b^2} [F(\alpha, r) - 2 E(\alpha, r)] + \frac{2b \cosh x \sqrt{a + b \sinh x}}{\sqrt{a^2 + b^2} + a + b \sinh x} \quad \text{BY (298.02)}$$

$$18. \quad \int \frac{\sqrt{a + b \sinh x}}{\cosh^2 x} \, dx = \sqrt[4]{a^2 + b^2} E(\alpha, r) - \frac{\sqrt{a^2 + b^2} - a}{2\sqrt[4]{a^2 + b^2}} F(\alpha, r) \\ - \frac{a + \sqrt{a^2 + b^2}}{b} \cdot \frac{\sqrt{a^2 + b^2} - a - b \sinh x}{\sqrt{a^2 + b^2} + a + b \sinh x} \cdot \frac{\sqrt{a + b \sinh x}}{\cosh x} \quad \text{BY (298.03)}$$

$$19. \quad \int \frac{\cosh^2 x \, dx}{[\sqrt{a^2 + b^2} + a + b \sinh x]^2 \sqrt{a + b \sinh x}} = \frac{1}{b^2 \sqrt[4]{a^2 + b^2}} E(\alpha, r) \quad \text{BY (298.01)}$$

$$20. \quad \int \frac{\sqrt{a + b \sinh x} \, dx}{[\sqrt{a^2 + b^2} - a - b \sinh x]^2} = -\frac{1}{\sqrt[4]{a^2 + b^2} (\sqrt{a^2 + b^2} - a)} E(\alpha, r) \\ + \frac{b}{\sqrt{a^2 + b^2} - a} \cdot \frac{\cosh x \sqrt{a + b \sinh x}}{a^2 + b^2 - (a + b \sinh x)^2} \quad \text{BY (298.04)}$$

Notation: In **2.464** 21–**2.464** 31, we set $\alpha = \arcsin \left(\tanh \frac{x}{2} \right)$, $r = \sqrt{\frac{a-b}{a+b}}$ [$0 < b < a$, $x > 0$]:

$$21. \quad \int \frac{dx}{\sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a+b}} F(\alpha, r) \quad \text{BY (297.25)}$$

$$22. \quad \int \sqrt{a + b \cosh x} \, dx = 2\sqrt{a+b} [F(\alpha, r) - E(\alpha, r)] + 2 \tanh \frac{x}{2} \sqrt{a + b \cosh x} \quad \text{BY (297.29)}$$

$$23. \quad \int \frac{\cosh x \, dx}{\sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a+b}} F(\alpha, r) - \frac{2\sqrt{a+b}}{b} E(\alpha, r) + \frac{2}{b} \tanh \frac{x}{2} \sqrt{a + b \cosh x} \quad \text{BY (297.33)}$$

$$24. \quad \int \frac{\tanh^2 \frac{x}{2}}{\sqrt{a + b \cosh x}} \, dx = \frac{2\sqrt{a+b}}{a-b} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (297.28)}$$

$$25.^{11} \quad \int \frac{\tanh^4 \frac{x}{2}}{\sqrt{a + b \cosh x}} \, dx = \frac{2\sqrt{a+b}}{3(a-b)^2} [(3a+b) F(\alpha, r) - 4a E(\alpha, r)] + \frac{2}{3(a-b)} \frac{\sinh \frac{x}{2} \sqrt{a + b \cosh x}}{\cosh^3 \frac{x}{2}} \quad \text{BY (297.28)}$$

$$26. \quad \int \frac{\cosh x - 1}{\sqrt{a + b \cosh x}} \, dx = \frac{2}{b} \left[\left(\tanh \frac{x}{2} \right) \sqrt{a + b \cosh x} - \sqrt{a+b} E(\alpha, r) \right] \quad \text{BY (297.31)}$$

$$27. \quad \int \frac{(\cosh x - 1)^2}{\sqrt{a + b \cosh x}} \, dx = \frac{4\sqrt{a+b}}{3b^2} [(a+3b) E(\alpha, r) - b F(\alpha, r)] \\ + \frac{4}{3b^2} \left[b \cosh^2 \frac{x}{2} - (a+3b) \right] \tanh \frac{x}{2} \sqrt{a + b \cosh x} \quad \text{BY (297.31)}$$

$$28. \quad \int \frac{\sqrt{a + b \cosh x}}{\cosh x + 1} \, dx = \sqrt{a+b} E(\alpha, r) \quad \text{BY (297.26)}$$

$$29. \quad \int \frac{dx}{(\cosh x + 1) \sqrt{a + b \cosh x}} = \frac{\sqrt{a+b}}{a-b} E(\alpha, r) - \frac{2b}{(a-b)\sqrt{a+b}} F(\alpha, r) \quad \text{BY (297.30)}$$

$$30. \quad \int \frac{dx}{(\cosh x + 1)^2 \sqrt{a + b \cosh x}} = \frac{1}{3(a-b)^2 \sqrt{a+b}} \left[b(5b-a) F(\alpha, r) + (a-3b)(a+b) E(\alpha, r) \right] + \frac{1}{6(a-b)} \cdot \frac{\sinh \frac{x}{2}}{\cosh^3 \frac{x}{2}} \sqrt{a + b \cosh x} \quad 297.30$$

$$31. \quad \int \frac{(1 + \cosh x) dx}{[1 + p^2 + (1 - p^2) \cosh x] \sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a+b}} \Pi(\alpha, p^2, r) \quad \text{BY (297.27)}$$

Notation: In **2.464** 32–**2.464** 40, we set:

$$\alpha = \arcsin \sqrt{\frac{a - b \cosh x}{a - b}}$$

$$r = \sqrt{\frac{a-b}{a+b}} \quad \left[0 < b < a, \quad 0 < x < \operatorname{arccosh} \frac{a}{b} \right]$$

$$32. \quad \int \frac{dx}{\sqrt{a-b \cosh x}} = \frac{2}{\sqrt{a+b}} F(\alpha, r) \quad \text{BY (297.50)}$$

$$33. \quad \int \sqrt{a-b \cosh x} dx = 2\sqrt{a+b} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (297.54)}$$

$$34. \quad \int \frac{\cosh x dx}{\sqrt{a-b \cosh x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r) - \frac{2}{\sqrt{a+b}} F(\alpha, r) \quad \text{BY (297.56)}$$

$$35. \quad \int \frac{\cosh^2 x dx}{\sqrt{a-b \cosh x}} = \frac{2(b-2a)}{3b\sqrt{a+b}} F(\alpha, r) + \frac{4a\sqrt{a+b}}{3b^2} E(\alpha, r) + \frac{2}{3b} \sinh x \sqrt{a-b \cosh x} \quad \text{BY (297.56)}$$

$$36. \quad \int \frac{(1 + \cosh x) dx}{\sqrt{a-b \cosh x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r) \quad \text{BY (297.51)}$$

$$37. \quad \int \frac{dx}{\cosh x \sqrt{a-b \cosh x}} = \frac{2b}{a\sqrt{a+b}} \Pi\left(\alpha, \frac{a-b}{a}, r\right) \quad \text{BY (297.57)}$$

$$38. \quad \int \frac{dx}{(1 + \cosh x) \sqrt{a-b \cosh x}} = \frac{1}{\sqrt{a+b}} E(\alpha, r) - \frac{1}{a+b} \tanh \frac{x}{2} \sqrt{a-b \cosh x} \quad \text{BY (297.58)}$$

$$39. \quad \int \frac{dx}{(1 + \cosh x)^2 \sqrt{a-b \cosh x}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+3b) E(\alpha, r) - b F(\alpha, r)] - \frac{1}{3(a+b)^2} \frac{\tanh \frac{x}{2} \sqrt{a-b \cosh x}}{\cosh x + 1} [2a + 4b + (a+3b) \cosh x] \quad \text{BY (297.58)}$$

$$40. \quad \int \frac{dx}{(a-b- ap^2 + bp^2 \cosh x) \sqrt{a-b \cosh x}} = \frac{2}{(a-b)\sqrt{a+b}} \Pi(\alpha, p^2, r) \quad \text{BY (297.52)}$$

Notation: In **2.464** 41–**2.464** 47, we set:

$$\alpha = \arcsin \sqrt{\frac{b(\cosh x - 1)}{b \cosh x - a}},$$

$$r = \sqrt{\frac{a+b}{2b}} \quad [0 < a < b, x > 0]$$

$$41. \quad \int \frac{dx}{\sqrt{b \cosh x - a}} = \sqrt{\frac{2}{b}} F(\alpha, r) \quad \text{BY (297.00)}$$

$$42. \quad \int \sqrt{b \cosh x - a} dx = (b - a) \sqrt{\frac{2}{b}} F(\alpha, r) - 2\sqrt{2b} E(\alpha, r) + \frac{2b \sinh x}{\sqrt{b \cosh x - a}} \quad \text{BY (297.05)}$$

$$43. \quad \int \frac{dx}{\sqrt{(b \cosh x - a)^3}} = \frac{1}{b^2 - a^2} \cdot \sqrt{\frac{2}{b}} [2b E(\alpha, r) - (b - a) F(\alpha, r)] \quad \text{BY (297.06)}$$

$$44. \quad \int \frac{dx}{\sqrt{(b \cosh x - a)^5}} = \frac{1}{3(b^2 - a^2)^2} \sqrt{\frac{2}{b}} [(b - 3a)(b - a) F(\alpha, r) + 8ab E(\alpha, r)] \\ + \frac{2b}{3(b^2 - a^2)} \cdot \frac{\sinh x}{\sqrt{(b \cosh x - a)^3}} \quad \text{BY (297.06)}$$

$$45. \quad \int \frac{\cosh x dx}{\sqrt{b \cosh x - a}} = \sqrt{\frac{2}{b}} [F(\alpha, r) - 2 E(\alpha, r)] + \frac{2 \sinh x}{\sqrt{b \cosh x - a}} \quad \text{BY (297.03)}$$

$$46. \quad \int \frac{(\cosh x + 1) dx}{\sqrt{(b \cosh x - a)^3}} = \frac{2}{b - a} \sqrt{\frac{2}{b}} E(\alpha, r) \quad \text{BY (297.01)}$$

$$47. \quad \int \frac{\sqrt{b \cosh x - a} dx}{p^2 b - a + b(1 - p^2) \cosh x} = \sqrt{\frac{2}{b}} \Pi(\alpha, p^2, r) \quad \text{BY (297.02)}$$

Notation: In **2.464** 48–**2.464** 55, we set $\alpha = \arcsin \sqrt{\frac{b \cosh x - a}{b(\cosh x - 1)}}$ and $r = \sqrt{\frac{2b}{a + b}}$ for

$$\left[0 < b < a, x > \operatorname{arccosh} \frac{a}{b} \right]:$$

$$48. \quad \int \frac{dx}{\sqrt{b \cosh x - a}} = \frac{2}{\sqrt{a + b}} F(\alpha, r) \quad \text{BY (297.75)}$$

$$49. \quad \int \sqrt{b \cosh x - a} dx = -2\sqrt{a + b} E(\alpha, r) + 2 \coth \frac{x}{2} \sqrt{b \cosh x - a} \quad \text{BY (297.79)}$$

$$50. \quad \int \frac{\coth^2 \frac{x}{2} dx}{\sqrt{b \cosh x - a}} = \frac{2\sqrt{a + b}}{a - b} E(\alpha, r) \quad \text{BY (297.76)}$$

$$51. \quad \int \frac{\sqrt{b \cosh x - a}}{\cosh x - 1} dx = \sqrt{a + b} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (297.77)}$$

$$52. \quad \int \frac{dx}{(\cosh x - 1) \sqrt{b \cosh x - a}} = \frac{\sqrt{a + b}}{a - b} E(\alpha, r) - \frac{1}{\sqrt{a + b}} F(\alpha, r) \quad \text{BY (297.78)}$$

$$53. \quad \int \frac{dx}{(\cosh x - 1)^2 \sqrt{b \cosh x - a}} = \frac{1}{3(a - b)^2 \sqrt{a + b}} \left[(a - 2b)(a - b) F(\alpha, r) \right. \\ \left. + (3a - b)(a + b) E(\alpha, r) \right] + \frac{a + b}{6b(a - b)} \cdot \frac{\cosh \frac{x}{2}}{\sinh^3 \frac{x}{2}} \sqrt{b \cosh x - a} \quad \text{BY (297.78)}$$

$$54. \quad \int \frac{dx}{(\cosh x + 1) \sqrt{b \cosh x - a}} = \frac{1}{\sqrt{a + b}} [F(\alpha, r) - E(\alpha, r)] + \frac{2\sqrt{b \cosh x - a}}{(a + b) \sinh x} \quad \text{BY (297.80)}$$

$$55. \quad \int \frac{dx}{(\cosh x + 1)^2 \sqrt{b \cosh x - a}} = \frac{1}{3\sqrt{(a+b)^3}} \left[(a+b) F(\alpha, r) - (a+3b) E(\alpha, r) \right] + \frac{\sqrt{b \cosh x - a}}{3(a+b) \sinh x} \left(2 \frac{a+3b}{a+b} - \tanh^2 \frac{x}{2} \right)$$

BY (297.80)

Notation: In **2.464** 56–**2.464** 60, we set

$$\alpha = \arccos \frac{\sqrt[4]{b^2 - a^2}}{\sqrt{a \sinh x + b \cosh x}},$$

$$r = \frac{1}{\sqrt{2}} \left[0 < a < b, \quad -\operatorname{arcsinh} \frac{a}{\sqrt{b^2 - a^2}} < x \right]$$

$$56. \quad \int \frac{dx}{\sqrt{a \sinh x + b \cosh x}} = \sqrt[4]{\frac{4}{b^2 - a^2}} F(\alpha, r) \quad \text{BY (299.00)}$$

$$57. \quad \int \sqrt{a \sinh x + b \cosh x} dx = \sqrt[4]{4(b^2 - a^2)} [F(\alpha, r) - 2 E(\alpha, r)] + \frac{2(a \cosh x + b \sinh x)}{\sqrt{a \sinh x + b \cosh x}}$$

BY (299.02)

$$58. \quad \int \frac{dx}{\sqrt{(a \sinh x + b \cosh x)^3}} = \sqrt[4]{\frac{4}{(b^2 - a^2)^3}} [2 E(\alpha, r) - F(\alpha, r)] \quad \text{BY (299.03)}$$

$$59. \quad \int \frac{dx}{\sqrt{(a \sinh x + b \cosh x)^5}} = \frac{1}{3} \sqrt[4]{\frac{4}{(b^2 - a^2)^5}} F(\alpha, r) + \frac{2}{3(b^2 - a^2)} \cdot \frac{a \cosh x + b \sinh x}{\sqrt{(a \sinh x + b \cosh x)^3}}$$

BY (299.03)

$$60. \quad \int \frac{(\sqrt{b^2 - a^2} + a \sinh x + b \cosh x) dx}{\sqrt{(a \sinh x + b \cosh x)^3}} = 2 \sqrt[4]{\frac{4}{b^2 - a^2}} E(\alpha, r) \quad \text{BY (299.01)}$$

2.47 Combinations of hyperbolic functions and powers

2.471

$$1. \quad \int x^r \sinh^p x \cosh^q x dx$$

$$= \frac{1}{(p+q)^2} \left[(p+q)x^r \sinh^{p-1} x \cosh^{q-1} x - rx^{r-1} \sinh^p x \cosh^q x + r(r+1) \int x^{r-2} \sinh^p x \cosh^q x dx \right. \\ \left. + rp \int x^{r-1} \sinh^{p-1} x \cosh^{q-1} x dx + (q-1)(p+q) \int x^r \sinh^p x \cosh^{q-2} x dx \right]$$

$$= \frac{1}{(p+q)^2} \left[(p+q)x^r \sinh^{p-1} x \cosh^{q+1} x - rx^{r-1} \sinh^p x \cosh^q x + r(r-1) \int x^{r-2} \sinh^p x \cosh^q x dx \right. \\ \left. - rq \int x^{r-1} \sinh^{p-1} x \cosh^{q-1} x dx - (p-1)(p+q) \int x^r \sinh^{p-2} x \cosh^q x dx \right]$$

GU (353)(1)

2. $\int x^n \sinh^{2m} x \, dx = (-1)^m \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cosh(2m-2k)x \, dx$
3. $\int x^n \sinh^{2m+1} x \, dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \int x^n \sinh(2m-2k+1)x \, dx$
4. $\int x^n \cosh^{2m} x \, dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cosh(2m-2k)x \, dx$
5. $\int x^n \cosh^{2m+1} x \, dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \int x^n \cosh(2m-2k+1)x \, dx$

2.472

1.
$$\begin{aligned} \int x^n \sinh x \, dx &= x^n \cosh x - n \int x^{n-1} \cosh x \, dx \\ &= x^n \cosh x - nx^{n-1} \sinh x + n(n-1) \int x^{n-2} \sinh x \, dx \end{aligned}$$
2.
$$\begin{aligned} \int x^n \cosh x \, dx &= x^n \sinh x - n \int x^{n-1} \sinh x \, dx \\ &= x^n \sinh x - nx^{n-1} \cosh x + n(n-1) \int x^{n-2} \cosh x \, dx \end{aligned}$$
3.
$$\int x^{2n} \sinh x \, dx = (2n)! \left\{ \sum_{k=0}^n \frac{x^{2k}}{(2k)!} \cosh x - \sum_{k=1}^n \frac{x^{2k-1}}{(2k-1)!} \sinh x \right\}$$
4.
$$\int x^{2n+1} \sinh x \, dx = (2n+1)! \sum_{k=0}^n \left\{ \frac{x^{2k+1}}{(2k+1)!} \cosh x - \frac{x^{2k}}{(2k)!} \sinh x \right\}$$
- 5.¹¹
$$\int x^{2n} \cosh x \, dx = (2n)! \left\{ \sum_{k=0}^n \frac{x^{2k}}{(2k)!} \sinh x - \sum_{k=1}^n \frac{x^{2k-1}}{(2k-1)!} \cosh x \right\}$$
6.
$$\int x^{2n+1} \cosh x \, dx = (2n+1)! \sum_{k=0}^n \left\{ \frac{x^{2k+1}}{(2k+1)!} \sinh x - \frac{x^{2k}}{(2k)!} \cosh x \right\}$$
7.
$$\int x \sinh x \, dx = x \cosh x - \sinh x$$
8.
$$\int x^2 \sinh x \, dx = (x^2 + 2) \cosh x - 2x \sinh x$$
9.
$$\int x \cosh x \, dx = x \sinh x - \cosh x$$
10.
$$\int x^2 \cosh x \, dx = (x^2 + 2) \sinh x - 2x \cosh x$$

2.473 Notation: $z_1 = a + bx$

1.
$$\int z_1 \sinh kx \, dx = \frac{1}{k} z_1 \cosh kx - \frac{b}{k^2} \sinh kx$$

2. $\int z_1 \cosh kx \, dx = \frac{1}{k} z_1 \sinh kx - \frac{b}{k^2} \cosh kx$
3. $\int z_1^2 \sinh kx \, dx = \frac{1}{k} \left(z_1^2 + \frac{2b^2}{k^2} \right) \cosh kx - \frac{2bz_1}{k^2} \sinh kx$
4. $\int z_1^2 \cosh kx \, dx = \frac{1}{k} \left(z_1^2 + \frac{2b^2}{k^2} \right) \sinh kx - \frac{2bz_1}{k^2} \cosh kx$
5. $\int z_1^3 \sinh kx \, dx = \frac{z_1}{k} \left(z_1^2 + \frac{6b^2}{k^2} \right) \cosh kx - \frac{3b}{k^2} \left(z_1^2 + \frac{2b^2}{k^2} \right) \sinh kx$
6. $\int z_1^3 \cosh kx \, dx = \frac{z_1}{k} \left(z_1^2 + \frac{6b^2}{k^2} \right) \sinh kx - \frac{3b}{k^2} \left(z_1^3 + \frac{2b^2}{k^2} \right) \cosh kx$
7. $\int z_1^4 \sinh kx \, dx = \frac{1}{k} \left(z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cosh kx - \frac{4bz_1}{k^2} \left(z_1^2 + \frac{6b^2}{k^2} \right) \sinh kx$
8. $\int z_1^4 \cosh kx \, dx = \frac{1}{k} \left(z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sinh kx - \frac{4bz_1}{k^2} \left(z_1^2 + \frac{6b^2}{k^2} \right) \cosh kx$
9. $\int z_1^5 \sinh kx \, dx = \frac{z_1}{k} \left(z_1^4 + \frac{20b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \cosh kx - \frac{5b}{k^2} \left(z_1^4 + 12 \frac{b^2}{k^2} z_1^2 + 24 \frac{b^4}{k^4} \right) \sinh kx$
10. $\int z_1^5 \cosh kx \, dx = \frac{z_1}{k} \left(z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \sinh kx - \frac{5b}{k^2} \left(z_1^4 + 12 \frac{b^2}{k^2} z_1^2 + 24 \frac{b^4}{k^4} \right) \cosh kx$
11. $\int z_1^6 \sinh kx \, dx = \frac{1}{k} \left(z_1^6 + 30 \frac{b^2}{k^2} z_1^4 + 360 \frac{b^4}{k^4} z_1^2 + 720 \frac{b^6}{k^6} \right) \cosh kx$
 $- \frac{6bz_1}{k^2} \left(z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \sinh kx$
12. $\int z_1^6 \cosh kx \, dx = \frac{1}{k} \left(z_1^6 + 30 \frac{b^2}{k^2} z_1^4 + 360 \frac{b^4}{k^4} z_1^2 + 720 \frac{b^6}{k^6} \right) \sinh kx$
 $- \frac{6bz_1}{k^2} \left(z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \cosh kx$

2.474

1. $\int x^n \sinh^2 x \, dx = -\frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{2^{2k}(n-2k)!} \sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cosh 2x \right\}$
 GU (353)(2b)
2. $\int x^n \cosh^2 x \, dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{2^{2k}(n-2k)!} \sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cosh 2x \right\}$
 GU (353)(3e)
3. $\int x \sinh^2 x \, dx = \frac{1}{4} x \sinh 2x - \frac{1}{8} \cosh 2x - \frac{x^2}{4}$
4. $\int x^2 \sinh^2 x \, dx = \frac{1}{4} \left(x^2 + \frac{1}{2} \right) \sinh 2x - \frac{x}{4} \cosh 2x - \frac{x^3}{6}$

$$5. \quad \int x \cosh^2 x \, dx = \frac{x}{4} \sinh 2x - \frac{1}{8} \cosh 2x + \frac{x^2}{4}$$

$$6. \quad \int x^2 \cosh^2 x \, dx = \frac{1}{4} \left(x^2 + \frac{1}{2} \right) \sinh 2x - \frac{x}{4} \cosh 2x + \frac{x^3}{6} \quad \text{MZ 261}$$

$$7. \quad \int x^n \sinh^3 x \, dx = \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{(n-2k)!} \left(\frac{\cosh 3x}{3^{2k+1}} - 3 \cosh x \right) - \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\sinh 3x}{3^{2k+2}} - 3 \sinh x \right) \right\} \quad \text{GU (353)(2f)}$$

$$8. \quad \int x^n \cosh^3 x \, dx = \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{(n-2k)!} \left(\frac{\sinh 3x}{3^{2k+1}} + 3 \sinh x \right) - \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\cosh 3x}{3^{2k+2}} + 3 \cosh x \right) \right\} \quad \text{GU (353)(3f)}$$

$$9. \quad \int x \sinh^3 x \, dx = \frac{3}{4} \sinh x - \frac{1}{36} \sinh 3x - \frac{3}{4} x \cosh x - \frac{x}{12} \cosh 3x$$

$$10. \quad \int x^2 \sinh^3 x \, dx = - \left(\frac{3x^2}{4} + \frac{3}{2} \right) \cosh x + \left(\frac{x^2}{12} + \frac{1}{54} \right) \cosh 3x + \frac{3x}{2} \sinh x - \frac{x}{18} \sinh 3x. \quad \text{MZ 257}$$

$$11. \quad \int x \cosh^3 x \, dx = -\frac{3}{4} \cosh x - \frac{1}{36} \cosh 3x + \frac{3}{4} x \sinh x + \frac{x}{12} \sinh 3x$$

$$12. \quad \int x^2 \cosh^3 x \, dx = \left(\frac{3}{4} x^2 + \frac{3}{2} \right) \sinh x + \left(\frac{x^2}{12} + \frac{1}{54} \right) \sinh 3x - \frac{3}{2} x \cosh x - \frac{x}{18} \cosh 3x \quad \text{MZ 262}$$

2.475

$$1. \quad \int \frac{\sinh^q x}{x^p} \, dx = -\frac{(p-2) \sinh^q x + qx \sinh^{q-1} x \cosh x}{(p-1)(p-2)x^{p-1}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\sinh^{q-2} x}{x^{p-2}} \, dx + \frac{q^2}{(p-1)(p-2)} \int \frac{\sinh^q x}{x^{p-2}} \, dx \quad [p > 2] \quad \text{GU (353)(6a)}$$

$$2. \quad \int \frac{\cosh^q x}{x^p} \, dx = -\frac{(p-2) \cosh^q x + qx \cosh^{q-1} x \sinh x}{(p-1)(p-2)x^{p-1}} - \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\cosh^{q-2} x}{x^{p-2}} \, dx + \frac{q^2}{(p-1)(p-2)} \int \frac{\cosh^q x}{x^{p-2}} \, dx \quad [p > 2] \quad \text{GU (353)(7a)}$$

$$3. \quad \int \frac{\sinh x}{x^{2n}} \, dx = -\frac{1}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \cosh x + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \sinh x \right\} + \frac{1}{(2n-1)!} \text{chi}(x) \quad \text{GU (353)(6b)}$$

$$4. \quad \int \frac{\sinh x}{x^{2n+1}} \, dx = -\frac{1}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \cosh x + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \sinh x \right\} + \frac{1}{(2n)!} \text{shi}(x) \quad \text{GU (353)(6b)}$$

$$5. \quad \int \frac{\cosh x}{x^{2n}} dx = -\frac{1}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \sinh x + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \cosh x \right\} + \frac{1}{(2n-1)!} \operatorname{shi}(x) \quad \text{GU (353)(7b)}$$

$$6. \quad \int \frac{\cosh x}{x^{2n+1}} dx = -\frac{1}{(2n)!x} \left\{ \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \sinh x + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \cosh x \right\} + \frac{1}{(2n)!} \operatorname{chi}(x) \quad \text{GU (353)(7b)}$$

$$7. \quad \int \frac{\sinh^{2m} x}{x} dx = \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \operatorname{chi}(2m-2k)x + \frac{(-1)^m}{2^{2m}} \binom{2m}{m} \ln x \quad \text{GU (353)(6c)}$$

$$8. \quad \int \frac{\sinh^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \operatorname{shi}(2m-2k+1)x \quad \text{GU (353)(6d)}$$

$$9. \quad \int \frac{\cosh^{2m} x}{x} dx = \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \operatorname{chi}(2m-2k)x + \frac{1}{2^{2m}} \binom{2m}{m} \ln x \quad \text{GU (353)(7c)}$$

$$10. \quad \int \frac{\cosh^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \operatorname{chi}(2m-2k+1)x \quad \text{GU (353)(7c)}$$

$$11. \quad \int \frac{\sinh^{2m} x}{x^2} dx = \frac{(-1)^{m-1}}{2^{2m}x} \binom{2m}{m} \\ + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\cosh(2m-2k)x}{x} - (2m-2k) \operatorname{shi}(2m-2k)x \right\}$$

$$12. \quad \int \frac{\sinh^{2m+1} x}{x^2} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^{k+1} \binom{2m+1}{k} \\ \times \left\{ \frac{\sinh(2m-2k+1)x}{x} - (2m-2k+1) \operatorname{chi}(2m-2k+1)x \right\}$$

$$13. \quad \int \frac{\cosh^{2m} x}{x^2} dx \\ = -\frac{1}{2^{2m}x} \binom{2m}{m} - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\cosh(2m-2k)x}{x} - (2m-2k) \operatorname{shi}(2m-2k)x \right\}$$

$$14. \quad \int \frac{\cosh^{2m+1} x}{x^2} dx \\ = -\frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \left\{ \frac{\cosh(2m-2k+1)x}{x} - (2m-2k+1) \operatorname{shi}(2m-2k+1)x \right\}$$

2.476

$$1. \quad \int \frac{\sinh kx}{a+bx} dx = \frac{1}{b} \left[\cosh \frac{ka}{b} \operatorname{shi}(u) - \sinh \frac{ka}{b} \operatorname{chi}(u) \right] \\ = \frac{1}{2b} \left[\exp \left(-\frac{ka}{b} \right) \operatorname{Ei}(u) - \exp \left(\frac{ka}{b} \right) \operatorname{Ei}(-u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right]$$

2.
$$\int \frac{\cosh kx}{a+bx} dx = \frac{1}{b} \left[\cosh \frac{ka}{b} \operatorname{chi}(u) - \sinh \frac{ka}{b} \operatorname{shi}(u) \right]$$

$$= \frac{1}{2b} \left[\exp \left(-\frac{ka}{b} \right) \operatorname{Ei}(u) + \exp \left(\frac{ka}{b} \right) \operatorname{Ei}(-u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right]$$
3.
$$\int \frac{\sinh kx}{(a+bx)^2} dx = -\frac{1}{b} \cdot \frac{\sinh kx}{a+bx} + \frac{k}{b} \int \frac{\cosh kx}{a+bx} dx \quad (\text{see } \mathbf{2.476} \ 2)$$
4.
$$\int \frac{\cosh kx}{(a+bx)^2} dx = -\frac{1}{b} \cdot \frac{\cosh kx}{a+bx} + \frac{k}{b} \int \frac{\sinh kx}{a+bx} dx \quad (\text{see } \mathbf{2.476} \ 1)$$
5.
$$\int \frac{\sinh kx}{(a+bx)^3} dx = -\frac{\sinh kx}{2b(a+bx)^2} - \frac{k \cosh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2} \int \frac{\sinh kx}{a+bx} dx$$

(see **2.476** 1)
6.
$$\int \frac{\cosh kx}{(a+bx)^3} dx = -\frac{\cosh kx}{2b(a+bx)^2} - \frac{k \sinh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2} \int \frac{\cosh kx}{a+bx} dx$$

(see **2.476** 2)
7.
$$\int \frac{\sinh kx}{(a+bx)^4} dx = -\frac{\sinh kx}{3b(a+bx)^3} - \frac{k \cosh kx}{6b^2(a+bx)^2} - \frac{k^2 \sinh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\cosh kx}{a+bx} dx$$

(see **2.476** 2)
8.
$$\int \frac{\cosh kx}{(a+bx)^4} dx = -\frac{\cosh kx}{3b(a+bx)^3} - \frac{k \sinh kx}{6b^2(a+bx)^2} - \frac{k^2 \cosh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\sinh kx}{a+bx} dx$$

(see **2.476** 1)
9.
$$\int \frac{\sinh kx}{(a+bx)^5} dx = -\frac{\sinh kx}{4b(a+bx)^4} - \frac{k \cosh kx}{12b^2(a+bx)^3} - \frac{k^2 \sinh kx}{24b^3(a+bx)^2}$$

$$- \frac{k^3 \cosh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\sinh kx}{a+bx} dx$$

(see **2.476** 1)
10.
$$\int \frac{\cosh kx}{(a+bx)^5} dx = -\frac{\cosh kx}{4b(a+bx)^4} - \frac{k \sinh kx}{12b^2(a+bx)^3} - \frac{k^2 \cosh kx}{24b^3(a+bx)^2}$$

$$- \frac{k^3 \sinh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\cosh kx}{a+bx} dx$$

(see **2.476** 2)
11.
$$\int \frac{\sinh kx}{(a+bx)^6} dx = -\frac{\sinh kx}{5b(a+bx)^5} - \frac{k \cosh kx}{20b^2(a+bx)^4} - \frac{k^2 \sinh kx}{60b^3(a+bx)^3} - \frac{k^3 \cosh kx}{120b^4(a+bx)^2}$$

$$- \frac{k^4 \sinh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\cosh kx}{a+bx} dx$$

(see **2.476** 2)
12.
$$\int \frac{\cosh kx}{(a+bx)^6} dx = -\frac{\cosh kx}{5b(a+bx)^5} - \frac{k \sinh kx}{20b^2(a+bx)^4} - \frac{k^2 \cosh kx}{60b^3(a+bx)^3} - \frac{k^3 \sinh kx}{120b^4(a+bx)^2}$$

$$- \frac{k^4 \cosh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\sinh kx}{a+bx} dx$$

(see **2.476** 1)

2.477

$$1. \quad \int \frac{x^p dx}{\sinh^q x} = \frac{-px^{p-1} \sinh x - (q-2)x^p \cosh x}{(q-1)(q-2) \sinh^{q-1} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2}}{\sinh^{q-2} x} dx \\ - \frac{q-2}{q-1} \int \frac{x^p dx}{\sinh^{q-2} x} \quad [q > 2] \quad \text{GU (353)(8a)}$$

$$2. \quad \int \frac{x^p dx}{\cosh^q x} = \frac{px^{p-1} \cosh x + (q-2)x^p \sinh x}{(q-1)(q-2) \cosh^{q-1} x} - \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\cosh^{q-2} x} \\ + \frac{q-2}{q-1} \int \frac{x^p dx}{\cosh^{q-2} x} \quad [q > 2] \quad \text{GU (353)(10a)}$$

$$3. \quad \int \frac{x^n}{\sinh x} dx = \sum_{k=0}^{\infty} \frac{(2-2^{2k}) B_{2k}}{(n+2k)(2k)!} x^{n+2k} \quad [|x| < \pi, \quad n > 0] \quad \text{GU(353)(8b)}$$

$$4. \quad \int \frac{x^n}{\cosh x} dx = \sum_{k=0}^{\infty} \frac{E_{2k} x^{n+2k+1}}{(n+2k+1)(2k)!} \quad \left[|x| < \frac{\pi}{2}, \quad n \geq 0\right] \quad \text{GU (353)(10b)}$$

$$5. \quad \int \frac{dx}{x^n \sinh x} = -[1 + (-1)^n] \frac{2^{n-1} - 1}{n!} B_n \ln x \\ + \sum_{\substack{k=0 \\ k \neq \frac{n}{2}}}^{\infty} \frac{2-2^{2k}}{(2k-n)(2k)!} B_{2k} x^{2k-n} \quad [|x| < \pi, \quad n \geq 1] \quad \text{GU (353)(9b)}$$

$$6.^{11} \quad \int \frac{dx}{x^n \cosh x} = \sum_{\substack{k=0 \\ k \neq \frac{n-1}{2}}}^{\infty} \frac{E_{2k}}{(2k-n+1)(2k)!} x^{2k-n+1} + \frac{1}{2} [1 + (-1)^n] + \frac{E_{n-1}}{(n-1)!} \ln x \\ \left[|x| < \frac{\pi}{2}\right] \quad \text{GU (353)(11b)}$$

$$7. \quad \int \frac{x^n}{\sinh^2 x} dx = -x^n \coth x + n \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(n+2k-1)(2k)!} x^{n+2k-1} \quad [n > 1, \quad |x| < \pi] \quad \text{GU (353)(8c)}$$

$$8. \quad \int \frac{x^n}{\cosh^2 x} dx = x^n \tanh x - n \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1) B_{2k}}{(n+2k-1)(2k)!} x^{n+2k-1} \quad \left[n > 1, \quad |x| < \frac{\pi}{2}\right] \quad \text{GU (353)(10c)}$$

$$9. \quad \int \frac{dx}{x^n \sinh^2 x} = -\frac{\coth x}{x^n} - [1 - (-1)^n] \frac{2^n n}{(n+1)!} B_{n+1} \ln x \\ - \frac{n}{x^{n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{B_{2k}}{(2k-n-1)(2k)!} (2x)^{2k} \quad [|x| < \pi] \quad \text{GU (353)(9c)}$$

10.
$$\int \frac{dx}{x^n \cosh^2 x} = \frac{\tanh x}{x^n} + [1 - (-1)^n] - \frac{2n(2^{n+1} - 1)n}{(n+1)!} B_{n+1} \ln x$$

$$+ \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(2^{2k} - 1) B_{2k}}{(2k - n - 1)(2k)!} (2x)^{2k}$$

$$\left[|x| < \frac{\pi}{2} \right] \quad \text{GU (353)(11c)}$$
11.
$$\int \frac{x}{\sinh^{2n} x} dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-2)(2n-4) \dots (2n-2k+2)}{(2n-1)(2n-3) \dots (2n-2k+1)}$$

$$\times \left\{ \frac{x \cosh x}{\sinh^{2n-2k+1} x} + \frac{1}{(2n-2k) \sinh^{2n-2k} x} \right\} + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x dx}{\sinh^2 x}$$

$$\text{(see 2.477 17)} \quad \text{GU (353)(8e)}$$
12.
$$\int \frac{x}{\sinh^{2n-1} x} dx$$

$$= \sum_{k=1}^{n-1} (-1)^k \frac{(2n-3)(2n-5) \dots (2n-2k+1)}{(2n-2)(2n-4) \dots (2n-2k)}$$

$$\times \left\{ \frac{x \cosh x}{\sinh^{2n-2k} x} + \frac{1}{(2n-2k-1) \sinh^{2n-2k-1} x} \right\} + (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x dx}{\sinh x}$$

$$\text{(see 2.477 15)} \quad \text{GU (353)(8e)}$$
13.
$$\int \frac{x}{\cosh^{2n} x} dx = \sum_{k=1}^{n-1} \frac{(2n-2)(2n-4) \dots (2n-2k+2)}{(2n-1)(2n-3) \dots (2n-2k+1)}$$

$$\times \left\{ \frac{x \sinh x}{\cosh^{2n-2k+1} x} + \frac{1}{(2n-2k) \cosh^{2n-2k} x} \right\} + \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x dx}{\cosh^2 x}$$

$$\text{(see 2.477 18)} \quad \text{GU (353)(10e)}$$
14.
$$\int \frac{x}{\cosh^{2n-1} x} dx = \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5) \dots (2n-2k+1)}{(2n-2)(2n-4) \dots (2n-2k)}$$

$$\times \left\{ \frac{x \sinh x}{\cosh^{2n-2k} x} + \frac{1}{(2n-2k-1) \cosh^{2n-2k-1} x} \right\} + \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x dx}{\cosh x}$$

$$\text{(see 2.477 16)} \quad \text{GU (353)(10e)}$$
15.
$$\int \frac{x dx}{\sinh x} = \sum_{k=0}^{\infty} \frac{2 - 2^{2k}}{(2k+1)(2k)!} B_{2k} x^{2k+1} \quad |x| < \pi \quad \text{GU (353)(8b)a}$$
16.
$$\int \frac{x dx}{\cosh x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k+2)(2k)!} \quad |x| < \frac{\pi}{2} \quad \text{GU (353)(10b)a}$$
17.
$$\int \frac{x dx}{\sinh^2 x} = -x \coth x + \ln \sinh x \quad \text{MZ 257}$$
18.
$$\int \frac{x dx}{\cosh^2 x} = x \tanh x - \ln \cosh x \quad \text{MZ 262}$$
19.
$$\int \frac{x dx}{\sinh^3 x} = -\frac{x \cosh x}{2 \sinh^2 x} - \frac{1}{2 \sinh x} - \frac{1}{2} \int \frac{x dx}{\sinh x} \quad \text{(see 2.477 15)} \quad \text{MZ 257}$$

20. $\int \frac{x dx}{\cosh^3 x} = \frac{x \sinh x}{2 \cosh^2 x} + \frac{1}{2 \cosh x} + \frac{1}{2} \int \frac{x dx}{\cosh x}$ (see **2.477** 16) MZ 262
21. $\int \frac{x dx}{\sinh^4 x} = -\frac{x \cosh x}{3 \sinh^3 x} - \frac{1}{6 \sinh^2 x} + \frac{2}{3} x \coth x - \frac{2}{3} \ln \sinh x$ MZ 258
22. $\int \frac{x dx}{\cosh^4 x} = \frac{x \sinh x}{3 \cosh^3 x} + \frac{1}{6 \cosh^2 x} + \frac{2}{3} x \tanh x - \frac{2}{3} \ln \cosh x$ MZ 262
23. $\int \frac{x dx}{\sinh^5 x} = -\frac{x \cosh x}{4 \sinh^4 x} - \frac{1}{12 \sinh^3 x} + \frac{3x \cosh x}{8 \sinh^2 x} + \frac{3}{8 \sinh x} + \frac{3}{8} \int \frac{x dx}{\sinh x}$
(see **2.477** 15) MZ 258
24. $\int \frac{x dx}{\cosh^5 x} = \frac{x \sinh x}{4 \cosh^4 x} + \frac{1}{12 \cosh^3 x} + \frac{3x \sinh x}{8 \cosh^2 x} + \frac{3}{8 \cosh x} + \frac{3}{8} \int \frac{x dx}{\cosh x}$
(see **2.477** 16) MZ 262

2.478

1. $\int \frac{x^n \cosh x dx}{(a + b \sinh x)^m} = -\frac{x^n}{(m-1)b(a + b \sinh x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a + b \sinh x)^{m-1}}$
[$m \neq 1$] MZ 263
2. $\int \frac{x^n \sinh x dx}{(a + b \cosh x)^m} = -\frac{x^n}{(m-1)b(a + b \cosh x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a + b \cosh x)^{m-1}}$
[$m \neq 1$] MZ 263
3. $\int \frac{x dx}{1 + \cosh x} = x \tanh \frac{x}{2} - 2 \ln \cosh \frac{x}{2}$
4. $\int \frac{x dx}{1 - \cosh x} = x \coth \frac{x}{2} - 2 \ln \sinh \frac{x}{2}$
5. $\int \frac{x \sinh x dx}{(1 + \cosh x)^2} = -\frac{x}{1 + \cosh x} + \tanh \frac{x}{2}$
6. $\int \frac{x \sinh x dx}{(1 - \cosh x)^2} = \frac{x}{1 - \cosh x} - \coth \frac{x}{2}$ MZ 262-264
7. $\int \frac{x dx}{\cosh 2x - \cos 2t} = \frac{1}{2 \sin 2t} [L(u+t) - L(u-t) - 2L(t)]$
[$u = \arctan(\tanh x \cot t)$, $t \neq \pm n\pi$] LO III 402
8. $\int \frac{x \cosh x dx}{\cosh 2x - \cos 2t} = \frac{1}{2 \sin t} \left[L\left(\frac{u+t}{2}\right) - L\left(\frac{u-t}{2}\right) + L\left(\pi - \frac{v+t}{2}\right) \right.$
 $\left. + L\left(\frac{v-t}{2}\right) - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right) \right]$
[$u = 2 \arctan\left(\tanh \frac{x}{2} \cdot \cot \frac{t}{2}\right)$, $v = 2 \arctan\left(\coth \frac{x}{2} \cdot \cot \frac{t}{2}\right)$; $t \neq \pm n\pi$] LO III 403

2.479

1. $\int x^p \frac{\sinh^{2m} x}{\cosh^n x} dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int \frac{x^p dx}{\cosh^{n-2k} x} \quad (\text{see } 4.477 \text{ 2})$
2. $\int x^p \frac{\sinh^{2m+1} x}{\cosh^n x} dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int x^p \frac{\sinh x}{\cosh^{n-2k} x} dx$
 $[n > 1] \quad (\text{see } 2.479 \text{ 3})$
3. $\int x^p \frac{\sinh x}{\cosh^n x} dx = -\frac{x^p}{(n-1) \cosh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} dx}{\cosh^{n-1} x}$
 $[n > 1] \quad (\text{see } 2.477 \text{ 2}) \quad \text{GU (353)(12)}$
4. $\int x^p \frac{\cosh^{2m} x}{\sinh^n x} dx = \sum_{k=0}^m \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x} dx \quad (\text{see } 2.477 \text{ 1})$
5. $\int x^p \frac{\cosh^{2m+1} x}{\sinh^n x} dx = \sum_{k=0}^m \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x} dx \quad (\text{see } 2.479 \text{ 6})$
6. $\int x^p \frac{\cosh x}{\sinh^n x} dx = -\frac{x^p}{(n-1) \sinh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} dx}{\sinh^{n-1} x}$
 $[n > 1] \quad (\text{see } 2.477 \text{ 1}) \quad \text{GU (353)(13c)}$
7. $\int x^p \tanh x dx = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1) B_{2k}}{(2k+p)(2k)!} x^{p+2k} \quad \left[p > -1, \quad |x| < \frac{\pi}{2} \right] \quad \text{GU (353)(12d)}$
8. $\int x^p \coth x dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \quad [p \geq +1, \quad |x| < \pi] \quad \text{GU (353)(13d)}$
9. $\int \frac{x \cosh x}{\sinh^2 x} dx = \ln \tanh \frac{x}{2} - \frac{x}{\sinh x}$
10. $\int \frac{x \sinh x}{\cosh^2 x} dx = -\frac{x}{\cosh x} + \arctan (\sinh x) \quad \text{MZ 263}$

2.48 Combinations of hyperbolic functions, exponentials, and powers

2.481

1. $\int e^{ax} \sinh(bx+c) dx = \frac{e^{ax}}{a^2 - b^2} [a \sinh(bx+c) - b \cosh(bx+c)]$
 $[a^2 \neq b^2]$
2. $\int e^{ax} \cosh(bx+c) dx = \frac{e^{ax}}{a^2 - b^2} [a \cosh(bx+c) - b \sinh(bx+c)]$
 $[a^2 \neq b^2]$

For $a^2 = b^2$:

$$3. \quad \int e^{ax} \sinh(ax + c) dx = -\frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

$$4. \quad \int e^{-ax} \sinh(ax + c) dx = \frac{1}{2}xe^c + \frac{1}{4a}e^{-(2ax+c)}$$

$$5. \quad \int e^{ax} \cosh(ax + c) dx = \frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

$$6. \quad \int e^{-ax} \cosh(ax + c) dx = \frac{1}{2}xe^c - \frac{1}{4a}e^{-(2ax+c)}$$

MZ 275-277

2.482

$$1. \quad \int x^p e^{ax} \sinh bx dx = \frac{1}{2} \left\{ \int x^p e^{(a+b)x} dx - \int x^p e^{(a-b)x} dx \right\}$$

$$[a^2 \neq b^2]$$

$$2. \quad \int x^p e^{ax} \cosh bx dx = \frac{1}{2} \left\{ \int x^p e^{(a+b)x} dx + \int x^p e^{(a-b)x} dx \right\}$$

$$[a^2 \neq b^2]$$

For $a^2 = b^2$:

$$3. \quad \int x^p e^{ax} \sinh ax dx = \frac{1}{2} \int x^p e^{2ax} dx - \frac{x^{p+1}}{2(p+1)} \quad (\text{see } \mathbf{2.321})$$

$$4. \quad \int x^p e^{-ax} \sinh ax dx = \frac{x^{p+1}}{2(p+1)} - \frac{1}{2} \int x^p e^{-2ax} dx \quad (\text{see } \mathbf{2.321})$$

$$5. \quad \int x^p e^{ax} \cosh ax dx = \frac{x^{p+1}}{2(p+1)} + \frac{1}{2} \int x^p e^{2ax} dx \quad (\text{see } \mathbf{2.321})$$

MZ 276, 278

2.483

$$1. \quad \int x e^{ax} \sinh bx dx = \frac{e^{ax}}{a^2 - b^2} \left[\left(ax - \frac{a^2 + b^2}{a^2 - b^2} \right) \sinh bx - \left(bx - \frac{2ab}{a^2 - b^2} \right) \cosh bx \right]$$

$$[a^2 \neq b^2]$$

$$2. \quad \int x e^{ax} \cosh bx dx = \frac{e^{ax}}{a^2 - b^2} \left[\left(ax - \frac{a^2 + b^2}{a^2 - b^2} \right) \cosh bx - \left(bx - \frac{2ab}{a^2 - b^2} \right) \sinh bx \right]$$

$$[a^2 \neq b^2]$$

$$3. \quad \int x^2 e^{ax} \sinh bx dx = \frac{e^{ax}}{a^2 - b^2} \left\{ \left[ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2}x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2} \right] \sinh bx \right. \\ \left. - \left[bx^2 - \frac{4ab}{a^2 - b^2}x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2} \right] \cosh bx \right\} \quad [a^2 \neq b^2]$$

$$4. \quad \int x^2 e^{ax} \cosh bx \, dx = \frac{e^{ax}}{a^2 - b^2} \left\{ \left[ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2} x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2} \right] \cosh bx - \left[bx^2 - \frac{4ab}{a^2 - b^2} x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2} \right] \sinh x \right\} \quad [a^2 \neq b^2]$$

For $a^2 = b^2$:

$$\begin{aligned} 5. \quad & \int x e^{ax} \sinh ax \, dx = \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right) - \frac{x^2}{4} \\ 6. \quad & \int x e^{-ax} \sinh ax \, dx = \frac{e^{-2ax}}{4a} \left(x + \frac{1}{2a} \right) + \frac{x^2}{4} \quad \text{MZ 276, 278} \\ 7. \quad & \int x e^{ax} \cosh ax \, dx = \frac{x^2}{4} + \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right) \\ 8. \quad & \int x e^{-ax} \cosh ax \, dx = \frac{x^2}{4} - \frac{e^{-2ax}}{4a} \left(x + \frac{1}{2a} \right) \\ 9. \quad & \int x^2 e^{ax} \sinh ax \, dx = \frac{e^{2ax}}{4a} \left(x^2 - \frac{x}{a} + \frac{1}{2a^2} \right) - \frac{x^3}{6} \\ 10. \quad & \int x^2 e^{-ax} \sinh ax \, dx = \frac{e^{-2ax}}{4a} \left(x^2 + \frac{x}{a} + \frac{1}{2a^2} \right) + \frac{x^3}{6} \\ 11. \quad & \int x^2 e^{ax} \cosh ax \, dx = \frac{x^3}{6} + \frac{e^{2ax}}{4a} \left(x^2 - \frac{x}{a} + \frac{1}{2a^2} \right) \end{aligned}$$

2.484

$$\begin{aligned} 1. \quad & \int e^{ax} \sinh bx \frac{dx}{x} = \frac{1}{2} \{ \text{Ei}[(a+b)x] - \text{Ei}[(a-b)x] \} \quad [a^2 \neq b^2] \\ 2. \quad & \int e^{ax} \cosh bx \frac{dx}{x} = \frac{1}{2} \{ \text{Ei}[(a+b)x] + \text{Ei}[(a-b)x] \} \quad [a^2 \neq b^2] \\ 3. \quad & \int e^{ax} \sinh bx \frac{dx}{x^2} = -\frac{e^{ax} \sinh bx}{2x} + \frac{1}{2} \{ (a+b) \text{Ei}[(a+b)x] - (a-b) \text{Ei}[(a-b)x] \} \\ & \quad [a^2 \neq b^2] \\ 4. \quad & \int e^{ax} \cosh bx \frac{dx}{x^2} = -\frac{e^{ax} \cosh bx}{2x} + \frac{1}{2} \{ (a+b) \text{Ei}[(a+b)x] + (a-b) \text{Ei}[(a-b)x] \} \\ & \quad [a^2 \neq b^2] \end{aligned}$$

For $a^2 = b^2$:

$$\begin{aligned} 5. \quad & \int e^{ax} \sinh ax \frac{dx}{x} = \frac{1}{2} [\text{Ei}(2ax) - \ln x] \\ 6. \quad & \int e^{-ax} \sinh ax \frac{dx}{x} = \frac{1}{2} [\ln x - \text{Ei}(-2ax)] \\ 7. \quad & \int e^{ax} \cosh ax \frac{dx}{x} = \frac{1}{2} [\ln x + \text{Ei}(2ax)] \end{aligned}$$

$$8. \quad \int e^{ax} \sinh ax \frac{dx}{x^2} = -\frac{1}{2x} (e^{2ax} - 1) + a \operatorname{Ei}(2ax)$$

$$9. \quad \int e^{-ax} \sinh ax \frac{dx}{x^2} = -\frac{1}{2x} (1 - e^{-2ax}) + a \operatorname{Ei}(-2ax)$$

$$10. \quad \int e^{ax} \cosh ax \frac{dx}{x^2} = -\frac{1}{2x} (e^{2ax} + 1) + a \operatorname{Ei}(2ax)$$

MZ 276, 278

2.5–2.6 Trigonometric Functions

2.50 Introduction

2.501 Integrals of the form $\int R(\sin x, \cos x) dx$ can always be reduced to integrals of rational functions by means of the substitution $t = \tan \frac{x}{2}$.

2.502 If $R(\sin x, \cos x)$ satisfies the relation

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

it is convenient to make the substitution $t = \cos x$.

2.503 If this function satisfies the relation

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

it is convenient to make the substitution $t = \sin x$.

2.504 If this function satisfies the relation

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

it is convenient to make the substitution $t = \tan x$.

2.51–2.52 Powers of trigonometric functions

$$\begin{aligned} \mathbf{2.510} \quad \int \sin^p x \cos^q x dx &= -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x dx \\ &= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx \\ &= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^q x dx \\ &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1} \int \sin^{p+2} x \cos^{q-2} x dx \\ &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x dx \\ &= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^p x \cos^{q+2} x dx \\ &= \frac{\sin^{p-1} x \cos^{q-1} x}{p+q} \left\{ \sin^2 x - \frac{q-1}{p+q-2} \right\} \\ &\quad + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x dx \end{aligned}$$

2.511

$$\begin{aligned}
 1. \quad & \int \sin^p x \cos^{2n} x \, dx \\
 &= \frac{\sin^{p+1} x}{2n+p} \left\{ \cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1) \cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \right\} \\
 &\quad + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \sin^p x \, dx
 \end{aligned}$$

This formula is applicable for arbitrary real p , except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have:

$$\begin{aligned}
 2. \quad & \int \sin^{2l} x \, dx \\
 &= -\frac{\cos x}{2l} \left\{ \sin^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \sin^{2l-2k-1} x \right\} \\
 &\quad + \frac{(2l-1)!!}{2^l l!} x \quad \text{(see also 2.513 1)} \\
 &\quad \text{TI (232)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int \sin^{2l+1} x \, dx = -\frac{\cos x}{2l+1} \left\{ \sin^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)} \sin^{2l-2k-2} x \right\} \\
 &\quad \text{(see also 2.513 2)} \quad \text{TI (233)}
 \end{aligned}$$

$$4. \quad \int \sin^p x \cos^{2n+1} x \, dx = \frac{\sin^{p+1} x}{2n+p+1} \left\{ \cos^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1)\dots(n-k+1) \cos^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right\}$$

This formula is applicable for arbitrary real p , except for the negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.512

$$\begin{aligned}
 1. \quad & \int \cos^p x \sin^{2n} x \, dx \\
 &= -\frac{\cos^{p+1} x}{2n+p} \left\{ \sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1) \sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \right\} \\
 &\quad + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \cos^p x \, dx
 \end{aligned}$$

This formula is applicable for arbitrary real p , except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

$$\begin{aligned}
 2. \quad & \int \cos^{2l} x \, dx = \frac{\sin x}{2l} \left\{ \cos^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \cos^{2l-2k-1} x \right\} \\
 &\quad + \frac{(2l-1)!!}{2^l l!} x \quad \text{(see also 2.513 3)} \\
 &\quad \text{TI (230)}
 \end{aligned}$$

$$3. \quad \int \cos^{2l+1} x \, dx = \frac{\sin x}{2l+1} \left\{ \cos^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1) \dots (l-k)}{(2l-1)(2l-3) \dots (2l-2k-1)} \cos^{2l-2k-2} x \right\}$$

(see also **2.513** 4) TI (231)

$$4. \quad \int \cos^p x \sin^{2n+1} x \, dx = -\frac{\cos^{p+1} x}{2n+p+1} \left\{ \sin^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1) \dots (n-k+1) \sin^{2n-2k} x}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \right\}$$

This formula is applicable for arbitrary real p , except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.513

$$1. \quad \int \sin^{2n} x \, dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(see also **2.511** 2) TI (226)

$$2. \quad \int \sin^{2n+1} x \, dx = \frac{1}{2^{2n}} (-1)^{n+1} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k}$$

(see also **2.511** 3) TI (227)

$$3. \quad \int \cos^{2n} x \, dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(see also **2.512** 2) TI (224)

$$4. \quad \int \cos^{2n+1} x \, dx = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1}$$

(see also **2.512** 3) TI (225)

$$5. \quad \int \sin^2 x \, dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x$$

$$6. \quad \int \sin^3 x \, dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x = \frac{1}{3} \cos^3 x - \cos x$$

$$7. \quad \int \sin^4 x \, dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$$

$$= -\frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x + \frac{3}{8} x$$

$$8. \quad \int \sin^5 x \, dx = -\frac{5}{8} \cos x + \frac{5}{48} \cos 3x - \frac{1}{80} \cos 5x$$

$$= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{15} \cos^3 x - \frac{4}{5} \cos x$$

$$9. \quad \int \sin^6 x \, dx = \frac{5}{16} x - \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x - \frac{1}{192} \sin 6x$$

$$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x$$

10.
$$\begin{aligned}\int \sin^7 x \, dx &= -\frac{35}{64} \cos x + \frac{7}{64} \cos 3x - \frac{7}{320} \cos 5x + \frac{1}{448} \cos 7x \\ &= -\frac{1}{7} \sin^6 x \cos x - \frac{6}{35} \sin^4 x \cos x + \frac{8}{35} \cos^3 x - \frac{24}{35} \cos x\end{aligned}$$
11.
$$\int \cos^2 x \, dx = \frac{1}{4} \sin 2x + \frac{x}{2} = \frac{1}{2} \sin x \cos x + \frac{1}{2} x$$
12.
$$\int \cos^3 x \, dx = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x = \sin x - \frac{1}{3} \sin^3 x$$
13.
$$\int \cos^4 x \, dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x = \frac{3}{8} x + \frac{3}{8} \sin x \cos x + \frac{1}{4} \sin x \cos^3 x$$
14.
$$\int \cos^5 x \, dx = \frac{5}{8} \sin x + \frac{5}{48} \sin 3x + \frac{1}{80} \sin 5x = \frac{4}{5} \sin x - \frac{4}{15} \sin^3 x + \frac{1}{5} \cos^4 x \sin x$$
15.
$$\begin{aligned}\int \cos^6 x \, dx &= \frac{5}{16} x + \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{192} \sin 6x \\ &= \frac{5}{16} x + \frac{5}{16} \sin x \cos x + \frac{5}{24} \sin x \cos^3 x + \frac{1}{6} \sin x \cos^5 x\end{aligned}$$
16.
$$\begin{aligned}\int \cos^7 x \, dx &= \frac{35}{64} \sin x + \frac{7}{64} \sin 3x + \frac{7}{320} \sin 5x + \frac{1}{448} \sin 7x \\ &= \frac{24}{35} \sin x - \frac{8}{35} \sin^3 x + \frac{6}{35} \sin x \cos^4 x + \frac{1}{7} \sin x \cos^6 x\end{aligned}$$
17.
$$\int \sin x \cos^2 x \, dx = -\frac{1}{4} \left(\frac{1}{3} \cos 3x + \cos x \right) = -\frac{\cos^3 x}{3}$$
18.
$$\int \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4}$$
19.
$$\int \sin x \cos^4 x \, dx = -\frac{\cos^5 x}{5}$$
20.
$$\int \sin^2 x \cos x \, dx = -\frac{1}{4} \left(\frac{1}{3} \sin 3x - \sin x \right) = \frac{\sin^3 x}{3}$$
21.
$$\int \sin^2 x \cos^2 x \, dx = -\frac{1}{8} \left(\frac{1}{4} \sin 4x - x \right)$$
22.
$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= -\frac{1}{16} \left(\frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - 2 \sin x \right) \\ &= \frac{\sin^3 x}{5} \left(\cos^2 x + \frac{2}{3} \right) = \frac{\sin^3 x}{5} \left(\frac{5}{3} - \sin^2 x \right)\end{aligned}$$
23.
$$\int \sin^2 x \cos^4 x \, dx = \frac{x}{16} + \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x - \frac{1}{192} \sin 6x$$
24.
$$\int \sin^3 x \cos x \, dx = \frac{1}{8} \left(\frac{1}{4} \cos 4x - \cos 2x \right) = \frac{\sin^4 x}{4}$$
25.
$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \frac{1}{16} \left(\frac{1}{5} \cos 5x - \frac{1}{3} \cos 3x - 2 \cos x \right) \\ &= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x\end{aligned}$$

$$26. \quad \int \sin^3 x \cos^3 x \, dx = \frac{1}{32} \left(\frac{1}{6} \cos 6x - \frac{3}{2} \cos 2x \right)$$

$$27. \quad \int \sin^3 x \cos^4 x \, dx = \frac{1}{7} \cos^3 x \left(-\frac{2}{5} - \frac{3}{5} \sin^2 x + \sin^4 x \right)$$

$$28. \quad \int \sin^4 x \cos x \, dx = \frac{\sin^5 x}{5}$$

$$29. \quad \int \sin^4 x \cos^2 x \, dx = \frac{1}{16} x - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x$$

$$30. \quad \int \sin^4 x \cos^3 x \, dx = \frac{1}{7} \sin^3 x \left(\frac{2}{5} + \frac{3}{5} \cos^2 x - \cos^4 x \right)$$

$$31. \quad \int \sin^4 x \cos^4 x \, dx = \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x$$

$$\begin{aligned} 2.514 \quad \int \frac{\sin^p x}{\cos^{2n} x} \, dx &= \frac{\sin^{p+1} x}{2n-1} \left\{ \sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right\} \\ &+ \frac{(2n-p-2)(2n-p-4) \dots (-p+2)(-p)}{(2n-1)!!} \int \sin^p x \, dx \end{aligned}$$

This formula is applicable for arbitrary real p . For $\int \sin^p x \, dx$, where p is a natural number, see **2.511** 2, 3 and **2.513** 1, 2. If $n = 0$ and p is a negative integer, we have for this integral:

2.515

$$1. \quad \int \frac{dx}{\sin^{2l} x} = -\frac{\cos x}{2l-1} \left\{ \operatorname{cosec}^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k(l-1)(l-2) \dots (l-k)}{(2l-3)(2l-5) \dots (2l-2k-1)} \operatorname{cosec}^{2l-2k-1} x \right\} \quad \text{TI (242)}$$

$$\begin{aligned} 2. \quad \int \frac{dx}{\sin^{2l+1} x} &= -\frac{\cos x}{2l} \left\{ \operatorname{cosec}^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3) \dots (2l-2k+1)}{2^k(l-1)(l-2) \dots (l-k)} \operatorname{cosec}^{2l-2k} x \right\} \\ &+ \frac{(2l-1)!!}{2^l l!} \ln \tan \frac{x}{2} \end{aligned} \quad \text{TI (243)}$$

2.516

$$\begin{aligned} 1. \quad \int \frac{\sin^p x \, dx}{\cos^{2n+1} x} &= \frac{\sin^{p+1} x}{2n} \left\{ \sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k(n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right\} \\ &+ \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\sin^p x}{\cos x} \, dx \end{aligned}$$

This formula is applicable for arbitrary real p . For $n = 0$ and p a natural number, we have

$$2. \quad \int \frac{\sin^{2l+1} x \, dx}{\cos x} = - \sum_{k=1}^l \frac{\sin^{2k} x}{2k} - \ln \cos x$$

$$3. \quad \int \frac{\sin^{2l} x \, dx}{\cos x} = - \sum_{k=1}^l \frac{\sin^{2k-1} x}{2k-1} + \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

2.517

$$1. \quad \int \frac{dx}{\sin^{2m+1} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+2) \sin^{2m-2k+2} x} + \ln \tan x$$

$$2. \quad \int \frac{dx}{\sin^{2m} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+1) \sin^{2m-2k+1} x} + \ln \tan \left(\frac{\pi}{4} - \frac{x}{2} \right)$$

2.518

$$1. \quad \int \frac{\sin^p x}{\cos^2 x} dx = \frac{\sin^{p-1} x}{\cos x} - (p-1) \int \sin^{p-2} x \, dx$$

$$2. \quad \int \frac{\cos^p x \, dx}{\sin^{2n} x} = \frac{\cos^{p+1} x}{2n-1} \left\{ \operatorname{cosec}^{2n-1} x \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x \right\} \\ + \frac{(2n-p-2)(2n-p-4) \dots (2-p)(-p)}{(2n-1)!!} \int \cos^p x \, dx$$

This formula is applicable for arbitrary real p . For $\int \cos^p x \, dx$ where p is a natural number, see **2.512** 2, 3 and **2.513** 3, 4. If $n = 0$ and p is a negative integer, we have for this integral:

2.519

$$1. \quad \int \frac{dx}{\cos^{2l} x} = \frac{\sin x}{2l-1} \left\{ \sec^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k(l-1)(l-2) \dots (l-k)}{(2l-3)(2l-5) \dots (2l-2k-1)} \sec^{2l-2k-1} x \right\} \quad \text{TI (240)}$$

$$2. \quad \int \frac{dx}{\cos^{2l+1} x} = \frac{\sin x}{2l} \left\{ \sec^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3) \dots (2l-2k+1)}{2^k(l-1)(l-2) \dots (l-k)} \sec^{2l-2k} x \right\} \\ + \frac{(2l-1)!!}{2^l l!} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \quad \text{TI (241)}$$

2.521

$$1. \quad \int \frac{\cos^p x \, dx}{\sin^{2n+1} x} = - \frac{\cos^{p+1} x}{2n} \left\{ \operatorname{cosec}^{2n} x \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k(n-1)(n-2) \dots (n-k)} \operatorname{cosec}^{2n-2k} x \right\} \\ + \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n \cdot n!} \int \frac{\cos^p x}{\sin x} dx$$

This formula is applicable for arbitrary real p . For $n = 0$ and p a natural number, we have

$$2. \quad \int \frac{\cos^{2l+1} x \, dx}{\sin x} = \sum_{k=1}^l \frac{\cos^{2k} x}{2k} + \ln \sin x$$

$$3. \quad \int \frac{\cos^{2l} x \, dx}{\sin x} = \sum_{k=1}^l \frac{\cos^{2k-1} x}{2k-1} + \ln \tan \frac{x}{2}$$

2.522

$$1. \quad \int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^m \frac{1}{(2m-2k+2) \cos^{2m-2k+2} x} + \ln \tan x$$

$$2. \quad \int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^m \frac{1}{(2m-2k+1) \cos^{2m-2k+1} x} + \ln \tan \frac{x}{2} \quad \text{GW (331)(15)}$$

$$\mathbf{2.523} \quad \int \frac{\cos^m x}{\sin^2 x} dx = -\frac{\cos^{m-1} x}{\sin x} - (m-1) \int \cos^{m-2} x \, dx$$

2.524 In formulas **2.524** 1 and **2.524** 2, $s = 1$ for m odd and $m < 2n + 1$; in other cases, $s = 0$.

$$1. \quad \int \frac{\sin^{2n+1} x}{\cos^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^{k+1} \binom{n}{k} \frac{\cos^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m+1}{2}} \binom{n}{\frac{m-1}{2}} \ln \cos x$$

GU (331)(11d)

$$2. \quad \int \frac{\cos^{2n+1} x}{\sin^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^k \binom{n}{k} \frac{\sin^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln \sin x$$

2.525

$$1. \quad \int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\tan^{2k-2m+1} x}{2k-2m+1} \quad \text{TI (267)}$$

$$2. \quad \int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\tan^{2k-2m} x}{2k-2m} + \binom{m+n}{m} \ln \tan x$$

TI (268), GU (331)(15f)

2.526

$$1. \quad \int \frac{dx}{\sin x} = \ln \tan \frac{x}{2}$$

$$2. \quad \int \frac{dx}{\sin^2 x} = -\cot x$$

$$3. \quad \int \frac{dx}{\sin^3 x} = -\frac{1}{2} \frac{\cos x}{\sin^2 x} + \frac{1}{2} \ln \tan \frac{x}{2}$$

$$4. \quad \int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3 \sin^3 x} - \frac{2}{3} \cot x = -\frac{1}{3} \cot^3 x - \cot x$$

$$5. \quad \int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4 \sin^4 x} - \frac{3}{8} \frac{\cos x}{\sin^2 x} + \frac{3}{8} \ln \tan \frac{x}{2}$$

6.
$$\begin{aligned}\int \frac{dx}{\sin^6 x} &= -\frac{\cos x}{5 \sin^5 x} - \frac{4}{15} \cot^3 x - \frac{4}{5} \cot x \\ &= -\frac{1}{5} \cot^5 x - \frac{2}{3} \cot^3 x - \cot x\end{aligned}$$
7.
$$\int \frac{dx}{\sin^7 x} = -\frac{\cos x}{6 \sin^2 x} \left(\frac{1}{\sin^4 x} + \frac{5}{4 \sin^2 x} + \frac{15}{8} \right) + \frac{5}{16} \ln \tan \frac{x}{2}$$
8.
$$\int \frac{dx}{\sin^8 x} = -\left(\frac{1}{7} \cot^7 x + \frac{3}{5} \cot^5 x + \cot^3 x + \cot x \right)$$
9.
$$\int \frac{dx}{\cos x} = \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) = \ln \cot \left(\frac{\pi}{4} - \frac{x}{2} \right) = \ln \sqrt{\frac{1 + \sin x}{1 - \sin x}}$$
10.
$$\int \frac{dx}{\cos^2 x} = \tan x$$
11.
$$\int \frac{dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$
12.
$$\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3 \cos^3 x} + \frac{2}{3} \tan x = \frac{1}{3} \tan^3 x + \tan x$$
13.
$$\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4 \cos^4 x} + \frac{3}{8} \frac{\sin x}{\cos^2 x} + \frac{3}{8} \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$
14.
$$\int \frac{dx}{\cos^6 x} = \frac{\sin x}{5 \cos^5 x} + \frac{4}{15} \tan^3 x + \frac{4}{5} \tan x = \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x$$
15.
$$\int \frac{dx}{\cos^7 x} = \frac{\sin x}{6 \cos^6 x} + \frac{5 \sin x}{24 \cos^4 x} + \frac{5 \sin x}{16 \cos^2 x} + \frac{5}{16} \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$
16.
$$\int \frac{dx}{\cos^8 x} = \frac{1}{7} \tan^7 x + \frac{3}{5} \tan^5 x + \tan^3 x + \tan x$$
17.
$$\int \frac{\sin x}{\cos x} dx = -\ln \cos x$$
18.
$$\int \frac{\sin^2 x}{\cos x} dx = -\sin x + \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$
19.
$$\int \frac{\sin^3 x}{\cos x} dx = -\frac{\sin^2 x}{2} - \ln \cos x = \frac{1}{2} \cos^2 x - \ln \cos x$$
20.
$$\int \frac{\sin^4 x}{\cos x} dx = -\frac{1}{3} \sin^3 x - \sin x + \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$
21.
$$\int \frac{\sin^2 x dx}{\cos^2 x} = \frac{1}{\cos x}$$
22.
$$\int \frac{\sin^2 x dx}{\cos^2 x} = \tan x - x$$
23.
$$\int \frac{\sin^3 x dx}{\cos^2 x} = \cos x + \frac{1}{\cos x}$$
24.
$$\int \frac{\sin^4 x dx}{\cos^2 x} = \tan x + \frac{1}{2} \sin x \cos x - \frac{3}{2} x$$

25. $\int \frac{\sin x \, dx}{\cos^3 x} = \frac{1}{2 \cos^2 x} = \frac{1}{2} \tan^2 x$
26. $\int \frac{\sin^2 x \, dx}{\cos^3 x} = \frac{\sin x}{2 \cos^2 x} - \frac{1}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$
27. $\int \frac{\sin^3 x \, dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \ln \cos x$
28. $\int \frac{\sin^4 x \, dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2} \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$
29. $\int \frac{\sin x \, dx}{\cos^4 x} = \frac{1}{3 \cos^3 x}$
30. $\int \frac{\sin^2 x \, dx}{\cos^4 x} = \frac{1}{3} \tan^3 x$
31. $\int \frac{\sin^3 x \, dx}{\cos^4 x} = -\frac{1}{\cos x} + \frac{1}{3 \cos^3 x}$
32. $\int \frac{\sin^4 x \, dx}{\cos^4 x} = \frac{1}{3} \tan^3 x - \tan x + x$
33. $\int \frac{\cos x \, dx}{\sin x} = \ln \sin x$
34. $\int \frac{\cos^2 x \, dx}{\sin x} = \cos x + \ln \tan \frac{x}{2}$
35. $\int \frac{\cos^3 x \, dx}{\sin x} = \frac{\cos^2 x}{2} + \ln \sin x$
36. $\int \frac{\cos^4 x \, dx}{\sin x} = \frac{1}{3} \cos^3 x + \cos x + \ln \tan \left(\frac{x}{2} \right)$
37. $\int \frac{\cos x}{\sin^2 x} \, dx = -\frac{1}{\sin x}$
38. $\int \frac{\cos^2 x}{\sin^2 x} \, dx = -\cot x - x$
39. $\int \frac{\cos^3 x}{\sin^2 x} \, dx = -\sin x - \frac{1}{\sin x}$
40. $\int \frac{\cos^4 x}{\sin^2 x} \, dx = -\cot x - \frac{1}{2} \sin x \cos x - \frac{3}{2} x$
41. $\int \frac{\cos x}{\sin^3 x} \, dx = -\frac{1}{2 \sin^2 x}$
42. $\int \frac{\cos^2 x}{\sin^3 x} \, dx = -\frac{\cos x}{2 \sin^2 x} - \frac{1}{2} \ln \tan \frac{x}{2}$
43. $\int \frac{\cos^3 x}{\sin^3 x} \, dx = -\frac{1}{2 \sin^2 x} - \ln \sin x$
44. $\int \frac{\cos^4 x}{\sin^3 x} \, dx = -\frac{1}{2} \frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2} \ln \tan \frac{x}{2}$

45. $\int \frac{\cos x}{\sin^4 x} dx = -\frac{1}{3\sin^3 x}$
46. $\int \frac{\cos^2 x}{\sin^4 x} dx = -\frac{1}{3} \cot^3 x$
47. $\int \frac{\cos^3 x}{\sin^4 x} dx = \frac{1}{\sin x} - \frac{1}{3\sin^3 x}$
48. $\int \frac{\cos^4 x}{\sin^4 x} dx = -\frac{1}{3} \cot^3 x + \cot x + x$
49. $\int \frac{dx}{\sin x \cos x} = \ln \tan x$
50. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \tan \frac{x}{2}$
51. $\int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2\cos^2 x} + \ln \tan x$
52. $\int \frac{dx}{\sin x \cos^4 x} = \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \ln \tan \frac{x}{2}$
53. $\int \frac{dx}{\sin^2 x \cos x} = \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) - \operatorname{cosec} x$
54. $\int \frac{dx}{\sin^2 x \cos^2 x} = -2 \cot 2x$
55. $\int \frac{dx}{\sin^2 x \cos^3 x} = \left(\frac{1}{2\cos^2 x} - \frac{3}{2} \right) \frac{1}{\sin x} + \frac{3}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$
56. $\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3\sin x \cos^3 x} - \frac{8}{3} \cot 2x$
57. $\int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2\sin^2 x} + \ln \tan x$
58. $\int \frac{dx}{\sin^3 x \cos^2 x} = -\frac{1}{\cos x} \left(\frac{1}{2\sin^2 x} - \frac{3}{2} \right) + \frac{3}{2} \ln \tan \frac{x}{2}$
59. $\int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2\cos 2x}{\sin^2 2x} + 2 \ln \tan x$
60. $\int \frac{dx}{\sin^3 x \cos^4 x} = \frac{2}{\cos x} + \frac{1}{3\cos^3 x} - \frac{\cos x}{2\sin^2 x} + \frac{5}{2} \ln \tan \frac{x}{2}$
61. $\int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$
62. $\int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3\cos x \sin^3 x} - \frac{8}{3} \cot 2x$
63. $\int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3\sin^3 x} + \frac{\sin x}{2\cos^2 x} + \frac{5}{2} \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$
64. $\int \frac{dx}{\sin^4 x \cos^4 x} = -8 \cot 2x - \frac{8}{3} \cot^3 2x$

2.527

$$1. \quad \int \tan^p x \, dx = \frac{\tan^{p-1} x}{p-1} - \int \tan^{p-2} x \, dx \quad [p \neq 1]$$

$$2. \quad \int \tan^{2n+1} x \, dx = \sum_{k=1}^n (-1)^{n+k} \binom{n}{k} \frac{1}{2k \cos^{2k} x} - (-1)^n \ln \cos x$$

$$= \sum_{k=1}^n \frac{(-1)^{k-1} \tan^{2n-2k+2} x}{2n-2k+2} - (-1)^n \ln \cos x$$

$$3. \quad \int \tan^{2n} x \, dx = \sum_{k=1}^n (-1)^{k-1} \frac{\tan^{2n-2k+1} x}{2n-2k+1} + (-1)^n x \quad \text{GU (331)(12)}$$

$$4. \quad \int \cot^p x \, dx = -\frac{\cot^{p-1} x}{p-1} - \int \cot^{p-2} x \, dx \quad [p \neq 1]$$

$$5. \quad \int \cot^{2n+1} x \, dx = \sum_{k=1}^n (-1)^{n+k+1} \binom{n}{k} \frac{1}{2k \sin^{2k} x} + (-1)^n \ln \sin x$$

$$= \sum_{k=1}^n (-1)^k \frac{\cot^{2n-2k+2} x}{2n-2k+2} + (-1)^n \ln \sin x$$

$$6. \quad \int \cot^{2n} x \, dx = \sum_{k=1}^n (-1)^k \frac{\cot^{2n-2k+1} x}{2n-2k+1} + (-1)^n x \quad \text{GU (331)(14)}$$

For special formulas for $p = 1, 2, 3, 4$, see **2.526** 17, **2.526** 33, **2.526** 22, **2.526** 38, **2.526** 27, **2.526** 43, **2.526** 32, and **2.526** 48.

2.53–2.54 Sines and cosines of multiple angles and of linear and more complicated functions of the argument

2.531

$$1. \quad \int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b)$$

$$2. \quad \int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b)$$

2.532

$$1. \quad \int \sin(ax+b) \sin(cx+d) \, dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} - \frac{\sin[(a+c)x+b+d]}{2(a+c)}$$

$$[a^2 \neq c^2]$$

$$2.^8 \quad \int \sin(ax+b) \cos(cx+d) \, dx = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)}$$

$$[a^2 \neq c^2]$$

$$3. \quad \int \cos(ax+b) \cos(cx+d) dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} + \frac{\sin[(a+c)x+b+d]}{2(a+c)} \\ [a^2 \neq c^2]$$

For $c = a$:

$$4. \quad \int \sin(ax+b) \sin(ax+d) dx = \frac{x}{2} \cos(b-d) - \frac{\sin(2ax+b+d)}{4a}$$

$$5. \quad \int \sin(ax+b) \cos(ax+d) dx = \frac{x}{2} \sin(b-d) - \frac{\cos(2ax+b+d)}{4a}$$

$$6. \quad \int \cos(ax+b) \cos(ax+d) dx = \frac{x}{2} \cos(b-d) + \frac{\sin(2ax+b+d)}{4a} \quad \text{GU (332)(3)}$$

2.533

$$1.^8 \quad \int \sin ax \cos bx dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \quad [a^2 \neq b^2]$$

$$2.^8 \quad \int \sin ax \sin bx \sin cx dx = -\frac{1}{4} \left\{ \frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} \right. \\ \left. + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right\}$$

PE (376)

$$3. \quad \int \sin ax \cos bx \cos cx dx = -\frac{1}{4} \left\{ \frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} \right. \\ \left. + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right\}$$

PE (378)

$$4. \quad \int \cos ax \sin bx \sin cx dx = \frac{1}{4} \left\{ \frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} \right. \\ \left. - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right\}$$

PE (379)

$$5. \quad \int \cos ax \cos bx \cos cx dx = \frac{1}{4} \left\{ \frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} \right. \\ \left. + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right\}$$

PE (377)

2.534

$$1. \quad \int \frac{\cos px + i \sin px}{\sin nx} dx = -2 \int \frac{z^{p+n-1}}{1 - z^{2n}} dz \quad [z = \cos x + i \sin x] \quad \text{Pe (374)}$$

$$2. \quad \int \frac{\cos px + i \sin px}{\cos nx} dx = -2i \int \frac{z^{p+n-1}}{1 - z^{2n}} dz \quad [z = \cos x + i \sin x] \quad \text{Pe (373)}$$

2.535

$$1. \quad \int \sin^p x \sin ax dx = \frac{1}{p+a} \left\{ -\sin^p x \cos ax + p \int \sin^{p-1} x \cos(a-1)x dx \right\} \quad \text{GU (332)(5a)}$$

$$2. \quad \int \sin^p x \sin(2n+1)x dx$$

$$= (2n+1) \left\{ \int \sin^{p+1} x dx + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2] [(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \right.$$

$$\left. \times \int \sin^{2k+p+1} x dx \right\}$$

TI (299)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^{k-1} \Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \cos(2n-2k+1)x \right. \right.$$

$$\left. + (-1)^k \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \sin(2n-2k)x \right]$$

$$\left. + \frac{(-1)^n \Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n+1} x dx \right\}$$

GU (332)(5c)

$$\begin{aligned}
3. \quad \int \sin^p x \sin 2nx \, dx &= 2n \left\{ \frac{\sin^{p+2} x}{p+2} \right. \\
&\quad \left. + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2 - 2^2)(4n^2 - 4^2) \dots [4n^2 - (2k)^2]}{(2k+1)!(2k+p+2)} \sin^{2k+p+2} x \right\} \\
&\qquad\qquad\qquad \text{TI (303)} \\
&= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k-1} \Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \cos(2n-2k)x \right. \\
&\quad \left. - \frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \sin(2n-2k-1)x \right\} \\
&\qquad\qquad\qquad [p \text{ is not equal to } -2, -4, \dots, -2n] \\
&\qquad\qquad\qquad \text{GU (332)(5c)}
\end{aligned}$$

2.536

$$\begin{aligned}
1. \quad \int \sin^p x \cos ax \, dx &= \frac{1}{p+1} \left\{ \sin^p x \sin ax - p \int \sin^{p-1} x \sin(a-1)x \, dx \right\} \qquad \text{GU (332)(6a)} \\
2. \quad \int \sin^p x \cos(2n+1)x \, dx \\
&= \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2] [(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k)!(2k+p+1)} \\
&\quad \times \sin^{2k+p+1} x \\
&\qquad\qquad\qquad \text{TI (301)} \\
&= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^k \Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k+1)x \right. \right. \\
&\quad \left. \left. + \frac{(-1)^k \Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k)x \right] \right. \\
&\quad \left. + \frac{(-1)^n \Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n} x \cos x \, dx \right\} \\
&\qquad\qquad\qquad [p \text{ is not equal to } -3, -5, \dots, -(2n+1)] \\
&\qquad\qquad\qquad \text{GU (332)(6c)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int \sin^p x \cos 2nx \, dx \\
&= \int \sin^p x \, dx + \sum_{k=1}^n (-1)^k \frac{4n^2 \cdot (4n^2 - 2^2) \cdots [4n^2 - (2k-2)^2]}{(2k)!} \int \sin^{2k+p} x \, dx
\end{aligned}$$

TI (300)

$$\begin{aligned}
&= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k)x \right. \right. \\
&\quad \left. \left. + \frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k-1)x \right] \right. \\
&\quad \left. + \frac{(-1)^n \Gamma\left(\frac{p}{2} - n + 1\right)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n} x \, dx \right\}
\end{aligned}$$

GU (332)(6c)

2.537

$$1. \quad \int \cos^p x \sin ax \, dx = \frac{1}{p+a} \left\{ -\cos^p x \cos ax + p \int \cos^{p-1} x \sin(a-1)x \, dx \right\}$$

GU (332)(7a)

$$\begin{aligned}
2. \quad & \int \cos^p x \sin(2n+1)x \, dx \\
&= (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} \right. \\
&\quad \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2] [(2n+1)^2 - 3^2] \cdots [(2n+1)^2 - (2k-1)^2]}{(2k)!(2k+p+1)} \cos^{2k+p+1} x \right\}
\end{aligned}$$

TI (295)

$$\begin{aligned}
&= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ - \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{2k+1} \Gamma(p-2k+1)} \cos^{p-k} x \cos(2n-k+1)x \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \sin(n+1)x \, dx \right\}
\end{aligned}$$

[p is not equal to $-3, -5, \dots, -(2n+1)$]
GU (332)(7b)a

$$\begin{aligned}
3. \quad \int \cos^p x \sin 2nx \, dx &= (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} \right. \\
&\quad \left. + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2 - 2^2)(4n^2 - 4^2) \dots [4n^2 - (2k)^2]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x \right\} \\
&\quad \text{TI (297)} \\
&= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ - \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \cos(2n-k)x \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \sin nx \, dx \right\} \\
&\quad [p \text{ is not equal to } -2, -4, \dots, -2n] \\
&\quad \text{GU (332)(7b)a}
\end{aligned}$$

2.538

$$\begin{aligned}
1. \quad \int \cos^p x \cos ax \, dx &= \frac{1}{p+a} \left\{ \cos^p x \sin ax + p \int \cos^{p-1} x \cos(a-1)x \, dx \right\} \\
&\quad \text{GU (332)(8a)} \\
2. \quad \int \cos^p x \cos(2n+1)x \, dx \\
&= (-1)^n (2n+1) \left\{ \int \cos^{p+1} x \, dx \right. \\
&\quad + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2] [(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \\
&\quad \times \int \cos^{2k+p+1} x \, dx \left. \right\} \\
&\quad \text{TI (293)} \\
&= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k+1)x \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \cos(n+1)x \, dx \right\} \\
&\quad \text{GU (332)(8b)a}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int \cos^p x \cos 2nx \, dx \\
&= (-1)^n \left\{ \int \cos^p x \, dx + \sum_{k=1}^n (-1)^k \frac{4n^2 [4n^2 - 2^2] \dots [4n^2 - (2k-2)^2]}{(2k)!} \int \cos^{2k+p} x \, dx \right\} \\
&\hspace{15em} \text{TI (294)} \\
&= \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2} + n + 1)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma(\frac{p}{2} + n - k)}{2^{k+1} \Gamma(p - k + 1)} \cos^{p-k} x \sin(2n - k)x \right. \\
&\quad \left. + \frac{\Gamma(\frac{p}{2} + 1)}{2^n \Gamma(p - n + 1)} \int \cos^{p-n} x \cos nx \, dx \right\} \\
&\hspace{15em} \text{GU (332)(8b)a}
\end{aligned}$$

2.539

$$\begin{aligned}
1. \quad & \int \frac{\sin(2n+1)x}{\sin x} \, dx = 2 \sum_{k=1}^n \frac{\sin 2kx}{2k} + x \\
2. \quad & \int \frac{\sin 2nx}{\sin x} \, dx = 2 \sum_{k=1}^n \frac{\sin(2k-1)x}{2k-1} \hspace{10em} \text{GU (332)(5e)} \\
3. \quad & \int \frac{\cos(2n+1)x}{\sin x} \, dx = 2 \sum_{k=1}^n \frac{\cos 2kx}{2k} + \ln \sin x \\
4. \quad & \int \frac{\cos 2nx}{\sin x} \, dx = 2 \sum_{k=1}^n \frac{\cos(2k-1)x}{2k-1} + \ln \tan \frac{x}{2} \hspace{10em} \text{GI (332)(6e)} \\
5. \quad & \int \frac{\sin(2n+1)x}{\cos x} \, dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos 2kx}{2k} + (-1)^{n+1} \ln \cos x \\
6. \quad & \int \frac{\sin 2nx}{\cos x} \, dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos(2k-1)x}{2k-1} \hspace{10em} \text{GU (332)(7d)} \\
7. \quad & \int \frac{\cos(2n+1)x}{\cos x} \, dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin 2kx}{2k} + (-1)^n x \\
8. \quad & \int \frac{\cos 2nx}{\cos x} \, dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin(2k-1)x}{2k-1} + (-1)^n \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right). \hspace{10em} \text{GU (332)(8d)}
\end{aligned}$$

2.541

$$\begin{aligned}
1. \quad & \int \sin(n+1)x \sin^{n-1} x \, dx = \frac{1}{n} \sin^n x \sin nx \hspace{10em} \text{BI (71)(1)a} \\
2. \quad & \int \sin(n+1)x \cos^{n-1} x \, dx = -\frac{1}{n} \cos^n x \cos nx \hspace{10em} \text{BI (71)(2)a}
\end{aligned}$$

$$3. \quad \int \cos(n+1)x \sin^{n-1} x \, dx = \frac{1}{n} \sin^n x \cos nx \quad \text{BI (71)(3)a}$$

$$4. \quad \int \cos(n+1)x \cos^{n-1} x \, dx = \frac{1}{n} \cos^n x \sin nx \quad \text{BI (71)(4)a}$$

$$5. \quad \int \sin \left[(n+1) \left(\frac{\pi}{2} - x \right) \right] \sin^{n-1} x \, dx = \frac{1}{n} \sin^n x \cos n \left(\frac{\pi}{2} - x \right) \quad \text{BI (71)(5)a}$$

$$6. \quad \int \cos \left[(n+1) \left(\frac{\pi}{2} - x \right) \right] \sin^{n-1} x \, dx = -\frac{1}{n} \sin^n x \sin n \left(\frac{\pi}{2} - x \right) \quad \text{BI (71)(6)a}$$

2.542

$$1. \quad \int \frac{\sin 2x}{\sin^n x} \, dx = -\frac{2}{(n-2) \sin^{n-2} x}$$

For $n = 2$:

$$2. \quad \int \frac{\sin 2x}{\sin^2 x} \, dx = 2 \ln \sin x$$

2.543

$$1. \quad \int \frac{\sin 2x \, dx}{\cos^n x} = \frac{2}{(n-2) \cos^{n-2} x}$$

For $n = 2$:

$$2. \quad \int \frac{\sin 2x}{\cos^2 x} \, dx = -2 \ln \cos x$$

2.544

$$1. \quad \int \frac{\cos 2x \, dx}{\sin x} = 2 \cos x + \ln \tan \frac{x}{2}$$

$$2. \quad \int \frac{\cos 2x \, dx}{\sin^2 x} = -\cot x - 2x$$

$$3. \quad \int \frac{\cos 2x \, dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} - \frac{3}{2} \ln \tan \frac{x}{2}$$

$$4. \quad \int \frac{\cos 2x \, dx}{\cos x} = 2 \sin x - \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$5. \quad \int \frac{\cos 2x \, dx}{\cos^2 x} = 2x - \tan x$$

$$6. \quad \int \frac{\cos 2x \, dx}{\cos^3 x} = -\frac{\sin x}{2 \cos^2 x} + \frac{3}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$7. \quad \int \frac{\sin 3x \, dx}{\sin x} = x + \sin 2x$$

$$8. \quad \int \frac{\sin 3x}{\sin^2 x} \, dx = 3 \ln \tan \frac{x}{2} + 4 \cos x$$

$$9. \quad \int \frac{\sin 3x}{\sin^3 x} \, dx = -3 \cot x - 4x$$

2.545

$$1. \quad \int \frac{\sin 3x}{\cos^n x} dx = \frac{4}{(n-3) \cos^{n-3} x} - \frac{1}{(n-1) \cos^{n-1} x}$$

For $n = 1$ and $n = 3$:

$$2. \quad \int \frac{\sin 3x}{\cos x} dx = 2 \sin^2 x + \ln \cos x$$

$$3. \quad \int \frac{\sin 3x}{\cos^3 x} dx = -\frac{1}{2 \cos^2 x} - 4 \ln \cos x$$

2.546

$$1. \quad \int \frac{\cos 3x}{\sin^n x} dx = \frac{4}{(n-3) \sin^{n-3} x} - \frac{1}{(n-1) \sin^{n-1} x}$$

For $n = 1$ and $n = 3$:

$$2. \quad \int \frac{\cos 3x}{\sin x} dx = -2 \sin^2 x + \ln \sin x$$

$$3. \quad \int \frac{\cos 3x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - 4 \ln \sin x$$

2.547

$$1. \quad \int \frac{\sin nx}{\cos^p x} dx = 2 \int \frac{\sin(n-1)x dx}{\cos^{p-1} x} - \int \frac{\sin(n-2)x dx}{\cos^p x}$$

$$2. \quad \int \frac{\cos 3x}{\cos x} dx = \sin 2x - x$$

$$3. \quad \int \frac{\cos 3x}{\cos^2 x} dx = 4 \sin x - 3 \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$4. \quad \int \frac{\cos 3x}{\cos^3 x} dx = 4x - 3 \tan x$$

2.548

$$1. \quad \int \frac{\sin^m x dx}{\sin(2n+1)x} = \frac{1}{2n+1} \sum_{k=0}^{2n} (-1)^{n+k} \cos^m \left[\frac{2k+1}{2(2n+1)} \pi \right] \ln \frac{\sin \left[\frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2} \right]}{\sin \left[\frac{k+n+1}{(2n+1)} \pi - \frac{x}{2} \right]}$$

[m a natural number $\leq 2n$] TI (378)

$$2. \quad \int \frac{\sin^{2m} x dx}{\sin 2nx} = \frac{(-1)^n}{2n} \left\{ \ln \cos x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left(\cos^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\}$$

[m a natural number $\leq n$] TI (379)

$$3. \quad \int \frac{\sin^{2m+1} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[\tan \left(\frac{n+k}{4n} \pi - \frac{x}{2} \right) \tan \left(\frac{n-k}{4n} \pi - \frac{x}{2} \right) \right] \right\}$$

[m a natural number $< n$]

$$4. \quad \int \frac{\sin^{2m} x \, dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) + \sum_{k=1}^n (-1)^k \right. \\ \left. \times \cos^{2m} \frac{k\pi}{2n+1} \ln \left[\tan \left(\frac{2n+2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \tan \left(\frac{2n-2k+1}{2(2n+1)} \pi - \frac{x}{2} \right) \right] \right\} \\ [m \text{ a natural number} \leq n] \quad \text{TI (381)}$$

$$5. \quad \int \frac{\sin^{2m+1} x \, dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \cos x + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left(\cos^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\} \\ [m \text{ a natural number} \leq n] \quad \text{TI (382)a}$$

$$6. \quad \int \frac{\sin^m x \, dx}{\cos 2nx} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^{n+k} \cos^m \left[\frac{2k+1}{4n} \pi \right] \ln \frac{\sin \left[\frac{2k-2n+1}{8n} \pi + \frac{x}{2} \right]}{\sin \left[\frac{2k+2n+1}{8n} \pi - \frac{x}{2} \right]} \\ [m \text{ a natural number} < 2n] \quad \text{TI (377)}$$

$$7. \quad \int \frac{\cos^{2m+1} x \, dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln \sin x + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left(\sin^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\} \\ [m \text{ a natural number} \leq n] \quad \text{TI (376)}$$

$$8. \quad \int \frac{\cos^{2m} x \, dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln \tan \frac{x}{2} \right. \\ \left. + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left[\tan \left(\frac{x}{2} + \frac{k\pi}{4n+2} \right) \tan \left(\frac{x}{2} - \frac{k\pi}{4n+2} \right) \right] \right\} \\ [m \text{ a natural number} \leq n] \quad \text{TI (375)}$$

$$9. \quad \int \frac{\cos^{2m+1} x \, dx}{\sin 2nx} = \frac{1}{2n} \left\{ \ln \tan \frac{x}{2} + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[\tan \left(\frac{x}{2} + \frac{k\pi}{4} \right) \tan \left(\frac{x}{2} - \frac{k\pi}{4n} \right) \right] \right\} \\ [m \text{ a natural number} < n] \quad \text{TI (374)}$$

$$10. \quad \int \frac{\cos^{2m} x \, dx}{\sin 2nx} = \frac{1}{2n} \left\{ \ln \sin x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left(\sin^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\} \\ [m \text{ a natural number} \leq n] \quad \text{TI (373)}$$

$$11. \quad \int \frac{\cos^m x \, dx}{\cos nx} = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{2k+1}{2n} \pi \ln \frac{\sin \left[\frac{2k+1}{4n} \pi + \frac{x}{2} \right]}{\sin \left[\frac{2k+1}{4n} \pi - \frac{x}{2} \right]} \\ [m \text{ is a natural number} \leq n] \quad \text{TI (372)}$$

2.549

$$1. \quad \int \sin x^2 \, dx = \sqrt{\frac{\pi}{2}} S(x)$$

$$2. \quad \int \cos x^2 dx = \sqrt{\frac{\pi}{2}} C(x)$$

$$3.^{11} \quad \int \sin(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{ac - b^2}{a} S\left(\frac{ax + b}{\sqrt{a}}\right) + \sin \frac{ac - b^2}{a} C\left(\frac{ax + b}{\sqrt{a}}\right) \right\}$$

$$[a > 0]$$

$$4.^{11} \quad \int \cos(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{ac - b^2}{a} C\left(\frac{ax + b}{\sqrt{a}}\right) - \sin \frac{ac - b^2}{a} S\left(\frac{ax + b}{\sqrt{a}}\right) \right\}$$

$$[a > 0]$$

$$5. \quad \int \sin \ln x dx = \frac{x}{2} (\sin \ln x - \cos \ln x) \quad \text{PE (444)}$$

$$6. \quad \int \cos \ln x dx = \frac{x}{2} (\sin \ln x + \cos \ln x) \quad \text{PE (445)}$$

2.55–2.56 Rational functions of the sine and cosine

2.551

$$1. \quad \int \frac{A + B \sin x}{(a + b \sin x)^n} dx = \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{(Ab - aB) \cos x}{(a + b \sin x)^{n-1}} \right. \\ \left. + \int \frac{(Aa - Bb)(n-1) + (aB - bA)(n-2) \sin x}{(a + b \sin x)^{n-1}} dx \right]$$

TI (358)a

For $n = 1$:

$$2. \quad \int \frac{A + B \sin x}{a + b \sin x} dx = \frac{B}{b} x + \frac{Ab - aB}{b} \int \frac{dx}{a + b \sin x} \quad (\text{see 2.551 3}) \quad \text{TI (342)}$$

$$3. \quad \int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{a^2 - b^2}} \arctan \frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \quad [a^2 > b^2]$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \tan \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \tan \frac{x}{2} + b + \sqrt{b^2 - a^2}} \quad [a^2 < b^2]$$

2.552

$$1. \quad \int \frac{A + B \cos x}{(a + b \sin x)^n} dx = -\frac{B}{(n-1)b(a + b \sin x)^{n-1}} + A \int \frac{dx}{(a + b \sin x)^n}$$

$$(\text{see 2.552 3}) \quad \text{TI (361)}$$

For $n = 1$:

$$2. \quad \int \frac{A + B \cos x}{a + b \sin x} dx = \frac{B}{b} \ln(a + b \sin x) + A \int \frac{dx}{a + b \sin x}$$

$$(\text{see 2.551 3}) \quad \text{TI (344)}$$

$$\begin{aligned}
 3. \quad \int \frac{dx}{(a + b \sin x)^n} &= \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{b \cos x}{(a + b \sin x)^{n-1}} \right. \\
 &\quad \left. + \int \frac{(n-1)a - (n-2)b \sin x}{(a + b \sin x)^{n-1}} dx \right] \quad (\text{see } \mathbf{2.551} \ 1)
 \end{aligned}$$

TI (359)

2.553

$$1. \quad \int \frac{A + B \sin x}{(a + b \cos x)^n} dx = \frac{B}{(n-1)b(a + b \cos x)^{n-1}} + A \int \frac{dx}{(a + b \cos x)^n}$$

(see **2.554** 3) TI (355)

For $n = 1$:

$$2. \quad \int \frac{A + B \sin x}{a + b \cos x} dx = -\frac{B}{b} \ln(a + b \cos x) + A \int \frac{dx}{a + b \cos x}$$

(see **2.553** 3*) TI (343)

$$\begin{aligned}
 3.^* \quad \int \frac{dx}{a + b \cos x} &= \frac{2}{\sqrt{a^2 - b^2}} \arctan \left(\frac{(a-b) \tan \left(\frac{x}{2} \right)}{\sqrt{a^2 - b^2}} \right) \quad [a^2 > b^2] \\
 &= \frac{2}{\sqrt{a^2 - b^2}} \ln \left| \frac{(b-a) \tan \left(\frac{x}{2} \right) + \sqrt{b^2 - a^2}}{(b-a) \tan \left(\frac{x}{2} \right) - \sqrt{b^2 - a^2}} \right| \quad [b^2 > a^2] \\
 &= \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arctanh} \left(\frac{(a-b) \tan \left(\frac{x}{2} \right)}{\sqrt{b^2 - a^2}} \right) \quad [b^2 > a^2, \quad |(b-a) \tan \left(\frac{x}{2} \right)| < \sqrt{b^2 - a^2}] \\
 &= \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arccoth} \left(\frac{(a-b) \tan \left(\frac{x}{2} \right)}{\sqrt{b^2 - a^2}} \right) \quad [b^2 > a^2, \quad |(b-a) \tan \left(\frac{x}{2} \right)| > \sqrt{b^2 - a^2}]
 \end{aligned}$$

(compare with **2.551** 3)

2.554

$$\begin{aligned}
 1. \quad \int \frac{A + B \cos x}{(a + b \cos x)^n} dx &= \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{(aB - Ab) \sin x}{(a + b \cos x)^{n-1}} \right. \\
 &\quad \left. + \int \frac{(Aa - bB)(n-1) + (n-2)(aB - bA) \cos x}{(a + b \cos x)^{n-1}} dx \right]
 \end{aligned}$$

TI (353)

For $n = 1$:

$$2. \quad \int \frac{A + B \cos x}{a + b \cos x} dx = \frac{B}{b}x + \frac{Ab - aB}{b} \int \frac{dx}{a + b \cos x} \quad (\text{see } \mathbf{2.553} \ 3) \quad \text{TI (341)}$$

$$3. \quad \int \frac{dx}{(a + b \cos x)^n} = -\frac{1}{(n-1)(a^2 - b^2)} \left\{ \frac{b \sin x}{(a + b \cos x)^{n-1}} - \int \frac{(n-1)a - (n-2)b \cos x}{(a + b \cos x)^{n-1}} dx \right\} \quad (\text{see } \mathbf{2.554} \ 1) \quad \text{TI (354)}$$

In integrating the functions in formulas **2.551** 3 and **2.553** 3, we may not take the integration over points at which the integrand becomes infinite, that is, over the points $x = \arcsin\left(-\frac{a}{b}\right)$ in formula **2.551** 3 or over the points $x = \arccos\left(-\frac{a}{b}\right)$ in formula **2.553** 3.

2.555 Formulas **2.551** 3 and **2.553** 3 are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

$$1. \quad \int \frac{A + B \sin x}{(1 \pm \sin x)^n} dx = -\frac{1}{2^{n-1}} \left\{ 2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\tan^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1} \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\tan^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1} \right\} \quad \text{TI (361)a}$$

$$2. \quad \int \frac{A + B \cos x}{(1 \pm \cos x)^n} dx = \frac{1}{2^{n-1}} \left\{ 2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\tan^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1} \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\tan^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1} \right\} \quad \text{TI (356)}$$

For $n = 1$:

$$3.^{11} \quad \int \frac{A + B \sin x}{1 \pm \sin x} dx = \pm Bx + (B \mp A) \tan\left(\frac{\pi}{4} \mp \frac{x}{2}\right) \quad \text{TI (250)}$$

$$4. \quad \int \frac{A + B \cos x}{1 \pm \cos x} dx = \pm Bx \pm (A \mp B) \tan\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right] \quad \text{TI (248)}$$

2.556

$$1. \quad \int \frac{(1 - a^2) dx}{1 - 2a \cos x + a^2} = 2 \arctan\left(\frac{1+a}{1-a} \tan \frac{x}{2}\right) \quad [0 < a < 1, \quad |x| < \pi] \quad \text{FI II 93}$$

$$2. \quad \int \frac{(1 - a \cos x) dx}{1 - 2a \cos x + a^2} = \frac{x}{2} + \arctan\left(\frac{1+a}{1-a} \tan \frac{x}{2}\right) \quad [0 < a < 1, \quad |x| < \pi] \quad \text{FI II 93}$$

2.557

$$1. \quad \int \frac{dx}{(a \cos x + b \sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}} \int \frac{dx}{\sin^n\left(x + \arctan \frac{a}{b}\right)} \quad (\text{see } \mathbf{2.515}) \quad \text{MZ 173a}$$

$$\begin{aligned}
2.^6 \quad & \int \frac{\sin x \, dx}{a \sin x + b \cos x} = \frac{ax - b \ln \sin \left(x + \arctan \frac{b}{a}\right)}{a^2 + b^2} \\
3. \quad & \int \frac{\cos x \, dx}{a \cos x + b \sin x} = \frac{ax + b \ln \sin \left(x + \arctan \frac{a}{b}\right)}{a^2 + b^2} \quad \text{MZ 174a} \\
4. \quad & \int \frac{dx}{a \cos x + b \sin x} = \frac{\ln \tan \left[\frac{1}{2} \left(x + \arctan \frac{a}{b}\right)\right]}{\sqrt{a^2 + b^2}} \\
5. \quad & \int \frac{dx}{(a \cos x + b \sin x)^2} = -\frac{\cot \left(x + \arctan \frac{a}{b}\right)}{a^2 + b^2} = +\frac{1}{a^2 + b^2} \cdot \frac{a \sin x - b \cos x}{a \cos x + b \sin x} \quad \text{MZ 174a}
\end{aligned}$$

2.558

$$\begin{aligned}
1. \quad & \int \frac{A + B \cos x + C \sin x}{(a + b \cos x + c \sin x)^n} dx \\
& = \frac{(Bc - Cb) + (Ac - Ca) \cos x - (Ab - Ba) \sin x}{(n-1)(a^2 - b^2 - c^2)(a + b \cos x + c \sin x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 - c^2)} \\
& \quad \times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba) \cos x - (Ac - Ca) \sin x]}{(a + b \cos x + c \sin x)^{n-1}} dx \\
& \quad \quad \quad [n \neq 1, \quad a^2 \neq b^2 + c^2] \\
& = \frac{Cb - Bc + Ca \cos x - Ba \sin x}{(n-1)a(a + b \cos x + c \sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2}\right)(-c \cos x + b \sin x) \\
& \quad \times \frac{(n-1)!}{(2n-1)!!} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k} \cdot \frac{1}{(a + b \cos x + c \sin x)^{n-k}} \\
& \quad \quad \quad [n \neq 1, \quad a^2 = b^2 + c^2]
\end{aligned}$$

For $n = 1$:

$$\begin{aligned}
2.^{11} \quad & \int \frac{A + B \cos x + C \sin x}{a + b \cos x + c \sin x} dx = \frac{Bc - Cb}{b^2 + c^2} \ln(a + b \cos x + c \sin x) + \frac{Bb + Cc}{b^2 + c^2} x \\
& \quad + \left(A - \frac{Bb + Cc}{b^2 + c^2} a\right) \int \frac{dx}{a + b \cos x + c \sin x} \quad (\text{see } \mathbf{2.558} \ 4) \\
& \quad \quad \quad \text{GU (331)(18)}
\end{aligned}$$

$$3. \quad \int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x - \alpha)}{[a + r \cos(x - \alpha)]^n},$$

where $b = r \cos \alpha$, $c = r \sin \alpha$ (see **2.554** 3)

$$\begin{aligned}
4. \quad & \int \frac{dx}{a + b \cos x + c \sin x} \\
& = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \arctan \frac{(a-b) \tan \frac{x}{2} + c}{\sqrt{a^2 - b^2 - c^2}} \quad [a^2 > b^2 + c^2] \quad \text{TI (253), FI II 94} \\
& = \frac{1}{\sqrt{b^2 + c^2 - a^2}} \ln \frac{(a-b) \tan \frac{x}{2} + c - \sqrt{b^2 + c^2 - a^2}}{(a-b) \tan \frac{x}{2} + c + \sqrt{b^2 + c^2 - a^2}} \quad [a^2 < b^2 + c^2] \quad \text{TI (253)a} \\
& = \frac{1}{c} \ln \left(a + c \cdot \tan \frac{x}{2}\right) \quad [a = b] \\
& = \frac{-2}{c + (a-b) \tan \frac{x}{2}} \quad [a^2 = b^2 + c^2] \quad \text{TI (253)a}
\end{aligned}$$

2.559

1.
$$\int \frac{dx}{[a(1 + \cos x) + c \sin x]^2} = \frac{1}{c^3} \left[\frac{c(a \sin x - c \cos x)}{a(1 + \cos x) + c \sin x} - a \ln \left(a + c \tan \frac{x}{2} \right) \right]$$
2.
$$\int \frac{A + B \cos x + C \sin x}{(a_1 + b_1 \cos x + c_1 \sin x)(a_2 + b_2 \cos x + c_2 \sin x)} dx$$

$$= A_0 \ln \frac{a_1 + b_1 \cos x + c_1 \sin x}{a_2 + b_2 \cos x + c_2 \sin x} + A_1 \int \frac{dx}{a_1 + b_1 \cos x + c_1 \sin x} + A_2 \int \frac{dx}{a_2 + b_2 \cos x + c_2 \sin x}$$

(see **2.558 4**) GU (331)(19)

where

$$A_0 = \frac{\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \quad A_1 = \frac{\begin{vmatrix} B & C \\ b_1 & c_1 \end{vmatrix} \begin{vmatrix} A & C \\ a_1 & c_1 \end{vmatrix} \begin{vmatrix} B & A \\ b_1 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} \begin{vmatrix} A & B \\ a_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \quad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \right]$$

3.
$$\int \frac{A \cos^2 x + 2B \sin x \cos x + C \sin^2 x}{a \cos^2 x + 2b \sin x \cos x + c \sin^2 x} dx$$

$$= \frac{1}{4b^2 + (a - c)^2} \left\{ [4Bb + (A - C)(a - c)]x + [(A - C)b - B(a - c)] \right.$$

$$\times \ln(a \cos^2 x + 2b \sin x \cos x + c \sin^2 x)$$

$$\left. + [2(A + C)b^2 - 2Bb(a + c) + (aC - Ac)(a - c)] f(x) \right\}$$

where

GU (331)(24)

$$f(x) = \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \tan x + b - \sqrt{b^2 - ac}}{c \tan x + b + \sqrt{b^2 - ac}} \quad [b^2 > ac]$$

$$= \frac{1}{\sqrt{ac - b^2}} \arctan \frac{c \tan x + b}{\sqrt{ac - b^2}} \quad [b^2 < ac]$$

$$= -\frac{1}{c \tan x + b} \quad [b^2 = ac]$$

2.561

1.
$$\int \frac{(A + B \sin x) dx}{\sin x (a + b \sin x)} = \frac{A}{a} \ln \tan \frac{x}{2} + \frac{Ba - Ab}{a} \int \frac{dx}{a + b \sin x}$$

(see **2.551 3**) TI (348)

$$2. \quad \int \frac{(A + B \sin x) dx}{\sin x (a + b \cos x)} = \frac{A}{a^2 - b^2} \left\{ a \ln \tan \frac{x}{2} + b \ln \frac{a + b \cos x}{\sin x} \right\} + B \int \frac{dx}{a + b \cos x} \quad (\text{see 2.553 3})$$

TI (349)

For $a^2 = b^2 (= 1)$:

$$3. \quad \int \frac{(A + B \sin x) dx}{\sin x (a + b \cos x)} = \frac{A}{2} \left\{ \ln \tan \frac{x}{2} + \frac{1}{1 + \cos x} \right\} + B \tan \frac{x}{2}$$

$$4. \quad \int \frac{(A + B \sin x) dx}{\sin x (1 - \cos x)} = \frac{A}{2} \left\{ \ln \tan \frac{x}{2} - \frac{1}{1 - \cos x} \right\} - B \cot \frac{x}{2}$$

$$5. \quad \int \frac{(A + B \sin x) dx}{\cos x (a + b \sin x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) - (Ab - aB) \ln \frac{a + b \sin x}{\cos x} \right\}$$

TI (346)

For $a^2 = b^2 (= 1)$:

$$6. \quad \int \frac{(A + B \sin x) dx}{\cos x (1 \pm \sin x)} = \frac{A \pm B}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp B}{2(1 \pm \sin x)}$$

$$7. \quad \int \frac{(A + B \sin x) dx}{\cos x (a + b \cos x)} = \frac{A}{a} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{B}{a} \ln \frac{a + b \cos x}{\cos x} - \frac{Ab}{a} \int \frac{dx}{a + b \cos x} \quad (\text{see 2.553 3})$$

TI (351)a

$$8. \quad \int \frac{(A + B \cos x) dx}{\sin x (a + b \sin x)} = \frac{A}{a} \ln \tan \frac{x}{2} - \frac{B}{a} \ln \frac{a + b \sin x}{\sin x} - \frac{Ab}{a} \int \frac{dx}{a + b \sin x} \quad (\text{see 2.551 3})$$

TI (352)

$$9. \quad \int \frac{(A + B \cos x) dx}{\sin x (a + b \cos x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \tan \frac{x}{2} + (Ab - Ba) \ln \frac{a + b \cos x}{\sin x} \right\}$$

TI (345)

For $a^2 = b^2 (= 1)$:

$$10. \quad \int \frac{(A + B \cos x) dx}{\sin x (1 \pm \cos x)} = \pm \frac{A \mp B}{2(1 \pm \cos x)} + \frac{A \pm B}{2} \ln \tan \frac{x}{2}$$

$$11. \quad \int \frac{(A + B \cos x) dx}{\cos x (a + b \sin x)} = \frac{A}{a^2 - b^2} \left\{ a \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) - b \ln \frac{a + b \sin x}{\cos x} \right\} + B \int \frac{dx}{a + b \sin x} \quad (\text{see 2.551 3})$$

TI (350)

For $a^2 = b^2 (= 1)$:

$$12. \quad \int \frac{(A + B \sin x) dx}{\cos x (1 \pm \sin x)} = \frac{A \pm B}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp B}{2(1 \pm \sin x)}$$

$$13. \quad \int \frac{(A + B \cos x) dx}{\cos x (a + b \cos x)} = \frac{A}{a} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{Ba - Ab}{a} \int \frac{dx}{a + b \cos x}$$

(see 2.553 3)

TI (347)

2.562

$$\begin{aligned}
1. \quad \int \frac{dx}{a + b \sin^2 x} &= \frac{\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan \left(\sqrt{\frac{a+b}{a}} \tan x \right) && \left[\frac{b}{a} > -1 \right] \\
&= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh} \left(\sqrt{-\frac{a+b}{a}} \tan x \right) && \left[\frac{b}{a} < -1, \quad \sin^2 x < -\frac{a}{b} \right] \\
&= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth} \left(\sqrt{-\frac{a+b}{a}} \tan x \right) && \left[\frac{b}{a} < -1, \quad \sin^2 x > -\frac{a}{b} \right]
\end{aligned}$$

MZ 155

$$\begin{aligned}
2. \quad \int \frac{dx}{a + b \cos^2 x} &= \frac{-\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan \left(\sqrt{\frac{a+b}{a}} \cot x \right) && \left[\frac{b}{a} > -1 \right] \\
&= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh} \left(\sqrt{-\frac{a+b}{a}} \cot x \right) && \left[\frac{b}{a} < -1, \quad \cos^2 x < -\frac{a}{b} \right] \\
&= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth} \left(\sqrt{-\frac{a+b}{a}} \cot x \right) && \left[\frac{b}{a} < -1, \quad \cos^2 x > -\frac{a}{b} \right]
\end{aligned}$$

MZ 162

$$\begin{aligned}
3. \quad \int \frac{dx}{1 + \sin^2 x} &= \frac{1}{\sqrt{2}} \arctan \left(\sqrt{2} \tan x \right) \\
4. \quad \int \frac{dx}{1 - \sin^2 x} &= \tan x \\
5. \quad \int \frac{dx}{1 + \cos^2 x} &= -\frac{1}{\sqrt{2}} \arctan \left(\sqrt{2} \cot x \right) \\
6. \quad \int \frac{dx}{1 - \cos^2 x} &= -\cot x
\end{aligned}$$

2.563

$$1. \quad \int \frac{dx}{(a + b \sin^2 x)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a + b \sin^2 x} + \frac{b \sin x \cos x}{a + b \sin^2 x} \right]$$

(see **2.562** 1)

MZ 155

$$2. \quad \int \frac{dx}{(a + b \cos^2 x)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a + b \cos^2 x} - \frac{b \sin x \cos x}{a + b \cos^2 x} \right]$$

(see **2.562** 2)

MZ 163

$$\begin{aligned}
3. \quad \int \frac{dx}{(a + b \sin^2 x)^3} &= \frac{1}{8pa^3} \left[\left(3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \tan x) \right. \\
&\quad \left. + \left(3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \tan x}{1 + p^2 \tan^2 x} + \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \tan^2 x \right) \frac{2p \tan x}{(1 + p^2 \tan^2 x)^2} \right] \\
&\quad \left[p^2 = 1 + \frac{b}{a} > 0 \right] \\
&= \frac{1}{8qa^3} \left[\left(3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{arctanh}(q \tan x) \right. \\
&\quad \left. + \left(3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \tan x}{1 - q^2 \tan^2 x} + \left(1 + \frac{2}{q^2} + \frac{1}{q^2} \tan^2 x \right) \frac{2q \tan x}{(1 - q^2 \tan^2 x)^2} \right] \\
&\quad \left[q^2 = -1 - \frac{b}{a} > 0, \quad \sin^2 x < -\frac{a}{b}; \quad \text{for } \sin^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}(q \tan x) \text{ to } \operatorname{arccoth}(q \tan x) \right]
\end{aligned}$$

MZ 156

$$\begin{aligned}
4. \quad \int \frac{dx}{(a + b \cos^2 x)^3} &= -\frac{1}{8pa^3} \left[\left(3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \cot x) \right. \\
&\quad \left. + \left(3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \cot x}{1 + p^2 \cot^2 x} + \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \cot^2 x \right) \frac{2p \cot x}{(1 + p^2 \cot^2 x)^2} \right] \\
&\quad \left[p^2 = 1 + \frac{b}{a} > 0 \right] \\
&= -\frac{1}{8qa^3} \left[\left(3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{arctanh}(q \cot x) \right. \\
&\quad \left. + \left(3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \cot x}{1 - q^2 \cot^2 x} + \left(1 + \frac{2}{q^2} + \frac{1}{q^2} \cot^2 x \right) \frac{2p \cot x}{(1 - q^2 \cot^2 x)^2} \right] \\
&\quad \left[q^2 = -1 - \frac{b}{a} > 0, \quad \cos^2 x < -\frac{a}{b}; \quad \text{for } \cos^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}(q \cot x) \text{ to } \operatorname{arccoth}(q \cot x) \right]
\end{aligned}$$

MZ 163a

2.564

$$1. \quad \int \frac{\tan x \, dx}{1 + m^2 \tan^2 x} = \frac{\ln(\cos^2 x + m^2 \sin^2 x)}{2(m^2 - 1)} \quad \text{LA 210 (10)}$$

$$2. \quad \int \frac{\tan \alpha - \tan x}{\tan \alpha + \tan x} dx = \sin 2\alpha \ln \sin(x + \alpha) - x \cos 2\alpha \quad \text{LA 210 (11)a}$$

$$3. \quad \int \frac{\tan x \, dx}{a + b \tan x} = \frac{1}{a^2 + b^2} \{bx - a \ln(a \cos x + b \sin x)\} \quad \text{PE (335)}$$

$$4. \quad \int \frac{dx}{a + b \tan^2 x} = \frac{1}{a - b} \left[x - \sqrt{\frac{b}{a}} \arctan \left(\sqrt{\frac{b}{a}} \tan x \right) \right] \quad \text{PE (334)}$$

2.57 Integrals containing $\sqrt{a \pm b \sin x}$ or $\sqrt{a \pm b \cos x}$

Notation:

$$\alpha = \arcsin \sqrt{\frac{1 - \sin x}{2}}, \quad \beta = \arcsin \sqrt{\frac{b(1 - \sin x)}{a + b}},$$

$$\gamma = \arcsin \sqrt{\frac{b(1 - \cos x)}{a + b}}, \quad \delta = \arcsin \sqrt{\frac{(a + b)(1 - \cos x)}{2(a - b \cos x)}}, \quad r = \sqrt{\frac{2b}{a + b}}$$

2.571

1.
$$\int \frac{dx}{\sqrt{a + b \sin x}} = \frac{-2}{\sqrt{a + b}} F(\alpha, r) \quad \left[a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right]$$

$$= -\sqrt{\frac{2}{b}} F\left(\beta, \frac{1}{r}\right) \quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right]$$

BY (288.00, 288.50)
2.
$$\int \frac{\sin x \, dx}{\sqrt{a + b \sin x}}$$

$$= \frac{2a}{b\sqrt{a + b}} F(\alpha, r) - \frac{2\sqrt{a + b}}{b} E(\alpha, r) \quad \left[a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right] \quad \text{BY (288.03)}$$

$$= \sqrt{\frac{2}{b}} \left\{ F\left(\beta, \frac{1}{r}\right) - 2 E\left(\beta, \frac{1}{r}\right) \right\} \quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right] \quad \text{BY (288.54)}$$
3.
$$\int \frac{\sin^2 x \, dx}{\sqrt{a + b \sin x}} = \frac{4a\sqrt{a + b}}{3b^2} E(\alpha, r) - \frac{2(2a^2 + b^2)}{3b^2\sqrt{a + b}} F(\alpha, r) - \frac{2}{3b} \cos x \sqrt{a + b \sin x}$$

$$\quad \left[a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right]$$

$$= \sqrt{\frac{2}{b}} \left\{ \frac{4a}{3b} E\left(\beta, \frac{1}{r}\right) - \frac{2a + b}{3b} F\left(\beta, \frac{1}{r}\right) \right\} - \frac{2}{3b} \cos x \sqrt{a + b \sin x}$$

$$\quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right]$$

BY (288.03, 288.54)
4.
$$\int \frac{dx}{\sqrt{a + b \cos x}} = \frac{2}{\sqrt{a + b}} F\left(\frac{x}{2}, r\right)$$

$$= \sqrt{\frac{2}{b}} F\left(\gamma, \frac{1}{r}\right) \quad \left[a > b > 0, \quad 0 \leq x \leq \pi \right]$$

$$\quad \left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right]$$

BY (289.00)
5.
$$\int \frac{dx}{\sqrt{a - b \cos x}} = \frac{2}{\sqrt{a + b}} F(\delta, r)$$

$$\quad \left[a > b > 0, \quad 0 \leq x \leq \pi \right] \quad \text{BY (291.00)}$$

$$6. \quad \int \frac{\cos x \, dx}{\sqrt{a+b \cos x}} = \frac{2}{b\sqrt{a+b}} \left\{ (a+b) E\left(\frac{x}{2}, r\right) - a F\left(\frac{x}{2}, r\right) \right\}$$

$$[a > b > 0, \quad 0 \leq x \leq \pi] \quad \text{BY (289.03)}$$

$$= \sqrt{\frac{2}{b}} \left\{ 2 E\left(\gamma, \frac{1}{r}\right) - F\left(\gamma, \frac{1}{r}\right) \right\}$$

$$\left[b > |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right] \quad \text{BY (290.04)}$$

$$7.^6 \quad \int \frac{\cos x \, dx}{\sqrt{a-b \cos x}} = \frac{2}{b\sqrt{a+b}} \{ (b-a) \Pi(\delta, r^2, r) + a F(\delta, r) \}$$

$$[a > b > 0, \quad 0 \leq x \leq \pi] \quad \text{BY (291.03)}$$

$$8. \quad \int \frac{\cos^2 x \, dx}{\sqrt{a+b \cos x}} = \frac{2}{3b^2\sqrt{a+b}} \left\{ (2a^2 + b^2) F\left(\frac{x}{2}, r\right) - 2a(a+b) E\left(\frac{x}{2}, r\right) \right\} + \frac{2}{3b} \sin x \sqrt{a+b \cos x}$$

$$[a > b > 0, \quad 0 \leq x \leq \pi] \quad \text{BY (289.03)}$$

$$= \frac{1}{3b} \sqrt{\frac{2}{b}} \left\{ (2a+b) F\left(\gamma, \frac{1}{r}\right) - 4a E\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2}{3b} \sin x \sqrt{a+b \cos x}$$

$$\left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right] \quad \text{BY (290.04)}$$

$$9. \quad \int \frac{\cos^2 x \, dx}{\sqrt{a-b \cos x}} = \frac{2}{3b^2\sqrt{a+b}} \{ (2a^2 + b^2) F(\delta, r) - 2a(a+b) E(\delta, r) \}$$

$$+ \frac{2}{3b} \sin x \frac{a+b \cos x}{\sqrt{a-b \cos x}} [a > b > 0,]$$

$$\text{BY (291.04)a}$$

$$2.572 \quad \int \frac{\tan^2 x \, dx}{\sqrt{a+b \sin x}}$$

$$= \frac{1}{\sqrt{a+b}} F(\alpha, r) + \frac{a}{(a-b)\sqrt{a+b}} E(\alpha, r)$$

$$- \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad \left[0 < b < a, -\frac{\pi}{2} < x < \frac{\pi}{2} \right]$$

$$= \sqrt{\frac{2}{b}} \left\{ \frac{2a+b}{2(a+b)} F\left(\beta, \frac{1}{r}\right) + \frac{ab}{a^2-b^2} E\left(\beta, \frac{1}{r}\right) \right\}$$

$$- \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right]$$

$$\text{BY (288.08, 288.58)}$$

2.573

$$1. \quad \int \frac{1-\sin x}{1+\sin x} \cdot \frac{dx}{\sqrt{a+b \sin x}} = \frac{2}{a-b} \left\{ \sqrt{a+b} E(\alpha, r) \right\} - \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \sqrt{a+b \sin x}$$

$$\left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right] \quad \text{BY (288.07)}$$

$$2. \quad \int \frac{1 - \cos x}{1 + \cos x} \frac{dx}{\sqrt{a + b \cos x}} = \frac{2}{a - b} \tan \frac{x}{2} \sqrt{a + b \cos x} - \frac{2\sqrt{a + b}}{a - b} E\left(\frac{x}{2}, r\right) \\ [a > b > 0, \quad 0 \leq x < \pi] \quad \text{BY (289.07)}$$

2.574

$$1. \quad \int \frac{dx}{(2 - p^2 + p^2 \sin x) \sqrt{a + b \sin x}} = -\frac{1}{a + b} \Pi(\alpha, p^2, r) \\ \left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right] \quad \text{BY (288.02)}$$

$$2. \quad \int \frac{dx}{(a + b - p^2 b + p^2 b \sin x) \sqrt{a + b \sin x}} = -\frac{1}{a + b} \sqrt{\frac{2}{b}} \Pi\left(\beta, p^2, \frac{1}{r}\right) \\ \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right] \quad \text{BY (288.52)}$$

$$3. \quad \int \frac{dx}{(2 - p^2 + p^2 \cos x) \sqrt{a + b \cos x}} = \frac{1}{\sqrt{a + b}} \Pi\left(\frac{x}{2}, p^2, r\right) \\ [a > b > 0, \quad 0 \leq x < \pi] \quad \text{BY (289.02)}$$

$$4. \quad \int \frac{dx}{(a + b - p^2 b + p^2 b \cos x) \sqrt{a + b \cos x}} = \frac{\sqrt{2}}{(a + b)\sqrt{b}} \Pi\left(\gamma, p^2, \frac{1}{r}\right) \\ \left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)\right] \quad \text{BY (290.02)}$$

2.575

$$1. \quad \int \frac{dx}{\sqrt{(a + b \sin x)^3}} = \frac{2b \cos x}{(a^2 - b^2) \sqrt{a + b \sin x}} - \frac{2}{(a - b)\sqrt{a + b}} E(\alpha, r) \\ \left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right] \quad \text{BY (288.05)}$$

$$= \sqrt{\frac{2}{b}} \left\{ \frac{2b}{b^2 - a^2} E\left(\beta, \frac{1}{r}\right) - \frac{1}{a + b} F\left(\beta, \frac{1}{r}\right) \right\} + \frac{2b}{b^2 - a^2} \cdot \frac{\cos x}{\sqrt{a + b \sin x}} \\ \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right] \quad \text{BY (288.56)}$$

$$\begin{aligned}
2. \quad \int \frac{dx}{\sqrt{(a+b\sin x)^5}} &= \frac{2}{3(a^2-b^2)^2\sqrt{a+b}} \left\{ (a^2-b^2) F(\alpha, r) - 4a(a+b) E(\alpha, r) \right\} \\
&\quad + \frac{2b(5a^2-b^2+4ab\sin x)}{3(a^2-b^2)^2\sqrt{(a+b\sin x)^3}} \cos x \\
&\quad \left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right] \quad \text{BY (288.05)} \\
&= -\frac{1}{3(a^2-b^2)^2} \sqrt{\frac{2}{b}} \left\{ (3a-b)(a-b) F\left(\beta, \frac{1}{r}\right) + 8ab E\left(\beta, \frac{1}{r}\right) \right\} \\
&\quad + \frac{2b[a^2-b^2+4a(a+b\sin x)]}{3(a^2-b^2)^2\sqrt{(a+b\sin x)^3}} \cos x \\
&\quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right] \quad \text{BY (288.56)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int \frac{dx}{\sqrt{(a+b\cos x)^3}} &= \frac{2}{(a-b)\sqrt{a+b}} E\left(\frac{x}{2}, r\right) - \frac{2b}{a^2-b^2} \cdot \frac{\sin x}{\sqrt{a+b\cos x}} \\
&\quad [a > b > 0, \quad 0 \leq x \leq \pi] \quad \text{BY (289.05)} \\
&= \frac{1}{a^2-b^2} \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\gamma, \frac{1}{r}\right) + 2b E\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2b}{b^2-a^2} \cdot \frac{\sin x}{\sqrt{a+b\cos x}} \\
&\quad \left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right] \quad \text{BY (290.06)}
\end{aligned}$$

$$4. \quad \int \frac{dx}{\sqrt{(a-b\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E(\delta, r) \quad [a > b > 0, \quad 0 \leq x \leq \pi] \quad (291.01)$$

$$\begin{aligned}
 5. \quad \int \frac{dx}{\sqrt{(a+b \cos x)^5}} &= \frac{2\sqrt{a+b}}{3(a^2-b^2)^2} \left\{ 4a E\left(\frac{x}{2}, r\right) - (a-b) F\left(\frac{x}{2}, r\right) \right\} \\
 &\quad - \frac{2b}{3(a^2-b^2)^2} \cdot \frac{5a^2-b^2+4ab \cos x}{\sqrt{(a+b \cos x)^3}} \sin x \\
 &\quad [a > b > 0, \quad 0 \leq x \leq \pi] \\
 &\quad \text{BY (289.05)} \\
 &= \frac{1}{3(a^2-b^2)^2} \sqrt{\frac{2}{b}} \left\{ (a-b)(3a-b) F\left(\gamma, \frac{1}{r}\right) + 8ab E\left(\gamma, \frac{1}{r}\right) \right\} \\
 &\quad + \frac{2b(5a^2-b^2+4ab \cos x) \sin x}{3(a^2-b^2)^2 \sqrt{(a+b \cos x)^3}} \\
 &\quad [b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)] \\
 &\quad \text{BY (290.06)}
 \end{aligned}$$

2.576

$$\begin{aligned}
 1. \quad \int \sqrt{a+b \cos x} dx &= 2\sqrt{a+b} E\left(\frac{x}{2}, r\right) \\
 &\quad [a > b > 0, \quad 0 \leq x \leq \pi] \\
 &\quad \text{BY (289.01)} \\
 &= \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\gamma, \frac{1}{r}\right) + 2b E\left(\gamma, \frac{1}{r}\right) \right\} \\
 &\quad [b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)] \\
 &\quad \text{BY (290.03)}
 \end{aligned}$$

$$2. \quad \int \sqrt{a-b \cos x} dx = 2\sqrt{a+b} E(\delta, r) - \frac{2b \sin x}{\sqrt{a-b \cos x}} \quad [a > b > 0, \quad 0 \leq x \leq \pi] \quad \text{BY (291.05)}$$

2.577

$$\begin{aligned}
 1.^3 \quad \int \frac{\sqrt{a-b \cos x}}{1+p \cos x} dx &= \frac{2(a-b)}{(1+p)\sqrt{a+b}} \Pi\left(\delta, \frac{2ap}{(a+b)(1+p)}, r\right) \\
 &\quad [a > b > 0, \quad 0 \leq x \leq \pi, \quad p \neq -1] \\
 &\quad \text{BY (291.02)}
 \end{aligned}$$

$$\begin{aligned}
 2.^3 \quad \int \sqrt{\frac{a-b \cos x}{1+p \cos x}} dx &= \frac{2(a-b)}{\sqrt{(1+p)(a+b)}} \Pi\left(\delta, -r^2, \sqrt{\frac{2(ap+b)}{(1+p)(a+b)}}\right) \\
 &\quad [a > b > 0, \quad 0 \leq x \leq \pi, \quad p \neq -1]
 \end{aligned}$$

$$\begin{aligned}
 2.578 \quad \int \frac{\tan x dx}{\sqrt{a+b \tan^2 x}} &= \frac{1}{\sqrt{b-a}} \arccos\left(\frac{\sqrt{b-a}}{\sqrt{b}} \cos x\right) \quad [b > a, \quad b > 0] \\
 &\quad \text{PE (333)}
 \end{aligned}$$

2.58–2.62 Integrals reducible to elliptic and pseudo-elliptic integrals

2.580

1.
$$\int \frac{d\varphi}{\sqrt{a+b\cos\varphi+c\sin\varphi}} = 2 \int \frac{d\psi}{\sqrt{a-p+2p\cos^2\psi}} \quad \left[\varphi = 2\psi + \alpha, \tan\alpha = \frac{c}{b}, p = \sqrt{b^2 + c^2} \right]$$
2.
$$\int \frac{d\varphi}{\sqrt{a+b\cos\varphi+c\sin\varphi+d\cos^2\varphi+e\sin\varphi\cos\varphi+f\sin^2\varphi}} = 2 \int \frac{dx}{\sqrt{A+Bx+Cx^2-Dx^3+Ex^4}}$$

$$\left[\tan \frac{\varphi}{2} = x, A = a+b+d, B = 2c+2e, C = 2a-2d+4f, D = 2c-2e, E = a-b+d \right]$$

Forms containing $\sqrt{1-k^2\sin^2 x}$

Notation: $\Delta = \sqrt{1-k^2\sin^2 x}$, $k' = \sqrt{1-k^2}$

2.581

1.
$$\begin{aligned} \int \sin^m x \cos^n x \Delta^r dx &= \frac{1}{(m+n+r)k^2} \left\{ \sin^{m-3} x \cos^{n+1} x \Delta^{r+2} + [(m+n-2) + (m+r-1)k^2] \right. \\ &\quad \left. \times \int \sin^{m-2} x \cos^n x \Delta^r dx - (m-3) \int \sin^{m-4} x \cos^n x \Delta^r dx \right\} \\ &= \frac{1}{(m+n+r)k^2} \left\{ \sin^{m+1} x \cos^{n-3} x \Delta^{r+2} + [(n+r-1)k^2 - (m+n-2)k'^2] \right. \\ &\quad \left. \times \int \sin^m x \cos^{n-2} x \Delta^r dx + (n-3)k'^2 \int \sin^m x \cos^{n-4} x \Delta^r dx \right\} \\ &\quad [m+n+r \neq 0] \end{aligned}$$

For $r = -3$ and $r = -5$:

2.
$$\int \frac{\sin^m x \cos^n x}{\Delta^3} dx = \frac{\sin^{m-1} x \cos^{n-1} x}{k^2 \Delta} - \frac{m-1}{k^2} \int \frac{\sin^{m-2} x \cos^n x}{\Delta} dx + \frac{n-1}{k^2} \int \frac{\sin^m x \cos^{n-2} x}{\Delta} dx$$
3.
$$\int \frac{\sin^m x \cos^n x}{\Delta^5} dx = \frac{\sin^{m-1} x \cos^{n-1} x}{3k^2 \Delta^3} - \frac{m-1}{3k^2} \int \frac{\sin^{m-2} x \cos^n x}{\Delta^3} dx + \frac{n-1}{3k^2} \int \frac{\sin^m x \cos^{n-2} x}{\Delta^3} dx$$

For $m = 1$ or $n = 1$:

4.
$$\int \sin x \cos^n x \Delta^r dx = -\frac{\cos^{n-1} x \Delta^{r+2}}{(n+r+1)k^2} - \frac{(n-1)k'^2}{(n+r+1)k^2} \int \cos^{n-2} x \sin x \Delta^r dx$$
5.
$$\int \sin^m x \cos x \Delta^r dx = -\frac{\sin^{m-1} x \Delta^{r+2}}{(m+r+1)k^2} + \frac{m-1}{(m+r+1)k^2} \int \sin^{m-2} x \cos x \Delta^r dx$$

For $m = 3$ or $n = 3$:

$$6. \quad \int \sin^3 x \cos^n x \Delta^r dx = \frac{(n+r+1)k^2 \cos^2 x - [(r+2)k^2 + n+1]}{(n+r+1)(n+r+3)k^4} \cos^{n-1} x \Delta^{r+2} \\ - \frac{[(r+2)k^2 + n+1](n-1)k'^2}{(n+r+1)(n+r+3)k^4} \int \cos^{n-2} x \sin x \Delta^r dx$$

$$7. \quad \int \sin^m x \cos^3 x \Delta^r dx \\ = \frac{(m+r+1)k^2 \sin^2 x - [(r+2)k^2 - (m+1)k'^2]}{(m+r+1)(m+r+3)k^4} \\ \times \sin^{m-1} x \cos^{m-1} x \Delta^{r+2} + \frac{[(r+2)k^2 - (m-1)k'^2](m-1)}{(m+r+1)(m+r+3)k^4} \int \sin^{m-2} x \cos x \Delta^r dx$$

2.582

$$1. \quad \int \Delta^n dx = \frac{n-1}{n} (2-k^2) \int \Delta^{n-2} dx - \frac{n-2}{n} (1-k^2) \int \Delta^{n-4} dx \\ + \frac{k^2}{n} \sin x \cos x \cdot \Delta^{n-2}$$

LA (316)(1)a

$$2. \quad \int \frac{dx}{\Delta^{n+1}} = -\frac{k^2 \sin x \cos x}{(n-1)k'^2 \Delta^{n-1}} + \frac{n-2}{n-1} \frac{2-k^2}{k'^2} \int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{n-1} \frac{1}{k'^2} \int \frac{dx}{\Delta^{n-3}}$$

LA 317(8)a

$$3. \quad \int \frac{\sin^n x}{\Delta} dx = \frac{\sin^{n-3} x}{(n-1)k^2} \cos x \cdot \Delta + \frac{n-2}{n-1} \frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta} dx \\ - \frac{n-3}{(n-1)k^2} \int \frac{\sin^{n-4} x}{\Delta} dx$$

LA 316(1)a

$$4. \quad \int \frac{\cos^n x}{\Delta} dx = \frac{\cos^{n-3} x}{(n-1)k^2} \sin x \cdot \Delta + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} dx \\ + \frac{n-3}{n-1} \frac{k'^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} dx$$

LA 316(2)a

$$5. \quad \int \frac{\tan^n x}{\Delta} dx = \frac{\tan^{n-3} x}{(n-1)k'^2} \frac{\Delta}{\cos^2 x} - \frac{(n-2)(2-k^2)}{(n-1)k'^2} \int \frac{\tan^{n-2} x}{\Delta} dx \\ - \frac{n-3}{(n-1)k'^2} \int \frac{\tan^{n-4} x}{\Delta} dx$$

LA 317(3)

$$6. \quad \int \frac{\cot^n x}{\Delta} dx = -\frac{\cot^{n-1} x}{n-1} \frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1} (2-k^2) \int \frac{\cot^{n-2} x}{\Delta} dx \\ - \frac{n-3}{n-1} k'^2 \int \frac{\cot^{n-4} x}{\Delta} dx$$

LA 317(6)

2.583

$$1. \quad \int \Delta dx = E(x, k)$$

2. $\int \Delta \sin x \, dx = -\frac{\Delta \cos x}{2} - \frac{k'^2}{2k} \ln(k \cos x + \Delta)$
3. $\int \Delta \cos x \, dx = \frac{\Delta \sin x}{2} + \frac{1}{2k} \arcsin(k \sin x)$
4. $\int \Delta \sin^2 x \, dx = -\frac{\Delta}{3} \sin x \cos x + \frac{k'^2}{3k^2} F(x, k) + \frac{2k^2 - 1}{3k^2} E(x, k)$
5. $\int \Delta \sin x \cos x \, dx = -\frac{\Delta^3}{3k^2}$
6. $\int \Delta \cos^2 x \, dx = \frac{\Delta}{3} \sin x \cos x - \frac{k'^2}{3k^2} F(x, k) + \frac{k^2 + 1}{3k^2} E(x, k)$
7. $\int \Delta \sin^3 x \, dx = -\frac{2k^2 \sin^2 x + 3k^2 - 1}{8k^2} \Delta \cos x + \frac{3k^4 - 2k^2 - 1}{8k^3} \ln(k \cos x + \Delta)$
8. $\int \Delta \sin^2 x \cos x \, dx = \frac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \frac{1}{8k^3} \arcsin(k \sin x)$
9. $\int \Delta \sin x \cos^2 x \, dx = -\frac{2k^2 \cos^2 x + k'^2}{8k^2} \Delta \cos x + \frac{k'^4}{8k^3} \ln(k \cos x + \Delta)$
10. $\int \Delta \cos^3 x \, dx = \frac{2k^2 \cos^2 x + 2k^2 + 1}{8k^2} \Delta \sin x + \frac{4k^2 - 1}{8k^3} \arcsin(k \sin x)$
11. $\int \Delta \sin^4 x \, dx = -\frac{3k^2 \sin^2 x + 4k^2 - 1}{15k^2} \Delta \sin x \cos x$
 $- \frac{2(2k^4 - k^2 - 1)}{15k^4} F(x, k) + \frac{8k^4 - 3k^2 - 2}{15k^4} E(x, k)$
12. $\int \Delta \sin^3 x \cos x \, dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta$
13. $\int \Delta \sin^2 x \cos^2 x \, dx = -\frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x$
 $- \frac{k'^2(1 + k'^2)}{15k^4} F(x, k) + \frac{2(k^4 - k^2 + 1)}{15k^4} E(x, k)$
14. $\int \Delta \sin x \cos^3 x \, dx = -\frac{3k^4 \sin^4 x - k^2(5k^2 + 1) \sin^2 x + 5k^2 - 2}{15k^4} \Delta$
15. $\int \Delta \cos^4 x \, dx = \frac{3k^2 \cos^2 x + 3k^2 + 1}{15k^2} \Delta \sin x \cos x$
 $+ \frac{2k'^2(k'^2 - 2k^2)}{15k^4} F(x, k) + \frac{3k^4 + 7k^2 - 2}{15k^4} E(x, k)$
16. $\int \Delta \sin^5 x \, dx = \frac{-8k^4 \sin^4 x - 2k^2(5k^2 - 1) \sin^2 x - 15k^4 + 4k^2 + 3}{48k^4} \Delta \cos x$
 $+ \frac{5k^6 - 3k^4 - k^2 - 1}{16k^5} \ln(k \cos x + \Delta)$
17. $\int \Delta \sin^4 x \cos x \, dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^5} \arcsin(k \sin x)$

18.
$$\int \Delta \sin^3 x \cos^2 x \, dx = \frac{8k^4 \sin^4 x - 2k^2 (k^2 + 1) \sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \Delta \cos x$$
$$+ \frac{k'^4 (k^2 + 1)}{16k^5} \ln (k \cos x + \Delta)$$
19.
$$\int \Delta \sin^2 x \cos^3 x \, dx = \frac{-8k^4 \sin^4 x + 2k^2 (6k^2 + 1) \sin^2 x - 6k^2 + 3}{48k^4} \Delta \sin x$$
$$+ \frac{2k^2 - 1}{16k^5} \arcsin (k \sin x)$$
20.
$$\int \Delta \sin x \cos^4 x \, dx = \frac{-8k^4 \sin^4 x + 2k^2 (7k^2 + 1) \sin^2 x - 3k^4 - 8k^2 + 3}{48k^4} \Delta \cos x$$
$$- \frac{k'^6}{16k^5} \ln (k \cos x + \Delta)$$
21.
$$\int \Delta \cos^5 x \, dx = \frac{8k^4 \sin^4 x - 2k^2 (12k^2 + 1) \sin^2 x + 24k^4 + 12k^2 - 3}{48k^4} \Delta \sin x$$
$$+ \frac{8k^4 - 4k^2 + 1}{16k^5} \arcsin (k \sin x)$$
22.
$$\int \Delta^3 \, dx = \frac{2}{3} (1 + k'^2) E(x, k) - \frac{k'^2}{3} F(x, k) + \frac{k^2}{3} \Delta \sin x \cos x$$
23.
$$\int \Delta^3 \sin x \, dx = \frac{2k^2 \sin^2 x + 3k^2 - 5}{8} \Delta \cos x - \frac{3k'^4}{8k} \ln (k \cos x + \Delta)$$
24.
$$\int \Delta^3 \cos x \, dx = \frac{-2k^2 \sin^2 x + 5}{8} \Delta \sin x + \frac{3}{8k} \arcsin (k \sin x)$$
25.
$$\int \Delta^3 \sin^2 x \, dx = \frac{3k^2 \sin^2 x + 4k^2 - 6}{15} \Delta \sin x \cos x + \frac{k'^2 (3 - 4k^2)}{15k^2} F(x, k)$$
$$- \frac{8k^4 - 13k^2 + 3}{15k^2} E(x, k)$$
26.
$$\int \Delta^3 \sin x \cos x \, dx = -\frac{\Delta^5}{5k^2}$$
27.
$$\int \Delta^3 \cos^2 x \, dx = \frac{-3k^2 \sin^2 x + k^2 + 5}{15} \Delta \sin x \cos x - \frac{k'^2 (k^2 + 3)}{15k^2} F(x, k)$$
$$- \frac{2k^4 - 7k^2 - 3}{15k^2} E(x, k)$$
28.
$$\int \Delta^3 \sin^3 x \, dx = \frac{8k^4 \sin^4 x + 2k^2 (5k^2 - 7) \sin^2 x + 15k^4 - 22k^2 + 3}{48k^2} \Delta \cos x$$
$$- \frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3} \ln (k \cos x + \Delta)$$
29.
$$\int \Delta^3 \sin^2 x \cos x \, dx = \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x$$
$$+ \frac{1}{16k^3} \arcsin (k \sin x)$$

30.
$$\int \Delta^3 \sin x \cos^2 x \, dx = \frac{-8k^4 \sin^4 x + 2k^2 (k^2 + 7) \sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \\ \times \Delta \cos x + \frac{k'^6}{16k^3} \ln (k \cos x + \Delta)$$
31.
$$\int \Delta^3 \cos^3 x \, dx = \frac{8k^4 \sin^4 x - 2k^2 (6k^2 + 7) \sin^2 x + 30k^2 + 3}{48k^2} \Delta \sin x \\ + \frac{6k^2 - 1}{16k^3} \arcsin (k \sin x)$$
32.
$$\int \frac{\Delta \, dx}{\sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + k \ln k (k \cos x + \Delta)$$
33.
$$\int \frac{\Delta \, dx}{\cos x} = \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} + k \arcsin (k \sin x)$$
34.
$$\int \frac{\Delta \, dx}{\sin^2 x} = k'^2 F(x, k) - E(x, k) - \Delta \cot x$$
35.
$$\int \frac{\Delta \, dx}{\sin x \cos x} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$
36.
$$\int \frac{\Delta \, dx}{\cos^2 x} = F(x, k) - E(x, k) + \Delta \tan x$$
37.
$$\int \frac{\sin x}{\cos x} \Delta \, dx = \int \Delta \tan x \, dx = -\Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$
38.
$$\int \frac{\cos x}{\sin x} \Delta \, dx = \int \Delta \cot x \, dx = \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$
39.
$$\int \frac{\Delta \, dx}{\sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$
40.
$$\int \frac{\Delta \, dx}{\sin^2 x \cos x} = \frac{-\Delta}{\sin x} - \frac{1 + k^2}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$
41.
$$\int \frac{\Delta \, dx}{\sin x \cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$
42.
$$\int \frac{\Delta \, dx}{\cos^3 x} = \frac{\Delta \sin x}{2 \cos^2 x} + \frac{1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$
43.
$$\int \frac{\Delta \sin x \, dx}{\cos^2 x} = \frac{\Delta}{\cos x} - k \ln (k \cos x + \Delta)$$
44.
$$\int \frac{\Delta \cos x \, dx}{\sin^2 x} = -\frac{\Delta}{\sin x} - k \arcsin (k \sin x)$$
45.
$$\int \frac{\Delta \sin^2 x \, dx}{\cos x} = -\frac{\Delta \sin x}{2} + \frac{2k^2 - 1}{2k} \arcsin (k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$
46.
$$\int \frac{\Delta \cos^2 x \, dx}{\sin x} = \frac{\Delta \cos x}{2} + \frac{k^2 + 1}{2k} \ln (k \cos x + \Delta) + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$
47.
$$\int \frac{\Delta \, dx}{\sin^4 x} = \frac{1}{3} \left\{ -\Delta \cot^3 x + (k^2 - 3) \Delta \cot x + 2k'^2 F(x, k) + (k^2 - 2) E(x, k) \right\}$$

48.
$$\int \frac{\Delta dx}{\sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'} + \frac{k^2 - 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$
49.
$$\int \frac{\Delta dx}{\sin^2 x \cos^2 x} = \left(\frac{1}{k'^2} \tan x - \cot x \right) \Delta + 2 F(x, k) - \frac{1 + k'^2}{k'^2} E(x, k)$$
50.
$$\int \frac{\Delta dx}{\sin x \cos^3 x} = \frac{\Delta}{2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$
51.
$$\int \frac{\Delta dx}{\cos^4 x} = \frac{1}{3k'^2} \left\{ \left[k'^2 \tan^2 x - (2k^2 - 3) \tan x \right] \Delta + 2k'^2 F(x, k) + (k^2 - 2) E(x, k) \right\}$$
52.
$$\int \frac{\sin x}{\cos^3 x} \Delta dx = \frac{\Delta}{2 \cos^2 x} + \frac{k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$
53.
$$\int \frac{\cos x}{\sin^3 x} \Delta dx = -\frac{\Delta}{2 \sin^2 x} + \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$
54.
$$\int \frac{\sin^2 x}{\cos^2 x} \Delta dx = \int \tan^2 x \Delta dx = \Delta \tan x + F(x, k) - 2 E(x, k)$$
55.
$$\int \frac{\cos^2 x}{\sin^2 x} \Delta dx = \int \cot^2 x \Delta dx = -\Delta \cot x + k'^2 F(x, k) - 2 E(x, k)$$
56.
$$\int \frac{\sin^3 x}{\cos x} \Delta dx = -\frac{k^2 \sin^2 x + 3k^2 - 1}{3k^2} \Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$
57.
$$\int \frac{\cos^3 x}{\sin x} \Delta dx = -\frac{k^2 \sin^2 x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$
58.
$$\int \frac{\Delta dx}{\sin^5 x} = \frac{(k^2 - 3) \sin^2 x + 2}{8 \sin^4 x} \cos x \Delta + \frac{k'^2 (k^2 + 3)}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$
59.
$$\int \frac{\Delta dx}{\sin^4 x \cos x} = -\frac{(3 - k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{k'}{2} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$
60.
$$\int \frac{\Delta dx}{\sin^3 x \cos^2 x} = \frac{3 \sin^2 x - 1}{2 \sin^2 x \cos x} \Delta + \frac{k^2 - 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$
61.
$$\int \frac{\Delta dx}{\sin^2 x \cos^3 x} = \frac{3 \sin^2 x - 2}{2 \sin x \cos^2 x} \Delta - \frac{2k^2 - 3}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$
62.
$$\int \frac{\Delta dx}{\sin x \cos^4 x} = \frac{(2k^2 - 3) \sin^2 x - 3k^2 + 4}{3k'^2 \cos^3 x} \Delta + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$
63.
$$\int \frac{\Delta dx}{\cos^5 x} = \frac{(2k^2 - 3) \sin^2 x - 4k^2 + 5}{8k'^2 \cos^4 x} \sin x \Delta - \frac{4k^2 - 3}{16k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$
64.
$$\int \frac{\sin x}{\cos^4 x} \Delta dx = \frac{-(2k^2 + 1) k^2 \sin^2 x + 3k^4 - k^2 + 1}{3k'^2 \cos^3 x} \Delta$$
65.
$$\int \frac{\cos x}{\sin^4 x} \Delta dx = -\frac{\Delta^3}{3 \sin^3 x}$$
66.
$$\int \frac{\sin^2 x}{\cos^3 x} \Delta dx = \frac{\sin x}{2 \cos^2 x} \Delta + \frac{2k^2 - 1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - k \arcsin(k \sin x)$$

$$67. \quad \int \frac{\cos^2 x}{\sin^3 x} \Delta dx = -\frac{\cos x}{2 \sin^2 x} \Delta - \frac{k^2 + 1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln (k \cos x + \Delta)$$

$$68. \quad \int \frac{\sin^3 x}{\cos^2 x} \Delta dx = -\frac{\sin^2 x - 3}{2 \cos x} \Delta - \frac{3k^2 - 1}{2k} \ln (k \cos x + \Delta)$$

$$69. \quad \int \frac{\cos^3 x}{\sin^2 x} \Delta dx = -\frac{\sin^2 x + 2}{2 \sin x} \Delta - \frac{2k^2 + 1}{2k} \arcsin (k \sin x)$$

$$70. \quad \int \frac{\sin^4 x}{\cos x} \Delta dx = -\frac{2k^2 \sin^2 x + 4k^2 - 1}{8k^2} \Delta \sin x \\ + \frac{8k^4 - 4k^2 - 1}{8k^3} \arcsin (k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

$$71. \quad \int \frac{\cos^4 x}{\sin x} \Delta dx = \frac{-2k^2 \sin^2 x + 5k^2 + 1}{8k^2} \Delta \cos x \\ + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^4 + 6k^2 - 1}{8k^3} \ln (k \cos x + \Delta)$$

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$$1. \quad \int \frac{dx}{\Delta} = F(x, k)$$

$$2. \quad \int \frac{\sin x dx}{\Delta} = \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x} = -\frac{1}{k} \ln (k \cos x + \Delta)$$

$$3. \quad \int \frac{\cos x dx}{\Delta} = \frac{1}{k} \arcsin (k \sin x) = \frac{1}{k} \arctan \frac{k \sin x}{\Delta}$$

$$4. \quad \int \frac{\sin^2 x dx}{\Delta} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k)$$

$$5. \quad \int \frac{\sin x \cos x dx}{\Delta} = -\frac{\Delta}{k^2}$$

$$6. \quad \int \frac{\cos^2 x dx}{\Delta} = \frac{1}{k^2} E(x, k) - \frac{k'^2}{k^2} F(x, k)$$

$$7. \quad \int \frac{\sin^3 x dx}{\Delta} = \frac{\cos x \Delta}{2k^2} - \frac{1 + k^2}{2k^3} \ln (k \cos x + \Delta)$$

$$8. \quad \int \frac{\sin^2 x \cos x dx}{\Delta} = -\frac{\sin x \Delta}{2k^2} + \frac{\arcsin (k \sin x)}{2k^3}$$

$$9. \quad \int \frac{\sin x \cos^2 x dx}{\Delta} = -\frac{\cos x \Delta}{2k^2} + \frac{k'^2}{2k^3} \ln (k \cos x + \Delta)$$

$$10. \quad \int \frac{\cos^3 x dx}{\Delta} = \frac{\sin x \Delta}{2k^2} + \frac{2k^2 - 1}{2k^3} \arcsin (k \sin x)$$

$$11. \quad \int \frac{\sin^4 x dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{2 + k^2}{3k^4} F(x, k) - \frac{2(1 + k^2)}{3k^4} E(x, k)$$

$$12. \quad \int \frac{\sin^3 x \cos x dx}{\Delta} = -\frac{1}{3k^4} (2 + k^2 \sin^2 x) \Delta$$

$$13. \quad \int \frac{\sin^2 x \cos^2 x \, dx}{\Delta} = -\frac{\sin x \cos x \Delta}{3k^2} + \frac{2-k^2}{3k^4} E(x, k) + \frac{2k^2-2}{3k^4} F(x, k)$$

$$14. \quad \int \frac{\sin x \cos^3 x \, dx}{\Delta} = -\frac{1}{3k^4} (k^2 \cos^2 x - 2k'^2) \Delta$$

$$15. \quad \int \frac{\cos^4 x \, dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{4k^2-2}{3k^4} E(x, k) + \frac{3k^4-5k^2+2}{3k^4} F(x, k)$$

$$16. \quad \int \frac{\sin^5 x \, dx}{\Delta} = \frac{2k^2 \sin^2 x + 3k^2 + 3}{8k^4} \cos x \Delta - \frac{3+2k^2+3k^4}{8k^5} \ln(k \cos x + \Delta)$$

$$17. \quad \int \frac{\sin^4 x \cos x \, dx}{\Delta} = -\frac{2k^2 \sin^2 x + 3}{8k^4} \sin x \Delta + \frac{3}{8k^5} \arcsin(k \sin x)$$

$$18. \quad \int \frac{\sin^3 x \cos x \, dx}{\Delta} = \frac{2k^2 \cos^2 x - k^2 - 3}{8k^4} \cos x \Delta - \frac{k^4 + 2k^2 - 3}{8k^5} \ln(k \cos x + \Delta)$$

$$19. \quad \int \frac{\sin^2 x \cos^3 x \, dx}{\Delta} = -\frac{2k^2 \cos^2 x + 2k^2 - 3}{8k^4} \sin x \Delta + \frac{4k^2 - 3}{8k^5} \arcsin(k \sin x)$$

$$20. \quad \int \frac{\sin x \cos^4 x \, dx}{\Delta} = \frac{3-5k^2+2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4-6k^2+3}{8k^5} \ln(k \cos x + \Delta)$$

$$21. \quad \int \frac{\cos^5 x \, dx}{\Delta} = \frac{2k^2 \cos^2 x + 6k^2 - 3}{8k^4} \sin x \Delta + \frac{8k^4-8k^2+3}{8k^5} \arcsin(k \sin x)$$

$$22. \quad \int \frac{\sin^6 x \, dx}{\Delta} = \frac{3k^2 \sin^2 x + 4k^2 + 4}{15k^4} \sin x \cos x \Delta \\ + \frac{4k^4 + 3k^2 + 8}{15k^6} F(x, k) - \frac{8k^4 + 7k^2 + 8}{15k^6} E(x, k)$$

$$23. \quad \int \frac{\sin^5 x \cos x \, dx}{\Delta} = -\frac{3k^4 \sin^4 x + 4k^2 \sin^2 x + 8}{15k^6} \Delta$$

$$24. \quad \int \frac{\sin^4 x \cos x \, dx}{\Delta} = \frac{3k^2 \cos^2 x - 2k^2 - 4}{15k^4} \sin x \cos x \Delta \\ + \frac{k^4 + 7k^2 - 8}{15k^6} F(x, k) - \frac{2k^4 + 3k^2 - 8}{15k^6} E(x, k)$$

$$25. \quad \int \frac{\sin^3 x \cos^3 x \, dx}{\Delta} = \frac{3k^4 \sin^4 x - (5k^4 - 4k^2) \sin^2 x - 10k^2 + 8}{15k^6} \Delta$$

$$26. \quad \int \frac{\sin^2 x \cos^4 x \, dx}{\Delta} = -\frac{3k^2 \cos^2 x + 3k^2 - 4}{15k^4} \sin x \cos x \Delta \\ + \frac{9k^4 - 17k^2 + 8}{15k^6} F(x, k) - \frac{3k^4 - 13k^2 + 8}{15k^6} E(x, k)$$

$$27. \quad \int \frac{\sin x \cos^5 x \, dx}{\Delta} = \frac{-3k^4 \cos^4 x + 4k^2 k'^2 \cos^2 x - 8k^4 + 16k^2 - 8}{15k^6} \Delta$$

$$28. \quad \int \frac{\cos^6 x \, dx}{\Delta} = \frac{3k^2 \cos^2 x + 8k^2 - 4}{15k^4} \sin x \cos x \Delta \\ + \frac{15k^6 - 34k^4 + 27k^2 - 8}{15k^6} F(x, k) + \frac{23k^4 - 23k^2 + 8}{15k^6} E(x, k)$$

- $$29. \quad \int \frac{\sin^7 x \, dx}{\Delta} = \frac{8k^4 \sin^4 x + 10k^2 (k^2 + 1) \sin^2 x + 15k^4 + 14k^2 + 15}{48k^6} \cos x \Delta - \frac{(5k^4 - 2k^2 + 5)(k^2 + 1)}{16k^7} \ln(k \cos x + \Delta)$$
- $$30. \quad \int \frac{\sin^6 x \cos x \, dx}{\Delta} = -\frac{8k^4 \sin^4 x + 10k^2 \sin^2 x + 15}{48k^6} \sin x \Delta + \frac{5}{16k^7} \arcsin(k \sin x)$$
- $$31. \quad \int \frac{\sin^5 x \cos^2 x \, dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2 (k^2 - 5) \sin^2 x + 3k^4 + 4k^2 - 15}{48k^6} \cos x \Delta - \frac{k^6 + k^4 + 3k^2 - 5}{16k^7} \ln(k \cos x + \Delta)$$
- $$32. \quad \int \frac{\sin^4 x \cos^3 x \, dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2 (6k^2 - 5) \sin^2 x - 18k^2 + 15}{48k^6} \sin x \Delta + \frac{6k^2 - 5}{16k^7} \arcsin(k \sin x)$$
- $$33. \quad \int \frac{\sin^3 x \cos^4 x \, dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2 (6k^2 - 5) \sin^2 x + 3k^4 - 22k^2 + 15}{48k^6} \cos x \Delta - \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7} \ln(k \cos x + \Delta)$$
- $$34. \quad \int \frac{\sin^2 x \cos^5 x \, dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2 (12k^2 - 5) \sin^2 x - 24k^4 + 36k^2 - 15}{48k^6} \sin x \Delta + \frac{8k^4 - 12k^2 + 5}{16k^7} \arcsin(k \sin x)$$
- $$35. \quad \int \frac{\sin x \cos^6 x \, dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2 (13k^2 - 5) \sin^2 x - 33k^4 + 40k^2 - 15}{48k^6} \cos x \Delta + \frac{5k^6}{16k^7} \ln(k \cos x + \Delta)$$
- $$36. \quad \int \frac{\cos^7 x \, dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2 (18k^2 - 5) \sin^2 x + 72k^4 - 54k^2 + 15}{48k^6} \sin x \Delta + \frac{16k^6 - 24k^4 + 18k^2 - 5}{16k^7} \arcsin(k \sin x)$$
- $$37. \quad \int \frac{dx}{\Delta^3} = \frac{1}{k'^2} E(x, k) - \frac{k^2}{k'^2} \frac{\sin x \cos x}{\Delta}$$
- $$38. \quad \int \frac{\sin x \, dx}{\Delta^3} = -\frac{\cos x}{k'^2 \Delta}$$
- $$39. \quad \int \frac{\cos x \, dx}{\Delta^3} = \frac{\sin x}{\Delta}$$
- $$40.^{11} \quad \int \frac{\sin^2 x \, dx}{\Delta^3} = \frac{1}{k'^2 k^2} E(x, k) - \frac{1}{k^2} F(x, k) - \frac{1}{k'^2} \frac{\sin x \cos x}{\Delta}$$
- $$41. \quad \int \frac{\sin x \cos x \, dx}{\Delta^3} = \frac{1}{k^2 \Delta}$$
- $$42. \quad \int \frac{\cos^2 x \, dx}{\Delta^3} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{\sin x \cos x}{\Delta}$$

43. $\int \frac{\sin^3 x \, dx}{\Delta^3} = -\frac{\cos x}{k^2 k'^2 \Delta} + \frac{1}{k^3} \ln(k \cos x + \Delta)$
44. $\int \frac{\sin^2 x \cos x \, dx}{\Delta^3} = \frac{\sin x}{k^2 \Delta} - \frac{1}{k^3} \arcsin(k \sin x)$
45. $\int \frac{\sin x \cos^2 x \, dx}{\Delta^3} = \frac{\cos x}{k^2 \Delta} - \frac{1}{k^3} \ln(k \cos x + \Delta)$
46. $\int \frac{\cos^3 x \, dx}{\Delta^3} = -\frac{k'^2 \sin x}{k^2 \Delta} + \frac{1}{k^3} \arcsin(k \sin x)$
47. $\int \frac{\sin^4 x \, dx}{\Delta^3} = \frac{k'^2 + 1}{k'^2 k^4} E(x, k) - \frac{2}{k^4} F(x, k) - \frac{\sin x \cos x}{k^2 k'^2 \Delta}$
48. $\int \frac{\sin^3 x \cos x \, dx}{\Delta^3} = \frac{2 - k^2 \sin^2 x}{k^4 \delta}$
49. $\int \frac{\sin^2 x \cos^2 x \, dx}{\Delta^3} = \frac{2 - k^2}{k^4} F(x, k) - \frac{2}{k^4} E(x, k) + \frac{\sin x \cos x}{k^2 \Delta}$
50. $\int \frac{\sin x \cos^3 x \, dx}{\Delta^3} = \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta}$
51. $\int \frac{\cos^4 x \, dx}{\Delta^3} = \frac{k'^2 + 1}{k^4} E(x, k) - \frac{2k'^2}{k^4} F(x, k) - \frac{k'^2 \sin x \cos x}{k^2 \Delta}$
- 52.⁹ $\int \frac{\sin^5 x \, dx}{\Delta^3} = \frac{k^2 k'^2 \sin^2 x + k^2 - 3}{2k^4 k'^2 \Delta} \cos x + \frac{k^2 + 3}{2k^5} \ln(k \cos x + \Delta)$
53. $\int \frac{\sin^4 x \cos x \, dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \arcsin(k \sin x)$
54. $\int \frac{\sin^3 x \cos^2 x \, dx}{\Delta} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \cos x + \frac{k^2 - 3}{2k^5} \ln(k \cos x + \Delta)$
55. $\int \frac{\sin^2 x \cos^3 x \, dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \sin x - \frac{2k^2 - 3}{2k^5} \arcsin(k \sin x)$
56. $\int \frac{\sin x \cos^4 x \, dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \cos x + \frac{3k'^2}{2k^5} \ln(k \cos x + \Delta)$
57. $\int \frac{\cos^5 x \, dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 2k^4 - 4k^2 + 3}{2k^4 \Delta} \sin x + \frac{4k^2 - 3}{2k^5} \arcsin(k \sin x)$
58. $\int \frac{dx}{\Delta^5} = \frac{-k^2 \sin x \cos x}{3k'^2 \Delta^3} - \frac{2k^2 (k'^2 + 1) \sin x \cos x}{3k'^4 \Delta} - \frac{1}{3k'^2} F(x, k)$
 $+ \frac{2(k'^2 + 1)}{3k'^4} E(x, k)$
59. $\int \frac{\sin x \, dx}{\Delta^5} = \frac{2k^2 \sin^2 x + k^2 - 3}{3k'^4 \Delta^3} \cos x$
60. $\int \frac{\cos x \, dx}{\Delta^5} = \frac{-2k^2 \sin^2 x + 3}{3\Delta^3} \sin x$

61.
$$\int \frac{\sin^2 x \, dx}{\Delta^5} = \frac{k^2 + 1}{3k'^4 k^2} E(x, k) - \frac{1}{3k'^2 k^2} F(x, k) + \frac{k^2 (k^2 + 1) \sin^2 x - 2}{3k'^4 \Delta^3} \sin x \cos x$$
62.
$$\int \frac{\sin x \cos x \, dx}{\Delta^5} = \frac{1}{3k^2 \Delta^3}$$
63.
$$\int \frac{\cos^2 x \, dx}{\Delta^5} = \frac{1}{3k^2} F(x, k) + \frac{2k^2 - 1}{3k^2 k'^2} E(x, k) + \frac{k^2 (2k^2 - 1) \sin^2 x - 3k^2 + 2}{2k'^2 \Delta} \sin x \cos x$$
64.
$$\int \frac{\sin^3 x}{\Delta^5} \, dx = \frac{(3k^2 - 1) \sin^2 x - 2}{3k'^4 \Delta^3} \cos x$$
65.
$$\int \frac{\sin^2 x \cos x}{\Delta^5} \, dx = \frac{\sin^3 x}{3\Delta^3}$$
66.
$$\int \frac{\sin x \cos^2 x}{\Delta^5} \, dx = -\frac{\cos^3 x}{3k'^2 \Delta^3}$$
67.
$$\int \frac{\cos^3 x \, dx}{\Delta^5} = \frac{-(2k^2 + 1) \sin^2 x + 3}{3\Delta^3} \sin x$$
68.
$$\int \frac{dx}{\Delta \sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$
69.
$$\int \frac{dx}{\Delta \cos x} = -\frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$
70.
$$\int \frac{dx}{\Delta \sin^2 x} = \int \frac{1 + \cot^2 x}{\Delta} \, dx = F(x, k) - E(x, k) - \Delta \cot x$$
71.
$$\int \frac{dx}{\Delta \sin x \cos x} = \int (\tan x + \cot x) \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$
72.
$$\int \frac{dx}{\Delta \cos^2 x} = \int (1 + \tan^2 x) \frac{dx}{\Delta} = F(x, k) - \frac{1}{k'^2} E(x, k) + \frac{1}{k'^2} \Delta \tan x$$
73.
$$\int \frac{\sin x}{\cos x} \frac{dx}{\Delta} = \int \tan x \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$
74.
$$\int \frac{\cos x}{\sin x} \frac{dx}{\Delta} = \int \cot x \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$
75.
$$\int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} - \frac{1 + k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$
76.
$$\int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$
77.
$$\int \frac{dx}{\Delta \sin x \cos^2 x} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$
78.
$$\int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} + \frac{2k^2 - 1}{4k'^3} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$
79.
$$\int \frac{\sin x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x}$$

80. $\int \frac{\cos x}{\sin^2 x} \frac{dx}{\Delta} = -\frac{\Delta}{\sin x}$
81. $\int \frac{\sin^2 x}{\cos x} \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{1}{k} \arcsin(k \sin x)$
82. $\int \frac{\cos^2 x}{\sin x} \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln(k \cos x + \Delta)$
83. $\int \frac{dx}{\Delta \sin^4 x} = \frac{1}{3} \{ -\Delta \cot^3 x - \Delta (2k^2 + 3) \cot x + (k^2 + 2) F(x, k) - 2(k^2 + 1) E(x, k) \}$
84. $\int \frac{dx}{\Delta \sin^3 x \cos x} = \int (\tan x + 2 \cot x + \cot^3 x) \frac{dx}{\Delta}$
 $= -\frac{\Delta}{2 \sin^2 x} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} - \frac{k^2 + 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$
85. $\int \frac{dx}{\Delta \sin^2 x \cos^2 x} = \int (\tan^2 x + 2 + \cot^2 x) \frac{dx}{\Delta}$
 $= \left(\frac{\tan x}{k'^2} - \cot x \right) \Delta + \frac{k^2 - 2}{k'^2} E(x, k) + 2 F(x, k)$
86. $\int \frac{dx}{\Delta \sin x \cos^3 x} = \int (\cot x + 2 \tan x + \tan^3 x) \frac{dx}{\Delta}$
 $= -\frac{\Delta}{2k'^2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - 3k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}$
87. $\int \frac{dx}{\Delta \cos^4 x} = \frac{1}{3k'^2} \left\{ \Delta \tan^3 x - \frac{5k^2 - 3}{k'^2} \Delta \tan x - (3k^2 - 2) F(x, k) \right.$
 $\left. + \frac{2(2k^2 - 1)}{k'^2} E(x, k) \right\}$
88. $\int \frac{\sin x}{\cos^3 x} \frac{dx}{\Delta} = \int \tan x (1 + \tan^2 x) \frac{dx}{\Delta} = \frac{\Delta}{2k'^2 \cos^2 x} - \frac{k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}$
89. $\int \frac{\cos x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta}{2 \sin^2 x} - \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$
90. $\int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \int \frac{\tan^2 x}{\Delta} dx = \frac{\Delta}{k'^2} \tan x - \frac{1}{k'^2} E(x, k)$
91. $\int \frac{\cos^2 x}{\sin^2 x} \frac{dx}{\Delta} = \int \frac{\cot^2 x}{\Delta} dx = -\Delta \cot x - E(x, k)$
92. $\int \frac{\sin^3 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$
93. $\int \frac{\cos^3 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta}$
94. $\int \frac{dx}{\Delta \sin^5 x} = -\frac{[3(1 + k^2) \sin^2 x + 2]}{8 \sin^2 x} \Delta \cos x + \frac{3k^4 + 2k^2 + 3}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$

95. $\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{(3+2k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$
96. $\int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{(3-k^2) \sin^2 x - k'^2}{2k'^2 \sin^2 x \cos x} \Delta + \frac{k^2+3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$
97. $\int \frac{dx}{\Delta \sin^2 x \cos^3 x} = \frac{(3-2k^2) \sin^2 x - 2k'^2}{2k'^2 \sin x \cos^2 x} \Delta - \frac{4k^2-3}{4k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$
98. $\int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{(5k^2-3) \sin^2 x - 6k^2 + 4}{3k'^4 \cos^3 x} \Delta - \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$
99. $\int \frac{dx}{\Delta \cos^5 x} = \frac{3(2k^2-1) \sin^2 x - 8k^2 + 5}{8k'^4 \cos^4 x} \Delta \sin x + \frac{8k^4 - 8k^2 + 3}{16k'^5} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$
100. $\int \frac{\sin x}{\cos^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \cos^2 x - k'^2}{2k'^4 \cos^3 x} \Delta$
101. $\int \frac{\cos x}{\sin^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \sin^2 x + 1}{3 \sin^3 x} \Delta$
102. $\int \frac{\sin^2 x}{\cos^3 x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} - \frac{1}{4k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$
103. $\int \frac{\cos^3 x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$
104. $\int \frac{\sin^3 x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{k} \ln (k \cos x + \Delta)$
105. $\int \frac{\cos^3 x}{\sin^2 x} \frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k} \arcsin (k \sin x)$
106. $\int \frac{\sin^4 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{2k^2+1}{2k^3} \arcsin (k \sin x)$
107. $\int \frac{\cos^4 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2-1}{2k^3} \ln (k \cos x + \Delta)$

2.585

$$\begin{aligned}
 1. \quad & \int \frac{(a + \sin x)^{p+3} dx}{\Delta} \\
 &= \frac{1}{(p+2)k^2} \left[(a + \sin x)^p \cos x \Delta \right. \\
 &\quad + 2(2p+3)ak^2 \int \frac{(a + \sin x)^{p+2} dx}{\Delta} + (p+1)(1+k^2-6a^2k^2) \int \frac{(a + \sin x)^{p+1} dx}{\Delta} \\
 &\quad - a(2p+1)(1+k^2-2a^2k^2) \int \frac{(a + b \sin x)^p dx}{\Delta} \\
 &\quad \left. - p(1-a^2)(1-a^2k^2) \int \frac{(a + \sin x)^{p-1} dx}{\Delta} \right] \\
 &\quad \left[p \neq -2, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k} \right]
 \end{aligned}$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

$$\begin{aligned}
2. \quad & \int \frac{a + \sin x}{\Delta} dx = a F(x, k) + \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x} \\
3. \quad & \int \frac{(a + \sin x)^2}{\Delta} dx = \frac{1 + k^2 a^2}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{a}{k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x} \\
4.^6 \quad & \int \frac{dx}{(a + \sin x) \Delta} = \frac{1}{a} \Pi\left(x, \frac{1}{a^2}, k\right) - \int \frac{\sin x dx}{(a^2 - \sin^2 x) \Delta},
\end{aligned}$$

where

$$5. \quad \int \frac{\sin x dx}{(a^2 - \sin^2 x) \Delta} = \frac{-1}{2\sqrt{(1-a^2)(1-a^2k^2)}} \ln \frac{\sqrt{1-a^2}\Delta - \sqrt{1-k^2a^2}\cos x}{\sqrt{1-a^2}\Delta + \sqrt{1-k^2a^2}\cos x}$$

2.586

$$\begin{aligned}
1. \quad & \int \frac{dx}{(a + \sin x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{(a + \sin x)^{n-1}} \right. \\
& \quad - (2n-3)(1+k^2-2a^2k^2) a \int \frac{dx}{(a + \sin x)^{n-1} \Delta} \\
& \quad - (n-2)(6a^2k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x)^{n-2} \Delta} \\
& \quad \left. - (10-4n)ak^2 \int \frac{dx}{(a + \sin x)^{n-3} \Delta} - (n-3)k^2 \int \frac{dx}{(a + \sin x)^{n-4} \Delta} \right] \\
& \quad \left[n \neq 1, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k} \right]
\end{aligned}$$

This integral can be reduced to the integrals:

$$\begin{aligned}
2. \quad & \int \frac{dx}{(a + \sin x)^2 \Delta} = \frac{1}{(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{a + \sin x} - a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x) \Delta} \right. \\
& \quad \left. - 2ak^2 \int \frac{(a + \sin x) dx}{\Delta} + k^2 \int \frac{(a + \sin x)^2 dx}{\Delta} \right] \\
& \quad \text{(see 2.585 2, 3, 4)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int \frac{dx}{(a + \sin x)^3 \Delta} = \frac{1}{2(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{(a + \sin x)^2} - 3a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x)^2 \Delta} \right. \\
& \quad \left. - (6a^2k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x) \Delta} + 2ak^2 F(x, k) \right] \\
& \quad \text{(see 2.585 4 and 2.586 2)}
\end{aligned}$$

For $a = \pm 1$, we have:

$$\begin{aligned}
4. \quad & \int \frac{dx}{(1 \pm \sin x)^n \Delta} = \frac{1}{(2n-1)k'^2} \left[\mp \frac{\cos x \Delta}{(1 \pm \sin x)^n} + (n-1)(1-5k^2) \int \frac{dx}{(1 \pm \sin x)^{n-1} \Delta} \right. \\
& \quad \left. + 2(2n-3)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-2} \Delta} - (n-2)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-3} \Delta} \right]
\end{aligned}$$

GU (241)(6a)

This integral can be reduced to the following integrals:

$$5. \quad \int \frac{dx}{(1 \pm \sin x) \Delta} = \frac{\mp \cos x \Delta}{k'^2 (1 \pm \sin x)} + F(x, k) - \frac{1}{k'^2} E(x, k) \quad \text{GU (241)(6c)}$$

$$6. \quad \int \frac{dx}{(1 \pm \sin x)^2 \Delta} = \frac{1}{3k'^4} \left\{ \mp \frac{k'^2 \cos x \Delta}{(1 \pm \sin x)^2} \mp \frac{(1 - 5k^2) \cos x \Delta}{1 \pm \sin x} \right. \\ \left. + (1 - 3k^2) k'^2 F(x, k) - (1 - 5k^2) E(x, k) \right\} \quad \text{GU (241)(6b)}$$

For $a = \pm \frac{1}{k}$, we have

$$7. \quad \int \frac{dx}{(1 \pm k \sin x)^n \Delta} = \frac{1}{(2n-1)k'^2} \left[\pm \frac{k \cos x \Delta}{(1 \pm k \sin x)^n} + (n-1)(5-k^2) \int \frac{dx}{(1 \pm k \sin x)^{n-1} \Delta} \right. \\ \left. - 2(2n-3) \int \frac{dx}{(1 \pm k \sin x)^{n-2} \Delta} + (n-2) \int \frac{dx}{(1 \pm k \sin x)^{n-3} \Delta} \right] \quad \text{GU (241)(7a)}$$

This integral can be reduced to the following integrals:

$$8. \quad \int \frac{dx}{(1 \pm k \sin x) \Delta} = \pm \frac{k \cos x \Delta}{k'^2 (1 \pm k \sin x)} + \frac{1}{k'^2} E(x, k) \quad \text{GU (241)(7b)}$$

$$9. \quad \int \frac{dx}{(1 \pm k \sin x)^2 \Delta} = \frac{1}{3k'^4} \left[\pm \frac{kk'^2 \cos x \Delta}{(1 \pm k \sin x)^2} \pm \frac{k(5-k^2) \cos x \Delta}{1 \pm k \sin x} \right. \\ \left. - 2k'^2 F(x, k) + (5-k^2) E(x, k) \right] \quad \text{GU(241)(7c)}$$

2.587

$$1. \quad \int \frac{(b + \cos x)^{p+3} dx}{\Delta} = \frac{1}{(p+2)k^2} \left[(b + \cos x)^p \sin x \Delta + 2(2p+3)bk^2 \int \frac{(b + \cos x)^{p+2} dx}{\Delta} \right. \\ - (p+1) \left(k'^2 - k^2 + 6b^2k^2 \right) \int \frac{(b + \cos x)^{p+1} dx}{\Delta} \\ + (2p+1)b \left(k'^2 - k^2 + b^2k^2 \right) \int \frac{(b + \cos x)^p dx}{\Delta} \\ \left. + p(1-b^2) \left(k'^2 + k^2b^2 \right) \int \frac{(b + \cos x)^{p-1} dx}{\Delta} \right] \\ \left[p \neq -2, \quad b \neq \pm 1, \quad b \neq \frac{ik'}{k} \right]$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

$$2. \quad \int \frac{b + \cos x}{\Delta} dx = b F(x, k) + \frac{1}{k} \arcsin(k \sin x)$$

$$3. \quad \int \frac{(b + \cos x)^2}{\Delta} dx = \frac{b^2k^2 - k'^2}{k^2} F(x, k) + \frac{1}{k^2} E(x, k) + \frac{2b}{k} \arcsin(k \sin x)$$

$$4. \quad \int \frac{dx}{(b + \cos x) \Delta} = \frac{b}{b^2 - 1} \Pi \left(x, \frac{1}{b^2 - 1}, k \right) + \int \frac{\cos x dx}{(1 - b^2 - \sin^2 x) \Delta},$$

where

$$5. \quad \int \frac{\cos x \, dx}{(1 - b^2 - \sin^2 x) \Delta} = \frac{1}{2\sqrt{(1 - b^2)(k'^2 + k^2 b^2)}} \ln \frac{\sqrt{1 - b^2} \Delta + k\sqrt{k'^2 + k^2 b^2} \sin x}{\sqrt{1 - b^2} \Delta - k\sqrt{k'^2 + k^2 b^2} \sin x}$$

2.588

$$1. \quad \int \frac{dx}{(b + \cos x)^n \Delta} = \frac{1}{(n - 1)(1 - b^2)(k'^2 + b^2 k^2)} \left[\frac{-k'^2 \sin x \Delta}{(b + \cos x)^{-1}} \right. \\ - (2n - 3)(1 - 2k^2 + 2b^2 k^2) b \int \frac{dx}{(b + \cos x)^{n-1} \Delta} \\ - (n - 2)(2k^2 - 1 - 6b^2 k^2) \int \frac{dx}{(b + \cos x)^{n-2} \Delta} \\ \left. - (4n - 10)bk^2 \int \frac{dx}{(b + \cos x)^{n-3} \Delta} + (n - 3)k^2 \int \frac{dx}{(b + \cos x)^{n-4} \Delta} \right] \\ \left[n \neq 1, \quad b \neq \pm 1, \quad b \neq \pm \frac{ik'}{k} \right]$$

This integral can be reduced to the following integrals:

$$2. \quad \int \frac{dx}{(b + \cos x)^2 \Delta} = \frac{1}{(1 - b^2)(k'^2 + b^2 k^2)} \left[\frac{-k'^2 \sin x \Delta}{b + \cos x} - (1 - 2k^2 + 2b^2 k^2) b \int \frac{dx}{(b + \cos x) \Delta} \right. \\ \left. + 2bk^2 \int \frac{b + \cos x}{\Delta} dx - k^2 \int \frac{(b + \cos x)^2}{\Delta} dx \right] \\ \text{(see 2.587 2, 3, 4)}$$

$$3. \quad \int \frac{dx}{(b + \cos x)^3 \Delta} = \frac{1}{2(1 - b^2)(k'^2 + b^2 k^2)} \left[\frac{-k'^2 \sin x \Delta}{(b + \cos x)^2} \right. \\ - 3b(1 - 2k^2 + 2k^2 b^2) \int \frac{dx}{(b + \cos x)^2 \Delta} \\ \left. - (2k^2 - 1 - 6b^2 k^2) \int \frac{dx}{(b + \cos x) \Delta} - 2bk^2 F(x, k) \right] \\ \text{(see 2.588 2 and 2.587 4)}$$

2.589

$$1. \quad \int \frac{(c + \tan x)^{p+3} dx}{\Delta} = \frac{1}{(p + 2)k'^2} \left[\frac{(c + \tan x)^p \Delta}{\cos^2 x} + 2(2n + 3)ck'^2 \int \frac{(c + \tan x)^{p+2} dx}{\Delta} \right. \\ - (p + 1)(1 + k'^2 + 6c^2 k'^2) \int \frac{(c + \tan x)^{p+1} dx}{\Delta} \\ + (2p + 1)c(1 + k'^2 + 2c^2 k'^2) \int \frac{(c + \tan x)^p dx}{\Delta} \\ \left. - p(1 + c^2)(1 + k'^2 c^2) \int \frac{(c + \tan x)^{p-1} dx}{\Delta} \right] \\ [p \neq -2]$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

$$\begin{aligned}
2. \quad & \int \frac{c + \tan x}{\Delta} dx = c F(x, k) + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} \\
3. \quad & \int \frac{(c + \tan x)^2}{\Delta} dx = \frac{1}{k'^2} \tan x \Delta + c^2 F(x, k) - \frac{1}{k'^2} E(x, k) + \frac{c}{k'} \ln \frac{\Delta + k'}{\Delta - k'} \\
4. \quad & \int \frac{dx}{(c + \tan x) \Delta} = \frac{c}{1 + c^2} F(x, k) + \frac{1}{c(1 + c^2)} \Pi \left(x, -\frac{1 + c^2}{c^2}, k \right) \\
& \quad - \int \frac{\sin x \cos x dx}{[c^2 - (1 + c^2) \sin^2 x] \Delta},
\end{aligned}$$

where

$$5. \quad \int \frac{\sin x \cos x dx}{[c^2 - (1 + c^2) \sin^2 x] \Delta} = \frac{1}{2\sqrt{(1 + c^2)(1 + c^2 k'^2)}} \ln \frac{\sqrt{1 + c^2 k'^2} + \sqrt{1 + c^2} \Delta}{\sqrt{1 + c^2 k'^2} - \sqrt{1 + c^2} \Delta}$$

2.591

$$\begin{aligned}
1. \quad & \int \frac{dx}{(c + \tan x)^n \Delta} = \frac{1}{(n - 1)(1 + c^2)(1 + k'^2 c^2)} \left[-\frac{\Delta}{(c + \tan x)^{n-1} \cos^2 x} \right. \\
& \quad + (2n - 3)c(1 + k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \tan x)^{n-1} \Delta} \\
& \quad - (n - 2)(1 + k'^2 + 6c^2 k'^2) \int \frac{dx}{(c + \tan x)^{n-2} \Delta} \\
& \quad \left. + (4n - 10)ck'^2 \int \frac{dx}{(c + \tan x)^{n-3} \Delta} - (n - 3)k'^2 \int \frac{dx}{(c + \tan x)^{n-4} \Delta} \right]
\end{aligned}$$

This integral can be reduced to the integrals:

$$\begin{aligned}
2. \quad & \int \frac{dx}{(c + \tan x)^2 \Delta} = \frac{1}{(1 + c^2)(1 + k'^2 c^2)} \left[\frac{-\Delta}{(c + \tan x) \cos^2 x} \right. \\
& \quad + c(1 + k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \tan x) \Delta} \\
& \quad \left. - 2ck'^2 \int \frac{c + \tan x}{\Delta} dx + k'^2 \int \frac{(c + \tan x)^2}{\Delta} dx \right] \\
& \quad \text{(see 2.589 2, 3, 4)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int \frac{dx}{(c + \tan x)^3 \Delta} = \frac{1}{2(1 + c^2)(1 + k'^2 c^2)} \left[\frac{-\Delta}{(c + \tan x)^2 \cos^2 x} \right. \\
& \quad + 3c(1 + k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \tan x)^2 \Delta} \\
& \quad \left. - (1 + k'^2 + 6c^2 k'^2) \int \frac{dx}{(c + \tan x) \Delta} + 2ck'^2 F(x, k) \right] \\
& \quad \text{(see 2.591 2 and 2.589 4)}
\end{aligned}$$

2.592

$$1. \quad P_n = \int \frac{(a + \sin^2 x)^n}{\Delta} dx$$

The recursion formula

$$P_{n+1} = \frac{1}{(2n+3)k^2} \left\{ (a + \sin^2 x)^n \sin x \cos x \Delta + (2n+2)(1+k^2+3ak^2) P_{n+1} - (2n+1)[1+2a(1+k^2)+3a^2k^2] P_n + 2na(1+a)(1+k^2a) P_{n-1} \right\}$$

reduces this integral (for n an integer) to the integrals:

$$2. \quad P_1 \quad (\text{see } \mathbf{2.584} \text{ 1 and } \mathbf{2.584} \text{ 4})$$

$$3. \quad P_0 \quad (\text{see } \mathbf{2.584} \text{ 1})$$

$$4. \quad P_{-1} = \int \frac{dx}{(a + \sin^2 x) \Delta} = \frac{1}{a} \Pi \left(x, \frac{1}{a}, k \right)$$

For $a = 0$

$$5. \quad \int \frac{dx}{\sin^2 x \Delta} \quad (\text{see } \mathbf{2.584} \text{ 70}) \quad \text{H (124)a}$$

$$6. \quad T_n = \int \frac{dx}{(h + g \sin^2 x)^n \Delta}$$

can be calculated by means of the recursion formula:

$$T_{n-3} = \frac{1}{(2n-5)k^2} \left\{ \frac{-g^2 \sin x \cos x \Delta}{(h + g \sin^2 x)^{n-1}} + 2(n-2)[g(1+k^2) + 3hk^2] T_{n-2} - (2n-3)[g^2 + 2hg(1+k^2) + 3h^2k^2] T_{n-1} + 2(n-1)h(g+h)(g+hk^2) T_n \right\}$$

2.593

$$1. \quad Q_n = \int \frac{(b + \cos^2 x)^n}{\Delta} dx$$

The recursion formula

$$Q_{n+2} = \frac{1}{(2n+3)k^2} \left\{ (b + \cos^2 x)^n \sin x \sin x \Delta - (2n+2)(1-2k^2-3bk^2) Q_{n+1} + (2n+1)[k'^2 + 2b(k'^2 - k^2) - 3b^2k^2] n - 2nb(1-b)(k'^2 - k^2b) Q_{n-1} \right\}$$

reduces this integral (for n an integer) to the integrals:

$$2. \quad Q_1 \quad (\text{see } \mathbf{2.584} \text{ 1 and } \mathbf{2.584} \text{ 6})$$

$$3. \quad Q_0 \quad (\text{see } \mathbf{2.584} \text{ 1})$$

$$4. \quad Q_{-1} = \int \frac{dx}{(b + \cos^2 x) \Delta} = \frac{1}{b+1} \Pi \left(x, -\frac{1}{b+1}, k \right)$$

For $b = 0$

$$5. \quad \int \frac{dx}{\cos^2 x \Delta} \quad (\text{see } \mathbf{2.584} \text{ 72}) \quad \text{H (123)}$$

2.594

$$1. \quad R_n = \int \frac{(c + \tan^2 x)^n dx}{\Delta}$$

The recursion formula

$$R_{n+2} = \frac{1}{(2n+3)k'^2} \left\{ \frac{(c + \tan^2 x)^n \tan x \Delta}{\cos^2 x} - (2n+2) (1 + k'^2 - 3ck'^2) R_{n+1} \right. \\ \left. + (2n-1) [1 - 2c(1 + k'^2) + 3c^2 k'^2] R_n + 2nc(1-c) (1 - k'^2 c) R_{n-1} \right\}$$

reduces this integral (for n an integer) to the integrals:

$$2. \quad R_1 \quad (\text{see } \mathbf{2.584} \text{ 1 and } \mathbf{2.584} \text{ 90})$$

$$3. \quad R_0 \quad (\text{see } \mathbf{2.584} \text{ 1})$$

$$4. \quad R_{-1} = \int \frac{dx}{(c + \tan^2 x) \Delta} = \frac{1}{c-1} F(x, k) + \frac{1}{c(1-c)} \Pi \left(x, \frac{1-c}{c}, k \right)$$

For $c = 0$, see **2.582** 5.

2.595 Integrals of the type $\int R \left(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x} \right) dx$ for $p^2 > 1$.

Notation: $\alpha = \arcsin(p \sin x)$.

Basic formulas

$$1. \quad \int \frac{dx}{\sqrt{1 - p^2 \sin^2 x}} = \frac{1}{p} F \left(\alpha, \frac{1}{p} \right) \quad [p^2 > 1] \quad \text{BY (283.00)}$$

$$2. \quad \int \sqrt{1 - p^2 \sin^2 x} dx = p E \left(\alpha, \frac{1}{p} \right) - \frac{p^2 - 1}{p} F \left(\alpha, \frac{1}{p} \right) \quad [p^2 > 1] \quad \text{BY (283.03)}$$

$$3. \quad \int \frac{dx}{(1 - r^2 \sin^2 x) \sqrt{1 - p^2 \sin^2 x}} = \frac{1}{p} \Pi \left(\alpha, \frac{r^2}{p^2}, \frac{1}{p} \right) \quad [p^2 > 1] \quad \text{BY (283.02)}$$

To evaluate integrals of the form $\int R \left(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x} \right) dx$ for $p^2 > 1$, we may use formulas

2.583 and **2.584**, making the following modifications in them. We replace

- (1) k with p ;
- (2) k'^2 with $1 - p^2$;

$$(3) \quad F(x, k) \text{ with } \frac{1}{p} F\left(\alpha, \frac{1}{p}\right);$$

$$(4) \quad E(x, k) \text{ with } p E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right).$$

For example (see **2.584** 15):

2.596

$$\begin{aligned} 1.^{10} \quad \int \frac{\cos^4 x \, dx}{\sqrt{1 - p^2 \sin^2 x}} &= \frac{\sin x \cos x \sqrt{1 - p^2 \sin^2 x}}{3p^2} + \frac{4p^2 - 2}{3p^4} \left[p E\left(\alpha, \frac{1}{p}\right) \right. \\ &\quad \left. - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] + \frac{2 - 5p^2 + 3p^4}{3p^4} \cdot \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) \\ &= \frac{\sin x \cos x \sqrt{1 - p^2 \sin^2 x}}{3p^2} - \frac{p^2 - 1}{3p^3} F\left(\alpha, \frac{1}{p}\right) + \frac{4p^2 - 2}{3p^3} E\left(\alpha, \frac{1}{p}\right) \quad [p^2 > 1] \end{aligned}$$

For example (see **2.583** 36):

$$\begin{aligned} 2. \quad \int \frac{\sqrt{1 - p^2 \sin^2 x}}{\cos^2 x} \, dx &= \tan x \sqrt{1 - p^2 \sin^2 x} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \left[p E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] \\ &= p \left[F\left(\alpha, \frac{1}{p}\right) - E\left(\alpha, \frac{1}{p}\right) \right] + \tan x \sqrt{1 - p^2 \sin^2 x} \\ &\quad [p^2 > 1] \end{aligned}$$

For example (see **2.584** 37):

$$\begin{aligned} 3. \quad \int \frac{dx}{\sqrt{(1 - p^2 \sin^2 x)^3}} &= \frac{-1}{p^2 - 1} \left[p E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] - \frac{p^2}{1 - p^2} \frac{\sin x \cos x}{\sqrt{1 - p^2 \sin^2 x}} \\ &= \frac{p^2}{p^2 - 1} \frac{\sin x \cos x}{\sqrt{1 - p^2 \sin^2 x}} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \frac{p}{p^2 - 1} E\left(\alpha, \frac{1}{p}\right) \\ &\quad [p^2 > 1] \end{aligned}$$

2.597 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1 + p^2 \sin^2 x}\right) dx$

Notation: $\alpha = \arcsin\left(\frac{\sqrt{1 + p^2 \sin x}}{\sqrt{1 + p^2 \sin^2 x}}\right)$

Basic formulas

$$1. \quad \int \frac{dx}{\sqrt{1 + p^2 \sin^2 x}} = \frac{1}{\sqrt{1 + p^2}} F\left(\alpha, \frac{p}{\sqrt{1 + p^2}}\right) \quad \text{BY (282.00)}$$

$$2. \quad \int \sqrt{1 + p^2 \sin^2 x} \, dx = \sqrt{1 + p^2} E\left(\alpha, \frac{p}{\sqrt{1 + p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1 + p^2 \sin^2 x}} \quad \text{BY (282.03)}$$

3. $\frac{\sqrt{1+p^2 \sin^2 x} \, dx}{1+(p^2-r^2 p^2-r^2) \sin^2 x} = \frac{1}{\sqrt{1+p^2}} \Pi\left(\alpha, r^2, \frac{p}{\sqrt{1+p^2}}\right)$ BY (282.02)
4. $\int \frac{\sin x \, dx}{\sqrt{1+p^2 \sin^2 x}} = -\frac{1}{p} \arcsin\left(\frac{p \cos x}{\sqrt{1+p^2}}\right)$
5. $\int \frac{\cos x \, dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{p} \ln\left(p \sin x + \sqrt{1+p^2 \sin^2 x}\right)$
6. $\int \frac{dx}{\sin x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{\sqrt{1+p^2 \sin^2 x} - \cos x}{\sqrt{1+p^2 \sin^2 x} + \cos x}$
7. $\int \frac{dx}{\cos x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2} \sin x}$
8. $\int \frac{\tan x \, dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2}}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2}}$
9. $\int \frac{\cot x \, dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{1 - \sqrt{1+p^2 \sin^2 x}}{1 + \sqrt{1+p^2 \sin^2 x}}$

2.598 To calculate integrals of the form $\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) \, dx$, we may use formulas **2.583** and **2.584**, making the following modifications in them. We replace

- (1) k^2 with $-p^2$;
- (2) k'^2 with $1+p^2$;
- (3) $F(x, k)$ with $\frac{1}{\sqrt{1+p^2}} F\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$;
- (4) $E(x, k)$ with $\sqrt{1+p^2} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}}$;
- (5) $\frac{1}{k} \ln(k \cos x + \Delta)$ with $\frac{1}{p} \arcsin \frac{p \cos x}{\sqrt{1+p^2}}$;
- (6) $\frac{1}{k} \arcsin(k \sin x)$ with $\frac{1}{p} \ln(p \sin x + \sqrt{1+p^2 \sin^2 x})$.

For example (see **2.584** 90):

$$\begin{aligned}
 1. \quad \int \frac{\tan^2 x \, dx}{\sqrt{1+p^2 \sin^2 x}} &= \frac{1}{(1+p^2)} \left[\tan x \sqrt{1+p^2 \sin^2 x} \right. \\
 &\quad \left. - \sqrt{1+p^2} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) + p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}} \right] \\
 &= -\frac{1}{\sqrt{1+p^2}} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) + \frac{\tan x}{\sqrt{1+p^2 \sin^2 x}}
 \end{aligned}$$

For example (see **2.584** 37):

$$2. \quad \int \frac{dx}{\sqrt{(1+p^2 \sin^2 x)^3}} = \frac{1}{\sqrt{1+p^2}} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$$

2.599 Integrals of the form $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx \quad [a^2 > 1]$

Notation: $\alpha = \arcsin\left(\frac{a \cos x}{\sqrt{a^2 - 1}}\right)$.

Basic formulas:

$$1. \quad \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{1}{a} F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1] \quad \text{BY (285.00)a}$$

$$2. \quad \int \sqrt{a^2 \sin^2 x - 1} dx = \frac{1}{a} F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) - a E\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1] \quad \text{BY (285.06)a}$$

$$3. \quad \int \frac{dx}{(1-r^2 \sin^2 x) \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a(r^2 - 1)} \Pi\left(\alpha, \frac{r^2(a^2 - 1)}{a^2(r^2 - 1)}, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1, \quad r^2 > 1] \quad \text{BY (285.02)a}$$

$$4. \quad \int \frac{\sin x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{\alpha}{a} \quad [a^2 > 1]$$

$$5. \quad \int \frac{\cos x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a} \ln\left(a \sin x + \sqrt{a^2 \sin^2 x - 1}\right) \quad [a^2 > 1]$$

$$6. \quad \int \frac{dx}{\sin x \sqrt{a^2 \sin^2 x - 1}} = -\arctan \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1]$$

$$7. \quad \int \frac{dx}{\cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 2}} \ln \frac{\sqrt{a^2 - 1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} \sin x - \sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1]$$

$$8. \quad \int \frac{\tan x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} - \sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1]$$

$$9. \quad \int \frac{\cot x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\arcsin\left(\frac{1}{a \sin x}\right) \quad [a^2 > 1]$$

2.611 To calculate integrals of the type $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx$ for $a^2 > 1$, we may use formulas **2.583** and **2.584**. In doing so, we should follow the procedure outlined below:

- (1) In the right members of these formulas, the following functions should be replaced with integrals equal to them:

$$\begin{aligned}
F(x, k) & \text{ should be replaced with } \int \frac{dx}{\Delta} \\
E(x, k) & \text{ should be replaced with } \int \Delta dx \\
-\frac{1}{k} \ln(k \cos x + \Delta) & \text{ should be replaced with } \int \frac{\sin x \, dx}{\Delta} \\
\frac{1}{k} \arcsin(k \sin x) & \text{ should be replaced with } \int \frac{\cos x \, dx}{\Delta} \\
\frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x} & \text{ should be replaced with } \int \frac{dx}{\Delta \sin x} \\
\frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} & \text{ should be replaced with } \int \frac{dx}{\Delta \cos x} \\
\frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} & \text{ should be replaced with } \int \frac{\tan x}{\Delta} dx \\
\frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} & \text{ should be replaced with } \int \frac{\cot x}{\Delta} dx
\end{aligned}$$

- (2) Then, on both sides of the equations, we should replace Δ with $i\sqrt{a^2 \sin^2 x - 1}$, k with a and k'^2 with $1 - a^2$.
- (3) Both sides of the resulting equations should be multiplied by i , as a result of which only real functions ($a^2 > 1$) should appear on both sides of the equations.
- (4) The integrals on the right sides of the equations should be replaced with their values found from formulas **2.599**.

Examples:

1. We rewrite equation **2.584 4** in the form

$$\int \frac{\sin^2 x}{i\sqrt{a^2 \sin^2 x - 1}} dx = \frac{1}{a^2} \int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{1}{a^2} \int i\sqrt{a^2 \sin^2 x - 1} dx,$$

from which we get

$$\int \frac{\sin^2 x \, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a^2} \left\{ \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} + \int \sqrt{a^2 \sin^2 x - 1} \, dx \right\} = -\frac{1}{a} E \left(\alpha, \frac{\sqrt{a^2 - 1}}{a} \right)$$

$[a^2 > 1]$

2. We rewrite equation **2.584 58** as follows:

$$\begin{aligned}
\int \frac{dx}{i^5 \sqrt{(a^2 \sin^2 x - 1)}^5} &= -\frac{2a^4 (a^2 - 2) \sin^2 x - (3a^2 - 5) a^2}{3(1 - a^2)^2 i^3 \sqrt{(a^2 \sin^2 x - 1)}^3} \sin x \cos x \\
&\quad - \frac{1}{3(1 - a^2)} \int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{2a^2 - 4}{3(1 - a^2)^2} \int i\sqrt{a^2 \sin^2 x - 1} \, dx
\end{aligned}$$

from which we obtain

$$\int \frac{dx}{\sqrt{(a^2 \sin^2 x - 1)^5}} = \frac{2a^4 (a^2 - 2) \sin^2 x - (3a^2 - 5) a^2}{3(1 - a^2)^2 \sqrt{(a^2 \sin^2 x - 1)^3}} \sin x \cos x + \frac{1}{3(1 - a^2)^2 a} \\ \times \left\{ (a^2 - 3) F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) - 2a^2 (a^2 - 2) E\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \right\} \\ [a^2 > 1]$$

3. We rewrite equation **2.584** 71 in the form

$$\int \frac{dx}{\sin x \cos x \sqrt{a^2 \sin^2 x - 1}} = \int \frac{\cot x dx}{i \sqrt{a^2 \sin^2 x - 1}} + \int \frac{\tan x dx}{i \sqrt{a^2 \sin^2 x - 1}},$$

from which we obtain

$$\int \frac{dx}{\sin x \cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} - \sqrt{a^2 \sin^2 x - 1}} - \arcsin\left(\frac{1}{a \sin x}\right) \\ [a^2 > 1]$$

2.612 Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx$.

To find integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx$, we make the substitution $x = \frac{\pi}{2} - y$, which yields

$$\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx = - \int R(\cos y, \sin y, \sqrt{1 - k^2 \sin^2 y}) dy.$$

The integrals $\int R(\cos y, \sin y, \sqrt{1 - k^2 \sin^2 y}) dy$ are found from formulas **2.583** and **2.584**. As a result of the use of these formulas (where it is assumed that the original integral can be reduced only to integrals of the first and second Legendre forms), when we replace the functions $F(x, k)$ and $E(x, k)$ with the corresponding integrals, we obtain an expression of the form

$$-g(\cos y, \sin y) - A \int \frac{dy}{\sqrt{1 - k^2 \sin^2 y}} - B \int \sqrt{1 - k^2 \sin^2 y} dy$$

Returning now to the original variable x , we obtain

$$\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx = -g(\sin x, \cos x) - A \int \frac{dx}{\sqrt{1 - k^2 \cos^2 x}} - B \int \sqrt{1 - k^2 \cos^2 x} dx$$

The integrals appearing in this expression are found from the formulas

1. $\int \frac{dx}{\sqrt{1 - k^2 \cos^2 x}} = F\left(\arcsin\left(\frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}}\right), k\right)$
2. $\int \sqrt{1 - k^2 \cos^2 x} dx = E\left(\arcsin\left(\frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}}\right), k\right) - \frac{k^2 \sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}}$

2.613 Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - p^2 \cos^2 x}) dx \quad [p > 1]$.

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{1 - p^2 \cos^2 x}) dx$, where $[p > 1]$, we proceed as in section **2.612**. Here, we use the formulas

1. $\int \frac{dx}{\sqrt{1-p^2 \cos^2 x}} = -\frac{1}{p} F\left(\arcsin(p \cos x), \frac{1}{p}\right) \quad [p > 1]$
2. $\int \sqrt{1-p^2 \cos^2 x} dx = \frac{p^2-1}{p} F\left(\arcsin(p \cos x), \frac{1}{p}\right) - p E\left(\arcsin(p \cos x), \frac{1}{p}\right)$

2.614 Integrals of the form $\int R(\sin x, \cos x, \sqrt{1+p^2 \cos^2 x}) dx$.

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{1+p^2 \cos^2 x}) dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R(\sin x, \cos x, \sqrt{1+p^2 \cos^2 x}) dx = -\int R(\cos y, \sin y, \sqrt{1+p^2 \sin^2 y}) dy.$$

To calculate the integrals $-\int R(\cos y, \sin y, \sqrt{1+p^2 \sin^2 y}) dy$, we need to use first what was said in **2.598** and **2.612** and then, after returning to the variable x , the formulas

1. $\int \frac{dx}{\sqrt{1+p^2 \cos^2 x}} = \frac{1}{\sqrt{1+p^2}} F\left(x, \frac{p}{\sqrt{1+p^2}}\right)$
2. $\int \sqrt{1+p^2 \cos^2 x} dx = \sqrt{1+p^2} E\left(x, \frac{p}{\sqrt{1+p^2}}\right)$

2.615 Integrals of the form $\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx \quad [a > 1]$.

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx = -\int R(\cos y, \sin y, \sqrt{a^2 \sin^2 y - 1}) dy$$

To calculate the integrals $-\int R(\cos y, \sin y, \sqrt{a^2 \sin^2 y - 1}) dy$, we use what was said in **2.611** and then, after returning to the variable x , we use the formulas

1. $\int \frac{dx}{\sqrt{a^2 \cos^2 x - 1}} = \frac{1}{a} F\left(\arcsin\left(\frac{a \sin x}{\sqrt{a^2 - 1}}\right), \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a > 1]$
2. $\int \sqrt{a^2 \cos^2 x - 1} dx = a E\left(\arcsin\left(\frac{a \sin x}{\sqrt{a^2 - 1}}\right), \frac{\sqrt{a^2 - 1}}{a}\right) - \frac{1}{a} F\left(\arcsin\left(\frac{a \sin x}{\sqrt{a^2 - 1}}\right), \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a > 1]$

2.616¹¹ Integrals of the form $\int R \left(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}, \sqrt{1 - q^2 \sin^2 x} \right) dx$.

Notation: $\alpha = \arcsin \left(\frac{\sqrt{1 - p^2} \sin x}{\sqrt{1 - p^2 \sin^2 x}} \right)$.

$$1. \quad \int \frac{dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} = \frac{1}{\sqrt{1 - p^2}} F \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.00)}$$

$$2. \quad \int \frac{\tan^2 x dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} = \frac{\tan x \sqrt{1 - q^2 \sin^2 x}}{(1 - q^2) \sqrt{1 - p^2 \sin^2 x}} \\ - \frac{1}{(1 - q^2) \sqrt{1 - p^2}} E \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.07)}$$

$$3. \quad \int \frac{\tan^4 x dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} \\ = \frac{1}{3(1 - q^2)^2(1 - p^2)^{\frac{3}{2}}} \times \left[2(2 - p^2 - q^2) E \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) - (1 - q^2) F \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \right] \\ + \frac{2p^2 + q^2 - 3 + \sin^2 x (4 - 3p^2 - 2q^2 + p^2 q^2)}{3(1 - p^2)(1 - q^2)^2} \frac{\sin x}{\cos^2 x} \sqrt{\frac{1 - q^2 \sin^2 x}{1 - p^2 \sin^2 x}} \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.07)}$$

$$4. \quad \int \frac{\sin^2 x dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}^3} \\ = \frac{\sqrt{1 - p^2}}{(1 - q^2)(q^2 - p^2)} E \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) - \frac{1}{(q^2 - p^2) \sqrt{1 - p^2}} F \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ - \frac{\sin x \cos x}{(1 - q^2) \sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.06)}$$

$$5. \quad \int \frac{\cos^2 x dx}{\sqrt{(1 - p^2 \sin^2 x)}^3 (1 - q^2 \sin^2 x)} \\ = \frac{\sqrt{1 - p^2}}{q^2 - p^2} E \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) - \frac{1 - q^2}{(q^2 - p^2) \sqrt{1 - p^2}} F \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.05)}$$

$$\begin{aligned}
6. \quad & \int \frac{\cos^4 x \, dx}{\sqrt{(1-p^2 \sin^2 x)^5} (1-q^2 \sin^2 x)} \\
&= \frac{(1-p^2)^{\frac{3}{2}}}{3(q^2-p^2)^2} \left[\frac{(2+p^2-3q^2)(1-q^2)}{(1-p^2)^2} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \right. \\
&\quad \left. + 2 \frac{2q^2-p^2-1}{1-p^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \right] + \frac{(1-p^2) \sin x \cos x \sqrt{1-q^2 \sin^2 x}}{3(q^2-p^2) \sqrt{(1-p^2 \sin^2 x)^3}} \\
&\quad \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.05)}
\end{aligned}$$

$$\begin{aligned}
7. \quad & \int \frac{dx}{1-p^2 \sin^2 x} \sqrt{\frac{1-q^2 \sin^2 x}{1-p^2 \sin^2 x}} = \frac{1}{\sqrt{1-p^2}} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \\
&\quad \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \\
&\quad \text{BY (284.01)}
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int \sqrt{\frac{1-p^2 \sin^2 x}{(1-q^2 \sin^2 x)^3}} \, dx = \frac{\sqrt{1-p^2}}{1-q^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{q^2-p^2}{1-q^2} \frac{\sin x \cos x}{\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)}} \\
&\quad \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right]. \\
&\quad \text{BY (284.04)}
\end{aligned}$$

$$\begin{aligned}
9. \quad & \int \frac{dx}{1+(p^2 r^2 - p^2 - r^2) \sin^2 x} \sqrt{\frac{1-p^2 \sin^2 x}{1-q^2 \sin^2 x}} = \frac{1}{\sqrt{1-p^2}} \Pi\left(\alpha, r^2, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \\
&\quad \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right]. \\
&\quad \text{BY (284.02)}
\end{aligned}$$

2.617 Notation: $\alpha = \arcsin \sqrt{\frac{\sqrt{b^2+c^2} - b \sin x - c \cos x}{2\sqrt{b^2+c^2}}}, \quad r = \sqrt{\frac{2\sqrt{b^2+c^2}}{a + \sqrt{b^2+c^2}}}.$

$$\begin{aligned}
1. \quad & \int \frac{dx}{\sqrt{a+b \sin x + c \cos x}} \\
&= -\frac{2}{\sqrt{a + \sqrt{b^2+c^2}}} F(\alpha, r) \\
&\quad \left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \\
&\quad \text{BY (294.00)} \\
&= -\frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}} F(\alpha, r) \\
&\quad \left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \\
&\quad \text{BY (293.00)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int \frac{\sin x \, dx}{\sqrt{a + b \sin x + c \cos x}} &= -\frac{\sqrt{2b}}{\sqrt[4]{(b^2 + c^2)^3}} \{2 E(\alpha, r) - F(\alpha, r)\} + \frac{2c}{b^2 + c^2} \sqrt{a + b \sin x + c \cos x} \\
&\left[0 < |a| < \sqrt{b^2 + c^2}, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \arccos \left(-\frac{a}{\sqrt{b^2 + c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right] \\
&\text{BY (293.05)}
\end{aligned}$$

$$3. \quad \int \frac{(b \cos x - c \sin x) \, dx}{\sqrt{a + b \sin x + c \cos x}} = 2\sqrt{a + b \sin x + c \cos x}$$

$$\begin{aligned}
4. \quad \int \frac{\sqrt{b^2 + c^2} + b \sin x + c \cos x}{\sqrt{a + b \sin x + c \cos x}} \, dx \\
&= -2\sqrt{a + \sqrt{b^2 + c^2}} E(\alpha, r) + \frac{2(a - \sqrt{b^2 + c^2})}{\sqrt{a + \sqrt{b^2 + c^2}}} F(\alpha, r) \\
&\left[0 < \sqrt{b^2 + c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right] \\
&\text{BY (294.04)} \\
&= -2\sqrt{2} \sqrt[4]{b^2 + c^2} E(\alpha, r) \\
&\left[0 < |a| < \sqrt{b^2 + c^2}, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \arccos \left(-\frac{a}{\sqrt{b^2 + c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right] \\
&\text{BY (293.01)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int \sqrt{a + b \sin x + c \cos x} \, dx \\
&= -2\sqrt{a + \sqrt{b^2 + c^2}} E(\alpha, r) \\
&\left[0 < \sqrt{b^2 + c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right] \\
&\text{BY (294.01)} \\
&= -2\sqrt{2} \sqrt[4]{b^2 + c^2} E(\alpha, r) + \frac{\sqrt{2}(\sqrt{b^2 + c^2} - a)}{\sqrt[4]{b^2 + c^2}} F(\alpha, r) \\
&\left[0 < |a| < \sqrt{b^2 + c^2}, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \arccos \left(\frac{-a}{\sqrt{b^2 + c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right] \\
&\text{BY (293.03)}
\end{aligned}$$

2.618 Integrals of the form $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx = \frac{1}{a} \int R(\sin t, \cos t, \sqrt{1 - 2\sin^2 t}) \, dt$ where the substitution $t = ax$ has been used.

Notation: $\alpha = \arcsin(\sqrt{2} \sin ax)$

The integrals $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx$ are special cases of the integrals **2.595.** for $(p = 2)$. We give some formulas:

$$\begin{aligned}
1. \quad \int \frac{dx}{\sqrt{\cos 2ax}} &= \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) & \left[0 < ax \leq \frac{\pi}{4} \right] \\
2. \quad \int \frac{\cos^2 ax}{\sqrt{\cos 2ax}} \, dx &= \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right) & \left[0 < ax \leq \frac{\pi}{4} \right]
\end{aligned}$$

3.
$$\int \frac{dx}{\cos^2 ax \sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\tan x}{a} \sqrt{\cos 2ax}$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$
4.
$$\int \frac{dx}{\cos^4 ax \sqrt{\cos 2ax}} = \frac{2\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{(6 \cos^2 ax + 1) \sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}$$

$$\left[0 < x \leq \frac{\pi}{4}\right]$$
5.
$$\int \frac{\tan^2 ax \, dx}{\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \tan ax \sqrt{\cos 2ax}$$

$$\left[0 < x \leq \frac{\pi}{2}\right]$$
6.
$$\int \frac{\tan^4 ax \, dx}{\sqrt{\cos 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$
7.
$$\int \frac{dx}{(1 - 2r^2 \sin^2 ax) \sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right]$$
8.
$$\int \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}}$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$
9.
$$\int \frac{\sin^2 ax \, dx}{\sqrt{\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{\cos 2ax}} - \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right]$$
10.
$$\int \frac{dx}{\sqrt{\cos^5 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{3a\sqrt{\cos^3 2ax}} \quad \left[0 < ax \leq \frac{\pi}{4}\right]$$
11.
$$\int \sqrt{\cos 2ax} \, dx = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$
12.
$$\int \frac{\sqrt{\cos 2ax}}{\cos^2 ax} \, dx = \frac{\sqrt{2}}{a} \left\{ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right\} + \frac{1}{a} \tan ax \sqrt{\cos 2ax}$$

$$\left[0 < x \leq \frac{\pi}{4}\right]$$

2.619 Integrals of the form $\int R(\sin ax, \cos ax, \sqrt{-\cos 2ax}) \, dx = \frac{1}{a} \int R(\sin x, \cos x, \sqrt{2\sin^2 x - 1}) \, dx$

Notation: $\alpha = \arcsin(\sqrt{2} \cos ax)$

The integrals $\int R(\sin x, \cos x, \sqrt{2\sin^2 x - 1}) \, dx$ are special cases of the integrals **2.599** and **2.611** for $(a = 2)$. We give some formulas:

1. $\int \frac{dx}{\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$
2. $\int \frac{\cos^2 ax \, dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right) \right]$
3. $\int \frac{\cos^4 ax \, dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a\sqrt{2}} \left[3 F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{5}{2} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] - \frac{1}{12a} \sin 2ax \sqrt{-\cos 2ax}$
4. $\int \frac{dx}{\sin^2 ax \sqrt{-\cos 2ax}} = \frac{1}{a} \cot ax \sqrt{-\cos 2ax} - \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right)$
5. $\int \frac{dx}{\sin^4 ax \sqrt{-\cos 2ax}} = \frac{2}{3a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 6 E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{3a} \frac{\cos ax}{\sin^3 ax} (6 \sin^2 ax + 1) \sqrt{-\cos 2ax}$
6. $\int \frac{\cot^2 ax \, dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2 E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{a} \cot ax \sqrt{-\cos 2ax}$
7. $\int \frac{dx}{(1 - 2r^2 \cos^2 ax) \sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}} \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right)$
8. $\int \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2 E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}$
9. $\int \frac{\cos^2 ax \, dx}{\sqrt{-\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}} - \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right)$
10. $\int \frac{dx}{\sqrt{-\cos^5 2ax}} = -\frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{3a\sqrt{-\cos^3 2ax}}$
11. $\int \sqrt{-\cos 2ax} \, dx = \frac{1}{a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2 E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right]$

2.621 Integrals of the form $\int R(\sin ax, \cos ax, \sqrt{\sin 2ax}) \, dx$.

Notation: $\alpha = \arcsin \sqrt{\frac{2 \sin ax}{1 + \sin ax + \cos ax}}$.

1. $\int \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$ BY (287.50)
2. $\int \frac{\sin ax \, dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[\frac{1+i}{2} \Pi\left(\alpha, \frac{1+i}{2}, \frac{1}{\sqrt{2}}\right) + \frac{1-i}{2} \Pi\left(\alpha, \frac{1-i}{2}, \frac{1}{\sqrt{2}}\right) + F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2 E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right]$ BY (287.57)
3. $\int \frac{\sin ax \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right]$ BY (287.54)

$$4. \quad \int \frac{\sin ax \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ \sqrt{\tan ax} - E \left(\alpha, \frac{1}{\sqrt{2}} \right) \right\} \quad \left[ax \neq \frac{\pi}{2} \right] \quad \text{BY (287.55)}$$

$$5. \quad \int \frac{(1 + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} E \left(\alpha, \frac{1}{\sqrt{2}} \right) \quad \text{BY (287.51)}$$

$$6. \quad \int \frac{(1 + \cos ax) \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ F \left(\alpha, \frac{1}{\sqrt{2}} \right) - E \left(\alpha, \frac{1}{\sqrt{2}} \right) + \sqrt{\tan ax} \right\} \quad \left[ax \neq \frac{\pi}{2} \right] \quad \text{BY (287.56)}$$

$$7. \quad \int \frac{(1 - \sin ax + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ 2 E \left(\alpha, \frac{1}{\sqrt{2}} \right) - F \left(\alpha, \frac{1}{\sqrt{2}} \right) \right\} \quad \text{BY (287.53)}$$

$$8. \quad \int \frac{(1 + \sin ax + \cos ax) \, dx}{[1 + \cos ax + (1 - 2r^2) \sin ax] \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \Pi \left(\alpha, r^2, \frac{1}{\sqrt{2}} \right). \quad \text{BY (287.52)}$$

2.63–2.65 Products of trigonometric functions and powers

2.631

$$1. \quad \int x^r \sin^p x \cos^q x \, dx = \frac{1}{(p+q)^2} \left[(p+q)x^r \sin^{p+1} x \cos^{q-1} x \right. \\ \left. + rx^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x \, dx \right. \\ \left. - rp \int x^{r-1} \sin^{p-1} x \cos^{q-1} x \, dx + (q-1)(p+q) \int x^r \sin^p x \cos^{q-2} x \, dx \right] \\ = \frac{1}{(p+q)^2} \left[-(p+q)x^r \sin^{p-1} x \cos^{q+1} x \right. \\ \left. + rx^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x \, dx \right. \\ \left. + rq \int x^{r-1} \sin^{p-1} x \cos^{q-1} x \, dx + (p-1)(p+q) \int x^r \sin^{p-2} x \cos^q x \, dx \right] \quad \text{GU (331)(1)}$$

$$2. \quad \int x^m \sin^n x \, dx = \frac{x^{m-1} \sin^{n-1} x}{n^2} \{ m \sin x - nx \cos x \} \\ + \frac{n-1}{n} \int x^m \sin^{n-2} x \, dx - \frac{m(m-1)}{n^2} \int x^{m-2} \sin^n x \, dx$$

$$3. \quad \int x^m \cos^n x \, dx = \frac{x^{m-1} \cos^{n-1} x}{n^2} \{ m \cos x + nx \sin x \} \\ + \frac{n-1}{n} \int x^m \cos^{n-2} x \, dx - \frac{m(m-1)}{n^2} \int x^{m-2} \cos^n x \, dx$$

$$\begin{aligned}
4. \quad \int x^n \sin^{2m} x \, dx &= \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} \\
&\quad + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cos(2m-2k)x \, dx \\
&\qquad\qquad\qquad (\text{see } \mathbf{2.633} \, 2) \qquad\qquad\qquad \text{TI } 333
\end{aligned}$$

$$\begin{aligned}
5. \quad \int x^n \sin^{2m+1} x \, dx &= \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \int x^n \sin(2m-2k+1)x \, dx \\
&\qquad\qquad\qquad (\text{see } \mathbf{2.633} \, 1) \qquad\qquad\qquad \text{TI } 333
\end{aligned}$$

$$\begin{aligned}
6. \quad \int x^n \cos^{2m} x \, dx &= \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} \\
&\quad + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cos(2m-2k)x \, dx \\
&\qquad\qquad\qquad (\text{see } \mathbf{2.633} \, 2) \qquad\qquad\qquad \text{TI } 333
\end{aligned}$$

$$\begin{aligned}
7. \quad \int x^n \cos^{2m+1} x \, dx &= \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \int x^n \cos(2m-2k+1)x \, dx \\
&\qquad\qquad\qquad (\text{see } \mathbf{2.633} \, 2) \qquad\qquad\qquad \text{TI } 333
\end{aligned}$$

2.632

$$\begin{aligned}
1. \quad \int x^{\mu-1} \sin \beta x \, dx &= \frac{i}{2} (i\beta)^{-\mu} \gamma(\mu, i\beta x) - \frac{i}{2} (-i\beta)^{-\mu} \gamma(\mu, -i\beta x) \\
&\qquad\qquad\qquad [\operatorname{Re} \mu > -1, \quad x > 0] \qquad\qquad\qquad \text{ET I } 317(2)
\end{aligned}$$

$$\begin{aligned}
2. \quad \int x^{\mu-1} \sin ax \, dx &= -\frac{1}{2a^\mu} \left\{ \exp \left[\frac{\pi i}{2} (\mu-1) \right] \Gamma(\mu, -iax) + \exp \left[\frac{\pi i}{2} (1-\mu) \right] \Gamma(\mu, iax) \right\} \\
&\qquad\qquad\qquad [\operatorname{Re} \mu < 1, \quad a > 0, \quad x > 0] \quad \text{ET I } 317(3)
\end{aligned}$$

$$\begin{aligned}
3. \quad \int x^{\mu-1} \cos \beta x \, dx &= \frac{1}{2} \left\{ (i\beta)^{-\mu} \gamma(\mu, i\beta x) + (-i\beta)^{-\mu} \gamma(\mu, -i\beta x) \right\} \\
&\qquad\qquad\qquad [\operatorname{Re} \mu > 0, \quad x > 0] \qquad\qquad\qquad \text{ET I } 319(22)
\end{aligned}$$

$$\begin{aligned}
4. \quad \int x^{\mu-1} \cos ax \, dx &= -\frac{1}{2a^\mu} \left\{ \exp \left(i\mu \frac{\pi}{2} \right) \Gamma(\mu, -iax) + \exp \left(-i\mu \frac{\pi}{2} \right) \Gamma(\mu, iax) \right\} \\
&\qquad\qquad\qquad \text{ET I } 319(23)
\end{aligned}$$

2.633

$$\begin{aligned}
1. \quad \int x^n \sin ax \, dx &= -\sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \cos \left(ax + \frac{1}{2} k\pi \right) \qquad\qquad\qquad \text{TI } (487)
\end{aligned}$$

$$\begin{aligned}
2.^8 \quad \int x^n \cos ax \, dx &= \sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \sin \left(ax + \frac{1}{2} k\pi \right) \qquad\qquad\qquad \text{TI } (486)
\end{aligned}$$

$$\begin{aligned}
3. \quad \int x^{2n} \sin x \, dx &= (2n)! \left\{ \sum_{k=0}^n (-1)^{k+1} \frac{x^{2n-2k}}{(2n-2k)!} \cos x + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \sin x \right\}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int x^{2n+1} \sin x \, dx = (2n+1)! \left\{ \sum_{k=0}^n (-1)^{k+1} \frac{x^{2n-2k+1}}{(2n-2k+1)!} \cos x + \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \sin x \right\} \\
5. \quad & \int x^{2n} \cos x \, dx = (2n)! \left\{ \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \sin x + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \cos x \right\} \\
6. \quad & \int x^{2n+1} \cos x \, dx = (2n+1)! \left\{ \sum_{k=0}^n (-1)^k \frac{x^{2n-2k+1}}{(2n-2k+1)!} \sin x + \sum_{k=0}^n \frac{x^{2n-2k}}{(2n-2k)!} \cos x \right\}
\end{aligned}$$

2.634

$$\begin{aligned}
1. \quad & \int P_n(x) \sin mx \, dx = -\frac{\cos mx}{m} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\sin mx}{m} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}} \\
2. \quad & \int P_n(x) \cos mx \, dx = \frac{\sin mx}{m} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\cos mx}{m} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}}
\end{aligned}$$

In formulas **2.634**, $P_n(x)$ is any n^{th} -degree polynomial, and $P_n^{(k)}(x)$ is its k^{th} derivative with respect to x .

2.635 Notation: $z_1 = a + bx$.

$$\begin{aligned}
1. \quad & \int z_1 \sin kx \, dx = -\frac{1}{k} z_1 \cos kx + \frac{b}{k^2} \sin kx \\
2. \quad & \int z_1 \cos kx \, dx = \frac{1}{k} z_1 \sin kx + \frac{b}{k^2} \cos kx \\
3. \quad & \int z_1^2 \sin kx \, dx = \frac{1}{k} \left(\frac{2b^2}{k^2} - z_1^2 \right) \cos kx + \frac{2bz_1}{k^2} \sin kx \\
4. \quad & \int z_1^2 \cos kx \, dx = \frac{1}{k} \left(z_1^2 - \frac{2b^2}{k^2} \right) \sin kx + \frac{2bz_1}{k^2} \cos kx \\
5. \quad & \int z_1^3 \sin kx \, dx = \frac{z_1}{k} \left(\frac{6b^2}{k^2} - z_1^2 \right) \cos kx + \frac{3b}{k^2} \left(z_1^2 - \frac{2b^2}{k^2} \right) \sin kx \\
6. \quad & \int z_1^3 \cos kx \, dx = \frac{z_1}{k} \left(z_1^2 - \frac{6b^2}{k^2} \right) \sin kx + \frac{3b}{k^2} \left(z_1^2 - \frac{2b^2}{k^2} \right) \cos kx \\
7. \quad & \int z_1^4 \sin kx \, dx = -\frac{1}{k} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cos kx + \frac{4bz_1}{k^2} \left(z_1^2 - \frac{6b^2}{k^2} \right) \sin kx \\
8. \quad & \int z_1^4 \cos kx \, dx = \frac{1}{k} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sin kx + \frac{4bz_1}{k^2} \left(z_1^2 - \frac{6b^2}{k^2} \right) \cos kx \\
9. \quad & \int z_1^5 \sin kx \, dx = \frac{5b}{k^2} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sin kx - \frac{z_1}{k} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \cos kx \\
10. \quad & \int z_1^5 \cos kx \, dx = \frac{5b}{k^2} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cos kx + \frac{z_1}{k} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \sin kx
\end{aligned}$$

$$11. \quad \int z_1^6 \sin kx \, dx = \frac{6bz_1}{k^2} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \sin kx \\ - \frac{1}{k} \left(z_1^6 - \frac{30b^2}{k^2} z_1^4 + \frac{360b^4}{k^4} z_1^2 - \frac{720b^6}{k^6} \right) \cos kx$$

$$12. \quad \int z_1^6 \cos kx \, dx = \frac{6bz_1}{k^2} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \cos kx \\ + \frac{1}{k} \left(z_1^6 - \frac{30b^2}{k^2} z_1^4 + \frac{360b^4}{k^4} z_1^2 - \frac{720b^6}{k^6} \right) \sin kx$$

2.636

$$1. \quad \int x^n \sin^2 x \, dx = \frac{x^{n+1}}{2(n+1)} \\ + \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k}(n-2k)!} \sin 2x + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cos 2x \right\}$$

GU (333)(2e)

$$2. \quad \int x^n \cos^2 x \, dx = \frac{x^{n+1}}{2(n+1)} \\ - \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k}(n-2k)!} \sin 2x + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cos 2x \right\}$$

GU (333)(3e)

$$3. \quad \int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x$$

$$4. \quad \int x^2 \sin^2 x \, dx = \frac{x^3}{6} - \frac{x}{4} \cos 2x - \frac{1}{4} \left(x^2 - \frac{1}{2} \right) \sin 2x$$

MZ 241

$$5. \quad \int x \cos^2 x \, dx = \frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x$$

$$6. \quad \int x^2 \cos^2 x \, dx = \frac{x^3}{6} + \frac{x}{4} \cos 2x + \frac{1}{4} \left(x^2 - \frac{1}{2} \right) \sin 2x$$

MZ 245

2.637

$$1.^{11} \quad \int x^n \sin^3 x \, dx = \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \left(\frac{\cos 3x}{3^{2k+1}} - 3 \cos x \right) \right. \\ \left. - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\sin 3x}{3^{2k+2}} - 3 \sin x \right) \right\}$$

GU(333)(2f)

$$2. \quad \int x^n \cos^3 x \, dx = \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \left(\frac{\sin 3x}{3^{2k+1}} + 3 \sin x \right) + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\cos 3x}{3^{2k+2}} + 3 \cos x \right) \right\}$$

GU(333)(3f)

$$3. \quad \int x \sin^3 x \, dx = \frac{3}{4} \sin x - \frac{1}{36} \sin 3x - \frac{3}{4} x \cos x + \frac{x}{12} \cos 3x$$

$$4. \quad \int x^2 \sin^3 x \, dx = - \left(\frac{3}{4} x^2 + \frac{3}{2} \right) \cos x + \left(\frac{x^2}{12} + \frac{1}{54} \right) \cos 3x + \frac{3}{2} x \sin x - \frac{x}{18} \sin 3x$$

MZ 241

$$5. \quad \int x \cos^3 x \, dx = \frac{3}{4} \cos x + \frac{1}{36} \cos 3x + \frac{3}{4} x \sin x + \frac{x}{12} \sin 3x$$

$$6. \quad \int x^2 \cos^3 x \, dx = \left(\frac{3}{4} x^2 - \frac{3}{2} \right) \sin x + \left(\frac{x^2}{12} - \frac{1}{54} \right) \sin 3x + \frac{3}{2} x \cos x + \frac{x}{18} \cos 3x$$

MZ 245, 246

2.638

$$1. \quad \int \frac{\sin^q x}{x^p} \, dx = - \frac{\sin^{q-1} x [(p-2) \sin x + qx \cos x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)} \int \frac{\sin^q x \, dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\sin^{q-2} x \, dx}{x^{p-2}}$$

[p ≠ 1, p ≠ 2] TI (496)

$$2. \quad \int \frac{\cos^q x}{x^p} \, dx = - \frac{\cos^{q-1} x [(p-2) \cos x - qx \sin x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)} \int \frac{\cos^q x \, dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\cos^{q-2} x \, dx}{x^{p-2}}$$

[p ≠ 1, p ≠ 2] TI (495)

$$3.^6 \quad \int \frac{\sin x \, dx}{x^p} = - \frac{\sin x}{(p-1)x^{p-1}} + \frac{1}{p-1} \int \frac{\cos x \, dx}{x^{p-1}} = - \frac{\sin x}{(p-1)x^{p-1}} - \frac{\cos x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\sin x \, dx}{x^{p-2}}$$

(p > 2) TI (492)

$$4.^6 \quad \int \frac{\cos x \, dx}{x^p} = - \frac{\cos x}{(p-1)x^{p-1}} - \frac{1}{p-1} \int \frac{\sin x \, dx}{x^{p-1}} = - \frac{\cos x}{(p-1)x^{p-1}} + \frac{\sin x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\cos x \, dx}{x^{p-2}}$$

(p > 2) TI (491)

2.639

$$1. \quad \int \frac{\sin x \, dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \cos x \right. \\ \left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \sin x \right\} + \frac{(-1)^{n+1}}{(2n-1)!} \operatorname{ci}(x)$$

GU (333)(6b)a

$$2. \quad \int \frac{\sin x}{x^{2n+1}} \, dx = \frac{(-1)^{n+1}}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \cos x \right. \\ \left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \sin x \right\} + \frac{(-1)^n}{(2n)!} \operatorname{si}(x)$$

GU (333)(6b)a

$$3. \quad \int \frac{\cos x \, dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \cos x \right. \\ \left. - \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \sin x \right\} + \frac{(-1)^n}{(2n-1)!} \operatorname{si}(x)$$

GU (333)(7b)

$$4. \quad \int \frac{\cos x \, dx}{x^{2n+1}} = \frac{(-1)^{n+1}}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \cos x \right. \\ \left. - \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \sin x \right\} + \frac{(-1)^n}{(2n)!} \operatorname{ci}(x)$$

GU (333)(7b)

2.641

$$1. \quad \int \frac{\sin kx}{a+bx} \, dx = \frac{1}{b} \left[\cos \frac{ka}{b} \operatorname{si}(u) - \sin \frac{ka}{b} \operatorname{ci}(u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right]$$

$$2. \quad \int \frac{\cos kx}{a+bx} \, dx = \frac{1}{b} \left[\cos \frac{ka}{b} \operatorname{ci}(u) + \sin \frac{ka}{b} \operatorname{si}(u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right]$$

$$3. \quad \int \frac{\sin kx}{(a+bx)^2} \, dx = -\frac{1}{b} \frac{\sin kx}{a+bx} + \frac{k}{b} \int \frac{\cos kx}{a+bx} \, dx \quad (\text{see } \mathbf{2.641} \, 2)$$

$$4. \quad \int \frac{\cos kx}{(a+bx)^2} \, dx = -\frac{1}{b} \frac{\cos kx}{a+bx} - \frac{k}{b} \int \frac{\sin kx}{a+bx} \, dx \quad (\text{see } \mathbf{2.641} \, 1)$$

$$5. \quad \int \frac{\sin kx}{(a+bx)^3} \, dx = -\frac{\sin kx}{2b(a+bx)^2} - \frac{k \cos kx}{2b^2(a+bx)} - \frac{k^2}{2b^2} \int \frac{\sin kx}{a+bx} \, dx$$

(see **2.641** 1)

$$6. \quad \int \frac{\cos kx}{(a+bx)^3} dx = -\frac{\cos kx}{2b(a+bx)^2} + \frac{k \sin kx}{2b^2(a+bx)} - \frac{k^2}{2b^2} \int \frac{\cos kx}{a+bx} dx$$

(see **2.641** 2)

$$7. \quad \int \frac{\sin kx}{(a+bx)^4} dx = -\frac{\sin kx}{3b(a+bx)^3} - \frac{k \cos kx}{6b^2(a+bx)^2} + \frac{k^2 \sin kx}{6b^2(a+bx)} - \frac{k^3}{6b^3} \int \frac{\cos kx}{a+bx} dx$$

(see **2.641** 2)

$$8. \quad \int \frac{\cos kx}{(a+bx)^4} dx = -\frac{\cos kx}{3b(a+bx)^3} + \frac{k \sin kx}{6b^2(a+bx)^2} + \frac{k^2 \cos kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\sin kx}{a+bx} dx$$

(see **2.641** 1)

$$9. \quad \int \frac{\sin kx}{(a+bx)^5} dx = -\frac{\sin kx}{4b(a+bx)^4} - \frac{k \cos kx}{12b^2(a+bx)^3} + \frac{k^2 \sin kx}{24b^3(a+bx)^2} + \frac{k^3 \cos kx}{24b^4(a+bx)} - \frac{k^4}{24b^4} \int \frac{\sin kx}{a+bx} dx$$

(see **2.641** 1)

$$10. \quad \int \frac{\cos kx}{(a+bx)^5} dx = -\frac{\cos kx}{4b(a+bx)^4} + \frac{k \sin kx}{12b^2(a+bx)^3} + \frac{k^2 \cos kx}{24b^3(a+bx)^2} - \frac{k^3 \sin kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\cos kx}{a+bx} dx$$

(see **2.641** 2)

$$11. \quad \int \frac{\sin kx}{(a+bx)^6} dx = -\frac{\sin kx}{5b(a+bx)^5} - \frac{k \cos kx}{20b^2(a+bx)^4} + \frac{k^2 \sin kx}{60b^3(a+bx)^3} + \frac{k^3 \cos kx}{120b^4(a+bx)^2} - \frac{k^4 \sin kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\cos kx}{a+bx} dx$$

(see **2.641** 2)

$$12. \quad \int \frac{\cos kx}{(a+bx)^6} dx = -\frac{\cos kx}{5b(a+bx)^5} + \frac{k \sin kx}{20b^2(a+bx)^4} + \frac{k^2 \cos kx}{60b^3(a+bx)^3} - \frac{k^3 \sin kx}{120b^4(a+bx)^2} - \frac{k^4 \cos kx}{120b^5(a+bx)} - \frac{k^5}{120b^5} \int \frac{\sin kx}{a+bx} dx$$

(see **2.641** 1)

2.642

$$1. \quad \int \frac{\sin^{2m} x}{x} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \text{ci}[(2m-2k)x]$$

$$2. \quad \int \frac{\sin^{2m+1} x}{x} dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \text{si}[(2m-2k+1)x]$$

$$3. \quad \int \frac{\cos^{2m} x}{x} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \text{ci}[(2m-2k)x]$$

$$4. \quad \int \frac{\cos^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \text{ci}[(2m-2k+1)x]$$

- $$\begin{aligned}
5. \quad & \int \frac{\sin^{2m} x}{x^2} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} \\
& + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + (2m-2k) \operatorname{si}[(2m-2k)x] \right\} \\
6. \quad & \int \frac{\sin^{2m+1} x}{x^2} dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^{k+1} \binom{2m+1}{k} \\
& \times \left\{ \frac{\sin(2m-2k+1)x}{x} - (2m-2k+1) \operatorname{ci}[(2m-2k+1)x] \right\} \\
7. \quad & \int \frac{\cos^{2m} x}{x^2} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} \\
& - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + (2m-2k) \operatorname{si}[(2m-2k)x] \right\} \\
8. \quad & \int \frac{\cos^{2m+1} x}{x^2} dx = - \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \left\{ \frac{\cos(2m-2k+1)x}{x} \right. \\
& \left. + (2m-2k+1) \operatorname{si}[(2m-2k+1)x] \right\}
\end{aligned}$$

2.643

- $$\begin{aligned}
1. \quad & \int \frac{x^p dx}{\sin^q x} = - \frac{x^{p-1} [p \sin x + (q-2)x \cos x]}{(q-1)(q-2) \sin^{q-1} x} + \frac{q-2}{q-1} \int \frac{x^p dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\sin^{q-2} x} \\
2. \quad & \int \frac{x^p dx}{\cos^q x} = - \frac{x^{p-1} [p \cos x - (q-2)x \sin x]}{(q-1)(q-2) \cos^{q-1} x} \\
& + \frac{q-2}{q-1} \int \frac{x^p dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\cos^{q-2} x} \\
3.^4 \quad & \int \frac{x^n}{\sin x} dx = \frac{x^n}{n} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1}-1)}{(n+2k)(2k)!} B_{2k} x^{n+2k} \\
& [|x| < \pi, \quad n > 0] \quad \text{TU (333)(8b)} \\
4. \quad & \int \frac{dx}{x^n \sin x} = - \frac{1}{n x^n} - [1 + (-1)^n] (-1)^{\frac{n}{2}} \frac{2^{n-1}-1}{n!} B_n \ln x - \sum_{\substack{k=1 \\ k \neq \frac{n}{2}}}^{\infty} (-1)^k \frac{2(2^{2n}-1)}{(2k-n) \cdot (2k)!} B_{2k} x^{2k-n} \\
& [n > 1, \quad |x| > \pi] \quad \text{GU (333)(9b)} \\
5.^8 \quad & \int \frac{x^n dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{n+2k+1}}{(n+2k+1)(2k)!} \\
& \left[|x| < \frac{\pi}{2}, \quad n > 0 \right] \quad \text{GU (333)(10b)} \\
6. \quad & \int \frac{dx}{x^n \cos x} = \frac{1}{2} [1 - (-1)^n] \frac{|E_{n-1}|}{(n-1)!} \ln x + \sum_{\substack{k=0 \\ k \neq \frac{n-1}{2}}}^{\infty} \frac{|E_{2k}| x^{2k-n+1}}{(2k-n+1) \cdot (2k)!} \\
& \left[|x| < \frac{\pi}{2} \right] \quad \text{GU (333)(11b)}
\end{aligned}$$

$$7. \quad \int \frac{x^n dx}{\sin^2 x} = -x^n \cot x + \frac{n}{n-1} x^{n-1} + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{n+2k-1}}{(n+2k-1)(2k)!} B_{2k} \\ [|x| < \pi, \quad n > 1] \quad \text{GU (333)(8c)}$$

$$8. \quad \int \frac{dx}{x^n \sin^2 x} = -\frac{\cot x}{x^n} + \frac{n}{(n+1)x^{n+1}} - [1 - (-1)^n] (-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} B_{n+1} \ln x \\ - \frac{n}{2^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \\ [|x| < \pi] \quad \text{GU (333)(9c)}$$

$$9. \quad \int \frac{x^n dx}{\cos^2 x} = x^n \tan x + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} (2^{2k} - 1) x^{n+2k-1}}{(n+2k-1) \cdot (2k)!} B_{2k} \\ [n > 1, \quad |x| < \frac{\pi}{2}] \quad \text{GU (333)(10c)}$$

$$10. \quad \int \frac{dx}{x^n \cos^2 x} = \frac{\tan x}{x^n} - [1 - (-1)^n] (-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} (2^{n+1} - 1) B_{n+1} \ln x \\ - \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2^{2k} - 1) (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \\ [|x| < \frac{\pi}{2}] \quad \text{GU (333)(11c)}$$

2.644

$$1. \quad \int \frac{x dx}{\sin^{2n} x} = - \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4) \dots (2n-2k+2)}{(2n-1)(2n-3) \dots (2n-2k+3)} \frac{\sin x + (2n-2k)x \cos x}{(2n-2k+1)(2n-2k) \sin^{2n-2k+1} x} \\ + \frac{2^{n-1}(n-1)!}{(2n-1)!!} (\ln \sin x - x \cot x)$$

$$2. \quad \int \frac{x dx}{\sin^{2n+1} x} = - \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2n(2n-2) \dots (2n-2k+2)} \frac{\sin x + (2n-2k-1)x \cos x}{(2n-2k)(2n-2k-1) \sin^{2n-2k} x} \\ + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\sin x} \\ \text{(see 2.644 5)}$$

$$3. \quad \int \frac{x dx}{\cos^{2n} x} = \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4) \dots (2n-2k+2)}{(2n-1)(2n-3) \dots (2n-2k+3)} \frac{(2n-2k)x \sin x - \cos x}{(2n-2k+1)(2n-2k) \cos^{2n-2k+1} x} \\ + \frac{2^{n-1}(n-1)!}{(2n-1)!!} (x \tan x + \ln \cos x)$$

$$4. \quad \int \frac{x dx}{\cos^{2n+1} x} = \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2n(2n-2) \dots (2n-2k+2)} \frac{(2n-2k+1)x \sin x - \cos x}{(2n-2k)(2n-2k-1) \cos^{2n-2k} x} \\ + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\cos x} \\ \text{(see 2.644 6)}$$

5. $\int \frac{x dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1} - 1)}{(2k+1)!} B_{2k} x^{2k+1}$
6. $\int \frac{x dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}$
7. $\int \frac{x dx}{\sin^2 x} = -x \cot x + \ln \sin x$
8. $\int \frac{x dx}{\cos^2 x} = x \tan x + \ln \cos x$
9. $\int \frac{x dx}{\sin^3 x} = -\frac{\sin x + x \cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{x}{\sin x} dx$ (see **2.644** 5)
10. $\int \frac{x dx}{\cos^3 x} = \frac{x \sin x - \cos x}{2 \cos^2 x} + \frac{1}{2} \int \frac{x dx}{\cos x}$ (see **2.644** 6)
11. $\int \frac{x dx}{\sin^4 x} = -\frac{x \cos x}{3 \sin^3 x} - \frac{1}{6 \sin^2 x} - \frac{2}{3} x \cot x + \frac{2}{3} \ln(\sin x)$
12. $\int \frac{x dx}{\cos^4 x} = \frac{x \sin x}{3 \cos^3 x} - \frac{1}{6 \cos^2 x} + \frac{2}{3} x \tan x - \frac{2}{3} \ln(\cos x)$
13. $\int \frac{x dx}{\sin^5 x} = -\frac{x \cos x}{4 \sin^4 x} - \frac{1}{12 \sin^3 x} - \frac{3x \cos x}{8 \sin^2 x} - \frac{3}{8 \sin x} + \frac{3}{8} \int \frac{x dx}{\sin x}$
(see **2.644** 5)
14. $\int \frac{x dx}{\cos^5 x} = \frac{x \sin x}{4 \cos^4 x} - \frac{1}{12 \cos^3 x} + \frac{3x \sin x}{8 \cos^2 x} - \frac{3}{8 \cos x} + \frac{3}{8} \int \frac{x dx}{\cos x}$
(see **2.644** 6)

2.645

1. $\int x^p \frac{\sin^{2m} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p dx}{\cos^{n-2k} x}$ (see **2.643** 2)
2. $\int x^p \frac{\sin^{2m+1} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x} dx$ (see **2.645** 3)
3. $\int x^p \frac{\sin x dx}{\cos^n x} = \frac{x^p}{(n-1) \cos^{n-1} x} - \frac{p}{n-1} \int \frac{x^{p-1}}{\cos^{n-1} x} dx$
[$n > 1$] (see **2.643** 2) GU (333)(12)
4. $\int x^p \frac{\cos^{2m} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p dx}{\sin^{n-2k} x}$ (see **2.643** 1)
5. $\int x^p \frac{\cos^{2m+1} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x} dx$ (see **2.645** 6)
6. $\int x^p \frac{\cos x}{\sin^n x} = -\frac{x^p}{(n-1) \sin^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} dx}{\sin^{n-1} x}$ [$n > 1$] (see **2.643** 1) GU (333)(13)

$$7. \quad \int \frac{x \cos x}{\sin^2 x} dx = -\frac{x}{\sin x} + \ln \tan \frac{x}{2}$$

$$8. \quad \int \frac{x \sin x}{\cos^2 x} dx = \frac{x}{\cos x} - \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$

2.646

$$1. \quad \int x^p \tan x dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k} (2^{2k-1} - 1)}{(p+2k) \cdot (2k)!} B_{2k} x^{p+2k} \quad \left[p \geq -1, \quad |x| < \frac{\pi}{2} \right] \quad \text{GU (333)(12d)}$$

$$2. \quad \int x^p \cot x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \quad [p \geq 1, \quad |x| < \pi] \quad \text{GU (333)(13d)}$$

$$3. \quad \int x^p \tan^2 x dx = x \tan x + \ln \cos x - \frac{x^2}{2}$$

$$4. \quad \int x \cot^2 x dx = -x \cot x + \ln \sin x - \frac{x^2}{2}$$

2.647

$$1. \quad \int \frac{x^n \cos x dx}{(a+b \sin x)^m} = -\frac{x^n}{(m-1)b(a+b \sin x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a+b \sin x)^{m-1}} \quad [m \neq 1] \quad \text{MZ 247}$$

$$2. \quad \int \frac{x^n \sin x dx}{(a+b \cos x)^m} = \frac{x^n}{(m-1)b(a+b \cos x)^{m-1}} - \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a+b \cos x)^{m-1}} \quad [m \neq 1] \quad \text{MZ 247}$$

$$3. \quad \int \frac{x dx}{1+\sin x} = -x \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \cos \left(\frac{\pi}{4} - \frac{x}{2} \right) \quad \text{PE (329)}$$

$$4. \quad \int \frac{x dx}{1-\sin x} = x \cot \left(\frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \sin \left(\frac{\pi}{4} - \frac{x}{2} \right) \quad \text{PE (330)}$$

$$5. \quad \int \frac{x dx}{1+\cos x} = x \tan \frac{x}{2} + 2 \ln \cos \frac{x}{2} \quad \text{PE (331)}$$

$$6. \quad \int \frac{x dx}{1-\cos x} = -x \cot \frac{x}{2} + 2 \ln \cos \frac{x}{2} \quad \text{PE (332)}$$

$$7. \quad \int \frac{x \cos x}{(1+\sin x)^2} dx = -\frac{x}{1+\sin x} + \tan \left(\frac{x}{2} - \frac{\pi}{4} \right)$$

$$8. \quad \int \frac{x \cos x}{(1-\sin x)^2} dx = \frac{x}{1-\sin x} + \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$

$$9. \quad \int \frac{x \sin x}{(1+\cos x)^2} dx = \frac{x}{1+\cos x} - \tan \frac{x}{2}$$

$$10. \quad \int \frac{x \sin x}{(1-\cos x)^2} dx = -\frac{x}{1-\cos x} - \cot \frac{x}{2} \quad \text{MZ 247a}$$

2.648

$$1. \quad \int \frac{x + \sin x}{1 + \cos x} dx = x \tan \frac{x}{2}$$

$$2. \quad \int \frac{x - \sin x}{1 - \cos x} dx = -x \cot \frac{x}{2}$$

GU (333)(16)

$$\mathbf{2.649} \quad \int \frac{x^2 dx}{[(ax - b) \sin x + (a + bx) \cos x]^2} = \frac{x \sin x + \cos x}{b [(ax - b) \sin x + (a + bx) \cos x]} \quad \text{GU (333)(17)}$$

$$\mathbf{2.651} \quad \int \frac{dx}{[a + (ax + b) \tan x]^2} = \frac{\tan x}{a [a + (ax + b) \tan x]} \quad \text{GU (333)(18)}$$

$$\mathbf{2.652} \quad \int \frac{x dx}{\cos(x+t) \cos(x-t)} = \operatorname{cosec} 2t \left\{ x \ln \frac{\cos(x-t)}{\cos(x+t)} - L(x+t) + L(x-t) \right\}$$

$$\left[t \neq n\pi; \quad |x| < \left| \frac{\pi}{2} - |t_0| \right| \right],$$

where t_0 is the value of the argument t , which is reduced by multiples of the argument π to lie in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. LO III 288

2.653

$$1. \quad \int \frac{\sin x}{\sqrt{x}} dx = \sqrt{2\pi} S(\sqrt{x}) \quad (\text{cf. } \mathbf{8.251} \text{ 21})$$

$$2. \quad \int \frac{\cos x}{\sqrt{x}} dx = \sqrt{2\pi} C(\sqrt{x}) \quad (\text{cf. } \mathbf{8.251} \text{ 3})$$

2.654 **Notation:** $\Delta = \sqrt{1 - k^2 \sin^2 x}$, $k' = \sqrt{1 - k^2}$:

$$1. \quad \int \frac{x \sin x \cos x}{\Delta} dx = -\frac{x\Delta}{k^2} + \frac{1}{k^2} E(x, k)$$

$$2. \quad \int \frac{x \sin^3 x \cos x}{\Delta} dx = -\frac{k'^2}{9k^4} F(x, k) + \frac{2k^2 + 5}{9k^4} E(x, k) - \frac{1}{9k^4} [3(3 - \Delta^2)x + k^2 \sin x \cos x] \Delta$$

$$3. \quad \int \frac{x \sin x \cos^3 x}{\Delta} dx = -\frac{k'^2}{9k^4} F(x, k) + \frac{7k^2 - 5}{9k^4} E(x, k) - \frac{1}{9k^4} [3(\Delta^2 - 3k'^2)x - k^2 \sin x \cos x] \Delta$$

$$4. \quad \int \frac{x \sin x dx}{\Delta^3} dx = -\frac{x \cos x}{k'^2 \Delta} + \frac{1}{kk'^2} \arcsin(k \sin x)$$

$$5. \quad \int \frac{x \cos x dx}{\Delta^3} = \frac{x \sin x}{\Delta} + \frac{1}{k} \ln(k \cos x + \Delta)$$

$$6. \quad \int \frac{x \sin x \cos x dx}{\Delta^3} = \frac{x}{k^2 \Delta} - \frac{1}{k^2} F(x, k)$$

$$7. \quad \int \frac{x \sin^3 x \cos x dx}{\Delta^3} = x \frac{2 - k^2 \sin^2 x}{k^4 \Delta} - \frac{1}{k^4} [E(x, k) + F(x, k)]$$

$$8. \quad \int \frac{x \sin x \cos^3 x dx}{\Delta^3} = x \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta} + \frac{k'^2}{k^4} F(x, k) + \frac{1}{k^4} E(x, k)$$

2.655 Integrals containing $\sin x^2$ and $\cos x^2$

In integrals containing $\sin x^2$ and $\cos x^2$, it is expedient to make the substitution $x^2 = u$.

1. $\int x^p \sin x^2 dx = -\frac{x^{p-1}}{2} \cos x^2 + \frac{p-1}{2} \int x^{p-2} \cos x^2 dx$
2. $\int x^p \cos x^2 dx = \frac{x^{p-1}}{2} \sin x^2 - \frac{p-1}{2} \int x^{p-2} \sin x^2 dx$
3.
$$\int x^n \sin x^2 dx = (n-1)!! \left\{ \sum_{k=1}^r (-1)^k \left[\frac{x^{n-4k+3} \cos x^2}{2^{2k-1}(n-4k+3)!!} - \frac{x^{n-4k+1} \sin x^2}{2^{2k}(n-4k+1)!!} \right] \right. \\ \left. + \frac{(-1)^r}{2^{2r}(n-4r-1)!!} \int x^{n-4r} \sin x^2 dx \right\} \\ \left[r = \left\lfloor \frac{n}{4} \right\rfloor \right] \quad \text{GU (336)(4a)}$$
4.
$$\int x^n \cos x^2 dx = (n-1)!! \left\{ \sum_{k=1}^r (-1)^{k-1} \left[\frac{x^{n-4k+3} \sin x^2}{2^{2k-1}(n-4k+3)!!} + \frac{x^{n-4k+1} \cos x^2}{2^{2k}(n-4k+1)!!} \right] \right. \\ \left. + \frac{(-1)^r}{2^{2r}(n-4r-1)!!} \int x^{n-4r} \cos x^2 dx \right\} \\ \left[r = \left\lfloor \frac{n}{4} \right\rfloor \right] \quad \text{GU (336)(5a)}$$
5. $\int x \sin x^2 dx = -\frac{\cos^2 x}{2}$
6. $\int x \cos x^2 dx = -\frac{\sin^2 x}{2}$
7. $\int x^2 \sin x^2 dx = -\frac{x}{2} \cos x^2 + \frac{1}{2} \sqrt{\frac{\pi}{2}} C(x)$
8. $\int x^2 \cos x^2 dx = \frac{x}{2} \sin x^2 - \frac{1}{2} \sqrt{\frac{\pi}{2}} S(x)$
9. $\int x^3 \sin x^2 dx = -\frac{x^2}{2} \cos x^2 + \frac{1}{2} \sin x^2$
10. $\int x^3 \cos x^2 dx = \frac{x^2}{2} \sin x^2 + \frac{1}{2} \cos x^2$

2.66 Combinations of trigonometric functions and exponentials

$$\begin{aligned}
 \mathbf{2.661} \quad \int e^{ax} \sin^p x \cos^q x \, dx &= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^p x \cos^{q-1} x [a \cos x + (p+q) \sin x] \right. \\
 &\quad \left. - pa \int e^{ax} \sin^{p-1} x \cos^{q-1} x \, dx + (q-1)(p+q) \int e^{ax} \sin^p x \cos^{q-2} x \, dx \right\} \\
 &\qquad \qquad \qquad \text{TI (523)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^q x [a \sin x - (p+q) \cos x] \right. \\
 &\quad \left. + qa \int e^{ax} \sin^{p-1} x \cos^{q-1} x \, dx + (p-1)(p+q) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\} \\
 &\qquad \qquad \qquad \text{TI (524)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x [a \sin x \cos x + q \sin^2 x - p \cos^2 x] \right. \\
 &\quad \left. + q(q-1) \int e^{ax} \sin^p x \cos^{q-2} x \, dx + p(p-1) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\} \\
 &\qquad \qquad \qquad \text{TI (525)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) \right. \\
 &\quad \left. + q(q-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x \, dx \right. \\
 &\quad \left. - (q-p)(p+q-1) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\} \\
 &\qquad \qquad \qquad \text{TI (526)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a^2 + (p+q)^2} \left[e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) \right. \\
 &\quad \left. + p(p-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x \, dx \right. \\
 &\quad \left. + (q-p)(p+q-1) \int e^{ax} \sin^p x \cos^{q-2} x \, dx \right]
 \end{aligned}$$

GU (334)(1a)

For $p = m$ and $q = n$ even integers, the integral $\int e^{ax} \sin^m x \cos^n x \, dx$ can be reduced by means of these formulas to the integral $\int e^{ax} \, dx$. However, when only m or only n is even, they can be reduced to

integrals of the form $\int e^{ax} \cos^n x \, dx$ or $\int e^{ax} \sin^m x \, dx$, respectively.

2.662

$$1. \quad \int e^{ax} \sin^n bx \, dx = \frac{1}{a^2 + n^2 b^2} \left[(a \sin bx - nb \cos bx) e^{ax} \sin^{n-1} bx \right. \\ \left. + n(n-1)b^2 \int e^{ax} \sin^{n-2} bx \, dx \right]$$

$$2. \quad \int e^{ax} \cos^n bx \, dx = \frac{1}{a^2 + n^2 b^2} \left[(a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx \right. \\ \left. + n(n-1)b^2 \int e^{ax} \cos^{n-2} bx \, dx \right]$$

$$3. \quad \int e^{ax} \sin^{2m} bx \, dx \\ = \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \sin^{2m-2k-1} bx}{(2m-2k)! [a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + (2m-2k)^2 b^2]} \\ \times [a \sin bx - (2m-2k)b \cos bx] + \frac{(2m)! b^{2m} e^{ax}}{[a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + 4b^2]} a \\ = \binom{2m}{m} \frac{e^{ax}}{2^{2m} a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \frac{1}{a^2 + 4b^2 k^2} (a \cos 2kx + 2bk \sin 2kx)$$

$$4. \quad \int e^{ax} \sin^{2m+1} bx \, dx \\ = \sum_{k=0}^m \frac{(2m+1)! b^{2k} e^{ax} \sin^{2m-2k} bx [a \sin bx - (2m-2k+1)b \cos bx]}{(2m-2k+1)! [a^2 + (2m+1)^2 b^2] [a^2 + (2m-1)^2 b^2] \cdots [a^2 + (2m-2k+1)^2 b^2]} \\ = \frac{e^{ax}}{2^{2m}} \sum_{k=0}^m \frac{(-1)^k}{a^2 + (2k+1)^2 b^2} \binom{2m+1}{m-k} [a \sin(2k+1)bx - (2k+1)b \cos(2k+1)bx]$$

$$5.8 \quad \int e^{ax} \cos^{2m} bx \, dx = \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \cos^{2m-2k-1} bx [a \cos bx + (2m-2k)b \sin bx]}{(2m-2k)! [a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + (2m-2k)^2 b^2]} \\ + \frac{(2m)! b^{2m} e^{ax}}{[a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + 4b^2]} a \\ = \binom{2m}{m} \frac{e^{ax}}{2^{2m} a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^m \binom{2m}{m-k} \frac{1}{a^2 + 4b^2 k^2} [a \cos 2kx + 2kb \sin 2kx]$$

$$6. \quad \int e^{ax} \cos^{2m+1} bx \, dx \\ = \sum_{k=0}^m \frac{(2m+1)! b^{2k} e^{ax} \cos^{2m-2k} bx}{(2m-2k+1)! [a^2 + (2m-1)^2 b^2] \cdots [a^2 + (2m-2k+1)^2 b^2]} \\ = \frac{e^{ax}}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{m-k} \frac{1}{a^2 + (2k+1)^2 b^2} [a \cos(2k+1)bx + (2k+1)b \sin(2k+1)bx]$$

2.663

$$1. \quad \int e^{ax} \sin bx \, dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$$

$$\begin{aligned}
2. \quad \int e^{ax} \sin^2 bx \, dx &= \frac{e^{ax} \sin bx (a \sin bx - 2b \cos bx)}{4b^2 + a^2} + \frac{2b^2 e^{ax}}{(4b^2 + a^2)a} \\
&= \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2 + 4b^2} \left(\frac{a}{2} \cos 2bx + b \sin 2bx \right) \\
3. \quad \int e^{ax} \cos bx \, dx &= \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2} \\
4. \quad \int e^{ax} \cos^2 bx \, dx &= \frac{e^{ax} \cos bx (a \cos bx + 2b \sin bx)}{4b^2 + a^2} + \frac{2b^2 e^{ax}}{(4b^2 + a^2)a} \\
&= \frac{e^{ax}}{2a} + \frac{e^{ax}}{a^2 + 4b^2} \left(\frac{a}{2} \cos 2bx + b \sin 2bx \right)
\end{aligned}$$

2.664

$$\begin{aligned}
1. \quad \int e^{ax} \sin bx \cos cx \, dx &= \frac{e^{ax}}{2} \left[\frac{a \sin(b+c)x - (b+c) \cos(b+c)x}{a^2 + (b+c)^2} \right. \\
&\quad \left. + \frac{a \sin(b-c)x - (b-c) \cos(b-c)x}{a^2 + (b-c)^2} \right] \quad \text{GU (334)(6b)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int e^{ax} \sin^2 bx \cos cx \, dx &= \frac{e^{ax}}{4} \left[2 \frac{a \cos cx + c \sin cx}{a^2 + c^2} - \frac{a \cos(2b+c)x + (2b+c) \sin(2b+c)x}{a^2 + (2b+c)^2} \right. \\
&\quad \left. - \frac{a \cos(2b-c)x + (2b-c) \sin(2b-c)x}{a^2 + (2b-c)^2} \right] \quad \text{GU (334)(6c)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int e^{ax} \sin bx \cos^2 cx \, dx &= \frac{e^{ax}}{4} \left[2 \frac{a \sin bx - b \cos bx}{a^2 + b^2} + \frac{a \sin(b+2c)x - (b+2c) \cos(b+2c)x}{a^2 + (b+2c)^2} \right. \\
&\quad \left. + \frac{a \sin(b-2c)x - (b-2c) \cos(b-2c)x}{a^2 + (b-2c)^2} \right] \quad \text{GU (334)(6d)}
\end{aligned}$$

2.665

$$1. \quad \int \frac{e^{ax} \, dx}{\sin^p bx} = -\frac{e^{ax} [a \sin bx + (p-2)b \cos bx]}{(p-1)(p-2)b^2 \sin^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} \, dx}{\sin^{p-2} bx} \quad \text{TI (530)a}$$

$$2. \quad \int \frac{e^{ax} \, dx}{\cos^p bx} = -\frac{e^{ax} [a \cos bx - (p-2)b \sin bx]}{(p-1)(p-2)b^2 \cos^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} \, dx}{\cos^{p-2} bx} \quad \text{TI (529)a}$$

By successive applications of formulas **2.665** for p a natural number, we obtain integrals of the form $\int \frac{e^{ax} \, dx}{\sin bx}$, $\int \frac{e^{ax} \, dx}{\sin^2 bx}$, $\int \frac{e^{ax} \, dx}{\cos bx}$, $\int \frac{e^{ax} \, dx}{\cos^2 bx}$, which are not expressible in terms of a finite combination of elementary functions.

2.666

$$1. \quad \int e^{ax} \tan^p x \, dx = \frac{e^{ax}}{p-1} \tan^{p-1} x - \frac{a}{p-1} \int e^{ax} \tan^{p-1} x \, dx - \int e^{ax} \tan^{p-2} x \, dx \quad \text{TI (527)}$$

$$2. \quad \int e^{ax} \cot^p x \, dx = -\frac{e^{ax} \cot^{p-1} x}{p-1} + \frac{a}{p-1} \int e^{ax} \cot^{p-1} x \, dx - \int e^{ax} \cot^{p-2} x \, dx \quad \text{TI (528)}$$

$$3. \quad \int e^{ax} \tan x \, dx = \frac{e^{ax} \tan x}{a} - \frac{1}{a} \int \frac{e^{ax} \, dx}{\cos^2 x} \quad (\text{see remark following } \mathbf{2.665})$$

4. $\int e^{ax} \tan^2 x \, dx = \frac{e^{ax}}{a} (a \tan x - 1) - a \int e^{ax} \tan x \, dx$ (see **2.666** 3) TI 355
5. $\int e^{ax} \cot x \, dx = \frac{e^{ax} \cot x}{a} + \frac{1}{a} \int \frac{e^{ax} \, dx}{\sin^2 x}$ (see remark following **2.665**)
6. $\int e^{ax} \cot^2 x \, dx = -\frac{e^{ax}}{a} (a \cot x + 1) + a \int e^{ax} \cot x \, dx$
- (see **2.666** 5)

Integrals of type $\int R(x, e^{ax}, \sin bx, \cos cx) \, dx$

Notation: $\sin t = -\frac{b}{\sqrt{a^2 + b^2}}; \quad \cos t = \frac{a}{\sqrt{a^2 + b^2}}.$

2.667

1.
$$\begin{aligned} \int x^p e^{ax} \sin bx \, dx &= \frac{x^p e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) - \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} (a \sin bx - b \cos bx) \, dx \\ &= \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \sin(bx + t) - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \sin(bx + t) \, dx \end{aligned}$$
2.
$$\begin{aligned} \int x^p e^{ax} \cos bx \, dx &= \frac{x^p e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) - \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} (a \cos bx + b \sin bx) \, dx \\ &= \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \cos(bx + t) - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \cos(bx + t) \, dx \end{aligned}$$
3.
$$\int x^n e^{ax} \sin bx \, dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \sin(bx + kt)$$
4.
$$\int x^n e^{ax} \cos bx \, dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \cos(bx + kt)$$
5.
$$\int x e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left[\left(ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \sin bx - \left(bx - \frac{2ab}{a^2 + b^2} \right) \cos bx \right]$$
6.
$$\int x e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left[\left(ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \cos bx + \left(bx - \frac{2ab}{a^2 + b^2} \right) \sin bx \right]$$
7.
$$\begin{aligned} \int x^2 e^{ax} \sin bx \, dx &= \frac{e^{ax}}{a^2 + b^2} \left\{ \left[ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2} x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \sin bx \right. \\ &\quad \left. - \left[bx^2 - \frac{4ab}{a^2 + b^2} x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \cos bx \right\} \end{aligned}$$

$$8. \quad \int x^2 e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left\{ \left[ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2}x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \cos bx \right. \\ \left. + \left[bx^2 - \frac{4ab}{a^2 + b^2}x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \sin bx \right\}$$

GU (335), MZ 274-275

2.67 Combinations of trigonometric and hyperbolic functions

2.671

$$1. \quad \int \sinh(ax + b) \sin(cx + d) \, dx = \frac{a}{a^2 + c^2} \cosh(ax + b) \sin(cx + d) \\ - \frac{c}{a^2 + c^2} \sinh(ax + b) \cos(cx + d)$$

$$2. \quad \int \sinh(ax + b) \cos(cx + d) \, dx = \frac{a}{a^2 + c^2} \cosh(ax + b) \cos(cx + d) \\ + \frac{c}{a^2 + c^2} \sinh(ax + b) \sin(cx + d)$$

$$3. \quad \int \cosh(ax + b) \sin(cx + d) \, dx = \frac{a}{a^2 + c^2} \sinh(ax + b) \sin(cx + d) \\ - \frac{c}{a^2 + c^2} \cosh(ax + b) \cos(cx + d)$$

$$4. \quad \int \cosh(ax + b) \cos(cx + d) \, dx = \frac{a}{a^2 + c^2} \sinh(ax + b) \cos(cx + d) \\ + \frac{c}{a^2 + c^2} \cosh(ax + b) \sin(cx + d)$$

GU (354)(1)

2.672

$$1. \quad \int \sinh x \sin x \, dx = \frac{1}{2} (\cosh x \sin x - \sinh x \cos x)$$

$$2. \quad \int \sinh x \cos x \, dx = \frac{1}{2} (\cosh x \cos x + \sinh x \sin x)$$

$$3. \quad \int \cosh x \sin x \, dx = \frac{1}{2} (\sinh x \sin x - \cosh x \cos x)$$

$$4. \quad \int \cosh x \cos x \, dx = \frac{1}{2} (\sinh x \cos x + \cosh x \sin x)$$

2.673

$$\begin{aligned}
1. \quad & \int \sinh^{2m}(ax+b) \sin^{2n}(cx+d) dx \\
&= \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \frac{(-1)^{m+n}}{2^{2m+2n-1}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k)c} \binom{2n}{k} \sin[(2n-2k)(cx+d)] \\
&\quad + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
&\quad + (2-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]
\end{aligned}$$

GU (354)(3a)

$$\begin{aligned}
2. \quad & \int \sinh^{2m}(ax+b) \sin^{2n-1}(cx+d) dx \\
&= \frac{(-1)^{m+n}}{2^{2m+2n-2}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k-1)c} \binom{2n-1}{k} \cos[(2n-2k-1)(cx+d)] \\
&\quad + \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)]\} \\
&\quad - (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)]
\end{aligned}$$

GU (354)(3b)

$$\begin{aligned}
3. \quad & \int \sinh^{2m-1}(ax+b) \sin^{2n}(cx+d) dx \\
&= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \cosh[(2m-2j-1)(ax+d)] \\
&\quad + \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{(2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
&\quad + (2n-2k)c \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)]
\end{aligned}$$

GU (354)(3c)

$$\begin{aligned}
4. \quad & \int \sinh^{2m-1}(ax+b) \sin^{2n-1}(cx+d) dx \\
&= \frac{(-1)^{n-1}}{2^{2m-2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{ (2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] \\
&\quad - (2n-2k-1)c \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)] \} \\
&\hspace{15em} \text{GU (354)(3d)}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int \sinh^{2m}(ax+b) \cos^{2n}(cx+d) dx \\
&= \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)] \\
&\quad + \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] \\
&\quad + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{ (2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] \} \\
&\quad + (2-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \\
&\hspace{15em} \text{GU (354)(4a)}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int \sinh^{2m}(ax+b) \cos^{2n-1}(cx+d) dx \\
&= \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c} \sin[(2n-2k-1)(cx+d)] \\
&\quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{ (2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)] \} \\
&\quad + (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)] \\
&\hspace{15em} \text{GU (354)(4a)}
\end{aligned}$$

$$\begin{aligned}
7. \quad & \int \sinh^{2m-1}(ax+b) \cos^{2n}(cx+d) dx \\
&= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \cosh[(2m-2j-1)(ax+d)] \\
&\quad + \frac{1}{2^{2m-2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{ (2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)] \} \\
&\quad + (2n-2k)c \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)] \\
&\hspace{15em} \text{GU (354)(4b)}
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int \sinh^{2m-1}(ax+b) \cos^{2n-1}(cx+d) dx \\
&= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{ (2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)] \} \\
&\quad + (2n-2k-1)c \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] \\
&\hspace{15em} \text{GU (354)(4b)}
\end{aligned}$$

$$\begin{aligned}
9. \quad & \int \cosh^{2m}(ax+b) \sin^{2n}(cx+d) dx \\
&= \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] \\
&\quad + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)] \\
&\quad + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{ (2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] \} \\
&\quad + (2n-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \\
&\hspace{15em} \text{GU (354)(5a)}
\end{aligned}$$

$$\begin{aligned}
10. \quad & \int \cosh^{2m-1}(ax+b) \sin^{2n}(cx+d) dx \\
&= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \sinh[(2m-2j-1)(ax+b)] \\
&+ \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \\
&\times \{ (2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)] \} \\
&+ (2n-2k)c \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)] \\
&\hspace{15em} \text{GU (354)(5a)}
\end{aligned}$$

$$\begin{aligned}
11. \quad & \int \cosh^{2m}(ax+b) \sin^{2n-1}(cx+d) dx \\
&= \frac{(-1)^{n-1} \binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{k+1} \binom{2n-1}{k}}{(2n-2k-1)c} \cos[(2n-2k-1)(cx+d)] \\
&+ \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\times \{ (2m-2j)a \sinh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)] \} \\
&- (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)] \\
&\hspace{15em} \text{GU (354)(5b)}
\end{aligned}$$

$$\begin{aligned}
12. \quad & \int \cosh^{2m-1}(ax+b) \sin^{2n-1}(cx+d) dx \\
&= \frac{(-1)^{n-1}}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\times \{ (2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] \} \\
&- (2n-2k-1)c \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)] \\
&\hspace{15em} \text{GU (354)(5b)}
\end{aligned}$$

$$\begin{aligned}
13. \quad & \int \cosh^{2m}(ax+b) \cos^{2n}(cx+d) dx \\
&= \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] \\
&\quad + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)] \\
&\quad + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{ (2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] \} \\
&\quad + (2n-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \\
&\hspace{15em} \text{GU (354)(6)}
\end{aligned}$$

$$\begin{aligned}
14. \quad & \int \cosh^{2m-1}(ax+b) \cos^{2n}(cx+d) dx \\
&= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \sinh[(2m-2j-1)(ax+b)] \\
&\quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{ (2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)] \} \\
&\quad + (2n-2k)c \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)] \\
&\hspace{15em} \text{GU (354)(6)}
\end{aligned}$$

$$\begin{aligned}
15. \quad & \int \cosh^{2m}(ax+b) \cos^{2n-1}(cx+d) dx \\
&= \frac{\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c} \sin[(2n-2k-1)(cx+d)] \\
&\quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{ (2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)] \} \\
&\quad + (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)] \\
&\hspace{15em} \text{GU (354)(6)}
\end{aligned}$$

$$\begin{aligned}
16. \quad & \int \cosh^{2m-1}(ax+b) \cos^{2n-1}(cx+d) dx \\
&= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{ (2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)] \} \\
&\quad + (2n-2k-1)c \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] \\
&\hspace{15em} \text{GU (354)(6)}
\end{aligned}$$

2.674

$$\begin{aligned}
1. \quad & \int e^{ax} \sinh bx \sin cx \, dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \sin cx - c \cos cx] \\
&\quad - \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \sin cx - c \cos cx] \\
2. \quad & \int e^{ax} \sinh bx \cos cx \, dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \cos cx + c \sin cx] \\
&\quad - \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \cos cx + c \sin cx] \\
3. \quad & \int e^{ax} \cosh bx \sin cx \, dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \sin cx - c \cos cx] \\
&\quad + \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \sin cx - c \cos cx] \\
4. \quad & \int e^{ax} \cosh bx \cos cx \, dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \cos cx + c \sin cx] \\
&\quad + \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \cos cx + c \sin cx]
\end{aligned}$$

MZ 379

2.7 Logarithms and Inverse-Hyperbolic Functions**2.71 The logarithm**

$$\begin{aligned}
2.711 \quad & \int \ln^m x \, dx = x \ln^m x - m \int \ln^{m-1} x \, dx \\
&= \frac{x}{m+1} \sum_{k=0}^m (-1)^k (m+1)m(m-1) \cdots (m-k+1) \ln^{m-k} x \\
&\hspace{15em} (m > 0)
\end{aligned}$$

TI (603)

2.72–2.73 Combinations of logarithms and algebraic functions

2.721

$$1. \quad \int x^n \ln^m x \, dx = \frac{x^{n+1} \ln^m x}{n+1} - \frac{m}{n+1} \int x^n \ln^{m-1} x \, dx \quad (\text{see } 2.722)$$

For $n = -1$

$$2. \quad \int \frac{\ln^m x \, dx}{x} = \frac{\ln^{m+1} x}{m+1}$$

For $n = -1$ and $m = -1$

$$3. \quad \int \frac{dx}{x \ln x} = \ln(\ln x)$$

$$2.722 \quad \int x^n \ln^m x \, dx = \frac{x^{n+1}}{m+1} \sum_{k=0}^m (-1)^k (m+1)m(m-1) \cdots (m-k+1) \frac{\ln^{m-k} x}{(n+1)^{k+1}} \quad \text{TI (604)}$$

2.723

$$1. \quad \int x^n \ln x \, dx = x^{n+1} \left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right] \quad \text{TI 375}$$

$$2. \quad \int x^n \ln^2 x \, dx = x^{n+1} \left[\frac{\ln^2 x}{n+1} - \frac{2 \ln x}{(n+1)^2} + \frac{2}{(n+1)^3} \right] \quad \text{TI 375}$$

$$3. \quad \int x^n \ln^3 x \, dx = x^{n+1} \left[\frac{\ln^3 x}{n+1} - \frac{3 \ln^2 x}{(n+1)^2} + \frac{6 \ln x}{(n+1)^3} - \frac{6}{(n+1)^4} \right]$$

2.724

$$1. \quad \int \frac{x^n \, dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1} \int \frac{x^n \, dx}{(\ln x)^{m-1}}$$

For $m = 1$

$$2. \quad \int \frac{x^n \, dx}{\ln x} = \text{li}(x^{n+1})$$

2.725

$$1. \quad \int (a+bx)^m \ln x \, dx = \frac{1}{(m+1)b} \left[(a+bx)^{m+1} \ln x - \int \frac{(a+bx)^{m+1} \, dx}{x} \right] \quad \text{TI 374}$$

$$2. \quad \int (a+bx)^m \ln x \, dx = \frac{1}{(m+1)b} \left[(a+bx)^{m+1} - a^{m+1} \right] \ln x - \sum_{k=0}^m \frac{\binom{m}{k} a^{m-k} b^k x^{k+1}}{(k+1)^2}$$

For $m = -1$, see 2.727 2.

2.726

$$1. \quad \int (a+bx) \ln x \, dx = \left[\frac{(a+bx)^2}{2b} - \frac{a^2}{2b} \right] \ln x - \left(ax + \frac{1}{4} bx^2 \right)$$

$$2. \quad \int (a+bx)^2 \ln x \, dx = \frac{1}{3b} \left[(a+bx)^3 - a^3 \right] \ln x - \left(a^2 x + \frac{abx^2}{2} + \frac{b^2 x^3}{9} \right)$$

$$3. \quad \int (a+bx)^3 \ln x \, dx = \frac{1}{4b} [(a+bx)^4 - a^4] \ln x - \left(a^3x + \frac{3}{4}a^2bx^2 + \frac{1}{3}ab^2x^3 + \frac{1}{16}b^3x^4 \right)$$

2.727

$$1.^8 \quad \int \frac{\ln x \, dx}{(a+bx)^m} = \frac{1}{b(m-1)} \left[-\frac{\ln x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}} \right]$$

TI 376

For $m = 1$

$$2.^8 \quad \int \frac{\ln x \, dx}{a+bx} = \frac{1}{b} \ln x \ln(a+bx) - \frac{1}{b} \int \frac{\ln(a+bx) \, dx}{x} \quad (\text{see } \mathbf{2.728} \, 2)$$

$$3. \quad \int \frac{\ln x \, dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab} \ln \frac{x}{a+bx}$$

$$4. \quad \int \frac{\ln x \, dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b} \ln \frac{x}{a+bx}$$

$$5. \quad \int \frac{\ln x \, dx}{\sqrt{a+bx}} = \frac{2}{b} \left\{ (\ln x - 2) \sqrt{a+bx} - 2\sqrt{a} \ln \left[\frac{(a+bx)^{1/2} - a^{1/2}}{x^{1/2}} \right] \right\} \quad [a > 0]$$

$$= \frac{2}{b} \left\{ (\ln x - 2) \sqrt{a+bx} + 2\sqrt{-a} \arctan \sqrt{\frac{a+bx}{-a}} \right\} \quad [a < 0]$$

2.728

$$1. \quad \int x^m \ln(a+bx) \, dx = \frac{1}{m+1} \left[x^{m+1} \ln(a+bx) - b \int \frac{x^{m+1} \, dx}{a+bx} \right]$$

$$2.^9 \quad \int \frac{\ln(a+bx)}{x} = \ln a \ln x + \frac{bx}{a} \Phi \left(-\frac{bx}{a}, 2, 1 \right) \quad [a > 0]$$

2.729

$$1. \quad \int x^m \ln(a+bx) \, dx = \frac{1}{m+1} \left[x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}} \right] \ln(a+bx) + \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} a^{k-1}}{(m-k+2)b^{k-1}}$$

$$2. \quad \int x \ln(a+bx) \, dx = \frac{1}{2} \left[x^2 - \frac{a^2}{b^2} \right] \ln(a+bx) - \frac{1}{2} \left[\frac{x^2}{2} - \frac{ax}{b} \right]$$

$$3. \quad \int x^2 \ln(a+bx) \, dx = \frac{1}{3} \left[x^3 + \frac{a^3}{b^3} \right] \ln(a+bx) - \frac{1}{3} \left[\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2} \right]$$

$$4. \quad \int x^3 \ln(a+bx) \, dx = \frac{1}{4} \left[x^4 - \frac{a^4}{b^4} \right] \ln(a+bx) - \frac{1}{4} \left[\frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2x^2}{2b^2} - \frac{a^3x}{b^3} \right]$$

$$\mathbf{2.731} \quad \int x^{2n} \ln(x^2 + a^2) \, dx = \frac{1}{2n+1} \left\{ x^{2n+1} \ln(x^2 + a^2) + (-1)^n 2a^{2n+1} \arctan \frac{x}{a} \right. \\ \left. - 2 \sum_{k=0}^n \frac{(-1)^{n-k}}{2k+1} a^{2n-2k} x^{2k+1} \right\}$$

$$2.732^7 \int x^{2n+1} \ln(x^2 + a^2) dx = \frac{1}{2n+2} \left\{ (x^{2n+2} + (-1)^n a^{2n+2}) \ln(x^2 + a^2) + \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} a^{2n-2k+2} x^{2k} \right\}$$

2.733

$$1. \int \ln(x^2 + a^2) dx = x \ln(x^2 + a^2) - 2x + 2a \arctan \frac{x}{a} \quad \text{DW}$$

$$2. \int x \ln(x^2 + a^2) dx = \frac{1}{2} [(x^2 + a^2) \ln(x^2 + a^2) - x^2] \quad \text{DW}$$

$$3. \int x^2 \ln(x^2 + a^2) dx = \frac{1}{3} \left[x^3 \ln(x^2 + a^2) - \frac{2}{3} x^3 + 2a^2 x - 2a^3 \arctan \frac{x}{a} \right] \quad \text{DW}$$

$$4. \int x^3 \ln(x^2 + a^2) dx = \frac{1}{4} \left[(x^4 - a^4) \ln(x^2 + a^2) - \frac{x^4}{2} + a^2 x^2 \right] \quad \text{DW}$$

$$5. \int x^4 \ln(x^2 + a^2) dx = \frac{1}{5} \left[x^5 \ln(x^2 + a^2) - \frac{2}{5} x^5 + \frac{2}{3} a^2 x^3 - 2a^4 x + 2a^5 \arctan \frac{x}{a} \right] \quad \text{DW}$$

$$2.734 \int x^{2n} \ln|x^2 - a^2| dx = \frac{1}{2n+1} \left\{ x^{2n+1} \ln|x^2 - a^2| + a^{2n+1} \ln \left| \frac{x+a}{x-a} \right| - 2 \sum_{k=0}^n \frac{1}{2k+1} a^{2n-2k} x^{2k+1} \right\}$$

$$2.735 \int x^{2n+1} \ln|x^2 - a^2| dx = \frac{1}{2n+2} \left\{ (x^{2n+2} - a^{2n+2}) \ln|x^2 - a^2| - \sum_{k=1}^{n+1} \frac{1}{k} a^{2n-2k+2} x^{2k} \right\}$$

2.736

$$1. \int \ln|x^2 - a^2| dx = x \ln|x^2 - a^2| - 2x + a \ln \left| \frac{x+a}{x-a} \right| \quad \text{DW}$$

$$2. \int x \ln|x^2 - a^2| dx = \frac{1}{2} \{ (x^2 - a^2) \ln|x^2 - a^2| - x^2 \} \quad \text{DW}$$

$$3. \int x^2 \ln|x^2 - a^2| dx = \frac{1}{3} \left\{ x^3 \ln|x^2 - a^2| - \frac{2}{3} x^3 - 2a^2 x + a^3 \ln \left| \frac{x+a}{x-a} \right| \right\} \quad \text{DW}$$

$$4. \int x^3 \ln|x^2 - a^2| dx = \frac{1}{4} \left\{ (x^4 - a^4) \ln|x^2 - a^2| - \frac{x^4}{2} - a^2 x^2 \right\} \quad \text{DW}$$

$$5. \int x^4 \ln|x^2 - a^2| dx = \frac{1}{5} \left\{ x^5 \ln|x^2 - a^2| - \frac{2}{5} x^5 - \frac{2}{3} a^2 x^3 - 2a^4 x + a^5 \ln \left| \frac{x+a}{x-a} \right| \right\} \quad \text{DW}$$

2.74 Inverse hyperbolic functions**2.741**

$$1. \int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2} \quad \text{DW}$$

$$\begin{aligned}
2. \quad \int \operatorname{arccosh} \frac{x}{a} dx &= x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 - a^2} && \left[\operatorname{arccosh} \frac{x}{a} > 0 \right] && \text{DW} \\
&= x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 - a^2} && \left[\operatorname{arccosh} \frac{x}{a} < 0 \right] && \text{DW}
\end{aligned}$$

$$3. \quad \int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln(a^2 - x^2) \quad \text{DW}$$

$$4. \quad \int \operatorname{arccoth} \frac{x}{a} dx = x \operatorname{arccoth} \frac{x}{a} + \frac{a}{2} \ln(x^2 - a^2) \quad \text{DW}$$

2.742

$$1. \quad \int x \operatorname{arcsinh} \frac{x}{a} dx = \left(\frac{x^2}{2} + \frac{a^2}{4} \right) \operatorname{arcsinh} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 + a^2} \quad \text{DW}$$

$$\begin{aligned}
2. \quad \int x \operatorname{arccosh} \frac{x}{a} dx &= \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{arccosh} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 - a^2} && \left[\operatorname{arccosh} \frac{x}{a} > 0 \right] \\
&= \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{arccosh} \frac{x}{a} + \frac{x}{4} \sqrt{x^2 - a^2} && \left[\operatorname{arccosh} \frac{x}{a} < 0 \right] \\
&&& \text{DW}
\end{aligned}$$

2.8 Inverse Trigonometric Functions**2.81 Arcsines and arccosines**

$$\begin{aligned}
2.811 \quad \int \left(\arcsin \frac{x}{a} \right)^n dx &= x \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arcsin \frac{x}{a} \right)^{n-2k} \\
&\quad + \sqrt{a^2 - x^2} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \binom{n}{2k-1} \cdot (2k-1)! \left(\arcsin \frac{x}{a} \right)^{n-2k+1}
\end{aligned}$$

$$\begin{aligned}
2.812 \quad \int \left(\arccos \frac{x}{a} \right)^n dx &= x \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arccos \frac{x}{a} \right)^{n-2k} \\
&\quad + \sqrt{a^2 - x^2} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^k \binom{n}{2k-1} \cdot (2k-1)! \left(\arccos \frac{x}{a} \right)^{n-2k+1}
\end{aligned}$$

2.813

$$1.^{11} \quad \int \arcsin \frac{x}{a} dx = \operatorname{sign}(a) \left[x \arcsin \frac{x}{|a|} + \sqrt{a^2 - x^2} \right]$$

$$2.^9 \quad \int \left(\arcsin \frac{x}{a} \right)^2 dx = x \left(\arcsin \frac{x}{|a|} \right)^2 + 2\sqrt{a^2 - x^2} \arcsin \frac{x}{|a|} - 2x$$

$$\begin{aligned}
3. \quad \int \left(\arcsin \frac{x}{a} \right)^3 dx &= \operatorname{sign}(a) \left[x \left(\arcsin \frac{x}{|a|} \right)^3 + 3\sqrt{a^2 - x^2} \left(\arcsin \frac{x}{|a|} \right)^2 \right. \\
&\quad \left. - 6x \arcsin \frac{x}{|a|} - 6\sqrt{a^2 - x^2} \right]
\end{aligned}$$

2.814

1. $\int \arccos \frac{x}{a} dx = x \arccos \frac{x}{a} - \sqrt{a^2 - x^2}$
2. $\int \left(\arccos \frac{x}{a} \right)^2 dx = x \left(\arccos \frac{x}{a} \right)^2 - 2\sqrt{a^2 - x^2} \arccos \frac{x}{a} - 2x$
3. $\int \left(\arccos \frac{x}{a} \right)^3 dx = x \left(\arccos \frac{x}{a} \right)^3 - 3\sqrt{a^2 - x^2} \left(\arccos \frac{x}{a} \right)^2 - 6x \arccos \frac{x}{a} + 6\sqrt{a^2 - x^2}$

2.82 The arcsecant, the arccosecant, the arctangent, and the arccotangent

2.821

1.
$$\begin{aligned} \int \operatorname{arccosec} \frac{x}{a} dx &= \int \arcsin \frac{a}{x} dx = x \arcsin \frac{x}{2} + a \ln \left(x + \sqrt{x^2 - a^2} \right) & \left[0 < \arcsin \frac{a}{x} < \frac{\pi}{2} \right] \\ &= x \arcsin \frac{a}{x} - a \ln \left(x + \sqrt{x^2 - a^2} \right) & \left[-\frac{\pi}{2} < \arcsin \frac{a}{x} < 0 \right] \end{aligned}$$
 DW
2.
$$\begin{aligned} \int \operatorname{arcsec} \frac{x}{a} dx &= \int \arccos \frac{a}{x} dx = x \arccos \frac{a}{x} - a \ln \left(x + \sqrt{x^2 - a^2} \right) & \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right] \\ &= x \arccos \frac{a}{x} - a \ln \left(x + \sqrt{x^2 - a^2} \right) & \left[-\frac{\pi}{2} < \arccos \frac{a}{x} < 0 \right] \end{aligned}$$
 DW

2.822

- 1.⁸ $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2)$ DW
2. $\int \operatorname{arccot} \frac{x}{a} dx = x \operatorname{arccot} \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2)$ DW
- 3.⁹ $\int x \arctan \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \arctan \frac{x}{a} - \frac{ax}{2}$
- 4.⁹ $\int x \operatorname{arccot} \frac{x}{a} dx = \frac{ax}{2} + \frac{\pi x^2}{4} - \frac{1}{2} (x^2 + a^2) \arctan \frac{x}{a}$
- 5.⁹ $\int x^2 \arctan \frac{x}{a} dx = \frac{1}{3} x^3 \arctan \frac{x}{a} + \frac{1}{6} a^3 \ln(x^2 + a^2) - \frac{ax^2}{6}$
- 6.⁹ $\int x^2 \operatorname{arccot} \frac{x}{a} dx = -\frac{1}{3} x^3 \arctan \frac{x}{a} - \frac{1}{6} a^3 \ln(x^2 + a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$

2.83 Combinations of arcsine or arccosine and algebraic functions

2.831 $\int x^n \arcsin \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arcsin \frac{x}{a} - \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}}$ (see 2.263 1, 2.264, 2.27)

2.832 $\int x^n \arccos \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arccos \frac{x}{a} + \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}}$ (see 2.263 1, 2.264, 2.27)

1. For $n = -1$, these integrals (that is, $\int \frac{\arcsin x}{x} dx$ and $\int \frac{\arccos x}{x} dx$) cannot be expressed as a finite combination of elementary functions.

$$2. \quad \int \frac{\arccos x}{x} dx = -\frac{\pi}{2} \ln \frac{1}{x} - \int \frac{\arcsin x}{x} dx$$

2.833⁹

$$\begin{aligned} 1. \quad & \int x \arcsin \frac{x}{a} dx = \text{sign}(a) \left[\left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arcsin \frac{x}{|a|} + \frac{x}{4} \sqrt{a^2 - x^2} \right] \\ 2. \quad & \int x \arccos \frac{x}{a} dx = \frac{\pi x^2}{4} - \text{sign}(a) \left[\frac{1}{4} (2x^2 - a^2) \arcsin \frac{x}{|a|} + \frac{x}{4} \sqrt{a^2 - x^2} \right] \\ 3. \quad & \int x^2 \arcsin \frac{x}{a} dx = \text{sign}(a) \left[\frac{x^3}{3} \arcsin \frac{x}{|a|} + \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2} \right] \\ 4. \quad & \int x^2 \arccos \frac{x}{a} dx = \frac{\pi x^3}{6} - \text{sign}(a) \left[\frac{x^3}{3} \arcsin \frac{x}{|a|} + \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2} \right] \\ 5. \quad & \int x^3 \arcsin \frac{x}{a} dx = \text{sign}(a) \left[\left(\frac{x^4}{4} - \frac{3a^4}{32} \right) \arcsin \frac{x}{|a|} + \frac{1}{32} x (2x^2 + 3a^2) \sqrt{a^2 - x^2} \right] \\ 6. \quad & \int x^3 \arccos \frac{x}{a} dx = \frac{\pi x^4}{8} - \text{sign}(a) \left[\frac{(8x^4 - 3a^4)}{32} \arcsin \frac{x}{|a|} + \frac{1}{32} x (2x^2 + 3a^2) \sqrt{a^2 - x^2} \right] \end{aligned}$$

2.834

$$\begin{aligned} 1. \quad & \int \frac{1}{x^2} \arcsin \frac{x}{a} dx = -\frac{1}{x} \arcsin \frac{x}{a} - \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x} \\ 2. \quad & \int \frac{1}{x^2} \arccos \frac{x}{a} dx = -\frac{1}{x} \arccos \frac{x}{a} - \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x} \end{aligned}$$

$$\begin{aligned} \mathbf{2.835} \quad & \int \frac{\arcsin x}{(a + bx)^2} dx = -\frac{\arcsin x}{b(a + bx)} - \frac{2}{b\sqrt{a^2 - b^2}} \arctan \sqrt{\frac{(a - b)(1 - x)}{(a + b)(1 + x)}} \quad [a^2 > b^2] \\ & = -\frac{\arcsin x}{b(a + bx)} - \frac{1}{b\sqrt{b^2 - a^2}} \ln \frac{\sqrt{(a + b)(1 + x)} + \sqrt{(b - a)(1 - x)}}{\sqrt{(a + b)(1 + x)} - \sqrt{(b - a)(1 - x)}} \quad [a^2 < b^2] \end{aligned}$$

$$\begin{aligned} \mathbf{2.836}^8 \quad & \int \frac{x \arcsin x}{(1 + cx^2)^2} dx = -\frac{\arcsin x}{2c(1 + cx^2)} + \frac{1}{2c\sqrt{c + 1}} \arctan \frac{\sqrt{c + 1}x}{\sqrt{1 - x^2}} \quad [c > -1] \\ & = -\frac{\arcsin x}{2c(1 + cx^2)} + \frac{1}{4c\sqrt{-(c + 1)}} \ln \frac{\sqrt{1 - x^2} + x\sqrt{-(c + 1)}}{\sqrt{1 - x^2} - x\sqrt{-(c + 1)}} \quad [c < -1] \end{aligned}$$

2.837

$$\begin{aligned} 1. \quad & \int \frac{x \arcsin x}{\sqrt{1 - x^2}} dx = x - \sqrt{1 - x^2} \arcsin x \\ 2. \quad & \int \frac{x \arcsin x}{\sqrt{1 - x^2}} dx = \frac{x^2}{4} - \frac{x}{2} \sqrt{1 - x^2} \arcsin x + \frac{1}{4} (\arcsin x)^2 \\ 3. \quad & \int \frac{x^3 \arcsin x}{\sqrt{1 - x^2}} dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3} (x^2 + 2) \sqrt{1 - x^2} \arcsin x \end{aligned}$$

2.838

$$1. \quad \int \frac{\arcsin x}{\sqrt{(1 - x^2)^3}} dx = \frac{x \arcsin x}{\sqrt{1 - x^2}} + \frac{1}{2} \ln(1 - x^2)$$

$$2. \quad \int \frac{x \arcsin x}{\sqrt{(1-x^2)^3}} dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln \frac{1-x}{1+x}$$

2.84 Combinations of the arcsecant and arccosecant with powers of x

2.841

$$1. \quad \int x \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = \frac{1}{2} \left\{ x^2 \arccos \frac{a}{x} - a \sqrt{x^2 - a^2} \right\} \quad \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right]$$

$$= \frac{1}{2} \left\{ x^2 \arccos \frac{a}{x} + a \sqrt{x^2 - a^2} \right\} \quad \left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi \right]$$

DW

$$2. \quad \int x^2 \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} - \frac{a}{2} x \sqrt{x^2 - a^2} - \frac{a^3}{2} \ln \left(x + \sqrt{x^2 - a^2} \right) \right\}$$

$$\quad \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right]$$

$$= \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} + \frac{a}{2} x \sqrt{x^2 - a^2} + \frac{a^3}{2} \ln \left(x + \sqrt{x^2 - a^2} \right) \right\}$$

$$\quad \left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi \right]$$

DW

$$3. \quad \int x \operatorname{arccosec} \frac{x}{a} dx = \int \arcsin \frac{a}{x} dx = \frac{1}{2} \left\{ x^2 \arcsin \frac{a}{x} + a \sqrt{x^2 - a^2} \right\} \quad \left[0 < \arcsin \frac{a}{x} < \frac{\pi}{2} \right]$$

$$= \frac{1}{2} \left\{ x^2 \arcsin \frac{a}{x} - a \sqrt{x^2 - a^2} \right\} \quad \left[-\frac{\pi}{2} < \arcsin \frac{a}{x} < 0 \right]$$

DW

2.85 Combinations of the arctangent and arccotangent with algebraic functions

$$2.851 \quad \int x^n \arctan \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arctan \frac{x}{a} - \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}$$

2.852

$$1. \quad \int x^n \operatorname{arccot} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}$$

For $n = -1$

$$2. \quad \int \frac{\arctan x}{x} dx \text{ cannot be expressed as a finite combination of elementary functions.}$$

$$3. \quad \int \frac{\operatorname{arccot} x}{x} dx = \frac{\pi}{2} \ln x - \int \frac{\arctan x}{x} dx$$

2.853

$$1. \quad \int x \arctan \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \arctan \frac{x}{a} - \frac{ax}{2}$$

$$2. \quad \int x \operatorname{arccot} \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \operatorname{arccot} \frac{x}{a} + \frac{ax}{2}$$

$$3.^9 \quad \int x^2 \arctan \frac{x}{a} dx = \frac{x^3}{3} \arctan \frac{x}{a} + \frac{a^3}{6} \ln(x^2 + a^2) - \frac{ax^2}{6}$$

$$4.^9 \quad \int x^2 \operatorname{arccot} \frac{x}{a} dx = -\frac{x^3}{3} \arctan \frac{x}{a} - \frac{a^3}{6} \ln(x^2 + a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$$

$$\mathbf{2.854} \quad \int \frac{1}{x^2} \arctan \frac{x}{a} dx = -\frac{1}{x} \arctan \frac{x}{a} - \frac{1}{2a} \ln \frac{a^2 + x^2}{x^2}$$

$$\mathbf{2.855} \quad \int \frac{\arctan x}{(\alpha + \beta x)^2} dx = \frac{1}{\alpha^2 + \beta^2} \left\{ \ln \frac{\alpha + \beta x}{\sqrt{1+x^2}} - \frac{\beta - \alpha x}{\alpha + \beta x} \arctan x \right\}$$

2.856

$$1. \quad \int \frac{x \arctan x}{1+x^2} dx = \frac{1}{2} \arctan x \ln(1+x^2) - \frac{1}{2} \int \frac{\ln(1+x^2)}{1+x^2} dx \quad \text{TI (689)}$$

$$2. \quad \int \frac{x^2 \arctan x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} (\arctan x)^2 \quad \text{TI (405)}$$

$$3. \quad \int \frac{x^3 \arctan x}{1+x^2} dx = -\frac{1}{2}x + \frac{1}{2}(1+x^2) \arctan x - \int \frac{x \arctan x}{1+x^2} dx$$

(see **2.8511**)

$$4. \quad \int \frac{x^4 \arctan x}{1+x^2} dx = -\frac{1}{6}x^2 + \frac{2}{3} \ln(1+x^2) + \left(\frac{x^3}{3} - x \right) \arctan x + \frac{1}{2} (\arctan x)^2$$

$$\mathbf{2.857} \quad \int \frac{\arctan x dx}{(1+x^2)^{n+1}} = \left[\sum_{k=1}^n \frac{(2n-2k)!!(2n-1)!!}{(2n)!!(2n-2k+1)!!} \frac{x}{(1+x^2)^{n-k+1}} + \frac{1}{2} \frac{(2n-1)!!}{(2)!!} \arctan x \right] \arctan x \\ + \frac{1}{2} \sum_{k=1}^n \frac{(2n-1)!!(2n-2k)!!}{(2n)!!(2n-2k+1)!!(n-k+1)} \frac{1}{(1+x^2)^{n-k+1}}$$

$$\mathbf{2.858} \quad \int \frac{x \arctan x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \arctan x + \sqrt{2} \arctan \frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x$$

$$\mathbf{2.859} \quad \int \frac{\arctan x}{\sqrt{(a+bx^2)^3}} dx = \frac{x \arctan x}{a\sqrt{a+bx^2}} - \frac{1}{a\sqrt{b-a}} \arctan \sqrt{\frac{a+bx^2}{b-a}} \quad [a < b] \\ = \frac{x \arctan x}{a\sqrt{a+bx^2}} + \frac{1}{2a\sqrt{a-b}} \ln \frac{\sqrt{a+bx^2} - \sqrt{a-b}}{\sqrt{a+bx^2} + \sqrt{a-b}} \quad [a > b]$$

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3–4 Definite Integrals of Elementary Functions

3.0 Introduction

3.01 Theorems of a general nature

3.011 Suppose that $f(x)$ is integrable[†] over the largest of the intervals (p, q) , (p, r) , (r, q) . Then (depending on the relative positions of the points p , q , and r) it is also integrable over the other two intervals, and we have

$$\int_p^q f(x) dx = \int_p^r f(x) dx + \int_r^q f(x) dx. \quad \text{FI II 126}$$

3.012 *The first mean-value theorem.* Suppose (1) that $f(x)$ is continuous and that $g(x)$ is integrable over the interval (p, q) , (2) that $m \leq f(x) \leq M$, and (3) that $g(x)$ does not change sign anywhere in the interval (p, q) . Then, there exists at least one point ξ (with $p \leq \xi \leq q$) such that

$$\int_p^q f(x)g(x) dx = f(\xi) \int_p^q g(x) dx. \quad \text{FI II 132}$$

3.013 *The second mean-value theorem.* If $f(x)$ is monotonic and non-negative throughout the interval (p, q) , where $p < q$, and if $g(x)$ is integrable over that interval, then there exists at least one point ξ (with $p \leq \xi \leq q$) such that

$$1. \quad \int_p^q f(x)g(x) dx = f(p) \int_p^\xi g(x) dx$$

Under the conditions of Theorem **3.013** 1, if $f(x)$ is nondecreasing, then

$$2. \quad \int_p^q f(x)g(x) dx = f(q) \int_\xi^q g(x) dx \quad [p \leq \xi \leq q].$$

If $f(x)$ is monotonic in the interval (p, q) , where $p < q$, and if $g(x)$ is integrable over that interval, then

*We omit the definition of definite and multiple integrals since they are widely known and can easily be found in any textbook on the subject. Here we give only certain theorems of a general nature which provide estimates, or which reduce the given integral to a simpler one.

[†]A function $f(x)$ is said to be integrable over the interval (p, q) , if the integral $\int_p^q f(x) dx$ exists. Here, we usually mean the existence of the integral in the sense of Riemann. When it is a matter of the existence of the integral in the sense of Stieltjes or Lebesgue, etc., we shall speak of integrability in the sense of Stieltjes or Lebesgue.

$$3. \quad \int_p^q f(x)g(x) dx = f(p) \int_p^\xi g(x) dx + f(q) \int_\xi^q g(x) dx \quad [p \leq \xi \leq q],$$

or

$$4. \quad \int_p^q f(x)g(x) dx = A \int_p^\xi g(x) dx + B \int_\xi^q g(x) dx \quad [p \leq \xi \leq q],$$

where A and B are any two numbers satisfying the conditions

$$\begin{aligned} A &\geq f(p+0) & \text{and} & & B &\leq f(q-0) & [\text{if } f \text{ decreases}], \\ A &\leq f(p+0) & \text{and} & & B &\geq f(q-0) & [\text{if } f \text{ increases}]. \end{aligned}$$

In particular,

$$5. \quad \int_p^q f(x)g(x) dx = f(p+0) \int_p^\xi g(x) dx + f(q-0) \int_\xi^q g(x) dx \quad \text{FI II 138}$$

3.02 Change of variable in a definite integral

$$3.020 \quad \int_\alpha^\beta f(x) dx = \int_\varphi^\psi f[g(t)]g'(t) dt; \quad x = g(t).$$

This formula is valid under the following conditions:

1. $f(x)$ is continuous on some interval $A \leq x \leq B$ containing the original limits of integration α and β .
2. The equalities $\alpha = g(\varphi)$ and $\beta = g(\psi)$ hold.
3. $g(t)$ and its derivative $g'(t)$ are continuous on the interval $\varphi \leq t \leq \psi$.
4. As t varies from φ to ψ , the function $g(t)$ always varies in the same direction from $g(\varphi) = \alpha$ to $g(\psi) = \beta$.*

3.021 The integral $\int_\alpha^\beta f(x) dx$ can be transformed into another integral with given limits φ and ψ by means of the linear substitution

$$x = \frac{\beta - \alpha}{\psi - \varphi}t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi} :$$

$$1. \quad \int_\alpha^\beta f(x) dx = \frac{\beta - \alpha}{\psi - \varphi} \int_\varphi^\psi f \left(\frac{\beta - \alpha}{\psi - \varphi}t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi} \right) dt$$

In particular, for $\varphi = 0$ and $\psi = 1$,

*If this last condition is not satisfied, the interval $\varphi \leq t \leq \psi$ should be partitioned into subintervals throughout each of which the condition is satisfied:

$$\int_\alpha^\beta f(x) dx = \int_\varphi^{\varphi_1} f[g(t)]g'(t) dt + \int_{\varphi_1}^{\varphi_2} f[g(t)]g'(t) dt + \cdots + \int_{\varphi_{n-1}}^\psi f[g(t)]g'(t) dt.$$

$$2. \quad \int_{\alpha}^{\beta} f(x) dx = (\beta - \alpha) \int_0^1 f((\beta - \alpha)t + \alpha) dt$$

For $\varphi = 0$ and $\psi = \infty$,

$$3. \quad \int_{\alpha}^{\beta} f(x) dx = (\beta - \alpha) \int_0^{\infty} f\left(\frac{\alpha + \beta t}{1 + t}\right) \frac{dt}{(1 + t)^2}$$

3.022 The following formulas also hold:

$$1. \quad \int_{\alpha}^{\beta} f(x) dx = \int_{\alpha}^{\beta} f(\alpha + \beta - x) dx$$

$$2. \quad \int_0^{\beta} f(x) dx = \int_0^{\beta} f(\beta - x) dx$$

$$3. \quad \int_{-\alpha}^{\alpha} f(x) dx = \int_{-\alpha}^{\alpha} f(-x) dx$$

3.03 General formulas

3.031

1. Suppose that a function $f(x)$ is integrable over the interval $(-p, p)$ and satisfies the relation $f(-x) = f(x)$ on that interval. (A function satisfying the latter condition is called an *even* function.) Then,

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx. \quad \text{FI II 159}$$

2. Suppose that $f(x)$ is a function that is integrable on the interval $(-p, p)$ and satisfies the relation $f(-x) = -f(x)$ on that interval. (A function satisfying the latter condition is called an *odd* function.) Then,

$$\int_{-p}^p f(x) dx = 0. \quad \text{FI II 159}$$

3.032

$$1. \quad \int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx,$$

where $f(x)$ is a function that is integrable on the interval $(0, 1)$.

FI II 159

$$2. \quad \int_0^{2\pi} f(p \cos x + q \sin x) dx = 2 \int_0^{\pi} f\left(\sqrt{p^2 + q^2} \cos x\right) dx,$$

where $f(x)$ is integrable on the interval $(-\sqrt{p^2 + q^2}, \sqrt{p^2 + q^2})$.

FI II 160

$$3. \quad \int_0^{\frac{\pi}{2}} f(\sin 2x) \cos x dx = \int_0^{\frac{\pi}{2}} f(\cos^2 x) \cos x dx,$$

where $f(x)$ is integrable on the interval $(0, 1)$.

FI II 161

3.033

1. If $f(x + \pi) = f(x)$ and $f(-x) = f(x)$, then

$$\int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) dx \quad \text{LO V 277(3)}$$

2. If $f(x + \pi) = -f(x)$ and $f(-x) = f(x)$, then

$$\int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) \cos x dx \quad \text{LO V 279(4)}$$

In formulas **3.033**, it is assumed that the integrals in the left members of the formulas exist.

$$\mathbf{3.034} \quad \int_0^{\infty} \frac{f(px) - f(qx)}{x} dx = [f(0) - f(+\infty)] \ln \frac{q}{p},$$

if $f(x)$ is continuous for $x \geq 0$ and if there exists a finite limit $f(+\infty) = \lim_{x \rightarrow +\infty} f(x)$. FI II 633

3.035

$$1. \quad \int_0^{\pi} \frac{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})}{1 + 2p \cos x + p^2} dx = \frac{2\pi}{1 - p^2} f(\alpha + p) \quad [|p| < 1] \quad \text{LA 230(16)}$$

$$2. \quad \int_0^{\pi} \frac{1 - p \cos x}{1 - 2p \cos x + p^2} \{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})\} dx = \pi \{f(\alpha + p) + f(\alpha)\} \\ [|p| < 1] \quad \text{BE 169}$$

$$3. \quad \int_0^{\pi} \frac{f(\alpha + e^{-xi}) - f(\alpha + e^{xi})}{1 - 2p \cos x + p^2} \sin x dx = \frac{\pi}{\pi} \{f(\alpha + p) - f(\alpha)\} \\ [|p| < 1] \quad \text{BE 169}$$

In formulas **3.035**, it is assumed that the function f is analytic in the closed unit circle with its center at the point α .

3.036

$$1.^{11} \quad \int_0^{\pi} f\left(\frac{\sin^2 x}{1 + 2p \cos x + p^2}\right) dx = \int_0^{\pi} f(\sin^2 x) dx \quad [p^2 < 1] \\ = \int_0^{\pi} f\left(\frac{\sin^2 x}{p^2}\right) dx \quad [p^2 \geq 1] \quad \text{LA 228(6)}$$

$$2. \quad \int_0^{\pi} F^{(n)}(\cos x) \sin^{2n} x dx = (2n - 1)!! \int_0^{\pi} F(\cos x) \cos nx dx \quad \text{B 174}$$

3.037 If f is analytic in the circle of radius r and if

$$f[r(\cos x + i \sin x)] = f_1(r, x) + i f_2(r, x),$$

then

$$1. \quad \int_0^{\infty} \frac{f_1(r, x)}{p^2 + x^2} dx = \frac{\pi}{2p} f(re^{-p}) \quad \text{LA 230(19)}$$

$$2. \quad \int_0^{\infty} f_2(r, x) \frac{x dx}{p^2 + x^2} = \frac{\pi}{2} [f(re^{-p}) - f(0)] \quad \text{LA 230(20)}$$

$$3. \quad \int_0^{\infty} \frac{f_2(r, x)}{x} dx = \frac{\pi}{2} [f(r) - f(0)] \quad \text{LA 230(21)}$$

$$4. \quad \int_0^\infty \frac{f_2(r, x)}{x(p^2 + x^2)} dx = \frac{\pi}{2p^2} [f(r) - f(re^{-p})] \quad \text{LA 230(22)}$$

$$\begin{aligned} 3.038 \quad \int_{-\infty}^\infty \frac{x dx}{\sqrt{1+x^2}} F(qx + p\sqrt{1+x^2}) &= \int_{-\infty}^\infty F(p \cosh x + q \sinh x) \sinh x dx \\ &= 2q \int_0^\infty F'(\operatorname{sign} p \cdot \sqrt{p^2 - q^2} \cosh x) \sinh^2 x dx \end{aligned}$$

[If F is a function with a continuous derivative in the interval $(-\infty, \infty)$, all these integrals converge.]

3.04 Improper integrals

3.041 Suppose that a function $f(x)$ is defined on an interval $(p, +\infty)$ and that it is integrable over an arbitrary finite subinterval of the form (p, P) . Then, by definition

$$\int_p^{+\infty} f(x) dx = \lim_{P \rightarrow +\infty} \int_p^P f(x) dx,$$

if this limit exists. If it does exist, we say that the integral $\int_p^{+\infty} f(x) dx$ exists or that it converges.

Otherwise, we say that the integral diverges.

3.042 Suppose that a function $f(x)$ is bounded and integrable in an arbitrary interval $(p, q - \eta)$ (for $0 < \eta < q - p$) but is unbounded in every interval $(q - \eta, q)$ to the left of the point q . The point q is then called a *singular point*. Then, by definition,

$$\int_p^q f(x) dx = \lim_{\eta \rightarrow 0} \int_p^{q-\eta} f(x) dx,$$

if this limit exists. In this case, we say that the integral $\int_p^q f(x) dx$ exists or that it converges.

3.043 If not only the integral of $f(x)$ but also the integral of $|f(x)|$ exists, we say that the integral of $f(x)$ converges *absolutely*.

3.044 The integral $\int_p^{+\infty} f(x) dx$ converges absolutely if there exists a number $\alpha > 1$ such that the limit

$$\lim_{x \rightarrow +\infty} \{x^\alpha |f(x)|\}$$

exists. On the other hand, if

$$\lim_{x \rightarrow +\infty} \{x |f(x)|\} = L > 0,$$

the integral $\int_p^{+\infty} |f(x)| dx$ diverges.

3.045 Suppose that the upper limit q of the integral $\int_p^q f(x) dx$ is a singular point. Then, this integral converges absolutely if there exists a number $\alpha < 1$ such that the limit

$$\lim_{x \rightarrow q} [(q - x)^\alpha |f(x)|]$$

exists. On the other hand, if

$$\lim_{x \rightarrow q} [(q - x) |f(x)|] = L > 0,$$

the integral $\int_p^q f(x) dx$ diverges.

3.046 Suppose that the functions $f(x)$ and $g(x)$ are defined on the interval $(p, +\infty)$, that $f(x)$ is integrable over every finite interval of the form (p, P) , that the integral

$$\int_p^P f(x) dx$$

is a bounded function of P , that $g(x)$ is monotonic, and that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$. Then, the integral

$$\int_p^{+\infty} f(x)g(x) dx$$

converges.

FI II 577

3.05 The principal values of improper integrals

3.051 Suppose that a function $f(x)$ has a singular point r somewhere inside the interval (p, q) , that $f(x)$ is defined at r , and that $f(x)$ is integrable over every portion of this interval that does not contain the point r . Then, by definition

$$\int_p^q f(x) dx = \lim_{\substack{\eta \rightarrow 0 \\ \eta' \rightarrow 0}} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta'}^q f(x) dx \right\}.$$

Here, the limit must exist for *independent* modes of approach of η and η' to zero. If this limit does not exist but the limit

$$\lim_{\eta \rightarrow 0} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta}^q f(x) dx \right\}$$

does exist, we say that this latter limit is the *principal value* of the improper integral $\int_p^q f(x) dx$, and we

say that the integral $\int_p^q f(x) dx$ exists in the sense of principal values.

FI II 603

3.052 Suppose that the function $f(x)$ is continuous over the interval (p, q) and vanishes at only one point r inside this interval. Suppose that the first derivative $f'(x)$ exists in a neighborhood of the point r . Suppose that $f'(r) \neq 0$ and that the second derivative $f''(r)$ exists at the point r itself. Then,

$$\int_p^q \frac{dx}{f(x)}$$

FI II 605

diverges, but exists in the sense of principal values.

3.053 A divergent integral of a positive function cannot exist in the sense of principal values.

3.054 Suppose that the function $f(x)$ has no singular points in the interval $(-\infty, +\infty)$. Then, by definition

$$\int_{-\infty}^{+\infty} f(x) dx = \lim_{\substack{P \rightarrow +\infty \\ Q \rightarrow +\infty}} \int_P^Q f(x) dx.$$

Here, the limit must exist for independent approach of P and Q to $\pm\infty$. If this limit does not exist but the limit

$$\lim_{P \rightarrow +\infty} \int_{-P}^{+P} f(x) dx$$

does exist, this last limit is called the principal value of the improper integral

$$\int_{-\infty}^{+\infty} f(x) dx.$$

FI II 607

3.055 The principal value of an improper integral of an even function exists only when this integral converges (in the ordinary sense).

FI II 607

3.1–3.2 Power and Algebraic Functions

3.11 Rational functions

1.
$$\int_{-\infty}^{\infty} \frac{p+qx}{r^2+2rx\cos\lambda+x^2} dx = \frac{\pi}{r\sin\lambda} (p-qr\cos\lambda) \quad (\text{principal value})$$

(see also **3.194** 8 and **3.252** 1 and 2) BI (22)(14)

3.112¹¹ Integrals of the form $\int_{-\infty}^{\infty} \frac{g_n(x) dx}{h_n(x)h_n(-x)}$, where

$$g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \cdots b_{n-1},$$

$$h_n(x) = a_0x^n + a_1x^{n-1} + \cdots a_n$$

[All roots of $h_n(x)$ lie in the upper half-plane.]

1.
$$\int_{-\infty}^{\infty} \frac{g_n(x) dx}{h_n(x)h_n(-x)} = \frac{\pi i}{a_0} \frac{M_n}{\Delta_n},$$

where

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & & 0 \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}, \quad M_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}.$$

2.
$$\int_{-\infty}^{\infty} \frac{g_1(x) dx}{h_1(x)h_1(-x)} = \frac{\pi i b_0}{a_0 a_1}$$
 JE

- 3.⁸
$$\int_{-\infty}^{\infty} \frac{g_2(x) dx}{h_2(x)h_2(-x)} = \pi i \frac{-b_0 + \frac{a_0 b_1}{a_2}}{a_0 a_1}$$

- 4.¹¹
$$\int_{-\infty}^{\infty} \frac{g_3(x) dx}{h_3(x)h_3(-x)} = \pi i \frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{a_0 (a_0 a_3 - a_1 a_2)}$$
 JE

5.
$$\int_{-\infty}^{\infty} \frac{g_4(x) dx}{h_4(x)h_4(-x)} = \pi i \frac{b_0 (-a_1 a_4 + a_2 a_3) - a_0 a_3 b_1 + a_0 a_1 b_2 + \frac{a_0 b_3}{a_4} (a_0 a_3 - a_1 a_2)}{a_0 (a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)}$$
 JE

6.
$$\int_{-\infty}^{\infty} \frac{g_5(x) dx}{h_5(x)h_5(-x)} = \pi i \frac{M_5}{a_0 \Delta_5},$$

where

$$M_5 = b_0 (-a_0 a_4 a_5 + a_1 a_4^2 + a_2^2 a_5 - a_2 a_3 a_4) + a_0 b_1 (-a_2 a_5 + a_3 a_4) \\ + a_0 b_2 (a_0 a_5 - a_1 a_4) + a_0 b_3 (-a_0 a_3 + a_1 a_2) + \frac{a_0 b_4}{a_5} (-a_0 a_1 a_5 + a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3),$$

$$\Delta_5 = a_0^2 a_5^2 - 2a_0 a_1 a_4 a_5 - a_0 a_2 a_3 a_5 + a_0 a_3^2 a_4 + a_1^2 a_4^2 + a_1 a_2^2 a_5 - a_1 a_2 a_3 a_4$$
 JE

3.12 Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials

3.121

$$1. \quad \int_0^1 \frac{1}{1 - 2x \cos \lambda + x^2} \frac{dx}{\sqrt{x}} = 2 \operatorname{cosec} \lambda \sum_{k=1}^{\infty} \frac{\sin k\lambda}{2k-1} \quad \text{BI (10)(17)}$$

$$2. \quad \int_0^1 \frac{1}{q - px} \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}} \quad [0 < p < q] \quad \text{BI (10)(9)}$$

$$3. \quad \int_0^1 \frac{dx}{1 - 2rx + r^2} \sqrt{\frac{1 \mp x}{1 \pm x}} = \pm \frac{\pi}{4r} \mp \frac{1}{r} \frac{1 \mp r}{1 \pm r} \arctan \frac{1+r}{1-r} \quad \text{LI (14)(5, 16)}$$

3.13–3.17 Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions

Notation: In 3.131–3.137 we set: $\alpha = \arcsin \sqrt{\frac{a-c}{a-u}}$, $\beta = \arcsin \sqrt{\frac{c-u}{b-u}}$,

$$\begin{aligned} \gamma &= \arcsin \sqrt{\frac{u-c}{b-c}}, & \delta &= \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \\ \kappa &= \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, & \lambda &= \arcsin \sqrt{\frac{a-u}{a-b}}, \\ \mu &= \arcsin \sqrt{\frac{u-a}{u-b}}, & \nu &= \arcsin \sqrt{\frac{a-c}{u-c}}, & p &= \sqrt{\frac{a-b}{a-c}}, & q &= \sqrt{\frac{b-c}{a-c}}. \end{aligned}$$

3.131

$$1. \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\alpha, p) \quad [a > b > c \geq u] \quad \text{BY (231.00)}$$

$$2. \quad \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\beta, p) \quad [a > b > c > u] \quad \text{BY (232.00)}$$

$$3. \quad \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\gamma, q) \quad [a > b \geq u > c] \quad \text{BY (233.00)}$$

$$4. \quad \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\delta, q) \quad [a > b > u \geq c] \quad \text{BY (234.00)}$$

$$5. \quad \int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\kappa, p) \quad [a \geq u > b > c] \quad \text{BY (235.00)}$$

$$6. \quad \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\lambda, p) \quad [a > u \geq b > c] \quad \text{BY (236.00)}$$

$$7. \quad \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\mu, q) \quad [u > a > b > c] \quad \text{BY (237.00)}$$

$$8. \quad \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\nu, q) \quad [u \geq a > b > c] \quad \text{BY (238.00)}$$

3.132

$$1. \quad \int_u^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} [c F(\beta, p) + (a-c) E(\beta, p)] - 2\sqrt{\frac{(a-u)(c-u)}{b-u}} \\ [a > b > c > u] \quad \text{BY (232.19)}$$

$$2. \quad \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\gamma, q) - 2\sqrt{a-c} E(\gamma, q) \\ [a > b \geq u > c] \quad \text{BY (233.17)}$$

$$3. \quad \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-a) \Pi(\delta, q^2, q) + a F(\delta, q)] \\ [a > b > u \geq c] \quad \text{BY (234.16)}$$

$$4. \quad \int_b^u \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-c) \Pi(\kappa, p^2, p) + c F(\kappa, p)] \\ [a \geq u > b > c] \quad \text{BY (235.16)}$$

$$5. \quad \int_u^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}} F(\lambda, p) + 2\sqrt{a-c} E(\lambda, p) \\ [a > u \geq b > c] \quad \text{BY (236.16)}$$

$$6. \quad \int_a^u \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}} [a(a-b) \Pi(\mu, 1, q) + b^2 F(\mu, q)] \\ [u > a > b > c] \quad \text{BY (237.16)}$$

3.133

$$1. \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\alpha, p) - E(\alpha, p)] \\ [a > b > c \geq u] \quad \text{BY (231.08)}$$

$$2. \quad \int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\beta, p) - E(\beta, p)] + \frac{2}{a-c} \sqrt{\frac{c-u}{(a-u)(b-u)}} \\ [a > b > c > u] \quad \text{BY (232.13)}$$

$$3. \quad \int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\gamma, q) - \frac{2}{(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{a-u}} \\ [a > b \geq u > c] \quad \text{BY (233.09)}$$

$$4. \quad \int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\delta, q) \quad [a > b > u \geq c] \quad \text{BY (234.05)}$$

$$5. \quad \int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\kappa, p) - E(\kappa, p)] + \frac{2}{a-b} \sqrt{\frac{u-b}{(a-u)(u-c)}} \\ [a > u > b > c] \quad \text{BY (235.04)}$$

6.
$$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}} E(\nu, q) + \frac{2}{a-b} \sqrt{\frac{u-b}{(u-a)(u-c)}}$$

[$u > a > b > c$] BY (238.05)
7.
$$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\alpha, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\alpha, p) - \frac{2}{b-c} \sqrt{\frac{c-u}{(a-u)(b-u)}}$$

[$a > b > c \geq u$] BY (231.09)
8.
$$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\beta, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\beta, p)$$

[$a > b > c > u$] BY (232.14)
9.
$$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} = \frac{2}{(b-c)\sqrt{a-c}} F(\gamma, q) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\gamma, q) + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{b-u}}$$

[$a > b > u > c$] BY (233.10)
10.
$$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} F(\lambda, p) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\lambda, p) + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{u-b}}$$

[$a > u > b > c$] BY (236.09)
11.
$$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\mu, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\mu, q)$$

[$u > a > b > c$] BY (237.12)
12.
$$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\nu, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\nu, q) - \frac{2}{a-b} \sqrt{\frac{u-a}{(u-b)(u-c)}}$$

[$u \geq a > b > c$] BY (238.04)
13.
$$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \frac{2}{(c-b)\sqrt{a-c}} E(\alpha, p) + \frac{2}{b-c} \sqrt{\frac{b-u}{(a-u)(c-u)}}$$

[$a > b > c > u$] BY (231.10)
14.
$$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\delta, q) - E(\delta, q)] + \frac{2}{b-c} \sqrt{\frac{b-u}{(a-u)(u-c)}}$$

[$a > b > u > c$] BY (234.04)
15.
$$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\kappa, p)$$

[$a \geq u > b > c$] BY (235.01)

$$16. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\lambda, p) - \frac{2}{(b-c)(a-c)} \sqrt{\frac{(a-u)(u-b)}{u-c}}$$

[a > u ≥ b > c] BY (236.10)

$$17. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\mu, q) - E(\mu, q)] + \frac{2}{a-c} \sqrt{\frac{u-a}{(u-b)(u-c)}}$$

[u > a > b > c] BY (237.13)

$$18. \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\nu, q) - E(\nu, q)]$$

[u ≥ a > b > c] BY (238.03)

3.134

$$1. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c) F(\alpha, p) - 2(2a-b-c) E(\alpha, p)]$$

$$+ \frac{2}{3(a-c)(a-b)} \sqrt{\frac{(c-u)(b-u)}{(a-u)^3}}$$

[a > b > c ≥ u] BY (231.08)

$$2. \int_u^c \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c) F(\beta, p) - 2(2a-b-c) E(\beta, p)]$$

$$+ \frac{2[4a^2-3ab-2ac+bc-u(3a-2b-c)]}{3(a-b)(a-c)^2} \sqrt{\frac{c-u}{(a-u)^3(b-u)}}$$

[a > b > c > u] BY (232.13)

$$3. \int_c^u \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^3\sqrt{(a-c)^3}} [2(2a-b-c) E(\gamma, q) - (a-b) F(\gamma, q)]$$

$$- \frac{2[5a^2-3ab-3ac+bc-2u(2a-b-c)]}{3(a-b)^2(a-c)^2} \sqrt{\frac{(b-u)(u-c)}{(a-u)^3}}$$

[a > b ≥ u > c] BY (233.09)

$$4. \int_u^b \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [2(2a-b-c) E(\delta, q) - (a-b) F(\delta, q)]$$

$$- \frac{2}{3(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{(a-u)^3}}$$

[a > b > u ≥ c] BY (234.05)

$$5. \int_b^u \frac{dx}{\sqrt{(a-x)^5(x-b)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c) F(\kappa, p) - 2(2a-b-c) E(\kappa, p)]$$

$$+ \frac{2[4a^2-2ab-3ac+bc-u(3a-b-2c)]}{3(a-b)^2(a-c)} \sqrt{\frac{u-b}{(a-u)^3(u-c)}}$$

[a > u > b > c] BY (235.04)

$$\begin{aligned}
6. \quad \int_u^\infty \frac{dx}{\sqrt{(x-a)^5(x-b)(x-c)}} &= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [2(2a-b-c)E(\nu, q) - (a-b)F(\nu, q)] \\
&\quad + \frac{2[4a^2-2ab-3ac+bc+u(b+2c-3a)]}{3(a-b)^2(a-c)} \sqrt{\frac{u-b}{(u-a)^3(u-c)}} \\
&\quad [u > a > b > c] \quad \text{BY (238.05)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} &= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\
&\quad \times [2(a-c)(a+c-2b)E(\alpha, p) + (b-c)(3b-a-2c)F(\alpha, p)] \\
&\quad - \frac{2[3ab-ac+2bc-4b^2-u(2a-3b+c)]}{3(a-b)(b-c)^2} \sqrt{\frac{c-u}{(a-u)(b-u)^3}} \\
&\quad [a > b > c \geq u] \quad \text{BY (231.09)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} &= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\
&\quad \times [(b-c)(3b-a-2c)F(\beta, p) + 2(a-c)(a-2b+c)E(\beta, p)] \\
&\quad + \frac{2}{3(a-b)(b-c)} \sqrt{\frac{(a-u)(c-u)}{(b-u)^3}} \\
&\quad [a > b > c > u] \quad \text{BY (232.14)}
\end{aligned}$$

$$\begin{aligned}
9. \quad \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^5(x-c)}} &= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\
&\quad \times [(a-b)(2a-3b+c)F(\gamma, q) + 2(a-c)(2b-a-c)E(\gamma, q)] \\
&\quad + \frac{2[3ab+3bc-ac-5b^2-2u(a-2b+c)]}{3(a-b)^2(b-c)^2} \sqrt{\frac{(a-u)(u-c)}{(b-u)^3}} \\
&\quad [a > b > u > c] \quad \text{BY (233.10)}
\end{aligned}$$

$$\begin{aligned}
10. \quad \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^5(x-c)}} &= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\
&\quad \times [(b-c)(3b-2c-a)F(\lambda, p) + 2(a-c)(a+c-2b)E(\lambda, p)] \\
&\quad + \frac{2[3ab+3bc-ac-5b^2+2u(2b-a-c)]}{3(a-b)^2(b-c)^2} \sqrt{\frac{(a-u)(u-c)}{(u-b)^3}} \\
&\quad [a > u > b > c] \quad \text{BY (236.09)}
\end{aligned}$$

$$\begin{aligned}
11. \quad \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} &= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\
&\quad \times [(a-b)(2a+c-3b)F(\mu, q) + 2(a-c)(2b-a-c)E(\mu, q)] \\
&\quad + \frac{2}{3(a-b)(b-c)} \sqrt{\frac{(u-a)(u-c)}{(u-b)^3}} \\
&\quad [u > a > b > c] \quad \text{BY (237.12)}
\end{aligned}$$

12.
$$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times [(a-b)(2a+c-3b)F(\nu, q) + 2(a-c)(2b-c-a)E(\nu, q)] - \frac{2[3bc+2ab-ac-4b^2+u(3b-a-2c)]}{3(a-b)^2(b-c)} \sqrt{\frac{u-a}{(u-b)^3(u-c)}} \quad [u \geq a > b > c] \quad \text{BY (238.04)}$$
13.
$$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\alpha, p) - (b-c)F(\alpha, p)] + \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(a-c)(b-c)^2} \sqrt{\frac{b-u}{(a-u)(c-u)^3}} \quad [a > b > c > u] \quad \text{By (231.10)}$$
14.
$$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\delta, q) - 2(a+b-2c)E(\delta, q)] + \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(b-c)^2(a-c)} \sqrt{\frac{b-u}{(a-u)(u-c)^3}} \quad [a > b > u > c] \quad \text{BY (234.04)}$$
15.
$$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\kappa, p) - (b-c)F(\kappa, p)] + \frac{2}{3(a-c)(b-c)} \sqrt{\frac{(a-u)(u-b)}{(u-c)^3}} \quad [a \geq u > b > c] \quad \text{BY (235.20)}$$
16.
$$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\lambda, p) - (b-c)F(\lambda, p)] - \frac{2[ab-3ac-3bc+5c^2+2u(a+b-2c)]}{3(b-c)^2(a-c)^2} \sqrt{\frac{(a-u)(u-b)}{(u-c)^3}} \quad [a > u \geq b > c] \quad \text{BY (236.10)}$$
17.
$$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\mu, q) - 2(a+b-2c)E(\mu, q)] + \frac{2[4c^2-ab-2ac-bc+u(3a+2b-5c)]}{3(b-c)(a-c)^2} \sqrt{\frac{u-a}{(u-b)(u-c)^3}} \quad [u > a > b > c] \quad \text{BY (237.13)}$$
18.
$$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\nu, q) - 2(a+b-2c)E(\nu, q)] + \frac{2}{3(a-c)(b-c)} \sqrt{\frac{(u-a)(u-b)}{(u-c)^3}} \quad [u \geq a > b > c] \quad \text{BY (238.03)}$$

3.135

$$1.^6 \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(b-c)F(\alpha, p) - (2a-b-c)E(\alpha, p)] \\ + \frac{2(b+c-2u)}{(b-c)^2\sqrt{(a-u)(b-u)(c-u)}} \\ [a > b > c > u] \quad \text{BY (231.13)}$$

$$2. \quad \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(b-c)F(\lambda, p) - 2(2a-b-c)E(\lambda, p)] \\ + \frac{2(a-b-c+u)}{(a-b)(b-c)(a-c)} \sqrt{\frac{a-u}{(u-b)(u-c)}} \\ [a > u > b > c] \quad \text{BY (236.15)}$$

$$3. \quad \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(2a-b-c)E(\mu, q) - 2(a-b)F(\mu, q)] \\ + \frac{2}{(a-c)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \\ [u > a > b > c] \quad \text{BY (236.14)}$$

$$4. \quad \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(2a-b-c)E(\nu, q) - 2(a-b)F(\nu, q)] \\ - \frac{2}{(a-b)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \\ [u \geq a > b > c] \quad \text{BY (238.13)}$$

$$5. \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(2b-a-c)E(\alpha, p) - (b-c)F(\alpha, p)] \\ + \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(c-u)}} \\ [a > b > c > u] \quad \text{BY (231.12)}$$

$$6. \quad \int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)^3}} = \frac{2}{(b-c)(a-b)\sqrt{(a-c)^3}} [(a-b)F(\delta, q) + (2b-a-c)E(\delta, q)] \\ + \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(u-c)}} \\ [a > b > u > c] \quad \text{BY (234.03)}$$

$$7. \quad \int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(b-c)F(\kappa, p) - (2b-a-c)E(\kappa, p)] \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{u-b}{(a-u)(u-c)}} \\ [a > u > b > c] \quad \text{BY (235.15)}$$

$$\begin{aligned}
 8. \quad \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)^3}} &= \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(a+c-2b)E(\nu, q) - (a-b)F(\nu, q)] \\
 &\quad + \frac{2}{(a-b)(a-c)} \sqrt{\frac{u-b}{(u-a)(u-c)}} \\
 &\quad [u > a > b > c] \qquad \text{BY (238.14)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} &= \frac{2}{(b-c)(a-b)^2\sqrt{a-c}} [(a+b-2c)E(\alpha, p) - 2(b-c)F(\alpha, p)] \\
 &\quad - \frac{2}{(a-b)(b-c)} \sqrt{\frac{c-u}{(a-u)(b-u)}} \\
 &\quad [a > b > c \geq u] \qquad \text{BY (231.11)}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} &= \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a+b-2c)E(\beta, p) - 2(b-c)F(\beta, p)] \\
 &\quad + \frac{2}{(a-b)(a-c)} \sqrt{\frac{c-u}{(a-u)(b-u)}} \\
 &\quad [a > b > c > u] \qquad \text{BY (232.15)}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} &= \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b)F(\gamma, q) - (a+b-2c)E(\gamma, q)] \\
 &\quad + \frac{2[a^2+b^2-ac-bc-u(a+b-2c)]}{(a-b)^2(b-c)(a-c)} \sqrt{\frac{u-c}{(a-u)(b-u)}} \\
 &\quad [a > b > u > c] \qquad \text{BY (233.11)}
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} &= \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b)F(\nu, q) - (a+b-2c)E(\nu, q)] \\
 &\quad + \frac{2u-a-b}{(a-b)^2\sqrt{(u-a)(u-b)(u-c)}} \\
 &\quad [u > a > b > c] \qquad \text{BY (238.15)}
 \end{aligned}$$

3.136

$$\begin{aligned}
 1. \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}} &= \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \\
 &\quad \times [(b-c)(a+b-2c)F(\alpha, p) - 2(c^2+a^2+b^2-ab-ac-bc)E(\alpha, p)] \\
 &\quad + \frac{2[c(a-c)+b(a-b)-u(2a-c-b)]}{(a-b)(a-c)(b-c)^2\sqrt{(a-u)(b-u)(c-u)}} \\
 &\quad [a > b > c > u] \qquad \text{BY (231.14)}
 \end{aligned}$$

$$\begin{aligned}
2. \quad \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)^3}} &= \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \\
&\times [(a-b)(2a-b-c)F(\nu, q) - 2(a^2+b^2+c^2-ab-ac-bc)E(\nu, q)] \\
&+ \frac{2[u(a+b-2c)-a(a-c)-b(b-c)]}{(a-b)^2(a-c)(b-c)\sqrt{(u-a)(u-b)(u-c)}} \\
&\quad [u > a > b > c] \quad \text{BY (238.16)}
\end{aligned}$$

3.137

$$\begin{aligned}
1.^6 \quad \int_{-\infty}^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} &= \frac{2}{(a-r)\sqrt{a-c}} \left[\Pi\left(\alpha, \frac{a-r}{a-c}, p\right) - F(\alpha, p) \right] \\
&\quad [a > b > c \geq u] \quad \text{BY (231.15)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} &= \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \\
&\times \Pi\left(\beta, \frac{r-b}{r-c}, p\right) + \frac{2}{(r-b)\sqrt{a-c}} F(\beta, p) \\
&\quad [a > b > c > u, \quad r \neq 0] \quad \text{BY (232.17)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_c^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} &= \frac{2}{(r-c)\sqrt{a-c}} \Pi\left(\gamma, \frac{b-c}{r-c}, q\right) \\
&\quad [a > b \geq u > c, \quad r \neq c] \quad \text{BY (233.02)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_u^b \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} &= \frac{2}{(r-a)(r-b)\sqrt{a-c}} \\
&\times \left[(b-a) \Pi\left(\delta, q^2 \frac{r-a}{r-b}, q\right) + (r-b) F(\delta, q) \right] \\
&\quad [a > b > u \geq c, \quad r \neq b] \quad \text{BY (234.18)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_b^u \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} &= \frac{2}{(c-r)(b-r)\sqrt{a-c}} \\
&\times \left[(c-b) \Pi\left(\kappa, p^2 \frac{c-r}{b-r}, p\right) + (b-r) F(\kappa, p) \right] \\
&\quad [a \geq u > b > c, \quad r \neq b] \quad \text{BY (235.17)}
\end{aligned}$$

$$\begin{aligned}
6.^8 \quad \int_u^a \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} &= \frac{2}{(a-r)\sqrt{a-c}} \Pi\left(\lambda, \frac{a-b}{a-r}, p\right) \\
&\quad [a > u \geq b > c, \quad r \neq a] \quad \text{BY (236.02)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} &= \frac{2}{(b-r)(a-r)\sqrt{a-c}} \\
&\times \left[(b-a) \Pi\left(\mu, \frac{b-r}{a-b}, q\right) + (a-p) F(\mu, q) \right] \\
&\quad [u > a > b > c, \quad r \neq a] \quad \text{BY (237.17)}
\end{aligned}$$

$$8. \quad \int_u^\infty \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}} \left[\Pi\left(\nu, \frac{r-c}{a-c}, q\right) - F(\nu, q) \right] \\ [u \geq a > b > c] \quad \text{BY (238.06)}$$

3.138

$$1. \quad \int_0^u \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2 F(\arcsin \sqrt{u}, k) \quad [0 < u < 1] \quad \text{PE (532), JA}$$

$$2. \quad \int_u^1 \frac{dx}{\sqrt{x(1-x)(k'^2+k^2x)}} = 2 F(\arccos \sqrt{u}, k) \quad [0 < u < 1] \quad \text{PE(533)}$$

$$3. \quad \int_u^1 \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} = 2 F\left(\arcsin \frac{\sqrt{1-u}}{k}, k\right) \quad [0 < u < 1] \quad \text{PE (534)}$$

$$4. \quad \int_0^u \frac{dx}{\sqrt{x(1+x)(1+k'^2x)}} = 2 F(\arctan \sqrt{u}, k) \quad [0 < u < 1] \quad \text{PE (535)}$$

$$5. \quad \int_0^u \frac{dx}{\sqrt{x[1+x^2+2(k'^2-k^2)x]}} = F(2 \arctan \sqrt{u}, k) \\ [0 < u < 1] \quad \text{JA}$$

$$6. \quad \int_u^1 \frac{dx}{\sqrt{x[k'^2(1+x^2)+2(1+k^2)x]}} = F\left(\frac{\pi}{2} - 2 \arctan \sqrt{u}, k\right) \\ [0 < u < 1] \quad \text{JA}$$

$$7. \quad \int_a^u \frac{dx}{\sqrt{(x-\alpha)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2 \arctan \sqrt{\frac{u-\alpha}{p}}, \sqrt{\frac{p+m-\alpha}{2p}}\right) \\ [\alpha < u],$$

$$8. \quad \int_u^a \frac{dx}{\sqrt{(\alpha-x)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2 \operatorname{arccot} \sqrt{\frac{\alpha-u}{p}}, \sqrt{\frac{p-m+\alpha}{2p}}\right) \\ [u < \alpha],$$

where $p = \sqrt{(m-\alpha)^2+n^2}$.

3.139 Notation $\alpha = \arccos \frac{1-\sqrt{3}-u}{1+\sqrt{3}-u}, \quad \beta = \arccos \frac{\sqrt{3}-1+u}{\sqrt{3}+1-u},$
 $\gamma = \arccos \frac{\sqrt{3}+1-u}{\sqrt{3}-1+u}, \quad \delta = \arccos \frac{u-1-\sqrt{3}}{u-1+\sqrt{3}}.$

$$1. \quad \int_{-\infty}^u \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\alpha, \sin 75^\circ) \quad \text{H 66 (285)}$$

$$2. \quad \int_u^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\beta, \sin 75^\circ) \quad \text{H 65 (284)}$$

3. $\int_1^u \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\gamma, \sin 15^\circ)$ H 65 (283)
4. $\int_u^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\delta, \sin 15^\circ)$ H 65 (282)
5. $\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}} \left\{ \Gamma\left(\frac{1}{3}\right) \right\}^3$ MO 9
6. $\int_0^1 \frac{x dx}{\sqrt{1-x^3}} = \frac{1}{\pi} \frac{\sqrt{3}}{\sqrt[3]{4}} \left\{ \Gamma\left(\frac{2}{3}\right) \right\}^3$ MO 9
7. $\int_u^1 \sqrt{1-x^3} dx = \frac{1}{5} \left\{ {}^4\sqrt{27} F(\beta, \sin 75^\circ) - 2u\sqrt{1-u^3} \right\}$ BY (244.01)
8. $\int_u^1 \frac{x dx}{\sqrt{1-x^3}} = \left(3^{-\frac{1}{4}} - 3^{\frac{1}{4}}\right) F(\beta, \sin 75^\circ) + 2\sqrt[4]{3} E(\beta, \sin 75^\circ) - \frac{2\sqrt{1-u^3}}{\sqrt{3}+1-u}$ BY (244.05)
9. $\int_u^1 \frac{x^m dx}{\sqrt{1-x^3}} = \frac{2u^{m-2}\sqrt{1-u^3}}{2m-1} + \frac{2(m-2)}{2m-1} \int_u^1 \frac{x^{m-3} dx}{\sqrt{1-x^3}}$ BY (244.07)
10. $\int_1^u \frac{x dx}{\sqrt{x^3-1}} = \left(3^{-\frac{1}{4}} + 3^{\frac{1}{4}}\right) F(\gamma, \sin 15^\circ) - 2\sqrt[4]{3} E(\gamma, \sin 15^\circ) + \frac{2\sqrt{u^3-1}}{\sqrt{3}-1+u}$ BY (240.05)
11. $\int_{-\infty}^u \frac{dx}{(1-x)\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{27}} [F(\alpha, \sin 75^\circ) - 2E(\alpha, \sin 75^\circ)] + \frac{2}{\sqrt{3}} \frac{\sqrt{1+u+u^2}}{(1+\sqrt{3}-u)\sqrt{1-u}}$
[$u \neq 1$] BY (246.06)
12. $\int_u^\infty \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}} [F(\delta, \sin 15^\circ) - 2E(\delta, \sin 15^\circ)] + \frac{2}{\sqrt{3}} \frac{\sqrt{1+u+u^2}}{(u-1+\sqrt{3})\sqrt{u-1}}$
[$u \neq 1$] BY (242.03)
13. $\int_{-\infty}^u \frac{(1-x) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}} [F(\alpha, \sin 75^\circ) - E(\alpha, \sin 75^\circ)]$ BY (246.07)
14. $\int_u^1 \frac{(1-x) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}} [F(\beta, \sin 75^\circ) - E(\beta, \sin 75^\circ)]$ BY (244.04)
15. $\int_1^u \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(\sqrt{3}-2)}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\gamma, \sin 15^\circ)$ BY (240.08)
16. $\int_u^\infty \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\delta, \sin 15^\circ)$ BY (242.07)
17. $\int_{-\infty}^u \frac{(1-x) dx}{(1-\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} \left[\frac{2\sqrt[4]{3}\sqrt{1-u^3}}{u^2-2u-2} - E(\alpha, \sin 75^\circ) \right]$ BY (246.08)
18. $\int_1^u \frac{(x-1) dx}{(1-\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\gamma, \sin 15^\circ) - E(\gamma, \sin 15^\circ)]$ BY (240.04)

$$19. \quad \int_u^\infty \frac{(x-1) dx}{(1-\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\delta, \sin 15^\circ) - E(\delta, \sin 15^\circ)] \quad \text{BY (242.05)}$$

$$20. \quad \int_{-\infty}^u \frac{(x^2+x+1) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\alpha, \sin 75^\circ) \quad \text{BY (246.01)}$$

$$21. \quad \int_u^1 \frac{(x^2+x+1) dx}{(x-1+\sqrt{3})^2 \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\beta, \sin 75^\circ) \quad \text{BY (244.02)}$$

$$22. \quad \int_1^u \frac{(x^2+x+1) dx}{(\sqrt{3}+x-1)^2 \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\gamma, \sin 15^\circ) \quad \text{BY (240.01)}$$

$$23. \quad \int_u^\infty \frac{(x^2+x+1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\delta, \sin 15^\circ) \quad \text{BY (242.01)}$$

$$24. \quad \int_1^u \frac{(x-1) dx}{(x^2+x+1) \sqrt{x^3-1}} = \frac{4}{\sqrt[4]{27}} E(\gamma, \sin 15^\circ) - \frac{2+\sqrt{3}}{\sqrt[4]{27}} F(\gamma, \sin 15^\circ) \\ - \frac{2-\sqrt{3}}{\sqrt{3}} \frac{2(u-1)(\sqrt{3}+1-u)}{(\sqrt{3}-1+u) \sqrt{u^3-1}} \quad \text{BY (240.09)}$$

$$25. \quad \int_{-\infty}^u \frac{(1+\sqrt{3}-x)^2 dx}{[(1+\sqrt{3}-x)^2 - 4\sqrt{3}p^2(1-x)] \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\alpha, p^2, \sin 75^\circ) \quad \text{BY (246.02)}$$

$$26. \quad \int_u^1 \frac{(1+\sqrt{3}-x)^2 dx}{[(1+\sqrt{3}-x)^2 - 4\sqrt{3}p^2(1-x)] \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\beta, p^2, \sin 75^\circ) \quad \text{BY (244.03)}$$

$$27. \quad \int_1^u \frac{(1-\sqrt{3}-x)^2 dx}{[(1-\sqrt{3}-x)^2 - 4\sqrt{3}p^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\gamma, p^2, \sin 15^\circ) \quad \text{BY (240.02)}$$

$$28. \quad \int_u^\infty \frac{(1-\sqrt{3}-x)^2 dx}{[(1-\sqrt{3}-x)^2 - 4\sqrt{3}p^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\delta, p^2, \sin 15^\circ) \quad \text{BY (242.02)}$$

3.141 Notation: In **3.141** and **3.142** we set:

$$\alpha = \arcsin \sqrt{\frac{a-c}{a-u}}, \quad \beta = \arcsin \sqrt{\frac{c-u}{b-u}}, \quad \gamma = \arcsin \sqrt{\frac{u-c}{b-c}}, \\ \delta = \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \quad \kappa = \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \quad \lambda = \arcsin \sqrt{\frac{a-u}{a-b}}, \\ \mu = \arcsin \sqrt{\frac{u-a}{u-b}}, \quad \nu = \arcsin \sqrt{\frac{a-c}{u-c}}, \quad p = \sqrt{\frac{a-b}{a-c}}, \quad q = \sqrt{\frac{b-c}{a-c}}.$$

1.
$$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)}} dx = 2\sqrt{a-c} [F(\beta, p) - E(\beta, p)] + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$

$[a > b > c > u]$ BY (232.06)
2.
$$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\gamma, q)$$

$[a > b \geq u > c]$ BY (233.01)
3.
$$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\delta, q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$

$[a > b > u \geq c]$ BY (234.06)
4.
$$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\kappa, p) - E(\kappa, p)] + 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$

$[a \geq u > b > c]$ BY (235.07)
5.
$$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\lambda, p) - E(\lambda, p)]$$

$[a > u \geq b > c]$ BY (236.04)
6.
$$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)}} dx = -2\sqrt{a-c} E(\mu, q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$

$[u > a > b > c]$ BY (237.03)
7.
$$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)}} dx = \frac{2(b-c)}{\sqrt{a-c}} F(\beta, p) - 2\sqrt{a-c} E(\beta, p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$

$[a > b > c > u]$ BY (232.07)
8.
$$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\gamma, q) - \frac{2(a-b)}{\sqrt{a-c}} F(\gamma, q)$$

$[a > b \geq u > c]$ BY (233.04)
9.
$$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\delta, q) - \frac{2(a-b)}{\sqrt{a-c}} F(\delta, q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$

$[a > b > u \geq c]$ BY (234.07)
10.
$$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\kappa, p) - \frac{2(b-c)}{\sqrt{a-c}} F(\kappa, p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$

$[a \geq u > b > c]$ BY (235.06)
11.
$$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\lambda, p) - \frac{2(b-c)}{\sqrt{a-c}} F(\lambda, p)$$

$[a > u \geq b > c]$ BY (236.03)
12.
$$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)}} dx = \frac{2(a-b)}{\sqrt{a-c}} F(\mu, q) - 2\sqrt{a-c} E(\mu, q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$

$[u > a > b > c]$ BY (237.04)

13.
$$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)}} dx = -2\sqrt{a-c} E(\beta, p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$

 $[a > b > c > u]$ BY (232.08)
14.
$$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c} [F(\gamma, q) - E(\gamma, q)]$$

 $[a > b \geq u > c]$ BY (233.03)
15.
$$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c} [F(\delta, q) - E(\delta, q)] + 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$

 $[a > b > u \geq c]$ BY (234.08)
16.
$$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\kappa, p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$

 $[a \geq u > b > c]$ BY (235.07)
17.
$$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\lambda, p)$$

 $[a > u \geq b > c]$ BY (236.01)
18.
$$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)}} dx = 2\sqrt{a-c} [F(\mu, q) - E(\mu, q)] + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$

 $[u > a > b > c]$ BY (237.05)
19.
$$\int_u^c \sqrt{\frac{(b-x)(c-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [(2a-b-c) E(\beta, p) - (b-c) F(\beta, p)]$$

 $+\frac{2}{3}(2b-2a+c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$
 $[a > b > c > u]$ BY (232.11)
20.
$$\int_c^u \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [(2a-b-c) E(\gamma, q) - 2(a-b) F(\gamma, q)]$$

 $-\frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$
 $[a > b \geq u > c]$ BY (233.06)
- 21.¹¹
$$\int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [2(b-a) F(\delta, q) + (2a-b-c) E(\delta, q)]$$

 $+\frac{2}{3}(b+c-a-u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$
 $[a > b > u \geq c]$ BY (234.11)
22.
$$\int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [(2a-b-c) E(\kappa, p) - (b-c) F(\kappa, p)]$$

 $+\frac{2}{3}(b+2c-2a-u)\sqrt{\frac{(a-u)(u-b)}{u-c}}$
 $[a \geq u > b > c]$ BY (235.10)

$$\begin{aligned}
23.^{11} \quad \int_u^a \sqrt{\frac{(x-b)(x-c)}{a-x}} dx &= \frac{2}{3} \sqrt{a-c} [(2a-b-c) E(\lambda, p) - (b-c) F(\lambda, p)] \\
&\quad + \frac{2}{3} \sqrt{(a-u)(u-b)(u-c)} \\
&\quad [a > u \geq b > c] \qquad \text{BY (236.07)}
\end{aligned}$$

$$\begin{aligned}
24. \quad \int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}} dx &= \frac{2}{3} \sqrt{a-c} [2(a-b) F(\mu, q) + (b+c-2a) E(\mu, q)] \\
&\quad + \frac{2}{3} (u+2a-2b-c) \sqrt{\frac{(u-a)(u-b)}{u-c}} \\
&\quad [u > a > b > c] \qquad \text{BY (237.08)}
\end{aligned}$$

$$\begin{aligned}
25. \quad \int_u^c \sqrt{\frac{(a-x)(c-x)}{b-x}} dx &= \frac{2}{3} \sqrt{a-c} [(2b-a-c) E(\beta, p) - (b-c) F(\beta, p)] \\
&\quad + \frac{2}{3} (a+c-b-u) \sqrt{\frac{(a-u)(c-u)}{b-u}} \\
&\quad [a > b > c > u] \qquad \text{BY (232.10)}
\end{aligned}$$

$$\begin{aligned}
26. \quad \int_c^u \sqrt{\frac{(a-x)(x-c)}{b-x}} dx &= \frac{2}{3} \sqrt{a-c} [(2b-a-c) E(\gamma, q) + (a-b) F(\gamma, q)] \\
&\quad - \frac{2}{3} \sqrt{(a-u)(b-u)(u-c)} \\
&\quad [a > b \geq u > c] \qquad \text{BY (233.05)}
\end{aligned}$$

$$\begin{aligned}
27. \quad \int_u^b \sqrt{\frac{(a-x)(x-c)}{b-x}} dx &= \frac{2}{3} \sqrt{a-c} [(a-b) F(\delta, q) + (2b-a-c) E(\delta, q)] \\
&\quad + \frac{2}{3} (2a+c-2b-u) \sqrt{\frac{(b-u)(u-c)}{a-u}} \\
&\quad [a > b > u \geq c] \qquad \text{BY (234.10)}
\end{aligned}$$

$$\begin{aligned}
28. \quad \int_b^u \sqrt{\frac{(a-x)(x-c)}{x-b}} dx &= \frac{2}{3} \sqrt{a-c} [(b-c) F(\kappa, p) + (a+c-2b) E(\kappa, p)] \\
&\quad + \frac{2}{3} (2b-a-2c+u) \sqrt{\frac{(a-u)(u-b)}{u-c}} \\
&\quad [a \geq u > b > c] \qquad \text{BY (235.11)}
\end{aligned}$$

$$\begin{aligned}
29. \quad \int_u^a \sqrt{\frac{(a-x)(x-c)}{x-b}} dx &= \frac{2}{3} \sqrt{a-c} [(a+c-2b) E(\lambda, p) + (b-c) F(\lambda, p)] \\
&\quad - \frac{2}{3} \sqrt{(a-u)(u-b)(u-c)} \\
&\quad [a > u \geq b > c] \qquad \text{BY (236.06)}
\end{aligned}$$

$$\begin{aligned}
30.^{11} \quad \int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}} dx &= \frac{2}{3} \sqrt{a-c} [(a+c-2b) E(\mu, q) - (a-b) F(\mu, q)] \\
&\quad + \frac{2}{3} (u+b-a-c) \sqrt{\frac{(u-a)(u-c)}{u-b}} \\
&\quad [u > a > b > c] \qquad \text{BY (237.06)}
\end{aligned}$$

$$\begin{aligned}
 31. \quad \int_u^c \sqrt{\frac{(a-x)(b-x)}{c-x}} dx &= \frac{2}{3} \sqrt{a-c} [2(b-c) F(\beta, p) + (2c-a-b) E(\beta, p)] \\
 &\quad + \frac{2}{3} (a+2b-2c-u) \sqrt{\frac{(a-u)(c-u)}{b-u}} \\
 &\quad [a > b > c > u] \qquad \text{BY (232.09)}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \int_c^u \sqrt{\frac{(a-x)(b-x)}{x-c}} dx &= \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\gamma, q) - (a-b) F(\gamma, q)] \\
 &\quad + \frac{2}{3} \sqrt{(a-u)(b-u)(u-c)} \\
 &\quad [a > b \geq u > c] \qquad \text{BY (233.07)}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \int_u^b \sqrt{\frac{(a-x)(b-x)}{x-c}} dx &= \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\delta, q) - (a-b) F(\delta, q)] \\
 &\quad + \frac{2}{3} (2c-2a-b+u) \sqrt{\frac{(b-u)(u-c)}{a-u}} \\
 &\quad [a > b > u \geq c] \qquad \text{BY (234.09)}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \int_b^u \sqrt{\frac{(a-x)(x-b)}{x-c}} dx &= \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\kappa, p) - 2(b-c) F(\kappa, p)] \\
 &\quad + \frac{2}{3} (u+c-a-b) \sqrt{\frac{(a-u)(u-b)}{u-c}} \\
 &\quad [a \geq u > b > c] \qquad \text{BY (235.09)}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \int_u^a \sqrt{\frac{(a-x)(x-b)}{x-c}} dx &= \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\lambda, p) - 2(b-c) F(\lambda, p)] \\
 &\quad - \frac{2}{3} \sqrt{(a-u)(u-b)(u-c)} \\
 &\quad [a > u \geq b > c] \qquad \text{BY (236.05)}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \int_a^u \sqrt{\frac{(x-a)(x-b)}{x-c}} dx &= \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\mu, q) - (a-b) F(\mu, q)] \\
 &\quad + \frac{2}{3} (u+2c-a-2b) \sqrt{\frac{(u-a)(u-c)}{u-b}} \\
 &\quad [u > a > b > c] \qquad \text{BY (237.07)}
 \end{aligned}$$

3.142

$$\begin{aligned}
 1. \quad \int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)(c-x)^3}} dx &= \frac{2}{\sqrt{a-c}} F(\alpha, p) - \frac{2\sqrt{a-c}}{b-c} E(\alpha, p) + \frac{2(a-c)}{b-c} \sqrt{\frac{b-u}{(a-u)(c-u)}} \\
 &\quad [a > b > c > u] \qquad \text{BY (231.05)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3}} dx &= 2 \frac{a-b}{(b-c)\sqrt{a-c}} F(\delta, q) - \frac{2\sqrt{a-c}}{b-c} E(\delta, q) \\
 &\quad + 2 \frac{a-c}{b-c} \sqrt{\frac{b-u}{(a-u)(u-c)}} \\
 &\quad [a > b > u > c] \qquad \text{BY (234.13)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx &= \frac{2\sqrt{a-c}}{b-c} E(\kappa, p) - \frac{2}{\sqrt{a-c}} F(\kappa, p) \\
 &\quad [a \geq u > b > c] \qquad \text{BY (235.12)}
 \end{aligned}$$

4.
$$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\lambda, p) - \frac{2}{\sqrt{a-c}} F(\lambda, p) - \frac{2}{b-c} \sqrt{\frac{(a-u)(u-b)}{u-c}}$$

[$a > u \geq b > c$] BY (236.12)
5.
$$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\mu, q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\mu, q) - 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$

[$u > a > b > c$] BY (237.10)
6.
$$\int_u^\infty \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\nu, q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\nu, q)$$

[$u \geq a > b > c$] BY (238.09)
7.
$$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\alpha, p) - 2\frac{a-b}{b-c} \sqrt{\frac{c-u}{(a-u)(b-u)}}$$

[$a > b > c \geq u$] BY (231.03)
8.
$$\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\beta, p)$$

[$a > b > c > u$] BY (232.01)
9.
$$\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\gamma, q) - E(\gamma, q)] + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{b-u}}$$

[$a > b > u > c$] BY (233.15)
10.
$$\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{c-b} E(\lambda, p) + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{u-b}}$$

[$a > u > b > c$] BY (236.11)
11.
$$\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\mu, q) - E(\mu, q)]$$

[$u > a > b > c$] BY (237.09)
12.
$$\int_u^\infty \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\nu, q) - E(\nu, q)] + 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$

[$u \geq a > b > c$] BY (238.10)
13.
$$\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\alpha, p)$$

[$a > b > c \geq u$] BY (231.01)
14.
$$\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\beta, p) - \frac{2(a-b)}{a-c} \sqrt{\frac{c-u}{(a-u)(b-u)}}$$

[$a > b > c > u$] BY (232.05)
15.
$$\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\gamma, q) - E(\gamma, q)] + \frac{2}{a-c} \sqrt{\frac{(b-u)(u-c)}{a-u}}$$

[$a > b \geq u > c$] BY (233.13)
16.
$$\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\delta, q) - E(\delta, q)]$$

[$a > b > u \geq c$] BY (234.15)

$$17. \quad \int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)}} dx = -\frac{2}{\sqrt{a-c}} E(\kappa, p) + 2\sqrt{\frac{u-b}{(a-u)(u-c)}} \\ [a > u > b > c] \quad \text{BY (235.08)}$$

$$18. \quad \int_u^\infty \sqrt{\frac{x-b}{(x-a)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\nu, q) - E(\nu, q)] + 2\sqrt{\frac{u-b}{(u-a)(u-c)}} \\ [u > a > b > c] \quad \text{BY (238.07)}$$

$$19. \quad \int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)(c-x)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\alpha, p) - E(\alpha, p)] + 2\sqrt{\frac{b-u}{(a-u)(c-u)}} \\ [a > b > c > u] \quad \text{BY (231.04)}$$

$$20. \quad \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3}} dx = -\frac{2}{\sqrt{a-c}} E(\delta, q) + 2\sqrt{\frac{b-u}{(a-u)(u-c)}} \\ [a > b > u > c] \quad \text{BY (234.14)}$$

$$21. \quad \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\kappa, p) - E(\kappa, p)] \\ [a \geq u > b > c] \quad \text{BY (235.03)}$$

$$22. \quad \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\lambda, p) - E(\lambda, p)] + \frac{2}{a-c} \sqrt{\frac{(a-u)(u-b)}{u-c}} \\ [a > u \geq b > c] \quad \text{BY (236.14)}$$

$$23. \quad \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\mu, q) - 2\frac{b-c}{a-c} \sqrt{\frac{u-a}{(u-b)(u-c)}} \\ [u > a > b > c] \quad \text{BY (237.11)}$$

$$24. \quad \int_u^\infty \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\nu, q) \\ [u \geq a > b > c] \quad \text{BY (238.01)}$$

$$25. \quad \int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\alpha, p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\alpha, p) \\ [a > b > c \geq u] \quad \text{BY (231.07)}$$

$$26. \quad \int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\beta, p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\beta, p) - 2\sqrt{\frac{c-u}{(a-u)(b-u)}} \\ [a > b > c > u] \quad \text{BY (232.03)}$$

$$27. \quad \int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\gamma, q) - \frac{2}{\sqrt{a-c}} F(\gamma, q) - \frac{2}{a-b} \sqrt{\frac{(b-u)(u-c)}{a-u}} \\ [a > b \geq u > c] \quad \text{BY (233.14)}$$

$$28. \quad \int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\delta, q) - \frac{2}{\sqrt{a-c}} F(\delta, q) \\ [a > b > u \geq c] \quad \text{BY (234.20)}$$

$$\begin{aligned}
 29. \quad \int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)}} dx &= \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\kappa, p) - \frac{2\sqrt{a-c}}{a-b} E(\kappa, p) \\
 &\quad + 2\frac{a-c}{a-b} \sqrt{\frac{u-b}{(a-u)(u-c)}} \\
 &\quad [a > u > b > c] \quad \text{BY (235.13)}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \int_u^\infty \sqrt{\frac{x-c}{(x-a)^3(x-b)}} dx &= \frac{2}{\sqrt{a-c}} F(\nu, q) - \frac{2\sqrt{a-c}}{a-b} E(\nu, q) + \frac{2(a-c)}{a-b} \sqrt{\frac{u-b}{(u-a)(u-c)}} \\
 &\quad [u > a > b > c] \quad \text{BY (238.08)}
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx &= \frac{2\sqrt{a-c}}{a-b} [F(\alpha, p) - E(\alpha, p)] + 2\sqrt{\frac{c-u}{(a-u)(b-u)}} \\
 &\quad [a > b > c \geq u] \quad \text{BY (231.06)}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx &= \frac{2\sqrt{a-c}}{a-b} [F(\beta, p) - E(\beta, p)] \\
 &\quad [a > b > c > u] \quad \text{BY (232.04)}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3}} dx &= -\frac{2\sqrt{a-c}}{a-b} E(\gamma, q) + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{b-u}} \\
 &\quad [a > b > u > c] \quad \text{BY (233.16)}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}} dx &= \frac{2\sqrt{a-c}}{a-b} [F(\lambda, p) - E(\lambda, p)] + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{u-b}} \\
 &\quad [a > u > b > c] \quad \text{BY (236.13)}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx &= \frac{2\sqrt{a-c}}{a-b} E(\mu, q) \\
 &\quad [u > a > b > c] \quad \text{BY (237.01)}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \int_u^\infty \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx &= \frac{2\sqrt{a-c}}{a-b} E(\nu, q) - 2\frac{b-c}{a-b} \sqrt{\frac{u-a}{(u-b)(u-c)}} \\
 &\quad [u \geq a > b > c] \quad \text{BY (238.11)}
 \end{aligned}$$

3.143

$$1.^6 \quad \int_u^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F\left(\arctan \frac{(1+\sqrt{2})(1-u)}{(1+u)}, 2^{\sqrt[4]{2}}(\sqrt{2}-1)\right) \quad \text{H 66 (286)}$$

$$2. \quad \int_u^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F\left(\arccos \frac{u^2-1}{u^2+1}, \frac{\sqrt{2}}{2}\right) \quad \text{H 66 (287)}$$

3.144 Notation: $\alpha = \arcsin \frac{1}{\sqrt{u^2-u+1}}.$

$$1. \quad \int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)}} = F\left(\alpha, \frac{\sqrt{3}}{2}\right) \quad [u \geq 1] \quad \text{BY (261.50)}$$

2.
$$\int_u^\infty \frac{dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = \frac{2(2u-1)}{\sqrt{u(u-1)(u^2-u+1)}} - 4E\left(\alpha, \frac{\sqrt{3}}{2}\right)$$

[$u > 1$] BY (261.54)
3.
$$\int_u^\infty \frac{(2x-1)^2 dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = 4 \left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2u-1}{2\sqrt{u(u-1)(u^2-u+1)}} \right]$$

[$u > 1$] BY (261.56)
4.
$$\int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = \frac{4}{3} \left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) \right]$$

[$u \geq 1$] BY (261.52)
5.
$$\int_u^\infty \frac{(2x-1)^2 dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = 4E\left(\alpha, \frac{\sqrt{3}}{2}\right)$$

[$u > 1$] BY (261.51)
6.
$$\int_u^\infty \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}} dx = \frac{4}{3}E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}F\left(\alpha, \frac{\sqrt{3}}{2}\right)$$

[$u > 1$] BY (261.53)
7.
$$\int_u^\infty \frac{dx}{(2x-1)^2} \sqrt{\frac{x(x-1)}{x^2-x+1}} = \frac{1}{3} \left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) \right] + \frac{1}{2(2u-1)} \sqrt{\frac{u(u-1)}{u^2-u+1}}$$

[$u > 1$] BY (261.57)
8.
$$\int_u^\infty \frac{dx}{(2x-1)^2} \sqrt{\frac{x^2-x+1}{x(x-1)}} = E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2u-1)} \sqrt{\frac{u(u-1)}{u^2-u+1}}$$

[$u > 1$] BY (261.58)
9.
$$\int_u^\infty \frac{dx}{(2x-1)^2 \sqrt{x(x-1)(x^2-x+1)}} = \frac{4}{3}E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2}{2u-1} \sqrt{\frac{u(u-1)}{u^2-u+1}}$$

[$u > 1$] BY (261.55)
10.
$$\int_u^\infty \frac{dx}{\sqrt{x^5(x-1)^5(x^2-x+1)}} = \frac{40}{3}E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{4}{3}F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2(2u-1)(9u^2-9u-1)}{3\sqrt{u^3(u-1)^3(u^2-u+1)}}$$

[$u > 1$] BY (261.54)
11.
$$\int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^5}} = \frac{44}{27}F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{56}{27}E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2(2u-1)\sqrt{u(u-1)}}{9\sqrt{(u^2-u+1)^3}}$$

[$u > 1$] BY (261.52)

$$12. \int_u^\infty \frac{dx}{(2x-1)^4 \sqrt{x(x-1)(x^2-x+1)}} = \frac{16}{27} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{27} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{8(5u^2-5u+2)}{9(2u-1)^3} \sqrt{\frac{u(u-1)}{u^2-u+1}}$$

BY (261.55)

$[u > 1]$

3.145

$$1. \int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \arctan \sqrt{\frac{q(u-\alpha)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2 + (\alpha-\beta)^2}{pq}}\right)$$

$[\beta < \alpha < u]$

$$2. \int_\beta^u \frac{dx}{\sqrt{(\alpha-x)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arccot} \sqrt{\frac{q(\alpha-u)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{-(p-q)^2 + (\alpha-\beta)^2}{pq}}\right)$$

$[\beta < u < \alpha]$

$$3. \int_u^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \arctan \sqrt{\frac{q(\beta-u)}{p(\alpha-u)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2 + (\alpha-\beta)^2}{pq}}\right)$$

$[u < \beta < \alpha]$

where $(m-\alpha)^2 + n^2 = p^2$, and $(m-\beta)^2 + n^2 = q^2$.*

4. Set

$$(m_1 - m)^2 + (n_1 + n)^2 = p^2, \quad (m_1 - m)^2 + (n_1 - n)^2 = p_1^2,$$

$$\cot \alpha = \sqrt{\frac{(p+p_1)^2 - 4n^2}{4n^2 - (p-p_1)^2}};$$

then

$$\int_{m-n \tan \alpha}^u \frac{dx}{\sqrt{[(x-m)^2+n^2][(xm_1)^2+n_1^2]}} = \frac{2}{p+p_1} F\left(\alpha + \arctan \frac{u-m}{n}, \frac{2\sqrt{pp_1}}{p+p_1}\right)$$

$[m-n \tan \alpha < u < m+n \cot \alpha]$

3.146

$$1. \int_0^1 \frac{1}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} + \frac{1}{4} \sqrt{2} K\left(\frac{\sqrt{2}}{2}\right) \quad \text{BI (13)(6)}$$

$$2. \int_0^1 \frac{x^2}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} \quad \text{BI (13)(7)}$$

*Formulas **3.145** are not valid for $\alpha + \beta = 2m$. In this case, we make the substitution $x - m = z$, which leads to one of the formulas in **3.152**.

$$3. \quad \int_0^1 \frac{x^4}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = -\frac{\pi}{8} + \frac{1}{4}\sqrt{2} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \quad \text{BI (13)(8)}$$

3.147 Notation: In **3.147–3.151** we set: $\alpha = \arcsin \sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\begin{aligned} \beta &= \arcsin \sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, & \gamma &= \arcsin \sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}}, \\ \delta &= \arcsin \sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, & \kappa &= \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \\ \lambda &= \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, & \mu &= \arcsin \sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}}, \\ \nu &= \arcsin \sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, & q &= \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, & r &= \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}. \end{aligned}$$

$$1. \quad \int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\alpha, q) \quad [a > b > c > d > u] \quad \text{BY (251.00)}$$

$$2. \quad \int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\beta, r) \quad [a > b > c \geq u > d] \quad \text{BY (254.00)}$$

$$3. \quad \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\gamma, r) \quad [a > b > c > u \geq d] \quad \text{BY (253.00)}$$

$$4. \quad \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\delta, q) \quad [a > b \geq u > c > d] \quad \text{BY (254.00)}$$

$$5. \quad \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\kappa, q) \quad [a > b > u \geq c > d] \quad \text{BY (255.00)}$$

$$6. \quad \int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\lambda, r) \quad [a \geq u > b > c > d] \quad \text{BY (256.00)}$$

$$7.^{11} \quad \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\mu, r) \quad [a > u \geq b > c > d] \quad \text{BY (257.00)}$$

$$8. \quad \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\nu, q)$$

[$u > a > b > c > d$] BY (258.00)

3.148

$$1.^8 \quad \int_u^d \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (d-c) \Pi \left(\alpha, \frac{a-d}{a-c}, q \right) + c F(\alpha, q) \right\}$$

[$a > b > c > d > u$] BY (251.03)

$$2. \quad \int_d^u \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (d-a) \Pi \left(\beta, \frac{d-c}{a-c}, r \right) + a F(\beta, r) \right\}$$

[$a > b > c \geq u > d$] BY (252.11)

$$3. \quad \int_u^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left(\gamma, \frac{c-d}{b-d}, r \right) + b F(\gamma, r) \right\}$$

[$a > b > c > u \geq d$] BY (253.11)

$$4. \quad \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-d) \Pi \left(\delta, \frac{b-c}{b-d}, q \right) + d F(\delta, q) \right\}$$

[$a > b \geq u > c > d$] BY (254.10)

$$5. \quad \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi \left(\kappa, \frac{b-c}{a-c}, q \right) + a F(\kappa, q) \right\}$$

[$a > b > u \geq c > d$] BY (255.17)

$$6.^8 \quad \int_b^u \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi \left(\lambda, \frac{a-b}{a-c}, r \right) + c F(\lambda, r) \right\}$$

[$a \geq u > b > c > d$] BY (256.11)

$$7. \quad \int_u^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-d) \Pi \left(\mu, \frac{b-a}{b-d}, r \right) + d F(\mu, r) \right\}$$

[$a > u \geq b > c > d$] BY (257.11)

$$8. \quad \int_a^u \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-b) \Pi \left(\nu, \frac{a-d}{b-d}, q \right) + b F(\nu, q) \right\}$$

[$u > a > b > c > d$] BY (258.11)

3.149

$$1. \quad \int_u^d \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{cd \sqrt{(a-c)(b-d)}} \left\{ (c-d) \Pi \left(\alpha, \frac{c(a-d)}{d(a-c)}, q \right) + d F(\alpha, q) \right\}$$

[$a > b > c > d > u$] BY (251.04)

$$\begin{aligned}
2. \quad & \int_d^u \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} \\
&= \frac{2}{ad\sqrt{(a-c)(b-d)}} \left\{ (a-d) \Pi \left(\beta, \frac{a(d-c)}{d(a-c)}, r \right) + d F(\beta, r) \right\} \\
&\quad [a > b > c \geq u > d] \quad \text{BY (252.12)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_u^c \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} \\
&= \frac{2}{bc\sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi \left(\gamma, \frac{b(c-d)}{c(b-d)}, r \right) + c F(\gamma, r) \right\} \\
&\quad [a > b > c > u \geq d] \quad \text{BY (253.12)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_c^u \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} \\
&= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{ (d-c) \Pi \left(\delta, \frac{d(b-c)}{c(b-d)}, q \right) + c F(\delta, q) \right\} \\
&\quad [a > b \geq u > c > d] \quad \text{BY (254.11)}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_u^b \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} \\
&= \frac{2}{ab\sqrt{(a-c)(b-d)}} \times \left\{ (a-b) \Pi \left(\kappa, \frac{a(b-c)}{b(a-c)}, q \right) + b F(\kappa, q) \right\} \\
&\quad [a > b > u \geq c > d] \quad \text{BY (255.18)}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int_b^u \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} \\
&= \frac{2}{bc\sqrt{(a-c)(b-d)}} \times \left\{ (c-b) \Pi \left(\lambda, \frac{c(a-b)}{b(a-c)}, r \right) + b F(\lambda, r) \right\} \\
&\quad [a \geq u > b > c > d] \quad \text{BY (256.12)}
\end{aligned}$$

$$\begin{aligned}
7. \quad & \int_u^a \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} \\
&= \frac{2}{ad\sqrt{(a-c)(b-d)}} \times \left\{ (d-a) \Pi \left(\mu, \frac{d(b-a)}{a(b-d)}, r \right) + a F(\mu, r) \right\} \\
&\quad [a > u \geq b > c > d] \quad \text{BY (257.12)}
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int_a^u \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}} \\
&= \frac{2}{ab\sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi \left(\nu, \frac{b(a-d)}{a(b-d)}, q \right) + a F(\nu, q) \right\} \\
&\quad [u > a > b > c > d] \quad \text{BY (258.12)}
\end{aligned}$$

3.151

$$\begin{aligned}
 1. \quad \int_u^d \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(d-x)}} &= \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \\
 &\times \left[(d-c) \Pi \left(\alpha, \frac{(a-d)(p-c)}{(a-c)(p-d)}, q \right) + (p-d) F(\alpha, q) \right] \\
 &\quad [a > b > c > d > u, \quad p \neq d] \quad \text{BY (251.39)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_d^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} &= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \\
 &\times \left[(d-a) \Pi \left(\beta, \frac{(d-c)(p-a)}{(a-c)(p-d)}, r \right) + (p-d) F(\beta, r) \right] \\
 &\quad [a > b > c \geq u > d, \quad p \neq d] \quad \text{BY (252.39)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_u^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} &= \frac{2}{(p-b)(p-c)\sqrt{(a-c)(b-d)}} \\
 &\times \left[(c-b) \Pi \left(\gamma, \frac{(c-d)(p-b)}{(b-d)(p-c)}, r \right) + (p-c) F(\gamma, r) \right] \\
 &\quad [a > b > c > u \geq d, \quad p \neq c] \quad \text{BY (253.39)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_c^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} &= \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \\
 &\times \left[(c-d) \Pi \left(\delta, \frac{(b-c)(p-d)}{(b-d)(p-c)}, q \right) + (p-c) F(\delta, q) \right] \\
 &\quad [a > b \geq u > c > d, \quad p \neq c] \quad \text{BY (254.39)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \int_u^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} &= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \\
 &\times \left[(b-a) \Pi \left(\kappa, \frac{(b-c)(p-a)}{(a-c)(p-b)}, q \right) + (p-b) F(\kappa, q) \right] \\
 &\quad [a > b > u \geq c > d, \quad p \neq b] \quad \text{BY (255.38)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_b^u \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)(x-d)}} &= \frac{2}{(b-p)(p-c)\sqrt{(a-c)(b-d)}} \\
 &\times \left[(b-c) \Pi \left(\lambda, \frac{(a-b)(p-c)}{(a-c)(p-b)}, r \right) + (p-b) F(\lambda, r) \right] \\
 &\quad [a \geq u > b > c > d, \quad p \neq b] \quad \text{BY (256.39)}
 \end{aligned}$$

$$\begin{aligned}
7. \quad \int_u^a \frac{dx}{(p-x)\sqrt{(a-x)(x-b)(x-c)(x-d)}} &= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \\
&\times \left[(a-d) \Pi \left(\mu, \frac{(b-a)(p-d)}{(b-d)(p-a)}, r \right) + (p-a) F(\mu, r) \right] \\
&\quad [a > u \geq b > c > d, \quad p \neq a] \quad \text{BY (257.39)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_a^u \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} &= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \\
&\times \left[(a-b) \Pi \left(\nu, \frac{(a-d)(p-b)}{(b-d)(p-a)}, q \right) + (p-a) F(\nu, q) \right] \\
&\quad [u > a > b > c > d, \quad p \neq a] \quad \text{BY (258.39)}
\end{aligned}$$

3.152 Notation: In **3.152–3.163** we set: $\alpha = \arctan \frac{u}{b}$, $\beta = \operatorname{arccot} \frac{u}{a}$

$$\begin{aligned}
\gamma &= \arcsin \frac{u}{b} \sqrt{\frac{a^2+b^2}{a^2+u^2}}, & \delta &= \arccos \frac{u}{b}, & \varepsilon &= \arccos \frac{b}{u}, & \xi &= \arcsin \sqrt{\frac{a^2+b^2}{a^2+u^2}}, \\
\eta &= \arcsin \frac{u}{b}, & \zeta &= \arcsin \frac{a}{b} \sqrt{\frac{b^2-u^2}{a^2-u^2}}, & \kappa &= \arcsin \frac{a}{u} \sqrt{\frac{u^2-b^2}{a^2-b^2}}, \\
\lambda &= \arcsin \sqrt{\frac{a^2-u^2}{a^2-b^2}}, & \mu &= \arcsin \sqrt{\frac{u^2-a^2}{u^2-b^2}}, & \nu &= \arcsin \frac{a}{u}, & q &= \frac{\sqrt{a^2-b^2}}{a}, \\
r &= \frac{b}{\sqrt{a^2+b^2}}, & s &= \frac{a}{\sqrt{a^2+b^2}}, & t &= \frac{b}{a}.
\end{aligned}$$

$$\begin{aligned}
1. \quad \int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} &= \frac{1}{a} F(\alpha, q) & [a > b > 0] & \quad \text{H 62(258), BY (221.00)} \\
2. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} &= \frac{1}{a} F(\beta, q) & [a > b > 0] & \quad \text{H 63 (259), BY (222.00)} \\
3. \quad \int_0^u \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} &= \frac{1}{\sqrt{a^2+b^2}} F(\gamma, r) & [b \geq u > 0] & \quad \text{H 63 (260)} \\
4. \quad \int_u^b \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} &= \frac{1}{\sqrt{a^2+b^2}} F(\delta, r) & [b > u \geq 0] & \quad \text{H 63 (261), BY (213.00)} \\
5. \quad \int_b^u \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} &= \frac{1}{\sqrt{a^2+b^2}} F(\varepsilon, s) & [u > b > 0] & \quad \text{H 63 (262), BY (211.00)} \\
6. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} &= \frac{1}{\sqrt{a^2+b^2}} F(\xi, s) & [u > b > 0] & \quad \text{H 63 (263), BY (212.00)}
\end{aligned}$$

7. $\int_0^u \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F(\eta, t) \quad [a > b \geq u > 0] \quad \text{H 63 (264), BY (219.00)}$
8. $\int_u^b \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F(\zeta, t) \quad [a > b > u \geq 0] \quad \text{H 63 (265), BY (220.00)}$
9. $\int_b^u \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\kappa, q) \quad [a \geq u > b > 0] \quad \text{H 63 (266), BY (217.00)}$
10. $\int_u^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\lambda, q) \quad [a > u \geq b > 0] \quad \text{H 63 (257), BY (218.00)}$
11. $\int_a^u \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\mu, t) \quad [u > a > b > 0] \quad \text{H 63 (268), BY (216.00)}$
12. $\int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\nu, t) \quad [u \geq a > b > 0] \quad \text{H 64(269), BY (215.00)}$

3.153

1. $\int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} - a E(\alpha, q) \quad [u > 0, \quad a > b] \quad \text{BY (221.09)}$
2. $\int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\gamma, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\gamma, r) - u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \quad [b \geq u > 0] \quad \text{BY (214.05)}$
3. $\int_u^b \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\delta, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\delta, r) \quad [b > u \geq 0] \quad \text{BY (213.06)}$
4. $\int_b^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)}} = \frac{b^2}{\sqrt{a^2 + b^2}} F(\varepsilon, s) - \sqrt{a^2 + b^2} E(\varepsilon, s) + \frac{1}{u} \sqrt{(u^2 + a^2)(u^2 - b^2)} \quad [u > b > 0] \quad \text{BY (211.09)}$
5. $\int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \{F(\eta, t) - E(\eta, t)\} \quad [a > b \geq u > 0] \quad \text{BY (219.05)}$
6. $\int_u^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \{F(\zeta, t) - E(\zeta, t)\} + u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u \geq 0] \quad \text{BY (220.06)}$
7. $\int_b^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = a E(\kappa, q) - \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0] \quad \text{BY (217.05)}$
8. $\int_u^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = a E(\lambda, q) \quad [a > u \geq b > 0] \quad \text{BY (218.06)}$

$$9.^6 \quad \int_a^u \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = a \{F(\mu, t) - E(\mu, t)\} + u \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}$$

[$u > a > b > 0$] BY (216.06)

$$10. \quad \int_0^1 \frac{x^2 dx}{\sqrt{(1+x^2)(1+k^2x^2)}} = \frac{1}{k^2} \left\{ \sqrt{\frac{1+k^2}{2}} - E\left(\frac{\pi}{4}, \sqrt{1-k^2}\right) \right\}$$

BI (14)(9)

3.154

$$1. \quad \int_0^u \frac{x^4 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{a}{3} \{2(a^2 + b^2) E(\alpha, q) - b^2 F(\alpha, q)\} + \frac{u}{3} (u^2 - 2a^2 - b^2) \sqrt{\frac{a^2 + u^2}{b^2 + u^2}}$$

[$a > b, \quad u > 0$] BY (221.09)

$$2. \quad \int_0^u \frac{x^4 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \frac{1}{3\sqrt{a^2 + b^2}} \{ (2a^2 - b^2) a^2 F(\gamma, r) - 2(a^4 - b^4) E(\gamma, r) \}$$

$$- \frac{u}{3} (2b^2 - a^2 + u^2) \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$

[$a \geq u > 0$] BY (214.05)

$$3. \quad \int_u^b \frac{x^4 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \frac{1}{3\sqrt{a^2 + b^2}} \{ (2a^2 - b^2) a^2 F(\delta, r) - 2(a^4 - b^4) E(\delta, r) \}$$

$$+ \frac{u}{3} \sqrt{(a^2 + u^2)(b^2 - u^2)}$$

[$b > u \geq 0$] BY (213.06)

$$4. \quad \int_b^u \frac{x^4 dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)}} = \frac{1}{3\sqrt{a^2 + b^2}} \{ (2b^2 - a^2) b^2 F(\varepsilon, s) + 2(a^4 - b^4) E(\varepsilon, s) \}$$

$$+ \frac{2b^2 - 2a^2 + u^2}{3u} \sqrt{(u^2 + a^2)(u^2 - b^2)}$$

[$u > b > 0$] BY (211.09)

$$5. \quad \int_0^u \frac{x^4 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{a}{3} \{ (2a^2 + b^2) F(\eta, t) - 2(a^2 + b^2) E(\eta, t) \} + \frac{u}{3} \sqrt{(a^2 - u^2)(b^2 - u^2)}$$

[$a > b \geq u > 0$] BY (219.05)

$$6. \quad \int_u^b \frac{x^4 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{a}{3} \{ (2a^2 + b^2) F(\zeta, t) - 2(a^2 + b^2) E(\zeta, t) \}$$

$$+ \frac{u}{3} (u^2 + a^2 + 2b^2) \sqrt{\frac{b^2 - u^2}{a^2 - u^2}}$$

[$a > b > u \geq 0$] BY (220.06)

$$7. \quad \int_b^u \frac{x^4 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{a}{3} \{ 2(a^2 + b^2) E(\kappa, q) - b^2 F(\kappa, q) \}$$

$$- \frac{u^2 + 2a^2 + 2b^2}{3u} \sqrt{(a^2 - u^2)(u^2 - b^2)}$$

[$a \geq u > b > 0$] BY (217.05)

$$8. \quad \int_u^a \frac{x^4 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{a}{3} \{ 2(a^2 + b^2) E(\lambda, q) - b^2 F(\lambda, q) \} + \frac{u}{3} \sqrt{(a^2 - u^2)(u^2 - b^2)}$$

[$a > u \geq b > 0$] BY (218.06)

$$9. \quad \int_a^u \frac{x^4 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{a}{3} \{ (2a^2 + b^2) F(\mu, t) - 2(a^2 + b^2) E(\mu, t) \} \\ + \frac{u}{3} (u^2 + 2a^2 + b^2) \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.06)}$$

3.155

$$1. \quad \int_u^a \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} \{ (a^2 + b^2) E(\lambda, q) - 2b^2 F(\lambda, q) \} - \frac{u}{3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a > u \geq b > 0] \quad \text{BY (218.11)}$$

$$2. \quad \int_a^u \sqrt{(x^2 - a^2)(x^2 - b^2)} dx = \frac{a}{3} \{ (a^2 + b^2) E(\mu, t) - (a^2 - b^2) F(\mu, t) \} \\ + \frac{u}{3} (u^2 - a^2 - 2b^2) \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.10)}$$

$$3. \quad \int_0^u \sqrt{(x^2 + a^2)(x^2 + b^2)} dx = \frac{a}{3} \{ 2b^2 F(\alpha, q) - (a^2 + b^2) E(\alpha, q) \} \\ + \frac{u}{3} (u^2 + a^2 + 2b^2) \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} \\ [a > b, \quad u > 0] \quad \text{BY (221.08)}$$

$$4. \quad \int_0^u \sqrt{(a^2 + x^2)(b^2 - x^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \{ a^2 F(\gamma, r) - (a^2 - b^2) E(\gamma, r) \} \\ + \frac{u}{3} (u^2 + 2a^2 - b^2) \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \\ [a \geq u > 0] \quad \text{BY (214.12)}$$

$$5.^9 \quad \int_u^b \sqrt{(a^2 + x^2)(b^2 - x^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \{ a^2 F(\delta, r) + (b^2 - a^2) E(\delta, r) \} \\ + \frac{u}{3} \sqrt{(a^2 + u^2)(b^2 - u^2)} \\ [b > u \geq 0] \quad \text{BY (213.13)}$$

$$6. \quad \int_b^u \sqrt{(a^2 + x^2)(x^2 - b^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \{ (b^2 - a^2) E(\varepsilon, s) - b^2 F(\varepsilon, s) \} \\ + \frac{u^2 + a^2 - b^2}{3u} \sqrt{(a^2 + u^2)(u^2 - b^2)} \\ [u > b > 0] \quad \text{BY (211.08)}$$

$$7. \quad \int_0^u \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \{ (a^2 + b^2) E(\eta, t) - (a^2 - b^2) F(\eta, t) \} \\ + \frac{u}{3} \sqrt{(a^2 - u^2)(b^2 - u^2)} \\ [a > b \geq u > 0] \quad \text{BY (219.11)}$$

$$8. \quad \int_u^b \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \{ (a^2 + b^2) E(\zeta, t) - (a^2 - b^2) F(\zeta, t) \} \\ + \frac{u}{3} (u^2 - 2a^2 - b^2) \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u \geq 0] \quad \text{BY (220.05)}$$

$$9. \quad \int_b^u \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} \{ (a^2 + b^2) E(\kappa, q) - 2b^2 F(\kappa, q) \} \\ + \frac{u^2 - a^2 - b^2}{3u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a \geq u > b > 0] \quad \text{BY (217.09)}$$

3.156

$$1.^6 \quad \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{ub^2} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} - \frac{1}{ab^2} E(\beta, q) \\ [a \geq b, \quad u > 0] \quad \text{BY (222.04)}$$

$$2. \quad \int_u^b \frac{dx}{x^2 \sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \{ a^2 F(\delta, r) - (a^2 + b^2) E(\delta, r) \} \\ + \frac{1}{a^2 b^2 u} \sqrt{(a^2 + u^2)(b^2 - u^2)} \\ [b > u > 0] \quad \text{BY (213.09)}$$

$$3. \quad \int_b^u \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \{ (a^2 + b^2) E(\varepsilon, s) - b^2 F(\varepsilon, s) \} \\ [u > b > 0] \quad \text{BY (211.11)}$$

$$4. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \{ (a^2 + b^2) E(\xi, s) - b^2 F(\xi, s) \} \\ - \frac{1}{b^2 u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \\ [u \geq b > 0] \quad \text{BY (212.06)}$$

$$5. \quad \int_u^b \frac{dx}{x^2 \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ab^2} \{ F(\zeta, t) - E(\zeta, t) \} + \frac{1}{b^2 u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u > 0] \quad \text{BY (220.09)}$$

$$6. \quad \int_b^u \frac{dx}{x^2 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{ab^2} E(\kappa, q) \quad [a \geq u > b > 0] \quad \text{BY (217.01)}$$

$$7. \quad \int_u^a \frac{dx}{x^2 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{ab^2} E(\lambda, q) - \frac{1}{a^2 b^2 u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a > u \geq b > 0] \quad \text{BY (218.12)}$$

$$8. \quad \int_a^u \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \{ F(\mu, t) - E(\mu, t) \} + \frac{1}{a^2 u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.09)}$$

$$9. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \{ F(\nu, t) - E(\nu, t) \} \\ [u \geq a > b > 0] \quad \text{BY (215.07)}$$

3.157

1.
$$\int_0^u \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a(p+b^2)} \left\{ \frac{b^2}{p} \Pi\left(\alpha, \frac{p+b^2}{p}, q\right) + F(\alpha, q) \right\}$$

[$p \neq 0$] BY (221.13)
2.
$$\int_u^\infty \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = -\frac{1}{a(a^2+p)} \left\{ \Pi\left(\beta, \frac{a^2+p}{a^2}, q\right) - F(\beta, q) \right\}$$

BY (222.11)
3.
$$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{p(p+a^2)\sqrt{a^2+b^2}} \left\{ a^2 \Pi\left(\gamma, \frac{b^2(p+a^2)}{p(a^2+b^2)}, r\right) + p F(\gamma, r) \right\}$$

[$b \geq u > 0, \quad p \neq 0$] BY (214.13)a
4.
$$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{(p-b^2)\sqrt{a^2+b^2}} \Pi\left(\delta, \frac{b^2}{b^2-p}, r\right)$$

[$b > u \geq 0, \quad p \neq b^2$] BY (213.02)
5.
$$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{p(p-b^2)\sqrt{a^2+b^2}} \left\{ b^2 \Pi\left(\varepsilon, \frac{p}{p-b^2}, s\right) + (p-b^2) F(\varepsilon, s) \right\}$$

[$u > b > 0, \quad p \neq b^2$] BY (211.14)
6.
$$\int_u^\infty \frac{dx}{(x^2-p)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{(a^2+p)\sqrt{a^2+b^2}} \left\{ \Pi\left(\xi, \frac{a^2+p}{a^2+b^2}, s\right) - F(\xi, s) \right\}$$

[$u \geq b > 0$] BY (212.12)
7.
$$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ap} \Pi\left(\eta, \frac{b^2}{p}, t\right)$$

[$a > b \geq u > 0; \quad p \neq b$] BY (219.02)
8.
$$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a(p-a^2)(p-b^2)} \times \left\{ (b^2-a^2) \Pi\left(\zeta, \frac{b^2(p-a^2)}{a^2(p-b^2)}, t\right) + (p-b^2) F(\zeta, t) \right\}$$

[$a > b > u \geq 0; \quad p \neq b^2$] BY (220.13)
9.
$$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ap(p-b^2)} \left\{ b^2 \Pi\left(\kappa, \frac{p(a^2-b^2)}{a^2(p-b^2)}, q\right) + (p-b^2) F(\kappa, q) \right\}$$

[$a \geq u > b > 0; \quad p \neq b^2$] BY (217.12)
10.
$$\int_u^a \frac{dx}{(x^2-p)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a(a^2-p)} \Pi\left(\lambda, \frac{a^2-b^2}{a^2-p}, q\right)$$

[$a > u \geq b > 0; \quad p \neq a^2$] BY (218.02)
11.
$$\int_a^u \frac{dx}{(p-x^2)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a(p-a^2)(p-b^2)} \left\{ (a^2-b^2) \Pi\left(\mu, \frac{p-b^2}{p-a^2}, t\right) + (p-a^2) F(\mu, t) \right\}$$

[$u > a > b > 0; \quad p \neq a^2, \quad p \neq b^2$] BY (216.12)

$$12. \quad \int_u^\infty \frac{dx}{(x^2 - p) \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ap} \left\{ \Pi \left(\nu, \frac{p}{a^2}, t \right) - F(\nu, t) \right\}$$

[$u \geq a > b > 0; \quad p \neq 0$] BY (215.12)

3.158

$$1. \quad \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)} \{a^2 E(\alpha, q) - b^2 F(\alpha, q)\}$$

[$a > b; \quad u > 0$] BY (221.05)

$$2. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)} \{a^2 E(\beta, q) - b^2 F(\beta, q)\} - \frac{u}{b^2 \sqrt{(a^2 + u^2)(b^2 + u^2)}}$$

[$a > b, \quad u \geq 0$] BY (222.05)

$$3. \quad \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{F(\alpha, q) - E(\alpha, q)\} + \frac{u}{a^2 \sqrt{(u^2 + a^2)(u^2 + b^2)}}$$

[$a > b; \quad u > 0$] BY (221.06)

$$4. \quad \int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{F(\beta, q) - E(\beta, q)\}$$

[$a > b, \quad u \geq 0$] BY (222.03)

$$5. \quad \int_0^u \frac{dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} E(\gamma, r) \quad [b \geq u > 0]$$

BY (214.01)a

$$6. \quad \int_u^b \frac{dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} E(\delta, r) - \frac{u}{a^2(a^2 + b^2)} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$

[$b > u \geq 0$] BY (213.08)

$$7. \quad \int_b^u \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} \{F(\varepsilon, s) - E(\varepsilon, s)\} + \frac{1}{(a^2 + b^2)u} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}}$$

[$u > b > 0$] BY (211.05)

$$8. \quad \int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} \{F(\xi, s) - E(\xi, s)\}$$

[$u \geq b > 0$] BY (212.03)

$$9. \quad \int_0^u \frac{dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)^3}} = \frac{1}{b^2 \sqrt{a^2 + b^2}} \{F(\gamma, r) - E(\gamma, r)\} + \frac{u}{b^2 \sqrt{(a^2 + u^2)(b^2 - u^2)}}$$

[$b > u > 0$] BY (214.10)

$$10. \quad \int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)^3}} = \frac{u}{b^2 \sqrt{(a^2 + u^2)(u^2 - b^2)}} - \frac{1}{b^2 \sqrt{a^2 + b^2}} E(\xi, s)$$

[$u \geq b > 0$] BY (212.04)

$$11. \quad \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a^2 (a^2 - b^2)} \left\{ a E(\eta, t) - u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\}$$

[$a > b \geq u > 0$] BY (219.07)

$$12. \quad \int_u^b \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a (a^2 - b^2)} E(\zeta, t) \quad [a > b > u \geq 0] \quad \text{BY (220.10)}$$

$$13. \quad \int_b^u \frac{dx}{\sqrt{(a^2 - x^2)^3 (x^2 - b^2)}} = \frac{1}{a (a^2 - b^2)} \left\{ F(\kappa, q) - E(\kappa, q) + \frac{a}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} \right\}$$

[$a > u > b > 0$] BY (217.10)

$$14. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)}} = \frac{1}{a (b^2 - a^2)} \left\{ E(\nu, t) - \frac{a}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} \right\}$$

[$u > a > b > 0$] BY (215.04)

$$15. \quad \int_0^u \frac{dx}{\sqrt{(a^2 - x^2) (b^2 - x^2)^3}} = \frac{1}{ab^2} F(\eta, t) - \frac{1}{b^2 (a^2 - b^2)} \left\{ a E(\eta, t) - u \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} \right\}$$

[$a > b > u > 0$] BY (219.06)

$$16. \quad \int_u^a \frac{dx}{\sqrt{(a^2 - x^2) (x^2 - b^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)} \left\{ b^2 F(\lambda, q) - a^2 E(\lambda, q) + au \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \right\}$$

[$a > u > b > 0$] BY (218.04)

$$17. \quad \int_a^u \frac{dx}{\sqrt{(x^2 - a^2) (x^2 - b^2)^3}} = \frac{a}{b^2 (a^2 - b^2)} E(\mu, t) - \frac{1}{ab^2} F(\mu, t)$$

[$u > a > b > 0$] BY (216.11)

$$18. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2) (x^2 - b^2)^3}} = \frac{1}{b^2 (a^2 - b^2)} \left\{ a E(\nu, t) - \frac{b^2}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\} - \frac{1}{ab^2} F(\nu, t)$$

[$u \geq a > b > 0$] BY (215.06)

3.159

$$1. \quad \int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^3}} = \frac{a}{a^2 - b^2} \{F(\alpha, q) - E(\alpha, q)\}$$

[$a > b, \quad u > 0$] BY (221.12)

$$2. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^3}} = \frac{a}{a^2 - b^2} \{F(\beta, q) - E(\beta, q)\} + \frac{u}{\sqrt{(a^2 + u^2) (b^2 + u^2)}}$$

[$a > b, \quad u \geq 0$] BY (222.10)

$$3. \quad \int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{a^2 E(\alpha, q) - b^2 F(\alpha, q)\} - \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}} \\ [a > b, \quad u > 0] \quad \text{BY (221.11)}$$

$$4. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{a^2 E(\beta, q) - b^2 F(\beta, q)\} \\ [a > b, \quad u \geq 0] \quad \text{BY (222.07)}$$

$$5. \quad \int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\gamma, r) - E(\gamma, r)\} \\ [b \geq u > 0] \quad \text{BY (214.04)}$$

$$6. \quad \int_u^b \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\delta, r) - E(\delta, r)\} + \frac{u}{a^2 + b^2} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \\ [b > u \geq 0] \quad \text{BY (213.07)}$$

$$7. \quad \int_b^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\varepsilon, s) - \frac{a^2}{u(a^2 + b^2)} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}} \\ [u > b > 0] \quad \text{BY (211.13)}$$

$$8. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\xi, s) \quad [u \geq b > 0] \quad \text{BY (212.01)}$$

$$9. \quad \int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (b^2 - x^2)}} = \frac{u}{\sqrt{(a^2 + u^2)(b^2 - u^2)}} - \frac{1}{\sqrt{a^2 + b^2}} E(\gamma, r) \\ [b > u > 0] \quad \text{BY (214.07)}$$

$$10. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\xi, s) - E(\xi, s)\} + \frac{u}{\sqrt{(a^2 + u^2)(u^2 - b^2)}} \\ [u > b > 0] \quad \text{BY (212.10)}$$

$$11. \quad \int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a^2 - b^2} \left\{ a E(\eta, t) - u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} - \frac{1}{a} F(\eta, t) \\ [a > b \geq u > 0] \quad \text{BY (219.04)}$$

$$12. \quad \int_u^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{a}{a^2 - b^2} E(\zeta, t) - \frac{1}{a} F(\zeta, t) \\ [a > b > u \geq 0] \quad \text{BY (220.08)}$$

$$13. \quad \int_b^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (x^2 - b^2)}} = \frac{1}{a(a^2 - b^2)} \left\{ b^2 F(\kappa, q) - a^2 E(\kappa, q) + \frac{a^3}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} \right\} \\ [a > u > b > 0] \quad \text{BY (217.06)}$$

$$14. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)}} = \frac{a}{a^2 - b^2} \left\{ \frac{a}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} - E(\nu, t) \right\} + \frac{1}{a} F(\nu, t)$$

[$u > a > b > 0$] BY (215.09)

$$15. \quad \int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^3}} = \frac{1}{a^2 - b^2} \left\{ u \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} - a E(\eta, t) \right\}$$

[$a > b > u > 0$] BY (219.12)

$$16. \quad \int_u^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)^3}} = \frac{1}{a^2 - b^2} \left\{ a F(\lambda, q) - a E(\lambda, q) + u \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \right\}$$

[$a > u > b > 0$] BY (218.07)

$$17. \quad \int_a^u \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^3}} = \frac{a}{a^2 - b^2} E(\mu, t)$$

[$u > a > b > 0$] BY (216.01)

$$18. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^3}} = \frac{1}{a^2 - b^2} \left\{ a E(\nu, t) - \frac{b^2}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\}$$

[$u \geq a > b > 0$] BY (215.11)

3.161

$$1. \quad \int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{3a^3b^4} \{ 2(a^2 + b^2) E(\beta, q) - b^2 F(\beta, q) \} + \frac{a^2b^2 - u^2(2a^2 + b^2)}{3a^2b^4u^3}$$

[$a > b, \quad u > 0$] BY (222.04)

$$2. \quad \int_u^b \frac{dx}{x^4 \sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{3a^4b^4\sqrt{a^2 + b^2}} \{ a^2(2a^2 - b^2) F(\delta, r) - 2(a^4 - b^4) E(\delta, r) \}$$

$$+ \frac{a^2b^2 + 2u^2(a^2 - b^2)}{3a^4b^4u^3} \sqrt{(b^2 - u^2)(a^2 + u^2)}$$

[$b > u > 0$] BY (213.09)

$$3. \quad \int_b^u \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{2b^2 - a^2}{3a^4b^2\sqrt{a^2 + b^2}} F(\varepsilon, s) + \frac{2(a^2 - b^2)\sqrt{a^2 + b^2}}{3a^4b^4} E(\varepsilon, s)$$

$$+ \frac{1}{3a^2b^2u^3} \sqrt{(u^2 + a^2)(u^2 - b^2)}$$

[$u > b > 0$] BY (211.11)

$$4. \quad \int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{3a^4b^4\sqrt{a^2 + b^2}} \{ 2(a^4 - b^4) E(\xi, s) + b^2(2b^2 - a^2) F(\xi, s) \}$$

$$- \frac{a^2b^2 + u^2(2a^2 - b^2)}{3a^2b^4u^3} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}}$$

[$u \geq b > 0$] BY (212.06)

$$\begin{aligned}
 5. \quad \int_u^b \frac{dx}{x^4 \sqrt{(a^2 - x^2)(b^2 - x^2)}} &= \frac{1}{3a^3b^4} \left\{ \{(2a^2 + b^2) F(\zeta, t) - 2(a^2 + b^2) E(\zeta, t)\} \right. \\
 &\quad \left. + \frac{[(2a^2 + b^2)u^2 + a^2b^2]a}{u^3} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \\
 &\quad [a > b > u > 0] \qquad \text{BY (220.09)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_b^u \frac{dx}{x^4 \sqrt{(a^2 - x^2)(x^2 - b^2)}} &= \frac{1}{3a^3b^4} \{2(a^2 + b^2) E(\kappa, q) - b^2 F(\kappa, q)\} \\
 &\quad + \frac{1}{3a^2b^2u^3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\
 &\quad [a \geq u > b > 0] \qquad \text{BY (217.14)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \int_u^a \frac{dx}{x^4 \sqrt{(a^2 - x^2)(x^2 - b^2)}} &= \frac{1}{3a^3b^4} \left\{ 2(a^2 + b^2) E(\lambda, q) - b^2 F(\lambda, q) \right. \\
 &\quad \left. - \frac{2(a^2 + b^2)u^2 + a^2b^2}{au^3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \right\} \\
 &\quad [a > u \geq b > 0] \qquad \text{BY (218.12)}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int_a^u \frac{dx}{x^4 \sqrt{(x^2 - a^2)(x^2 - b^2)}} &= \frac{1}{3a^3b^4} \left\{ \{(2a^2 + b^2) F(\mu, t) - 2(a^2 + b^2) E(\mu, t)\} \right. \\
 &\quad \left. + \frac{[(a^2 + 2b^2)u^2 + a^2b^2]b^2}{au^3} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\} \quad [u > a > b > 0] \\
 &\quad \text{BY (216.09)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 - a^2)(x^2 - b^2)}} &= \frac{1}{3a^3b^4} \left\{ (2a^2 + b^2) F(\nu, t) - 2(a^2 + b^2) E(\nu, t) \right. \\
 &\quad \left. + \frac{ab^2}{u^3} \sqrt{(u^2 - a^2)(u^2 - b^2)} \right\} \\
 &\quad [u \geq a > b > 0] \qquad \text{BY (215.07)}
 \end{aligned}$$

3.162

$$\begin{aligned}
 1. \quad \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^5(x^2 + b^2)}} &= \frac{1}{3a^3(a^2 - b^2)^2} \{ (3a^2 - b^2) F(\alpha, q) - 2(2a^2 - b^2) E(\alpha, q) \} \\
 &\quad + \frac{u[a^2(4a^2 - 3b^2) + u^2(3a^2 - 2b^2)]}{3a^4(a^2 - b^2)\sqrt{(u^2 + a^2)^3(u^2 + b^2)}} \\
 &\quad [a > b, \quad u > 0] \qquad \text{BY (221.06)}
 \end{aligned}$$

$$2. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^5 (x^2 + b^2)}} = \frac{1}{3a^3 (a^2 - b^2)^2} \{ (3a^2 - b^2) F(\beta, q) - 2(2a^2 - b^2) E(\beta, q) \} \\ + \frac{u}{3a^2 (a^2 - b^2)} \sqrt{\frac{u^2 + b^2}{(a^2 + u^2)^3}} \\ [a > b, \quad u \geq 0] \quad \text{BY (222.03)}$$

$$3. \quad \int_0^u \frac{dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^5}} = \frac{3b^2 - a^2}{3ab^2 (a^2 - b^2)^2} F(\alpha, q) + \frac{a(2a^2 - 4b^2)}{3b^4 (a^2 - b^2)^2} E(\alpha, q) \\ + \frac{u}{3b^2 (a^2 - b^2)} \sqrt{\frac{u^2 + a^2}{(u^2 + b^2)^3}} \\ [a > b, \quad u > 0] \quad \text{BY (221.05)}$$

$$4. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^5}} = \frac{1}{3ab^4 (a^2 - b^2)^2} \{ 2a^2 (a^2 - 2b^2) E(\beta, q) + b^2 (3b^2 - a^2) F(\beta, q) \} \\ - \frac{u [b^2 (3a^2 - 4b^2) + u^2 (2a^2 - 3b^2)]}{3b^4 (a^2 - b^2) \sqrt{(u^2 + a^2) (u^2 + b^2)^3}} \\ [a > b, \quad u \geq 0] \quad \text{BY (222.05)}$$

$$5. \quad \int_0^u \frac{dx}{\sqrt{(a^2 + x^2)^5 (b^2 - x^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \{ 2(b^2 + 2a^2) E(\gamma, r) - a^2 F(\gamma, r) \} \\ + \frac{u}{3a^2 (a^2 + b^2)} \sqrt{\frac{b^2 - u^2}{(a^2 + u^2)^3}} \\ [b \geq u > 0] \quad \text{BY (214.15)}$$

$$6. \quad \int_u^b \frac{dx}{\sqrt{(a^2 + x^2)^5 (b^2 - x^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \{ (4a^2 + 2b^2) E(\delta, r) - a^2 F(\delta, r) \} \\ - \frac{u [a^2 (5a^2 + 3b^2) + u^2 (4a^2 + 2b^2)]}{3a^4 (a^2 + b^2)^2} \sqrt{\frac{b^2 - u^2}{(a^2 + u^2)^3}} \\ [b > u > 0] \quad \text{BY (213.08)}$$

$$7. \quad \int_b^u \frac{dx}{\sqrt{(a^2 + x^3)^5 (x^2 - b^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \{ (3a^2 + 2b^2) F(\varepsilon, s) - (4a^2 + 2b^2) E(\varepsilon, s) \} \\ + \frac{(3a^2 + b^2) u^2 + 2(2a^2 + b^2) a^2}{3a^2 (a^2 + b^2)^2 u} \sqrt{\frac{u^2 - b^2}{(u^2 + a^2)^3}} \\ [u > b > 0] \quad \text{BY (211.05)}$$

$$8. \quad \int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^5 (x^2 - b^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \{ (3a^2 + 2b^2) F(\xi, s) - (4a^2 + 2b^2) E(\xi, s) \} \\ + \frac{u}{3a^2 (a^2 + b^2)} \sqrt{\frac{u^2 - b^2}{(a^2 + u^2)^3}} \\ [u > b > 0] \quad \text{BY (212.03)}$$

$$\begin{aligned}
 9. \quad \int_0^u \frac{dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)^5}} &= \frac{1}{3b^4 \sqrt{(a^2 + b^2)^3}} \{ (2a^2 + 3b^2) F(\gamma, r) - (2a^2 + 4b^2) E(\gamma, r) \} \\
 &\quad + \frac{u [(3a^3 + 4b^2) b^2 - (2a^2 + 3b^2) u^2]}{3b^4 (a^2 + b^2) \sqrt{(a^2 + u^2)(b^2 - u^2)^3}} \\
 &\quad [b > u > 0]
 \end{aligned}
 \tag{BY (214.10)}$$

$$\begin{aligned}
 10. \quad \int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)^5}} &= \frac{1}{3b^4 \sqrt{(a^2 + b^2)^3}} \{ (2a^2 + 4b^2) E(\xi, s) - b^2 F(\xi, s) \} \\
 &\quad + \frac{u [(3a^2 + 4b^2) b^2 - (2a^2 + 3b^2) u^2]}{3b^4 (a^2 + b^2) \sqrt{(a^2 + u^2)(u^2 - b^2)^3}} \\
 &\quad [u > b > 0]
 \end{aligned}
 \tag{BY (212.04)}$$

$$\begin{aligned}
 11. \quad \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^5}} &= \frac{2a^2 - 3b^2}{3ab^4 (a^2 - b^2)} F(\eta, t) + \frac{2a (2b^2 - a^2)}{3b^4 (a^2 - b^2)^2} E(\eta, t) \\
 &\quad + \frac{u [(3a^2 - 5b^2) b^2 - 2(a^2 - 2b^2) u^2]}{3b^4 (a^2 - b^2)^2 (b^2 - u^2)} \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} \\
 &\quad [a > b > a > 0]
 \end{aligned}
 \tag{BY (219.06)}$$

$$\begin{aligned}
 12. \quad \int_u^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)^5}} &= \frac{3b^2 - a^2}{3ab^4 (a^2 - b^2)^2} F(\lambda, q) + \frac{2a (a^2 - 2b^2)}{3b^4 (a^2 - b^2)^2} E(\lambda, q) \\
 &\quad + \frac{u [2(2b^2 - a^2) u^2 + (3a^2 - 5b^2) b^2]}{3b^4 (a^2 - b^2)^2 (u^2 - b^2)} \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \\
 &\quad [a > u > b > 0]
 \end{aligned}
 \tag{BY (218.04)}$$

$$\begin{aligned}
 13. \quad \int_a^u \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^5}} &= \frac{2a^2 - 3b^2}{3ab^4 (a^2 - b^2)} F(\mu, t) + \frac{2a (2b^2 - a^2)}{3b^4 (a^2 - b^2)^2} E(\mu, t) \\
 &\quad + \frac{u}{3b^2 (a^2 - b^2) (u^2 - b^2)} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\
 &\quad [u > a > b > 0]
 \end{aligned}
 \tag{BY (216.11)}$$

$$\begin{aligned}
 14. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^5}} &= \frac{(4b^2 - 2a^2) a}{3b^4 (a^2 - b^2)^2} E(\nu, t) + \frac{2a^2 - 3b^2}{3ab^4 (a^2 - b^2)} F(\nu, t) \\
 &\quad - \frac{(3b^2 - a^2) u^2 - (4b^2 - 2a^2) b^2}{3b^2 u (a^2 - b^2)^2 (u^2 - b^2)} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\
 &\quad [u \geq a > b > 0]
 \end{aligned}
 \tag{BY (215.06)}$$

$$\begin{aligned}
 15. \quad \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^5 (b^2 - x^2)}} &= \frac{1}{3a^3 (a^2 - b^2)^2} \{ (4a^2 - 2b^2) E(\eta, t) - (a^2 - b^2) F(\eta, t) \\
 &\quad - \frac{u [(5a^2 - 3b^2) a^2 - (4a^2 - 2b^2) u^2]}{a (a^2 - u^2)} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \} \\
 &\quad [a > b \geq u > 0]
 \end{aligned}
 \tag{BY (219.07)}$$

$$16. \quad \int_u^b \frac{dx}{\sqrt{(a^2 - x^2)^5 (b^2 - x^2)}} = \frac{2(2a^2 - b^2)}{3a^3(a^2 - b^2)^2} E(\zeta, r) - \frac{1}{3a^3(a^2 - b^2)} F(\zeta, t) \\ + \frac{u}{3a^2(a^2 - b^2)(a^2 - u^2)} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u \geq 0] \quad \text{BY (220.10)}$$

$$17. \quad \int_b^u \frac{dx}{\sqrt{(a^2 - x^2)^5 (x^2 - b^2)}} = \frac{1}{3a^3(a^2 - b^2)^2} \{ (3a^2 - b^2) F(\kappa, q) - (4a^2 - 2b^2) E(\kappa, q) \} \\ + \frac{2(2a^2 - b^2)a^2 + (b^2 - 3a^2)u^2}{3a^2u(a^2 - b^2)^2(a^2 - u^2)} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}}, \\ [a > u > b > 0] \quad \text{BY (217.10)}$$

$$18. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)^5 (x^2 - b^2)}} = \frac{1}{3a^3(a^2 - b^2)^2} \{ (4a^2 - 2b^2) E(\nu, t) - (a^2 - b^2) F(\nu, t) \} \\ + \frac{(4a^2 - 2b^2)a^2 + (b^2 - 3a^2)u^2}{3a^2u(a^2 - b^2)^2(u^2 - a^2)} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} \\ [u > a > b > 0] \quad \text{BY (215.04)}$$

3.163

$$1. \quad \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)^2} \{ (a^2 + b^2) E(\alpha, q) - 2b^2 F(\alpha, q) \} \\ - \frac{u}{a^2(a^2 - b^2) \sqrt{(a^2 + u^2)(b^2 + u^2)}} \\ [a > b, \quad u > 0] \quad \text{BY (221.07)}$$

$$2. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)^2} \{ (a^2 + b^2) E(\beta, q) - 2b^2 F(\beta, q) \} \\ - \frac{u}{b^2(a^2 - b^2) \sqrt{(a^2 + u^2)(b^2 + u^2)}} \\ [a > b, \quad u \geq 0] \quad \text{BY (222.12)}$$

$$3. \quad \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3 (b^3 - x^2)^3}} = \frac{1}{a^2b^2\sqrt{(a^2 + b^2)^3}} \{ a^2 F(\gamma, r) - (a^2 - b^2) E(\gamma, r) \} \\ + \frac{u}{b^2(a^2 + b^2) \sqrt{(a^2 + u^2)(b^2 - u^2)}} \\ [b > u > 0] \quad \text{BY (214.15)}$$

$$4. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 - b^2)^3}} = \frac{b^2 - a^2}{a^2b^2\sqrt{(a^2 + b^2)^3}} E(\xi, s) - \frac{1}{a^2\sqrt{(a^2 + b^2)^3}} F(\xi, s) \\ + \frac{u}{b^2(a^2 + b^2) \sqrt{(u^2 + a^2)(u^2 - b^2)}} \\ [u > b > 0] \quad \text{BY (212.05)}$$

$$5. \quad \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)} F(\eta, t) - \frac{a^2 + b^2}{ab^2 (a^2 - b^2)^2} E(\eta, t) \\ + \frac{[a^4 + b^4 - (a^2 + b^2) u^2] u}{a^2 b^2 (a^2 - b^2)^2 \sqrt{(a^2 - u^2) (b^2 - u^2)}} \\ [a > b > u > 0] \quad \text{BY (279.08)}$$

$$6. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)} F(\nu, t) - \frac{a^2 + b^2}{ab^2 (a^2 - b^2)^2} E(\nu, t) \\ + \frac{1}{u (a^2 - b^2) \sqrt{(u^2 - a^2) (u^2 - b^2)}} \\ [u > a > b > 0] \quad \text{BY (215.10)}$$

3.164 Notation: $\alpha = \arccos \frac{u^2 - \rho\bar{\rho}}{u^2 + \rho\bar{\rho}}, \quad r = \frac{1}{2} \sqrt{-\frac{(\rho - \bar{\rho})^2}{\rho\bar{\rho}}}.$

$$1. \quad \int_u^\infty \frac{dx}{\sqrt{(x^2 + \rho^2) (x^2 + \bar{\rho}^2)}} = \frac{1}{\sqrt{\rho\bar{\rho}}} F(\alpha, r) \quad \text{BY (225.00)}$$

$$2. \quad \int_u^\infty \frac{x^2 dx}{(x^2 - \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2) (x^2 + \bar{\rho}^2)}} = \frac{2u \sqrt{(u^2 + \rho^2) (u^2 + \bar{\rho}^2)}}{(\rho + \bar{\rho})^2 (u^4 - \rho^2 \bar{\rho}^2)} - \frac{1}{(\rho + \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} E(\alpha, r) \\ \text{BY (225.03)}$$

$$3. \quad \int_u^\infty \frac{x^2 dx}{(x^2 + \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2) (x^2 + \bar{\rho}^2)}} = -\frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (225.07)}$$

$$4. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho^2 - \bar{\rho}^2)^2} E(\alpha, r) + \frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} F(\alpha, r) \\ - \frac{2u (u^2 - \rho\bar{\rho})}{(\rho + \bar{\rho})^2 (u^2 + \rho\bar{\rho}) \sqrt{(u^2 + \rho^2) (u^2 + \bar{\rho}^2)}} \\ \text{BY (225.05)}$$

$$5. \quad \int_u^\infty \frac{(x^2 - \rho\bar{\rho})^2 dx}{\sqrt{(x^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho - \bar{\rho})^2} [F(\alpha, r) - E(\alpha, r)] \\ + \frac{2u (u^2 - \rho\bar{\rho})}{(u^2 + \rho\bar{\rho}) \sqrt{(u^2 + \rho^2) (u^2 + \bar{\rho}^2)}} \\ \text{BY (225.06)}$$

$$6. \quad \int_u^\infty \frac{\sqrt{(x^2 + \rho^2) (x^2 + \bar{\rho}^2)}}{(x^2 + \rho\bar{\rho})^2} dx = \frac{1}{\sqrt{\rho\bar{\rho}}} E(\alpha, r) \quad \text{BY (225.01)}$$

$$7. \quad \int_u^\infty \frac{(x^2 - \varrho\bar{\varrho})^2 dx}{(x^2 + \varrho\bar{\varrho})^2 \sqrt{(x^2 + \varrho^2) (x^2 + \bar{\varrho}^2)}} = -\frac{4\sqrt{\varrho\bar{\varrho}}}{(\varrho - \bar{\varrho})^2} E(\alpha, r) + \frac{(\varrho + \bar{\varrho})^2}{(\varrho - \bar{\varrho})^2 \sqrt{\varrho\bar{\varrho}}} F(\alpha, r) \quad \text{BY (225.08)}$$

$$8. \quad \int_u^\infty \frac{(x^2 + \varrho\bar{\varrho})^2 dx}{[(x^2 + \varrho\bar{\varrho})^2 - 4p^2 \varrho\bar{\varrho} x^2] \sqrt{(x^2 + \varrho^2) (x^2 + \bar{\varrho}^2)}} = \frac{1}{\sqrt{\varrho\bar{\varrho}}} \Pi(\alpha, p^2, r) \quad \text{BY (225.02)}$$

3.165 Notation: $\alpha = \arccos \frac{u^2 - a^2}{u^2 + a^2}, \quad r = \frac{\sqrt{a^2 - b^2}}{a\sqrt{2}}.$

$$1. \quad \int_u^a \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{\sqrt{2}}{a\sqrt{2} + \sqrt{a^2 + b^2}} \times F \left[\arctan \left(\frac{a\sqrt{2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2}} \frac{a - u}{a + u} \right), \frac{2\sqrt{a\sqrt{2}(a^2 - b^2)}}{a\sqrt{2} + \sqrt{a^2 - b^2}} \right] \\ [a > b, \quad a > u \geq 0] \quad \text{BY (264.00)}$$

$$2. \quad \int_u^\infty \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a} F(\alpha, r) \quad [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0] \\ \text{BY (263.00, 266.00)}$$

$$3. \quad \int_u^\infty \frac{dx}{x^2\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a^3} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sqrt{u^4 + 2b^2u^2 + a^4}}{a^2u(u^2 + a^2)} \\ [a > b > 0, \quad u > 0] \quad \text{BY (263.06)}$$

$$4. \quad \int_u^\infty \frac{x^2 dx}{(x^2 + a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{4a(a^2 - b^2)} [F(\alpha, r) - E(\alpha, r)] \\ [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0] \\ \text{BY (263.03, 266.05)}$$

$$5. \quad \int_u^\infty \frac{x^2 dx}{(x^2 - a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{u\sqrt{u^4 + 2b^2u^2 + a^4}}{2(a^2 + b^2)(u^4 - a^4)} - \frac{1}{4a(a^2 + b^2)} E(\alpha, r) \\ [a^2 > b^2 > -\infty, \quad u^2 > a^2 > 0] \\ \text{BY (263.05, 266.02)}$$

$$6. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{2(a^4 - b^4)} E(\alpha, r) - \frac{1}{4a(a^2 - b^2)} F(\alpha, r) \\ - \frac{u(u^2 - a^2)}{2(a^2 + b^2)(u^2 + a^2)\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [a^2 > b^2 > -\infty, a^2 > 0, \quad u \geq 0] \quad \text{BY (263.08, 266.03)}$$

$$7. \quad \int_u^\infty \frac{(x^2 - a^2)^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 - b^2} [F(\alpha, r) - E(\alpha, r)] + \frac{u^2 - a^2}{u^2 + a^2} \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [|b^2| < a^2, \quad u \geq 0] \quad \text{BY (266.08)}$$

$$8. \quad \int_u^\infty \frac{(x^2 + a^2)^2 dx}{\sqrt{(x^2 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 + b^2} E(\alpha, r) - \frac{a^2 - b^2}{a^2 + b^2} \cdot \frac{u^2 - a^2}{u^2 + a^2} \cdot \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [|b^2| < a^2, \quad u \geq 0] \quad \text{BY (266.06)a}$$

$$9. \quad \int_u^\infty \frac{(x^2 - a^2)^2 dx}{(x^2 + a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{a}{a^2 - b^2} E(\alpha, r) - \frac{a^2 + b^2}{2a(a^2 - b^2)} F(\alpha, r) \\ [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0] \\ \text{BY (263.04, 266.07)}$$

$$10. \quad \int_u^\infty \frac{\sqrt{x^4 + 2b^2x^2 + a^4}}{(x^2 + a^2)^2} dx = \frac{1}{2a} E(\alpha, r) \quad [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0]$$

BY (263.01, 266.01)

$$11. \quad \int_u^\infty \frac{\sqrt{x^4 + 2b^2x^2 + a^4}}{(x^2 - a^2)^2} dx = \frac{1}{2a} [F(\alpha, r) - E(\alpha, r)] + \frac{u}{u^4 - a^4} \sqrt{u^4 + 2b^2u^2 + a^4}$$

[a > b > 0, \quad u > a] \quad \text{BY (263)}

$$12. \quad \int_u^\infty \frac{(x^2 + a^2)^2 dx}{[(x^2 + a^2)^2 - 4a^2p^2x^2] \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a} \Pi(\alpha, p^2, r)$$

[a > b > 0, \quad u \geq 0] \quad \text{BY (263.02)}

3.166 Notation: $\alpha = \arccos \frac{u^2 - 1}{u^2 + 1}, \quad \beta = \arctan \left\{ \left(1 + \sqrt{2} \right) \frac{1 - u}{1 + u} \right\},$

$$\gamma = \arccos u, \quad \delta = \arccos \frac{1}{u}, \quad \varepsilon = \arccos \frac{1 - u^2}{1 + u^2},$$

$$r = \frac{\sqrt{2}}{2}, \quad q = 2\sqrt{3\sqrt{2} - 4} = 2^{\sqrt[4]{2}}(\sqrt{2} - 1) \approx 0.985171$$

$$1. \quad \int_u^\infty \frac{dx}{\sqrt{x^4 + 1}} = \frac{1}{2} F(\alpha, r) \quad [u \geq 0] \quad \text{H (287), BY (263.50)}$$

$$2. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{x^4 + 1}} = \frac{1}{2} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sqrt{u^4 + 1}}{u(u^2 + 1)}$$

[u > 0] \quad \text{BY (263.57)}

$$3. \quad \int_u^\infty \frac{x^2 dx}{(x^4 + 1) \sqrt{x^4 + 1}} = \frac{1}{2} E(\alpha, r) - \frac{1}{4} F(\alpha, r) - \frac{u(u^2 - 1)}{2(u^2 + 1) \sqrt{u^4 + 1}}$$

[u \geq 0] \quad \text{BY (263.59)}

$$4. \quad \int_u^\infty \frac{x^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = \frac{1}{4} [F(\alpha, r) - E(\alpha, r)] \quad [u \geq 0] \quad \text{BY (263.53)}$$

$$5. \quad \int_u^\infty \frac{x^2 dx}{(x^2 - 1)^2 \sqrt{x^4 + 1}} = \frac{u\sqrt{u^4 + 1}}{2(u^4 - 1)} - \frac{1}{4} E(\alpha, r) \quad [u > 1] \quad \text{BY (263.55)}$$

$$6. \quad \int_u^\infty \frac{\sqrt{x^4 + 1}}{(x^2 - 1)^2} dx = \frac{1}{2} [F(\alpha, r) - E(\alpha, r)] + \frac{u\sqrt{u^4 + 1}}{u^4 - 1}$$

[u > 1] \quad \text{BY (263.58)}

$$7. \quad \int_u^\infty \frac{(x^2 - 1)^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = E(\alpha, r) - \frac{1}{2} F(\alpha, r) \quad [u \geq 0] \quad \text{BY (263.54)}$$

$$8. \quad \int_u^\infty \frac{\sqrt{x^4 + 1}}{(x^2 + 1)^2} dx = \frac{1}{2} E(\alpha, r) \quad [u \geq 0] \quad \text{BY (263.51)}$$

9. $\int_u^\infty \frac{(x^2 + 1)^2 dx}{[(x^2 + 1)^2 - 4p^2 x^2] \sqrt{x^4 + 1}} = \frac{1}{2} \Pi(\alpha, p^2, r) \quad [u \geq 0] \quad \text{BY (263.52)}$
10. $\int_0^u \frac{dx}{\sqrt{x^4 + 1}} = \frac{1}{2} F(\varepsilon, r) \quad \text{H 66(288)}$
11. $\int_u^1 \frac{dx}{\sqrt{x^4 + 1}} = (2 - \sqrt{2}) F(\beta, q) \quad [0 \leq u < 1] \quad \text{BY (264.50)}$
12. $\int_u^1 \frac{(x^2 + x\sqrt{2} + 1) dx}{(x^2 - x\sqrt{2} + 1) \sqrt{x^4 + 1}} = (2 + \sqrt{2}) E(\beta, q) \quad [0 \leq u < 1] \quad \text{BY (264.51)}$
13. $\int_u^1 \frac{(1 - x)^2 dx}{(x^2 - x\sqrt{2} + 1) \sqrt{x^4 + 1}} = \frac{1}{\sqrt{2}} [F(\beta, q) - E(\beta, q)]$
 $[0 \leq u < 1] \quad \text{BY (264.55)}$
14. $\int_u^1 \frac{(1 + x)^2 dx}{(x^2 - x\sqrt{2} + 1) \sqrt{x^4 + 1}} = \frac{3\sqrt{2} + 4}{2} E(\beta, q) - \frac{3\sqrt{2} - 4}{2} F(\beta, q)$
 $[0 \leq u < 1] \quad \text{BY (264.56)}$
15. $\int_u^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{1}{\sqrt{2}} F(\gamma, r) \quad [u < 1] \quad \text{H 66 (290), BY (259.75)}$
16. $\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{1}{4\sqrt{2}\pi} \left\{ \Gamma\left(\frac{1}{4}\right) \right\}^2$
17. $\int_1^u \frac{dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F(\delta, r) \quad [u > 1] \quad \text{H 66 (289), BY (260.75)}$
- 18.⁸ $\int_u^1 \frac{x^2 dx}{\sqrt{1 - x^4}} = \sqrt{2} E(\gamma, r) - \frac{1}{\sqrt{2}} F(\gamma, r) \quad [u < 1]$
 $= \frac{1}{\sqrt{2}\pi} \left\{ \Gamma\left(\frac{3}{4}\right) \right\}^2 \quad [u = 0] \quad \text{BY (259.76)}$
19. $\int_1^u \frac{x^2 dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F(\delta, r) - \sqrt{2} E(\delta, r) + \frac{1}{u} \sqrt{u^4 - 1} \quad [u > 1] \quad \text{BY (260.77)}$
20. $\int_u^1 \frac{x^4 dx}{\sqrt{1 - x^4}} = \frac{1}{3\sqrt{2}} F(\gamma, r) + \frac{u}{3} \sqrt{1 - u^4} \quad [u < 1] \quad \text{BY (259.76)}$
- 21.³ $\int_1^u \frac{x^4 dx}{\sqrt{x^4 - 1}} = \frac{1}{3\sqrt{2}} F(\delta, r) + \frac{1}{3} u \sqrt{u^4 - 1} \quad [u > 1] \quad \text{BY (260.77)}$
22. $\int_0^u \frac{dx}{\sqrt{x(1 + x^3)}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos \frac{1 + (1 - \sqrt{3})u}{1 + (1 + \sqrt{3})u}, \frac{\sqrt{2 + \sqrt{3}}}{2}\right)$
 $[u > 0] \quad \text{BY (260.50)}$
23. $\int_0^u \frac{dx}{\sqrt{x(1 - x^3)}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u}, \frac{\sqrt{2 - \sqrt{3}}}{2}\right)$
 $[1 \geq u > 0] \quad \text{BY (259.50)}$

3.167 Notation: In **3.167** and **3.168** we set: $\alpha = \arcsin \sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\beta = \arcsin \sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, \quad \gamma = \arcsin \sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$

$$\delta = \arcsin \sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, \quad \kappa = \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\lambda = \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \quad \mu = \arcsin \sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$

$$\nu = \arcsin \sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \quad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \quad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

$$1. \quad \int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left\{ \Pi \left(\alpha, \frac{a-d}{a-c}, q \right) - F(\alpha, q) \right\}$$

[a > b > c > d > u] BY (251.05)

$$2. \quad \int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left\{ \Pi \left(\beta, \frac{d-c}{a-c}, r \right) - F(\beta, r) \right\}$$

[a > b > c ≥ u > d] BY (252.14)

$$3. \quad \int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left(\gamma, \frac{c-d}{b-d}, r \right) + (b-d) F(\gamma, r) \right\}$$

[a > b > c > u ≥ d] BY (253.14)

$$4. \quad \int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\delta, \frac{b-c}{b-d}, q \right)$$

[a > b ≥ u > c > d] BY (254.02)

$$5. \quad \int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi \left(\kappa, \frac{b-c}{a-c}, q \right) + (a-d) F(\kappa, q) \right\}$$

[a > b > u ≥ c > d] BY (255.20)

$$6. \quad \int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi \left(\lambda, \frac{a-b}{a-c}, r \right) + (c-d) F(\lambda, r) \right\}$$

[a ≥ u > b > c > d] BY (256.13)

$$7. \quad \int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\mu, \frac{b-a}{b-d}, r \right)$$

[a > u ≥ b > c > d] BY (257.02)

8.
$$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-b) \Pi \left(\nu, \frac{a-d}{b-d}, q \right) + (b-d) F(\nu, q) \right\}$$

[$u > a > b > c > d$] BY (258.14)
9.
$$\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\alpha, \frac{a-d}{a-c}, q \right)$$

[$a > b > c > d > u$] BY (251.02)
10.
$$\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\beta, \frac{d-c}{a-c}, r \right) - (a-c) F(\beta, r) \right]$$

[$a > b > c \geq u > d$] BY (252.13)
11.
$$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\gamma, \frac{c-d}{b-d}, r \right) - F(\gamma, r) \right]$$

[$a > b > c > u \geq d$] BY (253.13)
12.
$$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\delta, \frac{b-c}{b-d}, q \right) - F(\delta, q) \right]$$

[$a > b \geq u > c > d$] BY (254.12)
13.
$$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a) \Pi \left(\kappa, \frac{b-c}{a-c}, q \right) + (a-c) F(\kappa, q) \right]$$

[$a > b > u \geq c > d$] BY (259.19)
14.
$$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi \left(\lambda, \frac{a-b}{a-c}, r \right)$$

[$a \geq u > b > c > d$] BY (256.02)
15.
$$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\mu, \frac{b-a}{b-d}, r \right) + (d-c) F(\mu, r) \right]$$

[$a > u \geq b > c > d$] BY (257.13)
16.
$$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b) \Pi \left(\nu, \frac{a-d}{b-d}, q \right) + (b-c) F(\nu, q) \right]$$

[$u > a > b > c > d$] BY (258.13)
17.
$$\int_u^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi \left(\alpha, \frac{a-d}{a-c}, q \right) + (b-c) F(\alpha, q) \right]$$

[$a > b > c > d > u$] BY (251.07)
18.
$$\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\beta, \frac{d-c}{a-c}, r \right) - (a-b) F(\beta, r) \right]$$

[$a > b > c \geq u > d$] BY (252.15)

19.
$$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi\left(\gamma, \frac{c-d}{b-d}, r\right)$$

[$a > b > c > u \geq d$] BY (253.02)
20.
$$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (b-d) F(\delta, q) \right]$$

[$a > b \geq u > c > d$] BY (254.14)
21.
$$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) - F(\kappa, q) \right]$$

[$a > b > u \geq c > d$] BY (255.21)
22.
$$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) - F(\lambda, r) \right]$$

[$a \geq u > b > c > d$] BY (256.15)
- 23.⁸
$$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi\left(\mu, \frac{b-a}{b-d}, r\right) - (b-d) F(\mu, r) \right]$$

[$a > u \geq b > c > d$] BY (257.15)
24.
$$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\nu, \frac{a-d}{b-d}, q\right)$$

[$u > a > b > c > d$] BY (258.02)
25.
$$\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + (a-c) F(\alpha, q) \right]$$

[$a > b > c > d > u$] BY (251.06)
26.
$$\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\beta, \frac{d-c}{a-c}, r\right)$$

[$a > b > c \geq u > d$] BY (252.02)
27.
$$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c) \Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (a-b) F(\gamma, r) \right]$$

[$a > b > c > u \geq d$] BY (253.15)
28.
$$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (a-d) F(\delta, q) \right]$$

[$a > b \geq u > c > d$] BY (254.13)
29.
$$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\kappa, \frac{b-c}{a-c}, q\right)$$

[$a > b > u \geq c > d$] BY (255.02)

$$30. \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-b) \Pi \left(\lambda, \frac{a-b}{a-c}, r \right) + (a-c) F(\lambda, r) \right]$$

[$a \geq u > b > c > d$] BY (256.14)

$$31. \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\mu, \frac{b-a}{b-d}, r \right) - F(\mu, r) \right]$$

[$a > u \geq b > c > d$] BY (257.14)

$$32. \int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\nu, \frac{a-d}{b-d}, q \right) - F(\nu, q) \right]$$

[$u > a > b > c > d$] BY (258.15)

3.168

$$1. \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{d-a} \left[\sqrt{\frac{a-c}{b-d}} E(\gamma, r) - \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}} \right]$$

[$a > b > c > u > d$] BY (253.06)

$$2. \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\delta, q) - E(\delta, q)]$$

[$a > b \geq u > c > d$] BY (254.04)

$$3. \int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\kappa, q) - E(\kappa, q)] + \frac{2}{b-d} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$

[$a > b > u \geq c > d$] BY (255.09)

$$4. \int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \left[\sqrt{\frac{a-c}{b-d}} E(\lambda, r) - \frac{c-d}{b-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \right]$$

[$a \geq u > b > c > d$] BY (256.06)

$$5. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} E(\mu, r)$$

[$a > u \geq b > c > d$] BY (257.01)

$$6. \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\nu, q) - E(\nu, q)]$$

$$+ \frac{2}{a-d} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}}$$

[$u > a > b > c > d$] BY (258.10)

$$7. \int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}}$$

$$\times [(b-c)(a-d) F(\gamma, r) - (a-c)(b-d) E(\gamma, r)]$$

$$+ \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}$$

[$a > b > c > u > d$] BY (253.03)

$$\begin{aligned}
 8. \quad \int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}} dx &= \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \\
 &\times [(a-c)(b-d) E(\delta, q) - (a-b)(c-d) F(\delta, q)] \\
 &\quad [a > b \geq u > c > d] \quad \text{BY (254.15)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}} dx &= \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \\
 &\times [(a-c)(b-d) E(\kappa, q) - (a-b)(c-d) F(\kappa, q)] \\
 &\quad - \frac{2}{c-d} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \\
 &\quad [a > b > u \geq c > d] \quad \text{BY (255.06)}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx &= \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \\
 &\times [(a-c)(b-d) E(\lambda, r) - (a-d)(b-c) F(\lambda, r)] \\
 &\quad - \frac{2}{a-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \\
 &\quad [a \geq u > b > c > d] \quad \text{BY (256.03)}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx &= 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\mu, r) \\
 &\quad - \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}} F(\mu, r) \\
 &\quad [a > u \geq b > c > d] \quad \text{BY (257.09)}
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^3}} dx &= \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} + \frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}} F(\nu, q) \\
 &\quad + 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\nu, q) \\
 &\quad [u > a > b > c > d] \quad \text{BY (258.09)}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}} dx &= \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\gamma, r) - E(\gamma, r)] + \frac{2}{c-d} \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}} \\
 &\quad [a > b > c > u > d] \quad \text{BY (253.04)}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}} dx &= \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\delta, q) \\
 &\quad [a > b \geq u > c > d] \quad \text{BY (254.01)}
 \end{aligned}$$

15.
$$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\kappa, q) - \frac{2(a-d)}{(b-d)(c-d)} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$

[$a > b > u \geq c > d$] BY (255.08)
16.
$$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\lambda, r) - E(\lambda, r)] + \frac{2}{b-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$

[$a \geq u > b > c > d$] BY (256.05)
17.
$$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\mu, r) - E(\mu, r)]$$

[$a > u \geq b > c > d$] BY (257.06)
18.
$$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^3}} dx = \frac{-2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\nu, q) + \frac{2}{c-d} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}}$$

[$u > a > b > c > d$] BY (258.05)
19.
$$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\alpha, q) - E(\alpha, q)]$$

[$a > b > c > d > u$] BY (251.01)
20.
$$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)^3}} dx = \frac{-2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\beta, r) + \frac{2}{b-c} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$

[$a > b > c \geq u > d$] BY (252.06)
21.
$$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\kappa, q) - E(\kappa, q)] + \frac{2}{b-c} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$

[$a > b > u > c > d$] BY (255.05)
22.
$$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\lambda, r)$$

[$a \geq u > b > c > d$] BY (256.01)
23.
$$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\mu, r) - \frac{2(c-d)}{(a-c)(b-c)} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$

[$a > u \geq b > c > d$] BY (257.06)
24.
$$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\nu, q) - E(\nu, q)] + \frac{2}{a-c} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$

[$u > a > b > c > d$] BY (258.06)
25.
$$\int_u^a \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\alpha, q)$$

[$a > b > c > d > u$] BY (251.01)

$$26. \quad \int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\beta, r) - E(\beta, r)] + \frac{2}{c-d} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} \\ [a > b > c > u > d] \quad \text{BY (252.03)}$$

$$27. \quad \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{d-c} \sqrt{\frac{b-d}{a-c}} E(\kappa, q) + \frac{2}{c-d} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} \\ [a > b > u > c > d] \quad \text{BY (255.03)}$$

$$28. \quad \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\lambda, r) - E(\lambda, r)] \\ [a \geq u > b > c > d] \quad \text{BY (256.08)}$$

$$29. \quad \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\mu, r) - E(\mu, r)] + \frac{2}{a-c} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \\ [a > u \geq b > c > d] \quad \text{BY (257.03)}$$

$$30. \quad \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\nu, q) - \frac{2(b-c)}{(a-c)(c-d)} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} \\ [u > a > b > c > d] \quad \text{BY (258.03)}$$

$$31. \quad \int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)^3(d-x)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\alpha, q) - \frac{a-b}{b-c} \frac{2}{\sqrt{(a-c)(b-d)}} F(\alpha, q) \\ [a > b > c > d > u] \quad \text{BY (251.08)}$$

$$32. \quad \int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)^3(x-d)}} dx = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\beta, r) - 2 \frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\beta, r) \\ + 2 \frac{a-c}{(b-c)(c-d)} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} \\ [a > b > c > u > d] \quad \text{BY (252.04)}$$

$$33. \quad \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3(x-d)}} dx = \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-c)}} F(\kappa, q) - 2 \sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}} E(\kappa, q) \\ + \frac{2(a-c)}{(b-c)(c-d)} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} \\ [a > b > u > c > d] \quad \text{BY (255.04)}$$

$$34. \quad \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\lambda, r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\lambda, r) \\ [a \geq u > b > c > d] \quad \text{BY (256.09)}$$

$$35. \quad \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\mu, r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\mu, r) \\ - \frac{2}{b-c} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \\ [a > u \geq b > c > d] \quad \text{BY (257.04)}$$

$$\begin{aligned}
 36. \quad \int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3(x-d)}} dx &= \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\nu, q) - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu, q) \\
 &\quad - \frac{2}{c-d} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} \\
 &\quad [u > a > b > c > d] \quad \text{BY (258.04)}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad \int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)^3(c-x)}} dx &= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\alpha, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\alpha, q) \\
 &\quad - \frac{2}{a-b} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} \\
 &\quad [a > b > c > d > u] \quad \text{BY (251.11)}
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}} dx &= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\beta, r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\beta, r) \\
 &\quad + \frac{2}{b-c} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \\
 &\quad [a > b > c \geq u > d] \quad \text{BY (252.07)}
 \end{aligned}$$

$$\begin{aligned}
 39. \quad \int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}} dx &= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\gamma, r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma, r) \\
 &\quad [a > b > c > u \geq d] \quad \text{BY (253.07)}
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}} dx &= \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\delta, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\delta, q) \\
 &\quad + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \\
 &\quad [a > b > u > c > d] \quad \text{BY (254.05)}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)^3(x-c)}} dx &= \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\mu, r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\mu, r) \\
 &\quad + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \\
 &\quad [a > u > b > c > d] \quad \text{BY (257.07)}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}} dx &= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\nu, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu, q) \\
 &\quad [u > a > b > c > d] \quad \text{BY (258.07)}
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}} dx &= \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\alpha, q) - \frac{2(b-c)}{(a-b)(b-d)} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} \\
 &\quad [a > b > c > d > u]
 \end{aligned}$$

$$44. \quad \int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\beta, r) - E(\beta, r)] + \frac{2}{b-d} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \\ [a > b > c \geq u > d] \quad \text{BY (252.10)}$$

$$45. \quad \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\gamma, r) - E(\gamma, r)] \\ [a > b > c > u \geq d] \quad \text{BY (254.08)}$$

$$46. \quad \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{b-a} \sqrt{\frac{a-c}{b-d}} E(\delta, q) + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \\ [a > b \geq u > c > d] \quad \text{BY (254.08)}$$

$$47. \quad \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\mu, r) - E(\mu, r)] + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \\ [a > u \geq b > c > d] \quad \text{BY (257.10)}$$

$$48. \quad \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\nu, q) \\ [u > a > b > c > d] \quad \text{BY (258.01)}$$

$$49. \quad \int_u^d \sqrt{\frac{a-x}{(b-x)^3(c-x)(d-x)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\alpha, q) - E(\alpha, q)] + \frac{2}{b-d} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} \\ [a > b > c > d > u] \quad \text{BY (251.12)}$$

$$50. \quad \int_d^u \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\beta, r) - \frac{2(a-b)}{(b-c)(b-d)} \sqrt{\frac{(u-d)(c-u)}{(a-u)(b-u)}} \\ [a > b > c \geq u > d] \quad \text{BY (252.09)}$$

$$51. \quad \int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\gamma, r) \\ [a > b > c > u \geq d] \quad \text{BY (253.01)}$$

$$52. \quad \int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\delta, q) - E(\delta, q)] + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \\ [a > b > u > c > d] \quad \text{BY (254.06)}$$

$$53. \quad \int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)(x-d)}} dx = \frac{2}{c-b} \sqrt{\frac{a-c}{b-d}} E(\mu, r) + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \\ [a > u > b > c > d] \quad \text{BY (257.08)}$$

$$54. \quad \int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\nu, q) - E(\nu, q)] \\ [u > a > b > c > d] \quad \text{BY (258.08)}$$

$$55. \int_u^d \sqrt{\frac{d-x}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{b-a} \sqrt{\frac{b-d}{a-c}} E(\alpha, q) + \frac{2}{a-b} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \\ [a > b > c > d > u] \quad \text{BY (251.09)}$$

$$56. \int_d^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\beta, q) - E(\beta, q)] \\ [a > b > c \geq u > d] \quad \text{BY (252.05)}$$

$$57. \int_u^c \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\gamma, r) - E(\gamma, r)] + \frac{2}{a-c} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \\ [a > b > c > u \geq d] \quad \text{BY (253.05)}$$

$$58. \int_c^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\delta, q) - \frac{2(a-d)}{(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \\ [a > b \geq u > c > d] \quad \text{BY (254.03)}$$

$$59. \int_u^b \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\kappa, q) \\ [a > b > u \geq c > d] \quad \text{BY (255.01)}$$

$$60. \int_b^u \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\lambda, r) - E(\lambda, r)] + \frac{2}{a-b} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \\ [a > u > b > c > d] \quad \text{BY (256.10)}$$

$$61. \int_u^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}} dx = \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\alpha, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\alpha, q) \\ + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \\ [a > b > c > d > u] \quad \text{BY (251.15)}$$

$$62. \int_d^u \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\beta, r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\beta, r) \\ [a > b > c \geq u > d] \quad \text{BY (252.08)}$$

$$63. \int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\gamma, r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma, r) \\ - \frac{2}{a-d} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \\ [a > b > c > u \geq d] \quad \text{BY (253.10)}$$

$$64. \int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\delta, q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\delta, q) \\ - \frac{2}{a-b} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \\ [a > b \geq u > c > d] \quad \text{BY (254.09)}$$

$$65. \quad \int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\kappa, q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\kappa, q) \\ [a > b > u \geq c > d] \quad \text{BY (255.10)}$$

$$66. \quad \int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}} dx = \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\lambda, r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\lambda, r) \\ + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \\ [a > u > b > c > d] \quad \text{BY (256.07)}$$

$$67. \quad \int_u^d \sqrt{\frac{b-x}{(a-x)^3(c-x)(d-x)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\alpha, q) - E(\alpha, q)] + \frac{2}{a-d} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \\ [a > b > c > d > u] \quad \text{BY (251.13)}$$

$$68. \quad \int_d^u \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\beta, r) \\ [a > b > c \geq u > d] \quad \text{BY (252.01)}$$

$$69. \quad \int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\gamma, r) - \frac{2(a-b)}{(a-c)(a-d)} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \\ [a > b > c > u \geq d] \quad \text{BY (253.08)}$$

$$70. \quad \int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\delta, q) - E(\delta, q)] + \frac{2}{a-c} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \\ [a > b \geq u > c > d] \quad \text{BY (254.07)}$$

$$71. \quad \int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\kappa, q) - E(\kappa, q)] \\ [a > b > u \geq c > d] \quad \text{BY (255.07)}$$

$$72. \quad \int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}} dx = \frac{-2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\lambda, r) + \frac{2}{a-d} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \\ [a \geq u > b > c > d] \quad \text{BY (256.04)}$$

3.169 Notation: In **3.169–3.172**, we set: $\alpha = \arctan \frac{u}{b}, \quad \beta = \arctan \frac{a}{u},$

$$\gamma = \arcsin \frac{u}{b} \sqrt{\frac{a^2+b^2}{a^2+u^2}}, \quad \delta = \arccos \frac{u}{b}, \quad \varepsilon = \arccos \frac{b}{u}, \quad \xi = \arcsin \sqrt{\frac{a^2+b^2}{a^2+u^2}},$$

$$\eta = \arcsin \frac{u}{b}, \quad \zeta = \arcsin \frac{a}{b} \sqrt{\frac{b^2-u^2}{a^2-u^2}}, \quad \kappa = \arcsin \frac{a}{u} \sqrt{\frac{u^2-b^2}{a^2-b^2}},$$

$$\lambda = \arcsin \sqrt{\frac{a^2-u^2}{a^2-b^2}}, \quad \mu = \arcsin \sqrt{\frac{u^2-a^2}{u^2-b^2}}, \quad \nu = \arcsin \frac{a}{u}, \quad q = \frac{\sqrt{a^2-b^2}}{a},$$

$$r = \frac{b}{\sqrt{a^2+b^2}}, \quad s = \frac{a}{\sqrt{a^2+b^2}}, \quad t = \frac{b}{a}.$$

1.
$$\int_0^u \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} dx = a \{F(\alpha, q) - E(\alpha, q)\} + u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}}$$

[$a > b, \quad u > 0$] BY (221.03)
- 2.⁶
$$\int_0^u \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} dx = \frac{b^2}{a} F(\alpha, q) - a E(\alpha, q) + u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}}$$

[$a > b, \quad u > 0$] BY (221.04)
3.
$$\int_0^u \sqrt{\frac{x^2 + a^2}{b^2 - x^2}} dx = \sqrt{a^2 + b^2} E(\gamma, r) - u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$

[$b \geq u > 0$] BY (214.11)
4.
$$\int_u^b \sqrt{\frac{a^2 + x^2}{b^2 - x^2}} dx = \sqrt{a^2 + b^2} E(\delta, r)$$

[$b > u \geq 0$] BY (213.01), ZH 64 (273)
5.
$$\int_b^u \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} dx = \sqrt{a^2 + b^2} \{F(\varepsilon, s) - E(\varepsilon, s)\} + \frac{1}{u} \sqrt{(u^2 + a^2)(u^2 - b^2)}$$

[$u > b > 0$] BY (211.03)
6.
$$\int_0^u \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} dx = \sqrt{a^2 + b^2} \{F(\gamma, r) - E(\gamma, r)\} + u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$

[$b \geq u > 0$] BY (214.03)
7.
$$\int_u^b \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} dx = \sqrt{a^2 + b^2} \{F(\delta, r) - E(\delta, r)\}$$

[$b > u \geq 0$] BY (213.03)
8.
$$\int_b^u \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} dx = \frac{1}{u} \sqrt{(a^2 + u^2)(u^2 - b^2)} - \sqrt{a^2 + b^2} E(\varepsilon, s)$$

[$u > b > 0$] BY (211.04)
9.
$$\int_0^u \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = a E(\eta, t) - \frac{a^2 - b^2}{a} F(\eta, t)$$

[$a > b \geq u > 0$] BY (219.03)
10.
$$\int_u^b \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = a E(\zeta, t) - \frac{a^2 - b^2}{a} F(\zeta, t) - u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}}$$

[$a > b > u \geq 0$] BY (220.04)
11.
$$\int_b^u \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = a E(\kappa, q) - \frac{b^2}{a} F(\kappa, q) - \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)}$$

[$a \geq u > b > 0$] BY (217.04)
12.
$$\int_u^a \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = a E(\lambda, q) - \frac{b^2}{a} F(\lambda, q)$$

[$a > u \geq b > 0$] BY (218.03)
13.
$$\int_a^u \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} dx = \frac{a^2 - b^2}{a} F(\mu, t) - a E(\mu, t) + \mu \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}$$

[$u > a > b > 0$] BY (216.03)
14.
$$\int_0^u \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = a E(\eta, t)$$

[$a > b \geq u > 0$] H 64 (276), BY (219.01)

$$15. \quad \int_u^b \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = a \left\{ E(\zeta, t) - \frac{u}{a} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \quad [a > b > u \geq 0] \quad \text{BY (220.03)}$$

$$16. \quad \int_b^u \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a \{ F(\kappa, q) - E(\kappa, q) \} + \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a \geq u > b > 0] \quad \text{BY (217.03)}$$

$$17. \quad \int_u^a \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a \{ F(\lambda, q) - E(\lambda, q) \} \quad [a > u \geq b > 0] \quad \text{BY (218.09)}$$

$$18. \quad \int_a^u \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} dx = u \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} - a E(\mu, t) \quad [u > a > b > 0] \quad \text{BY (216.04)}$$

3.171

$$1. \quad \int_b^u \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} E(\varepsilon, s) \quad [u > b > 0] \quad \text{BY (211.01), ZH 64 (274)}$$

$$2. \quad \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} E(\xi, s) - \frac{a^2}{b^2 u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \\ [u \geq b > 0] \quad \text{BY (212.09)}$$

$$3. \quad \int_u^b \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} = \frac{a^2 - b^2}{ab^2} F(\zeta, t) - \frac{a}{b^2} E(\zeta, t) + \frac{a^2}{b^2 u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u > 0] \quad \text{BY (220.12)}$$

$$4. \quad \int_b^u \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} = \frac{a}{b^2} E(\kappa, q) - \frac{1}{a} F(\kappa, q) \quad [a \geq u > b > 0] \quad \text{BY (217.11)}$$

$$5. \quad \int_u^a \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} = \frac{a}{b^2} E(\lambda, q) - \frac{1}{a} f(\lambda, q) - \frac{\sqrt{(a^2 - u^2)(u^2 - b^2)}}{b^2 u} \\ [a > u \geq b > 0] \quad \text{BY (218.10)}$$

$$6. \quad \int_a^u \frac{dx}{x^2} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} = \frac{a}{b^2} E(\mu, t) - \frac{a^2 - b^2}{ab^2} F(\mu, t) - \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.08)}$$

$$7. \quad \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} = \frac{1}{a} F(\beta, q) - \frac{a}{b^2} E(\beta, q) + \frac{a^2}{b^2 u} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} \\ [a > b, \quad u > 0] \quad \text{BY (222.08)}$$

$$8. \quad \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} = \frac{1}{a} \{ F(\beta, q) - E(\beta, q) \} + \frac{1}{u} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} \\ [a > b, \quad u > 0] \quad \text{BY (222.09)}$$

$$9. \quad \int_u^b \frac{dx}{x^2} \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} = \frac{\sqrt{(b^2 - u^2)(a^2 + u^2)}}{a^2 u} - \frac{\sqrt{a^2 + b^2}}{a^2} E(\delta, r) \\ [b > u > 0] \quad \text{BY (213.10)}$$

$$10. \quad \int_b^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} = \frac{\sqrt{a^2 + b^2}}{a^2} \{F(\varepsilon, s) - E(\varepsilon, s)\} \quad [a > b > 0] \quad \text{BY (211.07)}$$

$$11. \quad \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} = \frac{\sqrt{a^2 + b^2}}{a^2} \{F(\xi, s) - E(\xi, s)\} + \frac{1}{u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \quad [u \geq b > 0] \quad \text{BY (212.11)}$$

$$12. \quad \int_u^b \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{b^2 - x^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} \{F(\delta, r) - E(\delta, r)\} + \frac{\sqrt{(b^2 - u^2)(a^2 + u^2)}}{b^2 u} \quad [b > u > 0] \quad \text{BY (213.05)}$$

$$13. \quad \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} = \frac{a}{b^2} E(\nu, t) - \frac{a^2 - b^2}{ab^2} F(\nu, t) \quad [u \geq a > b > 0] \quad \text{BY (215.08)}$$

$$14. \quad \int_u^b \frac{dx}{x^2} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} = \frac{1}{u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} - \frac{1}{a} E(\zeta, t) \quad [a > b > u > 0] \quad \text{BY (220.11)}$$

$$15. \quad \int_b^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} = \frac{1}{a} \{F(\kappa, q) - E(\kappa, q)\} \quad [a \geq u > b > 0] \quad \text{BY (217.08)}$$

$$16. \quad \int_u^a \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{u^2 - x^2}} = \frac{1}{a} \{F(\lambda, q) - E(\lambda, q)\} + \frac{\sqrt{(a^2 - u^2)(u^2 - b^2)}}{a^2 u} \quad [a > u \geq b > 0] \quad \text{BY (218.08)}$$

$$17. \quad \int_a^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} = \frac{1}{a} E(\mu, t) - \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0] \quad \text{BY (216.07)}$$

$$18. \quad \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} = \frac{1}{a} E(\nu, t) \quad [u \geq a > b > 0] \quad \text{BY (215.01), ZH 65 (281)}$$

3.172

$$1. \quad \int_0^u \sqrt{\frac{x^2 + b^2}{(x^2 + a^2)^3}} dx = \frac{1}{a} E(\alpha, q) - \frac{a^2 - b^2}{a^2} \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}} \quad [a > b, \quad u > 0] \quad \text{BY (221.10)}$$

$$2. \quad \int_u^\infty \sqrt{\frac{x^2 + b^2}{(x^2 + a^2)^3}} dx = \frac{1}{a} E(\beta, q) \quad [a > b, \quad u \geq 0] \quad \text{H 64 (271)}$$

$$3. \quad \int_0^u \sqrt{\frac{x^2 + a^2}{(x^2 + b^2)^3}} dx = \frac{a}{b^2} E(\alpha, q) \quad [a > b, \quad u > 0] \quad \text{H 64 (270)}$$

$$4. \quad \int_u^\infty \sqrt{\frac{x^2 + a^2}{(x^2 + b^2)^3}} dx = \frac{a}{b^2} E(\beta, q) - \frac{a^2 - b^2}{b^2} \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}} \quad [a > b, \quad u \geq 0] \quad \text{BY (222.06)}$$

$$5. \quad \int_0^u \sqrt{\frac{b^2 - x^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\gamma, r) - \frac{1}{\sqrt{a^2 + b^2}} F(\gamma, r) \quad [b \geq u > 0] \quad \text{BY (214.08)}$$

6.
$$\int_u^b \sqrt{\frac{b^2 - x^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\delta, r) - \frac{1}{\sqrt{a^2 + b^2}} F(\delta, r) - \frac{u}{a^2} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$

[$b > u \geq 0$] BY (213.04)
7.
$$\int_b^u \sqrt{\frac{x^2 - b^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\varepsilon, s) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\varepsilon, s) - \frac{1}{u} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}}$$

[$u > b > 0$] BY (211.06)
8.
$$\int_u^\infty \sqrt{\frac{x^2 - b^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\xi, s) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\xi, s)$$

[$u \geq b > 0$] BY (212.08)
9.
$$\int_0^u \sqrt{\frac{x^2 + a^2}{(b^2 - x^2)^3}} dx = \frac{a^2}{b^2 \sqrt{a^2 + b^2}} F(\gamma, r) - \frac{\sqrt{a^2 + b^2}}{b^2} E(\gamma, r) + \frac{(a^2 + b^2) u}{b^2 \sqrt{(a^2 + u^2)(b^2 - u^2)}}$$

[$b > u > 0$] BY (214.09)
10.
$$\int_u^\infty \sqrt{\frac{x^2 + a^2}{(x^2 - b^2)^3}} dx = \frac{1}{\sqrt{a^2 + b^2}} F(\xi, s) - \frac{\sqrt{a^2 + b^2}}{b^2} E(\xi, s) + \frac{(a^2 + b^2) u}{b^2 \sqrt{(a^2 + u^2)(u^2 - b^2)}}$$

[$u > b > 0$] BY (212.07)
11.
$$\int_0^u \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \left\{ F(\eta, t) - E(\eta, t) + \frac{u}{a} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\}$$

[$a > b \geq u > 0$] BY (219.09)
12.
$$\int_u^b \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \{ F(\zeta, t) - E(\zeta, t) \}$$

[$a > b > u \geq 0$] BY (220.07)
13.
$$\int_b^u \sqrt{\frac{x^2 - b^2}{(a^2 - x^2)^3}} dx = \frac{1}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} - \frac{1}{a} E(\kappa, q)$$

[$a > u > b > 0$] BY (217.07)
14.
$$\int_u^\infty \sqrt{\frac{x^2 - b^2}{(x^2 - a^2)^3}} dx = \frac{1}{a} [F(\nu, t) - E(\nu, t)] + \frac{1}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}}$$

[$u > a > b > 0$] BY (215.05)
15.
$$\int_0^u \sqrt{\frac{a^2 - x^2}{(b^2 - x^2)^3}} dx = \frac{a}{b^2} [F(\eta, t) - E(\eta, t)] + \frac{u}{b^2} \sqrt{\frac{a^2 - u^2}{b^2 - u^2}}$$

[$a > b > u > 0$] BY (219.10)
16.
$$\int_u^a \sqrt{\frac{a^2 - x^2}{(x^2 - b^2)^3}} dx = \frac{u}{b^2} \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} - \frac{a}{b^2} E(\lambda, q)$$

[$a > u > b > 0$] BY (218.05)
17.
$$\int_a^u \sqrt{\frac{x^2 - a^2}{(x^2 - b^2)^3}} dx = \frac{a}{b^2} [F(\mu, t) - E(\mu, t)]$$

[$u > a > b > 0$] BY (216.05)

$$18. \quad \int_u^\infty \sqrt{\frac{x^2 - a^2}{(x^2 - b^2)^3}} dx = \frac{a}{b^2} [F(\nu, t) - E(\nu, t)] + \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u \geq a > b > 0] \quad \text{BY (215.03)}$$

3.173

$$1. \quad \int_u^1 \frac{dx}{x^2} \sqrt{\frac{x^2 + 1}{1 - x^2}} = \sqrt{2} \left[F\left(\arccos u, \frac{\sqrt{2}}{2}\right) - E\left(\arccos u, \frac{\sqrt{2}}{2}\right) \right] + \frac{\sqrt{1 - u^4}}{u} \quad [u < 1] \quad \text{BY (259.77)}$$

$$2. \quad \int_1^u \frac{dx}{x^2} \sqrt{\frac{x^2 + 1}{x^2 - 1}} = \sqrt{2} E\left(\arccos \frac{1}{u}, \frac{\sqrt{2}}{2}\right) \quad [u > 1] \quad \text{BY (260.76)}$$

3.174 Notation: In **3.174** and **3.175**, we take: $\alpha = \arccos \frac{1 + (1 - \sqrt{3})u}{1 + (1 + \sqrt{3})u}$,

$$\beta = \arccos \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u}, \quad p = \frac{\sqrt{2 + \sqrt{3}}}{2}, \quad q = \frac{\sqrt{2 - \sqrt{3}}}{2}.$$

$$1. \quad \int_0^u \frac{dx}{[1 + (1 + \sqrt{3})x]^2} \sqrt{\frac{1 - x + x^2}{x(1 + x)}} = \frac{1}{\sqrt[4]{3}} E(\alpha, p) \quad [u > 0] \quad \text{BY (260.51)}$$

$$2. \quad \int_0^u \frac{dx}{[1 + (\sqrt{3} - 1)x]^2} \sqrt{\frac{1 + x + x^2}{x(1 - x)}} = \frac{1}{\sqrt[4]{3}} E(\beta, q) \quad [1 \geq u > 0] \quad \text{BY (259.51)}$$

$$3. \quad \int_0^u \frac{dx}{1 - x + x^2} \sqrt{\frac{x(1 + x)}{1 - x + x^2}} = \frac{1}{\sqrt[4]{27}} E(\alpha, p) + \frac{2 - \sqrt{3}}{\sqrt[4]{27}} F(\alpha, p) - \frac{2(2 + \sqrt{3})}{\sqrt{3}} \frac{1 + (1 - \sqrt{3})u}{1 + (1 + \sqrt{3})u} \\ \times \sqrt{\frac{u(1 + u)}{1 - u + u^2}} \quad [u > 0] \quad \text{BY (260.54)}$$

$$4. \quad \int_0^u \frac{dx}{1 + x + x^2} \sqrt{\frac{x(1 - x)}{1 + x + x^2}} = \frac{4}{\sqrt[4]{27}} E(\beta, q) - \frac{2 + \sqrt{3}}{\sqrt[4]{27}} F(\beta, q) - \frac{2(2 - \sqrt{3})}{\sqrt{3}} \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u} \\ \times \sqrt{\frac{u(1 - u)}{1 + u + u^2}} \quad [1 \geq u > 0] \quad \text{BY (259.55)}$$

3.175

$$1. \quad \int_0^u \frac{dx}{1 + x} \sqrt{\frac{x}{1 + x^3}} = \frac{1}{\sqrt[4]{27}} [F(\alpha, p) - 2E(\alpha, p)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1 - u + u^2)}}{\sqrt{1 + u} [1 + (1 + \sqrt{3})u]} \quad [u > 0] \quad \text{BY (260.55)}$$

$$2. \quad \int_0^u \frac{dx}{1 - x} \sqrt{\frac{x}{1 - x^3}} = \frac{1}{\sqrt[4]{27}} [F(\beta, q) - 2E(\beta, q)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1 + u + u^2)}}{\sqrt{1 - u} [1 + (\sqrt{3} - 1)u]} \quad [0 < u < 1] \quad \text{BY (259.52)}$$

3.18 Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions

3.181

$$1. \quad \int_b^u \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b} \left\{ 2 \left[E \left(\frac{1}{\sqrt{2}} \right) + E \left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}} \right) \right] \right. \\ \left. - \left[K \left(\frac{1}{\sqrt{2}} \right) + F \left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}} \right) \right] \right\} \\ [a \geq u > b] \quad \text{BY (271.05)}$$

$$2. \quad \int_a^u \frac{dx}{\sqrt[4]{(x-a)(x-b)}} \sqrt{\frac{a-b}{2}} F \left[\left(\arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) \right. \\ \left. - 2 E \left(\arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) \right] \\ + \frac{2(2u-a-b)\sqrt[4]{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}} \\ [u > a > b] \quad \text{BY (272.05)}$$

3.182

$$1. \quad \int_b^u \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}} \left[K \left(\frac{1}{\sqrt{2}} \right) + F \left(\arccos \sqrt{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}} \right) \right] \\ [a \geq u > b] \quad \text{BY (271.01)}$$

$$2. \quad \int_a^u \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}} F \left(\arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) \\ [u > a > b] \quad \text{BY (272.00)}$$

3.183 Notation: In **3.183–3.186** we set:

$$\alpha = \arccos \frac{1}{\sqrt[4]{u^2+1}}, \quad \beta = \arccos \sqrt[4]{1-u^2}, \quad \gamma = \arccos \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}}.$$

$$1. \quad \int_0^u \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2} \left[F \left(\alpha, \frac{1}{\sqrt{2}} \right) - 2 E \left(\alpha, \frac{1}{\sqrt{2}} \right) \right] + \frac{2u}{\sqrt[4]{u^2+1}} \\ [u > 0] \quad \text{BY (273.55)}$$

$$2. \quad \int_0^u \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2} \left[2 E \left(\beta, \frac{1}{\sqrt{2}} \right) - F \left(\beta, \frac{1}{\sqrt{2}} \right) \right] \quad [0 < u \leq 1] \quad \text{BY (271.55)}$$

$$3. \quad \int_1^u \frac{dx}{\sqrt[4]{x^2-1}} = F \left(\gamma, \frac{1}{\sqrt{2}} \right) - 2 E \left(\gamma, \frac{1}{\sqrt{2}} \right) + \frac{2u\sqrt[4]{u^2-1}}{1+\sqrt{u^2-1}} \\ [u > 1] \quad \text{BY (272.55)}$$

3.184

$$1. \quad \int_0^u \frac{x^2 dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5} \left[2E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right) \right] - \frac{2u}{5} \sqrt[4]{(1-u^2)^3} \quad [0 < u \leq 1] \quad \text{BY (271.59)}$$

$$2. \quad \int_1^u \frac{dx}{x^2 \sqrt[4]{x^2-1}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1}{2} F\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1 - \sqrt{u^2-1}}{1 + \sqrt{u^2-1}} \cdot \frac{\sqrt{u^2-1}}{u} \quad [u > 1] \quad \text{BY (272.54)}$$

3.185

$$1. \quad \int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^3}} = \sqrt{2} F\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad [u > 0] \quad \text{BY (273.50)}$$

$$2. \quad \int_0^u \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} F\left(\beta, \frac{1}{\sqrt{2}}\right) \quad [0 < u \leq 1] \quad \text{BY (271.51)}$$

$$3. \quad \int_1^u \frac{dx}{\sqrt[4]{(x^2-1)^3}} = F\left(\gamma, \frac{1}{\sqrt{2}}\right) \quad [u > 1] \quad \text{BY (272.50)}$$

$$4. \quad \int_0^u \frac{x^2 dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3} F\left(\beta, \frac{1}{\sqrt{2}}\right) - \frac{2}{3} u \sqrt[4]{1-u^2} \quad [0 < u \leq 1] \quad \text{BY (271.54)}$$

$$5. \quad \int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \sqrt{2} F\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad [u > 0] \quad \text{BY (273.54)}$$

$$6. \quad \int_0^u \frac{x^2 dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{2u}{\sqrt[4]{u^2+1}} \quad [u > 0] \quad \text{BY (273.56)}$$

$$7. \quad \int_0^u \frac{x^2 dx}{\sqrt[4]{(x^2+1)^7}} = \frac{1}{3\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{u}{6\sqrt[4]{(u^2+1)^3}} \quad [u > 0] \quad \text{BY (273.53)}$$

3.186

$$1. \quad \int_0^u \frac{1 + \sqrt{x^2+1}}{(x^2+1) \sqrt[4]{x^2+1}} dx = 2\sqrt{2} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad [u > 0] \quad \text{BY (273.51)}$$

$$2. \quad \int_0^u \frac{dx}{(1 + \sqrt{1-x^2}) \sqrt[4]{1-x^2}} = \sqrt{2} \left[F\left(\beta, \frac{1}{\sqrt{2}}\right) - E\left(\beta, \frac{1}{\sqrt{2}}\right) \right] + \frac{u \sqrt[4]{1-u^2}}{1 + \sqrt{1-u^2}} \quad [0 < u \leq 1] \quad \text{BY (271.58)}$$

$$3. \quad \int_1^u \frac{dx}{(x^2 + 2\sqrt{x^2-1}) \sqrt[4]{x^2-1}} = \frac{1}{2} \left[F\left(\gamma, \frac{1}{\sqrt{2}}\right) - E\left(\gamma, \frac{1}{\sqrt{2}}\right) \right] \quad [u > 1] \quad \text{BY (272.53)}$$

$$4. \quad \int_0^u \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \cdot \frac{dx}{\sqrt[4]{(1 - x^2)^3}} = \sqrt{2} \left[2 E \left(\beta, \frac{1}{\sqrt{2}} \right) - F \left(\beta, \frac{1}{\sqrt{2}} \right) \right] - \frac{2u \sqrt[4]{1 - u^2}}{1 + \sqrt{1 - u^2}} \quad [0 < u \leq 1] \quad \text{BY (271.57)}$$

$$5. \quad \int_1^u \frac{x^2 dx}{(x^2 + 2\sqrt{x^2 - 1}) \sqrt[4]{(x^2 - 1)^3}} = E \left(\gamma, \frac{1}{\sqrt{2}} \right) \quad [u > 1] \quad \text{BY (272.51)}$$

3.19–3.23 Combinations of powers of x and powers of binomials of the form $(\alpha + \beta x)$

3.191

$$1. \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} dx = u^{\mu+\nu-1} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 185(7)}$$

$$2. \quad \int_u^\infty x^{-\nu} (x-u)^{\mu-1} dx = u^{\mu-\nu} B(\nu - \mu, \mu) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{ET II 201(6)}$$

$$3. \quad \int_0^1 x^{\nu-1} (1-x)^{\mu-1} dx = \int_0^1 x^{\mu-1} (1-x)^{\nu-1} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 774(1)}$$

3.192

$$1. \quad \int_0^1 \frac{x^p dx}{(1-x)^p} = p\pi \operatorname{cosec} p\pi \quad [p^2 < 1] \quad \text{BI (3)(4)}$$

$$2. \quad \int_0^1 \frac{x^p dx}{(1-x)^{p+1}} = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0] \quad \text{BI (3)(5)}$$

$$3. \quad \int_0^1 \frac{(1-x)^p}{x^{p+1}} dx = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0] \quad \text{BI (4)(6)}$$

$$4. \quad \int_1^\infty (x-1)^{p-\frac{1}{2}} \frac{dx}{x} = \pi \sec p\pi \quad \left[-\frac{1}{2} < p < \frac{1}{2}\right] \quad \text{BI (23)(7)}$$

$$3.193 \quad \int_0^n x^{\nu-1} (n-x)^n dx = \frac{n! n^{\nu+n}}{\nu(\nu+1)(\nu+2) \dots (\nu+n)} \quad [\operatorname{Re} \nu > 0] \quad \text{EH I 2}$$

3.194

$$1. \quad \int_0^u \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu} {}_2F_1(\nu, \mu; 1+\mu; -\beta u) \quad [|\arg(1+\beta u)| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 310(20)}$$

$$2.^6 \quad \int_u^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \frac{u^{\mu-\nu}}{\beta^\nu (\nu-\mu)} {}_2F_1\left(\nu, \nu-\mu; \nu-\mu+1; -\frac{1}{\beta u}\right) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu] \quad \text{ET I 310(21)}$$

$$3. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \beta^{-\mu} B(\mu, \nu-\mu) \quad [|\arg \beta| < \pi, \quad \operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{FI II 775a, ET I 310(19)}$$

$$4.^{11} \quad \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^{n+1}} = (-1)^n \frac{\pi}{\beta^\mu} \binom{\mu-1}{n} \operatorname{cosec}(\mu\pi) \quad [|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < n+1]$$

ET I 308(6)

$$5. \quad \int_0^u \frac{x^{\mu-1} dx}{1+\beta x} = \frac{u^\mu}{\mu} {}_2F_1(1, \mu; 1+\mu; -\beta u) \quad [|\arg(1+u\beta)| < \pi, \quad \operatorname{Re} \mu > 0]$$

ET I 308(5)

$$6. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^2} = \frac{(1-\mu)\pi}{\beta^\mu} \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 2]$$

BI (16)(4)

$$7. \quad \int_0^\infty \frac{x^m dx}{(a+bx)^{n+\frac{1}{2}}} = 2^{m+1} m! \frac{(2n-2m-3)!!}{(2n-1)!!} \frac{a^{m-n+\frac{1}{2}}}{b^{m+1}} \quad [m < n - \frac{1}{2}, \quad a > 0, \quad b > 0]$$

BI (21)(2)

$$8. \quad \int_0^1 \frac{x^{n-1} dx}{(1+x)^m} = 2^{-n} \sum_{k=0}^\infty \binom{m-n-1}{k} \frac{(-2)^{-k}}{n+k}$$

BI (3)(1)

$$3.195^{11} \quad \int_0^\infty \frac{(1+x)^{p-1}}{(a+x)^{p+1}} dx = \frac{1-a^{-p}}{p(a-1)} \quad [p \neq 0, \quad a > 0, \quad a \neq 1]$$

$$= \frac{\ln a}{a-1} \quad [p = 0, \quad a > 0, \quad a \neq 1]$$

$$= 1 \quad [a = 1]$$

LI (19)(6)

3.196

$$1. \quad \int_0^u (x+\beta)^\nu (u-x)^{\mu-1} dx = \frac{\beta^\nu u^\mu}{\mu} {}_2F_1\left(1, -\nu; 1+\mu; -\frac{u}{\beta}\right) \quad \left[\left|\arg \frac{u}{\beta}\right| < \pi\right]$$

ET II 185(8)

$$2. \quad \int_u^\infty (x+\beta)^{-\nu} (x-u)^{\mu-1} dx = (u+\beta)^{\mu-\nu} B(\nu-\mu, \mu) \quad \left[\left|\arg \frac{u}{\beta}\right| < \pi, \quad \operatorname{Re} \nu > \operatorname{Re} \mu > 0\right]$$

ET II 201(7)

$$3. \quad \int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1} dx = (b-a)^{\mu+\nu-1} B(\mu, \nu) \quad [b > a, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$$

EH I 10(13)

$$4. \quad \int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b} \operatorname{cosec} \nu\pi \left(\frac{b}{b-a}\right)^\nu \quad [a < b, \quad b > 0, \quad 0 < \nu < 1]$$

LI (23)(5)

$$5. \quad \int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b} \operatorname{cosec} \nu\pi \left(\frac{b}{a-b}\right)^\nu \quad [a > b > 0, \quad 0 < \nu < 1]$$

LI (24)(10)

3.197

1.
$$\int_0^\infty x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho} dx = \beta^{-\mu}\gamma^{\nu-\varrho} B(\nu, \mu-\nu+\varrho) {}_2F_1\left(\mu, \nu; \mu+\varrho; 1-\frac{\gamma}{\beta}\right)$$

$$[|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > \operatorname{Re}(\nu-\varrho)] \quad \text{ET II 233(9)}$$
- 2.¹¹
$$\int_u^\infty x^{-\lambda}(x+\beta)^\nu(x-u)^{\mu-1} dx = u^{-\lambda}(\beta+u)^{\mu+\nu} B(\lambda-\mu-\nu, \mu) {}_2F_1\left(\lambda, \mu; \lambda-\mu; -\frac{\beta}{u}\right)$$

$$\left[\left|\arg \frac{u}{\beta}\right| < \pi \text{ or } \left|\frac{\beta}{u}\right| < 1, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda-\nu)\right] \quad \text{ET II 201(8)}$$
3.
$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu} dx = B(\lambda, \mu) {}_2F_1(\nu, \lambda; \lambda+\mu; \beta)$$

$$[\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0, \quad |\beta| < 1] \quad \text{WH}$$
4.
$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1+ax)^{-\mu-\nu} dx = (1+a)^{-\mu} B(\mu, \nu)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad a > -1] \quad \text{BI(5)4, EH I 10(11)}$$
5.
$$\int_0^\infty x^{\lambda-1}(1+x)^\nu(1+\alpha x)^\mu dx = B(\lambda, -\mu-\nu-\lambda) {}_2F_1(-\mu, \lambda; -\mu-\nu; 1-\alpha)$$

$$[|\arg \alpha| < \pi, \quad -\operatorname{Re}(\mu+\nu) > \operatorname{Re} \lambda > 0] \quad \text{EH I 60(12), ET I 310(23)}$$
6.
$$\int_1^\infty x^{\lambda-\nu}(x-1)^{\nu-\mu-1}(\alpha x-1)^{-\lambda} dx = \alpha^{-\lambda} B(\mu, \nu-\mu) {}_2F_1(\nu, \mu; \lambda; \alpha^{-1})$$

$$[1+\operatorname{Re} \nu > \operatorname{Re} \lambda > \operatorname{Re} \mu, \quad |\arg(\alpha-1)| < \pi] \quad \text{EH I 115(6)}$$
7.
$$\int_0^\infty x^{\mu-\frac{1}{2}}(x+a)^{-\mu}(x+b)^{-\mu} dx = \sqrt{\pi} \left(\sqrt{a} + \sqrt{b}\right)^{1-2\mu} \frac{\Gamma(\mu-\frac{1}{2})}{\Gamma(\mu)}$$

$$[\operatorname{Re} \mu > 0] \quad \text{BI 19(5)}$$
8.
$$\int_0^u x^{\nu-1}(x+\alpha)^\lambda(u-x)^{\mu-1} dx = \alpha^\lambda u^{\mu+\nu-1} B(\mu, \nu) {}_2F_1\left(-\lambda, \nu; \mu+\nu; -\frac{u}{\alpha}\right)$$

$$\left[\left|\arg\left(\frac{u}{\alpha}\right)\right| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0\right] \quad \text{ET II 186(9)}$$
9.
$$\int_0^\infty x^{\lambda-1}(1+x)^{-\mu+\nu}(x+\beta)^{-\nu} dx = B(\mu-\lambda, \lambda) {}_2F_1(\nu, \mu-\lambda; \mu; 1-\beta)$$

$$[\operatorname{Re} \mu > \operatorname{Re} \lambda > 0] \quad \text{EH I 205}$$
10.
$$\int_0^1 \frac{x^{q-1} dx}{(1-x)^q(1+px)} = \frac{\pi}{(1+p)^q} \operatorname{cosec} q\pi$$

$$[0 < q < 1, \quad p > -1] \quad \text{BI (5)(1)}$$
11.
$$\int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p(1+qx)^p} = \frac{2\Gamma(p+\frac{1}{2})\Gamma(1-p)}{\sqrt{\pi}} \cos^{2p}(\arctan \sqrt{q}) \frac{\sin[(2p-1)\arctan(\sqrt{q})]}{(2p-1)\sin[\arctan(\sqrt{q})]}$$

$$[-\frac{1}{2} < p < 1, \quad q > 0] \quad \text{BI (11)(1)}$$

$$12. \quad \int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p(1-qx)^p} = \frac{\Gamma(p+\frac{1}{2})\Gamma(1-p)}{\sqrt{\pi}} \frac{(1-\sqrt{q})^{1-2p} - (1+\sqrt{q})^{1-2p}}{(2p-1)\sqrt{q}} \\ \left[-\frac{1}{2} < p < 1, \quad 0 < q < 1\right] \quad \text{BI (11)(2)}$$

$$3.198 \quad \int_0^1 x^{\mu-1}(1-x)^{\nu-1}[ax+b(1-x)+c]^{-(\mu+\nu)} dx = (a+c)^{-\mu}(b+c)^{-\nu} B(\mu, \nu) \\ [a \geq 0, \quad b \geq 0, \quad c > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 787}$$

$$3.199 \quad \int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu} dx = (b-a)^{\mu+\nu-1}(b-c)^{-\mu}(a-c)^{-\nu} B(\mu, \nu) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad c < a < b] \\ \text{EH I 10(14)}$$

$$3.211 \quad \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-ux)^{-\varrho}(1-vx)^{-\sigma} dx = B(\mu, \lambda) F_1((\lambda, \varrho, \sigma, \lambda+\mu; u, v)) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0] \quad \text{EH I 231(5)}$$

$$3.212 \quad \int_0^\infty [(1+ax)^{-p} + (1+bx)^{-p}] x^{q-1} dx = 2(ab)^{-\frac{q}{2}} B(q, p-q) \cos \left\{ q \arccos \left[\frac{a+b}{2\sqrt{ab}} \right] \right\} \\ [p > q > 0] \quad \text{BI (19)(9)}$$

$$3.213 \quad \int_0^\infty [(1+ax)^{-p} - (1+bx)^{-p}] x^{q-1} dx = -2i(ab)^{-\frac{q}{2}} B(q, p-q) \sin \left\{ q \arccos \left[\frac{a+b}{2\sqrt{ab}} \right] \right\} \\ [p > q > 0] \quad \text{BI (19)(10)}$$

$$3.214 \quad \int_0^1 [(1+x)^{\mu-1}(1-x)^{\nu-1} + (1+x)^{\nu-1}(1-x)^{\mu-1}] dx = 2^{\mu+\nu-1} B(\mu, \nu) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \\ \text{LI(1)(15), EH I 10(10)}$$

$$3.215 \quad \int_0^1 \{a^\mu x^{\mu-1}(1-ax)^{\nu-1} + (1-a)^\nu x^{\nu-1}[1-(1-a)x]^{\mu-1}\} dx = B(\mu, \nu) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad |a| < 1] \\ \text{BI (1)(16)}$$

$$3.216 \quad 1. \quad \int_0^1 \frac{x^{\mu-1} + x^{\nu-1}}{(1+x)^{\mu+\nu}} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 775}$$

$$2. \quad \int_1^\infty \frac{x^{\mu-1} + x^{\nu-1}}{(1+x)^{\mu+\nu}} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 775}$$

$$3.217 \quad \int_0^\infty \left\{ \frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1} x^p} \right\} dx = \pi \cot p\pi \quad [0 < p < 1, \quad b > 0] \quad \text{BI(18)(13)}$$

$$3.218 \quad \int_0^\infty \frac{x^{2p-1} - (a+x)^{2p-1}}{(a+x)^p x^p} dx = \pi \cot p\pi \quad [p < 1] \quad (\text{cf. } 3.217) \quad \text{BI (18)(7)}$$

$$3.219 \quad \int_0^\infty \left\{ \frac{x^\nu}{(x+1)^{\nu+1}} - \frac{x^\mu}{(x+1)^{\mu+1}} \right\} dx = \psi(\mu+1) - \psi(\nu+1) \\ [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1] \quad \text{BI (19)(13)}$$

$$3.221 \quad 1. \quad \int_a^\infty \frac{(x-a)^{p-1}}{x-b} dx = \pi(a-b)^{p-1} \operatorname{cosec} p\pi \quad [a > b, \quad 0 < p < 1] \quad \text{LI (24)(8)}$$

$$2. \quad \int_{-\infty}^a \frac{(a-x)^{p-1}}{x-b} dx = -\pi(b-a)^{p-1} \operatorname{cosec} p\pi \quad [a < b, \quad 0 < p < 1] \quad \text{LI (24)(8)}$$

3.222

$$1. \quad \int_0^1 \frac{x^{\mu-1} dx}{1+x} = \beta(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{WH}$$

$$2. \quad \int_0^\infty \frac{x^{\mu-1} dx}{x+a} = \pi \operatorname{cosec}(\mu\pi) a^{\mu-1} \quad \text{for } a > 0 \quad \text{FI II 718, FI II 737}$$

$$= -\pi \cot(\mu\pi) (-a)^{\mu-1} \quad \text{for } a < 0 \quad \text{BI(18)(2), ET II 249(28)}$$

$$[0 < \operatorname{Re} \mu < 1]$$

3.223

$$1. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x)(\gamma+x)} = \frac{\pi}{\gamma-\beta} (\beta^{\mu-1} - \gamma^{\mu-1}) \operatorname{cosec}(\mu\pi) \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 309(7)}$$

$$2. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x)(\alpha-x)} = \frac{\pi}{\alpha+\beta} [\beta^{\mu-1} \operatorname{cosec}(\mu\pi) + \alpha^{\mu-1} \cot(\mu\pi)] \quad [|\arg \beta| < \pi, \quad \alpha > 0, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 309(8)}$$

$$3. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(a-x)(b-x)} = \pi \cot(\mu\pi) \frac{a^{\mu-1} - b^{\mu-1}}{b-a} \quad [a > b > 0, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 309(9)}$$

$$3.224 \quad \int_0^\infty \frac{(x+\beta)x^{\mu-1} dx}{(x+\gamma)(x+\delta)} = \pi \operatorname{cosec}(\mu\pi) \left\{ \frac{\gamma-\beta}{\gamma-\delta} \gamma^{\mu-1} + \frac{\delta-\beta}{\delta-\gamma} \delta^{\mu-1} \right\} \quad [|\arg \gamma| < \pi, \quad |\arg \delta| < \pi, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 309(10)}$$

3.225

$$1. \quad \int_1^\infty \frac{(x-1)^{p-1}}{x^2} dx = (1-p)\pi \operatorname{cosec} p\pi \quad [-1 < p < 1] \quad \text{BI (23)(8)}$$

$$2. \quad \int_1^\infty \frac{(x-1)^{1-p}}{x^3} dx = \frac{1}{2}p(1-p)\pi \operatorname{cosec} p\pi \quad [0 < p < 1] \quad \text{BI (23)(1)}$$

$$3. \quad \int_0^\infty \frac{x^p dx}{(1+x)^3} = \frac{\pi}{2}p(1-p) \operatorname{cosec} p\pi \quad [-1 < p < 2] \quad \text{BI (16)(5)}$$

3.226

$$1. \quad \int_0^1 \frac{x^n dx}{\sqrt{1-x}} = 2 \frac{(2n)!!}{(2n+1)!!} \quad \text{BI (8)(1)}$$

$$2. \quad \int_0^1 \frac{x^{n-\frac{1}{2}} dx}{\sqrt{1-x}} = \frac{(2n-1)!!}{(2n)!!} \pi. \quad \text{BI (8)(2)}$$

3.227

$$1. \quad \int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x} dx = \beta^{1-\mu}\gamma^{\nu-1} B(\nu, \mu-\nu) {}_2F_1\left(\mu-1, \nu; \mu; 1-\frac{\gamma}{\beta}\right) \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu] \quad \text{ET II 217(9)}$$

$$2. \quad \int_0^\infty \frac{x^{-\varrho}(\beta-x)^{-\sigma}}{\gamma+x} dx = \pi\gamma^{-\varrho}(\beta-\gamma)^{-\sigma} \operatorname{cosec}(\varrho\pi) I_{1-\gamma/\beta}(\sigma, \varrho) \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad -\operatorname{Re} \sigma < \operatorname{Re} \varrho < 1] \quad \text{ET II 217(10)}$$

3.228

$$1. \quad \int_a^b \frac{(x-a)^\nu(b-x)^{-\nu}}{x-c} dx = \pi \operatorname{cosec}(\nu\pi) \left[1 - \left(\frac{a-c}{b-c} \right)^\nu \right] \quad \text{for } c < a \\ = \pi \operatorname{cosec}(\nu\pi) \left[1 - \cos(\nu\pi) \left(\frac{c-a}{b-c} \right)^\nu \right] \quad \text{for } a < c < b \\ = \pi \operatorname{cosec}(\nu\pi) \left[1 - \left(\frac{c-a}{c-b} \right)^\nu \right] \quad \text{for } c > b \\ [|\operatorname{Re} \nu| < 1] \quad \text{ET II 250(31)}$$

$$2. \quad \int_a^b \frac{(x-a)^{\nu-1}(b-x)^{-\nu}}{x-c} dx = \frac{\pi \operatorname{cosec}(\nu\pi)}{b-c} \left| \frac{a-c}{b-c} \right|^{\nu-1} \quad \text{for } c < a \text{ or } c > b; \\ = -\frac{\pi(c-a)^{\nu-1}}{(b-c)^\nu} \cot(\nu\pi) \quad \text{for } a < c < b \\ [0 < \operatorname{Re} \nu < 1] \quad \text{ET II 250(32)}$$

$$3. \quad \int_a^b \frac{(x-a)^{\nu-1}(b-x)^{\mu-1}}{x-c} dx \\ = \frac{(b-a)^{\mu+\nu-1}}{b-c} B(\mu, \nu) {}_2F_1\left(1, \mu; \mu+\nu; \frac{b-a}{b-c}\right) \quad \text{for } c < a \text{ or } c > b; \\ = \pi(c-a)^{\nu-1}(b-c)^{\mu-1} \cot \mu\pi - (b-a)^{\mu+\nu-2} B(\mu-1, \nu) \\ \times {}_2F_1\left(2-\mu-\nu, 1; 2-\mu; \frac{b-c}{b-a}\right) \quad \text{for } a < c < b \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \mu+\nu \neq 1, \quad \mu \neq 1, 2, \dots] \quad \text{ET II 250(33)}$$

$$4. \quad \int_0^1 \frac{(1-x)^{\nu-1}x^{-\nu}}{a-bx} dx = \frac{\pi(a-b)^{\nu-1}}{a^\nu} \operatorname{cosec}(\nu\pi) \quad [0 < \operatorname{Re} \nu < 1, \quad 0 < b < a] \quad \text{BI (5)(8)}$$

$$5. \quad \int_0^\infty \frac{x^{\nu-1}(x+a)^{1-\mu}}{x-c} dx = a^{1-\mu}(-c)^{\nu-1} B(\mu-\nu, \nu) {}_2F_1\left(\mu-1, \nu; \mu; 1+\frac{c}{a}\right) \quad \text{for } c < 0; \\ = \pi c^{\nu-1}(a+c)^{1-\mu} \cot[(\mu-\nu)\pi] - \frac{a^{1-\mu-\nu}}{a+c} B(\mu-\nu-1, \nu) \\ \times {}_2F_1\left(2-\mu, 1; 2-\mu+\nu; \frac{a}{a+c}\right) \quad \text{for } c > 0 \\ [a > 0, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu] \quad \text{ET II 251(34)}$$

6.
$$\int_0^\infty x^{\nu-1} \frac{(\gamma+x)^{-n}}{x+\beta} dx = \frac{\pi}{\sin \pi \nu} \frac{\beta^{\nu-1}}{(\gamma-\beta)^n} \left[1 - \left(\frac{\gamma}{\beta} \right)^{\nu-1} \sum_{j=0}^{n-1} \frac{(1-\nu)_j}{j!} \left(\frac{\gamma-\beta}{\gamma} \right)^j \right]$$

$$[\arg \beta < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \nu < n] \quad \text{AS 256 (6.1.22)}$$
- 3.229**
$$\int_0^1 \frac{x^{\mu-1} dx}{(1-x)^\mu (1+ax)(1+bx)} = \frac{\pi \operatorname{cosec} \mu \pi}{a-b} \left[\frac{a}{(1+a)^\mu} - \frac{b}{(1+b)^\mu} \right]$$

$$[0 < \operatorname{Re} \mu < 1] \quad \text{BI (5)(7)}$$
- 3.231**
1.
$$\int_0^1 \frac{x^{p-1} - x^{-p}}{1-x} dx = \pi \cot p\pi$$

$$[p^2 < 1] \quad \text{BI (4)(4)}$$
- 2.¹¹
$$\int_0^1 \frac{x^{p-1} + x^{-p}}{1+x} dx = \pi \operatorname{cosec} p\pi$$

$$[p^2 < 1] \quad \text{BI (4)(1)}$$
3.
$$\int_0^1 \frac{x^p - x^{-p}}{x-1} dx = \frac{1}{p} - \pi \cot p\pi$$

$$[p^2 < 1] \quad \text{BI (4)(3)}$$
4.
$$\int_0^1 \frac{x^p - x^{-p}}{1+x} dx = \frac{1}{p} - \pi \operatorname{cosec} p\pi$$

$$[p^2 < 1] \quad \text{BI (4)(2)}$$
5.
$$\int_0^1 \frac{x^{\mu-1} - x^{\nu-1}}{1-x} dx = \psi(\nu) - \psi(\mu)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$$

$$\text{FI II 815, BI(4)(5)}$$
6.
$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x} dx = \pi (\cot p\pi - \cot q\pi)$$

$$[p > 0, \quad q > 0] \quad \text{FI II 718}$$
- 3.232**
$$\int_0^\infty \frac{(c+ax)^{-\mu} - (c+bx)^{-\mu}}{x} dx = c^{-\mu} \ln \frac{b}{a}$$

$$[\operatorname{Re} \mu > -1; \quad a > 0; \quad b > 0; \quad c > 0]$$

$$\text{BI (18)(14)}$$
- 3.233**
$$\int_0^\infty \left\{ \frac{1}{1+x} - (1+x)^{-\nu} \right\} \frac{dx}{x} = \psi(\nu) + C$$

$$[\operatorname{Re} \nu > 0] \quad \text{EH I 17, WH}$$
- 3.234**
- 1.¹¹
$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = \pi a^{-q} \cot q\pi$$

$$[0 < q < 1, \quad a > 0] \quad \text{BI (5)(11)}$$
2.
$$\int_0^1 \left(\frac{x^{q-1}}{1+ax} + \frac{x^{-q}}{a+x} \right) dx = \pi a^{-q} \operatorname{cosec} q\pi$$

$$[0 < q < 1, \quad a > 0] \quad \text{BI (5)(10)}$$
- 3.235**
$$\int_0^\infty \frac{(1+x)^\mu - 1}{(1+x)^\nu} \frac{dx}{x} = \psi(\nu) - \psi(\nu - \mu)$$

$$[\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{BI (18)(5)}$$
- 3.236**¹⁰
$$\int_0^1 \frac{x^{\frac{\mu}{2}} dx}{[(1-x)(1-a^2x)]^{\frac{\mu+1}{2}}} = \frac{(1-a)^{-\mu} - (1+a)^{-\mu}}{2a\mu\sqrt{\pi}} \Gamma\left(1 + \frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right)$$

$$[-2 < \mu < 1, \quad |a| < 1] \quad \text{BI (12)(32)}$$
- 3.237**
$$\sum_{n=0}^\infty (-1)^{n+1} \int_n^{n+1} \frac{dx}{x+u} = \ln \frac{u \left[\Gamma\left(\frac{u}{2}\right) \right]^2}{2 \left[\Gamma\left(\frac{u+1}{2}\right) \right]^2}$$

$$[|\arg u| < \pi] \quad \text{ET II 216(1)}$$

3.238

$$1. \quad \int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u} dx = -\pi \cot \frac{\nu\pi}{2} |u|^{\nu-1} \operatorname{sign} u \quad [0 < \operatorname{Re} \nu < 1 \quad u \text{ real}, \quad u \neq 0] \quad \text{ET II 249(29)}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u} \operatorname{sign} x dx = \pi \tan \frac{\nu\pi}{2} |u|^{\nu-1} \quad [0 < \operatorname{Re} \nu < 1 \quad u \text{ real}, \quad u \neq 0] \quad \text{ET II 249(30)}$$

$$3. \quad \int_a^b \frac{(b-x)^{\mu-1} (x-a)^{\nu-1}}{|x-u|^{\mu+\nu}} dx = \frac{(b-a)^{\mu+\nu-1}}{|a-u|^{\mu} |b-u|^{\nu}} \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu+\nu)} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad 0 < u < a < b \text{ and } 0 < a < b < u] \quad \text{MO 7}$$

3.24–3.27 Powers of x , of binomials of the form $\alpha + \beta x^p$ and of polynomials in x

3.241

$$1. \quad \int_0^1 \frac{x^{\mu-1} dx}{1+x^p} = \frac{1}{p} \beta \left(\frac{\mu}{p} \right) \quad [\operatorname{Re} \mu > 0, \quad p > 0] \quad \text{WH, BI (2)(13)}$$

$$2. \quad \int_0^{\infty} \frac{x^{\mu-1} dx}{1+x^{\nu}} = \frac{\pi}{\nu} \operatorname{cosec} \frac{\mu\pi}{\nu} = \frac{1}{\nu} B \left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu} \right) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{ET I 309(15)a, BI (17)(10)}$$

$$3.^{11} \quad \text{PV} \int_0^{\infty} \frac{x^{p-1} dx}{1-x^q} = \frac{\pi}{q} \cot \frac{p\pi}{q} \quad [p < q] \quad \text{BI (17)(11)}$$

$$4.^{11} \quad \int_0^{\infty} \frac{x^{\mu-1} dx}{(p+qx^{\nu})^{n+1}} = \frac{1}{\nu p^{n+1}} \left(\frac{p}{q} \right)^{\mu/\nu} \frac{\Gamma(\frac{\mu}{\nu}) \Gamma(1+n-\frac{\mu}{\nu})}{\Gamma(1+n)} \quad \left[0 < \frac{\mu}{\nu} < n+1, \quad p \neq 0, \quad q \neq 0 \right] \quad \text{BI (17)(22)a}$$

$$5. \quad \int_0^{\infty} \frac{x^{p-1} dx}{(1+x^q)^2} = \frac{(p-q)\pi}{q^2} \operatorname{cosec} \frac{(p-q)\pi}{q} \quad [p < 2q] \quad \text{BI (17)(18)}$$

$$6.^{10} \quad G(x) = \int_a^b \operatorname{sign} \left[\frac{x}{c} - \left(\frac{b-u}{b-a} \right)^p \right] du = (b-a) F \left[\left(\frac{x}{c} \right)^{1/p} \right]$$

where

$$F(x) = \int_0^1 \operatorname{sign}(x-t) dt = \begin{cases} -1 & x \leq 0 \\ 2x-1 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

3.242

$$1. \quad \int_{-\infty}^{\infty} \frac{x^{2m} dx}{x^{4n} + 2x^{2n} \cos t + 1} = \frac{\pi}{n} \sin \left[\frac{(2n-2m-1)}{2n} t \right] \operatorname{cosec} t \operatorname{cosec} \frac{(2m+1)\pi}{2n} \quad [m < n, \quad t^2 < \pi^2] \quad \text{FI II 642}$$

$$2.^{11} \int_0^\infty \left[\frac{x^2}{x^4 + 2ax^2 + 1} \right]^c \left(\frac{x^2 + 1}{x^b + 1} \right) \frac{dx}{x^2} = 2^{-1/2-c} (1+a)^{1/2-c} B \left(c - \frac{1}{2}, \frac{1}{2} \right)$$

$$3.243^{11} \int_0^\infty \frac{x^{\mu-1} dx}{(1+x^{2\nu})(1+x^{3\nu})}$$

$$= \frac{\pi}{48\nu} \left[8 \operatorname{cosec}(2\rho) + 12 \operatorname{cosec}(3\rho) - 8 \operatorname{cosec} \left(2\rho - \frac{4\pi}{3} \right) + 8 \operatorname{cosec} \left(2\rho - \frac{2\pi}{3} \right) \right. \\ \left. - 3 \operatorname{cosec} \left(\rho - \frac{\pi}{6} \right) \operatorname{cosec} \left(\rho + \frac{\pi}{6} \right) \sec(\rho) \right]$$

where $\rho = \frac{\mu\pi}{6\nu}$, $[0 < \operatorname{Re} \mu < 5 \operatorname{Re} \nu]$ ET I 312(34)

3.244

$$1. \int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p > 0] \quad \text{BI (2)(14)}$$

$$2. \int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{\pi}{q} \cot \frac{p\pi}{q} \quad [q > p > 0] \quad \text{BI (2)(16)}$$

$$3. \int_0^1 \frac{x^{\nu-1} - x^{\mu-1}}{1-x^\nu} dx = \frac{1}{\nu} \left[C + \psi \left(\frac{\mu}{\nu} \right) \right] \quad [\operatorname{Re} \mu > \operatorname{Re} \nu > 0] \quad \text{BI (2)(17)}$$

$$4. \int_{-\infty}^\infty \frac{x^{2m} - x^{2n}}{1-x^{2l}} dx = \frac{\pi}{l} \left[\cot \left(\frac{2m+1}{2l} \pi \right) - \cot \left(\frac{2n+1}{2l} \pi \right) \right]$$

$[m < l, \quad n < l] \quad \text{FI II 640}$

$$3.245 \int_0^\infty [x^{\nu-\mu} - x^\nu (1+x)^{-\mu}] dx = \frac{\nu}{\nu - \mu + 1} B(\nu, \mu - \nu)$$

$[\operatorname{Re} \mu > \operatorname{Re} \nu > 0] \quad \text{BI (16)(13)}$

$$3.246 \int_0^\infty \frac{1-x^q}{1-x^r} x^{p-1} dx = \frac{\pi}{r} \sin \frac{q\pi}{r} \operatorname{cosec} \frac{p\pi}{r} \operatorname{cosec} \frac{(p+q)\pi}{r}$$

$[p+q < r, \quad p > 0]$
ET I 331(33), BI (17)(12)

Integrals of the form $\int f(x^p \pm x^{-p}, x^q \pm x^{-q}, \dots) \frac{dx}{x}$ can be transformed by the substitution $x = e^t$ or $x = e^{-t}$. For example, instead of $\int_0^1 (x^{1+p} + x^{1-p})^{-1} dx$, we should seek to evaluate $\int_0^\infty \operatorname{sech} px dx$ and, instead of $\int_0^1 \frac{x^{n-m-1} + x^{n+m-1}}{1+2x^n \cos a + x^{2n}} dx$, we should seek to evaluate $\int_0^\infty \cosh mx (\cosh nx - \cos a)^{-1} dx$ (see 3.514 2).

3.247

$$1.^{11} \int_0^1 \frac{x^{\alpha-1} (1-x)^{n-1}}{1-\xi x^b} dx = (n-1)! \sum_{k=0}^\infty \frac{\xi^k}{(\alpha+kb)(\alpha+kb+1)\dots(\alpha+kb+n-1)}$$

$[b > 0, \quad |\xi| < 1] \quad \text{AD (6704)}$

$$2. \int_0^\infty \frac{(1-x^p) x^{\nu-1}}{1-x^{np}} dx = \frac{\pi}{np} \sin \left(\frac{\pi}{n} \right) \operatorname{cosec} \frac{(p+\nu)\pi}{np} \operatorname{cosec} \frac{\pi\nu}{np}$$

$[0 < \operatorname{Re} \nu < (n-1)p] \quad \text{ET I 311(33)}$

3.248

$$1. \quad \int_0^\infty \frac{x^{\mu-1} dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu} B\left(\frac{\mu}{\nu}, \frac{1}{2} - \frac{\mu}{\nu}\right) \quad [\operatorname{Re} \nu > \operatorname{Re} 2\mu > 0] \quad \text{BI (21)(9)}$$

$$2. \quad \int_0^1 \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!} \quad \text{BI (8)(14)}$$

$$3. \quad \int_0^1 \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \quad \text{BI (8)(13)}$$

$$4.^3 \quad \int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$$

$$6.^* \quad \int_{-\infty}^\infty \frac{dx}{(1+x^2)^2 \sqrt{b+ax^2}} = \begin{cases} \frac{2}{\sqrt{b-a}} \arctan\left(\sqrt{\frac{b}{a}-1}\right) & \text{if } a < b \\ \frac{2}{\sqrt{a}} & \text{if } a = b \\ \frac{1}{\sqrt{a-b}} \ln\left(\frac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right) & \text{if } a > b \end{cases}$$

3.249

$$1.^0 \quad \int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2 \cdot (2n-2)!!} \frac{\pi}{a^{2n-1}} \quad \text{FI II 743}$$

$$2.^9 \quad \int_0^a (a^2-x^2)^{n-\frac{1}{2}} dx = a^{2n} \frac{(2n-1)!!}{2(2n)!!} \pi. \quad \text{FI II 156}$$

$$3. \quad \int_{-1}^1 \frac{(1-x^2)^n dx}{(a-x)^{n+1}} = 2^{n+1} Q_n(a) \quad \text{EH II 181(31)}$$

$$4. \quad \int_0^1 \frac{x^\mu dx}{1+x^2} = \frac{1}{2} \beta\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1] \quad \text{BI (2)(7)}$$

$$5. \quad \int_0^1 (1-x^2)^{\mu-1} dx = 2^{2\mu-2} B(\mu, \mu) = \frac{1}{2} B\left(\frac{1}{2}, \mu\right) \quad [\operatorname{Re} \mu > 0] \quad \text{FI II 784}$$

$$6. \quad \int_0^1 (1-\sqrt{x})^{p-1} dx = \frac{2}{p(p+1)} \quad [p > 0] \quad \text{BI (7)(7)}$$

$$7. \quad \int_0^1 (1-x^\mu)^{-\frac{1}{\nu}} dx = \frac{1}{\mu} B\left(\frac{1}{\mu}, 1 - \frac{1}{\nu}\right) \quad [\operatorname{Re} \mu > 0, \quad |\nu| > 1]$$

$$8.^{11} \quad \int_{-\infty}^\infty \left(1 + \frac{x^2}{n-1}\right)^{-n/2} dx = \frac{\sqrt{\pi(n-1)}}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n-1}{2}\right) \quad [n > 1]$$

3.251

$$1. \quad \int_0^1 x^{\mu-1} (1-x^\lambda)^{\nu-1} dx = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \nu\right) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \lambda > 0]$$

FI II 787

$$2. \quad \int_0^\infty x^{\mu-1} (1+x^2)^{\nu-1} dx = \frac{1}{2} B\left(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2}\right) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\nu + \frac{1}{2}\mu) < 1]$$

3. $\int_1^\infty x^{\mu-1} (x^p - 1)^{\nu-1} dx = \frac{1}{p} \text{B} \left(1 - \nu - \frac{\mu}{p}, \nu \right)$ [$p > 0, \quad \text{Re } \nu > 0, \quad \text{Re } \mu < p - p \text{Re } \nu$]
ET I 311(32)
4. $\int_0^\infty \frac{x^{2m} dx}{(ax^2 + c)^n} = \frac{(2m-1)!!(2n-2m-3)!!\pi}{2 \cdot (2n-2)!!a^m c^{n-m-1} \sqrt{ac}}$ [$a > 0, \quad c > 0, \quad n > m + 1$]
GU (141)(8a)
5. $\int_0^\infty \frac{x^{2m+1} dx}{(ax^2 + c)^n} = \frac{m!(n-m-2)!}{2(n-1)!a^{m+1}c^{n-m-1}}$ [$ac > 0, \quad n > m + 1 \geq 1$] GU (141)(8b)
6. $\int_0^\infty \frac{x^{\mu+1}}{(1+x^2)^2} dx = \frac{\mu\pi}{4 \sin \frac{\mu\pi}{2}}$ [$-2 < \text{Re } \mu < 2$] WH
7. $\int_0^1 \frac{x^\mu dx}{(1+x^2)^2} = -\frac{1}{4} + \frac{\mu-1}{4} \beta \left(\frac{\mu-1}{2} \right)$ [$\text{Re } \mu > 1$] LI (3)(11)
8. $\int_0^1 x^{q+p-1} (1-x^q)^{-\frac{p}{q}} dx = \frac{p\pi}{q^2} \text{cosec } \frac{p\pi}{q}$ [$q > p$] BI (9)(22)
9. $\int_0^1 x^{\frac{q}{p}-1} (1-x^q)^{-\frac{1}{p}} dx = \frac{\pi}{q} \text{cosec } \frac{\pi}{p}$ [$p > 1, \quad q > 0$] BI (9)(23)a
10. $\int_0^1 x^{p-1} (1-x^q)^{-\frac{p}{q}} dx = \frac{\pi}{q} \text{cosec } \frac{p\pi}{q}$ [$q > p > 0$] BI (9)(20)
11. $\int_0^\infty x^{\mu-1} (1 + \beta x^p)^{-\nu} dx = \frac{1}{p} \beta^{-\frac{\mu}{p}} \text{B} \left(\frac{\mu}{p}, \nu - \frac{\mu}{p} \right)$
[$|\arg \beta| < \pi, \quad p > 0, \quad 0 < \text{Re } \mu < p \text{Re } \nu$] BI (17)(20), EH I 10(16)

3.252

1. $\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left[\frac{1}{\sqrt{ac-b^2}} \text{arccot } \frac{b}{\sqrt{ac-b^2}} \right]$
[$a > 0, \quad ac > b^2$] GW (131)(4)
2. $\int_{-\infty}^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(2n-3)!!\pi a^{n-1}}{(2n-2)!!(ac-b^2)^{n-\frac{1}{2}}}$ [$a > 0, \quad ac > b^2$] GW (131)(5)
3. $\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{(-2)^n}{(2n+1)!!} \frac{\partial^n}{\partial c^n} \left\{ \frac{1}{\sqrt{c}(\sqrt{ac}+b)} \right\}$
[$a \geq 0, \quad c > 0, \quad b > -\sqrt{ac}$] GW (213)(4)

$$\begin{aligned}
4. \quad \int_0^\infty \frac{x \, dx}{(ax^2 + 2bx + c)^n} &= \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left\{ \frac{1}{2(ac-b^2)} - \frac{b}{2(ac-b^2)^{\frac{3}{2}}} \operatorname{arccot} \frac{b}{\sqrt{ac-b^2}} \right\} \quad \text{for } ac > b^2; \\
&= \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left\{ \frac{1}{2(ac-b^2)} + \frac{b}{4(b^2-ac)^{\frac{3}{2}}} \ln \frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}} \right\} \quad \text{for } b^2 > ac > 0; \\
&= \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \quad \text{for } ac = b^2
\end{aligned}$$

[a > 0, b > 0, n ≥ 2] GW (141)(5)

$$5. \quad \int_{-\infty}^\infty \frac{x \, dx}{(ax^2 + 2bx + c)^n} = -\frac{(2n-3)!! \pi b a^{n-2}}{(2n-2)!! (ac-b^2)^{\frac{(2n-1)}{2}}} \quad [ac > b^2, a > 0, n \geq 2]$$

GW (141)(6)

$$\begin{aligned}
6. \quad \int_{-\infty}^\infty \frac{x^m \, dx}{(ax^2 + 2bx + c)^n} &= \frac{(-1)^m \pi a^{n-m-1} b^m}{(2n-2)!! (ac-b^2)^{n-\frac{1}{2}}} \\
&\quad \times \sum_{k=0}^{[m/2]} \binom{m}{2k} (2k-1)!! (2n-2k-3)!! \left(\frac{ac-b^2}{b^2} \right)^k \\
&\quad [ac > b^2, 0 \leq m \leq 2n-2] \quad \text{GW (141)(17)}
\end{aligned}$$

$$7. \quad \int_0^\infty \frac{x^n \, dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!! \sqrt{c} (\sqrt{ac} + b)^{n+1}}$$

[a ≥ 0, c > 0, b > -√ac] GW (213)(5a)

$$8. \quad \int_0^\infty \frac{x^{n+1} \, dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!! \sqrt{a} (\sqrt{ac} + b)^{n+1}}$$

[a > 0, c ≥ 0, b > -√ac] GW (213)(5b)

$$9. \quad \int_0^\infty \frac{x^{n+\frac{1}{2}} \, dx}{(ax^2 + 2bx + c)^{n+1}} = \frac{(2n-1)!! \pi}{2^{2n+\frac{1}{2}} (b+\sqrt{ac})^{n+\frac{1}{2}} n! \sqrt{a}}$$

[a > 0, c > 0, b + √ac > 0] LI (21)(19)

$$10.^6 \quad \int_0^\infty \frac{x^{\mu-1} \, dx}{(1+2x \cos t + x^2)^\nu} = 2^{\nu-\frac{1}{2}} (\sin t)^{\frac{1}{2}-\nu} t \Gamma\left(\nu + \frac{1}{2}\right) B(\mu, 2\nu - \mu) P_{\mu-\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos t)$$

[0 < t < π, 0 < Re μ < Re 2ν] ET I 310(22)

11.
$$\int_0^\infty (1 + 2\beta x + x^2)^{\mu - \frac{1}{2}} x^{-\nu-1} dx = 2^{-\mu} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma(1 - \mu) B(\nu - 2\mu + 1, -\nu) P_{\nu-\mu}^\mu(\beta)$$

$$[\operatorname{Re} \nu < 0, \quad \operatorname{Re}(2\mu - \nu) < 1, \quad |\arg(\beta \pm 1)| < \pi]$$

 EH I 160(33)

$$= -\pi \operatorname{cosec} \nu \pi C_{\nu}^{\frac{1}{2}-\mu}(\beta)$$

$$[-2 < \operatorname{Re}(\frac{1}{2} - \mu) < \operatorname{Re} \nu < 0, \quad |\arg(\beta \pm 1)| < \pi]$$

 EH I 178(24)
12.
$$\int_0^\infty \frac{x^{\mu-1} dx}{x^2 + 2ax \cos t + a^2} = -\pi a^{\mu-2} \operatorname{cosec} t \operatorname{cosec}(\mu\pi) \sin[(\mu-1)t]$$

$$[a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re} \mu < 2]$$

 FI II 738, BI(20)(3)
13.
$$\int_0^\infty \frac{x^{\mu-1} dx}{(x^2 + 2ax \cos t + a^2)^2} = \frac{\pi a^{\mu-4}}{2} \operatorname{cosec} \mu\pi \operatorname{cosec}^3 t$$

$$\times \{(\mu-1) \sin t \cos[(\mu-2)t] - \sin[(\mu-1)t]\}$$

$$[a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re} \mu < 4] \quad \text{LI(20)(8)a, ET I 309(13)}$$
14.
$$\int_0^\infty \frac{x^{\mu-1} dx}{\sqrt{1 + 2x \cos t + x^2}} = \pi \operatorname{cosec}(\mu\pi) P_{\mu-1}(\cos t) \quad [-\pi < t < \pi, \quad 0 < \operatorname{Re} \mu < 1]$$

 ET I 310(17)
- 3.253**
$$\int_{-1}^1 \frac{(1+x)^{2\mu-1} (1-x)^{2\nu-1}}{(1+x^2)^{\mu+\nu}} dx = 2^{\mu+\nu-2} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 787}$$
- 3.254**
1.
$$\int_0^u x^{\lambda-1} (u-x)^{\mu-1} (x^2 + \beta^2)^\nu dx$$

$$= \beta^{2\nu} u^{\lambda+\mu-1} B(\lambda, \mu) {}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; \frac{-u^2}{\beta^2}\right)$$

$$\left[\operatorname{Re}\left(\frac{u}{\beta}\right) > 0, \quad \lambda > 0, \quad \operatorname{Re} \mu > 0\right] \quad \text{ET II 186(10)}$$
- 2.⁶
$$\int_u^\infty (x^{-\lambda} (x-u)^{\mu-1} (x^2 + \beta^2)^\nu) dx$$

$$= u^{\mu-\lambda+2\nu} \frac{\Gamma(\mu) \Gamma(\lambda - \mu - 2\nu)}{\Gamma(\lambda - 2\nu)}$$

$$\times {}_3F_2\left(-\nu, \frac{\lambda - \mu}{2} - \nu, \frac{1 + \lambda - \mu}{2} - \nu; \frac{\lambda}{2} - \nu, \frac{1 + \lambda}{2} - \nu; -\frac{\beta^2}{u^2}\right)$$

$$\left[|u| > |\beta| \text{ and } \operatorname{Re}\left(\frac{\beta}{u}\right) > 0, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda - 2\nu)\right] \quad \text{ET II 202(9)}$$
- 3.255**
$$\int_0^1 \frac{x^{\mu+\frac{1}{2}} (1-x)^{\mu-\frac{1}{2}}}{(c + 2bx - ax^2)^{\mu+1}} dx = \frac{\sqrt{\pi}}{\left\{a + (\sqrt{c+2b-a} + \sqrt{c})^2\right\}^{\mu+\frac{1}{2}} \sqrt{c+2b-a}} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu+1)}$$

$$\left[a + (\sqrt{c+2b-a} + \sqrt{c})^2 > 0, \quad c + 2b - a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}\right]$$

 BI (14)(2)

3.256

$$1. \quad \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}} dx = \frac{1}{2} \cos\left(\frac{q-p}{4}\pi\right) \sec\left(\frac{q+p}{4}\pi\right) B\left(\frac{p}{2}, \frac{q}{2}\right) \\ [p > 0, \quad q > 0, \quad p+q < 2] \quad \text{BI (8)(25)}$$

$$2. \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}} dx = \frac{1}{2} \sin\left(\frac{q-p}{4}\pi\right) \operatorname{cosec}\left(\frac{q+p}{4}\pi\right) B\left(\frac{p}{2}, \frac{q}{2}\right) \\ [p > 0, \quad q > 0, \quad p+q < 2] \quad \text{BI (8)(26)}$$

$$3.257^9 \quad \int_0^\infty \left[\left(ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx \\ = \frac{\sqrt{\pi} \Gamma\left(p + \frac{1}{2}\right)}{2ac^{p+\frac{1}{2}} \Gamma(p+1)} \quad [a > 0, \quad b < 0, \quad c > 0, \quad p > -\frac{1}{2}] \quad \text{BI (20)(4)} \\ = \frac{1}{2} \frac{B\left(p + \frac{1}{2}, \frac{1}{2}\right)}{a(4ab+x)^{p+\frac{1}{2}}} \quad [a > 0, \quad b > 0, \quad c > -4ab, \quad p > -\frac{1}{2}]$$

3.258

$$1. \quad \int_b^\infty \left(x - \sqrt{x^2 - a^2} \right)^n dx = \frac{a^2}{2(n-1)} \left(b - \sqrt{b^2 - a^2} \right)^{n-1} - \frac{1}{2(n+1)} \left(b - \sqrt{b^2 - a^2} \right)^{n+1} \\ [0 < a \leq b, \quad n \geq 2] \quad \text{GW (215)(5)}$$

$$2. \quad \int_b^\infty \left(\sqrt{x^2 + 1} - x \right)^n dx = \frac{(\sqrt{b^2 + 1} - b)^{n-1}}{2(n-1)} + \frac{(\sqrt{b^2 + 1} - b)^{n+1}}{2(n+1)} \\ [n \geq 2] \quad \text{GW (214)(7)}$$

$$3. \quad \int_0^\infty \left(\sqrt{x^2 + a^2} - x \right)^n dx = \frac{na^{n+1}}{n^2 - 1} \quad [n \geq 2] \quad \text{GW (214)(6a)}$$

$$4. \quad \int_0^\infty \frac{dx}{(x + \sqrt{x^2 + a^2})^n} = \frac{n}{a^{n-1}(n^2 - 1)} \quad [n \geq 2] \quad \text{GW (214)(5a)}$$

$$5. \quad \int_0^\infty x^m \left(\sqrt{x^2 + a^2} - x \right)^n dx = \frac{n \cdot m! a^{m+n+1}}{(n-m-1)(n-m+1) \dots (m+n+1)} \\ [a > 0, \quad 0 \leq m \leq n-2] \quad \text{GW (214)(6)}$$

$$6. \quad \int_0^\infty \frac{x^m dx}{(x + \sqrt{x^2 + a^2})^n} = \frac{n \cdot m!}{(n-m-1)(n-m+1) \dots (m+n+1) a^{n-m-1}} \\ [a > 0, \quad 0 \leq m \leq n-2] \quad \text{GW (214)(5)}$$

$$7. \quad \int_a^\infty (x-a)^m \left(x - \sqrt{x^2 - a^2} \right)^n dx = \frac{n \cdot (n-m-2)!(2m+1)! a^{m+n+1}}{2^m (n+m+1)!} \\ [a > 0, \quad n \geq m+2] \quad \text{GH (215)(6)}$$

3.259

$$1.^6 \quad \int_0^1 x^{p-1} (1-x)^{n-1} (1+bx^m)^l dx = (n-1)! \sum_{k=0}^{\infty} \binom{l}{k} \frac{b^k \Gamma(p+km)}{\Gamma(p+n+km)} \\ [|b| < 1 \text{ unless } l = 0, 1, 2, \dots; \quad p, n, p+ml > 0] \quad \text{BI (1)(14)}$$

$$2.^{11} \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} (x^m + \beta^m)^\lambda dx \\ = \beta^{m\lambda} u^{\mu+\nu-1} B(\mu, \nu) \\ \times {}_{m+1}F_m \left(-\lambda, \frac{\nu}{m}, \frac{\nu+1}{m}, \dots, \frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}, \frac{\mu+\nu+1}{m}, \dots, \frac{\mu+\nu+m-1}{m}; \frac{-u^m}{\beta^m} \right) \\ \left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \left| \arg \left(\frac{u}{\beta} \right) \right| < \frac{\pi}{m} \right] \quad \text{ET II 186(11)}$$

$$3.^{11} \quad \int_0^\infty x^{\lambda-1} (1+\alpha x^p)^{-\mu} (1+\beta x^p)^{-\nu} dx = \frac{1}{p} \alpha^{-\lambda/p} B \left(\frac{\lambda}{p}, \mu+\nu-\frac{\lambda}{p} \right) {}_2F_1 \left(\nu, \frac{\lambda}{p}; \mu+\nu; 1-\frac{\beta}{\alpha} \right) \\ [|\arg \alpha| < \pi, \quad |\arg \beta| < \pi, \quad p > 0, \quad 0 < \operatorname{Re} \lambda < 2 \operatorname{Re}(\mu+\nu)] \quad \text{ET I 312(35)}$$

3.261

$$1.^{11} \quad \text{PV} \int_0^1 \frac{(1-x \cos t) x^{\mu-1} dx}{1-2x \cos t + x^2} = \sum_{k=0}^{\infty} \frac{\cos kt}{\mu+k} \quad [\operatorname{Re} \mu > 0, \quad t \neq 2n\pi] \quad \text{BI (6)(9)}$$

$$2. \quad \int_0^1 \frac{(x^\nu + x^{-\nu}) dx}{1+2x \cos t + x^2} = \frac{\pi \sin \nu t}{\sin t \sin \nu \pi} \quad [\nu^2 < 1, \quad t \neq (2n+1)\pi] \quad \text{BI (6)(8)}$$

$$3. \quad \int_0^1 \frac{(x^{1+p} + x^{1-p}) dx}{(1+2x \cos t + x^2)^2} = \frac{\pi (p \sin t \cos pt - \cos t \sin pt)}{2 \sin^3 t \sin p\pi} \\ [p^2 < 1, \quad t \neq (2n+1)\pi] \quad \text{BI (6)(18)}$$

$$4. \quad \int_0^1 \frac{x^{\mu-1}}{1+2ax \cos t + a^2 x^2} \cdot \frac{dx}{(1-x)^\mu} = \frac{\pi \operatorname{cosec} t \operatorname{cosec} \frac{\mu\pi}{2}}{(1+2a \cos t + a^2)^{\frac{\mu}{2}}} \sin \left(t - \mu \arctan \frac{a \sin t}{1+a \cos t} \right) \\ [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{BI (6)(21)}$$

$$3.262 \quad \int_0^\infty \frac{x^{-p} dx}{1+x^3} = \frac{\pi}{3} \operatorname{cosec} \frac{(1-p)\pi}{3} \quad [-2 < p < 1] \quad \text{LI (18)(3)}$$

$$3.263 \quad \int_0^\infty \frac{x^\nu dx}{(x+\gamma)(x^2+\beta^2)} = \frac{\pi}{2(\beta^2+\gamma^2)} \left[\gamma \beta^{\nu-1} \sec \frac{\nu\pi}{2} + \beta^\nu \operatorname{cosec} \frac{\nu\pi}{2} - 2\gamma^\nu \operatorname{cosec}(\nu\pi) \right] \\ [\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \pi, \quad -1 < \operatorname{Re} \nu < 2, \quad \nu \neq 0] \quad \text{ET II 216(7)}$$

3.264

$$1. \quad \int_0^\infty \frac{x^{p-1} dx}{(a^2+x^2)(b^2-x^2)} = \frac{\pi}{2} \frac{a^{p-2} + b^{p-2} \cos \frac{p\pi}{2}}{a^2 + b^2} \operatorname{cosec} \frac{p\pi}{2} \\ [0 < p < 4, \quad a > 0, \quad b > 0] \quad \text{BI (19)(14)}$$

$$\begin{aligned}
2. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x^2)(\gamma+x^2)} &= \frac{\pi}{2} \frac{\gamma^{\frac{\mu}{2}-1} - \beta^{\frac{\mu}{2}-1}}{\beta - \gamma} \operatorname{cosec} \frac{\mu\pi}{2} \\
&= \frac{\pi}{2(\gamma - \beta)} \left(\frac{1}{\sqrt{\beta}} - \frac{1}{\sqrt{\gamma}} \right) \quad [\mu = \tfrac{1}{2}] \\
&\quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 4] \quad \text{ET I 309(4)}
\end{aligned}$$

$$3. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^2} = \frac{\pi}{2} \left(\frac{1}{a^2 b^{1/2}} - \frac{1}{2a(a+b)^{3/2}} - \frac{1}{a^2(a+b)^{1/2}} \right) \quad \text{MC}$$

$$4. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^3} = \frac{\pi}{4} \left(\frac{2}{a^3 b^{1/2}} - \frac{3}{4a(a+b)^{5/2}} - \frac{1}{a^2(a+b)^{3/2}} - \frac{2}{a^3(a+b)^{1/2}} \right)$$

$$\begin{aligned}
5. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^4} &= \frac{\pi}{4} \left(\frac{2}{a^4 b^{1/2}} - \frac{5}{8a(a+b)^{7/2}} - \frac{3}{4a^2(a+b)^{5/2}} \right. \\
&\quad \left. - \frac{1}{a^3(a+b)^{3/2}} - \frac{2}{a^4(a+b)^{1/2}} \right)
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^n} &= \frac{\pi}{2} \frac{1}{a^n b^{1/2}} - \frac{1}{2a(a+b)^{n-1/2}} \operatorname{B} \left(n - \frac{1}{2}, \frac{1}{2} \right) {}_2F_1 \left(1 - n, 1; \frac{3}{2} - n; \frac{a+b}{a} \right) \\
&\quad \text{AS 263 (6.6.3.2)} \\
&= \frac{\pi}{2} \frac{1}{a^n b^{1/2}} - \frac{\pi}{2a^n(a+b)^{n-1/2}} \sum_{j=0}^{n-1} \frac{(\frac{1}{2})_j}{j!} \left(\frac{a}{a+b} \right)^j
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty \frac{x^2 dx}{(x^2 + \alpha^2)(x^2 + \beta^2)(x^2 + \gamma^2)} &= \frac{\pi}{2\alpha(\beta^2 - \gamma^2)} \left[\frac{\beta}{\beta + \alpha} - \frac{\gamma}{\gamma + \alpha} \right] = \frac{\pi}{2(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)} \\
&\quad [n > 0, \quad a + b > 0]
\end{aligned}$$

$$\begin{aligned}
3.265 \quad \int_0^1 \frac{1 - x^{\mu-1}}{1 - x} dx &= \psi(\mu) + \mathbf{C} \quad [\operatorname{Re} \mu > 0] \quad \text{FI II 796, WH, ET I 16(13)} \\
&= \psi(1 - \mu) + \mathbf{C} - \pi \cot(\mu\pi) \quad [\operatorname{Re} \mu > 0] \quad \text{EH I 16(15)a}
\end{aligned}$$

$$\begin{aligned}
3.266 \quad \int_0^\infty \frac{(x^\nu - a^\nu) dx}{(x - a)(\beta + x)} &= \frac{\pi}{a + \beta} \left\{ \beta^\nu \operatorname{cosec}(\nu\pi) - a^\nu \cot(\nu\pi) - \frac{a^\nu}{\pi} \ln \frac{\beta}{a} \right\} \\
&\quad [|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1, \quad \nu \neq 0] \\
&\quad \text{ET II 216(8)}
\end{aligned}$$

3.267

$$1. \quad \int_0^1 \frac{x^{3n} dx}{\sqrt[3]{1 - x^3}} = \frac{2\pi}{3\sqrt{3}} \frac{\Gamma(n + \frac{1}{3})}{\Gamma(\frac{1}{3}) \Gamma(n + 1)} \quad \text{BI (9)(6)}$$

$$2. \quad \int_0^1 \frac{x^{3n-1} dx}{\sqrt[3]{1 - x^3}} = \frac{(n-1)! \Gamma(\frac{2}{3})}{3 \Gamma(n + \frac{2}{3})} \quad \text{BI (9)(7)}$$

$$3.^* \quad \int_0^1 \frac{x^{3n-2} dx}{\sqrt[3]{1-x^3}} = \frac{\Gamma\left(n - \frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right)}{3 \Gamma\left(n + \frac{1}{3}\right)}$$

3.268

$$1. \quad \int_0^1 \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx = \ln p \quad \text{BI (5)(14)}$$

$$2. \quad \int_0^1 \frac{1-x^\mu}{1-x} x^{\nu-1} dx = \psi(\mu + \nu) - \psi(\nu) \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (2)(3)}$$

$$3. \quad \int_0^1 \left[\frac{n}{1-x} - \frac{x^{\mu-1}}{1-\sqrt[n]{x}} \right] dx = nC + \sum_{k=1}^n \psi \left(\mu + \frac{n-k}{n} \right) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (13)(10)}$$

3.269

$$1. \quad \int_0^1 \frac{x^p - x^{-p}}{1-x^2} x dx = \frac{\pi}{2} \cot \frac{p\pi}{2} - \frac{1}{p} \quad [p^2 < 1] \quad \text{BI (4)(12)}$$

$$2. \quad \int_0^1 \frac{x^p - x^{-p}}{1+x^2} x dx = \frac{1}{p} - \frac{\pi}{2} \operatorname{cosec} \frac{p\pi}{2} \quad [p^2 < 1] \quad \text{BI (4)(8)}$$

$$3. \quad \int_0^1 \frac{x^\mu - x^\nu}{1-x^2} dx = \frac{1}{2} \psi \left(\frac{\nu+1}{2} \right) - \frac{1}{2} \psi \left(\frac{\mu+1}{2} \right) \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1] \quad \text{BI (2)(9)}$$

3.271

$$1. \quad \int_0^\infty \frac{x^p - x^q}{x-1} \frac{dx}{x+a} = \frac{\pi}{1+a} \left(\frac{a^p - \cos p\pi}{\sin p\pi} - \frac{a^q - \cos q\pi}{\sin q\pi} \right) \quad [p^2 < 1, \quad q^2 < 1, \quad a > 0] \quad \text{BI (19)(2)}$$

$$2. \quad \int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^p - 1}{x-1} dx = \frac{\pi}{a-1} \left\{ \frac{a^{2p} - 1}{\sin(2p\pi)} - \frac{1}{\pi} a^p \ln a \right\} \quad \left[p^2 < \frac{1}{4} \right] \quad \text{BI (19)(3)}$$

$$3. \quad \int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^{-p} - 1}{x-1} dx = \frac{\pi}{a-1} \left\{ 2(a^p - 1) \cot p\pi - \frac{1}{\pi} (a^p + 1) \ln a \right\} \quad [p^2 < 1] \quad \text{BI (18)(9)}$$

$$4. \quad \int_0^\infty \frac{x^p - a^p}{x-a} \frac{1-x^{-p}}{1-x} x^q dx = \frac{\pi}{a-1} \left\{ \frac{a^{p+q} - 1}{\sin[(p+q)\pi]} + \frac{a^p - a^q}{\sin[(q-p)\pi]} \right\} \frac{\sin p\pi}{\sin q\pi} \quad \left[\frac{a^{p+q} - 1}{\sin[(p+q)\pi]} + \frac{a^p - a^q}{\sin[(q-p)\pi]} \right] \frac{\sin p\pi}{\sin q\pi} \quad [(p+q)^2 < 1, \quad (p-q)^2 < 1] \quad \text{BI (19)(4)}$$

$$5. \quad \int_0^\infty \left(\frac{x^p - x^{-p}}{1-x} \right)^2 dx = 2(1 - 2p\pi \cot 2p\pi) \quad [0 < p^2 < \frac{1}{4}] \quad \text{BI (16)(3)}$$

3.272

$$1. \quad \int_0^1 \frac{x^{n-1} + x^{n-\frac{1}{2}} - 2x^{2n-1}}{1-x} dx = 2 \ln 2 \quad \text{BI (8)(8)}$$

$$2. \quad \int_0^1 \frac{x^{n-1} + x^{n-\frac{2}{3}} + x^{n-\frac{1}{3}} - 3x^{3n-1}}{1-x} dx = 3 \ln 3 \quad \text{BI (8)(9)}$$

3.273

$$1. \quad \int_0^1 \frac{\sin t - a^n x^n \sin[(n+1)t] + a^{n+1} x^{n+1} \sin nt}{1 - 2ax \cos t + a^2 x^2} (1-x)^{p-1} dx = \Gamma(p) \sum_{k=1}^n \frac{(k-1)! a^{k-1} \sin kt}{\Gamma(p+k)} \quad [p > 0] \quad \text{BI (6)(13)}$$

$$2. \quad \int_0^1 \frac{\cos t - ax - a^n x^n \cos[(n+1)t] + a^{n+1} x^{n+1} \cos nt}{1 - 2ax \cos t + a^2 x^2} (1-x)^{p-1} dx = \Gamma(p) \sum_{k=1}^n \frac{(k-1)! a^{k-1} \cos kt}{\Gamma(p+k)} \quad [p > 0] \quad \text{BI (6)(14)}$$

$$3. \quad \int_0^1 x \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1} \sin nt}{1 - 2x \cos t + x^2} dx = \sum_{k=1}^n \frac{\sin kt}{k+1} \quad \text{BI (6)(12)}$$

$$4. \quad \int_0^1 \frac{1 - x \cos t - x^{n+1} \cos[(n+1)t] + x^{n+2} \cos nt}{1 - 2x \cos t + x^2} dx = \sum_{k=0}^n \frac{\cos kt}{k+1} \quad \text{BI (6)(11)}$$

3.274

$$1. \quad \int_0^\infty \frac{x^{\mu-1}(1-x)}{1-x^n} dx = \frac{\pi}{n} \sin \frac{\pi}{n} \operatorname{cosec} \frac{\mu\pi}{n} \operatorname{cosec} \frac{(\mu+1)\pi}{n} \quad [0 < \operatorname{Re} \mu < n-1] \quad \text{BI (20)(13)}$$

$$2. \quad \int_0^1 \frac{1-x^n}{(1+x)^{n+1}} \frac{dx}{1-x} = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k} \quad \text{BI (5)(3)}$$

$$3. \quad \int_0^\infty \frac{x^q - 1}{x^p - x^{-p}} \frac{dx}{x} = \frac{\pi}{2p} \tan \frac{q\pi}{2p} \quad [p > q] \quad \text{BI (18)(6)}$$

3.275

$$1. \quad \int_0^1 \left(\frac{x^{n-1}}{1-x^{1/p}} - \frac{px^{np-1}}{1-x} \right) dx = p \ln p \quad [p > 0] \quad \text{BI (13)(9)}$$

$$2. \quad \int_0^1 \left(\frac{nx^{n-1}}{1-x^n} - \frac{x^{mn-1}}{1-x} \right) dx = C + \frac{1}{n} \sum_{k=1}^n \psi \left(m + \frac{n-k}{n} \right) \quad \text{BI (5)(13)}$$

$$3. \quad \int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{qx^{pq-1}}{1-x^q} \right) dx = \ln q \quad [q > 0] \quad \text{BI (5)(12)}$$

$$4. \quad \int_0^\infty \left(\frac{1}{1+x^{2n}} - \frac{1}{1+x^{2m}} \right) \frac{dx}{x} = 0. \quad \text{BI (18)(17)}$$

3.276

$$1.^{10} \quad \int_0^\infty \frac{\left[\left(ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx}{x^2} = \frac{1}{2|b|} \frac{B \left(p + \frac{1}{2}, \frac{1}{2} \right)}{(2a(b+|b|) + c)^{p+\frac{1}{2}}} \quad [a > 0, \quad c > -4ac, \quad p > -\frac{1}{2}]$$

$$2.10 \quad \int_0^\infty \left(a + \frac{b}{x^2}\right) \left[\left(ax + \frac{b}{x}\right)^2 + c\right]^{-p-1} dx = \frac{B\left(p + \frac{1}{2}, \frac{1}{2}\right)}{(4ab + c)^{p+\frac{1}{2}}} \\ [a > 0, \quad b > 0, \quad c > -4ac, \quad p > -\frac{1}{2}]$$

3.277

$$1.11 \quad \int_0^\infty \frac{x^{\mu-1} [\sqrt{1+x^2} + \beta]^\nu}{\sqrt{1+x^2}} dx = 2^{\frac{\mu}{2}-1} (\beta^2 - 1)^{\frac{\nu}{2} + \frac{\mu}{4}} \Gamma\left(\frac{\mu}{2}\right) \Gamma(1 - \mu - \nu) P_{\frac{\mu}{2}-1}^{\nu+\frac{\mu}{2}}(\beta) \\ [\operatorname{Re} \beta > -1, \quad 0 < \operatorname{Re} \mu < 1 - \operatorname{Re} \nu] \quad \text{ET I 310(25)}$$

$$2. \quad \int_0^\infty \frac{x^{\mu-1} [\sqrt{\beta^2 + x^2} + x]^\nu}{\sqrt{\beta^2 + x^2}} dx = \frac{\beta^{\mu+\nu-1}}{2^\mu} B\left(\mu, \frac{1-\mu-\nu}{2}\right) \\ [\operatorname{Re} \beta > 0, \quad 0 < \operatorname{Re} \mu < 1 - \operatorname{Re} \nu] \quad \text{ET I 311(28)}$$

$$3. \quad \int_0^\infty \frac{x^{\mu-1} [\cos t \pm i \sin t \sqrt{1+x^2}]^\nu}{\sqrt{1+x^2}} dx = 2^{\frac{\mu-1}{2}} \sin^{\frac{1-\mu}{2}} t \frac{\Gamma\left(\frac{\mu}{2}\right) \Gamma(1 - \mu - \nu)}{\Gamma(-\nu)} \\ \times \left[\pi^{-\frac{1}{2}} Q_{-\frac{\mu+1}{2}-\nu}^{\frac{\mu+1}{2}}(\cos t) \mp \frac{i}{2} \pi^{\frac{1}{2}} P_{\frac{\mu-1}{2}}^{-\frac{\mu+1}{2}-\nu}(\cos t) \right] \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 311 (27)}$$

$$4. \quad \int_0^\infty \frac{x^{\mu-1} [\sqrt{(\beta^2 - 1)(x^2 + 1)} + \beta]^\nu}{\sqrt{x^2 + 1}} dx \\ = \frac{2^{\frac{\mu-1}{2}}}{\sqrt{\pi}} e^{-\frac{1}{2}i\pi(\mu-1)} \frac{\Gamma\left(\frac{\mu}{2}\right) \Gamma(1 - \mu - \nu)}{\Gamma(-\nu)} (\beta^2 - 1)^{\frac{1-\mu}{4}} Q_{\frac{\mu-1}{2}-\nu}^{\frac{\mu-1}{2}}(\beta) \\ [\operatorname{Re} \beta > 1, \quad \operatorname{Re} \nu < 0, \quad \operatorname{Re} \mu < 1 - \operatorname{Re} \nu] \quad \text{ET I 311(26)}$$

$$5. \quad \int_u^\infty \frac{(x-u)^{\mu-1} (\sqrt{x+1} - \sqrt{x-1})^{2\nu}}{\sqrt{x^2 - 1}} dx = \frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}} e^{(\mu-\frac{1}{2})\pi i} (u^2 - 1)^{\frac{2\mu-1}{4}} Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u) \\ [|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu] \quad \text{ET II 202(10)}$$

$$6. \quad \int_1^\infty \frac{x^{\mu-1} [(x - \sqrt{x^2 - 1})^\nu + (x - \sqrt{x^2 - 1})^{-\nu}]}{\sqrt{x^2 - 1}} dx = 2^{-\mu} B\left(\frac{1-\mu+\nu}{2}, \frac{1-\mu-\nu}{2}\right) \\ [\operatorname{Re} \mu < 1 + \operatorname{Re} \nu] \quad \text{ET I 311(29)}$$

$$7. \quad \int_0^u \frac{(u-x)^{\mu-1} [(\sqrt{x+2} + \sqrt{x})^{2\nu} + (\sqrt{x+2} - \sqrt{x})^{2\nu}]}{\sqrt{x(x+2)}} dx = 2^{\frac{2\mu+1}{2}} \sqrt{\pi[u(u+2)]^{\mu-\frac{1}{2}}} P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u) \\ 1) \quad [|\arg u| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 186(12)}$$

3.278⁸

$$1. \quad \int_0^\infty \left(\frac{x^p}{1+x^{2p}}\right)^q \frac{dx}{1-x^2} = 0 \quad [pq > 1]$$

3.3–3.4 Exponential Functions

3.31 Exponential functions

$$3.310^{11} \quad \int_0^{\infty} e^{-px} dx = \frac{1}{p} \quad [\operatorname{Re} p > 0]$$

3.311

$$1. \quad \int_0^{\infty} \frac{dx}{1 + e^{px}} = \frac{\ln 2}{p} \quad \text{LO III 284a}$$

$$2. \quad \int_0^{\infty} \frac{e^{-\mu x}}{1 + e^{-x}} dx = \beta(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{EH I 20(3), ET I 144(7)}$$

$$3.^{11} \quad \int_{-\infty}^{\infty} \frac{e^{-px}}{1 + e^{-qx}} dx = \frac{\pi}{|q|} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p > 0 \text{ or } 0 > p > q] \quad (\text{cf. } \mathbf{3.241} \text{ 2}) \quad \text{BI (28)(7)}$$

$$4. \quad \int_0^{\infty} \frac{e^{-qx} dx}{1 - ae^{-px}} = \sum_{k=0}^{\infty} \frac{a^k}{q + kp} \quad [0 < a < 1] \quad \text{BI (27)(7)}$$

$$5. \quad \int_0^{\infty} \frac{1 - e^{\nu x}}{e^x - 1} dx = \psi(\nu) + \mathbf{C} + \pi \cot(\pi\nu) \quad [\operatorname{Re} \nu < 1] \quad (\text{cf. } \mathbf{3.265}) \quad \text{EH I 16(16)}$$

$$6. \quad \int_0^{\infty} \frac{e^{-x} - e^{-\nu x}}{1 - e^{-x}} dx = \psi(\nu) + \mathbf{C} \quad [\operatorname{Re} \nu > 0] \quad \text{WH, EH I 16(14)}$$

$$7. \quad \int_0^{\infty} \frac{e^{-\mu x} - e^{-\nu x}}{1 - e^{-x}} dx = \psi(\nu) - \psi(\mu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad (\text{cf. } \mathbf{3.231} \text{ 5})$$

BI (27)(8)

$$8. \quad \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{b - e^{-x}} = \pi b^{\mu-1} \cot(\mu\pi) \quad [b > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 120(14)a}$$

$$9. \quad \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{b + e^{-x}} = \pi b^{\mu-1} \operatorname{cosec}(\mu\pi) \quad [|\arg b| < \pi, \quad 0 < \operatorname{Re} \mu < 1]$$

ET I 120(15)a

$$10.^{11} \quad \int_0^{\infty} \frac{e^{-px} - e^{-qx}}{1 - e^{-(p+q)x}} dx = \frac{\pi}{p+q} \cot \frac{p\pi}{p+q} \quad [p > 0, \quad q > 0] \quad \text{GW (311)(16c)}$$

$$11. \quad \int_0^{\infty} \frac{e^{px} - e^{qx}}{e^{rx} - e^{sx}} dx = \frac{1}{r-s} \left[\psi \left(\frac{r-q}{r-s} \right) - \psi \left(\frac{r-p}{r-s} \right) \right]$$

[$r > s, r > p, r > q$] GW (311)(16)

$$12. \quad \int_0^{\infty} \frac{a^x - b^x}{c^x - d^x} dx = \frac{1}{\ln \frac{c}{d}} \left[\psi \left(\frac{\ln \frac{c}{b}}{\ln \frac{c}{d}} \right) - \psi \left(\frac{\ln \frac{c}{a}}{\ln \frac{c}{d}} \right) \right] \quad [c > a > 0, \quad b > 0, \quad d > 0]$$

GW (311)(16a)

$$13.^* \quad \int_0^{\infty} \frac{e^{-px} + e^{-qx}}{1 + e^{-(p+q)x}} dx = \frac{\pi}{p+q} \operatorname{cosec} \left(\frac{\pi p}{p+q} \right)$$

3.312

$$1. \quad \int_0^\infty \left(1 - e^{-\frac{x}{\beta}}\right)^{\nu-1} e^{-\mu x} dx = \beta B(\beta\mu, \nu) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0] \\ \text{LI(25)(13), EH I 11(24)}$$

$$2. \quad \int_0^\infty (1 - e^{-x})^{-1} (1 - e^{-\alpha x}) (1 - e^{-\beta x}) e^{-px} dx = \psi(p + \alpha) + \psi(p + \beta) - \psi(p + \alpha + \beta) - \psi(p) \\ [\operatorname{Re} p > 0, \quad \operatorname{Re} p > -\operatorname{Re} \alpha, \quad \operatorname{Re} p > -\operatorname{Re} \beta, \quad \operatorname{Re} p > -\operatorname{Re}(\alpha + \beta)] \quad \text{ET I 145(15)}$$

$$3.^{11} \quad \int_0^\infty (1 - e^{-x})^{\nu-1} (1 - \beta e^{-x})^{-\varrho} e^{-\mu x} dx = B(\mu, \nu) {}_2F_1(\varrho, \mu; \mu + \nu; \beta) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad |\arg(1 - \beta)| < \pi] \quad \text{EH I 116(15)}$$

3.313

$$1.^7 \quad \text{PV} \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{1 - e^{-x}} = \pi \cot \pi \mu \quad [0 < \operatorname{Re} \mu < 1]$$

$$2.^7 \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(1 + e^{-x})^\nu} = B(\mu, \nu - \mu) \quad [0 < \operatorname{Re} \mu < \operatorname{Re} \nu]$$

$$3.314 \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(e^{\beta/\gamma} + e^{-x/\gamma})^\nu} = \gamma \exp \left[\beta \left(\mu - \frac{\nu}{\gamma} \right) \right] B(\gamma\mu, \nu - \gamma\mu) \\ \left[\operatorname{Re} \left(\frac{\nu}{\gamma} \right) > \operatorname{Re} \mu > 0, \quad |\operatorname{Im} \beta| < \pi \operatorname{Re} \gamma \right] \quad \text{ET I 120(21)}$$

3.315

$$1. \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(e^\beta + e^{-x})^\nu (e^\gamma + e^{-x})^\varrho} = \exp[\gamma(\mu - \varrho) - \beta\nu] B(\mu, \nu + \varrho - \mu) {}_2F_1(\nu, \mu; \nu + \varrho; 1 - e^{\nu-\beta}) \\ [|\operatorname{Im} \beta| < \pi, \quad |\operatorname{Im} \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\nu + \varrho)] \quad \text{ET I 121(22)}$$

$$2. \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(\beta + e^{-x})(\gamma + e^{-x})} = \frac{\pi(\beta^{\mu-1} - \gamma^{\mu-1})}{\gamma - \beta} \operatorname{cosec}(\mu\pi) \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \beta \neq \gamma, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 120(18)}$$

$$3.316 \quad \int_{-\infty}^\infty \frac{(1 + e^{-x})^\nu - 1}{(1 + e^{-x})^\mu} dx = \psi(\mu) - \psi(\mu - \nu) \quad [\operatorname{Re} \mu > \operatorname{Re} \nu > 0] \quad (\text{cf. 3.235}) \\ \text{BI (28)(8)}$$

3.317

$$1. \quad \int_{-\infty}^\infty \left(\frac{1}{1 + e^{-x}} - \frac{1}{(1 + e^{-x})^\mu} \right) dx = C + \psi(\mu) \quad [\operatorname{Re} \mu > 0] \quad (\text{cf. 3.233}) \quad \text{BI (28)(10)}$$

$$2. \quad \int_{-\infty}^\infty \left(\frac{1}{(1 + e^{-x})^\nu} - \frac{1}{(1 + e^{-x})^\mu} \right) dx = \psi(\mu) - \psi(\nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad (\text{cf. 3.219}) \\ \text{BI (28)(11)}$$

3.318

$$1. \quad \int_0^\infty \frac{[\beta + \sqrt{1 - e^{-x}}]^{-\nu} + [\beta - \sqrt{1 - e^{-x}}]^{-\nu}}{\sqrt{1 - e^{-x}}} e^{-\mu x} dx$$

$$= \frac{2^{\mu+1} e^{(\mu-\nu)\pi i} (\beta^2 - 1)^{(\mu-\nu)/2} \Gamma(\mu) Q_{\mu-1}^{\nu-\mu}(\beta)}{\Gamma(\nu)}$$

[Re $\mu > 0$] ET I 145(18)

$$2.^7 \quad \int_u^\infty \frac{1}{\sqrt{1 - e^{-2x}}} \left(e^{-u} \sqrt{1 - e^{-2x}} - e^{-x} \sqrt{1 - e^{-2u}} \right)^\nu e^{-\mu x} dx$$

$$= \frac{2^{-\frac{1}{2}(\mu+\nu)} \sqrt{\pi} e^{-\frac{u}{2}(\mu+\nu)} \Gamma(\mu) \Gamma(\nu+1) P_{-\frac{1}{2}(\mu-\nu)}^{-\frac{1}{2}(\mu+\nu)}(\sqrt{1 - e^{-2u}})}{\Gamma[(\mu+\nu+1)/2]}$$

[$u > 0, \quad \text{Re } \mu > 0, \quad \text{Re } \nu > -1$] ET I 145(19)

3.32–3.34 Exponentials of more complicated arguments

3.321

$$1.^{11} \quad \frac{\sqrt{\pi}}{2} \Phi(u) = \frac{\sqrt{\pi}}{2} \text{erf}(u) = \int_0^u e^{-x^2} dx = \sum_{k=0}^\infty \frac{(-1)^k u^{2k+1}}{k!(2k+1)}$$

$$= e^{-u^2} \sum_{k=0}^\infty \frac{2^k u^{2k+1}}{(2k+1)!!}$$

(cf. 8.25) AD 6.700

$$2. \quad \int_0^u e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \Phi(qu) \quad [q > 0]$$

$$3. \quad \int_0^\infty e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \quad [q > 0] \quad \text{FI II 624}$$

$$4.^* \quad \int_0^u x e^{-q^2 x^2} dx = \frac{1}{2q^2} [1 - e^{-q^2 u^2}]$$

$$5.^* \quad \int_0^u x^2 e^{-q^2 x^2} dx = \frac{1}{2q^3} \left[\frac{\sqrt{\pi}}{2} \Phi(qu) - q u e^{-q^2 u^2} \right]$$

$$6.^* \quad \int_0^u x^3 e^{-q^2 x^2} dx = \frac{1}{2q^4} [1 - (1 + q^2 u^2) e^{-q^2 u^2}]$$

$$7.^* \quad \int_0^u x^4 e^{-q^2 x^2} dx = \frac{1}{2q^5} \left[\frac{3\sqrt{\pi}}{4} \Phi(qu) - \left(\frac{3}{2} + q^2 u^2 \right) q u e^{-q^2 u^2} \right]$$

3.322

$$1.^{11} \quad \int_u^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left[1 - \Phi\left(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}\right) \right]$$

[Re $\beta > 0$] ET I 146(21)

$$2. \quad \int_0^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} \exp(\beta\gamma^2) [1 - \Phi(\gamma\sqrt{\beta})]$$

[Re $\beta > 0$] NT 27(1)a

$$3.^{11} \quad \text{PV} \int_0^\infty e^{\pm i\lambda x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} e^{\pm \pi i/4} \quad [\lambda > 0] \quad \text{PBM 343 (2.3.15)(2)}$$

3.323

$$1.^{11} \quad \int_1^\infty \exp(-qx - x^2) dx = \frac{\sqrt{\pi}}{2} e^{q^2/4} \left[1 - \Phi \left(1 + \frac{1}{2}q \right) \right] \quad \text{BI (29)(4)}$$

$$2.^{10} \quad \int_{-\infty}^\infty \exp(-p^2 x^2 \pm qx) dx = \exp\left(\frac{q^2}{4p^2}\right) \frac{\sqrt{\pi}}{p} \quad [\text{Re } p^2 > 0] \quad \text{BI (28)(1)}$$

$$3.^{11} \quad \int_0^\infty \exp(-\beta^2 x^4 - 2\gamma^2 x^2) dx = 2^{-\frac{3}{2}} \frac{\gamma}{\beta} e^{\frac{\gamma^4}{2\beta^2}} K_{\frac{1}{4}} \left(\frac{\gamma^4}{2\beta^2} \right) \quad \left[|\arg \beta| < \frac{\pi}{4}, \quad |\arg \gamma| < \frac{\pi}{4} \right] \\ \text{ET I 147(34)a}$$

3.324

$$1. \quad \int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right) dx = \sqrt{\frac{\beta}{\gamma}} K_1(\sqrt{\beta\gamma}) \quad [\text{Re } \beta \geq 0, \quad \text{Re } \gamma > 0] \quad \text{ET I 146(25)}$$

$$2.^{11} \quad \int_{-\infty}^\infty \exp\left[-\left(x - \frac{b}{x}\right)^{2n}\right] dx = \frac{1}{n} \Gamma\left(\frac{1}{2n}\right) \quad [b \geq 0]$$

$$3.^{325} \quad \int_0^\infty \exp\left(-ax^2 - \frac{b}{x^2}\right) dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp(-2\sqrt{ab}) \quad [a > 0, \quad b > 0] \quad \text{FI II 644}$$

3.326

$$1.^8 \quad \int_0^\infty \exp(-x^\mu) dx = \frac{1}{\mu} \Gamma\left(\frac{1}{\mu}\right) \quad [\text{Re } \mu > 0] \quad \text{BI (26)(4)}$$

$$2.^{10} \quad \int_0^\infty x^m \exp(-\beta x^n) dx = \frac{\Gamma(\gamma)}{n\beta^\gamma} \quad \gamma = \frac{m+1}{n} \quad [\text{Re } \beta > 0, \quad \text{Re } m > 0, \quad \text{Re } n > 0]$$

$$3.^* \quad \int_0^\infty (x-a) \exp(-\beta(x-b)^n) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right)}{n\beta^{2/n}} - (a-b) \frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right)}{n\beta^{1/n}} \\ [\text{Re } n > 0, \quad \text{Re } \beta > 0, \quad |\arg b| < \pi]$$

$$4.^* \quad \int_0^u (x-a) \exp(-\beta(x-b)^n) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{2}{n}, \beta(u-b)^n\right)}{n\beta^{2/n}} \\ - (a-b) \frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{1}{n}, \beta(u-b)^n\right)}{n\beta^{1/n}} \\ [\text{Re } n > 0, \quad \text{Re } \beta > 0, \quad |\arg b| < \pi, \quad |\arg(u-b)| < \pi]$$

$$5.^* \quad \int_u^\infty (x-a) \exp(-\beta(x-b)^n) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right)}{n\beta^{2/n}} - (a-b) \frac{\Gamma\left(\frac{1}{n}, \beta(u-b)^n\right)}{n\beta^{1/n}} \\ [\text{Re } n > 0, \quad \text{Re } \beta > 0, \quad |\arg(u-b)| < \pi]$$

Exponentials of exponentials

$$\mathbf{3.327} \quad \int_0^\infty \exp(-ae^{nx}) dx = -\frac{1}{n} \operatorname{Ei}(-a) \quad [n \geq 1, \quad \operatorname{Re} a \geq 0, \quad a \neq 0] \quad \text{LI (26)(5)}$$

$$\mathbf{3.328} \quad \int_{-\infty}^\infty \exp(-e^x) e^{\mu x} dx = \Gamma(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{NH 145(14)}$$

$$\mathbf{3.329} \quad \int_0^\infty \left[\frac{a \exp(-ce^{ax})}{1 - e^{-ax}} - \frac{b \exp(-ce^{bx})}{1 - e^{-bx}} \right] dx = e^{-c} \ln \frac{b}{a} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (27)(12)}$$

3.331

$$1. \quad \int_0^\infty \exp(-\beta e^{-x} - \mu x) dx = \beta^{-\mu} \gamma(\mu, \beta) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 147(36)}$$

$$2. \quad \int_0^\infty \exp(-\beta e^x - \mu x) dx = \beta^\mu \Gamma(-\mu, \beta) \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 147(37)}$$

$$3.^{11} \quad \int_0^\infty (1 - e^{-x})^{\nu-1} \exp(\beta e^{-x} - \mu x) dx = \operatorname{B}(\mu, \nu) \beta^{-\frac{\mu+\nu}{2}} e^{\frac{\beta}{2}} M_{\frac{\nu-\mu}{2}, \frac{\nu+\mu-1}{2}}(\beta) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 147(38)}$$

$$4. \quad \int_0^\infty (1 - e^{-x})^{\nu-1} \exp(-\beta e^x - \mu x) dx = \Gamma(\nu) \beta^{\frac{\mu-1}{2}} e^{-\frac{\beta}{2}} W_{\frac{1-\mu-2\nu}{2}, \frac{-\mu}{2}}(\beta) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 147(39)}$$

$$\mathbf{3.332} \quad \int_0^\infty (1 - e^{-x})^{\nu-1} (1 - \lambda e^{-x})^{-\varrho} \exp(\beta e^{-x} - \mu x) dx = \operatorname{B}(\mu, \nu) \Phi_1(\mu, \varrho, \nu, \lambda, \beta) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad |\arg(1 - \lambda)| < \pi] \quad \text{ET I 147(40)}$$

3.333

$$1.^3 \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{\exp(e^{-x}) - 1} = \Gamma(\mu) \zeta(\mu) \quad [\operatorname{Re} \mu > 1] \quad \text{ET I 121(24)}$$

$$2.^3 \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{\exp(e^{-x}) + 1} = (1 - 2^{1-\mu}) \Gamma(\mu) \zeta(\mu) \quad [\operatorname{Re} \mu > 0, \quad \mu \neq 1] \\ = \ln 2 \quad [\mu = 1] \quad \text{ET I 121(25)}$$

$$3.^* \quad \int_0^\infty \left(\frac{\tanh(x)}{x^3} - \frac{1}{x^2 \cosh^2(x)} \right) dx = \frac{7 \zeta(3)}{\pi^2}$$

$$\mathbf{3.334}^{11} \quad \int_0^\infty (e^x - 1)^{\nu-1} \exp \left[-\frac{\beta}{e^x - 1} - \mu x \right] dx = \Gamma(\mu - \nu + 1) e^{\frac{\beta}{2}} \beta^{\frac{\nu-1}{2}} W_{\frac{\nu-2\mu-1}{2}, -\frac{\nu}{2}}(\beta) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu - 1] \quad \text{ET I 137(41)}$$

Exponentials of hyperbolic functions

$$\mathbf{3.335} \quad \int_0^\infty (e^{\nu x} + e^{-\nu x} \cos \nu \pi) \exp(-\beta \sinh x) dx = -\pi [\mathbf{E}_\nu(\beta) + Y_\nu(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{EH II 35(34)}$$

3.336

1.
$$\int_0^\infty \exp(-\nu x - \beta \sinh x) dx = \pi \operatorname{cosec} \nu \pi [\mathbf{J}_\nu(\beta) - J_\nu(\beta)]$$

$$\left[|\arg \beta| < \frac{\pi}{2} \text{ and } |\arg \beta| = \frac{\pi}{2} \text{ for } \operatorname{Re} \nu > 0; \quad \nu \text{ is not an integer} \right] \quad \text{WA 341(2)}$$
2.
$$\int_0^\infty \exp(nx - \beta \sinh x) dx = \frac{1}{2} [S_n(\beta) - \pi \mathbf{E}_n(\beta) - \pi Y_n(\beta)]$$

$$[\operatorname{Re} \beta > 0; \quad n = 0, 1, 2, \dots] \quad \text{WA 342(6)}$$
3.
$$\int_0^\infty \exp(-nx - \beta \sinh x) dx = \frac{1}{2} (-1)^{n+1} [S_n(\beta) + \pi \mathbf{E}_n(\beta) + \pi Y_n(\beta)]$$

$$[\operatorname{Re} \beta > 0; \quad n = 0, 1, 2, \dots] \quad \text{EH II 84(47)}$$

3.337

1.
$$\int_{-\infty}^\infty \exp(-\alpha x - \beta \cosh x) dx = 2 K_\alpha(\beta) \quad \left[|\arg \beta| < \frac{\pi}{2} \right] \quad \text{WA 201(7)}$$
2.
$$\int_{-\infty}^\infty \exp(-\nu x + i\beta \cosh x) dx = i\pi e^{\frac{i\nu\pi}{2}} H_\nu^{(1)}(\beta) \quad [0 < \arg z < \pi] \quad \text{EH II 21(27)}$$
3.
$$\int_{-\infty}^\infty \exp(-\nu x - i\beta \cosh x) dx = -i\pi e^{-\frac{i\nu\pi}{2}} H_\nu^{(2)}(\beta) \quad [-\pi < \arg z < 0] \quad \text{EH II 21(30)}$$

Exponentials of trigonometric functions and logarithms**3.338**

1.
$$\int_0^\pi \{ \exp i[(\nu - 1)x - \beta \sin x] - \exp i[(\nu + 1)x - \beta \sin x] \} dx = 2\pi [\mathbf{J}'_\nu(\beta) + i \mathbf{E}'_\nu(\beta)]$$

$$[\operatorname{Re} \beta > 0] \quad \text{EH II 36}$$
2.
$$\int_0^\pi \exp[\pm i(\nu x - \beta \sin x)] dx = \pi [\mathbf{J}_\nu(\beta) \pm i \mathbf{E}_\nu(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{EH II 35(32)}$$
- 3.¹⁰
$$\int_0^\infty \exp[-\gamma(x - \beta \sin x)] dx = \frac{1}{\gamma} + 2 \sum_{k=1}^\infty \frac{\gamma J_k(k\beta)}{\gamma^2 + k^2} \quad [\operatorname{Re} \gamma > 0] \quad \text{WA 619(4)}$$
- 4.⁶
$$\int_{-\pi}^\pi \frac{\exp \left[\frac{a + b \sin x + c \cos x}{1 + p \sin x + q \cos x} \right] dx}{1 + p \sin x + q \cos x} = \frac{2\pi}{\sqrt{1 - p^2 - q^2}} e^{-\alpha} I_0(\beta),$$

$$\text{with } \alpha = \frac{bp + cq - a}{1 - p^2 - q^2}; \quad \beta = \sqrt{\alpha^2 - \frac{a^2 - b^2 - c^2}{1 - p^2 - q^2}}; \quad [p^2 + q^2 < 1]$$
- 5.*
$$\int_0^{\pi/4} \exp \left[- \sum_{n=1}^\infty \frac{\tan^{2n} x}{n + \frac{1}{2}} \right] dx = \ln \sqrt{2}$$

$$3.339^6 \int_0^\pi \exp(z \cos x) dx = \pi I_0(z) \quad \text{BI (277)(2)a}$$

$$3.341 \int_0^{\frac{\pi}{2}} \exp(-p \tan x) dx = \text{ci}(p) \sin p - \text{si}(p) \cos(p) \quad [p > 0] \quad \text{BI (271)(2)a}$$

$$3.342^{11} \int_0^1 \exp(-px \ln x) dx = \int_0^1 x^{-px} dx = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k^k} \quad \text{BI (29)(1)}$$

3.35 Combinations of exponentials and rational functions

3.351

$$1.^8 \int_0^u x^n e^{-\mu x} dx = \frac{n!}{\mu^{n+1}} - e^{-u\mu} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1} \gamma(n+1, \mu u) \\ [u > 0, \quad \text{Re } \mu > 0, n = 0, 1, 2, \dots] \quad \text{ET I 134(5)}$$

$$2.^{11} \int_u^\infty x^n e^{-\mu x} dx = e^{-u\mu} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1} \Gamma(n+1, \mu u) \\ [u > 0, \quad \text{Re } \mu > 0, n = 0, 1, 2, \dots] \quad \text{ET I 33(4)}$$

$$3. \int_0^\infty x^n e^{-\mu x} dx = n! \mu^{-n-1} \quad [\text{Re } \mu > 0] \quad \text{ET I 133(3)}$$

$$4. \int_u^\infty \frac{e^{-px} dx}{x^{n+1}} = (-1)^{n+1} \frac{p^n \text{Ei}(-pu)}{n!} + \frac{e^{-pu}}{u^n} \sum_{k=0}^{n-1} \frac{(-1)^k p^k u^k}{n(n-1) \dots (n-k)} \\ [p > 0] \quad \text{NT 21(3)}$$

$$5. \int_1^\infty \frac{e^{-\mu x} dx}{x} = -\text{Ei}(-\mu) \quad [\text{Re } \mu > 0] \quad \text{BI (104)(10)}$$

$$6. \int_{-\infty}^u \frac{e^x}{x} dx = \text{li}(e^u) = \text{Ei}(u) \quad [u < 0]$$

$$7.^9 \int_0^u x e^{-\mu x} dx = \frac{1}{\mu^2} - \frac{1}{\mu^2} e^{-\mu u} (1 + \mu u) \quad [u > 0]$$

$$8.^{11} \int_0^u x^2 e^{-\mu x} dx = \frac{2}{\mu^3} - \frac{1}{\mu^3} e^{-\mu u} (2 + 2\mu u + \mu^2 u^2) \quad [u > 0]$$

$$9.^7 \int_0^u x^3 e^{-\mu x} dx = \frac{6}{\mu^4} - \frac{1}{\mu^4} e^{-\mu u} (6 + 6\mu u + 3\mu^2 u^2 + \mu^3 u^3) \\ [u > 0]$$

3.352

$$1. \int_0^u \frac{e^{-\mu x} dx}{x + \beta} = e^{\mu\beta} [\text{Ei}(-\mu u - \mu\beta) - \text{Ei}(-\mu\beta)] \quad [u \geq 0, \quad |\arg \beta| < \pi] \quad \text{ET II 217(12)}$$

$$2. \int_u^\infty \frac{e^{-\mu x} dx}{x + \beta} = -e^{\beta\mu} \text{Ei}(-\mu u - \mu\beta) \quad [u \geq 0, \quad |\arg(u + \beta)| < \pi, \quad \text{Re } \mu > 0] \\ \text{ET I 134(6), JA}$$

$$3. \quad \int_u^v \frac{e^{-\mu x} dx}{x + \alpha} = e^{\alpha\mu} \{ \text{Ei}[-(\alpha + v)\mu] - \text{Ei}[-(\alpha + u)\mu] \} \quad [-\alpha < n, \text{ and } -\alpha > v, \quad \text{Re } \mu > 0] \\ \text{ET I 134 (7)}$$

$$4. \quad \int_0^\infty \frac{e^{-\mu x} dx}{x + \beta} = -e^{\beta\mu} \text{Ei}(-\mu\beta) \quad [|\arg \beta| < \pi, \quad \text{Re } \mu > 0] \quad \text{ET II 217(11)}$$

$$5.^7 \quad \int_u^\infty \frac{e^{-px} dx}{a - x} = e^{-pa} \text{Ei}(pa - pu) \\ [p > 0, \quad a < u; \text{ for } a > u, \text{ one should replace } \text{Ei}(pa - pu) \text{ in this formula with } \overline{\text{Ei}}(pa - pu)] \\ \text{ET II 251(37)}$$

$$6.^8 \quad \int_0^\infty \frac{e^{-\mu x} dx}{a - x} = e^{-\mu a} \text{Ei}(a\mu) \\ [a < 0, \quad \text{Re } \mu > 0] \quad \text{BI (91)(4)}$$

$$7. \quad \int_{-\infty}^\infty \frac{e^{ipx} dx}{x - a} = i\pi e^{iap} \quad [p > 0] \quad \text{ET II 251(38)}$$

3.353

$$1. \quad \int_u^\infty \frac{e^{-\mu x} dx}{(x + \beta)^n} = e^{-u\mu} \sum_{k=1}^{n-1} \frac{(k-1)!(-\mu)^{n-k-1}}{(n-1)!(u + \beta)^k} - \frac{(-\mu)^{n-1}}{(n-1)!} e^{\beta\mu} \text{Ei}[-(u + \beta)\mu] \\ [n \geq 2, \quad |\arg(u + \beta)| < \pi, \quad \text{Re } \mu > 0] \\ \text{ET I 134(10)}$$

$$2.^7 \quad \int_0^\infty \frac{e^{-\mu x} dx}{(x + \beta)^n} = \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (k-1)!(-\mu)^{n-k-1} \beta^{-k} - \frac{(-\mu)^{n-1}}{(n-1)!} e^{\beta\mu} \text{Ei}(-\beta\mu) \\ [n \geq 2, \quad |\arg \beta| < \pi, \quad \text{Re } \mu > 0] \\ \text{ET I 134(9), BI (92)(2)}$$

$$3. \quad \int_0^\infty \frac{e^{-px} dx}{(a + x)^2} = pe^{\alpha p} \text{Ei}(-ap) + \frac{1}{a} \\ [p > 0, \quad a > 0] \\ \text{LI (281)(28), LI (281)(29)}$$

$$4. \quad \int_0^1 \frac{xe^x}{(1+x)^2} dx = \frac{e}{2} - 1. \quad \text{BI (80)(6)}$$

$$5.^7 \quad \int_0^\infty \frac{x^n e^{-\mu x} dx}{x + \beta} = (-1)^{n-1} \beta^n e^{\beta\mu} \text{Ei}(-\beta\mu) + \sum_{k=1}^n (k-1)!(-\beta)^{n-k} \mu^{-k} \\ [|\arg \beta| < \pi, \quad \text{Re } \mu > 0] \\ \text{BI (91)(3)a, LET I 135(11)}$$

3.354

$$1. \quad \int_0^\infty \frac{e^{-\mu x} dx}{\beta^2 + x^2} = \frac{1}{\beta} [\text{ci}(\beta\mu) \sin \beta\mu - \text{si}(\beta\mu) \cos \beta\mu] \quad [\text{Re } \beta > 0, \quad \text{Re } \mu > 0] \quad \text{BI (91)(7)}$$

$$2. \quad \int_0^\infty \frac{xe^{-\mu x} dx}{\beta^2 + x^2} = -\text{ci}(\beta\mu) \cos \beta\mu - \text{si}(\beta\mu) \sin \beta\mu \quad [\text{Re } \beta > 0, \quad \text{Re } \mu > 0] \quad \text{BI (91)(8)}$$

$$3.^7 \quad \int_0^\infty \frac{e^{-\mu x} dx}{\beta^2 - x^2} = \frac{1}{2\beta} [e^{-\beta\mu} \text{Ei}(\beta\mu) - e^{\beta\mu} \text{Ei}(-\beta\mu)] \quad [|\arg(\pm\beta)| < \pi, \quad \text{Re } \mu > 0] \quad \text{BI (91)(14)}$$

$$4. \quad \int_0^\infty \frac{x e^{-\mu x} dx}{\beta^2 - x^2} = \frac{1}{2} [e^{-\beta\mu} \operatorname{Ei}(\beta\mu) + e^{\beta\mu} \operatorname{Ei}(-\beta\mu)]$$

BI (91)(15)

[$|\arg(\pm\beta)| < \pi$, $\operatorname{Re} \mu > 0$; for $\beta > 0$ one should replace $\operatorname{Ei}(\beta\mu)$ in this formula with $\overline{\operatorname{Ei}}(\beta\mu)$]

$$5.^8 \quad \int_{-\infty}^\infty \frac{e^{-ipx} dx}{a^2 + x^2} = \frac{\pi}{a} e^{-|ap|} \quad [a \neq 0, \quad p \text{ real}] \quad \text{ET I 118(1)a}$$

3.355

$$1. \quad \int_0^\infty \frac{e^{-\mu x} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^3} \{ \operatorname{ci}(\beta\mu) \sin \beta\mu - \operatorname{si}(\beta\mu) \cos \beta\mu \} - \beta\mu [\operatorname{ci}(\beta\mu) \cos \beta\mu + \operatorname{si}(\beta\mu) \sin \beta\mu]$$

LI (92)(6)

$$2. \quad \int_0^\infty \frac{x e^{-\mu x} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^2} \{ -\beta\mu [\operatorname{ci}(\beta\mu) \sin \beta\mu - \operatorname{si}(\beta\mu) \cos \beta\mu] \}$$

[$\operatorname{Re} \beta > 0$, $\operatorname{Re} \mu > 0$] BI (92)(7)

$$3.^3 \quad \int_0^\infty \frac{e^{-px} dx}{(a^2 - x^2)^2} = \frac{1}{4a^3} [(ap - 1)e^{ap} \operatorname{Ei}(-ap) + (1 + ap)e^{-ap} \operatorname{Ei}(ap)]$$

[$\operatorname{Im}(a^2) > 0$, $p > 0$] BI (92)(8)

$$4.^3 \quad \int_0^\infty \frac{x e^{-px} dx}{(a^2 - x^2)^2} = \frac{1}{4a^2} \{ -2 + ap [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap)] \}$$

[$\operatorname{Im}(a^2) > 0$, $p > 0$] LI (92)(9)

3.356

$$1. \quad \int_0^\infty \frac{x^{2n+1} e^{-px} dx}{a^2 + x^2} = (-1)^{n-1} a^{2n} [\operatorname{ci}(ap) \cos ap + \operatorname{si}(ap) \sin ap]$$

$$+ \frac{1}{p^{2n}} \sum_{k=1}^n (2n - 2k + 1)! (-a^2 p^2)^{k-1}$$

[$p > 0$] BI (91)(12)

$$2. \quad \int_0^\infty \frac{x^{2n} e^{-px} dx}{a^2 + x^2} = (-1)^n a^{2n-1} [\operatorname{ci}(ap) \sin ap - \operatorname{si}(ap) \cos ap] + \frac{1}{p^{2n-1}} \sum_{k=1}^n (2n - 2k)! (-a^2 p^2)^{k-1}$$

[$p > 0$] BI (91)(11)

$$3. \quad \int_0^\infty \frac{x^{2n+1} e^{-px} dx}{a^2 - x^2} = \frac{1}{2} a^{2n} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap)] - \frac{1}{p^{2n}} \sum_{k=1}^n (2n - 2k + 1)! (a^2 p^2)^{k-1}$$

[$p > 0$] BI (91)(17)

$$4. \quad \int_0^\infty \frac{x^{2n} e^{-px} dx}{a^2 - x^2} = \frac{1}{2} a^{2n-1} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap)] - \frac{1}{p^{2n-1}} \sum_{k=1}^n (2n - 2k)! (a^2 p^2)^{k-1}$$

[$p > 0$] BI (91)(16)

3.357

$$1. \quad \int_0^\infty \frac{e^{-\mu x} dx}{a^3 + a^2 x + ax^2 + x^3} = \frac{1}{2a^2} \{ \text{ci}(a\mu) (\sin a\mu + \cos a\mu) \\ + \text{si}(a\mu) (\sin a\mu - \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(18)}$$

$$2. \quad \int_0^\infty \frac{x e^{-\mu x} dx}{a^3 + a^2 x + ax^2 + x^3} = \frac{1}{2a} \{ \text{ci}(a\mu) (\sin a\mu - \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu + \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(19)}$$

$$3. \quad \int_0^\infty \frac{x^2 e^{-\mu x} dx}{a^3 + a^2 x + ax^2 + x^3} = \frac{1}{2} \{ -\text{ci}(a\mu) (\sin a\mu + \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu - \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(20)}$$

$$4. \quad \int_0^\infty \frac{e^{-\mu x} dx}{a^3 - a^2 x + ax^2 - x^3} = \frac{1}{2a^2} \{ \text{ci}(a\mu) (\sin a\mu - \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu + \cos a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(21)}$$

$$5. \quad \int_0^\infty \frac{x e^{-\mu x} dx}{a^3 - a^2 x + ax^2 - x^3} = \frac{1}{2a} \{ -\text{ci}(a\mu) (\sin a\mu + \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu - \cos a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(22)}$$

$$6. \quad \int_0^\infty \frac{x^2 e^{-\mu x} dx}{a^3 - a^2 x + ax^2 - x^3} = \frac{1}{2} \{ \text{ci}(a\mu) (\cos a\mu - \sin a\mu) \\ + \text{si}(a\mu) (\cos a\mu + \sin a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(23)}$$

3.358

$$1. \quad \int_0^\infty \frac{e^{-px}}{a^4 - x^4} dx = \frac{1}{4a^3} \{ e^{-ap} \text{Ei}(ap) - e^{ap} \text{Ei}(-ap) + 2 \text{ci}(ap) \sin ap - 2 \text{si}(ap) \cos ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(18)}$$

$$2. \quad \int_0^\infty \frac{x e^{-px}}{a^4 - x^4} dx = \frac{1}{4a^2} \{ e^{ap} \text{Ei}(-ap) + e^{-ap} \text{Ei}(ap) - 2 \text{ci}(ap) \cos ap - 2 \text{si}(ap) \sin ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(19)}$$

$$3. \quad \int_0^\infty \frac{x^2 e^{-px}}{a^4 - x^4} dx = \frac{1}{4a} \{ e^{-ap} \text{Ei}(ap) - e^{ap} \text{Ei}(-ap) - 2 \text{ci}(ap) \sin ap + 2 \text{si}(ap) \cos ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(20)}$$

$$4. \quad \int_0^\infty \frac{x^3 e^{-px}}{a^4 - x^4} dx = \frac{1}{4} \{ e^{ap} \text{Ei}(-ap) + e^{-ap} \text{Ei}(ap) + 2 \text{ci}(ap) \cos ap + 2 \text{si}(ap) \sin ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(21)}$$

$$\begin{aligned}
 5. \quad \int_0^\infty \frac{x^{4n} e^{-px}}{a^4 - x^4} dx &= \frac{1}{4} a^{4n-3} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) + 2 \operatorname{ci}(ap) \sin ap - 2 \operatorname{si}(ap) \cos ap] \\
 &\quad - \frac{1}{p^{4n-3}} \sum_{k=1}^n (4n-4k)! (a^4 p^4)^{k-1} \\
 &\quad [p > 0, \quad a > 0] \quad \text{BI (91)(22)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_0^\infty \frac{x^{4n+1} e^{-px}}{a^4 - x^4} dx &= \frac{1}{4} a^{4n-2} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) - 2 \operatorname{ci}(ap) \cos ap - 2 \operatorname{si}(ap) \sin ap] \\
 &\quad - \frac{1}{p^{4n-2}} \sum_{k=1}^n (4n-4k+1)! (a^4 p^4)^{k-1} \\
 &\quad [p > 0, \quad a > 0] \quad \text{BI (91)(23)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \int_0^\infty \frac{x^{4n+2} e^{-px}}{a^4 - x^4} dx &= \frac{1}{4} a^{4n-1} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) - 2 \operatorname{ci}(ap) \sin ap + 2 \operatorname{si}(ap) \cos ap] \\
 &\quad - \frac{1}{p^{4n-1}} \sum_{k=1}^n (4n-4k+2)! (a^4 p^4)^{k-1} \\
 &\quad [p > 0, \quad a > 0] \quad \text{BI (91)(24)}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int_0^\infty \frac{x^{4n+3} e^{-px}}{a^4 - x^4} dx &= \frac{1}{4} a^{4n} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) + 2 \operatorname{ci}(ap) \cos ap + 2 \operatorname{si}(ap) \sin ap] \\
 &\quad - \frac{1}{p^{4n}} \sum_{k=1}^n (4n-4k+3)! (a^4 p^4)^{k-1} \\
 &\quad [p > 0, \quad a > 0] \quad \text{BI (91)(25)}
 \end{aligned}$$

$$\begin{aligned}
 3.359 \quad \int_{-\infty}^\infty \frac{(i-x)^n}{(i+x)^n} \frac{e^{-ipx}}{i+x^2} dx &= (-1)^{n-1} 2\pi p e^{-p} L_{n-1}(2p) \quad \text{for } p > 0; \\
 &= 0 \quad \text{for } p < 0.
 \end{aligned}$$

ET I 118(2)

3.36–3.37 Combinations of exponentials and algebraic functions

3.361

$$\begin{aligned}
 1.^8 \quad \int_0^u \frac{e^{-qx}}{\sqrt{x}} dx &= \sqrt{\frac{\pi}{q}} \Phi(\sqrt{qu}) \quad [q > 0] \\
 2.^8 \quad \int_0^\infty \frac{e^{-qx}}{\sqrt{x}} dx &= \sqrt{\frac{\pi}{q}} \quad [q > 0] \quad \text{BI(98)(10)} \\
 3.^8 \quad \int_{-1}^\infty \frac{e^{-qx}}{\sqrt{1+x}} dx &= e^q \sqrt{\frac{\pi}{q}} \quad [q > 0] \quad \text{BI (104)(16)}
 \end{aligned}$$

3.362

$$\begin{aligned}
 1. \quad \int_1^\infty \frac{e^{-\mu x}}{\sqrt{x-1}} dx &= \sqrt{\frac{\pi}{\mu}} e^{-\mu} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (104)(11)a} \\
 2. \quad \int_0^\infty \frac{e^{-\mu x}}{\sqrt{x+\beta}} dx &= \sqrt{\frac{\pi}{\mu}} e^{\beta\mu} [1 - \Phi(\sqrt{\beta\mu})] \quad [\operatorname{Re} \mu > 0, \quad |\arg \beta| < \pi] \quad \text{ET I 135(18)}
 \end{aligned}$$

3.363

$$1. \quad \int_u^\infty \frac{\sqrt{x-u}}{x} e^{-\mu x} dx = \sqrt{\frac{\pi}{\mu}} e^{-u\mu} - \pi\sqrt{u} [1 - \Phi(\sqrt{u\mu})] \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 136(23)}$$

$$2. \quad \int_u^\infty \frac{e^{-\mu x} dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}} [1 - \Phi(\sqrt{u\mu})] \quad [u > 0, \quad \operatorname{Re} \mu \geq 0] \quad \text{ET I 136(26)}$$

3.364

$$1. \quad \int_0^2 \frac{e^{-px} dx}{\sqrt{x(2-x)}} = \pi e^{-p} I_0(p) \quad [p > 0] \quad \text{GW (312)(7a)}$$

$$2. \quad \int_{-1}^1 \frac{e^{2x} dx}{\sqrt{1-x^2}} = \pi I_0(2) \quad \text{BI (277)(2)a}$$

$$3. \quad \int_0^\infty \frac{e^{-px} dx}{\sqrt{x(x+a)}} = e^{\frac{ap}{2}} K_0\left(\frac{ap}{2}\right) \quad [a > 0, \quad p > 0] \quad \text{GW (312)(8a)}$$

3.365

$$1. \quad \int_0^u \frac{xe^{-\mu x} dx}{\sqrt{u^2-x^2}} = \frac{\pi u}{2} [\mathbf{L}_1(\mu u) - I_1(\mu u)] + u \quad [u > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 136(28)}$$

$$2. \quad \int_u^\infty \frac{xe^{-\mu x} dx}{\sqrt{x^2-u^2}} = u K_1(u\mu) \quad [u > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 136(29)}$$

3.366

$$1. \quad \int_0^{2u} \frac{(u-x)e^{-\mu x} dx}{\sqrt{2ux-x^2}} = \pi u e^{-u\mu} I_1(u\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 136(31)}$$

$$2. \quad \int_0^\infty \frac{(x+\beta)e^{-\mu x} dx}{\sqrt{x^2+2\beta x}} = \beta e^{\beta\mu} K_1(\beta\mu) \quad [\operatorname{Re} \mu > 0, \quad |\arg \beta| < \pi] \quad \text{ET I 136(30)}$$

$$3. \quad \int_0^\infty \frac{xe^{-\mu x} dx}{\sqrt{x^2+\beta^2}} = \frac{\beta\pi}{2} [\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)] - \beta \quad \left[|\arg \beta| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0\right] \quad \text{ET I 136(27)}$$

$$\mathbf{3.367} \quad \int_0^\infty \frac{e^{-\mu x} dx}{(1+\cos t+x)\sqrt{x^2+2x}} = \frac{\exp\left(2\mu \cos^2 \frac{t}{2}\right)}{\sin t} \left(t - \sin t \int_0^u K_0(v) e^{-v \cos t} dv\right) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 136(33)}$$

$$\mathbf{3.368} \quad \int_0^\infty \frac{e^{-\mu x} dx}{x+\sqrt{x^2+\beta^2}} = \frac{\pi}{2\beta\mu} [\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)] - \frac{1}{\beta^2\mu^2} \quad \left[|\arg \beta| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0\right] \quad \text{ET I 136(32)}$$

$$\mathbf{3.369}^{11} \quad \int_0^\infty \frac{e^{-\mu x} dx}{\sqrt{(x+a)^3}} = \frac{2}{\sqrt{a}} - 2\sqrt{\pi\mu} e^{a\mu} (1 - \Phi(\sqrt{a\mu})) \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 135(20)}$$

$$\mathbf{3.371}^{11} \quad \int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x} dx = \sqrt{\pi} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n-1}{2} \mu^{-n-\frac{1}{2}} \\ = \sqrt{\pi} 2^{-n} \mu^{-n-1/2} (2n-1)!! \quad [n \geq 0] \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 135(17)}$$

$$3.372 \quad \int_0^\infty x^{n-\frac{1}{2}} (2+x)^{n-\frac{1}{2}} e^{-px} dx = \frac{(2n-1)!!}{p^n} e^p K_n(p) \quad [p > 0, \quad n = 0, 1, 2, \dots] \quad \text{GW (312)(8)}$$

$$3.373 \quad \int_0^\infty \left[\left(x + \sqrt{x^2 + \beta^2} \right)^n + \left(x - \sqrt{x^2 + \beta^2} \right)^n \right] e^{-\mu x} dx = 2\beta^{n+1} O_n(\beta\mu) \\ [\operatorname{Re} \mu > 0] \quad \text{WA 05(1)}$$

3.374

$$1. \quad \int_0^\infty \frac{(x + \sqrt{1+x^2})^n}{\sqrt{1+x^2}} e^{-\mu x} dx = \frac{1}{2} [S_n(\mu) - \pi \mathbf{E}_n(\mu) - \pi Y_n(\mu)] \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 37(35)}$$

$$2. \quad \int_0^\infty \frac{(x - \sqrt{1+x^2})^n}{\sqrt{1+x^2}} e^{-\mu x} dx = -\frac{1}{2} [S_n(\mu) + \pi \mathbf{E}_n(\mu) + \pi Y_n(\mu)] \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 137(36)}$$

3.38–3.39 Combinations of exponentials and arbitrary powers

3.381

$$1. \quad \int_0^u x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \gamma(\nu, \mu u) \quad [\operatorname{Re} \nu > 0] \quad \text{EH I 266(22), EH II 133(1)}$$

$$2. \quad \int_0^u x^{p-1} e^{-x} dx = \sum_{k=0}^{\infty} (-1)^k \frac{u^{p+k}}{k!(p+k)} \\ = e^{-u} \sum_{k=0}^{\infty} \frac{u^{p+k}}{p(p+1) \dots (p+k)} \\ \text{AD 6.705}$$

$$3.^8 \quad \int_u^\infty x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \Gamma(\nu, \mu u) \quad [u > 0, \quad \operatorname{Re} \mu > 0] \\ \text{EH I 256(21), EH II 133(2)}$$

$$4. \quad \int_0^\infty x^{\nu-1} e^{-\mu x} dx = \frac{1}{\mu^\nu} \Gamma(\nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 779}$$

$$5. \quad \int_0^\infty x^{\nu-1} e^{-(p+iq)x} dx = \Gamma(\nu) (p^2 + q^2)^{-\frac{\nu}{2}} \exp \left(-i\nu \arctan \frac{q}{p} \right) \\ [p > 0, \quad \operatorname{Re} \nu > 0 \text{ and } p = 0, \quad 0 < \operatorname{Re} \nu < 1] \quad \text{EH I 12(32)}$$

$$6. \quad \int_u^\infty \frac{e^{-x}}{x^\nu} dx = u^{-\frac{\nu}{2}} e^{-\frac{u}{2}} W_{-\frac{\nu}{2}, \frac{(1-\nu)}{2}}(u) \quad [u > 0] \quad \text{WH}$$

$$7. \quad \int_0^\infty x^{k-1} e^{i\mu x} dx = \frac{\Gamma(k)}{(-i\mu)^k} \quad [0 < \operatorname{Re}(k) < 1, \quad \mu \neq 0] \\ \text{GH2 62 (313.14)}$$

$$8.^* \quad \int_0^u x^m e^{-\beta x^n} dx = \frac{\gamma(v, \beta u^n)}{n\beta^v} \quad v = \frac{m+1}{n} \quad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0]$$

$$9.^* \quad \int_u^\infty x^m e^{-\beta x^n} dx = \frac{\Gamma(v, \beta u^n)}{n\beta^v} \quad v = \frac{m+1}{n} \quad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0]$$

$$10.* \quad \int_0^\infty x^m e^{-\beta x^n} dx = \frac{\gamma(v, \beta u^n) + \Gamma(v, \beta u^n)}{n\beta^v}$$

$$v = \frac{m+1}{n} \quad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0] \quad \text{See also 3.326 1}$$

$$11.* \quad \int_{-\infty}^\infty x^{2m} e^{-\beta x^{2n}} dx = 2 \int_0^\infty x^{2m} e^{-\beta x^{2n}} dx = \frac{2(\gamma(v, \beta u^n) + \Gamma(v, \beta u^n))}{n\beta^v} = \frac{\Gamma(v)}{n\beta^v}$$

$$v = \frac{2m+1}{2n} \quad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0]$$

3.382

$$1.^6 \quad \int_0^u (u-x)^\nu e^{-\mu x} dx = (-\mu)^{-\nu-1} e^{-u\mu} \gamma(\nu+1, -u\mu) \quad [\operatorname{Re} \nu > -1, \quad u > 0] \quad \text{ET I 137(6)}$$

$$2. \quad \int_u^\infty (x-u)^\nu e^{-\mu x} dx = \mu^{-\nu-1} e^{-u\mu} \Gamma(\nu+1) \quad [u > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > 0]$$

ET I 137(5), ET II 202(11)

$$3. \quad \int_0^\infty (1+x)^{-\nu} e^{-\mu x} dx = \mu^{\frac{\nu}{2}-1} e^{\frac{\mu}{2}} W_{-\frac{\nu}{2}, \frac{(1-\nu)}{2}}(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{WH}$$

$$4. \quad \int_0^\infty (x+\beta)^\nu e^{-\mu x} dx = \mu^{-\nu-1} e^{\beta\mu} \Gamma(\nu+1, \beta\mu) \quad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0]$$

ET I 137(4), ET II 233(10)

$$5. \quad \int_0^u (a+x)^{\mu-1} e^{-x} dx = e^a [\gamma(\mu, a+u) - \gamma(\mu, a)] \quad [\operatorname{Re} \mu > 0] \quad \text{EH II 139}$$

$$6. \quad \int_{-\infty}^\infty (\beta + ix)^{-\nu} e^{-ipx} dx = 0 \quad [\text{for } p > 0]$$

$$= \frac{2\pi(-p)^{\nu-1} e^{\beta p}}{\Gamma(\nu)} \quad [\text{for } p < 0]$$

[$\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > 0$] ET I 118(4)

$$7. \quad \int_{-\infty}^\infty (\beta - ix)^{-\nu} e^{-ipx} dx = \frac{2\pi p^{\nu-1} e^{-\beta p}}{\Gamma(\nu)} \quad [\text{for } p > 0]$$

$$= 0 \quad [\text{for } p < 0]$$

[$\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > 0$] ET I 118(3)

3.383

$$1.^{11} \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} e^{\beta x} dx = \operatorname{B}(\mu, \nu) u^{\mu+\nu-1} {}_1F_1(\nu; \mu+\nu; \beta u)$$

[$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$] ET II 187(14)

$$2.^{11} \quad \int_0^u x^{\mu-1} (u-x)^{\mu-1} e^{\beta x} dx = \sqrt{\pi} \left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \exp\left(\frac{\beta u}{2}\right) \Gamma(\mu) I_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$

[$\operatorname{Re} \mu > 0$] ET II 187(13)

$$3. \quad \int_u^\infty x^{\mu-1} (x-u)^{\mu-1} e^{-\beta x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) K_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$

[$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta u > 0$] ET II 202(12)

$$4.11 \quad \int_u^\infty x^{\nu-1} (x-u)^{\mu-1} e^{-\beta x} dx = \beta^{-\frac{\mu+\nu}{2}} u^{\frac{\mu+\nu-2}{2}} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) W_{\frac{\nu-\mu}{2}, \frac{1-\mu-\nu}{2}}(\beta u) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta u > 0] \quad \text{ET II 202(13)}$$

$$5.11 \quad \int_0^\infty e^{-px} x^{q-1} (1+ax)^{-\nu} dx \\ = \frac{\pi^2}{p^q \Gamma(\nu) \sin[\pi(q-\nu)]} \left[\left(\frac{p}{a}\right)^\nu \frac{L_{-\nu}^{q-\nu}\left(\frac{p}{a}\right)}{\sin(\pi\nu) \Gamma(1-q)} - \left(\frac{p}{a}\right)^q \frac{L_{-q}^{q-\nu}\left(\frac{p}{a}\right)}{\sin(\pi q) \Gamma(1-\nu)} \right] \quad [\nu \neq \pm 1, \pm 2, \dots] \\ = \frac{\Gamma(q)}{p^q} \quad [\nu = 0] \\ [\operatorname{Re} q > 0, \quad \operatorname{Re} p > 0, \quad \operatorname{Re} a > 0]$$

$$6. \quad \int_0^\infty x^{\nu-1} (x+\beta)^{-\nu+\frac{1}{2}} e^{-\mu x} dx = 2^{\nu-\frac{1}{2}} \Gamma(\nu) \mu^{-\frac{1}{2}} e^{\frac{\beta\mu}{2}} D_{1-2\nu}(\sqrt{2\beta\mu}) \\ [|\arg \beta| < \pi, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu \geq 0, \quad \mu \neq 0] \quad \text{ET I 39(20), EH II 119(2)a}$$

$$7. \quad \int_0^\infty x^{\nu-1} (x+\beta)^{-\nu-\frac{1}{2}} e^{-\mu x} dx = 2^\nu \Gamma(\nu) \beta^{-\frac{1}{2}} e^{\frac{\beta\mu}{2}} D_{-2\nu}(\sqrt{2\beta\mu}) \\ [|\arg \beta| < \pi, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu \geq 0] \\ \text{ET I 139(21), EH II 119(1)a}$$

$$8. \quad \int_0^\infty x^{\nu-1} (x+\beta)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\frac{\beta\mu}{2}} \Gamma(\nu) K_{\frac{1}{2}-\nu}\left(\frac{\beta\mu}{2}\right) \\ [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \\ \text{ET II 233(11), EH II 19(16)a, EH II 82(22)a}$$

$$9. \quad \int_u^\infty \frac{(x-u)^\nu e^{-\mu x}}{x} dx = u^\nu \Gamma(\nu+1) \Gamma(-\nu, u\mu) \quad [u > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > 0] \\ \text{ET I 138(8)}$$

$$10. \quad \int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{x+\beta} dx = \beta^{\nu-1} e^{\beta\mu} \Gamma(\nu) \Gamma(1-\nu, \beta\mu) \quad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \\ \text{EH II 137(3)}$$

3.384

$$1. \quad \int_{-1}^1 (1-x)^{\nu-1} (1+x)^{\mu-1} e^{-ipx} dx = 2^{\mu+\nu-1} \operatorname{B}(\mu, \nu) e^{ip} {}_1F_1(\mu; \nu + \mu; -2ip) \\ [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 119(13)}$$

$$2. \quad \int_u^v (x-u)^{2\mu-1} (v-x)^{2\nu-1} e^{-px} dx \\ = \operatorname{B}(2\mu, 2\nu) (v-u)^{\mu+\nu-1} p^{-\mu-\nu} \exp\left(-p\frac{u+v}{2}\right) M_{\mu-\nu, \mu+\nu-\frac{1}{2}}(vp-up) \\ [v > u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 139(23)}$$

$$\begin{aligned}
3. \quad \int_u^\infty (x + \beta)^{2\nu-1} (x - u)^{2\rho-1} e^{-\mu x} dx \\
= \frac{(u + \beta)^{\nu+\rho-1}}{\mu^{\nu+\rho}} \exp \left[\frac{(\beta - u)\mu}{2} \right] \Gamma(2\rho) W_{\nu-\rho, \nu+\rho-\frac{1}{2}}(u\mu + \beta\mu) \\
[u > 0, \quad |\arg(\beta + u)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \rho > 0] \quad \text{ET I 139(22)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_u^\infty (x + \beta)^\nu (x - u)^{-\nu} e^{-\mu x} dx = \frac{1}{\mu} \nu \pi \operatorname{cosec}(\nu \pi) e^{-\frac{(\beta+u)\mu}{2}} k_{2\nu} \left[\frac{(\beta + u)\mu}{2} \right] \\
[\nu \neq 0, \quad u > 0, \quad |\arg(u + \beta)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET I 139(17)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_u^\infty (x - u)^{\nu-1} (x + u)^{-\nu+\frac{1}{2}} e^{-\mu x} dx = \frac{1}{\sqrt{\mu}} 2^{\nu-\frac{1}{2}} \Gamma(\nu) D_{1-2\nu}(2\sqrt{u\mu}) \\
[u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \\
\text{ET I 139(18)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_u^\infty (x - u)^{\nu-1} (x + u)^{-\nu-\frac{1}{2}} e^{-\mu x} dx = \frac{1}{\sqrt{u}} 2^{\nu-\frac{1}{2}} \Gamma(\nu) D_{-2\nu}(2\sqrt{u\mu}) \\
[u > 0, \quad \operatorname{Re} \mu \geq 0, \quad \operatorname{Re} \nu > 0] \\
\text{ET I 139(19)}
\end{aligned}$$

$$\begin{aligned}
7.^6 \quad \int_{-\infty}^\infty (\beta - ix)^{-\mu} (\gamma - ix)^{-\nu} e^{-ipx} dx = \frac{2\pi e^{-\beta p} p^{\mu+\nu-1}}{\Gamma(\mu + \nu)} {}_1F_1(\nu; \mu + \nu; (\beta - \gamma)p) \quad [\text{for } p > 0] \\
= 0 \quad [\text{for } p < 0] \\
[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu + \nu) > 1] \quad \text{ET I 119(10)}
\end{aligned}$$

$$\begin{aligned}
8.^6 \quad \int_{-\infty}^\infty (\beta + ix)^{-\mu} (\gamma + ix)^{-\nu} e^{-ipx} dx = 0 \quad [\text{for } p > 0] \\
= \frac{2\pi e^{\gamma p} (-p)^{\mu+\nu-1}}{\Gamma(\mu + \nu)} {}_1F_1[\mu; \mu + \nu; (\beta - \gamma)p] \quad [\text{for } p < 0] \\
[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu + \nu) > 1] \quad \text{ET I 19(11)}
\end{aligned}$$

$$\begin{aligned}
9.^6 \quad \int_{-\infty}^\infty (\beta + ix)^{-2\mu} (\gamma - ix)^{-2\nu} e^{-ipx} dx \\
= 2\pi(\beta + \gamma)^{-\mu-\nu} \frac{p^{\mu+\nu-1}}{\Gamma(2\nu)} \exp \left(\frac{\beta - \gamma}{2} p \right) W_{\nu-\mu, \frac{1}{2}-\nu-\mu}(\beta p + \gamma p) \quad [\text{for } p > 0] \\
= 2\pi(\beta + \gamma)^{-\mu-\nu} \frac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)} \exp \left(\frac{\beta - \gamma}{2} p \right) W_{\mu-\nu, \frac{1}{2}-\nu-\mu}(-\beta p - \gamma p) \quad [\text{for } p < 0] \\
[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu + \nu) > \frac{1}{2}] \quad \text{ET I 19(12)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{3.385}^{11} \quad \int_0^1 x^{\nu-1} (1 - x)^{\lambda-1} (1 - \beta x)^{-\rho} e^{-\mu x} dx = B(\nu, \lambda) \Phi_1(\nu, \rho, \lambda + \nu, -\mu, \beta) \\
[\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > 0, \quad |\arg(1 - \beta)| < \pi] \quad \text{ET I 39(24)}
\end{aligned}$$

3.386

$$1. \quad \int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^n (\beta_k + ix)^{\nu_k} e^{-ipx} dx}{\beta_0 - ix} = 2\pi e^{-\beta_0 p} \beta_0^{\nu_0} \prod_{k=1}^n (\beta_0 + \beta_k)^{\nu_k} \\ \left[\operatorname{Re} \nu_0 > -1, \quad \operatorname{Re} \beta_k > 0, \quad \sum_{k=0}^n \operatorname{Re} \nu_k < 1, \quad \arg ix = \frac{\pi}{2} \operatorname{sign} x, \quad p > 0 \right] \quad \text{ET I 118(8)}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^n (\beta_k + ix)^{\nu_k} e^{-ipx} dx}{\beta_0 + ix} = 0 \\ \left[\operatorname{Re} \nu_0 > -1, \quad \operatorname{Re} \beta_k > 0, \quad \sum_{k=0}^n \operatorname{Re} \nu_k < 1, \quad \arg ix = \frac{\pi}{2} \operatorname{sign} x, \quad p > 0 \right] \quad \text{ET I 119(9)}$$

3.387

$$1.^6 \quad \int_{-1}^1 (1-x^2)^{\nu-1} e^{-\mu x} dx = \sqrt{\pi} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) I_{\nu-\frac{1}{2}}(\mu) \\ \left[\operatorname{Re} \nu > 0, \quad |\arg \mu| < \frac{\pi}{2} \right] \quad \text{WA 172(2)a}$$

$$2.^6 \quad \int_{-1}^1 (1-x^2)^{\nu-1} e^{i\mu x} dx = \sqrt{\pi} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) J_{\nu-\frac{1}{2}}(\mu) \\ [\operatorname{Re} \nu > 0] \quad \text{WA 25(3), WA 48(4)a}$$

$$3. \quad \int_1^{\infty} (x^2-1)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) K_{\nu-\frac{1}{2}}(\mu) \\ \left[|\arg \mu| < \frac{\pi}{2}, \quad \operatorname{Re} \nu > 0 \right] \quad \text{WA 190(4)a}$$

$$4. \quad \int_1^{\infty} (x^2-1)^{\nu-1} e^{i\mu x} dx \\ = i \frac{\sqrt{\pi}}{2} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) H_{\frac{1}{2}-\nu}^{(1)}(\mu) \quad [\operatorname{Im} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH II 83(28)a} \\ = -i \frac{\sqrt{\pi}}{2} \left(-\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) H_{\frac{1}{2}-\nu}^{(2)}(-\mu) \quad [\operatorname{Im} \mu < 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH II 83(29)a}$$

$$5. \quad \int_0^u (u^2-x^2)^{\nu-1} e^{\mu x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) \left[I_{\nu-\frac{1}{2}}(u\mu) + \mathbf{L}_{\nu-\frac{1}{2}}(u\mu) \right] \\ [u > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 188(20)a}$$

$$6. \quad \int_u^{\infty} (x^2-u^2)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) K_{\nu-\frac{1}{2}}(u\mu) \\ [u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \\ \text{ET II 203(17)a}$$

$$7.11 \quad \int_0^\infty (x^2 + u^2)^{\nu-1} e^{-\mu x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) \left[\mathbf{H}_{\nu-\frac{1}{2}}(u\mu) - Y_{\nu-\frac{1}{2}}(u\mu) \right] \\ [|\arg u| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 138(10)}$$

3.388

$$1. \quad \int_0^{2u} (2ux - x^2)^{\nu-1} e^{-\mu x} dx = \sqrt{\pi} \left(\frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} e^{-u\mu} \Gamma(\nu) I_{\nu-\frac{1}{2}}(u\mu) \\ [u > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 138(14)}$$

$$2. \quad \int_0^\infty (2\beta x + x^2)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2\beta}{\mu} \right)^{\nu-\frac{1}{2}} e^{\beta\mu} \Gamma(\nu) K_{\nu-\frac{1}{2}}(\beta\mu) \\ [|\arg \beta| < \pi, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0] \\ \text{ET I 138(13)}$$

$$3. \quad \int_0^\infty (x^2 + ix)^{\nu-1} e^{-\mu x} dx = -\frac{i\sqrt{\pi}e^{\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}} \Gamma(\nu) H_{\nu-\frac{1}{2}}^{(2)}\left(\frac{\mu}{2}\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 138(15)}$$

$$4. \quad \int_0^\infty (x^2 - ix)^{\nu-1} e^{-\mu x} dx = \frac{i\sqrt{\pi}e^{-\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}} \Gamma(\nu) H_{\nu-\frac{1}{2}}^{(1)}\left(\frac{\mu}{2}\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 138(16)}$$

3.389

$$1. \quad \int_0^u x^{2\nu-1} (u^2 - x^2)^{\varrho-1} e^{\mu x} dx = \frac{1}{2} \mathbf{B}(\nu, \varrho) u^{2\nu+2\varrho-2} {}_1F_2\left(\nu; \frac{1}{2}, \nu + \varrho; \frac{\mu^2 u^2}{4}\right) \\ + \frac{\mu}{2} \mathbf{B}\left(\nu + \frac{1}{2}, \varrho\right) u^{2\nu+2\varrho-1} {}_1F_2\left(\nu + \frac{1}{2}; \frac{3}{2}, \nu + \varrho + \frac{1}{2}; \frac{\mu^2 u^2}{4}\right) \\ [\operatorname{Re} \varrho > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 188(21)}$$

$$2.7 \quad \int_0^\infty x^{2\nu-1} (u^2 + x^2)^{\varrho-1} e^{-\mu x} dx = \frac{u^{2\nu+2\varrho-2}}{2\sqrt{\pi}\Gamma(1-\varrho)} G_{13}^{31}\left(\frac{\mu^2 u^2}{4} \left| \begin{matrix} 1-\nu \\ 1-\varrho-\nu, 0, \frac{1}{2} \end{matrix} \right. \right) \\ \left[|\arg u| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right] \\ \text{ET II 234(15)a}$$

$$3.7 \quad \int_0^u x (u^2 - x^2)^{\nu-1} e^{\mu x} dx = \frac{u^{2\nu}}{2\nu} + \frac{\sqrt{\pi}}{2} \left(\frac{\mu}{2} \right)^{\frac{1}{2}-\nu} u^{\nu+\frac{1}{2}} \Gamma(\nu) \left[I_{\nu+\frac{1}{2}}(\mu u) + \mathbf{L}_{\nu+\frac{1}{2}}(\mu u) \right] \\ [\operatorname{Re} \nu > 0] \quad \text{ET II 188(19)a}$$

$$4. \quad \int_u^\infty x (x^2 - u^2)^{\nu-1} e^{-\mu x} dx = 2^{\nu-\frac{1}{2}} (\sqrt{\pi})^{-1} \mu^{\frac{1}{2}-\nu} u^{\nu+\frac{1}{2}} \Gamma(\nu) K_{\nu+\frac{1}{2}}(u\mu) \\ [\operatorname{Re}(u\mu) > 0] \quad \text{ET II 203(16)a}$$

$$5. \quad \int_{-\infty}^\infty \frac{(ix)^{-\nu} e^{-ipx} dx}{\beta^2 + x^2} = \pi \beta^{-\nu-1} e^{-|p|\beta} \\ \left[|\nu| < 1, \quad \operatorname{Re} \beta > 0, \quad \arg ix = \frac{\pi}{2} \operatorname{sign} x \right] \quad \text{ET I 118(5)}$$

$$6. \quad \int_0^\infty \frac{x^\nu e^{-\mu x}}{\beta^2 + x^2} dx = \frac{1}{2} \Gamma(\nu) \beta^{\nu-1} \left[\exp \left(i\mu\beta + i\frac{(\nu-1)\pi}{2} \right) \right. \\ \left. \times \Gamma(1-\nu, i\beta\mu) + \exp \left(-i\beta\mu - i\frac{(\nu-1)\pi}{2} \right) \Gamma(1-\nu, -i\beta\mu) \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 218(22)}$$

$$7. \quad \int_0^\infty \frac{x^{\nu-1} e^{-\mu x} dx}{1+x^2} = \pi \operatorname{cosec}(\nu\pi) V_\nu(2\mu, 0) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 138(9)}$$

$$8. \quad \int_{-\infty}^\infty \frac{(\beta + ix)^{-\nu} e^{-ipx}}{\gamma^2 + x^2} dx = \frac{\pi}{\gamma} (\beta + \gamma)^{-\nu} e^{-p\gamma} \\ [\operatorname{Re} \nu > -1, \quad p > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{ET I 118(6)}$$

$$9.^6 \quad \int_{-\infty}^\infty \frac{(\beta - ix)^{-\nu} e^{-ipx}}{\gamma^2 + x^2} dx = \frac{\pi}{\gamma} (\beta + \gamma)^{-\nu} e^{\gamma p} \\ [p < 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 118(7)}$$

$$3.391 \quad \int_0^\infty \left[\left(\sqrt{x+2\beta} + \sqrt{x} \right)^{2\nu} - \left(\sqrt{x+2\beta} - \sqrt{x} \right)^{2\nu} \right] e^{-\mu x} dx = 2^{\nu+1} \frac{\nu}{\mu} \beta^\nu e^{\beta\mu} K_\nu(\beta\mu) \\ [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 140(30)}$$

3.392

$$1. \quad \int_0^\infty \left(x + \sqrt{1+x^2} \right)^\nu e^{-\mu x} dx = \frac{1}{\mu} S_{1,\nu}(\mu) + \frac{\nu}{\mu} S_{0,\nu}(\mu) \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 140(25)}$$

$$2. \quad \int_0^\infty \left(\sqrt{1+x^2} - x \right)^\nu e^{-\mu x} dx = \frac{1}{\mu} S_{1,\nu}(\mu) - \frac{\nu}{\mu} S_{0,\nu}(\mu) \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 140(26)}$$

$$3. \quad \int_0^\infty \frac{(x + \sqrt{1+x^2})^\nu}{\sqrt{1+x^2}} e^{-\mu x} dx = \pi \operatorname{cosec} \nu\pi [\mathbf{J}_{-\nu}(\mu) - J_{-\nu}(\mu)] \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 140(27), EH II 35(33)}$$

$$4. \quad \int_0^\infty \frac{(\sqrt{1+x^2} - x)^\nu}{\sqrt{1+x^2}} e^{-\mu x} dx = S_{0,\nu}(\mu) - \nu S_{-1,\nu}(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 140(28)}$$

$$3.393 \quad \int_0^\infty \frac{(x + \sqrt{x^2 + 4\beta^2})^{2\nu}}{\sqrt{x^3 + 4\beta^2 x}} e^{-\mu x} dx \\ = \frac{\sqrt{\mu\pi^3}}{2^{2\nu+3/2} \beta^{2\nu}} [J_{\nu+1/4}(\beta\mu) Y_{\nu-1/4}(\beta\mu) - J_{\nu-1/4}(\beta\mu) Y_{\nu+1/4}(\beta\mu)] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 140(33)}$$

$$3.394 \quad \int_0^\infty \frac{(1 + \sqrt{1+x^2})^{\nu+1/2}}{x^{\nu+1} \sqrt{1+x^2}} e^{-\mu x} dx = \sqrt{2} \Gamma(-\nu) D_\nu(\sqrt{2i\mu}) D_\nu(\sqrt{-2i\mu}) \\ [\operatorname{Re} \mu \geq 0, \quad \operatorname{Re} \nu < 0] \quad \text{ET I 140(32)}$$

3.395

$$1. \quad \int_1^\infty \frac{(\sqrt{x^2-1}+x)^\nu + (\sqrt{x^2-1}+x)^{-\nu}}{\sqrt{x^2-1}} e^{-\mu x} dx = 2 K_\nu(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 140(29)}$$

$$2. \quad \int_1^\infty \frac{(x+\sqrt{x^2-1})^{2\nu} + (x-\sqrt{x^2-1})^{2\nu}}{\sqrt{x(x^2-1)}} e^{-\mu x} dx = \sqrt{\frac{2\mu}{\pi}} K_{\nu+1/4}\left(\frac{\mu}{2}\right) K_{\nu-1/4}\left(\frac{\mu}{2}\right) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 140(34)}$$

$$3. \quad \int_0^\infty \frac{(x+\sqrt{x^2+1})^\nu + \cos \nu \pi (x+\sqrt{x^2+1})^{-\nu}}{\sqrt{x^2+1}} e^{-\mu x} dx = -\pi [\mathbf{E}_\nu(\mu) + Y_\nu(\mu)] \quad [\operatorname{Re} \mu > 0] \quad \text{EH II 35(34)}$$

3.41–3.44 Combinations of rational functions of powers and exponentials

3.411

$$1. \quad \int_0^\infty \frac{x^{\nu-1} dx}{e^{\mu x} - 1} = \frac{1}{\mu^\nu} \Gamma(\nu) \zeta(\nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 1] \quad \text{FI II 792a}$$

$$2. \quad \int_0^\infty \frac{x^{2n-1} dx}{e^{px} - 1} = (-1)^{n-1} \left(\frac{2\pi}{p}\right)^{2n} \frac{B_{2n}}{4n} \quad [n = 1, 2, \dots] \quad \text{FI II 721a}$$

$$3. \quad \int_0^\infty \frac{x^{\nu-1} dx}{e^{\mu x} + 1} = \frac{1}{\mu^\nu} (1 - 2^{1-\nu}) \Gamma(\nu) \zeta(\nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 792a, WH}$$

$$4. \quad \int_0^\infty \frac{x^{2n-1} dx}{e^{px} + 1} = (1 - 2^{1-2n}) \left(\frac{2\pi}{p}\right)^{2n} \frac{|B_{2n}|}{4n} \quad [n = 1, 2, \dots] \quad \text{BI(83)(2), EH I 39(25)}$$

$$5. \quad \int_0^{\ln 2} \frac{x dx}{1 - e^{-x}} = \frac{\pi^2}{12} \quad \text{BI (104)(5)}$$

$$6.^8 \quad \int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{1 - \beta e^{-x}} dx = \Gamma(\nu) \sum_{n=0}^\infty (\mu + n)^{-\nu} \beta^n = \Gamma(\nu) \Phi(\beta, \nu, \mu) \quad [\operatorname{Re} \mu > 0 \text{ and either } |\beta| \leq 1, \quad \beta \neq 1, \quad \operatorname{Re} \nu > 0; \text{ or } \beta = 1, \quad \operatorname{Re} \nu > 1] \quad \text{EH I 27(3)}$$

$$7.^{11} \quad \int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{1 - e^{-\beta x}} dx = \frac{1}{\beta^\nu} \Gamma(\nu) \zeta\left(\nu, \frac{\mu}{\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 1] \quad \text{ET I 144(10)}$$

$$8. \quad \int_0^\infty \frac{x^{n-1} e^{-px}}{1 + e^x} dx = (n-1)! \sum_{k=1}^\infty \frac{(-1)^{k-1}}{(p+k)^n} \quad [p > -1; \quad n = 1, 2, \dots] \quad \text{BI (83)(9)}$$

$$9. \quad \int_0^\infty \frac{x e^{-x} dx}{e^x - 1} = \frac{\pi^2}{6} - 1 \quad (\text{cf. 4.231 3}) \quad \text{BI (82)(1)}$$

$$10. \quad \int_0^\infty \frac{x e^{-2x} dx}{e^{-x} + 1} = 1 - \frac{\pi^2}{12} \quad (\text{cf. 4.251 6}) \quad \text{BI (82)(2)}$$

11. $\int_0^\infty \frac{x e^{-3x}}{e^{-x} + 1} dx = \frac{\pi^2}{12} - \frac{3}{4}$ (cf. **4.251** 5) BI (82)(3)
- 12.¹¹ $\int_0^\infty \frac{x e^{-(2n-1)x}}{1 + e^x} dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k^2}$ (cf. **4.251** 6) BI (82)(5)
- 13.¹¹ $\int_0^\infty \frac{x e^{-2nx}}{1 + e^x} dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2}$ (cf. **4.251** 5) BI (82)(4)
- 14.⁷ $\int_0^\infty \frac{x^2 e^{-nx}}{1 - e^{-x}} dx = 2 \sum_{k=n}^\infty \frac{1}{k^3} = 2 \left(\zeta(3) - \sum_{k=1}^{n-1} \frac{1}{k^3} \right)$ [$n = 1, 2, \dots$] (cf. **4.261** 12)
BI (82)(9)
- 15.⁷ $\int_0^\infty \frac{x^2 e^{-nx}}{1 + e^{-x}} dx = 2 \sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^3} = (-1)^{n+1} \left(\frac{3}{2} \zeta(3) + 2 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^3} \right)$
[$n = 1, 2, \dots$] (cf. **4.261** 11)
LI (82)(10)
16. $\int_{-\infty}^\infty \frac{x^2 e^{-\mu x}}{1 + e^{-x}} dx = \pi^3 \csc^3 \mu \pi (2 - \sin^2 \mu \pi)$ [$0 < \operatorname{Re} \mu < 1$] ET I 120(17)a
17. $\int_0^\infty \frac{x^3 e^{-nx}}{1 - e^{-x}} dx = \frac{\pi^4}{15} - 6 \sum_{k=1}^{n-1} \frac{1}{k^4}$ (cf. **4.262** 5) BI (82)(12)
- 18.¹¹ $\int_0^\infty \frac{x^3 e^{-nx}}{1 + e^{-x}} dx = 6 \sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^4} = (-1)^{n+1} \left(\frac{7}{120} \pi^4 + 6 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^4} \right)$
(cf. **4.262** 4) LI (82)(13)
- 19.⁹ $\int_0^\infty e^{-px} (e^{-x} - 1)^n \frac{dx}{x} = - \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(p + n - k)$ LI (89)(10)
- 20.⁹ $\int_0^\infty e^{-px} (e^{-x} - 1)^n \frac{dx}{x^2} = \sum_{k=0}^n (-1)^k \binom{n}{k} (p + n - k) \ln(p + n - k)$ LI (89)(15)
21. $\int_0^\infty x^{n-1} \frac{1 - e^{-mx}}{1 - e^{-x}} dx = (n-1)! \sum_{k=1}^m \frac{1}{k^n}$ (cf. **4.272** 11) LI (83)(8)
- 22.⁷ $\int_0^\infty \frac{x^{p-1}}{e^{rx} - q} dx = \frac{1}{qr^p} \Gamma(p) \sum_{k=1}^\infty \frac{q^k}{k^p} = \Gamma(p) r^{-p} \Phi(q, p, 1)$
[$p > 0, \quad r > 0, \quad -1 < q < 1$]
BI (83)(5)
23. $\int_{-\infty}^\infty \frac{x e^{\mu x}}{\beta + e^x} dx = \pi \beta^{\mu-1} \operatorname{cosec}(\mu \pi) [\ln \beta - \pi \cot(\mu \pi)]$ [$|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < 1$]
BI (101)(5), ET I 120(16)a
24. $\int_{-\infty}^\infty \frac{x e^{\mu x}}{e^{\nu x} - 1} dx = \left(\frac{\pi}{\nu} \operatorname{cosec} \frac{\mu \pi}{\nu} \right)^2$ [$\operatorname{Re} \nu > \operatorname{Re} \mu > 0$] (cf. **4.254** 2)
LI (101)(3)

25. $\int_0^\infty x \frac{1+e^{-x}}{e^x-1} dx = \frac{\pi^2}{3} - 1$ (cf. **4.231** 4) BI (82)(6)
26. $\int_0^\infty x \frac{1-e^{-x}}{1+e^{-3x}} e^{-x} dx = \frac{2\pi^2}{27}$ LI (82)(7)a
27. $\int_0^\infty \frac{1-e^{-\mu x}}{1+e^x} \frac{dx}{x} = \ln \left[\frac{\Gamma(\frac{\mu}{2}+1)}{\Gamma(\frac{\mu+1}{2})} \sqrt{\pi} \right]$ $[\operatorname{Re} \mu > -1]$ BI (93)(4)
28. $\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{e^{-x} + 1} \frac{dx}{x} = \ln \frac{\Gamma(\frac{\nu}{2}) \Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu}{2}) \Gamma(\frac{\nu+1}{2})}$ $[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0]$ BI (93)(6)
29. $\int_{-\infty}^\infty \frac{e^{px} - e^{qx}}{1+e^{rx}} \frac{dx}{x} = \ln \left[\tan \frac{p\pi}{2r} \cot \frac{q\pi}{2r} \right]$ $[|r| > |p|, |r| > |q|, rp > 0, rq > 0]$ BI (103)(3)
30. $\int_{-\infty}^\infty \frac{e^{px} - e^{qx}}{1-e^{rx}} \frac{dx}{x} = \ln \left[\sin \frac{p\pi}{r} \operatorname{cosec} \frac{q\pi}{r} \right]$ $[|r| > |p|, |r| > |q|, rp > 0, rq > 0]$ BI (103)(4)
31. $\int_0^\infty \frac{e^{-qx} + e^{(q-p)x}}{1-e^{-px}} x dx = \left(\frac{\pi}{p} \operatorname{cosec} \frac{q\pi}{p} \right)^2$ $[0 < q < p]$ BI (82)(8)
32. $\int_0^\infty \frac{e^{-px} - e^{(p-q)x}}{e^{-qx} + 1} \frac{dx}{x} = \ln \cot \frac{p\pi}{2q}$ $[0 < p < q]$ BI (93)(7)
- 3.412** $\int_0^\infty \left\{ \frac{a+be^{-px}}{ce^{px}+g+he^{-px}} - \frac{a+be^{-qx}}{ce^{qx}+g+he^{-qx}} \right\} \frac{dx}{x} = \frac{a+b}{c+g+h} \ln \frac{p}{q}$ $[p > 0, q > 0]$ BI (96)(7)
- 3.413**
1. $\int_0^\infty \frac{(1-e^{-\beta x})(1-e^{-\gamma x})e^{-\mu x}}{1-e^{-x}} \frac{dx}{x} = \ln \frac{\Gamma(\mu)\Gamma(\beta+\gamma+\mu)}{\Gamma(\mu+\beta)\Gamma(\mu+\gamma)}$ $[\operatorname{Re} \mu > 0, \operatorname{Re} \mu > -\operatorname{Re} \beta, \operatorname{Re} \mu > -\operatorname{Re} \gamma, \operatorname{Re} \mu > -\operatorname{Re}(\beta+\gamma)]$ (cf. **4.267** 25) BI (93)(13)
2. $\int_0^\infty \frac{\{1-e^{(q-p)x}\}^2}{e^{qx}-e^{(q-2p)x}} \frac{dx}{x} = \ln \operatorname{cosec} \frac{q\pi}{2p}$ $[0 < q < p]$ BI (95)(6)
3. $\int_0^\infty \frac{e^{-px}-e^{-qx}}{1+e^{-x}} \frac{1+e^{-(2n+1)x}}{x} dx$
 $= \ln \left\{ \frac{q(q+2)(q+4)\cdots(q+2n)(p+1)(p+3)\cdots(p+2n-1)}{p(p+2)(p+4)\cdots(p+2n)(q+1)(q+3)\cdots(q+2n-1)} \right\}$
 $[\operatorname{Re} p > -2n, \operatorname{Re} q > -2n]$ (cf. **4.267** 14) BI (93)(11)
- 3.414** $\int_0^\infty \frac{(1-e^{-\beta x})(1-e^{-\gamma x})(1-e^{-\delta x})e^{-\mu x}}{1-e^{-x}} \frac{dx}{x} = \ln \frac{\Gamma(\mu)\Gamma(\mu+\beta+\gamma)\Gamma(\mu+\beta+\delta)\Gamma(\mu+\gamma+\delta)}{\Gamma(\mu+\beta)\Gamma(\mu+\gamma)\Gamma(\mu+\delta)\Gamma(\mu+\beta+\gamma+\delta)}$
 $[2\operatorname{Re} \mu > |\operatorname{Re} \beta| + |\operatorname{Re} \gamma| + |\operatorname{Re} \delta|]$ (cf. **4.267** 31) BI (93)(14), ET I 145(17)

3.415

$$1. \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)(e^{\mu x} - 1)} = \frac{1}{2} \left[\ln \left(\frac{\beta \mu}{2\pi} \right) - \frac{\pi}{\beta \mu} - \psi \left(\frac{\beta \mu}{2\pi} \right) \right] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0]$$

BI (97)(20), EH I 18(27)

$$2.^{11} \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)^2 (e^{2\pi x} - 1)} = -\frac{1}{8\beta^3} - \frac{1}{4\beta^2} + \frac{1}{4\beta} \psi'(\beta) \\ \sim \frac{1}{4\beta^4} \sum_{k=0}^\infty \frac{|B_{2k+2}|}{\beta^{2k}} \\ \text{[asymptotic expansion for } \operatorname{Re} \beta > 0 \text{]} \quad \text{BI(97)(22), EH I 22(12)}$$

$$3.^{11} \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)(e^{\mu x} + 1)} = \frac{1}{2} \left[\psi \left(\frac{\beta \mu}{2\pi} + \frac{1}{2} \right) - \ln \left(\frac{\beta \mu}{2\pi} \right) \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0]$$

$$4.^8 \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)^2 (e^{2\pi x} + 1)} = \frac{1}{4\beta^2} - \frac{1}{4\beta} \psi' \left(\beta + \frac{1}{2} \right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0]$$

3.416

$$1. \quad \int_0^\infty \frac{(1+ix)^{2n} - (1-ix)^{2n}}{i} \frac{dx}{e^{2\pi x} - 1} = \frac{1}{2} \frac{2n-1}{2n+1} \quad [n = 1, 2, \dots] \quad \text{BI (88)(4)}$$

$$2. \quad \int_0^\infty \frac{(1+ix)^{2n} - (1-ix)^{2n}}{i} \frac{dx}{e^{\pi x} + 1} = \frac{1}{2n+1} \quad [n = 1, 2, \dots] \quad \text{BI (87)(1)}$$

$$3.^8 \quad \int_0^\infty \frac{(1+ix)^{2n-1} - (1-ix)^{2n-1}}{i} \frac{dx}{e^{\pi x} + 1} = \frac{1}{2n} \left[1 - 2^{2^n} B_{2n} \right] \\ [n = 1, 2, \dots] \quad \text{BI (87)(2)}$$

3.417

$$1. \quad \int_{-\infty}^\infty \frac{x \, dx}{a^2 e^x + b^2 e^{-x}} = \frac{\pi}{2ab} \ln \frac{b}{a} \quad [ab > 0] \quad (\text{cf. 4.231 8}) \quad \text{BI (101)(1)}$$

$$2. \quad \int_{-\infty}^\infty \frac{x \, dx}{a^2 e^x - b^2 e^{-x}} = \frac{\pi^2}{4ab} \quad (\text{cf. 4.231 10}) \quad \text{LI (101)(2)}$$

3.418

$$1.^6 \quad \int_0^\infty \frac{x \, dx}{e^x + e^{-x} - 1} = \frac{1}{3} \left[\psi' \left(\frac{1}{3} \right) - \frac{2}{3} \pi^2 \right] = 1.1719536193 \dots \quad \text{LI (88)(1)}$$

$$2.^6 \quad \int_0^\infty \frac{x e^{-x} \, dx}{e^x + e^{-x} - 1} = \frac{1}{6} \left[\psi' \left(\frac{1}{3} \right) - \frac{5}{6} \pi^2 \right] = 0.3118211319 \dots \quad \text{LI (88)(2)}$$

$$3. \quad \int_0^{\ln 2} \frac{x \, dx}{e^x + 2e^{-x} - 2} = \frac{\pi}{8} \ln 2 \quad \text{BI (104)(7)}$$

3.419

$$1. \quad \int_{-\infty}^{\infty} \frac{x \, dx}{(\beta + e^x)(1 + e^{-x})} = \frac{(\ln \beta)^2}{2(\beta - 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. } \mathbf{4.232} \, 2) \quad \text{BI (101)(16)}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{x \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{\pi^2 + (\ln \beta)^2}{2(\beta + 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. } \mathbf{4.232} \, 3) \quad \text{BI (101)(17)}$$

$$3. \quad \int_{-\infty}^{\infty} \frac{x^2 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2] \ln \beta}{3(\beta + 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. } \mathbf{4.261} \, 4) \quad \text{BI (102)(6)}$$

$$4. \quad \int_{-\infty}^{\infty} \frac{x^3 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{4(\beta + 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. } \mathbf{4.262} \, 3) \quad \text{BI (102)(9)}$$

$$5. \quad \int_{-\infty}^{\infty} \frac{x^4 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{15(\beta + 1)} [7\pi^2 + 3(\ln \beta)^2] \ln \beta \quad (\text{cf. } \mathbf{4.263} \, 1) \quad \text{BI (102)(10)}$$

$$6.^{11} \quad \int_{-\infty}^{\infty} \frac{x^5 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{6(\beta + 1)} [3\pi^2 + (\ln \beta)^2] \quad (\text{cf. } \mathbf{4.264} \, 3) \quad \text{BI (102)(11)}$$

$$7. \quad \int_{-\infty}^{\infty} \frac{(x - \ln \beta) x \, dx}{(\beta - e^x)(1 - e^{-x})} = \frac{-[4\pi^2 + (\ln \beta)^2] \ln \beta}{6(\beta - 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. } \mathbf{4.257} \, 4) \quad \text{BI (102)(7)}$$

3.421

$$1. \quad \int_0^{\infty} (e^{-\nu x} - 1)^n (e^{-\rho x} - 1)^m e^{-\mu x} \frac{dx}{x^2} \\ = \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{l=0}^m (-1)^l \binom{m}{l} \\ \times \{(m-l)\rho + (n-k)\nu + \mu\} \ln [(m-l)\rho + (n-k)\nu + \mu] \\ [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \rho > 0] \quad \text{BI (89)(17)}$$

$$2. \quad \int_0^{\infty} (1 - e^{-\nu x})^n (1 - e^{-\rho x}) e^{-x} \frac{dx}{x^3} = \frac{1}{2} \sum_{k=0}^n (-1)^k \binom{n}{k} (\rho + k\nu + 1)^2 \\ \times \ln(\rho + k\nu + 1) + \frac{1}{2} \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (k\nu + 1)^2 \ln(k\nu + 1) \\ [n \geq 2, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \rho > 0] \quad \text{BI (89)(31)}$$

$$3. \quad \int_{-\infty}^{\infty} \frac{x e^{-\mu x} dx}{(\beta + e^{-x})(\gamma + e^{-x})} = \frac{\pi (\beta^{\mu-1} \ln \beta - \gamma^{\mu-1} \ln \gamma)}{(\beta - \gamma) \sin \mu \pi} + \frac{\pi^2 (\beta^{\mu-1} - \gamma^{\mu-1}) \cos \mu \pi}{(\gamma - \beta) \sin^2 \mu \pi}$$

$$[|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \beta \neq \gamma, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 120(19)}$$

$$4. \quad \int_0^{\infty} (e^{-px} - e^{-qx}) (e^{-rx} - e^{-sx}) e^{-x} \frac{dx}{x} = \ln \frac{(p+s+1)(q+r+1)}{(p+r+1)(q+s+1)}$$

$$[p+s > -1, \quad p+r > -1, \quad q > p] \quad (\text{cf. 4.267 24}) \quad \text{BI (89)(11)}$$

$$5. \quad \int_0^{\infty} (1 - e^{-px}) (1 - e^{-qx}) (1 - e^{-rx}) e^{-x} \frac{dx}{x}$$

$$= (p+q+1) \ln(p+q+1)$$

$$+ (p+r+1) \ln(p+r+1) + (q+r+1) \ln(q+r+1)$$

$$- (p+1) \ln(p+1) - (q+1) \ln(q+1) - (r+1) \ln(r+1)$$

$$- (p+q+r) \ln(p+q+r)$$

$$[p > 0, \quad q > 0, \quad r > 0] \quad (\text{cf. 4.268 3}) \quad \text{BI (89)(14)}$$

$$3.422 \quad \int_{-\infty}^{\infty} \frac{x(x-a)e^{\mu x} dx}{(\beta - e^x)(1 - e^{-x})} = \frac{-\pi^2}{e^a - 1} \operatorname{cosec}^2 \mu \pi [(e^{\alpha \mu} + 1) \ln \mu - 2\pi \cot \mu \pi (e^{\alpha \mu} - 1)]$$

$$[a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \mu| < 1] \quad (\text{cf. 4.257 5}) \quad \text{BI (102)(8)a}$$

3.423

$$1. \quad \int_0^{\infty} \frac{x^{\nu-1}}{(e^x - 1)^2} dx = \Gamma(\nu) [\zeta(\nu - 1) - \zeta(\nu)] \quad [\operatorname{Re} \nu > 2] \quad \text{ET I 313(10)}$$

$$2.^6 \quad \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{(e^x - 1)^2} dx = \Gamma(\nu) [\zeta(\nu - 1, \mu + 2) - (\mu + 1) \zeta(\nu, \mu + 2)]$$

$$[\operatorname{Re} \mu > -2, \quad \operatorname{Re} \nu > 2] \quad \text{ET I 313(11)}$$

$$3.^8 \quad \int_0^{\infty} \frac{x^q e^{-px} dx}{(1 - a e^{-px})^2} = \frac{\Gamma(q+1)}{a p^{q+1}} \sum_{k=1}^{\infty} \frac{a^k}{k^q}$$

$$[a < 1, \quad q > -1, \quad p > 0] \quad \text{BI (85)(13)}$$

$$4.^7 \quad \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{(1 - \beta e^{-x})^2} dx = \Gamma(\nu) [\Phi(\beta; \nu - 1; \mu) - (\mu - 1) \Phi(\beta; \nu; \mu)]$$

$$[\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0, \quad |\arg(1 - \beta)| < \pi] \quad (\text{cf. 9.550}) \quad \text{ET I 313(12)}$$

$$5. \quad \int_{-\infty}^{\infty} \frac{x e^x dx}{(\beta + e^x)^2} = \frac{1}{\beta} \ln \beta \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.231 5})$$

$$\text{BI (101)(10)}$$

$$6.^* \quad \int_0^t x^5 \frac{e^{-x}}{(1 - e^{-x})^2} dx = 120 \zeta(5) - \sum_{k=1}^{\infty} \frac{e^{-kt}}{k^5} (y^5 + 5y^4 + 20y^3 + 60y^2 + 120y + 120)$$

$$= 120 \zeta(5) - \frac{t^5 e^{-t/2}}{2 \sinh(t/2)} - 5 \sum_{k=1}^{\infty} \frac{e^{-kt}}{k^5} (y^4 + 4y^3 + 12y^2 + 24y + 24)$$

$$y = kt$$

3.424

- 1.⁷ $\int_0^\infty \frac{(1+a)e^x - a}{(1-e^x)^2} e^{-ax} x^n dx = n! \zeta(n, a) \quad [a > -1, \quad n = 1, 2, \dots] \quad \text{BI (85)(15)}$
2. $\int_0^\infty \frac{(1+a)e^x + a}{(1+e^x)^2} e^{-ax} x^n dx = n! \sum_{k=1}^\infty \frac{(-1)^k}{(a+k)^n} \quad [a > -1, \quad n = 1, 2, \dots] \quad \text{BI (85)(14)}$
3. $\int_{-\infty}^\infty \frac{a^2 e^x + b^2 e^{-x}}{(a^2 e^x - b^2 e^{-x})^2} x^2 dx = \frac{\pi^2}{2ab} \quad [ab > 0] \quad \text{BI (102)(3)a}$
4. $\int_{-\infty}^\infty \frac{a^2 e^x - b^2 e^{-x}}{(a^2 e^x + b^2 e^{-x})^2} x^2 dx = \frac{\pi}{ab} \ln \frac{b}{a} \quad [ab > 0] \quad \text{BI (102)(1)}$
5. $\int_0^\infty \frac{e^x - e^{-x} + 2}{(e^x - 1)^2} x^2 dx = \frac{2}{3} \pi^2 - 2 \quad \text{BI (85)(7)}$

3.425

- 1.⁷ $\int_{-\infty}^\infty \frac{x e^x dx}{(a^2 + b^2 e^{2x})^n} = \frac{\sqrt{\pi} \Gamma(n - \frac{1}{2})}{4a^{2n-1} b \Gamma(n)} \left[2 \ln \frac{a}{2b} - C - \psi\left(n - \frac{1}{2}\right) \right] \quad [ab > 0, \quad n > 0] \quad \text{BI(101)(13), LI(101)(13)}$
- 2.⁷ $\int_{-\infty}^\infty \frac{(a^2 e^x - e^{-x}) x^2 dx}{(a^2 e^x + e^{-x})^{p+1}} = -\frac{1}{a^{p+1}} B\left(\frac{p}{2}, \frac{p}{2}\right) \ln a \quad [a > 0, \quad p > 0] \quad \text{BI (102)(5)}$

3.426

1. $\int_{-\infty}^\infty \frac{(e^x - a e^{-x}) x^2 dx}{(a + e^x)^2 (1 + e^{-x})^2} = \frac{(\ln a)^2}{a - 1} \quad \text{BI (102)(12)}$
2. $\int_{-\infty}^\infty \frac{(e^x - a e^{-x}) x^2 dx}{(a + e^x)^2 (1 - e^{-x})^2} = \frac{\pi^2 + (\ln a)^2}{a + 1} \quad \text{BI (102)(13)}$

3.427

1. $\int_0^\infty \left(\frac{e^{-x}}{x} + \frac{e^{-\mu x}}{e^{-x} - 1} \right) dx = \psi(\mu) \quad [\operatorname{Re} \mu > 0] \quad (\text{cf. 4.281 4}) \quad \text{WH}$
- 2.⁷ $\int_0^\infty \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} dx = C \quad (\text{cf. 4.281 1}) \quad \text{BI (94)(1)}$
3. $\int_0^\infty \left(\frac{1}{2} - \frac{1}{1 + e^{-x}} \right) \frac{e^{-2x}}{x} dx = \frac{1}{2} \ln \frac{\pi}{4} \quad \text{BI (94)(5)}$
4. $\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-\mu x}}{x} dx = \ln \Gamma(\mu) - \left(\mu - \frac{1}{2} \right) \ln \mu + \mu - \frac{1}{2} \ln(2\pi) \quad [\operatorname{Re} \mu > 0] \quad \text{WH}$
5. $\int_0^\infty \left(\frac{1}{2} e^{-2x} - \frac{1}{e^x + 1} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi \quad \text{BI (94)(6)}$
6. $\int_0^\infty \left(\frac{e^{\mu x} - 1}{1 - e^{-x}} - \mu \right) \frac{e^{-x}}{x} dx = -\ln \Gamma(\mu) - \ln \sin(\pi \mu) + \ln \pi \quad [\operatorname{Re} \mu < 1] \quad \text{EH I 21(6)}$

$$7. \quad \int_0^\infty \left(\frac{e^{-\nu x}}{1 - e^{-x}} - \frac{e^{-\mu x}}{x} \right) dx = \ln \mu - \psi(\nu) \quad (\text{cf. 4.281 5}) \quad \text{BI (94)(3)}$$

$$8. \quad \int_0^\infty \left(\frac{n}{x} - \frac{e^{-\mu x}}{1 - e^{-x/n}} \right) e^{-x} dx = n \psi(n\mu + n) - n \ln n$$

$$[\operatorname{Re} \mu > 0, \quad n = 1, 2, \dots] \quad \text{BI (94)(4)}$$

$$9. \quad \int_0^\infty \left(\mu - \frac{1 - e^{-\mu x}}{1 - e^{-x}} \right) \frac{e^{-x}}{x} dx = \ln \Gamma(\mu + 1) \quad [\operatorname{Re} \mu > -1] \quad \text{WH}$$

$$10. \quad \int_0^\infty \left(\nu e^{-x} - \frac{e^{-\mu x} - e^{-(\mu+\nu)x}}{e^x - 1} \right) \frac{dx}{x} = \ln \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu + 1)}$$

$$[\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > 0] \quad \text{BI (94)(8)}$$

$$11. \quad \int_0^\infty \left[(1 - e^x)^{-1} + x^{-1} - 1 \right] e^{-xz} dx = \psi(z) - \ln z \quad [\operatorname{Re} z > 0] \quad \text{EH I 18(24)}$$

3.428

$$1. \quad \int_0^\infty \left(\nu e^{-\mu x} - \frac{1}{\mu} e^{-x} - \frac{1}{\mu} \frac{e^{-1} - e^{-\mu \nu x}}{1 - e^{-x}} \right) \frac{dx}{x} = \frac{1}{\mu} \ln \Gamma(\mu \nu) - \nu \ln \mu$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (94)(18)}$$

$$2. \quad \int_0^\infty \left(\frac{n-1}{2} + \frac{n-1}{1 - e^{-x}} + \frac{e^{(1-\mu)x}}{1 - e^{x/n}} + \frac{e^{-n\mu x}}{1 - e^{-x}} \right) e^{-x} \frac{dx}{x} = \frac{n-1}{2} \ln 2\pi - \left(n\mu + \frac{1}{2} \right) \ln n$$

$$[\operatorname{Re} \mu > 0, \quad n = 1, 2, \dots] \quad \text{BI (94)(14)}$$

$$3. \quad \int_0^\infty \left(n\mu - \frac{n-1}{2} - \frac{n}{1 - e^{-x}} - \frac{e^{(1-\mu)x}}{1 - e^{x/n}} \right) \frac{e^{-x}}{x} dx = \sum_{k=0}^{n-1} \ln \Gamma \left(\mu - \frac{k}{n} + 1 \right)$$

$$[\operatorname{Re} \mu > 0, \quad n = 1, 2, \dots] \quad \text{BI (94)(13)}$$

$$4. \quad \int_0^\infty \left(\frac{e^{-\nu x}}{1 - e^x} - \frac{e^{-\mu \nu x}}{1 - e^{\mu x}} - \frac{e^x}{1 - e^x} + \frac{e^{\mu x}}{1 - e^{\mu x}} \right) \frac{dx}{x} = \nu \ln \mu$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{LI (94)(15)}$$

$$5. \quad \int_0^\infty \left[\frac{1}{e^x - 1} - \frac{\mu e^{-\mu x}}{1 - e^{-\mu x}} + \left(a\mu - \frac{\mu+1}{2} \right) e^{-\mu x} + (1 - a\mu) e^{-x} \right] \frac{dx}{x}$$

$$= \frac{\mu-1}{2} \ln(2\pi) + \left(\frac{1}{2} - a\mu \right) \ln \mu$$

$$[\operatorname{Re} \mu > 0] \quad \text{BI (94)(16)}$$

$$6. \quad \int_0^\infty \left[\frac{e^{-\nu x}}{1 - e^{-x}} - \frac{e^{-\mu \nu x}}{1 - e^{-\mu x}} - \frac{(\mu-1)e^{-\mu x}}{1 - e^{-\mu x}} - \frac{\mu-1}{2} e^{-\mu x} \right] \frac{dx}{x} = \frac{\mu-1}{2} \ln(2\pi) + \left(\frac{1}{2} - \mu \nu \right) \ln \mu$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad (\text{cf. 4.267 37}) \quad \text{BI (94)(17)}$$

$$7. \quad \int_0^\infty \left[1 - e^{-x} - \frac{(1 - e^{-\nu x})(1 - e^{-\mu x})}{1 - e^{-x}} \right] \frac{dx}{x} = \ln B(\mu, \nu)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (94)(12)}$$

$$\mathbf{3.429} \quad \int_0^\infty [e^{-x} - (1+x)^{-\mu}] \frac{dx}{x} = \psi(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{NH 184(7)}$$

3.431

$$1. \quad \int_0^\infty \left(e^{-\mu x} - 1 + \mu x - \frac{1}{2} \mu^2 x^2 \right) x^{\nu-1} dx = \frac{-1}{\nu(\nu+1)(\nu+2)\mu^\nu} \Gamma(\nu+3) \\ [\operatorname{Re} \mu > 0, \quad -2 > \operatorname{Re} \nu > -3] \quad \text{LI (90)(5)}$$

$$2. \quad \int_0^\infty \left[x^{-1} - \frac{1}{2} x^{-2} (x+2) (1 - e^{-x}) \right] e^{-px} dx = -1 + \left(p + \frac{1}{2} \right) \ln \left(1 + \frac{1}{p} \right) \\ [\operatorname{Re} p > 0] \quad \text{ET I 144(6)}$$

3.432

$$1. \quad \int_0^\infty x^{\nu-1} e^{-mx} (e^{-x} - 1)^n dx = \Gamma(\nu) \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{(n+m-k)^\nu} \\ [n = 0, 1, \dots, \operatorname{Re} \nu > 0] \quad \text{LI (90)(10)}$$

$$2. \quad \int_0^\infty \left[x^{\nu-1} e^{-x} - e^{-\mu x} (1 - e^{-x})^{\nu-1} \right] dx = \Gamma(\nu) - \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)} \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{LI (81)(14)}$$

$$\mathbf{3.433} \quad \int_0^\infty x^{p-1} \left[e^{-x} + \sum_{k=1}^n (-1)^k \frac{x^{k-1}}{(k-1)!} \right] dx = \Gamma(p) \quad [-n < p < -n+1, \quad n = 0, 1, \dots]$$

FI II 805

3.434

$$1. \quad \int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}} dx = \frac{\mu^\rho - \nu^\rho}{\rho} \Gamma(1-\rho) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \rho < 1] \\ \text{BI (90)(6)}$$

$$2. \quad \int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x} dx = \ln \frac{\nu}{\mu} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 634}$$

3.435

$$1. \quad \int_0^\infty \left\{ (x+1)e^{-x} - e^{-\frac{x}{2}} \right\} \frac{dx}{x} = 1 - \ln 2 \quad \text{LI (89)(19)}$$

$$2.^{11} \quad \int_0^\infty \frac{1 - e^{-\mu x}}{x(x+\beta)} dx = \frac{1}{\beta} [\ln(\beta\mu) + \mathbf{C} - e^{\beta\mu} \operatorname{Ei}(-\beta\mu)] \quad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 217 (18)}$$

$$3. \quad \int_0^\infty \left(\frac{1}{1+x} - e^{-x} \right) \frac{dx}{x} = \mathbf{C} \quad \text{FI II 7 95, 802}$$

$$4. \quad \int_0^\infty \left(e^{-\mu x} - \frac{1}{1+a x} \right) \frac{dx}{x} = \ln \frac{a}{\mu} - \mathbf{C} \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (92)(10)}$$

$$\mathbf{3.436} \quad \int_0^\infty \left\{ \frac{e^{-npx} - e^{-nqx}}{n} - \frac{e^{-mpx} - e^{-mqx}}{m} \right\} \frac{dx}{x^2} = (q-p) \ln \frac{m}{n} \quad [p > 0, \quad q > 0] \quad \text{BI (89)(28)}$$

$$\mathbf{3.437} \quad \int_0^\infty \left\{ p e^{-x} - \frac{1 - e^{-px}}{x} \right\} \frac{dx}{x} = p \ln p - p \quad [p > 0] \quad \text{BI (89)(24)}$$

3.438

$$1. \quad \int_0^\infty \left\{ \left(\frac{1}{2} + \frac{1}{x} \right) e^{-x} - \frac{1}{x} e^{-\frac{x}{2}} \right\} \frac{dx}{x} = \frac{\ln 2 - 1}{2} \quad \text{BI (89)(19)}$$

$$2.7 \quad \int_0^\infty \left\{ \frac{p^2}{6} e^{-x} - \frac{p^2}{2x} - \frac{p}{x^2} - \frac{1 - e^{-px}}{x^3} \right\} \frac{dx}{x} = \frac{p^2}{6} \ln p - \frac{11}{36} p^3 \quad [p > 0] \quad \text{BI (89)(33)}$$

$$3. \quad \int_0^\infty \left(e^{-x} - e^{-2x} - \frac{1}{x} e^{-2x} \right) \frac{dx}{x} = 1 - \ln 2 \quad \text{BI (89)(25)}$$

$$4. \quad \int_0^\infty \left\{ \left(p - \frac{1}{2} \right) e^{-x} + \frac{x+2}{2x} (e^{-px} - e^{-\frac{x}{2}}) \right\} \frac{dx}{x} = \left(p - \frac{1}{2} \right) (\ln p - 1) \quad [p > 0] \quad \text{BI (89)(22)}$$

$$3.439 \quad \int_0^\infty \left\{ (p-q)e^{-rx} + \frac{1}{mx} (e^{-mpx} - e^{-mqx}) \right\} \frac{dx}{x} = p \ln p - q \ln q - (p-q) \left(1 + \ln \frac{r}{m} \right) \quad [p > 0, \quad q > 0, \quad r > 0] \quad \text{LI(89)(26), LI(89)(27)}$$

$$3.441 \quad \int_0^\infty \{ (p-r)e^{-qx} + (r-q)e^{-px} + (q-p)e^{-rx} \} \frac{dx}{x^2} = (r-q)p \ln p + (p-r)q \ln q + (q-p)r \ln r \quad [p > 0, \quad q > 0, \quad r > 0] \quad (\text{cf. 4.268 6}) \quad \text{BI (89)(18)}$$

3.442

$$1. \quad \int_0^\infty \left\{ 1 - \frac{x+2}{2x} (1 - e^{-x}) \right\} e^{-qx} \frac{dx}{x} = -1 + \left(q + \frac{1}{2} \right) \ln \frac{q+1}{q} \quad [q > 0] \quad \text{BI (89)(23)}$$

$$2. \quad \int_0^\infty \left(\frac{e^{-x} - 1}{x} + \frac{1}{1+x} \right) \frac{dx}{x} = C - 1 \quad \text{BI (92)(16)}$$

$$3. \quad \int_0^\infty \left(e^{-px} - \frac{1}{1+a^2 x^2} \right) \frac{dx}{x} = -C + \ln \frac{a}{p} \quad [p > 0] \quad \text{BI (92)(11)}$$

3.443

$$1. \quad \int_0^\infty \left\{ \frac{e^{-x} p^2}{2} - \frac{p}{x} + \frac{1 - e^{-px}}{x^2} \right\} \frac{dx}{x} = \frac{p^2}{2} \ln p - \frac{3}{4} p^2 \quad [p > 0] \quad \text{BI (89)(32)}$$

$$2. \quad \int_0^\infty \frac{(1 - e^{-px})^n e^{-qx}}{x^3} dx = \frac{1}{2} \sum_{k=2}^n (-1)^{k-1} \binom{n}{k} (q+kp)^2 \ln(q+kp) \quad [n > 2, \quad q > 0, \quad pn + q > 0] \quad (\text{cf. 4.268 4}) \quad \text{BI (89)(30)}$$

$$3. \quad \int_0^\infty (1 - e^{-px})^2 e^{-qx} \frac{dx}{x^2} = (2p+q) \ln(2p+q) - 2(p+q) \ln(p+q) + q \ln q \quad [q > 0, \quad 2p > -q] \quad (\text{cf. 4.268 2}) \quad \text{BI (89)(13)}$$

3.45 Combinations of powers and algebraic functions of exponentials

3.451

$$1. \quad \int_0^{\infty} x e^{-x} \sqrt{1 - e^{-x}} dx = \frac{4}{3} \left(\frac{4}{3} - \ln 2 \right) \quad \text{BI (99)(1)}$$

$$2. \quad \int_0^{\infty} x e^{-x} \sqrt{1 - e^{-2x}} dx = \frac{\pi}{4} \left(\frac{1}{2} + \ln 2 \right) \quad (\text{cf. 4.241 9}) \quad \text{BI (99)(2)}$$

3.452

$$1. \quad \int_0^{\infty} \frac{x dx}{\sqrt{e^x - 1}} = 2\pi \ln 2 \quad \text{FI II 643a, BI(99)(4)}$$

$$2. \quad \int_0^{\infty} \frac{x^2 dx}{\sqrt{e^x - 1}} = 4\pi \left\{ (\ln 2)^2 + \frac{\pi^2}{12} \right\} \quad \text{BI (99)(5)}$$

$$3. \quad \int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = \frac{\pi}{2} [2 \ln 2 - 1] \quad \text{BI (99)(6)}$$

$$4. \quad \int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^{2x} - 1}} = 1 - \ln 2 \quad \text{BI (99)(8)}$$

$$5. \quad \int_0^{\infty} \frac{x e^{-2x} dx}{\sqrt{e^x - 1}} = \frac{3}{4} \pi \left(\ln 2 - \frac{7}{12} \right) \quad \text{BI (99)(7)}$$

3.453

$$1. \quad \int_0^{\infty} \frac{x e^x}{a^2 e^x - (a^2 - b^2)} \frac{dx}{\sqrt{e^x - 1}} = \frac{2\pi}{ab} \ln \left(1 + \frac{b}{a} \right) \quad [ab > 0] \quad (\text{cf. 4.298 17}) \quad \text{BI (99)(16)}$$

$$2. \quad \int_0^{\infty} \frac{x e^x dx}{[a^2 e^x - (a^2 + b^2)] \sqrt{e^x - 1}} = \frac{2\pi}{ab} \arctan \frac{b}{a} \quad [ab > 0] \quad (\text{cf. 4.298 18}) \quad \text{BI (99)(17)}$$

3.454

$$1.^{11} \quad \int_0^{\infty} \frac{x e^{-2nx} dx}{\sqrt{e^{2x} - 1}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \left\{ \ln 2 + \sum_{k=1}^{2n} \frac{(-1)^k}{k} \right\} \quad \text{LI (99)(10)}$$

$$2. \quad \int_0^{\infty} \frac{x e^{-(2n-1)x} dx}{\sqrt{e^{2x} - 1}} = -\frac{(2n-2)!!}{(2n-1)!!} \left\{ \ln 2 + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k} \right\} \quad \text{LI (99)(9)}$$

3.455

$$1. \quad \int_0^{\infty} \frac{x^2 e^x dx}{\sqrt{(e^x - 1)^3}} = 8\pi \ln 2 \quad \text{BI (99)(11)}$$

$$2. \quad \int_0^{\infty} \frac{x^3 e^x dx}{\sqrt{(e^x - 1)^3}} = 24\pi \left[(\ln 2)^2 + \frac{\pi^2}{12} \right] \quad \text{BI (99)(12)}$$

3.456

$$1. \quad \int_0^{\infty} \frac{x dx}{\sqrt[3]{e^{3x} - 1}} = \frac{\pi}{3\sqrt{3}} \left[\ln 3 + \frac{\pi}{3\sqrt{3}} \right] \quad \text{BI (99)(13)}$$

$$2. \quad \int_0^\infty \frac{x dx}{\sqrt[3]{(e^{3x} - 1)^2}} = \frac{\pi}{3\sqrt{3}} \left[\ln 3 - \frac{\pi}{3\sqrt{3}} \right] \quad (\text{cf. 4.244 } 3) \quad \text{BI (99)(14)}$$

3.457

$$1. \quad \int_0^\infty x e^{-x} (1 - e^{-2x})^{n-1/2} dx = \frac{(2n-1)!!}{4 \cdot (2n)!!} \pi [C + \psi(n+1) + 2 \ln 2] \quad (\text{cf. 4.241 } 5) \quad \text{BI (99)(3)}$$

$$2. \quad \int_{-\infty}^\infty \frac{x e^x dx}{(a + e^x)^{n+3/2}} = \frac{2}{(2n+1)a^{n+1/2}} [\ln(4a) - 3C - 2\psi(2n) - \psi(n)] \quad \text{BI (101)(12)}$$

$$3. \quad \int_{-\infty}^\infty \frac{x dx}{(a^2 e^x + e^{-x})^\mu} = -\frac{1}{2a^\mu} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \ln a \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (101)(14)}$$

3.458

$$1.^7 \quad \int_0^{\ln 2} x e^x (e^x - 1)^{p-1} dx = \frac{1}{p} \left[\ln 2 + \sum_{k=0}^\infty \frac{(-1)^{k-1}}{p+k+1} \right] \quad \text{BI (104)(4)}$$

$$2. \quad \int_{-\infty}^\infty \frac{x e^x dx}{(a + e^x)^{\nu+1}} = \frac{1}{\nu a^\nu} [\ln a - C - \psi(\nu)] \quad [a > 0]$$

$$= \frac{1}{\nu a^\nu} \left[\ln a - \sum_{k=1}^{\nu-1} \frac{1}{k} \right] \quad [a > 0, \quad \nu = 1, 2, \dots]$$

BI (101)(11)

3.46–3.48 Combinations of exponentials of more complicated arguments and powers**3.461**

$$1. \quad \int_u^\infty \frac{e^{-p^2 x^2}}{x^{2n}} dx = \frac{(-1)^n 2^{n-1} p^{2n-1} \sqrt{\pi}}{(2n-1)!!} [1 - \Phi(pu)]$$

$$+ \frac{e^{-p^2 u^2}}{2u^{2n-1}} \sum_{k=0}^{n-1} \frac{(-1)^k 2^{k+1} (pu)^{2k}}{(2n-1)(2n-3) \cdots (2n-2k-1)}$$

[p > 0] NT 21(4)

$$2. \quad \int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}} \quad [p > 0, \quad n = 0, 1, \dots] \quad \text{FI II 743}$$

$$3. \quad \int_0^\infty x^{2n+1} e^{-px^2} dx = \frac{n!}{2p^{n+1}} \quad [p > 0] \quad \text{BI (81)(7)}$$

$$4. \quad \int_{-\infty}^\infty (x + ai)^{2n} e^{-x^2} dx = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \sum_{k=0}^n (-1)^k \frac{(2a)^{2k} n!}{(2k)!(n-k)!} \quad \text{BI (100)(12)}$$

$$5.^{11} \quad \int_u^\infty e^{-\mu x^2} \frac{dx}{x^2} = \frac{1}{u} e^{-\mu u^2} - \sqrt{\mu\pi} [1 - \Phi(u\sqrt{\mu})] \quad \left[\left| \arg \mu \right| < \frac{\pi}{2}, \quad u > 0 \right] \quad \text{ET I 135(19)a}$$

$$6.^* \quad \int_0^\infty \exp(-a\sqrt{x^2 + b^2}) dx = b K_1(ab) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$

$$7.* \quad \int_0^\infty x^2 \exp\left(-a\sqrt{x^2+b^2}\right) dx = \frac{2b}{a^2} K_1(ab) + \frac{b^2}{a} K_0(ab) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$

$$8.* \quad \int_0^\infty x^4 \exp\left(-a\sqrt{x^2+b^2}\right) dx = \frac{12b^2}{a^3} K_2(ab) + \frac{3b^3}{a^2} K_1(ab) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$

$$9.* \quad \int_0^\infty x^6 \exp\left(-a\sqrt{x^2+b^2}\right) dx = \frac{90b^3}{a^4} K_3(ab) + \frac{15b^4}{a^3} K_2(ab) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$

3.462

$$1. \quad \int_0^\infty x^{\nu-1} e^{-\beta x^2 - \gamma x} dx = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH II 119(3)a, ET I 313(13)}$$

$$2.^8 \quad \int_{-\infty}^\infty x^n e^{-px^2+2qx} dx = \frac{1}{2^{n-1}p} \sqrt{\frac{\pi}{p}} \frac{d^{n-1}}{dq^{n-1}} \left(qe^{q^2/p}\right) \quad [p > 0] \quad \text{BI (100)(8)} \\ = n! e^{q^2/p} \sqrt{\frac{\pi}{p}} \left(\frac{q}{p}\right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2k)!(k)!} \left(\frac{p}{4q^2}\right)^k \quad [p > 0] \quad \text{LI (100)(8)}$$

$$3.^{11} \quad \int_{-\infty}^\infty (ix)^\nu e^{-\beta^2 x^2 - iqx} dx = 2^{-\frac{\nu}{2}} \sqrt{\pi} \beta^{-\nu-1} \exp\left(-\frac{q^2}{8\beta^2}\right) D_\nu\left(\frac{q}{\beta\sqrt{2}}\right) \\ \left[\operatorname{Re} \beta^2 > 0, \quad \operatorname{Re} \nu > -1, \quad \arg ix = \frac{\pi}{2} \operatorname{sign} x\right] \quad \text{ET I 121(23)}$$

$$4. \quad \int_{-\infty}^\infty x^n \exp\left[-(x-\beta)^2\right] dx = (2i)^{-n} \sqrt{\pi} H_n(i\beta) \quad \text{EH II 195(31)}$$

$$5.^{11} \quad \int_0^\infty x e^{-\mu x^2 - 2\nu x} dx = \frac{1}{2\mu} - \frac{\nu}{2\mu} \sqrt{\frac{\pi}{\mu}} e^{\frac{\nu^2}{\mu}} \left[1 - \operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \\ \left[|\arg \nu| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0\right] \quad \text{ET I 146(31)a}$$

$$6. \quad \int_{-\infty}^\infty x e^{-px^2+2qx} dx = \frac{q}{p} \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{p}\right) \quad [\operatorname{Re} p > 0] \quad \text{BI (100)(7)}$$

$$7.^{11} \quad \int_0^\infty x^2 e^{-\mu x^2 - 2\nu x} dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}} \frac{2\nu^2 + \mu}{4} e^{\frac{\nu^2}{\mu}} \left[1 - \operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \\ \left[|\arg \nu| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0\right] \quad \text{ET I 146(32)}$$

$$8. \quad \int_{-\infty}^\infty x^2 e^{-\mu x^2 + 2\nu x} dx = \frac{1}{2\mu} \sqrt{\frac{\pi}{\mu}} \left(1 + 2\frac{\nu^2}{\mu}\right) e^{\frac{\nu^2}{\mu}} \quad [|\arg \nu| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{BI (100)(8)a}$$

$$9.* \quad \int_0^\infty e^{-\beta x^n \pm a} dx = \frac{e^{\pm a}}{n\beta^{1/n}} \Gamma\left(\frac{1}{n}\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} n > 0]$$

- $$10.^* \quad \int_0^\infty (x-a)e^{-\beta(x-a)} dx = e^{a\beta} \frac{(1-a\beta)}{\beta^2} \quad [\operatorname{Re} \beta > 0]$$
- $$11.^* \quad \int_0^\infty (x-a)e^{-\beta(x+a)} dx = e^{-a\beta} \frac{(1-a\beta)}{\beta^2} \quad [\operatorname{Re} \beta > 0]$$
- $$12.^* \quad \int_0^\infty (ax \pm b)^m e^{-px} dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}} \Gamma\left(m+1, \pm \frac{pb}{a}\right) \quad \left[p > 0, \quad \left|\arg\left(\frac{b}{a}\right)\right| < \pi\right]$$
- $$13.^* \quad \int_u^\infty (ax \pm b)^m e^{-px} dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}} \Gamma\left(m+1, pu \pm \frac{pb}{a}\right) \quad \left[p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$$
- $$14.^* \quad \int_0^u (ax \pm b)^m e^{-px} dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}} \left[\Gamma\left(m+1, \pm \frac{pb}{a}\right) - \Gamma\left(m+1, pu \pm \frac{pb}{a}\right)\right] \quad \left[u > 0, \quad p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$$
- $$15.^* \quad \int_0^\infty \frac{e^{-px}}{(ax \pm b)^n} dx = \frac{p^{n-1} e^{\pm pb/a}}{a^n} \Gamma\left(-n+1, \pm \frac{pb}{a}\right) \quad \left[p > 0, \quad \left|\arg\left(\frac{b}{a}\right)\right| < \pi\right]$$
- $$16.^* \quad \int_u^\infty \frac{e^{-px}}{(ax \pm b)^n} dx = \frac{p^{n-1} e^{\pm pb/a}}{a^n} \Gamma\left(-n+1, pu \pm \frac{pb}{a}\right) \quad \left[u > 0, \quad p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$$
- $$17.^* \quad \int_0^u \frac{e^{-px}}{(ax \pm b)^n} dx = \frac{p^{n-1} e^{\pm pb/a}}{a^n} \left[\Gamma\left(-n+1, \pm \frac{pb}{a}\right) - \Gamma\left(-n+1, pu \pm \frac{pb}{a}\right)\right] \quad \left[u > 0, \quad p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$$
- $$18.^* \quad \int_0^\infty \left(\frac{x-a}{b}\right)^j \exp\left(-\beta\left(\frac{x-a}{b}\right)^k\right) dx = \frac{b \Gamma\left(\frac{j+1}{k}, \beta\left(-\frac{a}{b}\right)^k\right)}{k \beta^{(j+1)/k}} \quad \left[\arg\left(-\frac{a}{b}\right) > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} k > 0\right]$$
- $$19.^* \quad \int_u^\infty \frac{e^{-\beta x^n}}{x^m} dx = \frac{\Gamma(z, \beta u^n)}{n \beta^z} \quad z = \frac{1-m}{n} \quad [u > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} z > 0]$$
- $$20.^* \quad \int_0^\infty \frac{\exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = K_0(ab) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$
- $$21.^* \quad \int_0^\infty \frac{x^2 \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = \frac{b}{a} K_1(ab) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$
- $$22.^* \quad \int_0^\infty \frac{x^4 \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = \frac{3b^2}{a^2} K_1(ab) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$
- $$23.^* \quad \int_0^\infty \frac{x^6 \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = \frac{15b^3}{a^3} K_3(ab) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$

- 24.* $\int_0^\infty \frac{x^{2n} \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = (2n-1)!! \left(\frac{b}{a}\right)^n K_n(ab)$
 $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$
- 25.* $\int_0^\infty \frac{\exp(-px^2)}{\sqrt{a^2+x^2}} dx = \frac{1}{2} \exp\left(\frac{a^2 p}{2}\right) K_0\left(\frac{a^2 p}{2}\right)$ $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$
- 3.463** $\int_0^\infty (e^{-x^2} - e^{-x}) \frac{dx}{x} = \frac{1}{2} C$ BI (89)(5)
- 3.464** $\int_0^\infty (e^{-\mu x^2} - e^{-\nu x^2}) \frac{dx}{x^2} = \sqrt{\pi} (\sqrt{\nu} - \sqrt{\mu})$ $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$ FI II 645
- 3.465** $\int_0^\infty (1 + 2\beta x^2) e^{-\mu x^2} dx = \frac{\mu + \beta}{2} \sqrt{\frac{\pi}{\mu^3}}$ $[\operatorname{Re} \mu > 0]$ ET I 136(24)a
- 3.466**
1. $\int_0^\infty \frac{e^{-\mu^2 x^2}}{x^2 + \beta^2} dx = [1 - \Phi(\beta\mu)] \frac{\pi}{2\beta} e^{\beta^2 \mu^2}$ $[\operatorname{Re} \beta > 0, \quad |\arg \mu| < \frac{\pi}{4}]$ NT 19(13)
 2. $\int_0^\infty \frac{x^2 e^{-\mu^2 x^2}}{x^2 + \beta^2} dx = \frac{\sqrt{\pi}}{2\mu} - \frac{\pi\beta}{2} e^{\mu^2 \beta^2} [1 - \Phi(\beta\mu)]$ $[\operatorname{Re} \beta > 0, \quad |\arg \mu| < \frac{\pi}{4}]$ ET II 217(16)
 3. $\int_0^1 \frac{e^{x^2} - 1}{x^2} dx = \sum_{k=1}^\infty \frac{1}{k!(2k-1)}$ FI II 683
- 3.467** $\int_0^\infty \left(e^{-x^2} - \frac{1}{1+x^2}\right) \frac{dx}{x} = -\frac{1}{2} C$ BI (92)(12)
- 3.468**
1. $\int_{u\sqrt{2}}^\infty \frac{e^{-x^2}}{\sqrt{x^2 - u^2}} \frac{dx}{x} = \frac{\pi}{4u} [1 - \Phi(u)]^2$ $[u > 0]$ NT 33(17)
 2. $\int_0^\infty \frac{x e^{-\mu x^2}}{\sqrt{a^2 + x^2}} dx = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} e^{a^2 \mu} [1 - \Phi(a\sqrt{\mu})]$ $[\operatorname{Re} \mu > 0, \quad a > 0]$ NT 19(11)
- 3.469**
1. $\int_0^\infty e^{-\mu x^4 - 2\nu x^2} dx = \frac{1}{4} \sqrt{\frac{2\nu}{\mu}} \exp\left(\frac{\nu^2}{2\mu}\right) K_{\frac{1}{4}}\left(\frac{\nu^2}{2\mu}\right)$ $[\operatorname{Re} \mu \geq 0]$ ET I 146(23)
 2. $\int_0^\infty (e^{-x^4} - e^{-x}) \frac{dx}{x} = \frac{3}{4} C$ BI (89)(7)
 3. $\int_0^\infty (e^{-x^4} - e^{-x^2}) \frac{dx}{x} = \frac{1}{4} C$ BI (89)(6)
- 3.471**
1. $\int_0^u \exp\left(-\frac{\beta}{x}\right) \frac{dx}{x^2} = \frac{1}{\beta} \exp\left(-\frac{\beta}{u}\right)$ ET II 188(22)
 2. $\int_0^u x^{\nu-1} (u-x)^{\mu-1} e^{-\frac{\beta}{x}} dx = \beta^{\frac{\nu-1}{2}} u^{\frac{2\mu+\nu-1}{2}} \exp\left(-\frac{\beta}{2u}\right) \Gamma(\mu) W_{\frac{1-2\mu-\nu}{2}, \frac{\nu}{2}}\left(\frac{\beta}{u}\right)$
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0, \quad u > 0]$ ET II 187(18)

$$3. \quad \int_0^u x^{-\mu-1} (u-x)^{\mu-1} e^{-\frac{\beta}{x}} dx = \beta^{-\mu} u^{\mu-1} \Gamma(\mu) \exp\left(-\frac{\beta}{u}\right) \\ [\operatorname{Re} \mu > 0, \quad u > 0] \quad \text{ET II 187(16)}$$

$$4. \quad \int_0^u x^{-2\mu} (u-x)^{\mu-1} e^{-\frac{\beta}{x}} dx = \frac{1}{\sqrt{\pi u}} \beta^{\frac{1}{2}-\mu} e^{-\frac{\beta}{2u}} \Gamma(\mu) K_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right) \\ [u > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0] \\ \text{ET II 187(17)}$$

$$5. \quad \int_u^\infty x^{\nu-1} (x-u)^{\mu-1} e^{\frac{\beta}{x}} dx = \operatorname{B}(1-\mu-\nu, \mu) u^{\mu+\nu-1} {}_1F_1\left(1-\mu-\nu; 1-\nu; \frac{\beta}{u}\right) \\ [0 < \operatorname{Re} \mu < \operatorname{Re}(1-\nu), \quad u > 0] \\ \text{ET II 203(15)}$$

$$6. \quad \int_u^\infty x^{-2\mu} (x-u)^{\mu-1} e^{\frac{\beta}{x}} dx = \sqrt{\frac{\pi}{u}} \beta^{\frac{1}{2}-\mu} \Gamma(\mu) \exp\left(\frac{\beta}{2u}\right) I_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right) \\ [\operatorname{Re} \mu > 0, \quad u > 0] \quad \text{ET II 202(14)}$$

$$7. \quad \int_0^\infty x^{\nu-1} (x+\gamma)^{\mu-1} e^{-\frac{\beta}{x}} dx = \beta^{\frac{\nu-1}{2}} \gamma^{\frac{\nu-1}{2}+\mu} \Gamma(1-\mu-\nu) e^{\frac{\beta}{2\gamma}} W_{\frac{\nu-1}{2}+\mu, -\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right) \\ [|\arg \gamma| < \pi, \quad \operatorname{Re}(1-\mu) > \operatorname{Re} \nu > 0] \\ \text{ET II 234(13)a}$$

$$8. \quad \int_0^u x^{-2\mu} (u^2-x^2)^{\mu-1} e^{-\frac{\beta}{x}} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}} \Gamma(\mu) K_{\mu-\frac{1}{2}}\left(\frac{\beta}{u}\right) \\ [\operatorname{Re} \beta > 0, \quad u > 0, \quad \operatorname{Re} \mu > 0] \\ \text{ET II 188(23)a}$$

$$9. \quad \int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x}-\gamma x} dx = 2 \left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\gamma}\right) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \\ \text{ET II 82(23)a, LET I 146(29)}$$

$$10. \quad \int_0^\infty x^{\nu-1} \exp\left[\frac{i\mu}{2}\left(x-\frac{\beta^2}{x}\right)\right] dx = 2\beta^\nu e^{\frac{i\nu\pi}{2}} K_{-\nu}(\beta\mu) \\ [\operatorname{Im} \mu > 0, \quad \operatorname{Im}(\beta^2\mu) < 0; \text{ note that } K_{-\nu} \equiv K_\nu] \quad \text{EH II 82(24)}$$

$$11. \quad \int_0^\infty x^{\nu-1} \exp\left[\frac{i\mu}{2}\left(x+\frac{\beta^2}{x}\right)\right] dx = i\pi\beta^\nu e^{-\frac{i\nu\pi}{2}} H_{-\nu}^{(1)}(\beta\mu) \\ [\operatorname{Im} \mu > 0, \quad \operatorname{Im}(\beta^2\mu) > 0] \\ \text{EH II 21(33)}$$

$$12. \quad \int_0^\infty x^{\nu-1} \exp\left(-x-\frac{\mu^2}{4x}\right) dx = 2\left(\frac{\mu}{2}\right)^\nu K_{-\nu}(\mu) \\ \left[|\arg \mu| < \frac{\pi}{2}, \operatorname{Re} \mu^2 > 0; \text{ note that } K_{-\nu} \equiv K_\nu\right] \quad \text{WA 203(15)}$$

$$13. \quad \int_0^\infty \frac{x^{\nu-1} e^{-\frac{\beta}{x}}}{x+\gamma} dx = \gamma^{\nu-1} e^{\frac{\beta}{\gamma}} \Gamma(1-\nu) \Gamma\left(\nu, \frac{\beta}{\gamma}\right) \\ [|\arg \gamma| < \pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < 1] \\ \text{ET II 218(19)}$$

14. $\int_0^1 \frac{\exp(1 - \frac{1}{x}) - x^\nu}{x(1-x)} dx = \psi(\nu)$ [Re $\nu > 0$] BI (80)(7)
15. $\int_0^\infty x^{-\frac{1}{2}} e^{-\gamma x - \beta/x} dx = \sqrt{\frac{\pi}{\gamma}} e^{-2\sqrt{\beta\gamma}}$ [Re $\beta \geq 0$, Re $\gamma > 0$] ET 245 (5.6.1)
16. $\int_0^\infty x^{n-\frac{1}{2}} e^{-px-q/x} dx = (-1)^n \sqrt{\pi} \frac{\partial^n}{\partial p^n} (p^{-1/2} e^{-2\sqrt{pq}})$
[Re $p > 0$, Re $q > 0$] PBM 344 (2.3.16(2))

3.472

1. $\int_0^\infty \left(\exp\left(-\frac{a}{x^2}\right) - 1 \right) e^{-\mu x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} [\exp(-2\sqrt{a\mu}) - 1]$
[Re $\mu > 0$, Re $a > 0$] ET I 146(30)
2. $\int_0^\infty x^2 \exp\left(-\frac{a}{x^2} - \mu x^2\right) dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu^3}} (1 + 2\sqrt{a\mu}) \exp(-2\sqrt{a\mu})$
[Re $\mu > 0$, Re $a > 0$] ET I 146(26)
3. $\int_0^\infty \exp\left(-\frac{a}{x^2} - \mu x^2\right) \frac{dx}{x^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp(-2\sqrt{a\mu})$ [Re $\mu > 0$, $a > 0$] ET I 146(28)a
4. $\int_0^\infty \exp\left[-\frac{1}{2a} \left(x^2 + \frac{1}{x^2}\right)\right] \frac{dx}{x^4} = \sqrt{\frac{a\pi}{2}} (1+a) e^{-1/a}$ [$a > 0$] BI (98)(14)
5. $\int_0^\infty x^{-n-1/2} e^{-px-q/x} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{-2\sqrt{pq}}$ [Re $p > 0$, Re $q > 0$] PBM 344 (2.3.16(3))

3.473 $\int_0^\infty \exp(-x^n) x^{(m+1/2)n-1} dx = \frac{(2m-1)!!}{2^m n} \sqrt{\pi}$ BI (98)(6)

3.474

1. $\int_0^1 \left\{ \frac{n \exp(1-x^{-n})}{1-x^n} - \frac{x^{np}}{1-x} \right\} \frac{dx}{x} = \frac{1}{n} \sum_{k=1}^n \psi\left(p + \frac{k-1}{n}\right)$
[$p > 0$] BI (80)(8)
2. $\int_0^1 \left\{ \frac{n \exp(1-x^{-n})}{1-x^n} - \frac{\exp(1-\frac{1}{x})}{1-x} \right\} \frac{dx}{x} = -\ln n$ BI (80)(9)

3.475

- 1.⁷ $\int_0^\infty \left\{ \exp(-x^{2^n}) - \frac{1}{1+x^{2^{n+1}}} \right\} \frac{dx}{x} = -\frac{1}{2^n} C$ [$n \in \mathbb{Z}$] BI (92)(14)
2. $\int_0^\infty \left\{ \exp(-x^{2^n}) - \frac{1}{1+x^2} \right\} \frac{dx}{x} = -2^{-n} C$ BI (92)(13)
3. $\int_0^\infty \left\{ \exp(-x^{2^n}) - e^{-x} \right\} \frac{dx}{x} = (1-2^{-n}) C$ BI (89)(8)

3.476

$$1. \quad \int_0^\infty [\exp(-\nu x^p) - \exp(-\mu x^p)] \frac{dx}{x} = \frac{1}{p} \ln \frac{\mu}{\nu} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (89)(3)}$$

$$2. \quad \int_0^\infty [\exp(-x^p) - \exp(-x^q)] \frac{dx}{x} = \frac{p-q}{pq} C \quad [p > 0, \quad q > 0] \quad \text{BI (89)(9)}$$

3.477

$$1.^{10} \quad \int_{-\infty}^\infty \frac{e^{-a|x|}}{x-u} dx = e^{-au} \gamma(0, -au) - e^{au} \gamma(0, au) \quad [\operatorname{Re} a > 0, \quad \operatorname{Im} u \neq 0, \quad \arg u \neq 0] \quad \text{MC}$$

$$2.^8 \quad \int_{-\infty}^\infty \frac{\operatorname{sign} x \exp(-a|x|)}{x-u} dx = -[\exp(a|u|) \operatorname{Ei}(-a|u|) - \exp(-a|u|) \operatorname{Ei}(a|u|)]$$

$$[a > 0] \quad \text{ET II 251(36)}$$

3.478

$$1. \quad \int_0^\infty x^{\nu-1} \exp(-\mu x^p) dx = \frac{1}{p} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad p > 0]$$

$$\text{BI(81)(8)a, ET I 313(15, 16)}$$

$$2. \quad \int_0^\infty x^{\nu-1} [1 - \exp(-\mu x^p)] dx = -\frac{1}{|p|} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right)$$

$$[\operatorname{Re} \mu > 0 \text{ and } -p < \operatorname{Re} \nu < 0 \text{ for } p > 0, \quad 0 < \operatorname{Re} \nu < -p \text{ for } p < 0] \quad \text{ET I 313(18, 19)}$$

$$3.^{11} \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} \exp(\beta x^n) dx = B(\mu, \nu) u^{\mu+\nu-1} {}_nF_n\left(\frac{\nu}{n}, \frac{\nu+1}{n}, \dots, \frac{\nu+n-1}{n}; \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, \dots, \frac{\mu+\nu+n-1}{n}; \beta u^n\right)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad n = 2, 3, \dots] \quad \text{ET II 187(15)}$$

$$4. \quad \int_0^\infty x^{\nu-1} \exp(-\beta x^p - \gamma x^{-p}) dx = \frac{2}{p} \left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}\left(2\sqrt{\beta\gamma}\right)$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{ET I 313(17)}$$

3.479

$$1. \quad \int_0^\infty \frac{x^{\nu-1} \exp(-\beta\sqrt{1+x})}{\sqrt{1+x}} dx = \frac{2}{\sqrt{\pi}} \left(\frac{\beta}{2}\right)^{\frac{1}{2}-\nu} \Gamma(\nu) K_{\frac{1}{2}-\nu}(\beta)$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 313(14)}$$

$$2.^{11} \quad \int_0^\infty \frac{x^{\nu-1} \exp(i\mu\sqrt{1+x^2})}{\sqrt{1+x^2}} dx = i\frac{\sqrt{\pi}}{2} \left(\frac{\mu}{2}\right)^{\frac{1-\nu}{2}} \Gamma\left(\frac{\nu}{2}\right) H_{\frac{1-\nu}{2}}^{(1)}(\mu)$$

$$[\operatorname{Im} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH II 83(30)}$$

3.481

$$1. \quad \int_{-\infty}^\infty x e^x \exp(-\mu e^x) dx = -\frac{1}{\mu} (C + \ln \mu) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (100)(13)}$$

$$2. \quad \int_{-\infty}^{\infty} x e^x \exp(-\mu e^{2x}) dx = -\frac{1}{4} [C + \ln(4\mu)] \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (100)(14)}$$

3.482

$$1.^3 \quad \int_0^{\infty} \exp(nx - \beta \sinh x) dx = \frac{1}{2} [S_n(\beta) - \pi \mathbf{E}_n(\beta) - \pi Y_n(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 168(11)}$$

$$2. \quad \int_0^{\infty} \exp(-nx - \beta \sinh x) dx = (-1)^{n+1} \frac{1}{2} [S_n(\beta) + \pi \mathbf{E}_n(\beta) + \pi Y_n(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 168(12)}$$

$$3. \quad \int_0^{\infty} \exp(-\nu x - \beta \sinh x) dx = \frac{\pi}{\sin \nu \pi} [\mathbf{J}_{\nu}(\beta) - J_{\nu}(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 168(13)}$$

$$3.483 \quad \int_{-\infty}^{\infty} \frac{\exp(\nu \operatorname{arcsinh} x - iax)}{\sqrt{1+x^2}} dx = \begin{cases} 2 \exp\left(-\frac{i\nu\pi}{2}\right) K_{\nu}(a) & \text{for } a > 0, \\ 2 \exp\left(\frac{i\nu\pi}{2}\right) K_{\nu}(-a) & \text{for } a < 0 \end{cases} \quad [|\operatorname{Re} \nu| < 1] \quad \text{ET I 122(32)}$$

$$3.484 \quad \int_0^{\infty} \left[\left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^{px} \right] \frac{dx}{x} = (e^a - 1) \ln \frac{q}{p} \quad [p > 0, \quad q > 0] \quad \text{BI (89)(34)}$$

$$3.485 \quad \int_0^{\pi/2} \exp(-\tan^2 x) dx = \frac{\pi e}{2} [1 - \Phi(1)]$$

$$3.486^6 \quad \int_0^1 x^{-x} dx = \int_0^1 e^{-x \ln x} dx = \sum_{k=1}^{\infty} k^{-k} = 1.2912859970627 \dots \quad \text{FI II 483}$$

3.487

$$1.^* \quad \int_0^{\pi/4} \exp \left[- \sum_{k=0}^{\infty} \left(\frac{\tan^{2k+1} x}{k + \frac{1}{2}} \right) \right] dx = \ln 2$$

3.5 Hyperbolic Functions**3.51 Hyperbolic functions****3.511**

$$1. \quad \int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a} \quad [a > 0]$$

$$2. \quad \int_0^{\infty} \frac{\sinh ax}{\sinh bx} dx = \frac{\pi}{2b} \tan \frac{a\pi}{2b} \quad [b > |a|] \quad \text{BI (27)(10)a}$$

$$3. \quad \int_0^{\infty} \frac{\sinh ax}{\cosh bx} dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} - \frac{1}{b} \beta \left(\frac{a+b}{2b} \right) \quad [b > |a|] \quad \text{GW (351)(3b)}$$

$$4. \quad \int_0^{\infty} \frac{\cosh ax}{\cosh bx} dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} \quad [b > |a|] \quad \text{BI (4)(14)a}$$

$$5. \quad \int_0^\infty \frac{\sinh ax \cosh bx}{\sinh cx} dx = \frac{\pi}{2c} \frac{\sin \frac{a\pi}{c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}} \quad [c > |a| + |b|] \quad \text{BI (27)(11)}$$

$$6. \quad \int_0^\infty \frac{\cosh ax \cosh bx}{\cosh cx} dx = \frac{\pi}{c} \frac{\cos \frac{a\pi}{2c} \cos \frac{b\pi}{2c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}} \quad [c > |a| + |b|] \quad \text{BI (27)(5)a}$$

$$7. \quad \int_0^\infty \frac{\sinh ax \sinh bx}{\cosh cx} dx = \frac{\pi}{c} \frac{\sin \frac{a\pi}{2c} \sin \frac{b\pi}{2c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}} \quad [c > |a| + |b|] \quad \text{BI (27)(6)a}$$

$$8.^{11} \quad \int_0^\infty \frac{dx}{\cosh^2 x} = 1 \quad \text{BI (98)(25)}$$

$$9. \quad \int_{-\infty}^\infty \frac{\sinh^2 ax}{\sinh^2 x} dx = 1 - a\pi \cot a\pi \quad [a^2 < 1] \quad \text{BI (16)(3)a}$$

$$10. \quad \int_0^\infty \frac{\sinh ax \sinh bx}{\cosh^2 bx} dx = \frac{a\pi}{2b^2} \sec \frac{a\pi}{2b} \quad [b > |a|] \quad \text{BI (27)(16)a}$$

3.512

$$1. \quad \int_0^\infty \frac{\cosh 2\beta x}{\cosh^{2\nu} ax} dx = \frac{4^{\nu-1}}{a} \text{B} \left(\nu + \frac{\beta}{a}, \nu - \frac{\beta}{a} \right) \quad [\text{Re}(\nu \pm \beta) > 0, \quad a > 0, \quad \beta > 0] \\ \text{LI(27)(17)a, EH I 11(26)}$$

$$2. \quad \int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x} dx = \frac{1}{2} \text{B} \left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2} \right) \quad [\text{Re} \mu > -1, \quad \text{Re}(\mu - \nu) < 0] \\ \text{EH I 11(23)}$$

3.513

$$1. \quad \int_0^\infty \frac{dx}{a + b \sinh x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a + b + \sqrt{a^2 + b^2}}{a + b - \sqrt{a^2 + b^2}} \quad [ab \neq 0] \quad \text{GW (351)(8)}$$

$$2. \quad \int_0^\infty \frac{dx}{a + b \cosh x} = \frac{2}{\sqrt{b^2 - a^2}} \arctan \frac{\sqrt{b^2 - a^2}}{a + b} \quad [b^2 > a^2] \\ = \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2}}{a + b - \sqrt{a^2 - b^2}} \quad [b^2 < a^2] \\ \text{GW (351)(7)}$$

$$3. \quad \int_0^\infty \frac{dx}{a \sinh x + b \cosh x} = \frac{2}{\sqrt{b^2 - a^2}} \arctan \frac{\sqrt{b^2 - a^2}}{a + b} \quad [b^2 > a^2] \\ = \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2}}{a + b - \sqrt{a^2 - b^2}} \quad [a^2 > b^2] \\ \text{GW (351)(9)}$$

$$\begin{aligned}
4. \quad \int_0^\infty \frac{dx}{a + b \cosh x + c \sinh x} &= \frac{2}{\sqrt{b^2 - a^2 - c^2}} \left[\arctan \frac{\sqrt{b^2 - a^2 - c^2}}{a + b + c} + \epsilon \pi \right] \\
&\quad \left[\text{when } b^2 > a^2 + c^2; \text{ and } \begin{cases} \epsilon = 0 & \text{for } (b-a)(a+b+c) > 0 \\ |\epsilon| = 1 & \text{for } (b-a)(a+b+c) < 0 \\ \epsilon = 1 & \text{for } a < b+c \\ \epsilon = -1 & \text{for } a > b+c \end{cases} \right] \\
&= \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{a + b + c + \sqrt{a^2 - b^2 + c^2}}{a + b + c - \sqrt{a^2 - b^2 + c^2}} \\
&\quad [b^2 < a^2 + c^2, \quad a^2 \neq b^2] \\
&= \frac{1}{c} \ln \frac{a+c}{a} \\
&\quad [a = b \neq 0, \quad c \neq 0] \\
&= \frac{2(a-b)}{c(a-b-c)} \\
&\quad [b^2 = a^2 + c^2, \quad c(a-b-c) < 0] \\
&\quad \text{GW (351)(6)}
\end{aligned}$$

3.514

$$\begin{aligned}
1. \quad \int_0^\infty \frac{dx}{\cosh ax + \cos t} &= \frac{t}{a} \operatorname{cosec} t && [0 < t < \pi, \quad a > 0] && \text{BI (27)(22)a} \\
2. \quad \int_0^\infty \frac{\cosh ax - \cos t_1}{\cosh bx - \cos t_2} dx &= \frac{\pi}{b} \frac{\sin \frac{a(\pi t_2)}{b}}{\sin t_2 \sin \frac{a}{b} \pi} - \frac{\pi t_2}{b \sin t_2} \cos t_1 \\
&&& [0 < |a| < b, \quad 0 < t_2 < \pi] && \text{BI (6)(20)a} \\
3. \quad \int_0^\infty \frac{\cosh ax \, dx}{(\cosh x + \cos t)^2} &= \frac{\pi (-\cos t \sin at + a \sin t \cos at)}{\sin^3 t \sin a\pi} \\
&&& [0 < a^2 < 1, \quad 0 < t < \pi] && \text{BI (6)(18)a} \\
4. \quad \int_0^\infty \frac{\sinh ax \sinh bx}{(\cosh ax + \cos t)^2} dx &= \frac{b\pi}{a^2} \operatorname{cosec} t \operatorname{cosec} \frac{b\pi}{a} \sin \frac{bt}{a} && [0 < |b| < a, \quad 0 < t < \pi] && \text{BI (27)(27)a}
\end{aligned}$$

$$\textbf{3.515} \quad \int_{-\infty}^\infty \left(1 - \frac{\sqrt{2} \cosh x}{\sqrt{\cosh 2x}} \right) dx = -\ln 2 \quad \text{BI (21)(12)a}$$

3.516

$$\begin{aligned}
1. \quad \int_0^\infty \frac{dx}{(z + \sqrt{z^2 - 1} \cosh x)^\mu} &= \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{(z + \sqrt{z^2 - 1} \cosh x)^\mu} = Q_{\mu-1}(z) \\
&&& [\operatorname{Re} \mu > -1]
\end{aligned}$$

For a suitable choice of a single-valued branch of the integrand, this formula is valid for arbitrary values of z in the z -plane cut from -1 to $+1$ provided $\mu < 0$. If $\mu > 0$, this formula ceases to be valid for points at which the denominator vanishes.

CO, WH

$$1. \quad \int_0^\infty \frac{dx}{(\beta + \sqrt{\beta^2 - 1} \cosh x)^{n+1}} = Q_n(\beta) \quad \text{EH II 181(32)}$$

$$2. \quad \int_0^\infty \frac{\cosh \gamma x \, dx}{(\beta + \sqrt{\beta^2 - 1} \cosh x)^{\nu+1}} = \frac{e^{-i\gamma\pi} \Gamma(\nu - \gamma + 1) Q_\nu^\gamma(\beta)}{\Gamma(\nu + 1)}$$

[Re($\nu \pm \gamma$) > -1, $\nu \neq -1, -2, -3, \dots$]
EH I 157(12)

$$3. \quad \int_0^\infty \frac{\sinh^{2\mu} x \, dx}{(\beta + \sqrt{\beta^2 - 1} \cosh x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi} \Gamma(\nu - 2\mu + 1) \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma(\nu + 1)} Q_{\nu-\mu}^\mu(\beta)$$

[Re($\nu - 2\mu + 1$) > 0, Re($\nu + 1$) > 0]
EH I 155(2)

3.517

$$1. \quad \int_0^\infty \frac{\cosh(\gamma + \frac{1}{2})x \, dx}{(\beta + \cosh x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}} (\beta^2 - 1)^{-\frac{\nu}{2}} \frac{\Gamma(\nu + \gamma + 1) \Gamma(\nu - \gamma) P_\gamma^{-\nu}(\beta)}{\Gamma(\nu + \frac{1}{2})}$$

[Re($\nu - \gamma$) > 0, Re($\nu + \gamma + 1$) > 0]
EH I 156(11)

$$2. \quad \int_0^a \frac{\cosh(\gamma + \frac{1}{2})x \, dx}{(\cosh a - \cosh x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{1}{2} - \nu)}{\sinh^\nu a} P_\gamma^\nu(\cosh a)$$

[Re $\nu < \frac{1}{2}$, $a > 0$]
EH I 156(8)

3.518

$$1. \quad \int_0^\infty \frac{\sinh^{2\mu} x \, dx}{(\cosh a + \sinh a \cosh x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi}}{\sqrt{\pi} \sinh^\mu a} \frac{\Gamma(\nu - 2\mu + 1) \Gamma(\mu + \frac{1}{2})}{\Gamma(\nu + 1)} Q_{\nu-\mu}^\mu(\cosh a)$$

[Re($\nu + 1$) > 0, Re($\nu - 2\mu + 1$) > 0, $a > 0$] EH I 155(3)a

$$2.10 \quad \int_0^\infty \frac{\sinh^{2\mu+1} x \, dx}{(\beta + \cosh x)^{\nu+1}} = 2^\mu (\beta^2 - 1)^{\frac{\mu-\nu}{2}} \Gamma(\nu - 2\mu) \Gamma(\mu + 1) P_\mu^{\mu-\nu}(\beta)$$

[Re($\nu - \mu$) > Re μ > -1, β does not lie on the ray $(-\infty, +1)$ of the real axis] EH I 155(1)

$$3. \quad \int_0^\infty \frac{\sinh^{2\mu-1} x \cosh x \, dx}{(1 + a \sinh^2 x)^\nu} = \frac{1}{2} a^{-\mu} B(\mu, \nu - \mu)$$

[Re $\nu > \text{Re } \mu > 0$, $a > 0$] EH I 11(22)

$$4.7 \quad \int_0^\infty \frac{\sinh^{\mu-1} x (\cosh x + 1)^{\nu-1} \, dx}{(\beta + \cosh x)^\varrho} = 2^{\mu+\nu-\varrho} B\left(\frac{1}{2}\mu, \varrho + 2 - \mu - \nu\right) \\ \times {}_2F_1\left(\varrho, \varrho + 2 - \mu - \nu; 2 - \frac{1}{2}\mu - \nu; \frac{1}{2} - \frac{1}{2}\beta\right)$$

[Re $\mu > 0$, Re($\varrho - \mu - \nu$) > -2, $|\arg(1 + \beta)| < \pi$] EH I 115(11)

$$5.6 \quad \int_0^\infty \frac{\sinh^{\mu-1} x (\cosh x - 1)^{\nu-1} \, dx}{(\beta + \cosh x)^\varrho} = 2^{-(2-\mu-\nu+\varrho)} {}_2F_1\left(\varrho, 2 - \mu - \nu + \varrho; 1 + \varrho - \frac{\mu}{2}; \frac{1-\beta}{2}\right) \\ \times B\left(2 - \mu - \nu + \varrho, -1 + \nu + \frac{\mu}{2}\right)$$

[$\beta \notin (-\infty, -1)$, Re($2 + \varrho$) Re($\mu + \nu$), Re($2\nu + \mu$) > 2] EH I 115(10)

$$6.7 \quad \int_0^\infty \frac{\sinh^{\mu-1} x \cosh^{\nu-1} x}{(\cosh^2 x - \beta)^e} dx = {}_2F_1 \left(\varrho, 1 + \varrho - \frac{\mu + \nu}{2}; 1 + \varrho - \frac{\nu}{2}; \beta \right) 2B \left(\frac{\mu}{2}, 1 + \varrho - \frac{\mu + \nu}{2} \right) \\ [\beta \notin (1, \infty), \quad \operatorname{Re} \mu > 0, \quad 2\operatorname{Re}(1 + \varrho) > \operatorname{Re}(\mu + \nu)] \quad \text{EH I 115(9)}$$

$$3.519 \quad \int_0^{\pi/2} \frac{\sinh[(r-p)] \tan x}{\sinh(r \tan x)} dx = \pi \sum_{k=1}^{\infty} \frac{1}{k\pi + r} \sin \frac{pk\pi}{r} \quad [p^2 < r^2] \quad \text{BI (274)(13)}$$

3.52–3.53 Combinations of hyperbolic functions and algebraic functions

3.521

1. $\int_0^\infty \frac{x dx}{\sinh ax} = \frac{\pi^2}{4a^2} \quad [a > 0] \quad \text{GW (352)(2b)}$
2. $\int_0^\infty \frac{x dx}{\cosh x} = 2\mathbf{G} = \pi \ln 2 - 4L\left(\frac{\pi}{4}\right) = 1.831931188\dots \quad \text{LI III 225(103a), BI(84)(1)a}$
3. $\int_1^\infty \frac{dx}{x \sinh ax} = -2 \sum_{k=0}^{\infty} \operatorname{Ei}[-(2k+1)a] \quad [a > 0] \quad \text{LI (104)(14)}$
4. $\int_1^\infty \frac{dx}{x \cosh ax} = 2 \sum_{k=0}^{\infty} (-1)^{k+1} \operatorname{Ei}[-(2k+1)a] \quad [a > 0] \quad \text{LI (104)(13)}$

3.522

1. $\int_0^\infty \frac{x dx}{(b^2 + x^2) \sinh ax} = \frac{\pi}{2ab} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k}{ab + k\pi} \quad [a > 0, \quad b > 0]$
2. $\int_0^\infty \frac{x dx}{(b^2 + x^2) \sinh \pi x} = \frac{1}{2b} - \beta(b+1) \quad [b > 0] \quad \text{BI(97)(16), GW(352)(8)}$
3. $\int_0^\infty \frac{dx}{(b^2 + x^2) \cosh ax} = \frac{2\pi}{b} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2ab + (2k-1)\pi} \quad [a > 0, \quad b > 0] \quad \text{BI (97)(5)}$
4. $\int_0^\infty \frac{dx}{(b^2 + x^2) \cosh \pi x} = \frac{1}{b} \beta\left(b + \frac{1}{2}\right) \quad [b > 0] \quad \text{BI (97)(4)}$
5. $\int_0^\infty \frac{x dx}{(1 + x^2) \sinh \pi x} = \ln 2 - \frac{1}{2} \quad \text{BI (97)(7)}$
6. $\int_0^\infty \frac{dx}{(1 + x^2) \cosh \pi x} = 2 - \frac{\pi}{2} \quad \text{BI (97)(1)}$
7. $\int_0^\infty \frac{x dx}{(1 + x^2) \sinh \frac{\pi x}{2}} = \frac{\pi}{2} - 1 \quad \text{BI (97)(8)}$
8. $\int_0^\infty \frac{dx}{(1 + x^2) \cosh \frac{\pi x}{2}} = \ln 2 \quad \text{BI (97)(2)}$
9. $\int_0^\infty \frac{x dx}{(1 + x^2) \sinh \frac{\pi x}{4}} = \frac{1}{\sqrt{2}} \left[\pi + 2 \ln(\sqrt{2} + 1) \right] - 2 \quad \text{BI (97)(9)}$

$$10. \quad \int_0^\infty \frac{dx}{(1+x^2) \cosh \frac{\pi x}{4}} = \frac{1}{\sqrt{2}} \left[\pi - 2 \ln \left(\sqrt{2} + 1 \right) \right] \quad \text{BI (97)(3)}$$

3.523

$$1. \quad \int_0^\infty \frac{x^{\beta-1}}{\sinh ax} dx = \frac{2^\beta - 1}{2^{\beta-1} a^\beta} \Gamma(\beta) \zeta(\beta) \quad [\operatorname{Re} \beta > 1, \quad a > 0] \quad \text{WH}$$

$$2. \quad \int_0^\infty \frac{x^{2n-1}}{\sinh ax} dx = \frac{2^{2n} - 1}{2n} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \quad [a > 0, \quad n = 1, 2, \dots] \quad \text{WH, GW(352)(2a)}$$

$$3. \quad \int_0^\infty \frac{x^{\beta-1}}{\cosh ax} dx = \frac{2}{(2a)^\beta} \Gamma(\beta) \Phi \left(-1, \beta, \frac{1}{2} \right) \\ = \frac{2}{(2a)^\beta} \Gamma(\beta) \sum_{k=0}^\infty (-1)^k \left(\frac{2}{2k+1} \right)^\beta \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{EH I 35, ET I 322(1)}$$

$$4. \quad \int_0^\infty \frac{x^{2n}}{\cosh ax} dx = \left(\frac{\pi}{2a} \right)^{2n+1} |E_{2n}| \quad [a > 0] \quad \text{BI(84)(12)a, GW(352)(1a)}$$

$$5. \quad \int_0^\infty \frac{x^2 dx}{\cosh x} = \frac{\pi^3}{8} \quad (\text{cf. 4.261 6}) \quad \text{BI (84)(3)}$$

$$6. \quad \int_0^\infty \frac{x^3 dx}{\sinh x} = \frac{\pi^4}{8} \quad (\text{cf. 4.262 1 and 2}) \quad \text{BI (84)(5)}$$

$$7. \quad \int_0^\infty \frac{x^4 dx}{\cosh x} = \frac{5}{32} \pi^5 \quad \text{BI (84)(7)}$$

$$8. \quad \int_0^\infty \frac{x^5}{\sinh x} dx = \frac{\pi^6}{4} \quad \text{BI (84)(8)}$$

$$9. \quad \int_0^\infty \frac{x^6}{\cosh x} dx = \frac{61}{128} \pi^7 \quad \text{BI (84)(9)}$$

$$10. \quad \int_0^\infty \frac{x^7}{\sinh x} dx = \frac{17}{16} \pi^8 \quad \text{BI (84)(10)}$$

$$11. \quad \int_0^\infty \frac{x^{1/2} dx}{\cosh x} = \sqrt{\pi} \sum_{k=0}^\infty (-1)^k \frac{1}{(2k+1)^{3/2}} \quad \text{BI (98)(7)a}$$

$$12. \quad \int_0^\infty \frac{dx}{x^{1/2} \cosh x} = 2\sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{1/2}} \quad \text{BI (98)(25)a}$$

3.524

$$1. \quad \int_0^\infty x^{\mu-1} \frac{\sinh \beta x}{\sinh \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \zeta \left[\mu, \frac{1}{2} \left(1 - \frac{\beta}{\gamma} \right) \right] - \zeta \left[\mu, \frac{1}{2} \left(1 + \frac{\beta}{\gamma} \right) \right] \right\} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \quad \operatorname{Re} \mu > -1] \quad \text{ET I 323(10)}$$

$$2.^{11} \quad \int_0^\infty x^{2m} \frac{\sinh ax}{\sinh bx} dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \left(\tan \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (112)(20)a}$$

3.
$$\int_0^\infty \frac{\sinh ax}{\sinh bx} \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^\infty \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} - \frac{1}{[b(2k+1)+a]^{1-p}} \right\}$$

$[b > |a|, \quad p < 1]$ Bl (131)(2)a
- 4.¹¹
$$\int_0^\infty x^{2m+1} \frac{\sinh ax}{\cosh bx} dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \left(\sec \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl (112)(18)a
5.
$$\int_0^\infty x^{\mu-1} \frac{\cosh \beta x}{\sinh \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \zeta \left[\mu, \frac{1}{2} \left(1 - \frac{\beta}{\gamma} \right) \right] + \zeta \left[\mu, \frac{1}{2} \left(1 + \frac{\beta}{\gamma} \right) \right] \right\}$$

$[\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \quad \operatorname{Re} \mu > 1]$ ET I 323(12)
6.
$$\int_0^\infty x^{2m} \frac{\cosh ax}{\cosh bx} dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \left(\sec \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl(112)(17)
7.
$$\int_0^\infty \frac{\cosh ax}{\cosh bx} \cdot \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^\infty (-1)^k \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} + \frac{1}{[b(2k+1)+a]^{1-p}} \right\}$$

$[b > |a|, \quad p < 1]$ Bl(131)(1)a
8.
$$\int_0^\infty x^{2m+1} \frac{\cosh ax}{\sinh bx} dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \left(\tan \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl (112)(19)a
- 9.⁸
$$\int_0^\infty x^2 \frac{\sinh ax}{\sinh bx} dx = \frac{\pi^3}{4b^3} \sin \frac{a\pi}{2b} \sec^3 \frac{a\pi}{2b}$$

$[b > |a|]$ Bl (84)(18)
10.
$$\int_0^\infty x^4 \frac{\sinh ax}{\sinh bx} dx = 8 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^5 \cdot \sin \frac{a\pi}{2b} \cdot \left(2 + \sin^2 \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl (82)(17)a
11.
$$\int_0^\infty x^6 \frac{\sinh ax}{\sinh bx} dx = 16 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^7 \sin \frac{a\pi}{2b} \left(45 - 30 \cos^2 \frac{a\pi}{2b} + 2 \cos^4 \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl (82)(21)a
12.
$$\int_0^\infty x \frac{\sinh ax}{\cosh bx} dx = \frac{\pi^2}{4b^2} \sin \frac{a\pi}{2b} \sec^2 \frac{a\pi}{2b}$$

$[b > |a|]$ Bl (84)(15)a
13.
$$\int_0^\infty x^3 \frac{\sinh ax}{\cosh bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^4 \sin \frac{a\pi}{2b} \cdot \left(6 - \cos^2 \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl (82)(14)a
14.
$$\int_0^\infty x^5 \frac{\sinh ax}{\cosh bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^6 \sin \frac{a\pi}{2b} \left(120 - 60 \cos^2 \frac{a\pi}{2b} + \cos^4 \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl (82)(18)a
15.
$$\int_0^\infty x^7 \frac{\sinh ax}{\cosh bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^8 \sin \frac{a\pi}{2b} \left(5040 - 4200 \cos^2 \frac{a\pi}{2b} + 546 \cos^4 \frac{a\pi}{2b} - \cos^6 \frac{a\pi}{2b} \right)$$

$[b > |a|]$ Bl (82)(22)a
16.
$$\int_0^\infty x \frac{\cosh ax}{\sinh bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^2$$

$[b > |a|]$ Bl (84)(16)a

$$17. \int_0^\infty x^3 \frac{\cosh ax}{\sinh bx} dx = 2 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^4 \left(1 + 2 \sin^2 \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (82)(15)a}$$

$$18. \int_0^\infty x^5 \frac{\cosh ax}{\sinh bx} dx = 8 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^6 \left(15 - 15 \cos^2 \frac{a\pi}{2b} + 2 \cos^4 \frac{a\pi}{2b} \right) \\ [b > |a|] \quad \text{BI (82)(19)a}$$

$$19. \int_0^\infty x^7 \frac{\cosh ax}{\sinh bx} dx = 16 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^8 \left(315 - 420 \cos^2 \frac{a\pi}{2b} + 126 \cos^4 \frac{a\pi}{2b} - 4 \cos^6 \frac{a\pi}{2b} \right) \\ [b > |a|] \quad \text{BI(82)(23)a}$$

$$20. \int_0^\infty x^2 \frac{\cosh ax}{\cosh bx} dx = \frac{\pi^3}{8b^3} \left(2 \sec^3 \frac{a\pi}{2b} - \sec \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (84)(17)a}$$

$$21. \int_0^\infty x^4 \frac{\cosh ax}{\cosh bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^5 \left(24 - 20 \cos^2 \frac{a\pi}{2b} + \cos^4 \frac{a\pi}{2b} \right) \\ [b > |a|] \quad \text{BI (82)(16)a}$$

$$22. \int_0^\infty x^6 \frac{\cosh ax}{\cosh bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^7 \left(720 - 840 \cos^2 \frac{a\pi}{2b} + 182 \cos^4 \frac{a\pi}{2b} - \cos^6 \frac{a\pi}{2b} \right) \\ [b > |a|] \quad \text{BI (82)(20)a}$$

$$23. \int_0^\infty \frac{\sinh ax}{\cosh bx} \cdot \frac{dx}{x} = \ln \tan \left(\frac{a\pi}{4b} + \frac{\pi}{4} \right) \quad [b > |a|] \quad \text{BI (95)(3)a}$$

3.525

$$1. \int_0^\infty \frac{\sinh ax}{\sinh \pi x} \cdot \frac{dx}{1+x^2} = -\frac{a}{2} \cos a + \frac{1}{2} \sin a \ln [2(1+\cos a)] \\ [\pi \geq |a|] \quad \text{BI (97)(10)a}$$

$$2. \int_0^\infty \frac{\sinh ax}{\sinh \frac{\pi}{2}x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{2} \sin a + \frac{1}{2} \cos a \ln \frac{1-\sin a}{1+\sin a} \quad [\pi \geq 2|a|] \quad \text{BI (97)(11)a}$$

$$3. \int_0^\infty \frac{\cosh ax}{\sinh \pi x} \cdot \frac{x dx}{1+x^2} = \frac{1}{2} (a \sin a - 1) + \frac{1}{2} \cos a \ln [2(1+\cos a)] \\ [\pi > |a|] \quad \text{BI (97)(12)a}$$

$$4. \int_0^\infty \frac{\cosh ax}{\sinh \frac{\pi}{2}x} \cdot \frac{x dx}{1+x^2} = \frac{\pi}{2} \cos a - 1 + \frac{1}{2} \sin a \ln \frac{1+\sin a}{1-\sin a} \\ \left[\frac{\pi}{2} > |a| \right] \quad \text{BI (97)(13)a}$$

$$5. \int_0^\infty \frac{\sinh ax}{\cosh \pi x} \cdot \frac{x dx}{1+x^2} = -2 \sin \frac{a}{2} + \frac{\pi}{2} \sin a - \cos a \ln \tan \frac{a+\pi}{4} \\ [\pi > |a|] \quad \text{GW (352)(12)}$$

$$6. \int_0^\infty \frac{\cosh ax}{\cosh \pi x} \cdot \frac{dx}{1+x^2} = 2 \cos \frac{a}{2} - \frac{\pi}{2} \cos a - \sin a \ln \tan \frac{a+\pi}{4} \\ [\pi > |a|] \quad \text{GW (352)(11)}$$

$$7. \quad \int_0^\infty \frac{\sinh ax}{\sinh bx} \cdot \frac{dx}{c^2 + x^2} = \frac{\pi}{c} \sum_{k=1}^\infty \frac{\sin \frac{k(b-a)}{b} \pi}{bc + k\pi} \quad [b \geq |a|] \quad \text{BI (97)(18)}$$

$$8. \quad \int_0^\infty \frac{\cosh ax}{\sinh bx} \cdot \frac{x dx}{c^2 + x^2} = \frac{\pi}{2bc} + \pi \sum_{k=1}^\infty \frac{\cos \frac{k(b-a)}{b} \pi}{bc + k\pi} \quad [b > |a|] \quad \text{BI (97)(19)}$$

3.526

$$1. \quad \int_0^\infty \frac{\sinh ax \cosh bx}{\cosh cx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \left\{ \tan \frac{(a+b+c)\pi}{4c} \cot \frac{(b+c-a)\pi}{4c} \right\} \\ [c > |a| + |b|] \quad \text{BI (93)(10)a}$$

$$2. \quad \int_0^\infty \frac{\sinh^2 ax}{\sinh bx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \sec \frac{a}{b} \pi \quad [b > |2a|] \quad \text{BI (95)(5)a}$$

$$3. \quad \int_0^\infty \frac{x^{\mu-1}}{\sinh \beta x \cosh \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \Phi \left[-1, \mu, \frac{1}{2} \left(1 + \frac{\beta}{\gamma} \right) \right] + \Phi \left[-1, \mu, \frac{1}{2} \left(1 - \frac{\beta}{\gamma} \right) \right] \right\} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 323(11)}$$

3.527

$$1. \quad \int_0^\infty \frac{x^{\mu-1}}{\sinh^2 ax} dx = \frac{4}{(2a)^\mu} \Gamma(\mu) \zeta(\mu-1) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 2] \quad \text{BI (86)(7)a}$$

$$2. \quad \int_0^\infty \frac{x^{2m}}{\sinh^2 ax} dx = \frac{\pi^{2m}}{a^{2m+1}} |B_{2m}| \quad [a > 0, \quad m = 1, 2, \dots] \quad \text{BI(86)(5)a}$$

$$3.^6 \quad \int_0^\infty \frac{x^{\mu-1}}{\cosh^2 ax} dx = \frac{4}{(2a)^\mu} (1 - 2^{2-\mu}) \Gamma(\mu) \zeta(\mu-1) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0, \quad \mu \neq 2] \\ = \frac{1}{a^2} \ln 2 \quad [\operatorname{Re} a > 0, \quad \mu = 2] \quad \text{BI (86)(6)a}$$

$$4. \quad \int_0^\infty \frac{x dx}{\cosh^2 ax} = \frac{\ln 2}{a^2} \quad [a \neq 0] \quad \text{LO III 396}$$

$$5. \quad \int_0^\infty \frac{x^{2m}}{\cosh^2 ax} dx = \frac{(2^{2m} - 2) \pi^{2m}}{(2a)^{2m} a} |B_{2m}| \quad [a > 0, \quad m = 1, 2, \dots] \quad \text{BI(86)(2)a}$$

$$6. \quad \int_0^\infty x^{\mu-1} \frac{\sinh ax}{\cosh^2 ax} dx = \frac{2 \Gamma(\mu)}{a^\mu} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\mu-1}} \quad [\operatorname{Re} \mu > 1, \quad a > 0] \quad \text{BI (86)(15)a}$$

$$7. \quad \int_0^\infty \frac{x \sinh ax}{\cosh^2 ax} dx = \frac{\pi}{2a^2} \quad [a > 0] \quad \text{BI (86)(8)a}$$

$$8. \quad \int_0^\infty x^{2m+1} \frac{\sinh ax}{\cosh^2 ax} dx = \frac{2m+1}{a} \left(\frac{\pi}{2a} \right)^{2m+1} |E_{2m}| \quad [a > 0, \quad m = 0, 1, \dots] \quad \text{BI (86)(12)a}$$

$$9. \quad \int_0^\infty x^{2m+1} \frac{\cosh ax}{\sinh^2 ax} dx = \frac{2^{2m+1} - 1}{a^2 (2a)^{2m}} (2m+1)! \zeta(2m+1) \\ [a \neq 0, \quad m = 1, 2, \dots] \quad \text{BI (86)(13)a}$$

$$10.^{11} \quad \int_0^\infty x^{2m} \frac{\cosh ax}{\sinh^2 ax} dx = \frac{2^{2m} - 1}{a} \left(\frac{\pi}{a}\right)^{2m} |B_{2m}| \quad [a > 0, \quad m = 1, 2, \dots] \quad \text{BI (86)(14)a}$$

$$11.^8 \quad \int_0^\infty \frac{x \sinh ax}{\cosh^{2\mu+1} ax} dx = \frac{\sqrt{\pi}}{4\mu a^2} \frac{\Gamma(\mu)}{\Gamma(\mu + \frac{1}{2})} \quad [\mu > 0, \quad a > 0] \quad \text{LI (86)(9)}$$

$$12. \quad \int_{-\infty}^\infty \frac{x^2 dx}{\sinh^2 x} = \frac{\pi^2}{3} \quad \text{BI (102)(2)a}$$

$$13. \quad \int_0^\infty x^2 \frac{\cosh ax}{\sinh^2 ax} dx = \frac{\pi^2}{2a^3} \quad [a > 0] \quad \text{BI (86)(11)a}$$

$$14.^{11} \quad \int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x} dx = 4G \quad [a \neq 0] \quad \text{BI (86)(10)a}$$

$$15.^{10} \quad \int_0^\infty \frac{\tanh \frac{x}{2} dx}{\cosh x} = \ln 2 \quad \text{BI (93)(17)a}$$

$$16.^* \quad \int_0^\infty x^{\mu-1} \frac{\cosh ax}{\sinh^2 ax} dx = \frac{2 \Gamma(\mu) \zeta(\mu-1)}{a^\mu} (1 - 2^{1-\mu})$$

3.528

$$1. \quad \int_0^\infty \frac{(1+xi)^{2n-1} - (1-xi)^{2n-1}}{i \sinh \frac{\pi x}{2}} dx = 2 \quad \text{BI (87)(8)}$$

$$2. \quad \int_0^\infty \frac{(1+xi)^{2n} - (1-xi)^{2n}}{i \sinh \frac{\pi x}{2}} dx = (-1)^{n+1} 2|E_{2n}| + 2 \quad [n = 0, 1, \dots] \quad \text{BI (87)(7)}$$

3.529

$$1. \quad \int_0^\infty \left(\frac{1}{\sinh x} - \frac{1}{x} \right) \frac{dx}{x} = -\ln 2 \quad \text{BI (94)(10)a}$$

$$2. \quad \int_0^\infty \frac{\cosh ax - 1}{\sinh bx} \cdot \frac{dx}{x} = -\ln \cos \frac{a\pi}{2b} \quad [b > |a|] \quad \text{GW (352)(66)}$$

$$3. \quad \int_0^\infty \left(\frac{a}{\sinh ax} - \frac{b}{\sinh bx} \right) \frac{dx}{x} = (b-a) \ln 2 \quad \text{BI (94)(11)a}$$

3.531

$$1.^7 \quad \int_0^\infty \frac{x dx}{2 \cosh x - 1} = \frac{4}{\sqrt{3}} \left[\frac{\pi}{3} \ln 2 - L\left(\frac{\pi}{3}\right) \right] = 1.1719536193 \dots$$

[see **8.26** for $L(x)$] LI (88)(1)

$$2.^{10} \quad \int_0^\infty \frac{x dx}{\cosh 2x + \cos 2t} = \frac{t \ln 2 - L(t)}{\sin 2t} \quad \text{LO III 402}$$

$$3. \quad \int_0^\infty \frac{x^2 dx}{\cosh x + \cos t} = \frac{t}{3} \cdot \frac{\pi^2 - t^2}{\sin t} \quad [0 < t < \pi] \quad \text{BI (88)(3)a}$$

$$4. \quad \int_0^\infty \frac{x^4 dx}{\cosh x + \cos t} = \frac{t}{15} \frac{(\pi^2 - t^2)(7\pi^2 - 3t^2)}{\sin t} \quad [0 < t < \pi] \quad \text{BI (88)(4)a}$$

$$\begin{aligned}
 5.^3 \quad \int_0^\infty \frac{x^{2m} dx}{\cosh x - \cos 2a\pi} &= 2(2m)! \operatorname{cosec} 2a\pi \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}} \quad [0 < a < 1, \quad a \neq \tfrac{1}{2}] \\
 &= 2(2^{2m-1} - 1) \pi^{2m} |B_{2m}| \quad [a = \tfrac{1}{2}]
 \end{aligned}$$

BI (88)(5)a

$$\begin{aligned}
 6.^3 \quad \int_0^\infty \frac{x^{\mu-1} dx}{\cosh x - \cos t} &= \frac{i\Gamma(\mu)}{\sin t} [e^{-it} \Phi(e^{-it}, \mu, 1) - e^{it} \Phi(e^{it}, \mu, 1)] \quad [\operatorname{Re} \mu > 0, \quad 0 < t < 2\pi, \quad t \neq \pi] \quad \text{ET I 323(5)} \\
 &= (2 - 2^{3-\mu}) \Gamma(\mu) \zeta(\mu - 1) \quad [\mu \neq 2, \quad t = \pi] \\
 &= 2 \ln 2 \quad [\mu = 2, \quad t = \pi]
 \end{aligned}$$

$$7. \quad \int_0^\infty \frac{x^\mu dx}{\cosh x + \cos t} = \frac{2\Gamma(\mu+1)}{\sin t} \sum_{k=1}^\infty (-1)^{k-1} \frac{\sin kt}{k^{\mu+1}} \quad [\mu > -1, \quad 0 < t < \pi] \quad \text{BII (96)(14)a}$$

$$\begin{aligned}
 8. \quad \int_0^u \frac{x dx}{\cosh 2x - \cos 2t} &= \frac{1}{2} \operatorname{cosec} 2t [L(\theta + t) - L(\theta - t) - 2L(t)] \\
 &\quad [\theta = \arctan(\tanh u \cot t), \quad t \neq n\pi] \quad \text{LO III 402}
 \end{aligned}$$

3.532

$$\begin{aligned}
 1.^{11} \quad \int_0^\infty \frac{x^n dx}{a \cosh x + b \sinh x} &= \frac{2n!}{a+b} \sum_{k=0}^\infty \frac{1}{(2k+1)^{n+1}} \left(\frac{b-a}{b+a} \right)^k \\
 &\quad [a > 0, \quad b > 0, \quad n > -1] \quad \text{GW (352)(5)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^u \frac{x \cosh x dx}{\cosh 2x - \cos 2t} &= \frac{1}{2} \operatorname{cosec} t \left\{ L\left(\frac{\theta+t}{2}\right) - L\left(\frac{\theta-t}{2}\right) + L\left(\pi - \frac{\psi+t}{2}\right) \right. \\
 &\quad \left. + L\left(\frac{\psi-t}{2}\right) - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right) \right\} \\
 &\quad \left[\tan \frac{\theta}{2} = \tanh \frac{u}{2} \cot \frac{t}{2}, \quad \tan \frac{\psi}{2} = \coth \frac{u}{2} \cot \frac{t}{2}; \quad t \neq n\pi \right] \quad \text{LO III 288a}
 \end{aligned}$$

3.533

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{x \cosh x dx}{\cosh 2x - \cos 2t} &= \operatorname{cosec} t \left[\frac{\pi}{2} \ln 2 - L\left(\frac{t}{2}\right) - L\left(\frac{\pi-t}{2}\right) \right] \\
 &\quad [t \neq m\pi] \quad \text{LO III 403}
 \end{aligned}$$

$$2.^6 \quad \int_0^\infty x \frac{\sinh ax dx}{(\cosh ax - \cos t)^2} = \frac{\pi - t}{a^2} \operatorname{cosec} t \quad [a > 0, \quad 0 < t < \pi] \quad (\text{cf. 3.5141})$$

BI (88)(11)a

$$3. \quad \int_0^\infty x^3 \frac{\sinh x dx}{(\cosh x + \cos t)^2} = \frac{t(\pi^2 - t^2)}{\sin t} \quad [0 < t < \pi] \quad (\text{cf. 3.531 3})$$

BI (88)(13)

$$\begin{aligned}
 4.^{11} \quad \int_0^\infty x^{2m+1} \frac{\sinh x \, dx}{(\cosh x - \cos 2a\pi)^2} &= 2(2m+1)! \operatorname{cosec} 2a\pi \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}} \quad [0 < a < 1, \quad a \neq \tfrac{1}{2}] \\
 &= 2(2m+1) (2^{2m-1} - 1) \pi^{2m} |B_{2m}| \quad [a = \tfrac{1}{2}]
 \end{aligned}$$

BI (88)(14)

3.534

$$1. \quad \int_0^1 \sqrt{1-x^2} \cosh ax \, dx = \frac{\pi}{2a} I_1(a) \quad \text{WA 94(9)}$$

$$2. \quad \int_0^1 \frac{\cosh ax}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} I_0(a) \quad \text{WA 94(9)}$$

$$3.535 \quad \int_0^1 \frac{x}{\sqrt{\cosh 2a - \cosh 2ax}} \cdot \frac{dx}{\sinh ax} = \frac{\pi}{2\sqrt{2}a^2} \cdot \frac{\arcsin(\tanh a)}{\sinh a} \quad [a > 0] \quad \text{BI (80)(11)}$$

3.536

$$1.^{11} \quad \int_0^\infty \frac{x^2}{\cosh^2 x} \, dx = \frac{\pi^2}{12} \quad \text{BI (98)(7)}$$

$$2. \quad \int_0^\infty \frac{x^2 \tanh x^2 \, dx}{\cosh^2 x} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}} \quad \text{BI (98)(8)}$$

$$\begin{aligned}
 3. \quad \int_0^\infty \sinh(\nu \operatorname{arcsinh} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}} \, dx &= \frac{\sin \frac{\mu\pi}{2} \sin \frac{\nu\pi}{2}}{2^\mu \pi} \Gamma(\mu) \Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right) \\
 & \quad [-1 < \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|] \quad \text{ET I 324(14)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_0^\infty \cosh(\nu \operatorname{arccosh} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}} \, dx &= \frac{\cos \frac{\mu\pi}{2} \cos \frac{\nu\pi}{2}}{2^\mu \pi} \Gamma(\mu) \Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right) \\
 & \quad [0 < \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|] \quad \text{ET I 324(15)}
 \end{aligned}$$

3.54 Combinations of hyperbolic functions and exponentials**3.541**

$$1. \quad \int_0^\infty e^{-\mu x} \sinh^\nu \beta x \, dx = \frac{1}{2^{\nu+1}\beta} B\left(\frac{\mu}{2\beta} - \frac{\nu}{2}, \nu+1\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1, \operatorname{Re} \mu > \operatorname{Re} \beta \nu] \\ \text{EH I 11(25), ET I 163(5)}$$

$$\begin{aligned}
 2. \quad \int_0^\infty e^{-\mu x} \frac{\sinh \beta x}{\sinh bx} \, dx &= \frac{1}{2b} \left[\psi\left(\frac{1}{2} + \frac{\mu+\beta}{2b}\right) - \psi\left(\frac{1}{2} + \frac{\mu-\beta}{2b}\right) \right] \\
 & \quad [\operatorname{Re}(\mu + b \pm \beta) > 0] \quad \text{EH I 16(14)a}
 \end{aligned}$$

$$3. \quad \int_{-\infty}^\infty e^{-\mu x} \frac{\sinh \mu x}{\sinh \beta x} \, dx = \frac{\pi}{2\beta} \tan \frac{\mu\pi}{\beta} \quad [\operatorname{Re} \beta > 2|\operatorname{Re} \mu|] \quad \text{BI (18)(6)}$$

$$4. \quad \int_0^\infty e^{-x} \frac{\sinh ax}{\sinh x} \, dx = \frac{1}{a} - \frac{\pi}{2} \cot \frac{a\pi}{2} \quad [0 < a < 2] \quad \text{BI (4)(3)}$$

$$5. \quad \int_0^\infty \frac{e^{-px} \, dx}{(\cosh px)^{2q+1}} = \frac{2^{2q-2}}{p} B(q, q) - \frac{1}{2qp} \quad [p > 0, \quad q > 0] \quad \text{LI (27)(19)}$$

6. $\int_0^\infty e^{-\mu x} \frac{dx}{\cosh x} = \beta \left(\frac{\mu+1}{2} \right)$ [Re $\mu > -1$] ET I 163(7)
7. $\int_0^\infty e^{-\mu x} \tanh x \, dx = \beta \left(\frac{\mu}{2} \right) - \frac{1}{\mu}$ [Re $\mu > 0$] ET I 163(9)
8. $\int_0^\infty \frac{e^{-\mu x}}{\cosh^2 x} \, dx = \mu \beta \left(\frac{\mu}{2} \right) - 1$ [Re $\mu > 0$] ET I 163(8)
9. $\int_0^\infty e^{-\mu x} \frac{\sinh \mu x}{\cosh^2 \mu x} \, dx = \frac{1}{\mu} (1 - \ln 2)$ [Re $\mu > 0$] LI (27)(15)
10. $\int_0^\infty e^{-qx} \frac{\sinh px}{\sinh qx} \, dx = \frac{1}{p} - \frac{\pi}{2q} \cot \frac{p\pi}{2q}$ [0 < p < 2q] BI (27)(9)a

3.542

1. $\int_0^\infty e^{-\mu x} (\cosh \beta x - 1)^\nu \, dx = \frac{1}{2^\nu \beta} B \left(\frac{\mu}{\beta} - \nu, 2\nu + 1 \right)$
[Re $\beta > 0$, Re $\nu > -\frac{1}{2}$, Re $\mu > \text{Re } \beta \nu$] ET I 163(6)
2. $\int_0^\infty e^{-\mu x} (\cosh x - \cosh u)^{\nu-1} \, dx = -i \sqrt{\frac{2}{\pi}} e^{i\pi\nu} \Gamma(\nu) \sinh^{\nu-\frac{1}{2}u} Q_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh u)$
[Re $\nu > 0$, Re $\mu > \text{Re } \nu - 1$] EH I 155(4), ET I 164(23)

3.543

1. $\int_{-\infty}^\infty \frac{e^{-ibx} \, dx}{\sinh x + \sinh t} = -\frac{i\pi e^{itb}}{\sinh \pi b \cosh t} (\cosh \pi b - e^{-2itb})$
[t > 0] ET I 121(30)
2. $\int_0^\infty \frac{e^{-\mu x}}{\cosh x - \cos t} \, dx = 2 \operatorname{cosec} t \sum_{k=1}^\infty \frac{\sin kt}{\mu + k}$ [Re $\mu > -1$, t $\neq 2n\pi$] BI (6)(10)a
3. $\int_0^\infty \frac{1 - e^{-x} \cos t}{\cosh x - \cos t} e^{-(\mu-1)x} \, dx = 2 \sum_{k=0}^\infty \frac{\cos kt}{\mu + k}$ [Re $\mu > 0$, t $\neq 2n\pi$] BI (6)(9)a
4. $\int_0^\infty \frac{e^{px} - \cos t}{(\cosh px + \cos t)^2} \, dx = \frac{1}{p} \left(t \operatorname{cosec} t + \frac{1}{1 + \cos t} \right)$ [p > 0] BI (27)(26)a

3.544 $\int_u^\infty \frac{\exp \left[-\left(n + \frac{1}{2} \right) x \right]}{\sqrt{2} (\cosh x - \cosh u)} \, dx = Q_n (\cosh u), \quad [u > 0]$ EH II 181(33)

3.545

1. $\int_0^\infty \frac{\sinh ax}{e^{px} + 1} \, dx = \frac{\pi}{2p} \operatorname{cosec} \frac{a\pi}{p} - \frac{1}{2a}$ [p > a, p > 0] BI (27)(3)
2. $\int_0^\infty \frac{\sinh ax}{e^{px} - 1} \, dx = \frac{1}{2a} - \frac{\pi}{2p} \cot \frac{a\pi}{p}$ [p > a, p > 0] BI (27)(9)

3.546

1. $\int_0^\infty e^{-\beta x^2} \sinh ax \, dx = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} \exp \frac{a^2}{4\beta} \Phi \left(\frac{a}{2\sqrt{\beta}} \right)$ $[\operatorname{Re} \beta > 0]$ ET I 166(38)a
2. $\int_0^\infty e^{-\beta x^2} \cosh ax \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \frac{a^2}{4\beta}$ $[\operatorname{Re} \beta > 0]$ FI II 720a
3. $\int_0^\infty e^{-\beta x^2} \sinh^2 ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left(\exp \frac{a^2}{\beta} - 1 \right)$ $[\operatorname{Re} \beta > 0]$ ET I 166(40)
4. $\int_0^\infty e^{-\beta x^2} \cosh^2 ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left(\exp \frac{a^2}{\beta} + 1 \right)$ $[\operatorname{Re} \beta > 0]$ ET I 166(41)

3.547

1. $\int_0^\infty \exp(-\beta \sinh x) \sinh \gamma x \, dx = \frac{\pi}{2} \cot \frac{\gamma\pi}{2} [J_\gamma(\beta) - \mathbf{J}_\gamma(\beta)] - \frac{\pi}{2} [\mathbf{E}_\gamma(\beta) + Y_\gamma(\beta)] = \gamma S_{-1,\gamma}(\beta)$
 $[\operatorname{Re} \beta > 0]$ WA 341(5), ET I 168(14)a
2. $\int_0^\infty \exp(-\beta \cosh x) \sinh \gamma x \sinh x \, dx = \frac{\gamma}{\beta} K_\gamma(\beta)$
3. $\int_0^\infty \exp(-\beta \sinh x) \cosh \gamma x \, dx = \frac{\pi}{2} \tan \frac{\pi\gamma}{2} [\mathbf{J}_\gamma(\beta) - J_\gamma(\beta)] - \frac{\pi}{2} [\mathbf{E}_\gamma(\beta) + Y_\gamma(\beta)] = S_{0,\gamma}(\beta)$
 $[\operatorname{Re} \beta > 0, \quad \gamma \text{ not an integer}]$
ET I 168(16)a, WA 341(4), EH II 84(50)
4. $\int_0^\infty \exp(-\beta \cosh x) \cosh \gamma x \, dx = K_\gamma(\beta)$ $[\operatorname{Re} \beta > 0]$ ET I 168(16)a, WA 201(5)
5. $\int_0^\infty \exp(-\beta \sinh x) \sinh \gamma x \cosh x \, dx = \frac{\gamma}{\beta} S_{0,\gamma}(\beta)$ $[\operatorname{Re} \beta > 0]$ ET I 168(7), EH II 85(51)
6. $\int_0^\infty \exp(-\beta \sinh x) \sinh[(2n+1)x] \cosh x \, dx = O_{2n+1}(\beta)$
 $[\operatorname{Re} \beta > 0]$ ET I 167(5)
7. $\int_0^\infty \exp(-\beta \sinh x) \cosh \gamma x \cosh x \, dx = \frac{1}{\beta} S_{1,\gamma}(\beta)$ $[\operatorname{Re} \beta > 0]$
8. $\int_0^\infty \exp(-\beta \sinh x) \cosh 2nx \cosh x \, dx = O_{2n}(\beta)$ $[\operatorname{Re} \beta > 0]$ ET I 168(6)
9. $\int_0^\infty \exp(-\beta \cosh x) \sinh^{2\nu} x \, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) K_\nu(\beta)$
 $[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$ EH II 82(20)
- 10.¹¹ $\int_0^\infty \exp[-2(\beta \coth x + \mu x)] \sinh^{2\nu} x \, dx = \frac{1}{2} \beta^\nu \Gamma(\mu - \nu) W_{-\mu, \nu - \frac{1}{2}}(4\beta)$
 $[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu]$
11. $\int_0^\infty \exp \left(-\frac{\beta^2}{2} \sinh x \right) \sinh^{\nu-1} x \cosh^\nu x \, dx = -\pi D_\nu \left(\beta e^{i\pi/4} \right) D_\nu \left(\beta e^{-i\pi/4} \right)$
 $[\operatorname{Re} \nu > 0, \quad |\arg \beta| \leq \frac{\pi}{4}]$ EH II 120(10)

$$12. \quad \int_0^\infty \frac{\exp(2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx = \frac{1}{2} \sqrt{\pi^3 \beta} \left[J_{\nu+\frac{1}{4}}(\beta) J_{\nu-\frac{1}{4}}(\beta) + Y_{\nu+\frac{1}{4}}(\beta) Y_{\nu-\frac{1}{4}}(\beta) \right] \\ [\operatorname{Re} \beta > 0] \quad \text{EH I 169(20)}$$

$$13. \quad \int_0^\infty \frac{\exp(-2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx = \frac{1}{2} \sqrt{\pi^3 \beta} \left[J_{\nu+\frac{1}{4}}(\beta) Y_{\nu-\frac{1}{4}}(\beta) - J_{\nu-\frac{1}{4}}(\beta) Y_{\nu+\frac{1}{4}}(\beta) \right] \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 169(21)}$$

$$14. \quad \int_0^\infty \frac{\exp(-2\beta \sinh x) \sinh 2\nu x}{\sqrt{\sinh x}} dx = \frac{1}{4i} \sqrt{\frac{\pi^3 \beta}{2}} \left\{ e^{\nu \pi i} H_{\frac{1}{2}+\nu}^{(1)}(\beta) H_{\frac{1}{2}-\nu}^{(2)}(\beta) \right. \\ \left. - e^{-\nu \pi i} H_{\frac{1}{2}-\nu}^{(1)}(\beta) H_{\frac{1}{2}+\nu}^{(2)}(\beta) \right\} \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 170(24)}$$

$$15. \quad \int_0^\infty \frac{\exp(-2\beta \sinh x) \cosh 2\nu x}{\sqrt{\sinh x}} dx = \frac{1}{4} \sqrt{\frac{\pi^3 \beta}{2}} \left\{ e^{\nu \pi i} H_{\frac{1}{2}+\nu}^{(1)}(\beta) H_{\frac{1}{2}-\nu}^{(2)}(\beta) \right. \\ \left. + e^{-\nu \pi i} H_{\frac{1}{2}-\nu}^{(1)}(\beta) H_{\frac{1}{2}+\nu}^{(2)}(\beta) \right\} \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 170(25)}$$

$$16. \quad \int_0^\infty \frac{\exp(-2\beta \cosh x) \cosh 2\nu x}{\sqrt{\cosh x}} dx = \sqrt{\frac{\beta}{\pi}} K_{\nu+\frac{1}{4}}(\beta) K_{\nu-\frac{1}{4}}(\beta) \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 170(26)}$$

$$17.^8 \quad \int_0^\infty \frac{\exp[-2\beta (\cosh x - 1)] \cosh 2\nu x}{\sqrt{\cosh x}} dx = \sqrt{\frac{\beta}{\pi}} \cdot e^{2\beta} K_{\nu+\frac{1}{4}}(\beta) K_{\nu-\frac{1}{4}}(\beta) \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 170(27)}$$

$$18. \quad \int_0^\infty \frac{\cos[(\nu + \frac{1}{4})\pi] \exp(-2\nu x - 2\beta \sinh x) + \sin[(\nu + \frac{1}{4})\pi] \exp(2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx \\ = \frac{1}{2} \sqrt{\pi^3 \beta} \left[J_{\frac{1}{4}+\nu}(\beta) J_{\frac{1}{4}-\nu}(\beta) + Y_{\frac{1}{4}+\nu}(\beta) Y_{\frac{1}{4}-\nu}(\beta) \right] \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 169(22)}$$

$$19. \quad \int_0^\infty \frac{\sin[(\nu + \frac{1}{4})\pi] \exp(-2\nu x - 2\beta \sinh x) - \cos[(\nu + \frac{1}{4})\pi] \exp(2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx \\ = \frac{1}{2} \sqrt{\pi^3 \beta} \left[J_{\frac{1}{4}+\nu}(\beta) Y_{\frac{1}{4}-\nu}(\beta) - J_{\frac{1}{4}-\nu}(\beta) Y_{\frac{1}{4}+\nu}(\beta) \right] \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 169(23)}$$

$$20. \quad \int_0^\infty \frac{\exp[-\beta (\cosh x - 1)] \cosh \nu x \sinh x}{\sqrt{\cosh x (\cosh x - 1)}} dx = e^\beta K_\nu(\beta) \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 169(19)}$$

3.548

$$1. \quad \int_0^\infty e^{-\mu x^4} \sinh ax^2 dx = \frac{\pi}{4} \sqrt{\frac{a}{2\mu}} \exp\left(\frac{a^2}{8\mu}\right) I_{\frac{1}{4}}\left(\frac{a^2}{8\mu}\right) \quad [\operatorname{Re} \mu > 0, \quad a \geq 0] \quad \text{ET I 166(42)}$$

$$2. \quad \int_0^\infty e^{-\mu x^4} \cosh ax^2 dx = \frac{\pi}{4} \sqrt{\frac{a}{2\mu}} \exp\left(\frac{a^2}{8\mu}\right) I_{-\frac{1}{4}}\left(\frac{a^2}{8\mu}\right) \\ [\operatorname{Re} \mu > 0, \quad a > 0] \quad \text{ET I 166(43)}$$

3.549

1. $\int_0^\infty e^{-\beta x} \sinh [(2n+1) \operatorname{arcsinh} x] dx = O_{2n+1}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.547} \ 6)$
ET I 167(5)
2. $\int_0^\infty e^{-\beta x} \cosh (2n \operatorname{arcsinh} x) dx = O_{2n}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.547} \ 8)$
ET I 168(6)
3. $\int_0^\infty e^{-\beta x} \sinh (\nu \operatorname{arcsinh} x) dx = \frac{\nu}{\beta} S_{0,\nu}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.5475}) \quad \text{ET I 168(7)}$
4. $\int_0^\infty e^{-\beta x} \cosh (\nu \operatorname{arcsinh} x) dx = \frac{1}{\beta} S_{1,\nu}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.547} \ 7)$

A number of other integrals containing hyperbolic functions and exponentials, depending on $\operatorname{arcsinh} x$ or $\operatorname{arccosh} x$, can be found by first making the substitution $x = \sinh t$ or $x = \cosh t$.

3.55–3.56 Combinations of hyperbolic functions, exponentials, and powers**3.551**

1. $\int_0^\infty x^{\mu-1} e^{-\beta x} \sinh \gamma x dx = \frac{1}{2} \Gamma(\mu) [(\beta - \gamma)^{-\mu} - (\beta + \gamma)^{-\mu}]$
[$\operatorname{Re} \beta > -1, \quad \operatorname{Re} \beta > |\operatorname{Re} \gamma|$]
ET I 164(18)
2. $\int_0^\infty x^{\mu-1} e^{-\beta x} \cosh \gamma x dx = \frac{1}{2} \Gamma(\mu) [(\beta - \gamma)^{-\mu} + (\beta + \gamma)^{-\mu}]$
[$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > |\operatorname{Re} \gamma|$]
ET I 164(19)
3. $\int_0^\infty x^{\mu-1} e^{-\beta x} \coth x dx = \Gamma(\mu) \left[2^{1-\mu} \zeta \left(\mu, \frac{\beta}{2} \right) - \beta^{-\mu} \right]$
[$\operatorname{Re} \mu > 1, \quad \operatorname{Re} \beta > 0$]
ET I 164(21)
4. $\int_0^\infty x^n e^{-(p+mq)x} \sinh^m qx dx = 2^{-m} n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(p+2kq)^{n+1}}$
[$p > 0, \quad q > 0, \quad m < p + qm$]
LI (81)(4)
- 5.¹¹ $\int_0^1 \frac{e^{-\beta x}}{x} \sinh \gamma x dx = \frac{1}{2} \left[\ln \frac{\beta + \gamma}{\beta - \gamma} + \operatorname{Ei}(\gamma - \beta) - \operatorname{Ei}(-\gamma - \beta) \right]$
[$\beta > \gamma$]
BI (80)(4)
6. $\int_0^\infty \frac{e^{-\beta x}}{x} \sinh \gamma x dx = \frac{1}{2} \ln \frac{\beta + \gamma}{\beta - \gamma}$
[$\operatorname{Re} \beta > |\operatorname{Re} \gamma|$]
ET I 163(12)
7. $\int_1^\infty \frac{e^{-\beta x}}{x} \cosh \gamma x dx = \frac{1}{2} [-\operatorname{Ei}(\gamma - \beta) - \operatorname{Ei}(-\gamma - \beta)]$
[$\operatorname{Re} \beta > |\operatorname{Re} \gamma|$]
ET I 164(15)
- 8.⁶ $\int_0^\infty x e^{-x} \coth x dx = \frac{\pi^2}{4} - 1$
BI (82)(6)

$$9. \quad \int_0^\infty e^{-\beta x} \tanh x \frac{dx}{x} = \ln \frac{\beta}{4} + 2 \ln \frac{\Gamma\left(\frac{\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4} + \frac{1}{2}\right)} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 164(16)}$$

$$10.^6 \quad \int_0^\infty x e^{-x} \coth(x/2) dx = \frac{\pi^2}{3} - 1$$

3.552

$$1. \quad \int_0^\infty \frac{x^{\mu-1} e^{-\beta x}}{\sinh x} dx = 2^{1-\mu} \Gamma(\mu) \zeta\left[\mu, \frac{1}{2}(\beta+1)\right] \quad [\operatorname{Re} \mu > 1, \quad \operatorname{Re} \beta > -1] \quad \text{ET I 164(20)}$$

$$2. \quad \int_0^\infty \frac{x^{2m-1} e^{-ax}}{\sinh ax} dx = \frac{1}{2m} |B_{2m}| \left(\frac{\pi}{a}\right)^{2m} \quad [a > 0, \quad m = 1, 2, \dots] \quad \text{EH I 38(24)a}$$

$$3. \quad \int_0^\infty \frac{x^{\mu-1} e^{-x}}{\cosh x} dx = 2^{1-\mu} (1 - 2^{1-\mu}) \Gamma(\mu) \zeta(\mu) \quad [\operatorname{Re} \mu > 0, \quad \mu \neq 1] \\ = \ln 2 \quad [\text{if } \mu = 1] \quad \text{EH I 32(5)}$$

$$4. \quad \int_0^\infty \frac{x^{2m-1} e^{-ax}}{\cosh ax} dx = \frac{1 - 2^{1-2m}}{2m} |B_{2m}| \left(\frac{\pi}{a}\right)^{2m} \quad [a > 0, \quad m = 1, 2, \dots] \quad \text{EH I 39(25)a}$$

$$5. \quad \int_0^\infty \frac{x^2 e^{-2nx}}{\sinh x} dx = 4 \sum_{k=n}^\infty \frac{1}{(2k+1)^3} \quad [n = 0, 1, 2, \dots] \quad (\text{cf. 4.261 13}) \\ \text{BI(84)(4)}$$

$$6.^{11} \quad \int_0^\infty \frac{x^3 e^{-2nx}}{\sinh x} dx = \frac{\pi^4}{8} - 12 \sum_{k=1}^n \frac{1}{(2k-1)^4} \quad [n = 0, 1, \dots] \quad (\text{cf. 4.262 6}) \\ \text{BI (84)(6)}$$

3.553

$$1. \quad \int_0^\infty \frac{\sinh^2 ax}{\sinh x} \frac{e^{-x} dx}{x} = \frac{1}{2} \ln(a\pi \operatorname{cosec} a\pi) \quad [a < 1] \quad \text{BI (95)(7)}$$

$$2.^{11} \quad \int_0^\infty \frac{\sinh^2 \frac{x}{2}}{\cosh x} \cdot \frac{e^{-x} dx}{x} = \frac{1}{2} \ln \frac{4}{\pi} \quad (\text{cf. 4.267 2}) \quad \text{BI (95)(4)}$$

3.554

$$1.^{11} \quad \int_0^\infty e^{-\beta x} (1 - \operatorname{sech} x) \frac{dx}{x} = 2 \ln \frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)} - \ln \frac{\beta}{4} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 164(17)}$$

$$2. \quad \int_0^\infty e^{-\beta x} \left(\frac{1}{x} - \operatorname{cosech} x\right) dx = \psi\left(\frac{\beta+1}{2}\right) - \ln \frac{\beta}{2} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 163(10)}$$

$$3. \quad \int_0^\infty \left[\frac{\sinh\left(\frac{1}{2} - \beta\right)x}{\sinh \frac{x}{2}} - (1 - 2\beta)e^{-x} \right] \frac{dx}{x} = 2 \ln \Gamma(\beta) - \ln \pi + \ln(\sin \pi \beta) \\ [0 < \operatorname{Re} \beta < 1] \quad \text{EH I 21(7)}$$

$$4. \quad \int_0^\infty e^{-\beta x} \left(\frac{1}{x} - \coth x \right) dx = \psi \left(\frac{\beta}{2} \right) - \ln \frac{\beta}{2} + \frac{1}{\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 163(11)}$$

$$5. \quad \int_0^\infty \left\{ -\frac{\sinh qx}{\sinh \frac{x}{2}} + 2qe^{-x} \right\} \frac{dx}{x} = 2 \ln \Gamma \left(q + \frac{1}{2} \right) + \ln \cos \pi q - \ln \pi$$

$$[q^2 < \frac{1}{2}] \quad \text{WH}$$

$$6. \quad \int_0^\infty x^{\mu-1} e^{-\beta x} (\coth x - 1) dx = 2^{1-\mu} \Gamma(\mu) \zeta \left(\mu, \frac{\beta}{2} + 1 \right)$$

$$[\operatorname{Re} \beta > 0; \quad \operatorname{Re} \mu > 1] \quad \text{ET I 164(22)}$$

3.555

$$1. \quad \int_0^\infty \frac{\sinh^2 ax}{1 - e^{px}} \cdot \frac{dx}{x} = \frac{1}{4} \ln \left(\frac{p}{2a\pi} \sin \frac{2a\pi}{p} \right)$$

$$[0 < 2|a| < p] \quad (\text{cf. 3.545 2})$$

$$\text{BI (93)(15)}$$

$$2. \quad \int_0^\infty \frac{\sinh^2 ax}{e^x + 1} \cdot \frac{dx}{x} = -\frac{1}{4} \ln (a\pi \cot a\pi)$$

$$[a < \frac{1}{2}] \quad (\text{cf. 3.545 1}) \quad \text{BI (93)(9)}$$

3.556

$$1. \quad \int_{-\infty}^\infty x \frac{1 - e^{px}}{\sinh x} dx = -\frac{\pi^2}{2} \tan^2 \frac{p\pi}{2}$$

$$[p < 1] \quad (\text{cf. 4.255 3}) \quad \text{BI (101)(4)}$$

$$2. \quad \int_0^\infty \frac{1 - e^{-px}}{\sinh x} \cdot \frac{1 - e^{-(p+1)x}}{x} dx = 2p \ln 2$$

$$[p > -1] \quad \text{BI (95)(8)}$$

3.557

$$1. \quad \int_0^\infty \frac{e^{-px} - e^{-qx}}{\cosh x - \cos \frac{m}{n}\pi} \cdot \frac{dx}{x}$$

$$= 2 \operatorname{cosec} \left(\frac{m}{n}\pi \right) \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \ln \frac{\Gamma \left(\frac{n+q+k}{2n} \right) \Gamma \left(\frac{p+k}{2n} \right)}{\Gamma \left(\frac{n+p+k}{2n} \right) \Gamma \left(\frac{q+k}{2n} \right)} \quad [m+n \text{ odd}]$$

$$= 2 \operatorname{cosec} \left(\frac{m}{n}\pi \right) \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \ln \frac{\Gamma \left(\frac{n+q-k}{n} \right) \Gamma \left(\frac{p+k}{n} \right)}{\Gamma \left(\frac{n+p-k}{n} \right) \Gamma \left(\frac{q+k}{n} \right)} \quad [m+n \text{ even}]$$

$$[p > -1, \quad q > -1] \quad \text{BI (96)(1)}$$

$$2. \quad \int_0^\infty \frac{(1 - e^{-x})^2}{\cosh x + \cos \frac{m}{n}\pi} \cdot \frac{dx}{x}$$

$$= 2 \operatorname{cosec} \left(\frac{m}{n}\pi \right) \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \times \ln \frac{[\Gamma \left(\frac{n+k+1}{2n} \right)]^2 \Gamma \left(\frac{k+2}{2n} \right) \Gamma \left(\frac{k}{2n} \right)}{[\Gamma \left(\frac{k+1}{2n} \right)]^2 \Gamma \left(\frac{n+k}{2n} \right) \Gamma \left(\frac{n+k+2}{2n} \right)} \quad [m+n \text{ odd}]$$

$$= 2 \operatorname{cosec} \left(\frac{m}{n}\pi \right) \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \times \ln \frac{[\Gamma \left(\frac{n-k+1}{n} \right)]^2 \Gamma \left(\frac{k+2}{n} \right) \Gamma \left(\frac{k}{n} \right)}{[\Gamma \left(\frac{k+1}{n} \right)]^2 \Gamma \left(\frac{n-k}{n} \right) \Gamma \left(\frac{n-k+2}{n} \right)} \quad [m+n \text{ even}]$$

$$\text{BI (96)(2)}$$

$$\begin{aligned}
3. \quad \int_0^\infty \left[e^{-x} \tan \frac{m}{2n} \pi - \frac{e^{-px} \sin \frac{m}{n} \pi}{\cosh x + \cos \frac{m}{n} \pi} \right] \cdot \frac{dx}{x} \\
= \tan \left(\frac{m}{2n} \pi \right) \ln(2n) + 2 \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left(\frac{km}{n} \pi \right) \ln \frac{\Gamma \left(\frac{p+n+k}{2n} \right)}{\Gamma \left(\frac{p+k}{2n} \right)} \quad [m+n \text{ odd}] \\
= \tan \left(\frac{m}{2n} \pi \right) \ln n + 2 \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left(\frac{km}{n} \pi \right) \ln \frac{\Gamma \left(\frac{p+n-k}{n} \right)}{\Gamma \left(\frac{p+k}{n} \right)} \quad [m+n \text{ even}]
\end{aligned}$$

BI (96)(3)

$$4. \quad \int_0^\infty \frac{1 + e^{-x}}{\cosh x + \cos a} \cdot \frac{dx}{x^{1-p}} = 2 \sec \frac{a}{2} \Gamma(p) \sum_{k=1}^\infty (-1)^{k-1} \frac{\cos \left(k - \frac{1}{2} \right) a}{k^p}$$

[p > 0] LI (96)(5)

$$5. \quad \int_0^\infty \frac{x^q e^{-\frac{x}{2}} \cosh \frac{x}{2}}{\cosh x + \cos \lambda} dx = \frac{\Gamma(q+1)}{\cos \frac{\lambda}{2}} \sum_{k=1}^\infty (-1)^{k-1} \frac{\cos \left(k - \frac{1}{2} \right) \lambda}{k^{q+1}}$$

[q > -1] LI (96)(5)a

$$6. \quad \int_0^\infty x \frac{e^{-x} - \cos a}{\cosh x - \cos a} dx = |a| \pi - \frac{a^2}{2} - \frac{\pi^2}{3}$$

BI (88)(8)

$$7. \quad \int_0^\infty x^{2m+1} \frac{e^{-x} - \cos a \pi}{\cosh x - \cos a \pi} dx = 2 \cdot (2m+1)! \sum_{k=1}^\infty \frac{\cos ka \pi}{k^{2m+2}}$$

BI (88)(6)

3.558

$$1. \quad \int_0^\infty x \frac{1 - e^{-nx}}{\sinh^2 \frac{x}{2}} dx = \frac{2n\pi^2}{3} - 4 \sum_{k=1}^{n-1} \frac{n-k}{k^2}$$

BI (85)(3)

$$2. \quad \int_0^\infty x \frac{1 - (-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} dx = \frac{n\pi^2}{3} + 4 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^2}$$

LI (85)(1)

$$3. \quad \int_0^\infty x^2 \frac{1 - e^{-nx}}{\sinh^2 \frac{x}{2}} dx = 8n \zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3}$$

BI (85)(5)

$$4. \quad \int_0^\infty x^2 e^x \frac{1 - e^{-2nx}}{\sinh^2 x} dx = 8n \sum_{k=1}^\infty \frac{1}{(2k-1)^3} - 8 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^3}$$

LI (85)(6)

$$5. \quad \int_0^\infty x^2 \frac{1 + (-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} dx = 6n \zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3}$$

LI (85)(4)

$$6. \quad \int_0^\infty x^3 \frac{1 - e^{-nx}}{\sinh^2 \frac{x}{2}} dx = \frac{4}{15} n \pi^4 - 24 \sum_{k=1}^{n-1} \frac{n-k}{k^4}$$

BI (85)(9)

$$7. \quad \int_0^\infty x^3 \frac{1 + (-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} dx = \frac{7}{30} n \pi^4 + 24 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^4}$$

BI (85)(8)

$$3.559 \quad \int_0^\infty e^{-x} \left[a - \frac{1}{2} + \frac{(1 - e^{-x})(1 - ax) - xe^{-x}}{4 \sinh^2 \frac{x}{2}} e^{(2-a)x} \right] \frac{dx}{x} = a - \frac{1}{2} + \ln \Gamma(a) - \frac{1}{2} \ln(2\pi) \quad [a > 0]$$

BI (96)(6)

$$3.561 \quad \int_0^\infty \frac{e^{-2x} \tanh \frac{x}{2}}{x \cosh x} dx = 2 \ln \frac{\pi}{2\sqrt{2}} \quad \text{BI (93)(18)}$$

3.562

$$1. \quad \int_0^\infty x^{2\mu-1} e^{-\beta x^2} \sinh \gamma x dx = \frac{1}{2} \Gamma(2\mu) (2\beta)^{-\mu} \exp\left(\frac{\gamma^2}{8\beta}\right) \left[D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) - D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \right] \\ [\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 166(44)}$$

$$2. \quad \int_0^\infty x^{2\mu-1} e^{-\beta x^2} \cosh \gamma x dx = \frac{1}{2} \Gamma(2\mu) (2\beta)^{-\mu} \exp\left(\frac{\gamma^2}{8\beta}\right) \left[D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) + D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \right] \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 166(45)}$$

$$3. \quad \int_0^\infty x e^{-\beta x^2} \sinh \gamma x dx = \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{BI(81)(12)a, ET I 165(34)}$$

$$4. \quad \int_0^\infty x e^{-\beta x^2} \cosh \gamma x dx = \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{1}{2\beta} \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 166(35)}$$

$$5. \quad \int_0^\infty x^2 e^{-\beta x^2} \sinh \gamma x dx = \frac{\sqrt{\pi} (2\beta + \gamma^2)}{8\beta^2 \sqrt{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{\gamma}{4\beta^2} \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 166(36)}$$

$$6. \quad \int_0^\infty x^2 e^{-\beta x^2} \cosh \gamma x dx = \frac{\sqrt{\pi} (2\beta + \gamma^2)}{8\beta^2 \sqrt{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 166(37)}$$

3.6–4.1 Trigonometric Functions

3.61 Rational functions of sines and cosines and trigonometric functions of multiple angles

3.611

$$1. \quad \int_0^{2\pi} (1 - \cos x)^n \sin nx dx = 0 \quad \text{BI (68)(10)}$$

$$2. \quad \int_0^{2\pi} (1 - \cos x)^n \cos nx dx = (-1)^n \frac{\pi}{2^{n-1}} \quad \text{BI (68)(11)}$$

$$3. \quad \int_0^\pi (\cos t + i \sin t \cos x)^n dx = \int_0^\pi (\cos t + i \sin t \cos x)^{-n-1} dx = \pi P_n(\cos t) \quad \text{EH I 158(23)a}$$

3.612

$$\begin{aligned}
 1.^6 \quad \int_0^\pi \frac{\sin nx \cos mx}{\sin x} dx &= 0 \quad \text{for } n \leq m; \\
 &= \pi \quad \text{for } n > m, \quad \text{if } m+n \text{ is odd and positive} \\
 &= 0 \quad \text{for } n > m, \quad \text{if } m+n \text{ is even}
 \end{aligned}$$

LI (64)(3)

$$\begin{aligned}
 2. \quad \int_0^\pi \frac{\sin nx}{\sin x} dx &= 0 \quad \text{for } n \text{ even} \\
 &= \pi \quad \text{for } n \text{ odd}
 \end{aligned}$$

BI (64)(1, 2)

$$3. \quad \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx = \frac{\pi}{2}$$

FI II 145

$$4. \quad \int_0^{\pi/2} \frac{\sin 2nx}{\sin x} dx = 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{k-1}}{2n-1} \right)$$

GW (332)(21b)

$$5. \quad \int_0^\pi \frac{\sin 2nx}{\cos x} dx = 2 \int_0^{\pi/2} \frac{\sin 2nx}{\cos x} dx = (-1)^{n-1} 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1} \right)$$

GW (332)(22a)

$$6. \quad \int_0^\pi \frac{\cos(2n+1)x}{\cos x} dx = 2 \int_0^{\pi/2} \frac{\cos(2n+1)x}{\cos x} dx = (-1)^n \pi$$

GW (332)(22b)

$$7. \quad \int_0^{\pi/2} \frac{\sin 2nx \cos x}{\sin x} dx = \frac{\pi}{2}$$

LI (45)(17)

3.613

$$1.^6 \quad \int_0^\pi \frac{\cos nx dx}{1 + a \cos x} = \frac{\pi}{\sqrt{1-a^2}} \left(\frac{\sqrt{1-a^2}-1}{a} \right)^n \quad [a^2 < 1, \quad n \geq 0]$$

BI (64)(12)

$$\begin{aligned}
 2.^6 \quad \int_0^\pi \frac{\cos nx dx}{1 - 2a \cos x + a^2} &= \frac{\pi a^n}{1 - a^2} \quad [a^2 < 1, \quad n \geq 0] \\
 &= \frac{\pi}{(a^2 - 1) a^n} \quad [a^2 > 1, \quad n \geq 0]
 \end{aligned}$$

BI (65)(3)

$$\begin{aligned}
 3. \quad \int_0^\pi \frac{\sin nx \sin x dx}{1 - 2a \cos x + a^2} &= \frac{\pi a^{n-1}}{2} \quad [a^2 < 1, \quad n \geq 1] \\
 &= \frac{\pi}{2a^{n+1}} \quad [a^2 > 1, \quad n \geq 1]
 \end{aligned}$$

BI(65)(4), GW(332)(34a)

$$\begin{aligned}
4.10 \quad \int_0^\pi \frac{\cos nx \cos x \, dx}{1 - 2a \cos x + a^2} &= \frac{\pi}{2} \cdot \frac{1 + a^2}{1 - a^2} a^{n-1} & [a^2 < 1, \quad n \geq 1] \\
&= \frac{\pi}{2a^{n+1}} \cdot \frac{a^2 + 1}{a^2 - 1} & [a^2 > 1, \quad n \geq 1] \\
&= \frac{\pi a}{1 - a^2} & [n = 0, \quad a^2 < 1] \\
&= \frac{\pi}{a(a^2 - 1)} & [n = 0, \quad a^2 > 1]
\end{aligned}$$

BI(65)(5), GW(332)(34b)

$$5. \quad \int_0^\pi \frac{\cos(2n-1)x \, dx}{1 - 2a \cos 2x + a^2} = \int_0^\pi \frac{\cos 2nx \cos x \, dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1] \quad \text{BI (65)(9, 10)}$$

$$6. \quad \int_0^\pi \frac{\cos(2n-1)x \cos 2x \, dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1] \quad \text{BI (65)(12)}$$

$$7. \quad \int_0^\pi \frac{\sin 2nx \sin x \, dx}{1 - 2a \cos 2x + a^2} = \int_0^\pi \frac{\sin(2n-1)x \sin 2x \, dx}{1 - 2a \cos 2x + a^2} = 0$$

$$[a^2 \neq 1] \quad \text{BI (65)(6, 7)}$$

$$\begin{aligned}
8. \quad \int_0^\pi \frac{\sin(2n-1)x \sin x \, dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2} \cdot \frac{a^{n-1}}{1+a} & [a^2 < 1] \\
&= \frac{\pi}{2} \cdot \frac{1}{(1+a)a^n} & [a^2 > 1]
\end{aligned}$$

BI (65)(8)

$$\begin{aligned}
9. \quad \int_0^\pi \frac{\cos(2n-1)x \cos x \, dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2} \cdot \frac{a^{n-1}}{1-a} & [a^2 < 1] \\
&= \frac{\pi}{2} \cdot \frac{1}{(a-1)a^n} & [a^2 > 1]
\end{aligned}$$

BI (65)(11)

$$\begin{aligned}
10. \quad \int_0^\pi \frac{\sin nx - a \sin(n-1)x}{1 - 2a \cos x + a^2} \sin mx \, dx &= 0 & \text{for } m < n \\
&= \frac{\pi}{2} a^{m-n} & \text{for } m \geq n
\end{aligned}$$

LI (65)(13)

$$11.6 \quad \int_0^\pi \frac{\cos nx - a \cos(n-1)x}{1 - 2a \cos x + a^2} \cos mx \, dx = \frac{\pi}{2} (a^{|m|-n} - 1)$$

BI (65)(14)

$$12. \quad \int_0^\pi \frac{\sin nx - a \sin[(n+1)x]}{1 - 2a \cos x + a^2} \, dx = 0 \quad [a^2 < 1] \quad \text{BI (68)(13)}$$

$$13. \quad \int_0^\pi \frac{\cos nx - a \cos[(n+1)x]}{1 - 2a \cos x + a^2} \, dx = \pi a^n \quad [a^2 < 1] \quad \text{BI (68)(14)}$$

$$\begin{aligned}
 3.614^7 \int_0^\pi \frac{\sin x}{a^2 - 2ab \cos x + b^2} \cdot \frac{\sin px \cdot dx}{1 - 2a^p \cos px + a^{2p}} \\
 &= \frac{\pi b^{p-1}}{2a^{p+1}(1 - b^p)} \quad [0 < b \leq a \leq 1, \quad p = 1, 2, 3, \dots] \\
 &= \frac{\pi a^{p-1}}{2b(b^p - a^{2p})} \quad [0 < a \leq 1, \quad a^2 < b, \quad p = 1, 2, 3, \dots]
 \end{aligned}$$

BI (66)(9)

3.615

$$\begin{aligned}
 1. \quad \int_0^{\pi/2} \frac{\cos 2nx \, dx}{1 - a^2 \sin^2 x} &= \frac{(-1)^n \pi}{2\sqrt{1 - a^2}} \left(\frac{1 - \sqrt{1 - a^2}}{a} \right)^{2n} \quad [a^2 < 1] && \text{BI (47)(27)} \\
 2. \quad \int_0^\pi \frac{\cos x \sin 2nx \, dx}{1 + (a + b \sin x)^2} &= -\frac{\pi}{b} \sin \left\{ 2n \arctan \sqrt{\frac{s}{2}} \right\} \tan^{2n} \left(\frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right) \\
 3. \quad \int_0^\pi \frac{\cos x \cos(2n+1)x \, dx}{1 + (a + b \sin x)^2} &= \frac{\pi}{b} \cos \left\{ (2n+1) \arctan \sqrt{\frac{s}{2}} \right\} \tan^{2n+1} \left(\frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right) \\
 &\quad \text{where } s = -(1 + b^2 - a^2) + \sqrt{(1 + b^2 - a^2)^2 + 4a^2} \quad \text{BI (65)(21, 22)}
 \end{aligned}$$

3.616

$$\begin{aligned}
 1. \quad \int_0^\pi (1 - 2a \cos x + a^2)^n \, dx &= \pi \sum_{k=0}^n \binom{n}{k}^2 a^{2k} && \text{BI (63)(1)} \\
 2.^{10} \quad \int_0^\pi \frac{dx}{(1 - 2a \cos x + a^2)^n} &= \frac{1}{2} \int_0^{2\pi} \frac{dx}{(1 - 2a \cos x + a^2)^n} \\
 &= \frac{\pi}{(1 - a^2)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2 (n-k-1)!} \left(\frac{a^2}{1 - a^2} \right)^k \quad [a^2 < 1] \\
 &= \frac{\pi}{(a^2 - 1)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2 (n-k-1)!} \frac{1}{(a^2 - 1)^k} \quad [a^2 > 1] \\
 &&& \text{BI (331)(63)} \\
 3. \quad \int_0^\pi (1 - 2a \cos x + a^2)^n \cos nx \, dx &= (-1)^n \pi a^n && \text{BI (63)(2)} \\
 4. \quad \int_0^\pi (1 - 2a \cos x + a^2)^n \cos mx \, dx \\
 &= \frac{1}{2} \int_0^{2\pi} (1 - 2a \cos x + a^2)^n \cos mx \, dx \\
 &= 0 \quad [n < m] \\
 &= \pi (-a)^m (1 + a^2)^{n-m} \sum_{k=0}^{[(n-m)/2]} \binom{n}{k} \binom{n-k}{m+k} \left(\frac{a}{1 + a^2} \right)^{2k} \quad [n \geq m] \\
 &&& \text{GW (332)(35a)} \\
 5. \quad \int_0^{2\pi} \frac{\sin nx \, dx}{(1 - 2a \cos 2x + a^2)^m} &= 0 && \text{GW (332)(32a)}
 \end{aligned}$$

$$6. \quad \int_0^\pi \frac{\sin x \, dx}{(1 - 2a \cos 2x + a^2)^m} = \frac{1}{2(m-1)a} \left[\frac{1}{(1-a)^{2m-2}} - \frac{1}{(1+a)^{2m-2}} \right] \quad [a \neq 0, \pm 1]$$

GW (332)(32c)

$$7. \quad \int_0^\pi \frac{\cos nx \, dx}{(1 - 2a \cos x + a^2)^m} = \frac{1}{2} \int_0^{2\pi} \frac{\cos nx \, dx}{(1 - 2a \cos x + a^2)^m}$$

$$= \frac{a^{2m+n-2}\pi}{(1-a^2)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} \left(\frac{1-a^2}{a^2} \right)^k \quad [a^2 < 1]$$

$$= \frac{\pi}{a^n (a^2 - 1)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} (a^2 - 1)^k \quad [a^2 > 1]$$

GW (332)(31)

$$8. \quad \int_0^{\pi/2} \frac{\cos 2nx \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \binom{2n}{n} \frac{(b^2 - a^2)^n}{(2ab)^{2n+1}} \pi$$

[a > 0, b > 0] GW (332)(30b)

$$3.617^{10} \int_0^\pi \frac{dx}{(1 - 2a \cos x + a^2)^{n+1/2}} = \frac{2}{|1+a|^{2n+1}} F_n \left(\frac{2\sqrt{|a|}}{|1+a|} \right), \quad |a| \neq 1$$

with

$$F_n(k) = \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{n+1/2}}$$

where the $F_n(k)$ satisfies the recurrence relation

$$F_{n+1}(k) = F_n(k) + \frac{k}{2n+1} \frac{dF_n(k)}{dk}, \quad n = 0, 1, 2, \dots$$

and

$$F_0(k) = \mathbf{K}(k) \equiv \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}}$$

is the complete elliptic integral of the first kind.

Introducing the complete elliptic integral of the second kind

$$\mathbf{E}(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 x)^{1/2} dx$$

the derivatives

$$\frac{d\mathbf{K}(k)}{dk} = \frac{\mathbf{E}(k)}{k(1-k^2)} - \frac{\mathbf{K}(k)}{k}, \quad \frac{d\mathbf{E}(k)}{dk} = \frac{\mathbf{E}(k) - \mathbf{K}(k)}{k}$$

combined with the recurrence relation lead to

$$F_1(k) = F_0(k) + k \frac{dF_0(k)}{dk}$$

$$= \mathbf{K}(k) + \frac{\mathbf{E}(k)}{1-k^2} - \mathbf{K}(k) = \frac{\mathbf{E}(k)}{1-k^2},$$

$$F_2(k) = \frac{\mathbf{E}(k)}{1-k^2} + \frac{k}{3} \frac{d}{dk} \left(\frac{\mathbf{E}(k)}{1-k^2} \right)$$

$$= \frac{1}{3(1-k^2)} \left[\left(\frac{4-2k^2}{1-k^2} \right) \mathbf{E}(k) - \mathbf{K}(k) \right]$$

3.62 Powers of trigonometric functions

3.621

$$1. \quad \int_0^{\pi/2} \sin^{\mu-1} x \, dx = \int_0^{\pi/2} \cos^{\mu-1} x \, dx = 2^{\mu-2} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \quad \text{FI II 789}$$

$$2. \quad \int_0^{\pi/2} \sin^{3/2} x \, dx = \int_0^{\pi/2} \cos^{3/2} x \, dx = \frac{1}{6\sqrt{2\pi}} \left[\Gamma\left(\frac{1}{4}\right) \right]^2$$

$$3. \quad \int_0^{\pi/2} \sin^{2m} x \, dx = \int_0^{\pi/2} \cos^{2m} x \, dx = \frac{(2m-1)!!}{(2m)!!} \frac{\pi}{2} \quad \text{FI II 151}$$

$$4. \quad \int_0^{\pi/2} \sin^{2m+1} x \, dx = \int_0^{\pi/2} \cos^{2m+1} x \, dx = \frac{(2m)!!}{(2m+1)!!} \quad \text{FI II 151}$$

$$5. \quad \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \, dx = \frac{1}{2} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0]$$

LO V 113(50), LO V 122, FI II 788

$$6.* \quad \int_0^{\pi/2} \sqrt{\sin x} \, dx = \sqrt{\frac{2}{\pi}} \left(\Gamma\left(\frac{3}{4}\right) \right)^2$$

$$7.* \quad \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \frac{(\Gamma(\frac{1}{4}))^2}{2\sqrt{2\pi}}$$

3.622

$$1. \quad \int_0^{\pi/2} \tan^{\pm\mu} x \, dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (42)(1)}$$

$$2. \quad \int_0^{\pi/4} \tan^{\mu} x \, dx = \frac{1}{2} \beta\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1] \quad \text{BI (34)(1)}$$

$$3. \quad \int_0^{\pi/4} \tan^{2n} x \, dx = (-1)^n \frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \quad \text{BI (34)(2)}$$

$$4.^{11} \quad \int_0^{\pi/4} \tan^{2n+1} x \, dx = (-1)^n \frac{\ln 2}{2} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} \quad \text{BI (34)(3)}$$

3.623

$$1. \quad \int_0^{\pi/2} \tan^{\mu-1} x \cos^{2\nu-2} x \, dx = \int_0^{\pi/2} \cot^{\mu-1} x \sin^{2\nu-2} x \, dx = \frac{1}{2} B\left(\frac{\mu}{2}, \nu - \frac{\mu}{2}\right) \quad [0 < \operatorname{Re} \mu < 2 \operatorname{Re} \nu] \quad \text{BI(42)(6), BI(45)(22)}$$

$$2.^6 \quad \int_0^{\pi/4} \tan^{\mu} x \sin^2 x \, dx = \frac{1+\mu}{4} \beta\left(\frac{\mu+1}{2}\right) - \frac{1}{4} \quad [\operatorname{Re} \mu > -1] \quad \text{BI (34)(4)}$$

$$3.^6 \quad \int_0^{\pi/4} \tan^{\mu} x \cos^2 x \, dx = \frac{1-\mu}{4} \beta\left(\frac{\mu+1}{2}\right) + \frac{1}{4} \quad [\operatorname{Re} \mu > -1] \quad \text{BI (34)(5)}$$

3.624

1. $\int_0^{\pi/4} \frac{\sin^p x}{\cos^{p+2} x} dx = \frac{1}{p+1} \quad [p > -1] \quad \text{GW (331)(34b)}$
- 2.³ $\int_0^{\pi/2} \frac{\sin^{\mu-\frac{1}{2}} x}{\cos^{2\mu-1} x} dx = \int_0^{\pi/2} \frac{\cos^{\mu-\frac{1}{2}} x}{\sin^{2\mu-1} x} dx = \frac{1}{2} \left\{ \frac{\Gamma(\frac{\mu}{2} + \frac{1}{4}) \Gamma(1-\mu)}{\Gamma(\frac{5}{4} - \frac{\mu}{2})} \right\}$
 $[-\frac{1}{2} < \operatorname{Re} \mu < 1] \quad \text{LI (55)(12)}$
- 3.¹¹ $\int_0^{\pi/4} \frac{\cos^{n-\frac{1}{2}}(2x)}{\cos^{2n+1}(x)} dx = \pi \frac{(2n)!!}{2^{2n+1} (n!)^2} \quad \text{BI (38)(3)}$
- 4.⁸ $\int_0^{\pi/4} \frac{\cos^\mu 2x}{\cos^{2(\mu+1)} x} dx = 2^{2\mu} \text{B}(\mu+1, \mu+1) \quad [\operatorname{Re} \mu > -1] \quad \text{BI (35)(1)}$
5. $\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{\cos^\mu 2x} dx = 2^{1-2\mu} \text{B}(2\mu-1, 1-\mu) = \frac{\Gamma(\mu-\frac{1}{2}) \Gamma(1-\mu)}{2\sqrt{\pi}} \quad [\frac{1}{2} < \operatorname{Re} \mu < 1] \quad \text{BI (35)(4)}$
- 6.⁶ $\int_0^{\pi/2} \left(\frac{\sin ax}{\sin x} \right)^2 dx = \frac{a\pi}{2} - \frac{1}{2} \sin \pi a [2a \beta(a) - 1], \quad [a > 0]$

3.625

1. $\int_0^{\pi/4} \frac{\sin^{2n-1} x \cos^p 2x}{\cos^{2p+2n+1} x} dx = \frac{(n-1)!}{2} \cdot \frac{\Gamma(p+1)}{\Gamma(p+n+1)}$
 $= \frac{(n-1)!}{2(p+n)(p+n-1) \cdots (p+1)} = \frac{1}{2} \text{B}(n, p+1)$
 $[p > -1] \quad (\text{cf. 3.251 1}) \quad \text{BI (35)(2)}$
2. $\int_0^{\pi/4} \frac{\sin^{2n} x \cos^p 2x}{\cos^{2p+2n+2} x} dx = \frac{1}{2} \text{B}(n + \frac{1}{2}, p+1) \quad [p > -1] \quad (\text{cf. 3.251 1}) \quad \text{BI (35)(3)}$
3. $\int_0^{\pi/4} \frac{\sin^{2n-1} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m} x} dx = \frac{(2n-2)!!(2m-1)!!}{(2n+2m-1)!!} \quad \text{BI (38)(6)}$
- 4.⁸ $\int_0^{\pi/4} \frac{\sin^{2n} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m+1} x} dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} \cdot \frac{\pi}{2} \quad \text{BI (38)(7)}$

3.626

1. $\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x} \sqrt{\cos 2x} dx = \frac{(2n-2)!!}{(2n+1)!!} \quad (\text{cf. 3.251 1}) \quad \text{BI (38)(4)}$
2. $\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x} \sqrt{\cos 2x} dx = \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{\pi}{2} \quad (\text{cf. 3.251 1}) \quad \text{BI (38)(5)}$

3.627 $\int_0^{\pi/2} \frac{\tan^\mu x}{\cos^\mu x} dx = \int_0^{\pi/2} \frac{\cot^\mu x}{\sin^\mu x} dx = \frac{\Gamma(\mu) \Gamma(\frac{1}{2} - \mu)}{2^\mu \sqrt{\pi}} \sin \frac{\mu\pi}{2}$
 $[-1 < \operatorname{Re} \mu < \frac{1}{2}] \quad \text{BI (55)(12)a}$

3.628¹¹ $\int_0^{\frac{\pi}{2}} \sec^{2p} x \sin^{2p-1} x dx = \frac{1}{2\sqrt{\pi}} \Gamma(p) \Gamma(\frac{1}{2} - p)$
 $[0 < p < \frac{1}{2}] \quad \text{WA 691}$

3.63 Powers of trigonometric functions and trigonometric functions of linear functions

3.631

1.
$$\int_0^\pi \sin^{\nu-1} x \sin ax \, dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu \, \text{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$

[Re $\nu > 0$] LO V 121(67a), WA 337a
- 2.⁷
$$\int_0^{\pi/2} 2 \sin^{\nu-2} x \sin \nu x \, dx = \frac{1}{1-\nu} \cos \frac{\nu\pi}{2}$$

[Re $\nu > 1$] GW(332)(16d), FI I 152
- 3.⁶
$$\int_0^\pi \sin^\nu x \sin \nu x \, dx = 2^{-\nu} \pi \sin \frac{\nu\pi}{2}$$

[Re $\nu > -1$] LO V 121(69)
4.
$$\int_0^\pi \sin^n x \sin 2mx \, dx = 0$$

GW (332)(11a)
5.
$$\int_0^\pi \sin^{2n} x \sin(2m+1)x \, dx = \int_0^{\pi/2} \sin^{2n} x \sin(2m+1)x \, dx$$

$$= \frac{(-1)^m 2^{n+1} n! (2n-1)!!}{(2n-2m-1)!! (2m+2n+1)!!} \quad [m \leq n]^*$$

$$= \frac{(-1)^n 2^{n+1} n! (2m-2n-1)!! (2n-1)!!}{(2m+2n+1)!!} \quad [m \geq n]^*$$

GW (332)(11b)
6.
$$\int_0^\pi \sin^{2n+1} x \sin(2m+1)x \, dx = 2 \int_0^{\pi/2} \sin^{2n+1} x \sin(2m+1)x \, dx$$

$$= \frac{(-1)^m \pi}{2^{2n+1}} \binom{2n+1}{n-m} \quad [n \geq m]$$

$$= 0 \quad [n < m]$$

BI(40)(12), GW(332)(11c)
7.
$$\int_0^\pi \sin^n x \cos(2m+1)x \, dx = 0$$

GW (332)(12a)
8.
$$\int_0^\pi \sin^{\nu-1} x \cos ax \, dx = \frac{\pi \cos \frac{a\pi}{2}}{2^{\nu-1} \nu \, \text{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$

[Re $\nu > 0$] LO V 121(68)a, WA 337a
9.
$$\int_0^{\pi/2} \cos^{\nu-1} x \cos ax \, dx = \frac{\pi}{2^\nu \nu \, \text{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$

[Re $\nu > 0$] GW (332)(9c)
10.
$$\int_0^{\pi/2} \sin^{\nu-2} x \cos \nu x \, dx = \frac{1}{\nu-1} \sin \frac{\nu\pi}{2}$$

[Re $\nu > 1$] GW(332)(16b), FI II 15 2

*In 3.631.5, for $m = n$ we should set $(2n-2m-1)!! = 1$

$$11. \quad \int_0^\pi \sin^\nu x \cos \nu x \, dx = \frac{\pi}{2^\nu} \cos \frac{\nu\pi}{2} \quad [\operatorname{Re} \nu > -1] \quad \text{LO V 121(70)a}$$

$$12. \quad \int_0^\pi \sin^{2n} x \cos 2mx \, dx = 2 \int_0^{\pi/2} \sin^{2n} x \cos 2mx \, dx = \frac{(-1)^m}{2^{2n}} \binom{2n}{n-m} \pi \quad [n \geq m]$$

$$= 0 \quad [n < m]$$

BI(40)(16), GW(332)(12b)

$$13.^7 \quad \int_0^\pi \sin^{2n+1} x \cos 2mx \, dx$$

$$= 2 \int_0^{\pi/2} \sin^{2n+1} x \cos 2mx \, dx = \frac{(-1)^m 2^{n+1} n! (2n+1)!!}{(2m-2n-3)!! (2m+2n+1)!!} \quad [n \geq m-1]$$

$$= \frac{(-1)^{n+1} 2^{n+1} n! (2m-2n+3)!! (2n+1)!!}{(2m+2n+1)!!} \quad [n < m-1]$$

GW (332)(12c)

$$14. \quad \int_0^{\pi/2} \cos^{\nu-2} x \sin \nu x \, dx = \frac{1}{\nu-1} \quad [\operatorname{Re} \nu > 1] \quad \text{GW(332)(16c), FI II 152}$$

$$15. \quad \int_0^\pi \cos^m x \sin nx \, dx = [1 - (-1)^{m+n}] \int_0^{\pi/2} \cos^m x \sin nx \, dx$$

$$= [1 - (-1)^{m+n}] \left\{ \sum_{k=0}^{r-1} \frac{m!}{(m-k)!} \frac{(m+n-2k-2)!!}{(m+n)!!} + s \frac{m!(n-m-2)!!}{(m+n)!!} \right\}$$

$$\left[r = \begin{cases} m & \text{if } m \leq n \\ n & \text{if } m \geq n \end{cases} \quad s = \begin{cases} 2 & \text{if } n-m = 4l+2 > 0 \\ 1 & \text{if } n-m = 2l+1 > 0 \\ 0 & \text{if } n-m = 4l \text{ or } n-m < 0 \end{cases} \right] \quad \text{GW (332)(13a)}$$

$$16. \quad \int_0^{\pi/2} \cos^n x \sin nx \, dx = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k} \quad \text{FI II 153}$$

$$17.^{11} \quad \int_0^\pi \cos^n x \sin mx \, dx = \begin{cases} [1 + (-1)^{m+n}] \frac{\pi}{2^{n+1}} \binom{n}{k} & \text{if } m \leq n \text{ and } n-m = 2k \\ 0 & \text{otherwise} \end{cases}$$

GW (332)(15a)

$$18.^6 \quad \int_0^\pi \cos^m x \cos ax \, dx = \frac{(-1)^m \sin a\pi}{2^m(m+a)} {}_2F_1 \left(-m, -\frac{a+m}{2}; 1 - \frac{a+m}{2}; -1 \right)$$

[a ≠ 0, ±1, ±2, ...] WA 313

$$19. \quad \int_0^{\pi/2} \cos^{\nu-2} x \cos \nu x \, dx = 0 \quad [\operatorname{Re} \nu > 1] \quad \text{GW(332)(16a), FI II 152}$$

$$20.^{10} \quad \int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}} \quad [\operatorname{Re} n > -1] \quad \text{LO V 122(78), FI II 153}$$

3.632

$$1. \quad \int_0^\pi \sin^{p-1} x \cos \left[a \left(\frac{\pi}{2} - x \right) \right] dx = 2^{p-1} \frac{\Gamma \left(\frac{p-a}{2} \right) \Gamma \left(\frac{p+a}{2} \right)}{\Gamma(p-a) \Gamma(p+a)} \Gamma(p)$$

[p² < a²] BI (62)(11)

$$2. \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\nu-1} x \sin \left[a \left(x + \frac{\pi}{2} \right) \right] dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu \operatorname{B} \left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2} \right)} \quad [\operatorname{Re} \nu > 0] \quad \text{WA 337a}$$

$$3.^{10} \quad \int_0^{\pi/2} \cos^p x \sin[(p+2n)x] dx = (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k 2^k}{p+k+1} \binom{n-1}{k} \quad [n > 0] \quad \text{LI (41)(12)}$$

$$4. \quad \int_{-\pi}^{\pi} \cos^{n-1} x \cos[m(x-a)] dx = [1 - (-1)^{n+m}] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x \cos[m(x-a)] dx \\ = \frac{[1 - (-1)^{n+m}] \pi \cos ma}{2^{n-1} n \operatorname{B} \left(\frac{n+m+1}{2}, \frac{n-m+1}{2} \right)} \quad [n \geq m] \quad \text{LO V 123(80), LO V 139(94a)}$$

$$5. \quad \int_0^{\pi/2} \cos^{p+q-2} x \cos[(p-q)x] dx = \frac{\pi}{2^{p+q-1} (p+q-1) \operatorname{B}(p, q)} \quad [p+q > 1] \quad \text{WH}$$

3.633

$$1. \quad \int_0^{\pi/2} \cos^{p-1} x \sin ax \sin x dx = \frac{a\pi}{2^{p+1} p(p+1) \operatorname{B} \left(\frac{p+a}{2} + 1, \frac{p-a}{2} + 1 \right)} \quad \text{LO V 150(110)}$$

$$2. \quad \int_0^{\pi/2} \cos^n x \sin nx \sin 2mx dx = \int_0^{\pi/2} \cos^n x \cos nx \cos 2mx dx = \frac{\pi}{2^{n+2}} \binom{n}{m} \quad \text{BI (42)(19, 20)}$$

$$3. \quad \int_0^{\pi/2} \cos^{n-1} x \cos[(n+1)x] \cos 2mx dx = \frac{\pi}{2^{n+1}} \binom{n-1}{m-1} \quad [n > m-1] \quad \text{BI (42)(21)}$$

$$4. \quad \int_0^{\pi/2} \cos^{p+q} x \cos px \cos qx dx = \frac{\pi}{2^{p+q+2}} \left[1 + \frac{1}{(p+q+1) \operatorname{B}(p+1, q+1)} \right] \quad [p+q > -1] \quad \text{GW (332)(10c)}$$

$$5.^6 \quad \int_0^{\pi/2} \cos^{p+q} x \sin px \sin qx dx = \frac{\pi}{2^{p+q+2}} \sum_{k=1}^{\infty} \binom{p}{k} \binom{q}{k} = \frac{\pi}{2^{p+q+2}} \left[\frac{\Gamma(p+q+1)}{\Gamma(p+1) \Gamma(q+1)} - 1 \right] \quad [p+q > -1] \quad \text{BI (42)(16)}$$

3.634

$$1. \quad \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \sin(\mu+\nu)x dx = \sin \frac{\mu\pi}{2} \operatorname{B}(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI(42)(23), FI II 814a}$$

$$2. \quad \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \cos(\mu + \nu)x \, dx = \cos \frac{\mu\pi}{2} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$$

BI(42)(24), FI II 814a

$$3. \quad \int_0^{\pi/2} \cos^{p+n-1} x \sin px \cos[(n+1)x] \sin x \, dx = \frac{\pi}{2^{p+n+1}} \frac{\Gamma(p+n)}{n! \Gamma(p)}$$

[$p > -n$] BI (42)(15)

3.635

$$1. \quad \int_0^{\pi/4} \cos^{\mu-1} 2x \tan x \, dx = \frac{1}{4} \left[\psi \left(\frac{\mu+1}{2} \right) - \psi \left(\frac{\mu}{2} \right) \right] \quad [\operatorname{Re} \mu > 0] \quad \text{BI (34)(7)}$$

$$2.^7 \quad \int_0^{\pi/2} \cos^{p+2n} x \sin px \tan x \, dx = \frac{\pi}{2^{p+2n+1} \Gamma(p)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(p+n-k)}{(n-k)!}$$

$$= \frac{p\pi}{2^{p+2n+1}} \frac{\Gamma(p+2n)}{\Gamma(n+1) \Gamma(p+n+1)} \quad [p > -2n] \quad \text{BI (42)(22)}$$

$$3. \quad \int_0^{\pi/2} \cos^{n-1} x \sin[(n+1)x] \cot x \, dx = \frac{\pi}{2} \quad \text{BI (45)(18)}$$

3.636

$$1. \quad \int_0^{\pi/2} \tan^{\pm\mu} x \sin 2x \, dx = \frac{\mu\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \quad [0 < \operatorname{Re} \mu < 2] \quad \text{BI (45)(20)a}$$

$$2. \quad \int_0^{\pi/2} \tan^{\pm\mu} x \cos 2x \, dx = \mp \frac{\mu\pi}{2} \sec \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (45)(21)}$$

$$3.^{11} \quad \int_0^{\pi/2} \frac{\tan^{2\mu} x}{\cos x} \, dx = \int_0^{\pi/2} \frac{\cot^{2\mu} x}{\sin x} \, dx = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(-\mu)}{2\sqrt{\pi}}$$

[$-\frac{1}{2} < \operatorname{Re} \mu < 1$] (cf. **3.251 1**)
BI (45)(13, 14)

3.637

$$1. \quad \int_0^{\pi/2} \tan^p x \sin^{q-2} x \sin qx \, dx = -\cos \frac{(p+q)\pi}{2} B(p+q-1, 1-p)$$

[$p+q > 1 > p$] GW (332)(15d)

$$2. \quad \int_0^{\pi/2} \tan^p x \sin^{q-2} x \cos qx \, dx = \sin \frac{(p+q)\pi}{2} B(p+q-1, 1-p)$$

[$p+q > 1 > p$] GW (332)(15b)

$$3. \quad \int_0^{\pi/2} \cot^p x \cos^{q-2} x \sin qx \, dx = \cos \frac{p\pi}{2} B(p+q-1, 1-p)$$

[$p+q > 1 > p$] GW (332)(15c)

$$4. \quad \int_0^{\pi/2} \cot^p x \cos^{q-2} x \cos qx \, dx = \sin \frac{p\pi}{2} B(p+q-1, 1-p) \quad [p+q > 1 > p] \quad \text{GW (332)(15a)}$$

3.638

$$1. \quad \int_0^{\pi/4} \frac{\sin^{2\mu} x \, dx}{\cos^{\mu+\frac{1}{2}} 2x \cos x} = \frac{\pi}{2} \sec \mu\pi \quad [|\operatorname{Re} \mu| < \tfrac{1}{2}] \quad (\text{cf. } \mathbf{3.192} \, 2) \quad \text{BI (38)(8)}$$

$$2. \quad \int_0^{\pi/4} \frac{\sin^{\mu-\frac{1}{2}} 2x \, dx}{\cos^\mu 2x \cos x} = \frac{2}{2\mu-1} \cdot \frac{\Gamma(\mu+\frac{1}{2}) \Gamma(1-\mu)}{\sqrt{\pi}} \sin\left(\frac{2\mu-1}{4}\pi\right) \quad [-\tfrac{1}{2} < \operatorname{Re} \mu < 1] \quad \text{BI (38)(17)}$$

$$3. \quad \int_0^{\pi/2} \frac{\cos^{p-1} x \sin px}{\sin x} \, dx = \frac{\pi}{2} \quad [p > 0] \quad \text{GW(332)(17), BI(45)(5)}$$

3.64–3.65 Powers and rational functions of trigonometric functions**3.641**

$$1. \quad \int_0^{\pi/2} \frac{\sin^{p-1} x \cos^{-p} x}{a \cos x + b \sin x} \, dx = \int_0^{\pi/2} \frac{\sin^{-p} x \cos^{p-1} x}{a \sin x + b \cos x} \, dx = \frac{\pi \operatorname{cosec} p\pi}{a^{1-p} b^p} \quad [ab > 0, \quad 0 < p < 1] \quad \text{GW (331)(62)}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^{1-p} x \cos^p x}{(\sin x + \cos x)^3} \, dx = \int_0^{\pi/2} \frac{\sin^p x \cos^{1-p} x}{(\sin x + \cos x)^3} \, dx = \frac{(1-p)p}{2} \pi \operatorname{cosec} p\pi \quad [-1 < p < 2] \quad \text{BI(48)(5)}$$

3.642

$$1. \quad \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x \, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{\mu+\nu}} = \frac{1}{2a^{2\mu} b^{2\nu}} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (48)(28)}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^{n-1} x \cos^{n-1} x \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^n} = \frac{B(\frac{n}{2}, \frac{n}{2})}{2(ab)^n} \quad [ab > 0] \quad \text{GW (331)(59a)}$$

$$3. \quad \begin{aligned} \int_0^{\pi/2} \frac{\sin^{2n} x \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} &= \frac{1}{2} \int_0^\pi \frac{\sin^{2n} x \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} \\ &= \int_0^{\pi/2} \frac{\cos^{2n} x \, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{n+1}} = \frac{1}{2} \int_0^\pi \frac{\cos^{2n} x \, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{n+1}} = \frac{(2n-1)!!\pi}{2^{n+1} n! ab^{2n+1}} \end{aligned} \quad [ab > 0] \quad \text{GW (331)(58)}$$

$$4. \quad \int_0^{\pi/2} \frac{\cos^{p+2n} x \cos px \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \pi \sum_{k=0}^n \binom{2n-k}{n} \binom{p+k-1}{k} \frac{b^{p-1}}{(2a)^{2n-k+1} (a+b)^{p+k}} \quad [a > 0, \quad b > 0, \quad p > -2n-1] \quad \text{GW (332)(30)}$$

3.643

1.
$$\int_0^{\pi/2} \frac{\cos^p x \cos px \, dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{2^{p+1}} \cdot \frac{(1+a)^{p-1}}{1-a} \quad [a^2 < 1, \quad p > -1] \quad \text{GW (332)(33c)}$$
2.
$$\begin{aligned} \int_0^{\pi/2} \frac{\sin^{2n} x \cos^\mu x \cos \beta x}{(1 - 2a \cos 2x + a^2)^m} dx &= \frac{(-1)^n \pi (1-a)^{2n-2m+1}}{2^{2m-\beta-1} (1+a)^{2m+\beta+1}} \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} \binom{\beta}{k} \binom{2n}{l} \\ &\quad \times \binom{2m-k-l-2}{m-1(-2)^l} (a-1)^k \\ &\quad [a^2 < 1, \quad \beta = 2m - 2n - \mu - 2, \quad \mu > -1] \quad \text{GW (332)(33)} \end{aligned}$$

3.644

1.
$$\int_0^\pi \frac{\sin^m x}{p+q \cos x} dx = 2^{m-2} \frac{p}{q^2} \sum_{\nu=1}^k \left(\frac{p^2 - q^2}{-4q^2} \right)^{\nu-1} B \left(\frac{m+1-2\nu}{2}, \frac{m+1-2\nu}{2} \right) + \left(\frac{p^2 - q^2}{-q^2} \right)^k A$$

where $A = \begin{cases} \frac{\pi p}{q^2} \left(1 - \sqrt{1 - \frac{q^2}{p^2}} \right) & \text{if } m = 2k + 2 \\ \frac{1}{q} \ln \frac{p+q}{p-q} & \text{if } m = 2k + 1 \end{cases} \quad [k \geq 1, \quad q \neq 0, \quad p^2 - q^2 \geq 0]$
2.
$$\int_0^\pi \frac{\sin^m x}{1 + \cos x} dx = 2^{m-1} B \left(\frac{m-1}{2}, \frac{m+1}{2} \right) \quad [m \geq 2]$$
3.
$$\int_0^\pi \frac{\sin^m x}{1 - \cos x} dx = 2^{m-1} B \left(\frac{m-1}{2}, \frac{m+1}{2} \right) \quad [m \geq 2]$$
4.
$$\int_0^\pi \frac{\sin^2 x}{p+q \cos x} dx = \frac{p\pi}{q^2} \left(1 - \sqrt{1 - \frac{q^2}{p^2}} \right)$$
5.
$$\int_0^\pi \frac{\sin^3 x}{p+q \cos x} dx = 2 \frac{p}{q^2} + \frac{1}{q} \left(1 - \frac{p^2}{q^2} \right) \ln \frac{p+q}{p-q}$$

3.645
$$\int_0^\pi \frac{\cos^n x \, dx}{(a+b \cos x)^{n+1}} = \frac{\pi}{2^n (a+b)^n \sqrt{a^2 - b^2}} \sum_{k=0}^n (-1)^k \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!} \left(\frac{a+b}{a-b} \right)^k$$

[a^2 > b^2] LI (64)(16)

3.646

1.
$$\int_0^{\pi/2} \frac{\cos^n x \sin nx \sin 2x}{1 - 2a \cos 2x + a^2} dx = \frac{\pi}{4a} \left[\left(\frac{1+a}{2} \right)^n - \frac{1}{2^n} \right] \quad [a^2 < 1] \quad \text{BI (50)(6)}$$
2.
$$\int_0^{\pi/2} \frac{1 - a \cos 2nx}{1 - 2a \cos 2nx + a^2} \cos^m x \cos mx \, dx = \frac{\pi}{2^{m+2}} \sum_{k=1}^{\infty} \binom{m}{kn} a^k + \frac{\pi}{2^{m+1}}$$

[a^2 < 1] LI (50)(7)

3.647
$$\int_0^{\pi/2} \frac{\cos^p x \cos px \, dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2b} \cdot \frac{a^{p-1}}{(a+b)^p} \quad [p > -1, \quad a > 0, \quad b > 0] \quad \text{BI (47)(20)}$$

3.648

$$\begin{aligned}
1. \quad & \int_0^{\pi/4} \frac{\tan^l x \, dx}{1 + \cos \frac{m}{n} \pi \sin 2x} \\
&= \frac{1}{2n} \operatorname{cosec} \frac{m}{n} \pi \sum_{k=0}^{n-1} (-1)^{k-1} \sin \frac{km}{n} \pi \left[\psi \left(\frac{n+l+k}{2n} \right) - \psi \left(\frac{l+k}{2n} \right) \right] \quad [m+n \text{ is odd}] \\
&= \frac{1}{n} \operatorname{cosec} \frac{m}{n} \pi \sum_{k=0}^{\frac{n-1}{2}} (-1)^{k-1} \sin \frac{km}{n} \pi \left[\psi \left(\frac{n+l-k}{n} \right) - \psi \left(\frac{l+k}{n} \right) \right] \quad [m+n \text{ is even}] \\
&\quad [l \text{ is a natural number}] \quad \text{BI (36)(5)}
\end{aligned}$$

$$2. \quad \int_0^{\pi/2} \frac{\tan^{\pm\mu} x \, dx}{1 + \cos t \sin 2x} = \pi \operatorname{cosec} t \sin \mu t \operatorname{cosec}(\mu\pi) \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2] \quad \text{BI (47)(4)}$$

3.649

$$\begin{aligned}
1. \quad & \int_0^{\pi/2} \frac{\tan^{\pm\mu} x \sin 2x \, dx}{1 \mp 2a \cos 2x + a^2} = \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[1 - \left(\frac{1-a}{1+a} \right)^\mu \right] \quad [a^2 < 1] \\
&= \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[1 + \left(\frac{a-1}{a+1} \right)^\mu \right] \quad [a^2 > 1] \\
&\quad [-2 < \operatorname{Re} \mu < 1] \quad \text{BI (50)(3)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^{\pi/2} \frac{\tan^{\pm\mu} x (1 \mp a \cos 2x)}{1 \mp 2a \cos 2x + a^2} \, dx = \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[1 + \left(\frac{1-a}{1+a} \right)^\mu \right] \quad [a^2 < 1] \\
&= \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[1 - \left(\frac{a-1}{a+1} \right)^\mu \right] \quad [a^2 > 1] \\
&\quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (50)(4)}
\end{aligned}$$

3.651

$$\begin{aligned}
1. \quad & \int_0^{\pi/4} \frac{\tan^\mu x \, dx}{1 + \sin x \cos x} = \frac{1}{3} \left[\psi \left(\frac{\mu+2}{3} \right) - \psi \left(\frac{\mu+1}{3} \right) \right] \quad [\operatorname{Re} \mu > -1] \quad \text{BI (36)(3)} \\
2. \quad & \int_0^{\pi/4} \frac{\tan^\mu x \, dx}{1 - \sin x \cos x} = \frac{1}{3} \left[\beta \left(\frac{\mu+2}{3} \right) + \beta \left(\frac{\mu+1}{3} \right) \right] \quad [\operatorname{Re} \mu > -1] \quad \text{BI (36)(4)a}
\end{aligned}$$

3.652

$$1. \quad \int_0^{\pi/2} \frac{\tan^\mu x \, dx}{(\sin x + \cos x) \sin x} = \int_0^{\pi/2} \frac{\cot^\mu x \, dx}{(\sin x + \cos x) \cos x} = \pi \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (49)(1)}$$

$$2. \quad \int_0^{\pi/2} \frac{\tan^\mu x \, dx}{(\sin x - \cos x) \sin x} = \int_0^{\pi/2} \frac{\cot^\mu x \, dx}{(\cos x - \sin x) \cos x} = -\pi \cot \mu\pi \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (49)(2)}$$

$$3. \quad \int_0^{\pi/2} \frac{\cot^{\mu+\frac{1}{2}} x \, dx}{(\sin x + \cos x) \cos x} = \int_0^{\pi/2} \frac{\tan^{\mu-\frac{1}{2}} x \, dx}{(\sin x + \cos x) \cos x} = \pi \sec \mu\pi \quad [|\operatorname{Re} \mu| < \tfrac{1}{2}] \quad \text{BI (61)(1, 2)}$$

3.653

$$1. \quad \int_0^{\pi/2} \frac{\tan^{1-2\mu} x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi/2} \frac{\cot^{1-2\mu} x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2a^{2\mu} b^{2-2\mu} \sin \mu\pi} \quad [0 < \operatorname{Re} \mu < 1] \quad \text{GW (331)(59b)}$$

$$2.^{11} \quad \int_0^{\pi/2} \frac{\tan^\mu x \, dx}{1 - a \sin^2 x} = \int_0^{\pi/2} \frac{\cot^\mu x \, dx}{1 - a \cos^2 x} = \frac{\pi \sec \frac{\mu\pi}{2}}{2\sqrt{(1-a)^{\mu+1}}} \quad [|\operatorname{Re} \mu| < 1, \quad a < 1] \quad \text{BI (49)(6)}$$

$$3. \quad \int_0^{\pi/2} \frac{\tan^{\pm\mu} x \, dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \operatorname{cosec} t \sec \frac{\mu\pi}{2} \cos \left[\left(\frac{\pi}{2} - t \right) \mu \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2] \quad \text{BI(49)(7), BI(47)(21)}$$

$$4. \quad \int_0^{\pi/2} \frac{\tan^{\pm\mu} x \sin 2x}{1 - \cos^2 t \sin^2 2x} \, dx = \pi \operatorname{cosec} 2t \operatorname{cosec} \frac{\mu\pi}{2} \sin \left[\left(\frac{\pi}{2} - t \right) \mu \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2] \quad \text{BI (47)(22)a}$$

$$5. \quad \int_0^{\pi/2} \frac{\tan^\mu x \sin^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\pi/2} \frac{\cot^\mu x \cos^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \operatorname{cosec} 2t \sec \frac{\mu\pi}{2} \cos \left[\frac{\mu\pi}{2} - (\mu+1)t \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2] \quad \text{BI(47)(23)a, BI(49)(10)}$$

$$6. \quad \int_0^{\pi/2} \frac{\tan^\mu x \cos^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\pi/2} \frac{\cot^\mu x \sin^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \operatorname{cosec} 2t \sec \frac{\mu\pi}{2} \cos \left[\frac{\mu\pi}{2} - (\mu-1)t \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2] \quad \text{BI(47)(24)a, BI(49)(9)}$$

3.654

$$1. \quad \int_0^{\pi/2} \frac{\tan^{\mu+1} x \cos^2 x \, dx}{(1 + \cos t \sin 2x)^2} = \int_0^{\pi/2} \frac{\cot^{\mu+1} x \sin^2 x \, dx}{(1 + \cos t \sin 2x)^2} = \frac{\pi (\mu \sin t \cos \mu t - \cos t \sin \mu t)}{2 \sin \mu\pi \sin^3 t} \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2] \quad \text{BI(48)(3), BI(49)(22)}$$

$$2. \quad \int_0^{\pi/2} \frac{\tan^{\pm\mu} x \, dx}{(\sin x + \cos x)^2} = \frac{\mu\pi}{\sin \mu\pi} \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (56)(9)a}$$

$$3. \quad \int_0^{\pi/2} \frac{\tan^{\pm(\mu-1)x} \, dx}{\cos^2 x - \sin^2 x} = \pm \frac{\pi}{2} \cot \frac{\mu\pi}{2} \quad [0 < \operatorname{Re} \mu < 2] \quad \text{BI (45)(27, 29)}$$

3.655

$$\begin{aligned} \int_0^{\pi/2} \frac{\tan^{2\mu-1} x \, dx}{1 - 2a (\cos t_1 \sin^2 x + \cos t_2 \cos^2 x) + a^2} &= \int_0^{\pi/2} \frac{\cot^{2\mu-1} x \, dx}{1 - 2a (\cos t_1 \cos^2 x + \cos t_2 \sin^2 x) + a^2} \\ &= \frac{\pi \operatorname{cosec} \mu\pi}{(1 - 2a \cos t_2 + a^2)^\mu (1 - 2a \cos t_1 + a^2) 1 - \mu} \\ &\quad [0 < \operatorname{Re} \mu < 1, \quad t_1^2 < \pi^2, \quad t_2^2 < \pi^2] \quad \text{BI (50)(18)} \end{aligned}$$

3.656

$$1. \quad \int_0^{\pi/4} \frac{\tan^\mu x \, dx}{1 - \sin^2 x \cos^2 x} = \frac{1}{12} \left\{ -\psi\left(\frac{\mu+1}{6}\right) - \psi\left(\frac{\mu+2}{6}\right) \right. \\ \left. + \psi\left(\frac{\mu+4}{6}\right) + \psi\left(\frac{\mu+5}{6}\right) + 2\psi\left(\frac{\mu+2}{3}\right) - 2\psi\left(\frac{\mu+1}{3}\right) \right\} \\ [\operatorname{Re} \mu > -1] \quad (\text{cf. } \mathbf{3.651} \text{ 1 and 2}) \quad \text{LI (36)(10)}$$

$$2. \quad \int_0^{\pi/2} \frac{\tan^{\mu-1} x \cos^2 x \, dx}{1 - \sin^2 x \cos^2 x} = \int_0^{\pi/2} \frac{\cot^{\mu-1} x \sin^2 x \, dx}{1 - \sin^2 x \cos^2 x} = \frac{\pi}{4\sqrt{3}} \operatorname{cosec} \frac{\mu\pi}{6} \operatorname{cosec} \left(\frac{2+\mu}{6} \pi \right) \\ [0 < \operatorname{Re} \mu < 4] \quad \text{LI (47)(26)}$$

3.66 Forms containing powers of linear functions of trigonometric functions

3.661

$$1. \quad \int_0^{2\pi} (a \sin x + b \cos x)^{2n+1} \, dx = 0 \quad \text{BI (68)(9)}$$

$$2. \quad \int_0^{2\pi} (a \sin x + b \cos x)^{2n} \, dx = \frac{(2n-1)!!}{(2n)!!} \cdot 2\pi (a^2 + b^2)^n \quad \text{BI (68)(8)}$$

$$3. \quad \int_0^\pi (a + b \cos x)^n \, dx = \frac{1}{2} \int_0^{2\pi} (a + b \cos x)^n \, dx = \pi (a^2 - b^2)^{\frac{n}{2}} P_n \left(\frac{a}{\sqrt{a^2 - b^2}} \right) \\ = \frac{\pi}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} a^{n-2k} (a^2 - b^2)^k \\ [a^2 > b^2] \quad \text{GW (332)(37a)}$$

$$4. \quad \int_0^\pi \frac{dx}{(a + b \cos x)^{n+1}} = \frac{1}{2} \int_0^{2\pi} \frac{dx}{(a + b \cos x)^{n+1}} = \frac{\pi}{(a^2 - b^2)^{\frac{n+1}{2}}} P_n \left(\frac{a}{\sqrt{a^2 - b^2}} \right) \\ = \frac{\pi}{2^n (a+b)^n \sqrt{a^2 - b^2}} \sum_{k=0}^n \frac{(2n-2k-1)!! (2k-1)!!}{(n-k)! k!} \cdot \left(\frac{a+b}{a-b} \right)^k \\ [a > |b|] \quad \text{GW(332)(38), LI(64)(14)}$$

3.662

$$1. \quad \int_0^{\pi/2} (\sec x - 1)^\mu \sin x \, dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cos x \, dx = \mu\pi \operatorname{cosec} \mu\pi \\ [|\operatorname{Re} \mu| < 1] \quad \text{BI (55)(13)}$$

$$2. \quad \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \sin 2x \, dx = (1 - \mu) \mu\pi \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 2] \quad \text{BI (48)(7)}$$

$$3. \quad \int_0^{\pi/2} (\sec x - 1)^\mu \tan x \, dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cot x \, dx = -\pi \operatorname{cosec} \mu\pi \\ [-1 < \operatorname{Re} \mu < 0] \quad \text{BI (46)(4,6)}$$

$$4. \quad \int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\sin 2x} = -\frac{\pi}{2} \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 0] \quad \text{BI (38)(22)a}$$

$$5. \quad \int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (38)(11)a}$$

3.663

$$1. \quad \int_0^u (\cos x - \cos u)^{\nu-\frac{1}{2}} \cos ax \, dx = \sqrt{\frac{\pi}{2}} \sin^\nu u \, \Gamma\left(\nu + \frac{1}{2}\right) P_{a-\frac{1}{2}}^{-\nu}(\cos u) \\ [\operatorname{Re} \nu > -\frac{1}{2}; \quad a > 0, \quad 0 < u < \pi] \\ \text{EH I 159(27), ET I 22(28)}$$

$$2. \quad \int_0^u (\cos x - \cos u)^{\nu-1} \cos[(\nu + \beta)x] \, dx = \frac{\sqrt{\pi} \, \Gamma(\beta + 1) \, \Gamma(\nu) \, \Gamma(2\nu) \sin^{2\nu-1} u}{2^\nu \, \Gamma(\beta + 2\nu) \, \Gamma(\nu + \frac{1}{2})} C_\beta^\nu(\cos u) \\ [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > -1, \quad 0 < u < \pi] \\ \text{EH I 178(23)}$$

3.664

$$1. \quad \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos x\right)^q \, dx = \pi P_q(z) \\ \left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2 - 1} \cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right] \quad \text{SM 482}$$

$$2. \quad \int_0^\pi \frac{dx}{\left(z + \sqrt{z^2 - 1} \cos x\right)^q} = \pi P_{q-1}(z) \\ \left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2 - 1} \cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right] \quad \text{WH}$$

$$3. \quad \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos x\right)^q \cos nx \, dx = \frac{\pi}{(q+1)(q+2) \cdots (q+n)} P_q^n(z) \\ \left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2 - 1} \cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}, \right. \\ \left. z \text{ lies outside the interval } (-1, 1) \text{ of the real axis}\right] \\ \text{WH, SM 483(15)}$$

$$4. \quad \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos x\right)^\mu \sin^{2\nu-1} x \, dx \\ = \frac{2^{2\nu-1} \Gamma(\mu+1) [\Gamma(\nu)]^2}{\Gamma(2\nu+\mu)} C_\mu^\nu(z) \\ = \frac{\sqrt{\pi} \Gamma(\nu) \Gamma(2\nu) \Gamma(\mu+1)}{\Gamma(2\nu+\mu) \Gamma(\nu+\frac{1}{2})} C_\mu^\nu(z) = 2^\nu \sqrt{\frac{\pi}{2}} (z^2 - 1)^{\frac{1}{4}-\frac{\nu}{2}} \Gamma(\nu) P_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(z) \\ [\operatorname{Re} \nu > 0] \quad \text{EH I 155(6)a, EH I 178(22)}$$

$$5. \quad \int_0^{2\pi} \left[\beta + \sqrt{\beta^2 - 1} \cos(a-x)\right]^\nu \left(\gamma + \sqrt{\gamma^2 - 1} \cos x\right)^{\nu-1} \, dx \\ = 2\pi P_\nu\left(\beta\gamma - \sqrt{\beta^2 - 1} \sqrt{\gamma^2 - 1} \cos a\right) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{EH I 157(18)}$$

3.665

1.
$$\int_0^\pi \frac{\sin^{\mu-1} x \, dx}{(a + b \cos x)^\mu} = \frac{2^{\mu-1}}{\sqrt{(a^2 - b^2)^\mu}} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \quad [\operatorname{Re} \mu > 0, \quad 0 < b < a] \quad \text{FI II 790a}$$
2.
$$\int_0^\pi \frac{\sin^{2\mu-1} x \, dx}{(1 + 2a \cos x + a^2)^\nu} = B\left(\mu, \frac{1}{2}\right) F\left(\nu, \nu - \mu + \frac{1}{2}; \mu + \frac{1}{2}; a^2\right) \quad [\operatorname{Re} \mu > 0, \quad |a| < 1] \quad \text{EH I 81(9)}$$

3.666

1.
$$\int_0^\pi (\beta + \cos x)^{\mu-\nu-\frac{1}{2}} \sin^{2\nu} x \, dx = \frac{2^{\nu+\frac{1}{2}} e^{-i\mu\pi} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma\left(\nu + \frac{1}{2}\right) Q_{\nu-\frac{1}{2}}^\mu(\beta)}{\Gamma\left(\nu + \mu + \frac{1}{2}\right)} \quad [\operatorname{Re}(\nu + \mu + \frac{1}{2}) > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{EH I 155(5)a}$$
- 2.⁶
$$\int_0^\pi (\cosh \beta + \sinh \beta \cos x)^{\mu+\nu} \sin^{-2\nu} x \, dx = \frac{\sqrt{\pi}}{2^\nu} \sinh^\nu(\beta) \Gamma\left(\frac{1}{2} - \nu\right) P_\mu^\nu(\cosh \beta) \quad [\operatorname{Re} \nu < \frac{1}{2}] \quad \text{EH I 156(7)}$$
3.
$$\int_0^\pi (\cos t + i \sin t \cos x)^\mu \sin^{2\nu-1} x \, dx = 2^{\nu-\frac{1}{2}} \sqrt{\pi} \sin^{\frac{1}{2}-\nu} t \Gamma(\nu) P_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos t) \quad [\operatorname{Re} \nu > 0, \quad t^2 < \pi^2] \quad \text{EH I 158(23)}$$
4.
$$\int_0^{2\pi} [\cos t + i \sin t \cos(a-x)]^\nu \cos mx \, dx = \frac{i^{3m} 2\pi \Gamma(\nu+1)}{\Gamma(\nu+m+1)} \cos ma P_\nu^m(\cos t) \quad \left[0 < t < \frac{\pi}{2}\right] \quad \text{EH I 159(25)}$$
- 5.¹⁰
$$\int_0^{2\pi} [\cos t + i \sin t \cos(a-x)]^\nu \sin mx \, dx = \frac{i^{3m} 2\pi \Gamma(\nu+1)}{\Gamma(\nu+m+1)} \sin ma P_\nu^m(\cos t) \quad \left[0 < t < \frac{\pi}{2}\right] \quad \text{EH I 159(26)}$$

3.667

1.
$$\int_0^{\pi/4} \frac{\sin^{\mu-1} 2x \, dx}{(\cos x + \sin x)^{2\mu}} = \frac{\sqrt{\pi}}{2^{\mu+1}} \frac{\Gamma(\mu)}{\Gamma\left(\mu + \frac{1}{2}\right)} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (37)(1)}$$
2.
$$\int_0^{\pi/4} \frac{\sin^\mu x \, dx}{(\cos x - \sin x)^{\mu+1} \cos x} = -\pi \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 0] \quad (\text{cf. } \mathbf{3.192} \, 2) \quad \text{BI (37)(16)}$$
3.
$$\int_0^{\pi/4} \frac{(\cos x - \sin x)^\mu}{\sin^\mu x \sin 2x} \, dx = -\frac{\pi}{2} \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 0] \quad \text{BI (35)(27)}$$
4.
$$\int_0^{\pi/4} \frac{\sin^\mu x \, dx}{(\cos x - \sin x)^\mu \sin 2x} = \frac{\pi}{2} \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 1] \quad \text{LI (37)(20)a}$$
5.
$$\int_0^{\pi/4} \frac{\sin^\mu x \, dx}{(\cos x - \sin x)^\mu \cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (37)(17)}$$

$$6. \quad \int_0^{\pi/4} \frac{\sin^\mu x \, dx}{(\cos x - \sin x)^{\mu-1} \cos^3 x} = \frac{1-\mu}{2} \mu \pi \operatorname{cosec} \mu \pi \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI(35)(24), BI(37)(18)}$$

$$7. \quad \int_0^{\pi/2} \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{(\sin x + \cos x)^{\mu+\nu}} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (48)(8)}$$

3.668

$$1. \quad \int_{-\pi/4}^{\pi/4} \left(\frac{\cos x + \sin x}{\cos x - \sin x} \right)^{\cos 2t} dx = \frac{\pi}{2 \sin(\pi \cos^2 t)} \quad \text{FI II 788}$$

$$2. \quad \int_u^v \frac{(\cos u - \cos x)^{\mu-1}}{(\cos x - \cos v)^\mu} \cdot \frac{\sin x \, dx}{1 - 2a \cos x + a^2} = \frac{(1 - 2a \cos u + a^2)^{\mu-1}}{(1 - 2a \cos v + a^2)^\mu} \cdot \frac{\pi}{\sin \mu \pi} \\ [0 < \operatorname{Re} \mu < 1, \quad a^2 < 1] \quad \text{BI (73)(2)}$$

$$3.669 \quad \int_0^{\pi/2} \frac{\sin^{p-1} x \cos^{q-p-1} x \, dx}{(a \cos x + b \sin x)^q} = \int_0^{\pi/2} \frac{\sin^{q-p-1} x \cos^{p-1} x}{(a \sin x + b \cos x)^q} dx = \frac{B(p, q-p)}{a^{q-p} b^p} \\ [q > p > 0, \quad ab > 0] \quad \text{BI (331)(9)}$$

3.670

$$1. \quad \int_0^\pi \sqrt{a \pm b \cos x} \, dx = \int_{-\pi/2}^{\pi/2} \sqrt{a \pm b \cos x} \, dx = 2\sqrt{a+b} \mathbf{K} \left(\sqrt{\frac{2b}{a+b}} \right) \\ [a > b > 0]$$

$$2.* \quad \int_0^\pi \frac{dx}{\sqrt{a \pm b \cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a \pm b \sin x}} = \frac{2}{\sqrt{a+b}} \mathbf{E} \left(\sqrt{\frac{2b}{a+b}} \right) \\ [a > b > 0]$$

3.67 Square roots of expressions containing trigonometric functions**3.671**

$$1. \quad \int_0^{\pi/2} \sin^\alpha x \cos^\beta x \sqrt{1 - k^2 \sin^2 x} \, dx = \frac{1}{2} B \left(\frac{\alpha+1}{2}, \frac{\beta+1}{2} \right) F \left(\frac{\alpha+1}{2}, -\frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2 \right) \\ [\alpha > -1, \quad \beta > -1, \quad |k| < 1] \quad \text{GW (331)(93)}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^\alpha x \cos^\beta x}{\sqrt{1 - k^2 \sin^2 x}} dx = \frac{1}{2} B \left(\frac{\alpha+1}{2}, \frac{\beta+1}{2} \right) F \left(\frac{\alpha+1}{2}, \frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2 \right) \\ [\alpha > -1, \quad \beta > -1, \quad |k| < 1] \quad \text{GW (331)(92)}$$

$$3. \quad \int_0^\pi \frac{\sin^{2n} x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2^n} \sum_{j=0}^{\infty} \frac{(2j-1)!! (2n+2j-1)!!}{2^{2j} j! (n+j)!} k^{2j} \quad [k^2 < 1] \\ = \frac{(2n-1)!! \pi}{2^n \sqrt{1-k^2}} \sum_{j=0}^{\infty} \frac{[(2j-1)!!]^2}{2^{2j} j! (n+j)!} \left(\frac{k^2}{k^2-1} \right)^j \quad [k^2 < \frac{1}{2}]$$

LI (67)(2)

$$4.^* \quad \int_0^\pi \sqrt{a+b \cos x} \, dx = \int_{-\pi/2}^{\pi/2} \sqrt{a+b \sin x} \, dx = 2\sqrt{a+b} \mathbf{E} \left(\sqrt{\frac{2b}{a+b}} \right) \\ [a > b]$$

$$5.^* \quad \int_0^\pi \frac{dx}{\sqrt{a \pm b \cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a \pm b \sin x}} = \frac{2}{a+b} \mathbf{K} \left(\sqrt{\frac{2b}{a+b}} \right) \\ [a > b]$$

3.672

$$1. \quad \int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\cos x (\cos x - \sin x)}} = 2 \cdot \frac{(2n)!!}{(2n+1)!!} \quad \text{BI (39)(5)}$$

$$2. \quad \int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\sin x (\cos x - \sin x)}} = \frac{(2n-1)!!}{(2n)!!} \pi \quad \text{BI (39)(6)}$$

$$3.673 \quad \int_u^{\pi/2} \frac{dx}{\sqrt{\sin x - \sin u}} = \sqrt{2} \mathbf{K} \left(\sin \frac{\pi - 2u}{4} \right) \quad \text{BI (74)(11)}$$

3.674

$$1.^8 \quad \int_0^{\pi/2} \frac{dx}{\sqrt{1 - (p^2/2)(1 - \cos 2x)}} = \mathbf{K}(p), \quad [1 > p > 0] \quad \text{BI (67)(5)}$$

$$2. \quad \int_0^\pi \frac{\sin x \, dx}{\sqrt{1 - 2p \cos x + p^2}} = 2 \quad [p^2 \leq 1] \\ = \frac{2}{p} \quad [p^2 \geq 1] \\ \text{BI (67)(6)}$$

$$3.^8 \quad \int_0^\pi \frac{\cos x \, dx}{\sqrt{1 - 2p \cos x + p^2}} = \frac{1}{p} \left[\frac{1+p^2}{1+p} \mathbf{K} \left(\frac{2\sqrt{p}}{1+p} \right) - (1+p) \mathbf{E} \left(\frac{2\sqrt{p}}{1+p} \right) \right] \\ [p^2 < 1] \quad \text{BI (67)(7)}$$

3.675

$$1. \quad \int_u^\pi \frac{\sin \left(n + \frac{1}{2} \right) x \, dx}{\sqrt{2 (\cos u - \cos x)}} = \frac{\pi}{2} P_n (\cos u) \quad \text{WH}$$

$$2. \quad \int_0^u \frac{\cos \left(n + \frac{1}{2} \right) x \, dx}{\sqrt{2 (\cos x - \cos u)}} = \frac{\pi}{2} P_n (\cos u) \quad \text{FI II 684, WH}$$

3.676

$$1. \quad \int_0^{\pi/2} \frac{\sin x \, dx}{\sqrt{1 + p^2 \sin^2 x}} = \frac{1}{p} \arctan p \quad \text{BI (60)(5)}$$

$$2. \quad \int_0^{\pi/2} \tan^2 x \sqrt{1 - p^2 \sin^2 x} \, dx = \infty \quad \text{BI (53)(8)}$$

$$3. \quad \int_0^{\pi/2} \frac{dx}{\sqrt{p^2 \cos^2 x + q^2 \sin^2 x}} = \frac{1}{p} \mathbf{K} \left(\frac{\sqrt{p^2 - q^2}}{p} \right) \quad [0 < q < p] \quad \text{FI II 165}$$

3.677

$$1. \quad \int_0^{\pi/2} \frac{\sin^2 x \, dx}{\sqrt{1 + \sin^2 x}} = \sqrt{2} \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \frac{1}{\sqrt{2}} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \quad \text{BI (60)(2)}$$

$$2. \quad \int_0^{\pi/2} \frac{\cos^2 x \, dx}{\sqrt{1 + \sin^2 x}} = \sqrt{2} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right] \quad \text{BI (60)(3)}$$

3.678

$$1. \quad \int_0^{\pi/4} (\sec^{1/2} 2x - 1) \frac{dx}{\tan x} = \ln 2 \quad \text{BI (38)(23)}$$

$$2. \quad \int_0^{\pi/4} \frac{\tan^2 x \, dx}{\sqrt{1 - k^2 \sin^2 2x}} = \sqrt{1 - k^2} - \mathbf{E}(k) + \frac{1}{2} \mathbf{K}(k) \quad \text{BI (39)(2)}$$

$$3. \quad \int_0^u \sqrt{\frac{\cos 2x - \cos 2u}{\cos 2x + 1}} \, dx = \frac{\pi}{2} (1 - \cos u) \quad \left[u^2 < \frac{\pi^2}{4} \right] \quad \text{LI (74)(6)}$$

$$4. \quad \int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x} \sqrt{\operatorname{cosec} x} \, dx = \frac{(2n-1)!!}{(2n)!!} \pi \quad \text{BI (38)(24)}$$

$$5. \quad \int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x} \tan^m x \sqrt{\operatorname{cosec} x} \, dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} \pi \quad \text{BI (38)(25)}$$

3.679

$$1. \quad \int_0^{\pi/2} \frac{\cos^2 x}{1 - \cos^2 \beta \cos^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{\sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - \mathbf{K} \mathbf{E}(\beta, k') - \mathbf{E} \mathbf{F}(\beta, k') + \mathbf{K} \mathbf{F}(\beta, k') \right\}^* \quad \text{MO 138}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^2 x}{1 - (1 - k'^2 \sin^2 \beta) \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k'^2 \sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - \mathbf{K} \mathbf{E}(\beta, k') - \mathbf{E} \mathbf{F}(\beta, k') + \mathbf{K} \mathbf{F}(\beta, k') \right\}^* \quad \text{MO 138}$$

$$3. \quad \int_0^{\pi/2} \frac{\sin^2 x}{1 - k^2 \sin^2 \beta \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\mathbf{K} \mathbf{E}(\beta, k) - \mathbf{E} \mathbf{F}(\beta, k)}{k^2 \sin \beta \cos \beta \sqrt{1 - k^2 \sin^2 \beta}} \quad \text{MO 138}$$

*In 3.679, $k' = \sqrt{1 - k^2}$.

3.68 Various forms of powers of trigonometric functions

3.681

$$1. \quad \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(1 - k^2 \sin^2 x)^{\varrho}} = \frac{1}{2} B(\mu, \nu) F(\varrho, \mu; \mu + \nu; k^2) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH I 115(7)}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(1 - k^2 \sin^2 x)^{\mu+\nu}} = \frac{B(\mu, \nu)}{2(1 - k^2)^{\mu}} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH I 10(20)}$$

$$3. \quad \int_0^{\pi/2} \frac{\sin^{\mu} x dx}{\cos^{\mu-3} x (1 - k^2 \sin^2 x)^{\frac{\mu}{2}-1}} = \frac{\Gamma\left(\frac{\mu+1}{2}\right) \Gamma\left(2 - \frac{\mu}{2}\right)}{k^3 \sqrt{\pi}(\mu-1)(\mu-3)(\mu-5)} \left\{ \frac{1 + (\mu-3)k + k^2}{(1+k)^{\mu-3}} - \frac{1 - (\mu-3)k + k^2}{(1-k)^{\mu-3}} \right\} \\ [-1 < \operatorname{Re} \mu < 4] \quad \text{BI (54)(10)}$$

$$4.^8 \quad \int_0^{\pi/2} \frac{\sin^{\mu+1} x dx}{\cos^{\mu} x (1 - k^2 \sin^2 x)^{\frac{\mu+1}{2}}} = \frac{(1-k)^{-\mu} - (1+k)^{-\mu}}{2k\mu\sqrt{\pi}} \Gamma\left(1 + \frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right) \\ [-2 < \operatorname{Re} \mu < 1] \quad \text{BI (61)(5)}$$

$$3.682 \quad \int_0^{\pi/2} \frac{\sin^{\mu} x \cos^{\nu} x}{(a - b \cos^2 x)^{\varrho}} dx = \frac{1}{2a^{\varrho}} B\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) F\left(\frac{\nu+1}{2}, \varrho; \frac{\mu+\nu}{2} + 1; \frac{b}{a}\right) \\ [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1, \quad a > |b| \geq 0] \quad \text{GW (331)(64)}$$

3.683

$$1. \quad \int_0^{\pi/4} (\sin^n 2x - 1) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^n 2x - 1) \cot x dx = -\frac{1}{2} \sum_{k=1}^n \frac{1}{k} \\ = -\frac{1}{2} [C + \psi(n+1)] \quad [n \geq 0] \quad \text{BI(34)(8), BI(35)(11)}$$

$$2. \quad \int_0^{\pi/4} (\sin^{\mu} 2x - 1) \operatorname{cosec}^{\mu} 2x \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^{\mu} 2x - 1) \sec^{\mu} 2x \cot x dx \\ = \frac{1}{2} [C + \psi(1 - \mu)] \quad [\operatorname{Re} \mu < 1] \quad \text{BI (35)(20)}$$

$$3. \quad \int_0^{\frac{\pi}{4}} (\sin^{2\mu} 2x - 1) \operatorname{cosec}^{\mu} 2x \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^{2\mu} 2x - 1) \sec^{\mu} 2x \cot x dx \\ = -\frac{1}{2\mu} + \frac{\pi}{2} \cot \mu\pi \quad \text{BI (35)(21)}$$

$$4. \quad \int_0^{\pi/4} (1 - \sec^{\mu} 2x) \cot x dx = \int_0^{\pi/4} (1 - \operatorname{cosec}^{\mu} 2x) \tan\left(\frac{\pi}{4} + x\right) dx = \frac{1}{2} [C + \psi(1 - \mu)] \\ [\operatorname{Re} \mu < 1] \quad \text{BI (35)(13)}$$

$$\mathbf{3.684} \quad \int_0^{\pi/4} \frac{(\cot^\mu x - 1) dx}{(\cos x - \sin x) \sin x} = \int_0^{\pi/2} \frac{(\tan^\mu x - 1) dx}{(\sin x - \cos x) \cos x} = -C - \psi(1 - \mu) \quad [\operatorname{Re} \mu < 1]$$

BI (37)(9)

3.685

$$\begin{aligned} 1. \quad \int_0^{\pi/4} (\sin^{\mu-1} 2x - \sin^{\nu-1} 2x) \tan\left(\frac{\pi}{4} + x\right) dx &= \int_0^{\pi/4} (\cos^{\mu-1} 2x - \cos^{\nu-1} 2x) \cot x dx \\ &= \frac{1}{2} [\psi(\nu) - \psi(\mu)] \\ &\quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{BI(34)(9), BI(35)(12)} \end{aligned}$$

$$\begin{aligned} 2. \quad \int_0^{\pi/2} (\sin^{\mu-1} x - \sin^{\nu-1} x) \frac{dx}{\cos x} &= \int_0^{\pi/2} (\cos^{\mu-1} x - \cos^{\nu-1} x) \frac{dx}{\sin x} = \frac{1}{2} \left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\mu}{2}\right) \right] \\ &\quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{BI (46)(2)} \end{aligned}$$

$$\begin{aligned} 3. \quad \int_0^{\pi/2} (\sin^\mu x - \operatorname{cosec}^\mu x) \frac{dx}{\cos x} &= \int_0^{\pi/2} (\cos^\mu x - \sec^\mu x) \frac{dx}{\sin x} = -\frac{\pi}{2} \tan \frac{\mu\pi}{2} \\ &\quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (46)(1, 3)} \end{aligned}$$

$$\begin{aligned} 4. \quad \int_0^{\pi/4} (\sin^\mu 2x - \operatorname{cosec}^\mu 2x) \cot\left(\frac{\pi}{4} + x\right) dx &= \int_0^{\pi/4} (\cos^\mu 2x - \sec^\mu 2x) \tan x dx \\ &= \frac{1}{2\mu} - \frac{\pi}{2} \operatorname{cosec} \mu\pi \\ &\quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (35)(19, 22)} \end{aligned}$$

$$\begin{aligned} 5. \quad \int_0^{\pi/4} (\sin^\mu 2x - \operatorname{cosec}^\mu 2x) \tan\left(\frac{\pi}{4} + x\right) dx &= \int_0^{\pi/4} (\cos^\mu 2x - \sec^\mu 2x) \cot x dx \\ &= -\frac{1}{2\mu} + \frac{\pi}{2} \cot \mu\pi \\ &\quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (35)(14)} \end{aligned}$$

$$\begin{aligned} 6. \quad \int_0^{\pi/4} (\sin^{\mu-1} 2x + \operatorname{cosec}^\mu 2x) \cot\left(\frac{\pi}{4} + x\right) dx \\ &= \int_0^{\pi/4} (\cos^{\mu-1} 2x + \sec^\mu 2x) \tan x dx = \frac{\pi}{4} \operatorname{cosec} \mu\pi \\ &\quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (35)(18, 8)} \end{aligned}$$

$$\begin{aligned} 7. \quad \int_0^{\pi/4} (\sin^{\mu-1} 2x - \operatorname{cosec}^\mu 2x) \tan\left(\frac{\pi}{4} + x\right) dx &= \int_0^{\pi/4} (\cos^{\mu-1} 2x - \sec^\mu 2x) \cot x dx = \frac{\pi}{2} \cot \mu\pi \\ &\quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI(35)(7), LI(34)(10)} \end{aligned}$$

$$\mathbf{3.686} \quad \int_0^{\pi/2} \frac{\tan x dx}{\cos^\mu x + \sec^\mu x} = \int_0^{\pi/2} \frac{\cot x dx}{\sin^\mu x + \operatorname{cosec}^\mu x} = \frac{\pi}{4\mu} \quad \text{BI(47)(28), BI(49)(14)}$$

3.687

$$\begin{aligned} 1. \quad \int_0^{\pi/2} \frac{\sin^{\mu-1} x + \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx &= \int_0^{\pi/2} \frac{\cos^{\mu-1} x + \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\cos\left(\frac{\nu-\mu}{4}\pi\right)}{2 \cos\left(\frac{\nu+\mu}{4}\pi\right)} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right) \\ &\quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, \operatorname{Re}(\mu + \nu) < 2] \\ &\quad \text{BI (46)(7)} \end{aligned}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^{\mu-1} x - \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx = \int_0^{\pi/2} \frac{\cos^{\mu-1} x - \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\sin\left(\frac{\nu-\mu}{4}\pi\right)}{2 \sin\left(\frac{\nu+\mu}{4}\pi\right)} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re}(\mu + \nu) < 4] \quad \text{BI(46)(8)}$$

$$3. \quad \int_0^{\pi/2} \frac{\sin^{\mu} x + \sin^{\nu} x}{\sin^{\mu+\nu} x + 1} \cot x dx = \int_0^{\pi/2} \frac{\cos^{\mu} x + \cos^{\nu} x}{\cos^{\mu+\nu} x + 1} \tan x dx = \frac{\pi}{\mu + \nu} \sec\left(\frac{\mu - \nu}{\mu + \nu} \cdot \frac{\pi}{2}\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (49)(15)a, BI (47)(29)}$$

$$4. \quad \int_0^{\pi/2} \frac{\sin^{\mu} x - \sin^{\nu} x}{\sin^{\mu+\nu} x - 1} \cot x dx = \int_0^{\pi/2} \frac{\cos^{\mu} x - \cos^{\nu} x}{\cos^{\mu+\nu} x - 1} \tan x dx = \frac{\pi}{\mu + \nu} \tan\left(\frac{\mu - \nu}{\mu + \nu} \cdot \frac{\pi}{2}\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI(149)(16)a, BI(47)(30)}$$

$$5. \quad \int_0^{\pi/2} \frac{\cos^{\mu} x + \sec^{\mu} x}{\cos^{\nu} x + \sec^{\nu} x} \tan x dx = \frac{\pi}{2\nu} \sec\left(\frac{\mu}{\nu} \cdot \frac{\pi}{2}\right) \quad [|\operatorname{Re} \nu| > |\operatorname{Re} \mu|] \quad \text{BI (49)(12)}$$

$$6. \quad \int_0^{\pi/2} \frac{\cos^{\mu} x - \sec^{\mu} x}{\cos^{\nu} x - \sec^{\nu} x} \tan x dx = \frac{\pi}{2\nu} \tan\left(\frac{\mu}{\nu} \cdot \frac{\pi}{2}\right) \quad [|\operatorname{Re} \nu| > |\operatorname{Re} \mu|] \quad \text{BI (49)(13)}$$

3.688

$$1. \quad \int_0^{\pi/4} \frac{\tan^{\nu} x - \tan^{\mu} x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \psi(\mu) - \psi(\nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (37)(10)}$$

$$2. \quad \int_0^{\pi/4} \frac{\tan^{\mu} x - \tan^{1-\mu} x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \pi \cot \mu\pi \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (37)(11)}$$

$$3. \quad \int_0^{\pi/4} (\tan^{\mu} x + \cot^{\mu} x) dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (35)(9)}$$

$$4. \quad \int_0^{\pi/4} (\tan^{\mu} x - \cot^{\mu} x) \tan x dx = \frac{1}{\mu} - \frac{\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \quad [0 < \operatorname{Re} \mu < 2] \quad \text{BI (35)(15)}$$

$$5. \quad \int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^{\mu-1} x}{\cos 2x} dx = \frac{\pi}{2} \cot \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 2] \quad \text{BI (35)(10)}$$

$$6. \quad \int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{\cos 2x} \tan x dx = -\frac{1}{\mu} + \frac{\pi}{2} \cot \frac{\mu\pi}{2} \quad [-2 < \operatorname{Re} \mu < 0] \quad \text{BI (35)(23)}$$

$$7. \quad \int_0^{\pi/4} \frac{\tan^{\mu} x + \cot^{\mu} x}{1 + \cos t \sin 2x} dx = \pi \operatorname{cosec} t \operatorname{cosec} \mu\pi \sin \mu t \quad [t \neq n\pi, \quad |\operatorname{Re} \mu| < 1] \quad \text{BI (36)(6)}$$

$$8. \quad \int_0^{\pi/4} \frac{\tan^{\mu-1} x + \cot^{\mu} x}{(\sin x + \cos x) \cos x} dx = \pi \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (37)(3)}$$

$$9. \quad \int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{(\sin x + \cos x) \cos x} dx = -\pi \operatorname{cosec} \mu\pi + \frac{1}{\mu} \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (37)(4)}$$

$$10. \quad \int_0^{\pi/4} \frac{\tan^{\nu} x - \cot^{\mu} x}{(\cos x - \sin x) \cos x} dx = \psi(1 - \mu) - \psi(1 + \nu) \quad [\operatorname{Re} \mu < 1, \quad \operatorname{Re} \nu > -1] \quad \text{BI (37)(5)}$$

$$11. \int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^{\mu} x}{(\cos x - \sin x) \cos x} dx = \pi \cot \mu \pi \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (37)(7)}$$

$$12. \int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{(\cos x - \sin x) \cos x} dx = \pi \cot \mu \pi - \frac{1}{\mu} \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (37)(8)}$$

$$13. \int_0^{\pi/4} \frac{1}{\tan^{\mu} x + \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{8\mu} \quad [\operatorname{Re} \mu \neq 0] \quad \text{BI (37)(12)}$$

$$14. \int_0^{\pi/2} \frac{1}{(\tan^{\mu} x + \cot^{\mu} x)^{\nu}} \cdot \frac{dx}{\tan x} = \int_0^{\pi/2} \frac{1}{(\tan^{\mu} x + \cot^{\mu} x)^{\nu}} \cdot \frac{dx}{\sin 2x} = \frac{\sqrt{\pi}}{2^{2\nu+1} \mu} \frac{\Gamma(\nu)}{\Gamma(\nu + \frac{1}{2})} \quad [\nu > 0] \quad \text{BI(49)(25), BI(49)(26)}$$

$$15. \int_0^{\pi/4} (\tan^{\mu} x - \cot^{\mu} x) (\tan^{\nu} x - \cot^{\nu} x) dx = \frac{2\pi \sin \frac{\mu\pi}{2} \sin \frac{\nu\pi}{2}}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, \quad |\operatorname{Re} \nu| < 1] \quad \text{BI (35)(17)}$$

$$16. \int_0^{\pi/4} (\tan^{\mu} x + \cot^{\mu} x) (\tan^{\nu} x + \cot^{\nu} x) dx = \frac{2\pi \cos \frac{\mu\pi}{2} \cos \frac{\nu\pi}{2}}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, \quad |\operatorname{Re} \nu| < 1] \quad \text{BI (35)(16)}$$

$$17. \int_0^{\pi/4} \frac{(\tan^{\mu} x - \cot^{\mu} x) (\tan^{\nu} x + \cot^{\nu} x)}{\cos 2x} dx = -\pi \frac{\sin \mu\pi}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, \quad |\operatorname{Re} \nu| < 1] \quad \text{BI (35)(25)}$$

$$18. \int_0^{\pi/4} \frac{\tan^{\nu} x - \cot^{\nu} x}{\tan^{\mu} x - \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \tan \frac{\nu\pi}{2\mu} \quad [0 < \operatorname{Re} \nu < 1] \quad \text{BI (37)(14)}$$

$$19. \int_0^{\pi/4} \frac{\tan^{\nu} x + \cot^{\nu} x}{\tan^{\mu} x + \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \sec \frac{\nu\pi}{2\mu} \quad [0 < \operatorname{Re} \nu < 1] \quad \text{BI (37)(13)}$$

$$20. \int_0^{\pi/2} \frac{(1 + \tan x)^{\nu} - 1}{(1 + \tan x)^{\mu+\nu}} \frac{dx}{\sin x \cos x} = \psi(\mu + \nu) - \psi(\mu) \quad [\mu > 0, \quad \nu > 0] \quad \text{BI (49)(29)}$$

3.689

$$1. \int_0^{\pi/2} \frac{(\sin^{\mu} x + \operatorname{cosec}^{\mu} x) \cot x dx}{\sin^{\nu} x - 2 \cos t + \operatorname{cosec}^{\nu} x} = \frac{\pi}{\nu} \operatorname{cosec} t \operatorname{cosec} \frac{\mu\pi}{\nu} \sin \frac{\mu t}{\nu} \quad [\mu < \nu] \quad \text{LI (50)(14)}$$

$$2. \int_0^{\pi/2} \frac{\sin^{\mu} x - 2 \cos t_1 + \operatorname{cosec}^{\mu} x}{\sin^{\nu} x + 2 \cos t_2 + \operatorname{cosec}^{\nu} x} \cdot \cot x dx = \frac{\pi}{\nu} \operatorname{cosec} t_2 \operatorname{cosec} \frac{\mu\pi}{\nu} \sin \frac{\mu t_2}{\nu} - \frac{t_2}{\nu} \operatorname{cosec} t_2 \cos t_1 \quad [(\nu > \mu > 0) \text{ or } (\nu < \mu < 0) \text{ or } (\mu > 0, \nu < 0, \text{ and } \mu + \nu < 0) \text{ or } (\mu < 0, \nu > 0, \text{ and } \mu + \nu > 0)] \quad \text{BI (50)(15)}$$

3.69–3.71 Trigonometric functions of more complicated arguments

3.691

1. $\int_0^\infty \sin(ax^2) dx = \int_0^\infty \cos ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \quad [a > 0] \quad \text{FI II 743a, ET I 64(7)a}$
2. $\int_0^1 \sin(ax^2) dx = \sqrt{\frac{\pi}{2a}} S(\sqrt{a}) \quad [a > 0]$
3. $\int_0^1 \cos(ax^2) dx = \sqrt{\frac{\pi}{2a}} C(\sqrt{a}) \quad [a > 0] \quad \text{ET I 8(5)a}$
4. $\int_0^\infty \sin(ax^2) \sin 2bx dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} C\left(\frac{b}{\sqrt{a}}\right) + \sin \frac{b^2}{a} S\left(\frac{b}{\sqrt{a}}\right) \right\} \quad [a > 0, \quad b > 0] \quad \text{ET I 82(1)a}$
5. $\int_0^\infty \sin(ax^2) \cos 2bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} - \sin \frac{b^2}{a} \right\} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \cos\left(\frac{b^2}{a} + \frac{\pi}{4}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 82(18), BI(70)(13) GW(334)(5a)}$
6. $\int_0^\infty \cos ax^2 \sin 2bx dx = \sqrt{\frac{\pi}{2a}} \left\{ \sin \frac{b^2}{a} C\left(\frac{b}{\sqrt{a}}\right) - \cos \frac{b^2}{a} S\left(\frac{b}{\sqrt{a}}\right) \right\} \quad [a > 0, \quad b > 0] \quad \text{ET I 83(3)a}$
7. $\int_0^\infty \cos ax^2 \cos 2bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} + \sin \frac{b^2}{a} \right\} \quad [a > 0, \quad b > 0] \quad \text{GW(334)(5a), BI(70)(14), ET I 24(7)}$
8. $\int_0^\infty (\cos ax + \sin ax) \sin(b^2 x^2) dx = \frac{1}{2b} \sqrt{\frac{\pi}{2}} \left\{ \left(1 + 2 C\left(\frac{a}{2b}\right)\right) \cos\left(\frac{a^2}{4b^2}\right) - \left(1 - 2 S\left(\frac{a}{2b}\right)\right) \sin\left(\frac{a^2}{4b^2}\right) \right\} \quad [a > 0, \quad b > 0] \quad \text{ET I 85(22)}$
9. $\int_0^\infty (\cos ax + \sin ax) \cos(b^2 x^2) dx = \frac{1}{2b} \sqrt{\frac{\pi}{2}} \left\{ \left(1 + 2 C\left(\frac{a}{2b}\right)\right) \sin\left(\frac{a^2}{4b^2}\right) + \left(1 - 2 S\left(\frac{a}{2b}\right)\right) \cos\left(\frac{a^2}{4b^2}\right) \right\} \quad [a > 0, \quad b > 0] \quad \text{ET I 25(21)}$
10. $\int_0^\infty \sin(a^2 x^2) \sin 2bx \sin 2cx dx = \frac{\sqrt{\pi}}{2a} \sin \frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} - \frac{\pi}{4}\right) \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{ET I 84(15)}$
11. $\int_0^\infty \sin(a^2 x^2) \cos 2bx \cos 2cx dx = \frac{\sqrt{\pi}}{2a} \cos \frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} + \frac{\pi}{4}\right) \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{ET I 84(21)}$

$$12. \quad \int_0^\infty \cos(a^2 x^2) \sin 2bx \sin 2cx \, dx = \frac{\sqrt{\pi}}{2a} \sin \frac{2bc}{a^2} \sin \left(\frac{b^2 + c^2}{a^2} - \frac{\pi}{4} \right) \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{ET I 25(19)}$$

$$13. \quad \int_0^\infty \sin(ax^2) \cos(bx^2) \, dx = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{a-b}} \right) \quad [a > b > 0]$$

$$= \frac{1}{4} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{b+a}} - \frac{1}{\sqrt{b-a}} \right) \quad [b > a > 0]$$

BI (177)(21)

$$14. \quad \int_0^\infty (\sin^2 ax^2 - \sin^2 bx^2) \, dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}} \right) \quad [a > 0, \quad b > 0] \quad \text{BI (178)(1)}$$

$$15. \quad \int_0^\infty (\cos^2 ax^2 - \sin^2 bx^2) \, dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{b}} + \sqrt{\frac{\pi}{a}} \right) \quad [a > 0, \quad b > 0] \quad \text{BI (178)(3)}$$

$$16. \quad \int_0^\infty (\cos^2 ax^2 - \cos^2 bx^2) \, dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{a}} - \sqrt{\frac{\pi}{b}} \right) \quad [a > 0, \quad b > 0] \quad \text{BI (178)(5)}$$

$$17. \quad \int_0^\infty (\sin^4 ax^2 - \sin^4 bx^2) \, x = \frac{1}{64} (8 - \sqrt{2}) \left(\sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}} \right) \quad [a > 0, \quad b > 0] \quad \text{BI (178)(2)}$$

$$18. \quad \int_0^\infty (\cos^4 ax^2 - \sin^4 bx^2) \, dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{b}} \right) + \frac{1}{32} \left(\sqrt{\frac{\pi}{2a}} - \sqrt{\frac{\pi}{2b}} \right) \quad [a > 0, \quad b > 0] \quad \text{BI (178)(4)}$$

$$19. \quad \int_0^\infty (\cos^4 ax^2 - \cos^4 bx^2) \, dx = \frac{1}{64} (8 + \sqrt{2}) \left(\sqrt{\frac{\pi}{a}} - \sqrt{\frac{\pi}{b}} \right) \quad [a > 0, \quad b > 0] \quad \text{BI (178)(6)}$$

$$20. \quad \int_0^\infty \sin^{2n} ax^2 \, dx = \int_0^\infty \cos^{2n} ax^2 \, dx = \infty \quad \text{BI (177)(5, 6)}$$

$$21. \quad \int_0^\infty \sin^{2n+1}(ax^2) \, dx = \frac{1}{2^{2n+1}} \sum_{k=0}^n (-1)^{n+k} \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}} \quad [a > 0] \quad \text{BI (70)(9)}$$

$$22. \quad \int_0^\infty \cos^{2n+1}(ax^2) \, dx = \frac{1}{2^{2n+1}} \sum_{k=0}^n \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}} \quad [a > 0] \quad \text{BI(177)(7)a, BI(70)(10)}$$

3.692

$$1. \quad \int_0^\infty [\sin(a - x^2) + \cos(a - x^2)] \, dx = \sqrt{\frac{\pi}{a}} \sin a \quad \text{GW(333)(30c), BI(178)(7)a}$$

$$2. \quad \int_0^\infty \cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right) \cos ax \, dx = \sqrt{\frac{\pi}{2}} \cos\left(\frac{a^2}{2} - \frac{\pi}{8}\right) \quad [a > 0] \quad \text{ET I 24(8)}$$

$$3. \quad \int_0^\infty \sin[a(1-x^2)] \cos bx \, dx = -\frac{1}{2} \sqrt{\frac{\pi}{a}} \cos\left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) \quad [a > 0] \quad \text{ET I 23(2)}$$

$$4. \quad \int_0^\infty \cos[a(1-x^2)] \cos bx \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \sin\left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) \quad [a > 0] \quad \text{ET I 24(10)}$$

$$5. \quad \int_0^\infty \sin\left(ax^2 + \frac{b^2}{a}\right) \cos 2bx \, dx = \int_0^\infty \cos\left(ax^2 + \frac{b^2}{a}\right) \cos 2bx \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \quad [a > 0] \quad \text{BI (70)(19, 20)}$$

$$6.^8 \quad \int_{-\infty}^\infty [\cos \sqrt{x^2-1} - \cos \sqrt{x^2+1}] \, dx = \sum_{n=0}^\infty \frac{\pi}{\{2^{4n+1} [(2n)!]^2 (n + \frac{1}{2})\}}$$

3.693

$$1. \quad \int_0^\infty \sin(ax^2 + 2bx) \, dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) - \sin \frac{b^2}{a} \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \right\} \quad [a > 0] \quad \text{BI (70)(3)}$$

$$2. \quad \int_0^\infty \cos(ax^2 + 2bx) \, dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) + \sin \frac{b^2}{a} \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \right\} \quad [a > 0] \quad \text{BI (70)(4)}$$

3.694

$$1. \quad \int_0^\infty \sin(ax^2 + 2bx + c) \, dx = \sqrt{\frac{\pi}{2a}} \cos \frac{b^2}{a} \left\{ \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \sin c + \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \cos c \right\} \\ + \sqrt{\frac{\pi}{2a}} \sin \frac{b^2}{a} \left\{ \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \sin c - \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \cos c \right\} \quad [a > 0] \quad \text{GW (334)(4a)}$$

$$2. \quad \int_0^\infty \cos(ax^2 + 2bx + c) \, dx = \sqrt{\frac{\pi}{2a}} \cos \frac{b^2}{a} \left\{ \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \cos c - \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \sin c \right\} \\ + \sqrt{\frac{\pi}{2a}} \sin \frac{b^2}{a} \left\{ \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \cos c + \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \sin c \right\} \quad [a > 0] \quad \text{GW (334)(4b)}$$

3.695

$$1. \quad \int_0^\infty \sin(a^3 x^3) \sin(bx) \, dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left\{ J_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) + J_{-\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) - \frac{\sqrt{3}}{\pi} K_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) \right\} \quad [a > 0, \quad b > 0] \quad \text{ET I 83(5)}$$

$$2. \quad \int_0^\infty \cos(a^3 x^3) \cos(bx) \, dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left\{ J_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) + J_{-\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) + \frac{\sqrt{3}}{\pi} K_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) \right\} \quad [a > 0, \quad b > 0] \quad \text{ET I 24(11)}$$

3.696

$$1. \quad \int_0^\infty \sin(ax^4) \sin(bx^2) dx = -\frac{\pi}{4} \sqrt{\frac{b}{2a}} \sin\left(\frac{b^2}{8a} - \frac{3}{8}\pi\right) J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 83(2)}$$

$$2. \quad \int_0^\infty \sin(ax^4) \cos(bx^2) dx = -\frac{\pi}{4} \sqrt{\frac{b}{2a}} \sin\left(\frac{b^2}{8a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 84(19)}$$

$$3. \quad \int_0^\infty \cos(ax^4) \sin(bx^2) dx = \frac{\pi}{4} \sqrt{\frac{b}{2a}} \cos\left(\frac{b^2}{8a} - \frac{3}{8}\pi\right) J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 83(4), ET I 25(24)}$$

$$4. \quad \int_0^\infty \cos(ax^4) \cos(bx^2) dx = \frac{\pi}{4} \sqrt{\frac{b}{2a}} \cos\left(\frac{b^2}{8a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 25(25)}$$

$$\mathbf{3.697} \quad \int_0^\infty \sin\left(\frac{a^2}{x}\right) \sin(bx) dx = \frac{a\pi}{2\sqrt{b}} J_1(2a\sqrt{b}) \quad [a > 0, \quad b > 0] \quad \text{ET I 83(6)}$$

3.698

$$1. \quad \int_0^\infty \sin\left(\frac{a^2}{x^2}\right) \sin(b^2x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab - \cos 2ab + e^{-2ab}] \quad [a > 0, \quad b > 0] \quad \text{ET I 83(9)}$$

$$2.^8 \quad \int_0^\infty \sin\left(\frac{a^2}{x^2}\right) \cos(b^2x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab + \cos 2ab - e^{-2ab}] \quad \text{ET I 24(13)}$$

$$3. \quad \int_0^\infty \cos\left(\frac{a^2}{x^2}\right) \sin(b^2x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab + \cos 2ab + e^{-2ab}] \quad [a > 0, \quad b > 0] \quad \text{ET I 84(12)}$$

$$4. \quad \int_0^\infty \cos\left(\frac{a^2}{x^2}\right) \cos(b^2x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\cos 2ab - \sin 2ab + e^{-2ab}] \quad [a > 0, \quad b > 0] \quad \text{ET I 24(14)}$$

3.699

$$1. \quad \int_0^\infty \sin\left(a^2x^2 + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} (\cos 2ab + \sin 2ab) \quad [a > 0, \quad b > 0] \quad \text{BI (70)(27)}$$

$$2. \quad \int_0^\infty \cos\left(a^2x^2 + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} (\cos 2ab - \sin 2ab) \quad [a > 0, \quad b > 0] \quad \text{BI (70)(28)}$$

$$3. \quad \int_0^\infty \sin\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right) dx = \int_0^\infty \cos\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} \quad [a > 0, \quad b > 0]$$

BI(179)(11, 12)a, ET I 83(6)

$$4. \quad \int_0^\infty \sin \left(a^2 x^2 - \frac{b^2}{x^2} \right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \quad [a > 0, \quad b > 0] \quad \text{GW (334)(9b)a}$$

$$5. \quad \int_0^\infty \cos \left(a^2 x^2 - \frac{b^2}{x^2} \right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \quad [a > 0, \quad b > 0] \quad \text{GW (334)(9b)a}$$

$$\mathbf{3.711} \quad \int_0^u \sin \left(a \sqrt{u^2 - x^2} \right) \cos bx \, dx = \frac{\pi au}{2\sqrt{a^2 + b^2}} J_1 \left(u \sqrt{a^2 + b^2} \right) \quad [a > 0, \quad b > 0, \quad u > 0] \\ \text{ET I 27(37)}$$

3.712

$$1. \quad \int_0^\infty \sin(ax^p) \, dx = \frac{\Gamma\left(\frac{1}{p}\right) \sin \frac{\pi}{2p}}{pa^{\frac{1}{p}}} \quad [a > 0, \quad p > 1] \quad \text{EH I 13(40)}$$

$$2. \quad \int_0^\infty \cos(ax^p) \, dx = \frac{\Gamma\left(\frac{1}{p}\right) \cos \frac{\pi}{2p}}{pa^{\frac{1}{p}}} \quad [a > 0, \quad p > 1] \quad \text{EH I 13(39)}$$

3.713

$$1. \quad \int_0^\infty \sin(ax^p + bx^q) \, dx = \frac{1}{p} \sum_{k=0}^\infty \frac{(-b)^k}{k!} a^{-\frac{kq+1}{p}} \Gamma\left(\frac{kq+1}{p}\right) \sin \left[\frac{k(q-p)+1}{2p} \pi \right] \\ [a > 0, \quad b > 0, \quad p > 0, \quad q > 0] \\ \text{BI (70)(7)}$$

$$2. \quad \int_0^\infty \cos(ax^p + bx^q) \, dx = \frac{1}{p} \sum_{k=0}^\infty \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \cos \left[\frac{k(q-p)+1}{2p} \pi \right] \\ [a > 0, \quad b > 0, \quad p > 0, \quad q > 0] \\ \text{BI (70)(8)}$$

3.714

$$1. \quad \int_0^\infty \cos(z \sinh x) \, dx = K_0(z) \quad [\operatorname{Re} z > 0] \quad \text{WA 202(14)}$$

$$2. \quad \int_0^\infty \sin(z \cosh x) \, dx = \frac{\pi}{2} J_0(z) \quad [\operatorname{Re} z > 0] \quad \text{MO 36}$$

$$3. \quad \int_0^\infty \cos(z \cosh x) \, dx = -\frac{\pi}{2} Y_0(z) \quad [\operatorname{Re} z > 0] \quad \text{MO 37}$$

$$4. \quad \int_0^\infty \cos(z \sinh x) \cosh \mu x \, dx = \cos \frac{\mu\pi}{2} K_\mu(z) \quad [\operatorname{Re} z > 0, \quad |\operatorname{Re} \mu| < 1] \quad \text{WA 202(13)}$$

$$5. \quad \int_0^\pi \cos(z \cosh x) \sin^{2\mu} x \, dx = \sqrt{\pi} \left(\frac{2}{z} \right)^\mu \Gamma\left(\mu + \frac{1}{2}\right) I_\mu(z) \\ [\operatorname{Re} z > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{WH}$$

3.715

$$1. \quad \int_0^\pi \sin(z \sin x) \sin ax \, dx = \sin a\pi s_{0,a}(z) = \sin a\pi \sum_{k=1}^\infty \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \\ [a > 0] \quad \text{WA 338(13)}$$

$$\begin{aligned}
2. \quad \int_0^\pi \sin(z \sin x) \sin nx \, dx &= \frac{1}{2} \int_{-\pi}^\pi \sin(z \sin x) \sin nx \, dx \\
&= [1 - (-1)^n] \int_0^{\pi/2} \sin(z \sin x) \sin nx \, dx = [1 - (-1)^n] \frac{\pi}{2} J_n(z) \\
&\quad [n = 0, \pm 1, \pm 2, \dots] \quad \text{WA 30(6), GW(334)(153a)}
\end{aligned}$$

$$3. \quad \int_0^{\pi/2} \sin(z \sin x) \sin 2x \, dx = \frac{2}{z^2} (\sin z - z \cos z) \quad \text{LI (43)(14)}$$

$$\begin{aligned}
4. \quad \int_0^\pi \sin(z \sin x) \cos ax \, dx &= (1 + \cos a\pi) s_{0,a}(z) \\
&= (1 + \cos a\pi) \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \\
&\quad [a > 0] \quad \text{WA 338(14)}
\end{aligned}$$

$$5. \quad \int_0^\pi \sin(z \sin x) \cos[(2n+1)x] \, dx = 0 \quad \text{GW (334)(53b)}$$

$$\begin{aligned}
6. \quad \int_0^\pi \cos(z \sin x) \sin ax \, dx &= -a(1 - \cos a\pi) s_{-1,a}(z) \\
&= -a(1 - \cos a\pi) \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\
&\quad [a > 0] \quad \text{WA 338(12)}
\end{aligned}$$

$$7. \quad \int_0^\pi \cos(z \sin x) \sin 2nx \, dx = 0 \quad \text{GW (334)(54a)}$$

$$\begin{aligned}
8. \quad \int_0^\pi \cos(z \sin x) \cos ax \, dx &= -a \sin a\pi s_{-1,a}(z) \\
&= -a \sin a\pi \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\
&\quad [a > 0] \quad \text{WA 338(11)}
\end{aligned}$$

$$\begin{aligned}
9. \quad \int_0^\pi \cos(z \sin x) \cos nx \, dx &= \frac{1}{2} \int_{-\pi}^\pi \cos(z \sin x) \cos nx \, dx \\
&= [1 + (-1)^n] \int_0^{\pi/2} \cos(z \sin x) \cos nx \, dx = [1 + (-1)^n] \frac{\pi}{2} J_n(z) \\
&\quad \text{GW (334)(54b)}
\end{aligned}$$

$$10.^8 \quad \int_0^{\pi/2} \cos(z \sin x) \cos^{2n} x \, dx = \frac{\pi}{2} \frac{(2n-1)!!}{z^n} J_n(z) \quad [n = 0, 1, 2, \dots] \quad \text{FI II 486, WA 35a}$$

$$11. \quad \int_0^{\pi/2} \sin(z \cos x) \sin 2x \, dx = \frac{2}{z^2} (\sin z - z \cos z) \quad \text{LI (43)(15)}$$

- 12.⁸ $\int_0^{\pi/2} \sin(z \cos x) \cos ax \, dx = \cos \frac{a\pi}{2} s_{0,a}(z) = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{J}_a(z) - \mathbf{J}_{-a}(z)]$
 $= -\frac{\pi}{4} \sec \frac{a\pi}{4} [\mathbf{E}_a(z) + \mathbf{E}_{-a}(z)]$
 $= \cos \frac{a\pi}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]}$
 $[a > 0]$ WA 339
13. $\int_0^{\pi} \sin(z \cos x) \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(z \cos x) \cos nx \, dx = \pi \sin \frac{n\pi}{2} J_n(z)$ GW (334)(55b)
14. $\int_0^{\pi/2} \sin(z \cos x) \cos[(2n+1)x] \, dx = (-1)^n \frac{\pi}{2} J_{2n+1}(z)$ WA 30(8)
- 15.¹¹ $\int_0^{\pi/2} \sin(a \cos x) \tan x \, dx = \operatorname{si}(a) + \frac{\pi}{2}$ $[a > 0]$ BI (43)(17)
16. $\int_0^{\pi/2} \sin(z \cos x) \sin^{2\nu} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_{\nu}(z)$
 $[\operatorname{Re} \nu > -\frac{1}{2}]$ WA 358(1)
- 17.⁷ $\int_0^{\pi/2} \cos(z \cos x) \cos ax \, dx = -a \sin \frac{a\pi}{2} s_{-1,a}(z)$
 $= \frac{\pi}{4} \sec \frac{a\pi}{2} [\mathbf{J}_a(z) + \mathbf{J}_{-a}(z)] = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{E}_a(z) - \mathbf{E}_{-a}(z)]$
 $= -a \sin \frac{a\pi}{2} \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\}$
 $[a > 0]$ WA 339
18. $\int_0^{\pi} \cos(z \cos x) \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(z \cos x) \cos nx \, dx = \pi \cos \frac{n\pi}{2} J_n(z)$ GW (334)(56b)
19. $\int_0^{\pi/2} \cos(z \cos x) \cos 2nx \, dx = (-1)^n \cdot \frac{\pi}{2} J_{2n}(z)$ WA 30(9)
20. $\int_0^{\pi/2} \cos(z \cos x) \sin^{2\nu} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}(z)$
 $[\operatorname{Re} \nu > -\frac{1}{2}]$ WA 35, WH
21. $\int_0^{\pi} \cos(z \cos x) \sin^{2\mu} x \, dx = \sqrt{\pi} \left(\frac{2}{z}\right)^{\mu} \Gamma\left(\mu + \frac{1}{2}\right) J_{\mu}(z)$
 $[\operatorname{Re} \mu > -\frac{1}{2}]$ WH

3.716

1. $\int_0^{\pi/2} \sin(a \tan x) \, dx = \frac{1}{2} [e^{-a} \overline{\operatorname{Ei}}(a) - e^a \operatorname{Ei}(-a)]$ $[a > 0]$ (cf. **3.723** 1) BI (43)(1)
2. $\int_0^{\pi/2} \cos(a \tan x) \, dx = \frac{\pi}{2} e^{-a}$ $[a \geq 0]$ BI (43)(2)

3. $\int_0^{\pi/2} \sin(a \tan x) \sin 2x \, dx = \frac{a\pi}{2} e^{-a} \quad [a \geq 0] \quad \text{BI (43)(7)}$
4. $\int_0^{\pi/2} \cos(a \tan x) \sin^2 x \, dx = \frac{1-a}{4} \pi e^{-a} \quad [a \geq 0] \quad \text{BI (43)(8)}$
5. $\int_0^{\pi/2} \cos(a \tan x) \cos^2 x \, dx = \frac{1+a}{4} \pi e^{-a} \quad [a \geq 0] \quad \text{BI (43)(9)}$
6. $\int_0^{\pi/2} \sin(a \tan x) \tan x \, dx = \frac{\pi}{2} e^{-a} \quad [a > 0] \quad \text{BI (43)(5)}$
7. $\int_0^{\pi/2} \cos(a \tan x) \tan x \, dx = -\frac{1}{2} [e^{-a} \overline{\text{Ei}}(a) + e^a \text{Ei}(-a)]$
 $[a > 0] \quad (\text{cf. } \mathbf{3.723} \text{ 5}) \quad \text{BI (43)(6)}$
8. $\int_0^{\pi/2} \sin(a \tan x) \sin^2 x \tan x \, dx = \frac{2-a}{4} \pi e^{-a} \quad [a > 0] \quad \text{BI (43)(11)}$
9. $\int_0^{\pi/2} \sin^2(a \tan x) \, dx = \frac{\pi}{4} (1 - e^{-2a}) \quad [a \geq 0] \quad (\text{cf. } \mathbf{3.742} \text{ 1}) \quad \text{BI (43)(3)}$
10. $\int_0^{\pi/2} \cos^2(a \tan x) \, dx = \frac{\pi}{4} (1 + e^{-2a}) \quad [a \geq 0] \quad (\text{cf. } \mathbf{3.742} \text{ 3}) \quad \text{BI (43)(4)}$
11. $\int_0^{\pi/2} \sin^2(a \tan x) \cot^2 x \, dx = \frac{\pi}{4} (e^{-2a} + 2a - 1) \quad [a \geq 0] \quad \text{BI (43)(19)}$
12. $\int_0^{\pi/2} [1 - \sec^2 x \cos(\tan x)] \frac{dx}{\tan x} = C \quad \text{BI (51)(14)}$
13. $\int_0^{\pi/2} \sin(a \cot x) \sin 2x \, dx = \frac{a\pi}{2} e^{-a} \quad [a \geq 0] \quad (\text{cf. } \mathbf{3.716} \text{ 3})$

In general, formulas **3.716** remain valid if we replace $\tan x$ in the argument of the sine or cosine with $\cot x$ if we also replace $\sin x$ with $\cos x$, $\cos x$ with $\sin x$, hence $\tan x$ with $\cot x$, $\cot x$ with $\tan x$, $\sec x$ with $\operatorname{cosec} x$, and $\operatorname{cosec} x$ with $\sec x$ in the factors. Analogously,

$$\mathbf{3.717} \quad \int_0^{\pi/2} \sin(a \operatorname{cosec} x) \sin(a \cot x) \frac{dx}{\cos x} = \int_0^{\pi/2} \sin(a \sec x) \sin(a \tan x) \frac{dx}{\sin x} = \frac{\pi}{2} \sin a \quad [a \geq 0]$$

BI (52)(11, 12)

3.718

1. $\int_0^{\pi/2} \sin\left(\frac{\pi}{2}p - a \tan x\right) \tan^{p-1} x \, dx = \int_0^{\pi/2} \cos\left(\frac{\pi}{2}p - a \tan x\right) \tan^p x \, dx = \frac{\pi}{2} e^{-a}$
 $[p^2 < 1, \quad p \neq 0, \quad a \geq 0] \quad \text{BI (44)(5, 6)}$
2. $\int_0^{\pi/2} \sin(a \tan x - \nu x) \sin^{\nu-2} x \, dx = 0 \quad [\operatorname{Re} \nu > 0, \quad a > 0] \quad \text{NH 157(15)}$
3. $\int_0^{\pi/2} \sin(n \tan x + \nu x) \frac{\cos^{\nu-1} x}{\sin x} \, dx = \frac{\pi}{2} \quad [\operatorname{Re} \nu > 0] \quad \text{BI (51)(15)}$

4. $\int_0^{\pi/2} \cos(a \tan x - \nu x) \cos^{\nu-2} x \, dx = \frac{\pi e^{-a} a^{\nu-1}}{\Gamma(\nu)} \quad [\operatorname{Re} \nu > 1, \quad a > 0]$
LO V 153(112), NT 157(14)
5. $\int_0^{\pi/2} \cos(a \tan x + \nu x) \cos^{\nu} x \, dx = 2^{-\nu-1} \pi e^{-a} \quad [\operatorname{Re} \nu > -1, \quad a \geq 0]$ BI (44)(4)
6. $\int_0^{\pi/2} \cos(a \tan x - \gamma x) \cos^{\nu} x \, dx = \frac{\pi a^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}+1}} \cdot \frac{W_{\frac{\gamma}{2}, -\frac{\nu+1}{2}}(2a)}{\Gamma(1 + \frac{\gamma+\nu}{2})}$
 $\left[a > 0, \quad \operatorname{Re} \nu > -1, \quad \frac{\nu+\gamma}{2} \neq -1, -2, \dots \right]$ EH I 274(13)a
7. $\int_0^{\pi/2} \frac{\sin nx - \sin(nx - a \tan x)}{\sin x} \cos^{n-1} x \, dx = \begin{cases} \pi/2 & [n = 0, \quad a > 0], \\ \pi(1 - e^{-a}) & [n = 1, \quad a \geq 0] \end{cases}$
LO V 153(114)

3.719

- 1.⁶ $\int_0^{\pi} \sin(\nu x - z \sin x) \, dx = \pi \mathbf{E}_{\nu}(z)$ WA 336(2)
2. $\int_0^{\pi} \cos(nx - z \sin x) \, dx = \pi J_n(z)$ WH
3. $\int_0^{\pi} \cos(\nu x - z \sin x) \, dx = \pi \mathbf{J}_{\nu}(z)$ WA 336(1)

3.72–3.74 Combinations of trigonometric and rational functions**3.721**

1. $\int_0^{\infty} \frac{\sin(ax)}{x} \, dx = \frac{\pi}{2} \operatorname{sign} a$ FI II 645
2. $\int_1^{\infty} \frac{\sin(ax)}{x} \, dx = -\operatorname{si}(a)$ BI 203(1)
- 3.⁸ $\int_1^{\infty} \frac{\cos(ax)}{x} \, dx = -\operatorname{ci}(a)$ BI 203(5)

3.722

1. $\int_0^{\infty} \frac{\sin(ax)}{x + \beta} \, dx = \operatorname{ci}(a\beta) \sin(a\beta) - \cos(a\beta) \operatorname{si}(a\beta) \quad [|\arg \beta| < \pi, \quad a > 0]$
BI(16)(1), FI II 646a
- 2.¹¹ $\int_{-\infty}^{\infty} \frac{\sin(ax)}{x + \beta} \, dx = \pi e^{ia\beta} \quad [a > 0, \quad \operatorname{Im} \beta > 0]$
3. $\int_0^{\infty} \frac{\cos(ax)}{x + \beta} \, dx = -\sin(a\beta) \operatorname{si}(a\beta) - \cos(a\beta) \operatorname{ci}(a\beta) \quad [|\arg \beta| < \pi, \quad a > 0]$
ET I 8(7), BI(160)(2)

$$4.^8 \quad \int_{-\infty}^{\infty} \frac{\cos(ax)}{x + \beta} dx = -i\pi e^{ia\beta} \quad [a > 0, \quad \text{Im } \beta > 0]$$

$$5.^{10} \quad \int_0^{\infty} \frac{\sin(ax)}{\beta - x} dx = \sin(\beta a) \text{ci}(\beta a) - \cos(\beta a) [\text{si}(\beta a) + \pi] \\ [a > 0, \quad \beta \text{ not real and positive}] \\ \text{FI II 646, BI(161)(1)}$$

$$6.^8 \quad \int_{-\infty}^{\infty} \frac{\sin(ax)}{\beta - x} dx = -\pi e^{ia\beta} \quad [a > 0, \quad \text{Im } \beta > 0]$$

$$7.^{10} \quad \int_0^{\infty} \frac{\cos(ax)}{\beta - x} dx = -\cos(a\beta) \text{ci}(a\beta) + \sin(a\beta) [\text{si}(a\beta) + \pi] \\ [a > 0, \quad \beta \text{ not real and positive}] \\ \text{ET I 8(8), BI(161)(2)a}$$

$$8.^{11} \quad \int_{-\infty}^{\infty} \frac{\cos(ax)}{\beta - x} dx = -i\pi e^{ia\beta} \quad [a > 0, \quad \text{Im } \beta > 0]$$

3.723

$$1.^{11} \quad \int_0^{\infty} \frac{\sin(ax)}{\beta^2 + x^2} dx = \frac{1}{2\beta} [e^{-a\beta} \overline{\text{Ei}}(a\beta) - e^{a\beta} \text{Ei}(-a\beta)] \quad [a > 0, \quad \beta > 0] \quad \text{ET I 65(14), BI(160)(3)}$$

$$2. \quad \int_0^{\infty} \frac{\cos(ax)}{\beta^2 + x^2} dx = \frac{\pi}{2\beta} e^{-a\beta} \quad [a \geq 0, \quad \text{Re } \beta > 0] \\ \text{FI II 741, 750, ET I 8(11), WH}$$

$$3. \quad \int_0^{\infty} \frac{x \sin(ax)}{\beta^2 + x^2} dx = \frac{\pi}{2} e^{-a\beta} \quad [a > 0, \quad \text{Re } \beta > 0] \\ \text{FI II 741, 750, ET I 65(15), WH}$$

$$4. \quad \int_{-\infty}^{\infty} \frac{x \sin(ax)}{\beta^2 + x^2} dx = \pi e^{-a\beta} \quad [a > 0, \quad \text{Re } \beta > 0] \quad \text{BI (202)(10)}$$

$$5.^{11} \quad \int_0^{\infty} \frac{x \cos(ax)}{\beta^2 + x^2} dx = -\frac{1}{2} [e^{-a\beta} \overline{\text{Ei}}(a\beta) + e^{a\beta} \text{Ei}(-a\beta)] \quad [a > 0, \quad \beta > 0] \quad \text{BI (160)(6)}$$

$$6. \quad \int_{-\infty}^{\infty} \frac{\sin[a(b-x)]}{c^2 + x^2} dx = \frac{\pi}{c} e^{-ac} \sin(ab) \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{LI (202)(9)}$$

$$7. \quad \int_{-\infty}^{\infty} \frac{\cos[a(b-x)]}{c^2 + x^2} dx = \frac{\pi}{c} e^{-ac} \cos(ab) \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{LI (202)(11)a}$$

$$8. \quad \int_0^{\infty} \frac{\sin(ax)}{\beta^2 - x^2} dx = \frac{1}{\beta} \left[\sin(a\beta) \text{ci}(a\beta) - \cos(a\beta) \left(\text{si}(a\beta) + \frac{\pi}{2} \right) \right] \\ [|\arg \beta| < \pi, \quad a > 0] \quad \text{BI (161)(3)}$$

$$9. \quad \int_0^{\infty} \frac{\cos(ax)}{b^2 - x^2} dx = \frac{\pi}{2b} \sin(ab) \quad [a > 0, \quad b > 0] \quad \text{BI(161)(5), ET I 9(15)}$$

$$10. \quad \int_0^{\infty} \frac{x \sin(ax)}{b^2 - x^2} dx = -\frac{\pi}{2} \cos(ab) \quad [a > 0] \quad \text{FI II 647, ET II 252(45)}$$

$$11. \quad \int_0^{\infty} \frac{x \cos(ax)}{\beta^2 + x^2} dx = \cos(a\beta) \text{ci}(a\beta) + \sin(a\beta) \left[\text{si}(a\beta) + \frac{\pi}{2} \right] \\ [|\arg \beta| < \pi, \quad a > 0] \quad \text{BI (161)(6)}$$

$$12. \quad \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x-b)} dx = \pi \frac{\cos(ab) - 1}{b} \quad [a > 0, \quad b > 0] \quad \text{ET II 252(44)}$$

3.724

$$1. \quad \int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2} \sin(ax) dx = \left(\frac{cq-b}{\sqrt{p-q^2}} \sin(aq) + c \cos(aq) \right) \pi e^{-a\sqrt{p-q^2}} \\ [a > 0, \quad p > q^2] \quad \text{BI (202)(12)}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2} \cos(ax) dx = \left(\frac{b-cq}{\sqrt{p-q^2}} \cos(aq) + c \sin(aq) \right) \pi e^{-a\sqrt{p-q^2}} \\ [a > 0, \quad p > q^2] \quad \text{BI (202)(13)}$$

$$3. \quad \int_{-\infty}^{\infty} \frac{\cos[(b-1)t] - x \cos(bt)}{1-2x \cos t + x^2} \cos(ax) dx = \pi e^{-a \sin t} \sin(bt + a \cos t) \\ [a > 0, \quad t^2 < \pi^2] \quad \text{BI (202)(14)}$$

3.725

$$1. \quad \int_0^{\infty} \frac{\sin(ax) dx}{x(\beta^2 + x^2)} = \frac{\pi}{2\beta^2} (1 - e^{-a\beta}) \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (172)(1)}$$

$$2. \quad \int_0^{\infty} \frac{\sin(ax) dx}{x(b^2 - x^2)} = \frac{\pi}{2b^2} (1 - \cos(ab)) \quad [a > 0] \quad \text{BI (172)(4)}$$

$$3. \quad \int_0^{\infty} \frac{\sin(ax) \cos(bx)}{x(x^2 + \beta^2)} dx = \frac{\pi}{2\beta^2} e^{-\beta b} \sinh(a\beta) \quad [0 < a < b] \\ = -\frac{\pi}{2\beta^2} e^{-a\beta} \cosh(b\beta) + \frac{\pi}{2\beta^2} \quad [a > b > 0] \\ \text{ET I 19(4)}$$

3.726

$$1.^{11} \quad \int_0^{\infty} \frac{x \sin(ax) dx}{b^3 \pm b^2 x + bx^2 \pm x^3} \\ = \pm \frac{1}{4b} \left[e^{-ab} \overline{\operatorname{Ei}}(ab) - e^{ab} \operatorname{Ei}(-ab) - 2 \operatorname{ci}(ab) \sin(ab) + 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) \right] \\ + \frac{\pi e^{-ab} - \pi \cos(ab)}{4b} \\ [a > 0, \quad b > 0; \quad \text{if the lower sign is taken, then the integral is a principal value integral}] \\ \text{ET I 65(21)a, BI(176)(10, 13)}$$

$$2.^7 \quad \int_0^{\infty} \frac{x^2 \sin(ax) dx}{b^3 \pm b^2 x + bx^2 \pm x^3} \\ = \frac{1}{4} \left[e^{ab} \operatorname{Ei}(-ab) - e^{-ab} \overline{\operatorname{Ei}}(ab) + 2 \operatorname{ci}(ab) \sin(ab) - 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) \right] \\ \pm \pi (e^{-ab} + \cos(ab)) \\ [a > 0, \quad b > 0; \quad \text{if the lower sign is taken, then the integral is a principal value integral}] \\ \text{ET I 66(22), BI(176)(11, 14)}$$

3.727

1.
$$\int_0^\infty \frac{\cos(ax)}{b^4 + x^4} dx = \frac{\pi\sqrt{2}}{4b^3} \exp\left(-\frac{ab}{\sqrt{2}}\right) \left(\cos \frac{ab}{\sqrt{2}} + \sin \frac{ab}{\sqrt{2}}\right)$$

 $[a > 0, \quad b > 0] \quad \text{BI(160)(25)a, ET I 9(19)}$
- 2.⁸
$$\int_0^\infty \frac{\sin(ax)}{b^4 - x^4} dx = \frac{1}{4b^3} \left[2 \sin(ab) \operatorname{ci}(ab) - 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) \right. \\ \left. + e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab) \right]$$

 $[a > 0, \quad b > 0] \quad \text{BI (161)(12)}$
3.
$$\int_0^\infty \frac{\cos(ax)}{b^4 - x^4} dx = \frac{\pi}{4b^3} [e^{-ab} + \sin(ab)]$$

 $[a > 0, \quad b > 0] \quad (\text{cf. } \mathbf{3.723} \text{ 2 and } \mathbf{3.723} \text{ 9}) \quad \text{BI (161)(16)}$
4.
$$\int_0^\infty \frac{x \sin(ax)}{b^4 + x^4} dx = \frac{\pi}{2b^2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \sin \frac{ab}{\sqrt{2}}$$

 $[a > 0, \quad b > 0] \quad \text{BI (160)(23)a}$
5.
$$\int_0^\infty \frac{x \sin(ax)}{b^4 - x^4} dx = \frac{\pi}{4b^2} [e^{-ab} - \cos(ab)]$$

 $[a > 0, \quad b > 0] \quad \text{BI (161)(13)}$
- 6.¹¹
$$\int_0^\infty \frac{x \cos(ax)}{b^4 - x^4} dx = \frac{1}{4b^2} \left[2 \cos(ab) \operatorname{ci}(ab) + 2 \sin(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) \right. \\ \left. - e^{-ab} \overline{\operatorname{Ei}}(ab) - e^{ab} \operatorname{Ei}(-ab) \right]$$

 $[a > 0, \quad b > 0] \quad (\text{cf. } \mathbf{3.723} \text{ 5 and } \mathbf{3.723} \text{ 11}) \quad \text{BI (161)(17)}$
7.
$$\int_0^\infty \frac{x^2 \cos(ax)}{b^4 + x^4} dx = \frac{\pi\sqrt{2}}{4b} \exp\left(-\frac{ab}{\sqrt{2}}\right) \left(\cos \frac{ab}{\sqrt{2}} - \sin \frac{ab}{\sqrt{2}}\right)$$

 $[a > 0, \quad b > 0] \quad \text{BI (160)(26)a}$
- 8.¹¹
$$\int_0^\infty \frac{x^2 \sin(ax)}{b^4 - x^4} dx = \frac{1}{4b} \left[2 \sin(ab) \operatorname{ci}(ab) \right. \\ \left. - 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) - e^{-ab} \overline{\operatorname{Ei}}(ab) + e^{ab} \operatorname{Ei}(-ab) \right]$$

 $[a > 0, \quad b > 0] \quad \text{BI (161)(14)}$
9.
$$\int_0^\infty \frac{x^2 \cos(ax)}{b^4 - x^4} dx = \frac{\pi}{4b} (\sin(ab) - e^{-ab})$$

 $[a > 0, \quad b > 0] \quad \text{BI (161)(18)}$
10.
$$\int_0^\infty \frac{x^3 \sin(ax)}{b^4 + x^4} dx = \frac{\pi}{2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos \frac{ab}{\sqrt{2}}$$

 $[a > 0, \quad b > 0] \quad \text{BI (160)(24)}$
11.
$$\int_0^\infty \frac{x^3 \sin(ax)}{b^4 - x^4} dx = \frac{-\pi}{4} [e^{-ab} - \cos(ab)]$$

 $[a > 0, \quad b > 0] \quad \text{BI (161)(15)}$
- 12.⁷
$$\int_0^\infty \frac{x^3 \cos(ax)}{b^4 - x^4} dx = \frac{1}{4} \left[2 \cos(ab) \operatorname{ci}(ab) + 2 \sin(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) \right. \\ \left. + e^{-ab} \overline{\operatorname{Ei}}(ab) + e^{ab} \operatorname{Ei}(-ab) \right]$$

 $[a > 0, \quad b > 0] \quad \text{BI(161)(19)}$

$$13. \quad \int_0^\infty \frac{x^3 \sin ax}{(x^2 + b^2)^3} dx = \frac{\pi e^{-ab}}{16b} (3a - ba^2)$$

$$14. \quad \int_0^\infty \frac{x^3 \sin ax}{(x^2 + b^2)^4} dx = \frac{\pi e^{-ab} a}{96b^3} (3 + 3ab - a^2 b^2)$$

3.728

$$1. \quad \int_0^\infty \frac{\cos(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (\beta e^{-a\gamma} - \gamma e^{-a\beta})}{2\beta\gamma(\beta^2 - \gamma^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{BI (175)(1)}$$

$$2. \quad \int_0^\infty \frac{x \sin(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (e^{-a\beta} - e^{-a\gamma})}{2(\gamma^2 - \beta^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{BI (174)(1)}$$

$$3. \quad \int_0^\infty \frac{x^2 \cos(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (\beta e^{-a\beta} - \gamma e^{-a\gamma})}{2(\beta^2 - \gamma^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{BI (175)(2)}$$

$$4. \quad \int_0^\infty \frac{x^3 \sin(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (\beta^2 e^{-a\beta} - \gamma^2 e^{-a\gamma})}{2(\beta^2 - \gamma^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{BI (174)(2)}$$

$$5. \quad \int_0^\infty \frac{\cos(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (b \sin(ac) - c \sin(ab))}{2bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (175)(3)}$$

$$6. \quad \int_0^\infty \frac{x \sin(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (\cos(ab) - \cos(ac))}{2(b^2 - c^2)} \quad [a > 0] \quad \text{BI (174)(3)}$$

$$7. \quad \int_0^\infty \frac{x^2 \cos(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (c \sin(ac) - b \sin(ab))}{2(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (175)(4)}$$

$$8. \quad \int_0^\infty \frac{x^3 \sin(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (b^2 \cos(ab) - c^2 \cos(ac))}{2(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (174)(4)}$$

$$9. \quad \int_0^\infty \frac{x \sin ax}{(b^2 - x^2)(c^2 + x^2)} dx = \frac{\pi}{2} \frac{e^{-ac} - \cos ba}{a^2 + c^2} \quad [a > 0, \quad c > 0, \quad b \text{ real}]$$

3.729

$$1. \quad \int_0^\infty \frac{\cos(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4b^3} (1 + ab) e^{-ab} \quad [a > 0, \quad b > 0] \quad \text{BI (170)(7)}$$

$$2. \quad \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4b} a e^{-ab} \quad [a > 0, \quad b > 0] \quad \text{BI (170)(3)}$$

$$3. \quad \int_0^\infty \cos(px) \frac{1 - x^2}{(1 + x^2)^2} dx = \frac{\pi p}{2} e^{-p} \quad \text{BI (43)(10)a}$$

$$4. \quad \int_0^\infty \frac{x^3 \sin(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4} (2 - ab) e^{-ab} \quad [a > 0, \quad b > 0] \quad \text{BI (170)(4)}$$

3.731 Notation: $2A^2 = \sqrt{b^4 + c^2} + b^2$, $2B^2 = \sqrt{b^4 + c^2} - b^2$,

$$1. \quad \int_0^\infty \frac{\cos(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2c} \frac{e^{-aA} (B \cos(aB) + A \sin(aB))}{\sqrt{b^4 + c^2}} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (176)(3)}$$

$$2. \quad \int_0^\infty \frac{x \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2c} e^{-aA} \sin(aB) \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (176)(1)}$$

$$3. \quad \int_0^\infty \frac{(x^2 + b^2) \cos(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2} \frac{e^{-aA} (A \cos(aB) - B \sin(aB))}{\sqrt{b^4 + c^2}} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (176)(4)}$$

$$4. \quad \int_0^\infty \frac{x (x^2 + b^2) \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2} e^{-aA} \cos(aB) \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (176)(2)}$$

3.732

$$1. \quad \int_0^\infty \left[\frac{1}{\beta^2 + (\gamma - x)^2} - \frac{1}{\beta^2 + (\gamma + x)^2} \right] \sin(ax) dx = \frac{\pi}{\beta} e^{-a\beta} \sin(a\gamma) \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma + i\beta \text{ is not real}]$$

ET I 65(16)

$$2. \quad \int_0^\infty \left[\frac{1}{\beta^2 + (\gamma - x)^2} + \frac{1}{\beta^2 + (\gamma + x)^2} \right] \cos(ax) dx = \frac{\pi}{\beta} e^{-a\beta} \cos(a\gamma) \quad [a > 0, \quad |\operatorname{Im} \gamma| < \operatorname{Re} \beta] \quad \text{ET I 8(13)}$$

$$3. \quad \int_0^\infty \left[\frac{\gamma + x}{\beta^2 + (\gamma + x)^2} - \frac{\gamma - x}{\beta^2 + (\gamma - x)^2} \right] \sin(ax) dx = \pi e^{-a\beta} \cos(a\gamma) \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma + i\beta \text{ is not real}]$$

LI (175)(17)

$$4. \quad \int_0^\infty \left[\frac{\gamma + x}{\beta^2 + (\gamma + x)^2} + \frac{\gamma - x}{\beta^2 + (\gamma - x)^2} \right] \cos(ax) dx = \pi e^{-a\beta} \sin(a\gamma) \quad [a > 0, \quad |\operatorname{Im} a| < \operatorname{Re} \beta] \quad \text{LI (176)(21)}$$

3.733

$$1. \quad \int_0^\infty \frac{\cos(ax) dx}{x^4 + 2b^2 x^2 \cos 2t + b^4} = \frac{\pi}{2b^3} \exp(-ab \cos t) \frac{\sin(t + ab \sin t)}{\sin 2t} \quad \left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right] \quad \text{BI (176)(7)}$$

$$2. \quad \int_0^\infty \frac{x \sin(ax) dx}{x^4 + 2b^2 x^2 \cos 2t + b^4} = \frac{\pi}{2b^2} \exp(-ab \cos t) \frac{\sin(ab \sin t)}{\sin 2t} \quad \left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right]$$

BI(176)(5), ET I 66(23)

$$3. \quad \int_0^\infty \frac{x^2 \cos(ax) dx}{x^4 + 2b^2 x^2 \cos 2t + b^4} = \frac{\pi}{2b} \exp(-ab \cos t) \frac{\sin(t - ab \sin t)}{\sin 2t} \quad \left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right] \quad \text{BI (176)(8)}$$

$$4. \quad \int_0^\infty \frac{x^3 \sin(ax) dx}{x^4 + 2b^2 x^2 \cos 2t + b^4} = \frac{\pi}{2} \exp(-ab \cos t) \frac{\sin(2t - ab \sin t)}{\sin 2t} \quad \left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right] \quad \text{BI (176)(6)}$$

$$5. \quad \int_0^\infty \frac{\sin(ax) dx}{x(x^4 + 2b^2 x^2 \cos 2t + b^4)} = \frac{\pi}{2b^4} \left[1 - \exp(-ab \cos t) \frac{\sin(2t + ab \sin t)}{\sin 2t} \right] \quad \left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right] \quad \text{BI (176)(22)}$$

3.734

$$1. \quad \int_0^\infty \frac{\sin(ax) dx}{x(b^4 + x^4)} = \frac{\pi}{2b^4} \left[1 - \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos \frac{ab}{\sqrt{2}} \right] \quad [a > 0, \quad b > 0] \quad \text{BI (172)(7)}$$

$$2. \quad \int_0^\infty \frac{\sin(ax) dx}{x(b^4 - x^4)} = \frac{\pi}{4b^4} [2 - e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0] \quad \text{BI (172)(10)}$$

$$\mathbf{3.735} \quad \int_0^\infty \frac{\sin(ax) dx}{x(b^2 + x^2)^2} = \frac{\pi}{2b^4} \left[1 - \frac{1}{2} e^{-ab} (2 + ab) \right] \quad [a > 0, \quad b > 0] \quad \text{WH, BI (172)(22)}$$

3.736

$$1. \quad \int_0^\infty \frac{\cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^5} [\sin(ab) + (2 + ab)e^{-ab}] \quad [a > 0, \quad b > 0] \quad \text{BI (176)(5)}$$

$$2. \quad \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^4} [(1 + ab)e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0] \quad \text{BI (174)(5)}$$

$$3. \quad \int_0^\infty \frac{x^2 \cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^3} [\sin(ab) - abe^{-ab}] \quad [a > 0, \quad b > 0] \quad \text{BI (175)(6)}$$

$$4. \quad \int_0^\infty \frac{x^3 \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^2} [(1 - ab)e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0] \quad \text{BI (174)(6)}$$

$$5. \quad \int_0^\infty \frac{x^4 \cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b} [\sin(ab) + (ab - 2)e^{-ab}] \quad [a > 0, \quad b > 0] \quad \text{BI (175)(7)}$$

$$6. \quad \int_0^\infty \frac{x^5 \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8} [(ab - 3)e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0] \quad \text{BI (174)(7)}$$

3.737

$$\begin{aligned}
 1.^8 \quad \int_0^\infty \frac{\cos(ax) dx}{(b^2 + x^2)^n} &= \frac{\pi e^{-ab}}{(2b)^{2n-1}(n-1)!} \sum_{k=0}^{n-1} \frac{(2n-k-2)!(2ab)^k}{k!(n-k-1)!} \\
 &= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!} \left[\frac{d^{n-1}}{dp^{n-1}} \left(\frac{e^{-ab\sqrt{p}}}{\sqrt{p}} \right) \right]_{p=1} \\
 &= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!} \left[\frac{d^{n-1}}{dp^{n-1}} \left(\frac{e^{-abp}}{(1+p)^n} \right) \right]_{p=1} \\
 &\quad [a > 0, \quad b > 0] \quad \text{GW(333)(67b), WA 209, WA 192}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{x \sin(ax) dx}{(x^2 + \beta^2)^{n+1}} &= \frac{\pi a e^{-a\beta}}{2^{2n} n! \beta^{2n-1}} \sum_{k=0}^{n-1} \frac{(2n-k-2)!(2a\beta)^k}{k!(n-k-1)!} \\
 &= \frac{\pi}{2} e^{-a\beta} \quad [n = 0, \quad \beta \geq 0] \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{GW (333)(66c)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty \frac{\sin(ax) dx}{x(\beta^2 + x^2)^{n+1}} &= \frac{\pi}{2\beta^{2n+2}} \left[1 - \frac{e^{-a\beta}}{2^n n!} F_n(a\beta) \right] \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad F_0(z) = 1, \quad F_1(z) = z + 2, \dots, F_n(z) = (z + 2n)F_{n-1}(z) - zF'_{n-1}(z)] \\
 &\quad \text{GW (333)(66e)}
 \end{aligned}$$

$$4. \quad \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^3} = \frac{\pi a}{16b^3} (1 + ab) e^{-ab} \quad [a > 0, \quad b > 0] \quad \text{BI(170)(5), ET I 67(35)a}$$

$$5. \quad \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^4} = \frac{\pi a}{96b^5} (3 + 3ab + a^2b^2) e^{-ab} \quad [a > 0, \quad b > 0] \quad \text{BI(170)(6), ET I 67(35)a}$$

$$\begin{aligned}
 6. \quad \int_0^\infty \frac{x^3 \sin ax}{(x^2 + \beta^2)^{n+1}} dx &= \frac{\pi e^{-a\beta}}{2^{2n} n! \beta^{2n-2}} \left[2^{n-1} (2n-3)!! (2 - \beta a) \right. \\
 &\quad \left. - \sum_{k=1}^{n-1} \frac{(2n-k-2)! 2^k (\beta a)^{k-1}}{k!(n-k-1)!} [k(k+1) - 2(k+1)\beta a + \beta^2 a^2] \right]
 \end{aligned}$$

3.738

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{x^{m-1} \sin(ax)}{x^{2n} + \beta^{2n}} dx &= -\frac{\pi \beta^{m-2n}}{2n} \sum_{k=1}^n \exp \left[-a\beta \sin \frac{(2k-1)\pi}{2n} \right] \\
 &\quad \times \cos \left\{ \frac{(2k-1)m\pi}{2n} + a\beta \cos \frac{(2k-1)\pi}{2n} \right\} \\
 &\quad [m \text{ is even}], \quad \left[a > 0, \quad |\arg \beta| < \frac{\pi}{2n}, \quad 0 < m \leq 2n \right] \quad \text{ET I 67(38)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{x^{m-1} \cos(ax)}{x^{2n} + \beta^{2n}} dx &= \frac{\pi \beta^{m-2n}}{2n} \sum_{k=1}^n \exp \left[-a\beta \sin \frac{(2k-1)\pi}{2n} \right] \\
 &\quad \times \sin \left\{ \frac{(2k-1)m\pi}{2n} + a\beta \cos \frac{(2k-1)\pi}{2n} \right\} \\
 &\quad [m \text{ is odd}], \quad \left[a > 0, \quad |\arg \beta| < \frac{\pi}{2n}, \quad 0 < m < 2n + 1 \right] \quad \text{BI(160)(29)a, ET I 10(29)}
 \end{aligned}$$

3.739

$$1. \quad \int_0^\infty \frac{\sin(ax) dx}{x(x^2+2^2)(x^2+4^2)\dots(x^2+4n^2)} = \frac{\pi(-1)^n}{(2n)!2^{2n+1}} \left[2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} e^{2(k-n)a} + (-1)^n \binom{2n}{n} \right] \\ [a > 0, \quad n \geq 0] \quad \text{LI(174)(8)}$$

$$2. \quad \int_0^\infty \frac{\cos(ax) dx}{(x^2+1^2)(x^2+3^2)\dots[x^2+(2n+1)^2]} = \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{(2k-2n-1)a} \quad [a \geq 0, \quad n \geq 0] \\ = \frac{\pi 2^{-2n-1}}{(2n+1)(n!)^2} \quad [a = 0, \quad n \geq 0] \\ \text{BI(175)(8)}$$

$$3. \quad \int_0^\infty \frac{x \sin(ax) dx}{(x^2+1^2)(x^2+3^2)\dots[x^2+(2n+1)^2]} = \frac{\pi(-1)^n}{(2n+1)!2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n-2k+1) e^{(2k-2n-1)a} \\ [a > 0, \quad n \geq 0] \quad \text{LI (174)(9)}$$

$$4. \quad \int_0^\infty \frac{\cos ax dx}{(x^2+2^2)(x^2+4^2)\dots(x^2+4n^2)} = \frac{\pi 2^{1-2n}}{(2n)!} \sum_{k=1}^n (-1)^k k \binom{2n}{n-k} e^{-2ak} \\ [n \geq 1, \quad a \geq 0]$$

3.741

$$1. \quad \int_0^\infty \frac{\sin(ax) \sin(bx)}{x} dx = \frac{1}{4} \ln \left(\frac{a+b}{a-b} \right)^2 \quad [a > 0, \quad b > 0, \quad a \neq b] \quad \text{FI II 647}$$

$$2. \quad \int_0^\infty \frac{\sin(ax) \cos(bx)}{x} dx = \frac{\pi}{2} \quad [a > b \geq 0] \\ = \frac{\pi}{4} \quad [a = b > 0] \\ = 0 \quad [b > a \geq 0]$$

FI II 645

$$3. \quad \int_0^\infty \frac{\sin(ax) \sin(bx)}{x^2} dx = \frac{a\pi}{2} \quad [0 < a \leq b] \\ = \frac{b\pi}{2} \quad [0 < b \leq a]$$

BI (157)(1)

3.742

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin(ax) \sin(bx)}{\beta^2 + x^2} dx &= \frac{\pi}{4\beta} \left(e^{-|a-b|\beta} - e^{-(a+b)\beta} \right) & [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0] \\
 &= \frac{\pi}{2\beta} e^{-a\beta} \sinh b\beta & [\beta > 0, \quad a \geq b \geq 0] \\
 &= \frac{\pi}{2\beta} e^{-b\beta} \sinh a\beta & [\beta > 0, \quad b \geq a \geq 0]
 \end{aligned}$$

BI(162)(1)a, GW(333)(71a)

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{\sin(ax) \cos(bx)}{\beta^2 + x^2} dx &= \frac{1}{4\beta} e^{-a\beta} \{ e^{b\beta} \operatorname{Ei} [\beta(a-b)] + e^{-b\beta} \operatorname{Ei} [\beta(a+b)] \} \\
 &\quad - \frac{1}{4\beta} e^{a\beta} \{ e^{b\beta} \operatorname{Ei} [-\beta(a+b)] + e^{-b\beta} \operatorname{Ei} [\beta(b-a)] \}
 \end{aligned}$$

BI (162)(3)

$$\begin{aligned}
 3. \quad \int_0^\infty \frac{\cos(ax) \cos(bx)}{\beta^2 + x^2} dx &= \frac{\pi}{4\beta} \left[e^{-|a-b|\beta} + e^{-(a+b)\beta} \right] & [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0] \\
 &= \frac{\pi}{2\beta} e^{-a\beta} \cosh b\beta & [\beta > 0, \quad a \geq b \geq 0] \\
 &= \frac{\pi}{2\beta} e^{-b\beta} \cosh a\beta & [\beta > 0, \quad b \geq a \geq 0]
 \end{aligned}$$

BI(163)(1)a, GW(333)(71c)

$$\begin{aligned}
 4. \quad \int_0^\infty \frac{x \cos(ax) \cos(bx)}{\beta^2 + x^2} dx &= -\frac{1}{4} e^{a\beta} \{ e^{b\beta} \operatorname{Ei} [-\beta(a+b)] + e^{-b\beta} \operatorname{Ei} [\beta(b-a)] \} \\
 &\quad - \frac{1}{4} e^{-a\beta} \{ e^{b\beta} \operatorname{Ei} [\beta(a-b)] + e^{-b\beta} \operatorname{Ei} [\beta(a+b)] \} & [a \neq b] \\
 &= \infty & [a = b]
 \end{aligned}$$

BI (163)(2)

$$\begin{aligned}
 5. \quad \int_0^\infty \frac{x \sin(ax) \cos(bx)}{x^2 + \beta^2} dx &= \frac{\pi}{2} e^{-a\beta} \cosh(b\beta) & [0 < b < a] \\
 &= \frac{\pi}{4} e^{-2a\beta} & [0 < b = a] \\
 &= -\frac{\pi}{2} e^{-b\beta} \sinh(a\beta) & [0 < a < b]
 \end{aligned}$$

BI (162)(4)

$$\begin{aligned}
 6. \quad \int_0^\infty \frac{\sin(ax) \sin(bx)}{p^2 - x^2} dx &= -\frac{\pi}{2p} \cos(ap) \sin(bp) & [a > b > 0] \\
 &= -\frac{\pi}{4p} \sin(2ap) & [a = b > 0] \\
 &= -\frac{\pi}{2p} \sin(ap) \cos(bp) & [b > a > 0]
 \end{aligned}$$

BI (166)(1)

$$\begin{aligned}
 7. \quad \int_0^\infty \frac{\sin(ax) \cos(bx)}{p^2 - x^2} x dx &= -\frac{\pi}{2} \cos(ap) \cos(bp) & [a > b > 0] \\
 &= -\frac{\pi}{4} \cos(2ap) & [a = b > 0] \\
 &= \frac{\pi}{2} \sin(ap) \sin(bp) & [b > a > 0]
 \end{aligned}$$

BI (166)(2)

$$\begin{aligned}
 8. \quad \int_0^\infty \frac{\cos(ax) \cos(bx)}{p^2 - x^2} dx &= \frac{\pi}{2p} \sin(ap) \cos(bp) & [a > b > 0] \\
 &= \frac{\pi}{4p} \sin(2ap) & [a = b > 0] \\
 &= \frac{\pi}{2p} \cos(ap) \sin(bp) & [b > a > 0]
 \end{aligned}$$

BI (166)(3)

3.743

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin(ax)}{\sin(bx)} \cdot \frac{dx}{x^2 + \beta^2} &= \frac{\pi}{2\beta} \cdot \frac{\sinh(a\beta)}{\sinh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 80(21)} \\
 2. \quad \int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{x dx}{x^2 + \beta^2} &= -\frac{\pi}{2} \cdot \frac{\sinh(a\beta)}{\cosh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 81(30)} \\
 3. \quad \int_0^\infty \frac{\cos(ax)}{\sin(bx)} \cdot \frac{x dx}{x^2 + \beta^2} &= \frac{\pi}{2} \cdot \frac{\cosh(a\beta)}{\sinh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 23(37)} \\
 4. \quad \int_0^\infty \frac{\cos(ax)}{\cos(bx)} \cdot \frac{dx}{x^2 + \beta^2} &= \frac{\pi}{2\beta} \cdot \frac{\cosh(a\beta)}{\cosh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 23(36)} \\
 5.^6 \quad \text{PV} \int_0^\infty \frac{\sin(ax)}{\sin x} \cdot \frac{dx}{b^2 - x^2} &= 0 & \text{if } 0 \leq a \leq 1 \\
 &= \frac{\pi}{b} \sin(a-1)b & \text{if } 1 \leq a \leq 2 \\
 & & [b \text{ real, } b/\pi \notin \mathbb{Z}]
 \end{aligned}$$

$$\mathbf{3.744}^3 \quad \int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(x^2 + \beta^2)} = \frac{\pi}{2\beta^2} \cdot \frac{\sinh(a\beta)}{\cosh(b\beta)} \quad [0 < a < b, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 82(32)}$$

$$\mathbf{3.745}^3 \quad \int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(c^2 - x^2)} = 0 \quad [0 < a < b, \quad c > 0] \quad \text{ET I 82(31)}$$

3.746

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{dx}{x^{n+1}} \prod_{k=1}^n \sin(a_k x) &= \frac{\pi}{2} \prod_{k=1}^n a_k & \left[a_0 > \sum_{k=1}^n a_k, \quad a_k > 0 \right] & \text{FI II 646} \\
 2. \quad \int_0^\infty \frac{\sin(ax)}{x^{n+1}} dx \prod_{k=1}^n \sin(a_k x) \prod_{j=1}^m \cos(b_j x) &= \frac{\pi}{2} \prod_{k=1}^n a_k & \left[a > \sum_{k=1}^n |a_k| + \sum_{j=1}^m |b_j| \right] & \text{WH}
 \end{aligned}$$

3.747

$$1.^7 \quad \int_0^{\pi/2} \frac{x^m}{\sin x} dx = \left(\frac{\pi}{2}\right)^m \left[\frac{1}{m} + \sum_{k=1}^\infty \frac{2^{2k-1} - 1}{4^{2k-1}(m+2k)} \zeta(2k) \right] = 2\pi \mathbf{G} - \frac{7}{2} \zeta(3)$$

[m = 2] LI (206)(2)

$$2. \quad \int_0^{\pi/2} \frac{x dx}{\sin x} = \int_0^{\pi/2} \frac{(\frac{\pi}{2} - x) dx}{\cos x} = 2\mathbf{G} \quad \text{BI(204)(18), BI(206)(1), GW(333)(32)}$$

$$3. \quad \int_0^\infty \frac{x dx}{(x^2 + b^2) \sin(ax)} = \frac{\pi}{2 \sinh(ab)} \quad [b > 0] \quad \text{GW (333)(79c)}$$

$$4. \quad \int_0^\pi x \tan x dx = -\pi \ln 2 \quad \text{BI (218)(4)}$$

$$5. \quad \int_0^{\pi/2} x \tan x \, dx = \infty \quad \text{BI (205)(2)}$$

$$6. \quad \int_0^{\pi/4} x \tan x \, dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G} = 0.1857845358 \dots \quad \text{BI (204)(1)}$$

$$7. \quad \int_0^{\pi/2} x \cot x \, dx = \frac{\pi}{2} \ln 2 \quad \text{FI II 623}$$

$$8. \quad \int_0^{\pi/4} x \cot x \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G} = 0.7301810584 \dots \quad \text{BI (204)(2)}$$

$$9. \quad \int_0^{\pi/2} \left(\frac{\pi}{2} - x \right) \tan x \, dx = \frac{1}{2} \int_0^{\pi} \left(\frac{\pi}{2} - x \right) \tan x \, dx = \frac{\pi}{2} \ln 2 \quad \text{GW(333)(33b), BI(218)(12)}$$

$$10. \quad \int_0^{\infty} \tan ax \frac{dx}{x} = \frac{\pi}{2} \quad [a > 0] \quad \text{LO V 279(5)}$$

$$11. \quad \int_0^{\pi/2} \frac{x \cot x}{\cos 2x} \, dx = \frac{\pi}{4} \ln 2 \quad \text{BI (206)(12)}$$

3.748

$$1. \quad \int_0^{\pi/4} x^m \tan x \, dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^m \sum_{k=1}^{\infty} \frac{(4^k - 1) \zeta(2k)}{4^{2k-1}(m+2k)} \quad \text{LI (204)(5)}$$

$$2. \quad \int_0^{\pi/2} x^p \cot x \, dx = \left(\frac{\pi}{2} \right)^p \left(\frac{1}{p} - 2 \sum_{k=1}^{\infty} \frac{1}{4^k(p+2k)} \zeta(2k) \right) \quad \text{LI (205)(7)}$$

$$3. \quad \int_0^{\pi/4} x^m \cot x \, dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^m \left(\frac{2}{m} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(m+2k)} \right) \quad \text{LI (204)(6)}$$

3.749

$$1. \quad \int_0^{\infty} \frac{x \tan(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} + 1} \quad [a > 0, \quad b > 0] \quad \text{GW (333)(79a)}$$

$$2. \quad \int_0^{\infty} \frac{x \cot(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} - 1} \quad [a > 0, \quad b > 0] \quad \text{GW (333)(79b)}$$

$$3. \quad \int_0^{\infty} \frac{x \tan(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \cot(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \operatorname{cosec}(ax) \, dx}{b^2 - x^2} = \infty \quad \text{BI (161)(7, 8, 9)}$$

3.75 Combinations of trigonometric and algebraic functions**3.751**

$$1. \quad \int_0^{\infty} \frac{\sin(ax) \, dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}} \left[\cos(a\beta) - \sin(a\beta) + 2 C \left(\sqrt{a\beta} \right) \sin(a\beta) - 2 S \left(\sqrt{a\beta} \right) \cos(a\beta) \right] \\ [a > 0, \quad |\arg \beta| < \pi] \quad \text{ET I 65(12)a}$$

$$2.^9 \quad \int_0^{\infty} \frac{\cos(ax) \, dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}} \left[\cos(a\beta) + \sin(a\beta) - 2 C \left(\sqrt{a\beta} \right) \cos(a\beta) - 2 S \left(\sqrt{a\beta} \right) \sin(a\beta) \right] \\ [a > 0, \quad |\arg \beta| < \pi] \quad \text{ET I 8(9)a}$$

$$3. \quad \int_u^\infty \frac{\sin(ax)}{\sqrt{x-u}} dx = \sqrt{\frac{\pi}{2a}} [\sin(au) + \cos(au)] \quad [a > 0, \quad u > 0] \quad \text{ET I 65(13)}$$

$$4. \quad \int_u^\infty \frac{\cos(ax)}{\sqrt{x-u}} dx = \sqrt{\frac{\pi}{2a}} [\cos(au) - \sin(au)] \quad [a > 0, \quad u > 0] \quad \text{ET I 8(10)}$$

3.752

$$1.^8 \quad \int_0^1 \sin(ax) \sqrt{1-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k-1)!!(2k+3)!!} = \frac{\pi}{2a} \mathbf{H}_1(a) \quad [a > 0] \quad \text{BI (149)(6)}$$

$$2. \quad \int_0^1 \cos(ax) \sqrt{1-x^2} dx = \frac{\pi}{2a} J_1(a) \quad \text{KU 65(6)a}$$

3.753

$$1.^8 \quad \int_0^1 \frac{\sin(ax) dx}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{[(2k+1)!!]^2} = \frac{\pi}{2} \mathbf{H}_0(a) \quad [a > 0] \quad \text{BI (149)(9)}$$

$$2. \quad \int_0^1 \frac{\cos(ax) dx}{\sqrt{1-x^2}} = \frac{\pi}{2} J_0(a) \quad \text{WA 30(7)a}$$

$$3. \quad \int_1^\infty \frac{\sin(ax) dx}{\sqrt{x^2-1}} = \frac{\pi}{2} J_0(a) \quad [a > 0] \quad \text{WA 200(14)}$$

$$4. \quad \int_1^\infty \frac{\cos(ax) dx}{\sqrt{x^2-1}} = -\frac{\pi}{2} Y_0(a) \quad \text{WA 200(15)}$$

$$5. \quad \int_0^1 \frac{x \sin(ax)}{\sqrt{1-x^2}} dx = \frac{\pi}{2} J_1(a) \quad [a > 0] \quad \text{WA 30(6)}$$

3.754

$$1. \quad \int_0^\infty \frac{\sin(ax) dx}{\sqrt{\beta^2 + x^2}} = \frac{\pi}{2} [I_0(a\beta) - \mathbf{L}_0(a\beta)] \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 66(26)}$$

$$2. \quad \int_0^\infty \frac{\cos(ax) dx}{\sqrt{\beta^2 + x^2}} = K_0(a\beta) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{WA 191(1), GW(333)(78a)}$$

$$3. \quad \int_0^\infty \frac{x \sin(ax)}{\sqrt{(\beta^2 + x^2)^3}} dx = a K_0(a\beta) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 66(27)}$$

3.755

$$1. \quad \int_0^\infty \frac{\sqrt{\sqrt{x^2 + \beta^2} - \beta} \sin(ax) dx}{\sqrt{x^2 + \beta^2}} = \sqrt{\frac{\pi}{2a}} e^{-a\beta} \quad [a > 0] \quad \text{ET I 66(31)}$$

$$2. \quad \int_0^\infty \frac{\sqrt{\sqrt{x^2 + \beta^2} + \beta} \cos(ax) dx}{\sqrt{x^2 + \beta^2}} = \sqrt{\frac{\pi}{2a}} e^{-a\beta} \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 10(25)}$$

3.756

$$1. \quad \int_0^\infty \frac{\sin(ax)}{x^{\frac{n}{2}-1}} \prod_{k=2}^n \sin(a_k x) dx = 0 \quad \left[a_k > 0, \quad a > \sum_{k=2}^n a_k \right] \quad \text{ET I 80(22)}$$

$$2. \quad \int_0^\infty x^{\frac{n}{2}-1} \cos(ax) \prod_{k=1}^n \cos(a_k x) dx = 0 \quad \left[a_k > 0, \quad a > \sum_{k=1}^n a_k \right] \quad \text{ET I 22(26)}$$

3.757

$$1.^{11} \quad \int_0^\infty \frac{\sin(ax)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2a}} \quad [a > 0] \quad \text{BI (177)(1)}$$

$$2.^{11} \quad \int_0^\infty \frac{\cos(ax)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2a}} \quad [a > 0] \quad \text{BI (177)(2)}$$

3.76–3.77 Combinations of trigonometric functions and powers

3.761

$$1. \quad \int_0^1 x^{\mu-1} \sin(ax) dx = \frac{-i}{2\mu} [{}_1F_1(\mu; \mu+1; ia) - {}_1F_1(\mu; \mu+1; -ia)] \quad [a > 0, \quad \operatorname{Re} \mu > -1, \quad \mu \neq 0] \quad \text{ET I 68(2)a}$$

$$2.^8 \quad \int_u^\infty x^{\mu-1} \sin x dx = \frac{i}{2} [e^{-\frac{\pi}{2}i\mu} \Gamma(\mu, iu) - e^{\frac{\pi}{2}i\mu} \Gamma(\mu, -iu)] \quad [\operatorname{Re} \mu < 1] \quad \text{EH II 149(2)}$$

$$3. \quad \int_1^\infty \frac{\sin(ax)}{x^{2n}} dx = \frac{a^{2n-1}}{(2n-1)!} \left[\sum_{k=1}^{2n-1} \frac{(2n-k-1)!}{a^{2n-k}} \sin\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^n \operatorname{ci}(a) \right] \quad [a > 0] \quad \text{LI (203)(15)}$$

$$4. \quad \int_0^\infty x^{\mu-1} \sin(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2} = \frac{\pi \sec \frac{\mu\pi}{2}}{2a^\mu \Gamma(1-\mu)} \quad [a > 0; \quad 0 < |\operatorname{Re} \mu| < 1] \quad \text{FI II 809a, BI(150)(1)}$$

$$5.^{10} \quad \int_0^\pi x^m \sin(nx) dx = \frac{(-1)^{n+1}}{n^{m+1}} \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m!}{(m-2k)!} (n\pi)^{m-2k} - (-1)^{\lfloor m/2 \rfloor} \frac{m! \lfloor m-2\lfloor \frac{m}{2} \rfloor - 1 \rfloor}{n^{m+1}} \quad \text{GW(333)(6)}$$

$$6.^8 \quad \int_0^1 x^{\mu-1} \cos(ax) dx = \frac{1}{2\mu} [{}_1F_1(\mu; \mu+1; ia) + {}_1F_1(\mu; \mu+1; -ia)] \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 11(2)}$$

$$7. \quad \int_u^\infty x^{\mu-1} \cos x dx = \frac{1}{2} [e^{-\frac{\pi}{2}i\mu} \Gamma(\mu, iu)s + e^{\frac{\pi}{2}i\mu} \Gamma(\mu, -iu)] \quad [\operatorname{Re} \mu < 1] \quad \text{EH II 149(1)}$$

$$8. \quad \int_1^\infty \frac{\cos(ax)}{x^{2n+1}} dx = \frac{a^{2n}}{(2n)!} \left[\sum_{k=1}^{2n} \frac{(2n-k)!}{a^{2n-k+1}} \cos\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^{n+1} \operatorname{ci}(a) \right] \\ [a > 0] \quad \text{LI (203)(16)}$$

$$9.^8 \quad \int_0^\infty x^{\mu-1} \cos(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \cos \frac{\mu\pi}{2} = \frac{\pi \operatorname{cosec} \frac{\mu\pi}{2}}{2a^\mu \Gamma(1-\mu)} \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \\ \text{FI II 809a, BI(150)(2)}$$

$$10. \quad \int_0^\pi x^m \cos(nx) dx = \frac{(-1)^n}{n^{m+1}} \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} (-1)^k \frac{m!}{(m-2k-1)!} (n\pi)^{m-2k-1} \\ + (-1)^{\lfloor (m+1)/2 \rfloor} \frac{2[(m+1)/2] - m}{n^{m+1}} \cdot m! \\ \text{GW (333)(7)}$$

$$11. \quad \int_0^{\pi/2} x^m \cos x dx = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m!}{(m-2k)!} \left(\frac{\pi}{2}\right)^{m-2k} + (-1)^{\lfloor m/2 \rfloor} \left(2 \left\lfloor \frac{m}{2} \right\rfloor - m\right) m! \\ \text{GW (333)(9c)}$$

$$12. \quad \int_0^{2n\pi} x^m \cos kx dx = - \sum_{j=0}^{m-1} \frac{j!}{k^{j+1}} \binom{m}{j} (2n\pi)^{m-j} \cos \frac{j+1}{2} \pi \\ \text{BI (226)(2)}$$

3.762

$$1. \quad \int_0^\infty x^{\mu-1} \sin(ax) \sin(bx) dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) \left[|b-a|^{-\mu} - (b+a)^{-\mu} \right] \\ [a > 0, \quad b > 0, \quad a \neq b, \quad -2 < \operatorname{Re} \mu < 1] \\ (\text{for } \mu = 0, \text{ see } \mathbf{3.741} \text{ 1, for } \mu = -1, \text{ see } \mathbf{3.741} \text{ 3}) \\ \text{BI(149)(7), ET I 321(40)}$$

$$2. \quad \int_0^\infty x^{\mu-1} \sin(ax) \cos(bx) dx = \frac{1}{2} \sin \frac{\mu\pi}{2} \Gamma(\mu) \left[(a+b)^{-\mu} + |a-b|^{-\mu} \operatorname{sign}(a-b) \right] \\ [a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1] \quad (\text{for } \mu = 0 \text{ see } \mathbf{3.741} \text{ 2}) \quad \text{BI(159)(8)a, ET I 321(41)}$$

$$3. \quad \int_0^\infty x^{\mu-1} \cos(ax) \cos(bx) dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) \left[(a+b)^{-\mu} + |a-b|^{-\mu} \right] \\ [a > 0, \quad b > 0, \quad 0 < \operatorname{Re} \mu < 1] \\ \text{ET I 20(17)}$$

3.763

$$1. \quad \int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x^\nu} dx = \frac{1}{4} \cos \frac{\nu\pi}{2} \Gamma(1-\nu) \left\{ (c+a-b)^{\nu-1} - (c+a+b)^{\nu-1} \right. \\ \left. - |c-a+b|^{\nu-1} \operatorname{sign}(a-b-c) + |c-a-b|^{\nu-1} \operatorname{sign}(a+b-c) \right\} \\ [c > 0, \quad 0 < \operatorname{Re} \nu < 4, \quad \nu \neq 1, 2, 3, \quad a \geq b > 0] \quad \text{GW(333)(26a)a, ET I 79(13)}$$

$$\begin{aligned}
2. \quad \int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x} dx &= 0 & [c < a - b \text{ and } c > a + b] \\
&= \frac{\pi}{8} & [c = a - b \text{ and } c = a + b] \\
&= \frac{\pi}{4} & [a - b < c < a + b] \\
& & [a \geq b > 0, \quad c > 0] \quad \text{FI II 645}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x^2} dx &= \frac{1}{4}(c + a + b) \ln(c + a + b) \\
&\quad - \frac{1}{4}(c + a - b) \ln(c + a - b) - \frac{1}{4}|c - a - b| \ln|c - a - b| \\
&\quad \times \operatorname{sign}(a + b - c) + \frac{1}{4}|c - a + b| \ln|c - a + b| \operatorname{sign}(a - b - c) \\
& \quad [a \geq b > 0, \quad c > 0] \quad \text{BI(157)(8)a, ET I 79(11)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x^3} dx &= \frac{\pi bc}{2} & [0 < c < a - b \text{ and } c > a + b] \\
&= \frac{\pi bc}{2} - \frac{\pi(a - b - c)^2}{8} & [a - b < c < a + b] \\
& \quad [a \geq b > 0, \quad c > 0] \quad \text{BI(157)(20), ET I 79(12)}
\end{aligned}$$

3.764

$$\begin{aligned}
1. \quad \int_0^\infty x^p \sin(ax + b) dx &= \frac{1}{a^{p+1}} \Gamma(1 + p) \cos\left(b + \frac{p\pi}{2}\right) & [a > 0, \quad -1 < p < 0] \quad \text{GW (333)(30a)} \\
2. \quad \int_0^\infty x^p \cos(ax + b) dx &= -\frac{1}{a^{p+1}} \Gamma(1 + p) \sin\left(b + \frac{p\pi}{2}\right) \\
& \quad [a > 0, \quad -1 < p < 0] \quad \text{GW (333)(30b)}
\end{aligned}$$

3.765

$$\begin{aligned}
1.^{10} \quad \int_0^\infty \frac{\sin ax}{x^\nu(x + b)} dx &= a^{1+\nu} b \cos \frac{\pi\nu}{2} \Gamma(-1 - \nu) {}_1F_2\left(1; 1 + \frac{\nu}{2}, \frac{3}{2} + \frac{\nu}{2}; -\frac{1}{4}a^2b^2\right) \operatorname{sign}(a) \\
&\quad - \frac{\pi \operatorname{cosec}(\pi\nu) \sin(ab)}{b^\nu} - a^\nu \Gamma(-\nu) {}_1F_2\left(1; 1 + \frac{\nu}{2}, 1 + \frac{\nu}{2}; -\frac{1}{4}a^2b^2\right) \operatorname{sign}(a) \sin \frac{\pi\nu}{2} \\
& \quad [\operatorname{Im} a = 0, \quad -1 < \operatorname{Re} b < 2, \quad \arg b \neq \pi] \quad \text{MC} \\
2. \quad \int_0^\infty \frac{\cos(ax)}{x^\nu(x + \beta)} dx &= \frac{\Gamma(1 - \nu)}{2\beta^\nu} [e^{ia\beta} \Gamma(\nu, ia\beta) + e^{-ia\beta} \Gamma(\nu, -ia\beta)] \\
& \quad [a > 0, \quad |\operatorname{Re} \nu| < 1, \quad |\arg \beta| < \pi] \\
& \quad \text{ET II 221(52)}
\end{aligned}$$

3.766

$$\begin{aligned}
1.^{10} \quad \int_0^\infty \frac{x^{\mu-1} \sin ax}{1 + x^2} dx &= -a^{2-\mu} \Gamma(\mu - 2) {}_1F_2\left(1; \frac{3-\mu}{2}, \frac{4-\mu}{2}; \frac{a^2}{4}\right) \operatorname{sign}(a) \sin \frac{\pi\mu}{2} + \frac{\pi}{2} \sec \frac{\pi\mu}{2} \sinh(a) \\
& \quad [\operatorname{Im} a = 0, \quad -1 < \operatorname{Re} \mu < 3] \quad \text{MC}
\end{aligned}$$

$$2. \quad \int_0^\infty \frac{x^{\mu-1} \cos(ax)}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \cosh a + \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) \{ \exp[-a + i\pi(1-\mu)] \gamma(1-\mu, -a) - e^a \gamma(1-\mu, a) \} \\ [a > 0, \quad 0 < \operatorname{Re} \mu < 3] \quad \text{ET I 319(24)}$$

$$3.^9 \quad \int_0^\infty \frac{x^{2\mu+1} \sin(ax) dx}{x^2 + b^2} = -\frac{\pi}{2} b^{2\mu} \sec(\mu\pi) \sinh(ab) + \frac{\sin(\mu\pi)}{2a^{2\mu}} \Gamma(2\mu) [{}_1F_1(1; 1-2\mu; ab) + {}_1F_1(1; 1-2\mu; -ab)] \\ [a > 0, \quad -\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 220(39)}$$

$$4.^9 \quad \int_0^\infty \frac{x^{2\mu+1} \cos(ax) dx}{x^2 + b^2} = -\frac{\pi}{2} b^{2(\mu+\frac{1}{2})} \operatorname{cosec} \left[\left(\mu + \frac{1}{2} \right) \pi \right] \cosh(ab) + \frac{\cos \left[\left(\mu + \frac{1}{2} \right) \pi \right]}{2a^{2(\mu+\frac{1}{2})}} \Gamma \left[2 \left(\mu + \frac{1}{2} \right) \right] \left\{ {}_1F_1 \left(1; 1-2 \left(\mu + \frac{1}{2} \right); ab \right) + {}_1F_1 \left(1; 1-2 \left(\mu + \frac{1}{2} \right); -ab \right) \right\} \\ [a > 0, \quad -1 < \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 221(56)}$$

3.767

$$1. \quad \int_0^\infty \frac{x^{\beta-1} \sin \left(ax - \frac{\beta\pi}{2} \right)}{\gamma^2 + x^2} dx = -\frac{\pi}{2} \gamma^{\beta-2} e^{-a\gamma} \quad [a > 0, \quad \operatorname{Re} \gamma > 0, \quad 0 < \operatorname{Re} \beta < 2] \\ \text{BI (160)(20)}$$

$$2. \quad \int_0^\infty \frac{x^\beta \cos \left(ax - \frac{\beta\pi}{2} \right)}{\gamma^2 + x^2} dx = \frac{\pi}{2} \gamma^{\beta-1} e^{-a\gamma} \quad [a > 0, \quad \operatorname{Re} \gamma > 0, \quad |\operatorname{Re} \beta| < 1] \\ \text{BI (160)(21)}$$

$$3. \quad \int_0^\infty \frac{x^{\beta-1} \sin \left(ax - \frac{\beta\pi}{2} \right)}{x^2 - b^2} dx = \frac{\pi}{2} b^{\beta-2} \cos \left(ab - \frac{\pi\beta}{2} \right) \quad [a > 0, \quad b > 0, \quad 0 < \operatorname{Re} \beta < 2] \\ \text{BI (161)(11)}$$

$$4. \quad \int_0^\infty \frac{x^\beta \cos \left(ax - \frac{\beta\pi}{2} \right)}{x^2 - b^2} dx = -\frac{\pi}{2} b^{\beta-1} \sin \left(ab - \frac{\pi\beta}{2} \right) \quad [a > 0, \quad b > 0, \quad |\beta| < 1] \\ \text{GW (333)(82)}$$

3.768

$$1. \quad \int_u^\infty (x-u)^{\mu-1} \sin(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \sin \left(au + \frac{\mu\pi}{2} \right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET II 203(19)}$$

$$2. \quad \int_u^\infty (x-u)^{\mu-1} \cos(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \cos \left(au + \frac{\mu\pi}{2} \right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET II 204(24)}$$

$$3.^{11} \quad \int_0^1 (1-x)^\nu \sin(ax) dx = \frac{1}{a} - \frac{\Gamma(\nu+1)}{a^{\nu+1}} C_\nu(a) = a^{-\nu-1/2} s_{\nu+1/2, 1/2}(a) \\ [a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 11(3)a}$$

Here $C_\nu(a)$ is the Young's function given by:

$$C_\nu(a) = \frac{\frac{1}{2}a^\nu}{\Gamma(\nu+1)} [{}_1F_1(1; \nu+1; ia) + {}_1F_1(1; \nu+1; -ia)] = \sum_{n=0}^{\infty} \frac{(-1)^n a^{\nu+2n}}{\Gamma(\nu+2n+1)}.$$

$$\begin{aligned} 4.^3 \quad \int_0^1 (1-x)^\nu \cos(ax) dx &= \frac{i}{2} a^{-\nu-1} \left\{ \exp \left[\frac{i}{2} (\nu\pi - 2a) \right] \gamma(\nu+1, -ia) \right. \\ &\quad \left. - \exp \left[-\frac{i}{2} (\nu\pi - 2a) \right] \gamma(\nu+1, ia) \right\} \\ &= \Gamma(\nu+1) \sum_{n=0}^{\infty} \frac{(-a^2)^n}{\Gamma(\nu+2+2n)} \\ &\quad [a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 11(3)a} \end{aligned}$$

$$\begin{aligned} 5. \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} \sin(ax) dx &= \frac{u^{\mu+\nu-1}}{2i} B(\mu, \nu) [{}_1F_1(\nu; \mu+\nu; iau) - {}_1F_1(\nu; \mu+\nu; -iau)] \\ &\quad [a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1, \quad \nu \neq 0] \quad \text{ET II 189(26)} \end{aligned}$$

$$\begin{aligned} 6. \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} \cos(ax) dx &= \frac{u^{\mu+\nu-1}}{2} B(\mu, \nu) [{}_1F_1(\nu; \mu+\nu; iau) + {}_1F_1(\nu; \mu+\nu; -iau)] \\ &\quad [a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \\ &\quad \text{ET II 189(32)} \end{aligned}$$

$$\begin{aligned} 7. \quad \int_0^u x^{\mu-1} (u-x)^{\mu-1} \sin(ax) dx &= \sqrt{\pi} \left(\frac{u}{a} \right)^{\mu-1/2} \sin \frac{au}{2} \Gamma(\mu) J_{\mu-1/2} \left(\frac{au}{2} \right) \\ &\quad [\operatorname{Re} \mu > 0] \quad \text{ET II 189(25)} \end{aligned}$$

$$\begin{aligned} 8. \quad \int_u^\infty x^{\mu-1} (x-u)^{\mu-1} \sin(ax) dx \\ &= \frac{\sqrt{\pi}}{2} \left(\frac{u}{a} \right)^{\mu-1/2} \Gamma(\mu) \left[\cos \frac{au}{2} J_{1/2-\mu} \left(\frac{au}{2} \right) - \sin \frac{au}{2} Y_{1/2-\mu} \left(\frac{au}{2} \right) \right] \\ &\quad [a > 0, \quad 0 < \operatorname{Re} \mu < \tfrac{1}{2}] \quad \text{ET II 203(20)} \end{aligned}$$

$$\begin{aligned} 9. \quad \int_0^u x^{\mu-1} (u-x)^{\mu-1} \cos(ax) dx &= \sqrt{\pi} \left(\frac{u}{a} \right)^{\mu-1/2} \cos \frac{au}{2} \Gamma(\mu) J_{\mu-1/2} \left(\frac{au}{2} \right) \\ &\quad [\operatorname{Re} \mu > 0] \quad \text{ET II 189(31)} \end{aligned}$$

$$\begin{aligned} 10. \quad \int_u^\infty x^{\mu-1} (x-u)^{\mu-1} \cos(ax) dx &= -\frac{\sqrt{\pi}}{2} \left(\frac{u}{a} \right)^{\mu-1/2} \Gamma(\mu) \left[\sin \frac{au}{2} J_{1/2-\mu} \left(\frac{au}{2} \right) - \cos \frac{au}{2} Y_{1/2-\mu} \left(\frac{au}{2} \right) \right] \\ &\quad [a > 0, \quad 0 < \operatorname{Re} \mu < \tfrac{1}{2}] \quad \text{ET II 204(25)} \end{aligned}$$

$$\begin{aligned} 11.^3 \quad \int_0^1 x^{\nu-1} (1-x)^{\mu-1} \sin(ax) dx &= -\frac{i}{2} B(\mu, \nu) [{}_1F_1(\nu; \nu+\mu; ia) - {}_1F_1(\nu; \nu+\mu; -ia)] \\ &\quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1, \quad \nu \neq 0] \\ &\quad \text{ET I 68 (5)a, ET I 317(5)} \end{aligned}$$

$$\begin{aligned} 12.^3 \quad \int_0^1 x^{\nu-1} (1-x)^{\mu-1} \cos(ax) dx &= \frac{1}{2} B(\mu, \nu) [{}_1F_1(\nu; \nu+\mu; ia) + {}_1F_1(\nu; \nu+\mu; -ia)] \\ &\quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 11(5)} \end{aligned}$$

$$13. \quad \int_0^1 x^\mu (1-x)^\mu \sin(2ax) dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a) \sin a$$

$$[a > 0, \quad \operatorname{Re} \mu > -1] \quad \text{ET I 68(4)}$$

$$14. \quad \int_0^1 x^\mu (1-x)^\mu \cos(2ax) dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a) \cos a$$

$$[a > 0, \quad \operatorname{Re} \mu > -1] \quad \text{ET I 11(4)}$$

3.769

$$1. \quad \int_0^\infty [(\beta + ix)^{-\nu} - (\beta - ix)^{-\nu}] \sin(ax) dx = -\frac{\pi i a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)}$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 70(15)}$$

$$2. \quad \int_0^\infty [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \cos(ax) dx = \frac{\pi a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)}$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 13(19)}$$

$$3. \quad \int_0^\infty x [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \sin(ax) dx = -\frac{\pi a^{\nu-2} (\nu - 1 - a\beta) e^{-a\beta}}{\Gamma(\nu)}$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 70(16)}$$

$$4. \quad \int_0^\infty x^{2n} [(\beta - ix)^{-\nu} - (\beta + ix)^{-\nu}] \sin(ax) dx = \frac{(-1)^n i}{\Gamma(\nu)} (2n)! \pi a^{\nu-2n-1} e^{-a\beta} L_{2n}^{\nu-2n-1}(a\beta)$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq 2n < \operatorname{Re} \nu] \quad \text{ET I 70(17)}$$

$$5. \quad \int_0^\infty x^{2n} [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \cos(ax) dx = \frac{(-1)^n}{\Gamma(\nu)} (2n)! \pi a^{\nu-2n-1} e^{-a\beta} L_{2n}^{\nu-2n-1}(a\beta)$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq 2n < \operatorname{Re} \nu] \quad \text{ET I 13(20)}$$

$$6. \quad \int_0^\infty x^{2n+1} [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \sin(ax) dx = \frac{(-1)^{n+1}}{\Gamma(\nu)} (2n+1)! \pi a^{\nu-2n-2} e^{-a\beta} L_{2n+1}^{\nu-2n-2}(a\beta)$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 \leq 2n+1 < \operatorname{Re} \nu] \quad \text{ET I 70(18)}$$

$$7. \quad \int_0^\infty x^{2n+1} [(\beta + ix)^{-\nu} - (\beta - ix)^{-\nu}] \cos(ax) dx = \frac{(-1)^{n+1}}{\Gamma(\nu)} (2n+1)! \pi a^{\nu-2n-2} e^{-a\beta} L_{2n+1}^{\nu-2n-2}(a\beta)$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq 2n < \operatorname{Re} \nu - 1] \quad \text{ET I 13(21)}$$

3.771

$$1. \quad \int_0^\infty (\beta^2 + x^2)^{\nu-\frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a} \right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) [I_{-\nu}(a\beta) - \mathbf{L}_\nu(a\beta)]$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2}, \quad \nu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots] \quad \text{EH II 38a, ET I 68(6)}$$

$$2. \quad \int_0^\infty (\beta^2 + x^2)^{\nu-\frac{1}{2}} \cos(ax) dx = \frac{1}{\sqrt{\pi}} \left(\frac{2\beta}{a} \right)^\nu \cos(\pi\nu) \Gamma\left(\nu + \frac{1}{2}\right) K_{-\nu}(a\beta) \\ \left[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2} \right] \\ \text{WA 191(1)a, GW(333)(78)a}$$

$$3. \quad \int_0^u x^{2\nu-1} (u^2 - x^2)^{\mu-1} \sin(ax) dx \\ = \frac{a}{2} u^{2\mu+2\nu-1} \operatorname{B}\left(\mu, \nu + \frac{1}{2}\right) {}_1F_2\left(\nu + \frac{1}{2}; \frac{3}{2}, \mu + \nu + \frac{1}{2}; -\frac{a^2 u^2}{4}\right) \\ \left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 189(29)}$$

$$4. \quad \int_0^u x^{2\nu-1} (u^2 - x^2)^{\mu-1} \cos(ax) dx = \frac{1}{2} u^{2\mu+2\nu-2} \operatorname{B}(\mu, \nu) {}_1F_2\left(\nu; \frac{1}{2}, \mu + \nu; -\frac{a^2 u^2}{4}\right) \\ \left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right] \quad \text{ET II 190(35)}$$

$$5.7 \quad \int_0^\infty x (x^2 + \beta^2)^{\nu-\frac{1}{2}} \sin(ax) dx = \frac{1}{\sqrt{\pi}} \beta \left(\frac{2\beta}{a} \right)^\nu \cos \nu \pi \Gamma\left(\nu + \frac{1}{2}\right) K_{\nu+1}(a\beta) \\ = \sqrt{\pi} \beta \left(\frac{2\beta}{a} \right)^\nu \frac{1}{\Gamma\left(\frac{1}{2} - \nu\right)} K_{\nu+1}(a\beta) \\ \left[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < 0 \right] \quad \text{ET I 69(11)}$$

$$6. \quad \int_0^u (u^2 - x^2)^{\nu-\frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{a} \right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_\nu(au) \\ \left[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \\ \text{ET I 69(7), WA 358(1)a}$$

$$7. \quad \int_u^\infty (x^2 - u^2)^{\nu-\frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{a} \right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_{-\nu}(au) \\ \left[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right] \\ \text{EH II 81(12)a, ET I 69(8), WA 187(3)a}$$

$$8. \quad \int_0^u (u^2 - x^2)^{\nu-\frac{1}{2}} \cos(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{a} \right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(au) \\ \left[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \\ \text{ET I 11(8)}$$

$$9. \quad \int_u^\infty (x^2 - u^2)^{\nu-\frac{1}{2}} \cos(ax) dx = -\frac{\sqrt{\pi}}{2} \left(\frac{2u}{a} \right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) Y_{-\nu}(au) \\ \left[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right] \\ \text{WA 187(4)a, EH II 82(13)a, ET I 11(9)}$$

$$10. \quad \int_0^u x (u^2 - x^2)^{\nu-\frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} u \left(\frac{2u}{a} \right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu+1}(au) \\ \left[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \\ \text{ET I 69(9)}$$

$$11. \quad \int_u^\infty x (x^2 - u^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} u \left(\frac{2u}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) Y_{-\nu-1}(au) \\ [a > 0, \quad u > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 0] \quad \text{ET I 69(10)}$$

$$12.^7 \quad \int_0^u x (u^2 - x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = -\frac{u^{\nu+1}}{a^\nu} s_{(\nu-1)\nu+1}(au) \\ = \frac{1}{2} \left(\nu + \frac{1}{2} \right)^{-1} u^{2\nu+1} - \frac{\sqrt{\pi}}{2} u \left(\frac{2u}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) \mathbf{H}_{\nu+1}(au) \\ [a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 12(10)}$$

$$13. \quad \int_u^\infty x (x^2 - u^2)^{\nu-1/2} \cos(ax) dx \frac{\sqrt{\pi} u}{2} \left(\frac{2u}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) J_{-\nu-1}(au) \\ [a > 0, \quad u > 0, \quad 0 < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET I 12(11)}$$

3.772

$$1. \quad \int_0^\infty (x^2 + 2\beta x)^{\nu-1/2} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) [J_{-\nu}(a\beta) \cos(a\beta) + Y_{-\nu}(a\beta) \sin(a\beta)] \\ [a > 0, \quad |\arg \beta| < \pi, \quad \frac{1}{2} > \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET I 69(12)}$$

$$2. \quad \int_0^\infty (x^2 + 2\beta x)^{\nu-1/2} \cos(ax) dx \\ = -\frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) [Y_{-\nu}(a\beta) \cos(a\beta) - J_{-\nu}(a\beta) \sin(a\beta)] \\ [a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET I 12(13)}$$

$$3. \quad \int_0^{2u} (2ux - x^2)^{\nu-1/2} \sin(ax) dx = \sqrt{\pi} \left(\frac{2u}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) \sin(au) J_\nu(au) \\ [a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 69(13)a}$$

$$4. \quad \int_{2u}^\infty (x^2 - 2ux)^{\nu-1/2} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) [J_{-\nu}(au) \cos(au) - Y_{-\nu}(au) \sin(au)] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET I 70(14)}$$

$$5. \quad \int_0^{2u} (2ux - x^2)^{\nu-1/2} \cos(ax) dx = \sqrt{\pi} \left(\frac{2u}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) J_\nu(au) \cos(au) \\ [a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 12(4)}$$

$$6. \quad \int_{2u}^\infty (x^2 - 2ux)^{\nu-1/2} \cos(ax) dx \\ = -\frac{\sqrt{\pi}}{2} \left(\frac{2u}{a} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) [J_{-\nu}(au) \sin(au) + Y_{-\nu}(au) \cos(au)] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET I 12(12)}$$

3.773

$$\begin{aligned}
1.^8 \quad \int_0^\infty \frac{x^{2\nu}}{(x^2 + \beta^2)^{\mu+1}} \sin(ax) dx &= \frac{1}{2} \beta^{2\nu-2\mu} a \operatorname{B}\left(1 + \nu, \mu - \nu\right) {}_1F_2\left(\nu + 1; \nu + 1 - \mu, \frac{3}{2}; \frac{\beta^2 a^2}{4}\right) \\
&+ \frac{\sqrt{\pi} a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}} \frac{\Gamma(\nu - \mu)}{\Gamma\left(\mu - \nu + \frac{3}{2}\right)} {}_1F_2\left(\mu + 1; \mu - \nu + \frac{3}{2}, \mu - \nu + 1; \frac{\beta^2 a^2}{4}\right) \\
&= \frac{\sqrt{\pi}}{2\Gamma(\mu + 1)} \beta^{2\nu-2\mu-1} G_{13}^{21}\left(\frac{a^2 \beta^2}{4} \left| \begin{matrix} -\nu + \frac{1}{2} \\ \mu - \nu + \frac{1}{2}, \frac{1}{2}, 0 \end{matrix} \right. \right) \\
&\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} \mu + 1] \quad \text{ET I 71(28)a, ET II 234(17)}
\end{aligned}$$

$$\begin{aligned}
2.^8 \quad \int_0^\infty \frac{x^{2m+1} \sin(ax)}{(z + x^2)^{n+1}} dx &= \frac{(-1)^{n+m}}{n!} \cdot \frac{\pi}{2} \frac{d^n}{dz^n} \left(z^m e^{-a\sqrt{z}} \right) \\
&\quad [a > 0, \quad 0 \leq m \leq n, \quad |\arg z| < \pi] \\
&\quad \text{ET I 68(39)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty \frac{x^{2m+1} \sin(ax) dx}{(\beta^2 + x^2)^{n+\frac{1}{2}}} &= \frac{(-1)^{m+1} \sqrt{\pi}}{2^n \beta^n \Gamma\left(n + \frac{1}{2}\right)} \frac{d^{2m+1}}{da^{2m+1}} [a^n K_n(a\beta)] \\
&\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 \leq m \leq n] \\
&\quad \text{ET I 67(37)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty \frac{x^{2\nu} \cos(ax) dx}{(x^2 + \beta^2)^{\mu+1}} &= \frac{1}{2} \beta^{2\nu-2\mu-1} \operatorname{B}\left(\nu + \frac{1}{2}, \mu - \nu + \frac{1}{2}\right) {}_1F_2\left(\nu + \frac{1}{2}; \nu - \mu + \frac{1}{2}, \frac{1}{2}; \frac{\beta^2 a^2}{4}\right) \\
&+ \frac{\sqrt{\pi} a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}} \frac{\Gamma\left(\nu - \mu - \frac{1}{2}\right)}{\Gamma(\mu - \nu + 1)} {}_1F_2\left(\mu + 1; \mu - \nu + 1, \mu - \nu + \frac{3}{2}; \frac{\beta^2 a^2}{4}\right) \\
&= \frac{\sqrt{\pi}}{2\Gamma(\mu + 1)} \beta^{2\nu-2\mu-1} G_{13}^{21}\left(\frac{a^2 \beta^2}{4} \left| \begin{matrix} -\nu + \frac{1}{2} \\ \mu - \nu + \frac{1}{2}, 0, \frac{1}{2} \end{matrix} \right. \right) \\
&\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \operatorname{Re} \mu + 1] \quad \text{ET I 14(29)a, ET II 235(19)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty \frac{x^{2m} \cos(ax) dx}{(z + x^2)^{n+1}} &= (-1)^{m+n} \frac{\pi}{2 \cdot n!} \cdot \frac{d^n}{dz^n} \left(z^{m-\frac{1}{2}} e^{-a\sqrt{z}} \right) \\
&\quad [a > 0, \quad n + 1 > m \geq 0, \quad |\arg z| < \pi] \\
&\quad \text{ET I 10(28)}
\end{aligned}$$

$$\begin{aligned}
6.^7 \quad \int_0^\infty \frac{x^{2m} \cos(ax) dx}{(\beta^2 + x^2)^{n+\frac{1}{2}}} &= \frac{(-1)^m \sqrt{\pi}}{2^n \beta^n \Gamma\left(n + \frac{1}{2}\right)} \cdot \frac{d^{2m}}{da^{2m}} \{a^n K_n(a\beta)\} \\
&\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq m < n + \frac{1}{2}] \\
&\quad \text{ET I 14(28)}
\end{aligned}$$

3.774

$$\begin{aligned}
1. \quad \int_0^\infty \frac{\sin(ax) dx}{\sqrt{x^2 + b^2} (x + \sqrt{x^2 + b^2})^\nu} &= \frac{\pi}{b^\nu \sin(\nu\pi)} \left[\sin \frac{\nu\pi}{2} I_\nu(ab) + \frac{i}{2} \mathbf{J}_\nu(iab) - \frac{i}{2} \mathbf{J}_\nu(-iab) \right] \\
&\quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \\
&\quad \text{ET I 70(19)}
\end{aligned}$$

$$2. \quad \int_0^\infty \frac{\cos(ax) dx}{\sqrt{x^2 + b^2} (x + \sqrt{x^2 + b^2})^\nu} = \frac{\pi}{b^\nu \sin(\nu\pi)} \left[\frac{1}{2} \mathbf{J}_\nu(iab) + \frac{1}{2} \mathbf{J}_\nu(-iab) - \cos \frac{\nu\pi}{2} I_\nu(ab) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 12(15)}$$

$$3. \quad \int_0^\infty \frac{(x + \sqrt{x^2 + \beta^2})^\nu}{\sqrt{x(x^2 + \beta^2)}} \sin(ax) dx = \sqrt{\frac{a\pi}{2}} \beta^\nu I_{\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a\beta}{2} \right) K_{\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a\beta}{2} \right) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{3}{2}] \quad \text{ET I 71(23)}$$

$$4. \quad \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x(x^2 + \beta^2)}} \cos(ax) dx = \sqrt{\frac{a\pi}{2}} \beta^\nu I_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a\beta}{2} \right) K_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a\beta}{2} \right) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET I 12(17)}$$

$$5. \quad \int_0^\infty \frac{(\beta + \sqrt{x^2 + \beta^2})^\nu}{x^{\nu + \frac{1}{2}} \sqrt{x^2 + \beta^2}} \sin(ax) dx = \frac{1}{\beta} \sqrt{\frac{2}{a}} \Gamma \left(\frac{3}{4} - \frac{\nu}{2} \right) W_{\frac{\nu}{2}, \frac{1}{4}}(a\beta) M_{-\frac{\nu}{2}, \frac{1}{4}}(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{3}{2}] \quad \text{ET I 71(27)}$$

$$6. \quad \int_0^\infty \frac{(\beta + \sqrt{x^2 + \beta^2})^\nu}{x^{\nu + \frac{1}{2}} \sqrt{\beta^2 + x^2}} \cos(ax) dx = \frac{1}{\beta \sqrt{2a}} \Gamma \left(\frac{1}{4} - \frac{\nu}{2} \right) W_{\frac{\nu}{2}, -\frac{1}{4}}(a\beta) M_{-\frac{\nu}{2}, -\frac{1}{4}}(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET I 12(18)}$$

3.775

$$1. \quad \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} + x)^\nu - (\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x^2 + \beta^2}} \sin(ax) dx = 2\beta^\nu \sin \frac{\nu\pi}{2} K_\nu(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 70(20)}$$

$$2. \quad \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} + x)^\nu + (\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x^2 + \beta^2}} \cos(ax) dx = 2\beta^\nu \cos \frac{\nu\pi}{2} K_\nu(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 13(22)}$$

$$3. \quad \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x^2 - u^2}} \sin(ax) dx = \pi u^\nu \left[J_\nu(au) \cos \frac{\nu\pi}{2} - Y_\nu(au) \sin \frac{\nu\pi}{2} \right] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 70(22)}$$

$$4. \quad \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x^2 - u^2}} \cos(ax) dx = -\pi u^\nu \left[Y_\nu(au) \cos \frac{\nu\pi}{2} + J_\nu(au) \sin \frac{\nu\pi}{2} \right] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 13(25)}$$

$$5. \quad \int_0^u \frac{(x + i\sqrt{u^2 - x^2})^\nu + (x - i\sqrt{u^2 - x^2})^\nu}{\sqrt{u^2 - x^2}} \sin(ax) dx = \frac{\pi}{2} u^\nu \operatorname{cosec} \frac{\nu\pi}{2} [\mathbf{J}_\nu(au) - \mathbf{J}_{-\nu}(au)]$$

$$[a > 0, \quad u > 0] \quad \text{ET I 70(21)}$$

$$6. \quad \int_0^u \frac{(x + i\sqrt{u^2 - x^2})^\nu + (x - i\sqrt{u^2 - x^2})^\nu}{\sqrt{u^2 - x^2}} \cos(ax) dx = \frac{\pi}{2} u^\nu \sec \frac{\nu\pi}{2} [\mathbf{J}_\nu(au) + \mathbf{J}_{-\nu}(au)]$$

$$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 13(24)}$$

$$7.^6 \quad \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x(x^2 - u^2)}} \sin(ax) dx$$

$$= -\sqrt{\left(\frac{\pi}{2}\right)^3} au^\nu \left[J_{1/4+\nu/2} \left(\frac{au}{2}\right) Y_{1/4-\nu/2} \left(\frac{au}{2}\right) + J_{1/4-\nu/2} \left(\frac{au}{2}\right) Y_{1/4+\nu/2} \left(\frac{au}{2}\right) \right]$$

$$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2}] \quad \text{ET I 71(25)}$$

$$8.^6 \quad \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x(x^2 - u^2)}} \cos(ax) dx$$

$$= -\sqrt{\left(\frac{\pi}{2}\right)^3} au^\nu \left[J_{-1/4+\nu/2} \left(\frac{au}{2}\right) Y_{-1/4-\nu/2} \left(\frac{au}{2}\right) + J_{-1/4-\nu/2} \left(\frac{au}{2}\right) Y_{-1/4+\nu/2} \left(\frac{au}{2}\right) \right]$$

$$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2}] \quad \text{ET I 13(26)}$$

$$9. \quad \int_0^\infty \frac{(x + \beta + \sqrt{x^2 + 2\beta x})^\nu + (x + \beta - \sqrt{x^2 + 2\beta x})^\nu}{\sqrt{x^2 + 2\beta x}} \sin(ax) dx$$

$$= \pi\beta^\nu \left[Y_\nu(\beta a) \sin \left(\beta a - \frac{\nu\pi}{2} \right) + J_\nu(\beta a) \cos \left(\beta a - \frac{\nu\pi}{2} \right) \right]$$

$$[a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 71(26)}$$

$$10. \quad \int_0^\infty \frac{(x + \beta + \sqrt{x^2 + 2\beta x})^\nu + (x + \beta - \sqrt{x^2 + 2\beta x})^\nu}{\sqrt{x^2 + 2\beta x}} \cos(ax) dx$$

$$= \pi\beta^\nu \left[J_\nu(\beta a) \sin \left(\beta a - \frac{\nu\pi}{2} \right) - Y_\nu(\beta a) \cos \left(\beta a - \frac{\nu\pi}{2} \right) \right]$$

$$[a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 13(23)}$$

$$11. \quad \int_0^{2u} \frac{(\sqrt{2u+x} + i\sqrt{2u-x})^{4\nu} + (\sqrt{2u+x} - i\sqrt{2u-x})^{4\nu}}{\sqrt{4u^2x - x^3}} \cos(ax) dx$$

$$= (4u)^{2\nu} \pi^{3/2} \sqrt{\frac{a}{2}} J_{\nu-1/4}(au) J_{-\nu-1/4}(au)$$

$$[a > 0, \quad u > 0] \quad \text{ET I 14(27)}$$

3.776

$$1. \quad \int_0^\infty \frac{a^2(b+x)^2 + p(p+1)}{(b+x)^{p+2}} \sin(ax) dx = \frac{a}{b^p} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (170)(1)}$$

$$2. \quad \int_0^\infty \frac{a^2(b+x)^2 + p(p+1)}{(b+x)^{p+2}} \cos(ax) dx = \frac{p}{b^{p+1}} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (170)(2)}$$

3.78–3.81 Rational functions of x and of trigonometric functions

3.781

$$1. \quad \int_0^\infty \left(\frac{\sin x}{x} - \frac{1}{1+x} \right) \frac{dx}{x} = 1 - C \quad (\text{cf. 3.784 4 and 3.781 2}) \quad \text{BI (173)(7)}$$

$$2. \quad \int_0^\infty \left(\cos x - \frac{1}{1+x} \right) \frac{dx}{x} = -C \quad \text{BI (173)(8)}$$

3.782

$$1. \quad \int_0^u \frac{1 - \cos x}{x} dx - \int_u^\infty \frac{\cos x}{x} dx = C + \ln u \quad [u > 0] \quad \text{GW (333)(31)}$$

$$2. \quad \int_0^\infty \frac{1 - \cos ax}{x^2} dx = \frac{a\pi}{2} \quad [a \geq 0] \quad \text{BI (158)(1)}$$

$$3. \quad \int_{-\infty}^\infty \frac{1 - \cos ax}{x(x-b)} dx = \pi \frac{\sin ab}{b} \quad [a > 0, \quad b \text{ real}, \quad b \neq 0] \quad \text{ET II 253(48)}$$

3.783

$$1. \quad \int_0^\infty \left[\frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)} \right] \frac{dx}{x} = \frac{1}{2} C - \frac{3}{4} \quad \text{BI (173)(19)}$$

$$2. \quad \int_0^\infty \left(\cos x - \frac{1}{1+x^2} \right) \frac{dx}{x} = -C \quad \text{EH I 17, BI(273)(21)}$$

3.784

$$1. \quad \int_0^\infty \frac{\cos ax - \cos bx}{x} dx = \ln \frac{b}{a} \quad [a > 0, \quad b > 0] \quad \text{FI II 635, GW(333)(20)}$$

$$2. \quad \int_0^\infty \frac{a \sin bx - b \sin ax}{x^2} dx = ab \ln \frac{a}{b} \quad [a > 0, \quad b > 0] \quad \text{FI II 647}$$

$$3. \quad \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{(b-a)\pi}{2} \quad [a \geq 0, \quad b \geq 0] \quad \text{BI(158)(12), FI II 645}$$

$$4. \quad \int_0^\infty \frac{\sin x - x \cos x}{x^2} dx = 1 \quad \text{BI (158)(3)}$$

$$5. \quad \int_0^\infty \frac{\cos ax - \cos bx}{x(x+\beta)} dx = \frac{1}{\beta} \left[\text{ci}(a\beta) \cos a\beta + \text{si}(a\beta) \sin a\beta - \text{ci}(b\beta) \cos b\beta - \text{si}(b\beta) \sin b\beta + \ln \frac{b}{a} \right] \\ [a > 0, \quad b > 0, \quad |\arg \beta| < \pi] \quad \text{ET II 221(49)}$$

$$6. \quad \int_0^\infty \frac{\cos ax + x \sin ax}{1+x^2} dx = \pi e^{-a} \quad [a > 0] \quad \text{GW (333)(73)}$$

$$7. \quad \int_0^\infty \frac{\sin ax - ax \cos ax}{x^3} dx = \frac{\pi}{4} a^2 \text{sign } a \quad \text{LI (158)(5)}$$

$$8. \quad \int_0^\infty \frac{\cos ax - \cos bx}{x^2(x^2 + \beta^2)} dx = \frac{\pi [(b-a)\beta + e^{-b\beta} - e^{-a\beta}]}{2\beta^3} \\ [a > 0, \quad b > 0, \quad |\arg \beta| < \pi] \\ \text{BI(173)(20)a, ET II 222(59)}$$

$$9.^{10} \quad \int_0^\infty \frac{\cos mx}{1+a^2 T_n(x)} dx = \frac{\pi}{2n\sqrt{1+a^2}} \sum_{k=1}^n e^{-m \sin u \sinh \phi} (\cos \beta \sin u \cosh \phi + \sin \beta \cos u \sinh \phi)$$

$$[u = (2k-1)\pi/(2n), \quad \phi = \operatorname{arcsinh}(1/a), \quad \beta = m \cos u \cosh \phi, \quad 0 < |a| < 1]$$

$$3.785 \quad \int_0^\infty \frac{1}{x} \sum_{k=1}^n a_k \cos b_k x dx = - \sum_{k=1}^n a_k \ln b_k \quad \left[b_k > 0, \quad \sum_{k=1}^n a_k = 0 \right] \quad \text{FI II 649}$$

3.786

$$1. \quad \int_0^\infty \frac{(1 - \cos ax) \sin bx}{x^2} dx = \frac{b}{2} \ln \frac{b^2 - a^2}{b^2} + \frac{a}{2} \ln \frac{a+b}{a-b}$$

$$[a > 0, \quad b > 0] \quad \text{ET I 81(29)}$$

$$2.^{11} \quad \int_0^\infty \frac{(1 - \cos ax) \cos bx}{x} dx = \ln \frac{\sqrt{a^2 - b^2}}{b}$$

$$[a > 0, \quad b > 0, \quad a \neq b] \quad \text{FI II 647}$$

$$3.^{11} \quad \int_0^\infty \frac{(1 - \cos ax) \cos bx}{x^2} dx = \frac{\pi}{2}(a-b)$$

$$= 0$$

$$[a < b \leq 0]$$

$$[0 < a \leq b]$$

$$\text{ET I 20(16)}$$

3.787

$$1. \quad \int_0^\infty \frac{(\cos a - \cos na x) \sin mx}{x} dx = \frac{\pi}{2} (\cos a - 1)$$

$$= \frac{\pi}{2} \cos a$$

$$[m > na > 0]$$

$$[na > m]$$

$$\text{BI(155)(7)}$$

$$2. \quad \int_0^\infty \frac{\sin^2 ax - \sin^2 bx}{x} dx = \frac{1}{2} \ln \frac{a}{b}$$

$$[a > 0, \quad b > 0]$$

$$\text{GW (333)(20b)}$$

$$3. \quad \int_0^\infty \frac{x^3 - \sin^3 x}{x^5} dx = \frac{13}{32} \pi$$

$$\text{BI (158)(6)}$$

$$4. \quad \int_0^\infty \frac{(3 - 4 \sin^2 ax) \sin^2 ax}{x} dx = \frac{1}{2} \ln 2$$

$$[a \text{ real}, a \neq 0]$$

$$\text{HBI (155)(6)}$$

$$3.788 \quad \int_0^{\pi/2} \left(\frac{1}{x} - \cot x \right) dx = \ln \frac{\pi}{2}$$

$$\text{GW (333)(61)a}$$

$$3.789 \quad \int_0^{\pi/2} \frac{4x^2 \cos x + (\pi - x)x}{\sin x} dx = \pi^2 \ln 2$$

$$\text{LI (206)(10)}$$

3.791

$$1. \quad \int_0^{\pi/2} \frac{x dx}{1 + \sin x} = \ln 2$$

$$\text{GW (333)(55a)}$$

$$2. \quad \int_0^\pi \frac{x \cos x}{1 + \sin x} dx = \pi \ln 2 - 4G$$

$$\text{GW (333)(55c)}$$

$$3. \quad \int_0^{\pi/2} \frac{x \cos x}{1 + \sin x} dx = \pi \ln 2 - 2G$$

$$\text{GW (333)(55b)}$$

4. $\int_0^\pi \frac{\left(\frac{\pi}{2} - x\right) \cos x}{1 - \sin x} dx = 2 \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x}{1 - \sin x} dx = \pi \ln 2 + 4\mathbf{G} = 5.8414484669 \dots$
BI(207)(3), GW(333)(56c)
 5. $\int_0^{\pi/2} \frac{x^2 dx}{1 - \cos x} = -\frac{\pi^2}{4} + \pi \ln 2 + 4\mathbf{G} = 3.3740473667 \dots$
BI (207)(3)
 6. $\int_0^\pi \frac{x^2 dx}{1 - \cos x} = 4\pi \ln 2$
BI (219)(1)
 7. $\int_0^{\pi/2} \frac{x^{p+1} dx}{1 - \cos x} = -\left(\frac{\pi}{2}\right)^{p+1} + \left(\frac{\pi}{2}\right)^p (p+1) \left\{ \frac{2}{p} - \sum_{k=1}^\infty \frac{1}{4^{2k-1}(p+2k)} \zeta(2k) \right\}$
[$p > 0$] LI (207)(4)
 8. $\int_0^{\pi/2} \frac{x dx}{1 + \cos x} = \frac{\pi}{2} - \ln 2$
GW (333)(55a)
 9. $\int_0^{\pi/2} \frac{x \sin x dx}{1 - \cos x} = \frac{\pi}{2} \ln 2 + 2\mathbf{G}$
GW (333)(56a)
 10. $\int_0^\pi \frac{x \sin x dx}{1 - \cos x} = 2\pi \ln 2$
GW (333)(56b)
 11. $\int_0^\pi \frac{x - \sin x}{1 - \cos x} dx = \frac{\pi}{2} + \int_0^{\pi/2} \frac{x - \sin x}{1 - \cos x} dx = 2$
GW (333)(57a)
 12. $\int_0^{\pi/2} \frac{x \sin x}{1 + \cos x} dx = -\frac{\pi}{2} \ln 2 + 2\mathbf{G}$
GW (333)(55b)
- 3.792**
1. $\int_{-\pi}^\pi \frac{dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{1 - a^2}$ [$a^2 < 1$] FI II 485
 2. $\int_0^{\pi/2} \frac{x \cos x dx}{1 + 2a \sin x + a^2} = \frac{\pi}{2a} \ln(1+a) - \sum_{k=0}^\infty (-1)^k \frac{a^{2k}}{(2k+1)^2}$
[$a^2 < 1$] LI (241)(2)
 3. $\int_0^\pi \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{\pi}{a} \ln(1+a)$ [$a^2 < 1, a \neq 0$]
 $= \frac{\pi}{a} \ln\left(1 + \frac{1}{a}\right)$ [$a^2 < 1$]
BI (221)(2)
 4. $\int_0^{2\pi} \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{a} \ln(1-a)$ [$a^2 < 1, a \neq 0$]
 $= \frac{2\pi}{a} \ln\left(1 - \frac{1}{a}\right)$ [$a^2 > 1$]
BI (223)(4)
 5. $\int_0^{2\pi} \frac{x \sin nx dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{1 - a^2} \left[(a^{-n} - a^n) \ln(1-a) + \sum_{k=1}^{n-1} \frac{a^{-k} - a^k}{n-k} \right]$
[$a^2 < 1, a \neq 0$] BI (223)(5)

6. $\int_0^\infty \frac{\sin x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{4a} \left[\left| \frac{1+a}{1-a} \right| - 1 \right] \quad [a \text{ real}, \quad a \neq 0, \quad a \neq 1]$
 GW (333)(62b)
- 7.⁸ $\int_0^\infty \frac{\sin bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{1+a-2a^{[b]+1}}{(1-a^2)(1-a)} \quad [b \neq 0, 1, 2, \dots]$
 $= \frac{\pi}{2} \frac{1+a-a^b-a^{b+1}}{(1-a^2)(1-a)} \quad [b = 1, 2, \dots]; \quad [0 < a < 1]$
 ET I 81(26)
8. $\int_0^\infty \frac{\sin x \cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2(1-a)} a^{[b]} \quad [b \neq 0, 1, 2, \dots]$
 $= \frac{\pi}{2(1-a)} a^b + \frac{\pi}{4} a^{b-1} \quad [b = 1, 2, 3, \dots];$
 $[0 < a < 1, \quad b > 0]; (\text{for } b = 0, \text{ see } \mathbf{3.792} \text{ 6}) \quad \text{ET I 19(5)}$
9. $\int_0^\infty \frac{(1-a \cos x) \sin bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{1-a^{[b]+1}}{1-a} \quad [b \neq 1, 2, 3, \dots]$
 $= \frac{\pi}{2} \cdot \frac{1-a^b}{1-a} + \frac{\pi a^b}{4} \quad [b = 1, 2, 3, \dots]$
 $[0 < a < 1, \quad b > 0] \quad \text{ET I 82(33)}$
- 10.³ $\int_0^\infty \frac{1}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta(1-a^2)} \frac{1+ae^{-b\beta}}{1-ae^{-b\beta}}$
 $[a^2 < 1, \quad b \geq 0] \quad \text{BI (192)(1)}$
11. $\int_0^\infty \frac{1}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 - x^2} = \frac{a\pi}{\beta(1-a^2)} \frac{\sin b\beta}{1 - 2a \cos b\beta + a^2}$
 $[a^2 < 1, \quad b > 0] \quad \text{BI (193)(1)}$
12. $\int_0^\infty \frac{\sin bcx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \frac{e^{-\beta bc} - a^c}{(1 - ae^{-b\beta})(1 - ae^{b\beta})}$
 $[a^2 < 1, \quad b > 0, \quad c > 0] \quad \text{BI (192)(8)}$
13. $\int_0^\infty \frac{\sin bx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \frac{1}{e^{b\beta} - a} \quad [a^2 < 1, \quad b > 0]$
 $= \frac{\pi}{2a} \frac{1}{ae^{b\beta} - 1} \quad [a^2 > 1, \quad b > 0]$
 BI (192)(2)
14. $\int_0^\infty \frac{\sin bcx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 - x^2} = \frac{\pi}{2} \frac{a^c - \cos \beta bc}{1 - 2a \cos \beta b + a^2}$
 $[a^2 < 1, \quad b > 0, \quad c > 0] \quad \text{BI (193)(5)}$
15. $\int_0^\infty \frac{\cos bcx}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 - x^2} = \frac{\pi}{2\beta(1-a^2)} \frac{(1-a^2) \sin \beta bc + 2a^{c+1} \sin \beta b}{1 - 2a \cos \beta b + a^2}$
 $[a^2 < 1, \quad b > 0, \quad c > 0] \quad \text{BI (193)(9)}$
16. $\int_0^\infty \frac{1 - a \cos bx}{1 - 2a \cos bx + a^2} \frac{dx}{1+x^2} = \frac{\pi}{2} \frac{e^b}{e^b - a} \quad [a^2 < 1, \quad b > 0] \quad \text{FI II 719}$

$$17. \int_0^\infty \frac{\cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi (e^{\beta - \beta b} + a e^{\beta b})}{2\beta (1 - a^2) (e^\beta - a)} \quad [0 \leq b < 1, \quad |a| < 1, \quad \operatorname{Re} \beta > 0]$$

ET I 21(21)

$$18. \int_0^\infty \frac{\sin bx \sin x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi \sinh b\beta}{2\beta e^\beta - a} \quad [0 \leq b < 1]$$

$$= \frac{\pi}{4\beta (ae^\beta - 1)} \left[a^m e^{\beta(m+1-b)} - e^{(1-b)\beta} \right] - \frac{\pi}{4\beta (ae^{-\beta} - 1)} \left[a^m e^{-(m+1-b)\beta} - e^{-(1-b)\beta} \right] \quad [m \leq b \leq m+1]$$

[0 < a < 1, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 81(27)}

$$19. \int_0^\infty \frac{(\cos x - a) \cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi \cosh \beta b}{2\beta (e^\beta - a)} \quad [0 \leq b < 1, \quad |a| < 1, \quad \operatorname{Re} \beta > 0]$$

ET I 21(23)

$$20. \int_0^\infty \frac{\sin x}{(1 - 2a \cos 2x + a^2)^{n+1}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{(1 - 2a \cos 2x + a^2)^{n+1}} \frac{dx}{x}$$

$$= \int_0^\infty \frac{\tan x}{(1 - 2a \cos 4x + a^2)^{n+1}} \frac{dx}{x} = \frac{\pi}{2(1 - a^2)^{2n+1}} \sum_{k=0}^n \binom{n}{k}^2 a^{2k}$$

BI (187)(14)

3.793

$$1.^3 \int_0^{2\pi} \frac{\sin nx - a \sin[(n+1)x]}{1 - 2a \cos x + a^2} x dx = -2\pi a^n \left[\ln(1 - a) + \sum_{k=1}^n \frac{1}{ka^k} \right]$$

[|a| < 1] \quad \text{BI (223)(9)}

$$2. \int_0^{2\pi} \frac{\cos nx - a \cos[(n+1)x]}{1 - 2a \cos x + a^2} x dx = 2\pi a^n \quad [a^2 < 1] \quad \text{BI (223)(13)}$$

3.794

$$1.^3 \int_0^\pi \frac{x dx}{1 + a^2 + 2a \cos x} = \frac{\pi^2}{2(1 - a^2)} + \frac{4}{(1 - a^2)} \sum_{k=0}^\infty \frac{a^{2k+1}}{(2k+1)^2}$$

[a^2 < 1]

$$2. \int_0^{2\pi} \frac{x \sin nx}{1 \pm a \cos x} dx = \frac{2\pi}{\sqrt{1 - a^2}} \left[(\mp 1)^n \frac{(1 + \sqrt{1 - a^2})^n - (1 - \sqrt{1 - a^2})^n}{a^n} \right.$$

$$\left. \times \ln \frac{2\sqrt{1 \pm a}}{\sqrt{1 + a} + \sqrt{1 - a}} + \sum_{k=0}^{n-1} \frac{(\mp 1)^k}{n - k} \frac{(1 + \sqrt{1 - a^2})^k - (1 - \sqrt{1 - a^2})^k}{a^k} \right]$$

[a^2 < 1] \quad \text{BI (223)(2)}

$$3.^3 \int_0^{2\pi} \frac{x \cos nx}{1 \pm a \cos x} dx = \frac{2\pi^2}{\sqrt{1 - a^2}} \left(\frac{1 - \sqrt{1 - a^2}}{\mp a} \right)^n \quad [a^2 < 1] \quad \text{BI (223)(3)}$$

$$4. \quad \int_0^\pi \frac{x \sin x \, dx}{a + b \cos x} = \frac{\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a - b)} \quad [a > |b| > 0] \quad \text{GW (333)(53a)}$$

$$5. \quad \int_0^{2\pi} \frac{x \sin x \, dx}{a + b \cos x} = \frac{2\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a + b)} \quad [a > |b| > 0] \quad \text{GW (333)(53b)}$$

$$6. \quad \int_0^\infty \frac{\sin x}{a \pm b \cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \quad [a^2 > b^2]$$

$$= 0 \quad [a^2 < b^2]$$

BI (181)(1)

$$3.795 \quad \int_{-\infty}^\infty \frac{(b^2 + c^2 + x^2) x \sin ax - (b^2 - c^2 - x^2) c \sinh ac}{[x^2 + (b - c)^2][x^2 + (b + c)^2](\cos ax + \cosh ac)} dx = \pi \quad [c > b > 0]$$

$$= \frac{2\pi}{e^{ab} + 1} \quad [b > c > 0]$$

$$[a > 0] \quad \text{BI (202)(18)}$$

3.796

$$1. \quad \int_0^{\pi/2} \frac{\cos x \pm \sin x}{\cos x \mp \sin x} x \, dx = \mp \frac{\pi}{4} \ln 2 - G \quad \text{BI (207)(8, 9)}$$

$$2. \quad \int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} x \, dx = \frac{\pi}{4} \ln 2 - \frac{1}{2} G \quad \text{BI (204)(23)}$$

3.797

$$1. \quad \int_0^{\pi/4} \left(\frac{\pi}{4} - x \tan x \right) \tan x \, dx = \frac{1}{2} \ln 2 + \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi}{8} \ln 2 \quad \text{BI (204)(8)}$$

$$2. \quad \int_0^{\pi/4} \frac{\left(\frac{\pi}{4} - x \right) \tan x \, dx}{\cos 2x} = -\frac{\pi}{8} \ln 2 + \frac{1}{2} G \quad \text{BI (204)(19)}$$

$$3. \quad \int_0^{\pi/4} \frac{\frac{\pi}{4} - x \tan x}{\cos 2x} \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} G \quad \text{BI (204)(20)}$$

3.798

$$1.^8 \quad \int_0^\infty \frac{\tan x}{a + b \cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \quad [0 < b < a]$$

$$= 0 \quad [0 < a < b]$$

BI (181)(2)

$$2.^8 \quad \int_0^\infty \frac{\tan x}{a + b \cos 4x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \quad [0 < b < a]$$

$$= 0 \quad [0 < a < b]$$

BI (181)(3)

3.799

$$1. \quad \int_0^{\pi/2} \frac{x \, dx}{(\sin x + a \cos x)^2} = \frac{a}{1 + a^2} \frac{\pi}{2} - \frac{\ln a}{1 + a^2} \quad [a > 0] \quad \text{BI (208)(5)}$$

$$2. \quad \int_0^{\pi/4} \frac{x \, dx}{(\cos x + a \sin x)^2} = \frac{1}{1+a^2} \ln \frac{1+a}{\sqrt{2}} + \frac{\pi}{4} \cdot \frac{1-a}{(1+a)(1+a^2)} \quad [a > 0] \quad \text{BI (204)(24)}$$

$$3. \quad \int_0^{\pi} \frac{a \cos x + b}{(a + b \cos x)^2} x^2 \, dx = \frac{2\pi}{b} \ln \frac{2(a-b)}{a + \sqrt{a^2 - b^2}} \quad [a > |b| > 0] \quad \text{GW (333)(58a)}$$

3.811

$$1. \quad \int_0^{\pi} \frac{\sin x}{1 - \cos t_1 \cos x} \cdot \frac{x \, dx}{1 - \cos t_2 \cos x} = \pi \operatorname{cosec} \frac{t_1 + t_2}{2} \operatorname{cosec} \frac{t_1 - t_2}{2} \ln \frac{1 + \tan \frac{t_1}{2}}{1 + \tan \frac{t_2}{2}} \quad (\text{cf. } \mathbf{3.794} \, 4) \quad \text{BI (222)(5)}$$

$$2. \quad \int_0^{\pi/2} \frac{x \, dx}{(\cos x \pm \sin x) \sin x} = \frac{\pi}{4} \ln 2 + \mathbf{G} \quad \text{BI (208))(16, 17)}$$

$$3. \quad \int_0^{\pi/4} \frac{x \, dx}{(\cos x + \sin x) \sin x} = -\frac{\pi}{8} \ln 2 + \mathbf{G} \quad \text{BI (204)(29)}$$

$$4. \quad \int_0^{\pi/4} \frac{x \, dx}{(\cos x + \sin x) \cos x} = \frac{\pi}{8} \ln 2 \quad \text{BI (204)(28)}$$

$$5. \quad \int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x} \frac{x \, dx}{\cos^2 x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2 \quad \text{BI (204)(30)}$$

3.812

$$1. \quad \int_0^{\pi} \frac{x \sin x \, dx}{a + b \cos^2 x} = \frac{\pi}{\sqrt{ab}} \arctan \sqrt{\frac{b}{a}} \quad [a > 0, \quad b > 0]$$

$$= \frac{\pi}{2\sqrt{-ab}} \ln \frac{\sqrt{a} + \sqrt{-b}}{\sqrt{a} - \sqrt{-b}} \quad [a > -b > 0]$$

GW (333)(60a)

$$2. \quad \int_0^{\pi/2} \frac{x \sin 2x \, dx}{1 + a \cos^2 x} = \frac{\pi}{a} \ln \frac{1 + \sqrt{1+a}}{2} \quad [a > -1, \quad a \neq 0] \quad \text{BI (207)(10)}$$

$$3. \quad \int_0^{\pi/2} \frac{x \sin 2x \, dx}{1 + a \sin^2 x} = \frac{\pi}{a} \ln \frac{2(1+a - \sqrt{1+a})}{2} \quad [a > -1, \quad a \neq 0] \quad \text{BI (207)(2)}$$

$$4.^{11} \quad \int_0^{\pi} \frac{x \, dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2 - 1}} \quad [a^2 > 1]$$

$$= 0 \quad [\text{principal value for } 0 < a^2 < 1]$$

$$= \text{divergent} \quad [a = 0]$$

BI (219)(10)

$$5.^7 \quad \int_0^{\pi} \frac{x \sin x \, dx}{a^2 - \cos^2 x} = \frac{\pi}{2a} \ln \left| \frac{1+a}{1-a} \right| \quad [0 < a < 1] \quad \text{divergent if } a = 0$$

BI (219)(13)

$$\begin{aligned}
6.^{11} \quad \int_0^\pi \frac{x \sin 2x \, dx}{a^2 - \cos^2 x} &= \pi \ln \{4(1 - a^2)\} && [\text{principal value for } 0 \leq a^2 < 1] \\
&= 2\pi \ln \left[2 \left(1 - a^2 + a\sqrt{a^2 - 1} \right) \right] && [a^2 > 1] \\
&= \text{divergent} && [|a| = 1]
\end{aligned}$$

BI (219)(19)

$$7. \quad \int_0^{\pi/2} \frac{x \sin x \, dx}{\cos^2 t - \sin^2 x} = -2 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2} \quad \text{BI (207)(1)}$$

$$8. \quad \int_0^\pi \frac{x \sin x \, dx}{1 - \cos^2 t \sin^2 x} = \pi(\pi - 2t) \operatorname{cosec} 2t \quad \text{BI (219)(12)}$$

$$9. \quad \int_0^\pi \frac{x \cos x \, dx}{\cos^2 t - \cos^2 x} = 4 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2} \quad \text{BI (219)(17)}$$

$$10. \quad \int_0^\pi \frac{x \sin x \, dx}{\tan^2 t + \cos^2 x} = \frac{\pi}{2}(\pi - 2t) \cot t \quad \text{BI (219)(14)}$$

$$11. \quad \int_0^\infty \frac{x(a \cos x + b) \sin x \, dx}{\cot^2 t + \cos^2 x} = 2a\pi \ln \cos \frac{t}{2} + \pi b t \tan t \quad \text{BI (219)(18)}$$

$$12.^* \quad \int_0^\pi \frac{x \sin x \cos x}{a - \sin^2 x} \, dx = -\pi \ln 2 + \ln \left[1 + \sqrt{\frac{a-1}{a}} \right] \quad [a > 1]$$

$$13.^* \quad \int_0^{\pi/2} \ln(a - \sin^2 x) \, dx = -\pi \ln 2 + i\pi \ln \arccos \sqrt{a} \quad [0 < a < 1]$$

$$14.^* \quad \text{PV} \int_0^{\pi/2} \ln(|a - \sin^2 x|) \, dx = -\pi \ln 2 \quad [0 < a < 1]$$

$$15.^* \quad \text{PV} \int_0^{\pi/2} \ln(|a - \cos^2 x|) \, dx = -\pi \ln 2 \quad [0 < a < 1]$$

3.813

$$1. \quad \int_0^\pi \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{4} \int_0^{2\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab} \quad [a > 0, \quad b > 0] \quad \text{GW (333)(36)}$$

$$\begin{aligned}
2. \quad \int_0^\infty \frac{1}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{dx}{x^2 + \delta^2} &= \frac{\pi \sinh(2a\delta)}{4\delta(\beta^2 \sinh^2(a\delta) - \gamma^2 \cosh^2(a\delta))} \left[\frac{\beta}{\gamma} - \frac{\gamma}{\beta} - \frac{2}{\sinh(2a\delta)} \right] \\
&\quad \left[\left| \arg \frac{\beta}{\gamma} \right| < \pi, \quad \operatorname{Re} \delta > 0, \quad a > 0 \right] \\
&\quad \text{GW(333)(81), ET II 222(63)}
\end{aligned}$$

$$3. \quad \int_0^\infty \frac{\sin x \, dx}{x(a^2 \sin^2 x + b^2 \cos^2 x)} = \frac{\pi}{2ab} \quad [ab > 0] \quad \text{BI (181)(8)}$$

$$4. \quad \int_0^\infty \frac{\sin^2 x \, dx}{x(a^2 \cos^2 x + b^2 \sin^2 x)} = \frac{\pi}{2b(a+b)} \quad [a > 0, \quad b > 0] \quad \text{BI (181)(11)}$$

5. $\int_0^{\pi/2} \frac{x \sin 2x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b}$ $[a > 0, \quad b > 0, \quad a \neq b]$ GW (333)(52a)
6. $\int_0^{\pi} \frac{x \sin 2x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a}$ $[a > 0, \quad b > 0, \quad a \neq b]$ GW (333)(52b)
7. $\int_0^{\infty} \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{a(a+b)}$ $[a > 0, \quad b > 0]$ BI (182)(3)
8. $\int_0^{\infty} \frac{\sin 2ax}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{x \, dx}{x^2 + \delta^2} = \frac{\pi}{2(\beta^2 \sinh^2(a\delta) - \gamma^2 \cosh^2(a\delta))} \left[\frac{\beta - \gamma}{\beta + \gamma} - e^{-2a\delta} \right]$
 $\left[a > 0, \quad \left| \arg \frac{\beta}{\gamma} \right| < \pi, \quad \operatorname{Re} \delta > 0 \right]$
ET II 222(64), GW(333)(80)
9. $\int_0^{\infty} \frac{(1 - \cos x) \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)}$ $[a > 0, \quad b > 0]$ BI (182)(7)a
10. $\int_0^{\infty} \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a(a+b)}$ $[a > 0, \quad b > 0]$ BI (182)(4)
11. $\int_0^{\infty} \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{2}{a+b}$ $[a > 0, \quad b > 0]$ BI (182)(1)

3.814

1. $\int_0^{\pi/2} \frac{(1 - x \cot x) \, dx}{\sin^2 x} = \frac{\pi}{4}$ BI (206)(9)
2. $\int_0^{\pi/4} \frac{x \tan x \, dx}{(\sin x + \cos x) \cos x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2$ BI (204)(30)
3. $\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \frac{dx}{x} = \frac{\pi}{2ab}$ $[a > 0, \quad b > 0]$ BI (181)(9)
4. $\int_0^{\pi/2} \frac{x \cot x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2a^2} \ln \frac{a+b}{b}$ $[a > 0, \quad b > 0]$ LI (208)(20)
5. $\int_0^{\pi/2} \frac{(\frac{\pi}{2} - x) \tan x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{2} \int_0^{\pi} \frac{(\frac{\pi}{2} - x) \tan x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$
 $= \frac{\pi}{2b^2} \ln \frac{a+b}{a}$
 $[a > 0, \quad b > 0]$ GW (333)(59)
6. $\int_0^{\infty} \frac{\sin^2 x \tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)}$ $[a > 0, \quad b > 0]$ BI (182)(6)
7. $\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2ab}$ $[a > 0, \quad b > 0]$ BI (181)(10)a
8. $\int_0^{\infty} \frac{\sin^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{1}{a+b}$ $[a > 0, \quad b > 0]$ BI (182)(2)a
9. $\int_0^{\infty} \frac{\cos^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \cdot \frac{1}{a+b}$ $[a > 0, \quad b > 0]$ BI (182)(5)a

10. $\int_0^\infty \frac{\sin^2 x \cos x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x \cos 4x} = -\frac{\pi}{8b} \frac{a}{a^2 + b^2} \quad [a > 0, \quad b > 0] \quad \text{BI (186)(12)a}$
11. $\int_0^\infty \frac{\sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2 - a^2}{b^2 + a^2} \quad [a > 0, \quad b > 0] \quad \text{BI (186)(4)a}$
12. $\int_0^\infty \frac{\sin x \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2a} \cdot \frac{b}{a^2 + b^2} \quad [a > 0, \quad b > 0] \quad \text{BI (186)(7)a}$
13. $\int_0^\infty \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2}{a^2 + b^2} \quad [a > 0, \quad b > 0] \quad \text{BI (186)(8)a}$
14. $\int_0^\infty \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = -\frac{\pi}{2b} \cdot \frac{a}{a^2 + b^2} \quad [a > 0, \quad b > 0] \quad \text{BI (186)(10)}$
15. $\int_0^\infty \frac{1 - \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \sin x} = \frac{\pi}{2ab} \quad [a > 0, \quad b > 0] \quad \text{BI (186)(3)a}$

3.815

1. $\int_0^{\pi/2} \frac{x \sin 2x \, dx}{(1 + a \sin^2 x)(1 + b \sin^2 x)} = \frac{\pi}{a - b} \ln \left\{ \frac{1 + \sqrt{1 + b}}{1 + \sqrt{1 + a}} \cdot \frac{\sqrt{1 + a}}{\sqrt{1 + b}} \right\} \quad [a > 0, \quad b > 0] \quad \text{(cf. 3.812 3)}$
BI (208)(22)
2. $\int_0^{\pi/2} \frac{x \sin 2x \, dx}{(1 + a \sin^2 x)(1 + b \cos^2 x)} = \frac{\pi}{a + ab + b} \ln \frac{(1 + \sqrt{1 + n}) \sqrt{1 + a}}{1 + \sqrt{1 + a}} \quad [a > 0, \quad b > 0] \quad \text{(cf. 3.812 2 and 3)}$
BI (208)(24)
3. $\int_0^{\pi/2} \frac{x \sin 2x \, dx}{(1 + a \cos^2 x)(1 + b \cos^2 x)} = \frac{\pi}{a - b} \ln \frac{1 + \sqrt{1 + a}}{1 + \sqrt{1 + b}} \quad [a > 0, \quad b > 0] \quad \text{(cf. 3.812 2)}$
BI (208)(23)
4. $\int_0^{\pi/2} \frac{x \sin 2x \, dx}{(1 - \sin^2 t_1 \cos^2 x)(1 - \sin^2 t_2 \cos^2 x)} = \frac{2\pi}{\cos^2 t_1 - \cos^2 t_2} \ln \frac{\cos \frac{t_1}{2}}{\cos \frac{t_2}{2}} \quad [-\pi < t_1 < \pi, \quad -\pi < t_2 < \pi]$
BI (208)(21)

3.816

1. $\int_0^\pi \frac{x^2 \sin 2x}{(a^2 - \cos^2 x)^2} \, dx = \pi^2 \frac{\sqrt{a^2 - 1} - a}{a(a^2 - 1)} \quad [a > 1] \quad \text{LI (220)(9)}$
- 2.⁷ $\int_0^\pi \frac{(a^2 - 1 - \sin^2 x) \cos x}{(a^2 - \cos^2 x)^2} x^2 \, dx = \frac{\pi}{2} \ln \left| \frac{1 - a}{1 + a} \right| \quad [a^2 > 1] \quad \text{(cf. 3.812 5)} \quad \text{BI (220)(12)}$
- 3.¹¹ $\int_0^\pi \frac{a \cos 2x - \sin^2 x}{(a + \sin^2 x)^2} x^2 \, dx = -2\pi \ln [2(-a + \sqrt{a}\sqrt{a+1})]$
 $\left[a < -1 \text{ and } a > 0. \text{ When } a > 0, \text{ can write } \sqrt{a}\sqrt{a+1} \text{ as } \sqrt{a(a+1)}. \right] \quad \text{LI (220)(10)}$

$$4.11 \quad \int_0^\pi \frac{a \cos 2x + \sin^2 x}{(a - \sin^2 x)^2} x^2 dx = 2\pi \ln [2(a - \sqrt{a}\sqrt{a+1})]$$

[$a < 0$ and $a > 1$. When $a > 1$, can write $\sqrt{a}\sqrt{a+1}$ as $\sqrt{a(a+1)}$.] (cf. **3.812** 6)

LI (220)(11)

3.817

1. $\int_0^\infty \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0] \quad \text{BI (181)(12)}$
2. $\int_0^\infty \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0] \quad \text{BI (182)(8)}$
3. $\int_0^\infty \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \quad [ab > 0] \quad \text{BI (181)(15)}$
4. $\int_0^\infty \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0] \quad \text{BI (182)(9)}$
5. $\int_0^\infty \frac{\tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0] \quad \text{BI (181)(13)}$
6. $\int_0^\infty \frac{\tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0] \quad \text{BI (181)(14)}$
7. $\int_0^\infty \frac{\sin^2 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \quad [ab > 0] \quad \text{BI (182)(11)}$
8. $\int_0^\infty \frac{\tan x \cos^2 2x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0] \quad \text{BI (182)(10)}$

3.818

1. $\int_0^\infty \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^5 b^5} \quad [ab > 0] \quad \text{BI (181)(16)}$
2. $\int_0^\infty \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \quad [ab > 0] \quad \text{BI (182)(13)}$
3. $\int_0^\infty \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \quad [ab > 0] \quad \text{BI (182)(14)}$
4. $\int_0^\infty \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3 b^5} \quad [ab > 0] \quad \text{LI (181)(19)}$
5. $\int_0^\infty \frac{\sin^3 x \cos x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{64} \cdot \frac{3a^2 + b^2}{a^3 b^5} \quad [ab > 0] \quad \text{BI (182)(17)}$

$$6. \quad \int_0^\infty \frac{\tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5b^5} \quad [ab > 0] \quad \text{BI (181)(17)}$$

$$7. \quad \int_0^\infty \frac{\sin^2 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3b^5} \quad [ab > 0] \quad \text{BI (182)(16)}$$

$$8. \quad \int_0^\infty \frac{\tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5b^5} \quad [ab > 0] \quad \text{BI (181)(18)}$$

$$9. \quad \int_0^\infty \frac{\tan x \cos^2 2x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5b^3} \quad [ab > 0] \quad \text{BI (182)(15)}$$

3.819

$$1. \quad \int_0^\infty \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^6 + 3a^4b^2 + 3a^2b^4 + 5b^6}{a^7b^7} \quad [ab > 0] \quad \text{BI (181)(20)}$$

$$2. \quad \int_0^\infty \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7b^5} \quad [ab > 0] \quad \text{BI (182)(18)}$$

$$3. \quad \int_0^\infty \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7b^5} \quad [ab > 0] \quad \text{BI (182)(19)}$$

$$4. \quad \int_0^\infty \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + a^2b^2 + b^4}{a^5b^7} \quad [ab > 0] \quad \text{BI (181)(23)}$$

$$5. \quad \int_0^\infty \frac{\sin^3 x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5b^5} \quad [ab > 0] \quad \text{BI (182)(26)}$$

$$6. \quad \int_0^\infty \frac{\sin x \cos^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7b^3} \quad [ab > 0] \quad \text{BI (182)(23)}$$

$$7. \quad \int_0^\infty \frac{\sin^3 x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5b^5} \quad [ab > 0] \quad \text{BI (182)(27)}$$

$$8. \quad \int_0^\infty \frac{\sin x \cos^4 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7b^3} \quad [ab > 0] \quad \text{BI (182)(24)}$$

$$9. \quad \int_0^\infty \frac{\sin^5 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3b^7} \quad [ab > 0] \quad \text{BI (181)(24)}$$

10.
$$\int_0^\infty \frac{\sin^3 x \cos x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{128} \cdot \frac{5a^4 + 2a^2b^2 + b^4}{a^5b^7} \quad [ab > 0] \quad \text{BI (182)(22)}$$
11.
$$\int_0^\infty \frac{\sin^5 x \cos^3 x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{512} \cdot \frac{5a^2 + b^2}{a^3b^7} \quad [ab > 0] \quad \text{BI (182)(30)}$$
12.
$$\int_0^\infty \frac{\sin^2 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + 2a^2b^2 + b^4}{a^5b^7} \quad [ab > 0] \quad \text{BI (182)(21)}$$
13.
$$\int_0^\infty \frac{\sin^4 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3b^7} \quad [ab > 0] \quad \text{BI (182)(29)}$$
14.
$$\int_0^\infty \frac{\cos^2 2x \tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7b^5} \quad [ab > 0] \quad \text{BI (182)(29)}$$
15.
$$\int_0^\infty \frac{\sin^3 4x \tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{8} \cdot \frac{a^2 + b^2}{a^5b^5} \quad [ab > 0] \quad \text{BI (182)(28)}$$
16.
$$\int_0^\infty \frac{\cos^4 2x \tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7b^3} \quad [ab > 0] \quad \text{BI (182)(25)}$$

3.82–3.83 Powers of trigonometric functions combined with other powers

3.821

1.
$$\int_0^\pi x \sin^p x \, dx = \frac{\pi^2}{2^{p+1}} \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2} + 1\right)\right]^2} \quad [p > -1] \quad \text{BI(218)(7), LO V 121(71)}$$
2.
$$\begin{aligned} \int_0^{r\pi} x \sin^n x \, dx &= \frac{\pi^2}{2} \cdot \frac{(2m-1)!!}{(2m)!!} r^2 & [n = 2m] \\ &= (-1)^{r+1} \pi \frac{(2m)!!}{(2m+1)!!} r & [n = 2m+1] \end{aligned} \quad [r \text{ is a natural number}] \quad \text{GW (333)(8c)}$$
- 3.¹¹
$$\begin{aligned} \int_0^{\pi/2} x \cos^n x \, dx &= \frac{\pi^2}{8} \frac{(n-1)!!}{(n)!!} - \frac{1}{2^{n-2}} \sum_{k=0, m-k \text{ odd}}^{m-1} \binom{n}{k} \frac{1}{(n-2k)^2} & [n = 2m] \\ &= \frac{\pi}{2} \frac{(n-1)!!}{(n)!!} - \frac{1}{2^{n-1}} \sum_{k=0}^{m-1} \binom{n}{k} \frac{1}{(n-2k)^2} & [n = 2m-1] \end{aligned} \quad \text{GW (333)(9b)}$$
4.
$$\int_0^\pi x \cos^{2m} x \, dx = \frac{\pi^2}{2} \frac{(2m-1)!!}{(2m)!!} \quad \text{BI (218)(10)}$$
5.
$$\int_{r\pi}^{s\pi} x \cos^{2m} x \, dx = \frac{\pi^2}{2} (s^2 - r^2) \frac{(2m-1)!!}{(2m)!!} \quad \text{BI (226)(3)}$$

$$6. \quad \int_0^\infty \frac{\sin^p x}{x} dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} = 2^{p-2} B\left(\frac{p}{2}, \frac{p}{2}\right) \\ [p \text{ is a fraction with odd numerator and denominator}] \quad \text{LO V 278, FI II 808}$$

$$7. \quad \int_0^\infty \frac{\sin^{2n+1} x}{x} dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \quad \text{BI (151)(4)}$$

$$8. \quad \int_0^\infty \frac{\sin^{2n} x}{x} dx = \infty \quad \text{BI (151)(3)}$$

$$9. \quad \int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{a\pi}{2} \quad [a > 0] \quad \text{LO V 307, 312, FI II 632}$$

$$10. \quad \int_0^\infty \frac{\sin^{2m} ax}{x^2} dx = \frac{(2m-3)!!}{(2m-2)!!} \cdot \frac{a\pi}{2} \quad [a > 0] \quad \text{GW (333)(14b)}$$

$$11. \quad \int_0^\infty \frac{\sin^{2m+1} ax}{x^3} dx = \frac{(2m-3)!!}{(2m)!!} (2m+1) \frac{a^2 \pi}{4} \quad [a > 0] \quad \text{GW (333)(14d)}$$

$$12. \quad \int_0^\infty \frac{\sin^p x}{x^m} dx \\ = \frac{p}{m-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{m-1}} \cos x dx \quad [p > m-1 > 0] \\ = \frac{p(p-1)}{(m-1)(m-2)} \int_0^\infty \frac{\sin^{p-2} x}{x^{m-2}} dx - \frac{p^2}{(m-1)(m-2)} \int_0^\infty \frac{\sin^p x}{x^{m-2}} dx \quad [p > m-1 > 1] \\ \text{GW (333)(17)}$$

$$13. \quad \int_0^\infty \frac{\sin^{2n} px}{\sqrt{x}} dx = \infty \quad \text{BI (177)(5)}$$

$$14. \quad \int_0^\infty \sin^{2n+1} px \frac{dx}{\sqrt{x}} = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}} \quad \text{BI (177)(7)}$$

3.822

$$1. \quad \int_0^{\pi/2} x^p \cos^m x dx = -\frac{p(p-1)}{m^2} \int_0^{\pi/2} x^{p-2} \cos^m x dx + \frac{m-1}{m} \int_0^{\pi/2} x^p \cos^{m-2} x dx \\ [m > 1, \quad p > 1] \quad \text{GW (333)(9a)}$$

$$2. \quad \int_0^\infty x^{-1/2} \cos^{2n+1}(px) dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}} \quad \text{BI (177)(8)}$$

$$3.823 \quad \int_0^\infty x^{\mu-1} \sin^2 ax dx = -\frac{\Gamma(\mu) \cos \frac{\mu\pi}{2}}{2^{\mu+1} a^\mu} \quad [a > 0, \quad -2 < \operatorname{Re} \mu < 0] \\ \text{ET I 319(15), GW(333)(19c)a}$$

3.824

$$1. \quad \int_0^\infty \frac{\sin^2 ax}{x^2 + \beta^2} dx = \frac{\pi}{4\beta} (1 - e^{-2a\beta}) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{BI (160)(10)}$$

$$2. \quad \int_0^\infty \frac{\cos^2 ax}{x^2 + \beta^2} dx = \frac{\pi}{4\beta} (1 + e^{-2a\beta}) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{BI (160)(11)}$$

$$3.^7 \quad \int_0^\infty \sin^{2m} x \frac{dx}{a^2 + x^2} = \frac{(-1)^m}{2^{2m+1}} \cdot \frac{\pi}{2} \left\{ 2^{2m} \sinh^{2m} a - 2 \sum_{k=0}^m (-1)^k \binom{2m}{k} \sinh[2(m-k)a] \right\} \\ [a > 0] \quad \text{BI (160)(12)}$$

$$4.^7 \quad \int_0^\infty \sin^{2m+1} x \frac{dx}{a^2 + x^2} = \frac{(-1)^{m-1}}{2^{2m+2}a} \left\{ e^{(2m+1)a} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} e^{-2ka} \text{Ei}[(2k-2m-1)a] \right. \\ \left. + e^{-(2m+1)a} \sum_{k=0}^{2m+1} (-1)^{k-1} \binom{2m+1}{k} e^{2ka} \text{Ei}[(2m+1-2k)a] \right\} \\ [a > 0] \quad \text{BI (160)(14)}$$

$$5.^7 \quad \int_0^\infty \sin^{2m+1} x \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1}} e^{-(2m+1)a} \sum_{k=0}^m (-1)^{m+k} \binom{2m+1}{k} e^{2ka} \\ \left[|\arg a| < \frac{\pi}{2} \right], \quad m = 0, 1, 2, \dots$$

$$6.^7 \quad \int_0^\infty \cos^{2m} x \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1}a} \binom{2m}{m} + \frac{\pi}{2^{2m}} \sum_{k=1}^m \binom{2m}{m+k} e^{-2ka} \\ [a > 0] \quad \text{BI (160)(16)}$$

$$7. \quad \int_0^\infty \cos^{2m+1} x \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1}a} \sum_{k=1}^m \binom{2m+1}{m+k+1} e^{-(2k+1)a} \\ [a > 0] \quad \text{BI (160)(17)}$$

$$8. \quad \int_0^\infty \cos^{2m+1} x \frac{x dx}{a^2 + x^2} = -\frac{e^{-(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{2ka} \text{Ei}[(2m-2k+1)a] \\ - \frac{e^{(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{-2ka} \text{Ei}[(2k-2m-1)a] \\ \text{BI (160)(18)}$$

$$9. \quad \int_0^\infty \frac{\cos^2 ax}{b^2 + x^2} dx = \frac{\pi}{4b} \sin 2ab \quad [a > 0, \quad b > 0] \quad \text{BI (161)(10)}$$

$$10. \quad \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{\beta^2 + x^2} dx = \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(b-a)\beta} - e^{-2a\beta} \right] \quad [a > b] \\ = \frac{\pi}{16\beta} [1 - e^{-4a\beta}] \quad [a = b] \\ = \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(a-b)\beta} - e^{-2a\beta} \right] \quad [a < b] \\ [a > 0, \quad b > 0], \quad (\text{cf. } \mathbf{3.824} \text{ 1 and 3}) \quad \text{BI (162)(6)}$$

$$\begin{aligned}
 11. \quad \int_0^\infty \frac{x \sin 2ax \cos^2 bx}{\beta^2 + x^2} dx &= \frac{\pi}{8} \left[2e^{-2a\beta} + e^{-2(a+b)\beta} + e^{2(b-a)\beta} \right] & [a > 0] \\
 &= \frac{\pi}{8} \left[e^{-4a\beta} + 2e^{-2a\beta} \right] & [a = b] \\
 &= \frac{\pi}{8} \left[2e^{-2a\beta} + e^{-2(a+b)\beta} - e^{2(a-b)\beta} \right] & [a < b]
 \end{aligned}$$

LI (162)(5)

3.825

$$1. \quad \int_0^\infty \frac{\sin^2 ax \, dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi (b - c + ce^{-2ab} - be^{-2ac})}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (174)(15)}$$

$$2. \quad \int_0^\infty \frac{\cos^2 ax \, dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi (b - c + be^{-2ac} - ce^{-2ab})}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (175)(14)}$$

$$3.^3 \quad \int_0^\infty \frac{\sin^2 ax \, dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (c \sin 2ab - b \sin 2ac)}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c] \quad \text{LI (174)(16)}$$

$$4.^3 \quad \int_0^\infty \frac{\cos^2 ax \, dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (b \sin 2ac - c \sin 2ab)}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c] \quad \text{LI (175)(15)}$$

3.826

$$1. \quad \int_0^\infty \frac{\sin^2 ax \, dx}{x^2 (b^2 + x^2)} = \frac{\pi}{4b^2} \left[2a - \frac{1}{b} (1 - e^{-2ab}) \right] \quad [a > 0, \quad b > 0] \quad \text{BI (172)(13)}$$

$$2. \quad \int_0^\infty \frac{\sin^2 ax \, dx}{x^2 (b^2 - x^2)} = \frac{\pi}{4b^2} \left(2a - \frac{1}{b} \sin 2ab \right) \quad [a > 0, \quad b > 0] \quad \text{BII (172)(14)}$$

3.827

$$1.^8 \quad \int_0^\infty \frac{\sin^3 ax}{x^\nu} dx = \frac{3 - 3^{\nu-1}}{4} a^{\nu-1} \cos \frac{\nu\pi}{2} \Gamma(1 - \nu) \quad [a < \operatorname{Re} \nu < 4, \nu \neq 1, 2, 3] \quad \text{GW (333)(19f)}$$

$$2.^8 \quad \int_0^\infty \frac{\sin^3 ax}{x} dx = \frac{\pi}{4} \quad \text{LO V 277}$$

$$3. \quad \int_0^\infty \frac{\sin^3 ax}{x^2} dx = \frac{3}{4} a \ln 3 \quad \text{BI (156)(2)}$$

$$4.^8 \quad \int_0^\infty \frac{\sin^3 ax}{x^3} dx = \frac{3}{8} a^2 \pi \quad \text{BI(156)(7)a, LO V 312}$$

$$5. \quad \int_0^\infty \frac{\sin^4 ax}{x^2} dx = \frac{a\pi}{4} \quad [a > 0] \quad \text{BI (156)(3)}$$

$$6. \quad \int_0^\infty \frac{\sin^4 ax}{x^3} dx = a^2 \ln 2 \quad \text{BI (156)(8)}$$

7. $\int_0^\infty \frac{\sin^4 ax}{x^4} dx = \frac{a^3 \pi}{3}$ [$a > 0$] BI(156)(11), LO V 312
8. $\int_0^\infty \frac{\sin^5 ax}{x^2} dx = \frac{5}{16} a (3 \ln 3 - \ln 5)$ BI (156)(4)
9. $\int_0^\infty \frac{\sin^5 ax}{x^3} dx = \frac{5}{32} a^2 \pi$ [$a > 0$] BI (156)(9)
10. $\int_0^\infty \frac{\sin^5 ax}{x^4} dx = \frac{5}{96} a^3 (25 \ln 5 - 27 \ln 3)$ BI (156)(12)
11. $\int_0^\infty \frac{\sin^5 ax}{x^5} dx = \frac{115}{384} a^4 \pi$ [$a > 0$] BI(156)(13), LO V 312
12. $\int_0^\infty \frac{\sin^6 ax}{x^2} dx = \frac{3}{16} a \pi$ [$a > 0$] BI (156)(5)
13. $\int_0^\infty \frac{\sin^6 ax}{x^3} dx = \frac{3}{16} a^2 (8 \ln 2 - 3 \ln 3)$ BI (156)(10)
14. $\int_0^\infty \frac{\sin^6 ax}{x^5} dx = \frac{1}{16} a^4 (27 \ln 3 - 32 \ln 2)$ BI (156)(14)
15. $\int_0^\infty \frac{\sin^6 ax}{x^6} dx = \frac{11}{40} a^5 \pi$ [$a > 0$] LO V 312

3.828 In **3.828** 1–21 the restrictions $a > 0$, $b > 0$, $c > 0$ apply.

- 1.⁸ $\int_0^\infty \frac{\sin ax \sin bx}{x} dx = \frac{1}{2} \ln \left| \frac{a+b}{a-b} \right|$ [$a \neq b$] FI II 647
- 2.⁸ $\int_0^\infty \sin ax \sin bx \frac{dx}{x^2} = \frac{1}{2} \pi \min(a, b)$ BI (157)(1)
- 3.⁸ $\int_0^\infty \frac{\sin^2 ax \sin bx}{x} dx = \frac{\pi}{4}$ [$b < 2a$]
 $= \frac{\pi}{8}$ [$b = 2a$]
 $= 0$ [$b > 2a$]
- BI (151)(10)
- 4.⁸ $\int_0^\infty \frac{\sin^2 ax \cos bx}{x} dx = \frac{1}{4} \ln \frac{4a^2 - b^2}{b^2}$ [$2a \neq b$] BI (151)(12)
- 5.⁸ $\int_0^\infty \frac{\sin^2 ax \cos 2bx}{x^2} dx = \frac{1}{2} \pi \max(0, a - b)$
6. $\int_0^\infty \frac{\sin 2ax \cos^2 bx}{x} dx = \frac{\pi}{2}$ [$a > b$]
 $= \frac{3}{8} \pi$ [$a = b$]
 $= \frac{\pi}{4}$ [$a < b$]
- BI (151)(9)

$$7.^8 \quad \int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x^2} dx = \frac{\pi}{16} (|b - 2a - c| - |2a - b - c| + 2c) \quad [a > 0, \quad 0 < c \leq b]$$

BI(157)(9)a, ET I 79(15)

$$8.^8 \quad \int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x} dx = \frac{1}{8} \ln \left| \frac{(b+c)^2(2a-b+c)(2a+b-c)}{(b-c)^2(2a+b+c)(2a-b-c)} \right|$$

[$b \neq c, \quad 2a+c \neq b, \quad 2a+b \neq c, \quad 2a \neq b+c$] LI (152)(2)

$$9. \quad \int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^2} dx = \frac{\pi}{4} a \quad [0 \leq a \leq b]$$

$$= \frac{\pi}{4} b \quad [0 \leq b \leq a]$$

BI (157)(3)

$$10.^8 \quad \int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^4} dx = \frac{1}{6} \pi \min(a^2, b^2) [3 \max(a, b) - \min(a, b)]$$

BI (157)(27)

$$11.^8 \quad \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{x^2} dx = \frac{1}{4} \pi [a + \max(0, a - b)]$$

BI (157)(6)

$$12. \quad \int_0^\infty \frac{\sin^3 ax \sin 3bx}{x^4} dx = \frac{a^3 \pi}{2} \quad [b > a]$$

$$= \frac{\pi}{16} [8a^3 - 9(a - b)^3] \quad [a \leq 3b \leq 3a] \quad \text{BI (157)(28)}$$

$$= \frac{9b\pi}{8} (a^2 - b^2) \quad [3b \leq a] \quad \text{LI (157)(28)}$$

$$13. \quad \int_0^\infty \frac{\sin^3 ax \cos bx}{x} dx = 0 \quad [b > 3a]$$

$$= -\frac{\pi}{16} \quad [b = 3a]$$

$$= -\frac{\pi}{8} \quad [3a > b > a]$$

$$= \frac{\pi}{16} \quad [b = a]$$

$$= \frac{\pi}{4} \quad [a > b]$$

$$[a > 0, \quad b > 0] \quad \text{BI (151)(15)}$$

$$14.^{10} \quad \int_0^\infty \frac{\sin^3 ax \cos 3bx}{x^2} dx = \frac{3}{16} \left(a \ln 81 - 2(a - 3b) \ln(a - 3b) + 2(a - b) \ln(a - b) \right.$$

$$\left. + 2(a + b) \ln(a + b) - 2(a + 3b) \ln(a + 3b) \right)$$

[Im $a = 0, \quad \text{Im } b = 0$] MC

15. $\int_0^\infty \frac{\sin^3 ax \cos bx}{x^3} dx = \frac{\pi}{8} (3a^2 - b^2) \quad [b < a]$
 $= \frac{\pi b^2}{4} \quad [a = b]$
 $= \frac{\pi}{16} (3a - b)^2 \quad [a < b < 3a]$
 $= 0 \quad [3a < b]$
 $[a > 0, \quad b > 0] \quad \text{BI(157)(19), ET I 19(10)}$
16. $\int_0^\infty \frac{\sin^3 ax \sin bx}{x^4} dx = \frac{b\pi}{24} (9a^2 - b^2) \quad [0 < b \leq a]$
 $= \frac{\pi}{48} [24a^3 - (3a - b)^3] \quad [0 < a \leq b \leq 3a]$
 $= \frac{\pi a^3}{2} \quad [0 < 3a \leq b]$
 ET I 79(16)
17. $\int_0^\infty \frac{\sin^3 ax \sin^2 bx}{x} dx = \frac{\pi}{8} \quad [2b > 3a]$
 $= \frac{5\pi}{32} \quad [2b = 3a]$
 $= \frac{3\pi}{32} \quad [3a > 2b > a]$
 $= \frac{16}{3\pi} \quad [2b = a]$
 $= 0 \quad [a > 2b]$
 $[a > 0, \quad b > 0] \quad \text{BI (151)(14)}$
- 18.⁸ $\int_0^\infty \frac{\sin^2 ax \cos^3 bx}{x} dx = \frac{1}{16} \ln \left| \frac{(2a+b)^3(b-2a)^3(2a+3b)(3b-2a)}{9b^8} \right|$
 $[2a \neq b, \quad 2a \neq 3b] \quad \text{BI (151)(13)}$
- 19.¹¹ $\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin^2 cx}{x} dx$
 $= \frac{\pi}{32} \left(4 \operatorname{sign}(c) - 2 \operatorname{sign}(2b+c) + 2 \operatorname{sign}(2b-c) + \operatorname{sign}(2a-2b+c) - \operatorname{sign}(2a-2b-c) \right.$
 $\left. + 2 \operatorname{sign}(2a-c) + \operatorname{sign}(2a+2b+c) - \operatorname{sign}(2a+2b-c) - 2 \operatorname{sign}(2a+c) \right)$
 $[\operatorname{Im} a = 0, \quad \operatorname{Im} b = 0, \quad \operatorname{Im} c = 0] \quad \text{MC}$
20. $\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx}{x^2} dx$
 $= \frac{a-b-c}{16} \ln 4(a-b-c)^2 - \frac{a+b+c}{16} \ln 4(a+b+c)^2 + \frac{a+b-c}{16} \ln 4(a+b-c)^2$
 $- \frac{a-b+c}{16} \ln 4(a-b+c)^2 + \frac{a+c}{8} \ln 4(a+c)^2 - \frac{a-c}{8} \ln 4(a-c)^2$
 $+ \frac{b+c}{8} \ln 4(b+c)^2 - \frac{b-c}{8} \ln 4(b-c)^2 - \frac{1}{2} c \ln 2c$
 $[a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (157)(10)}$

$$\begin{aligned}
 21.^8 \quad \int_0^\infty \frac{\sin^2 ax \sin^3 bx}{x^3} dx &= \frac{3b^2\pi}{16} & [2a > 3b] \\
 &= \frac{a^2\pi}{12} & [2a = 3b] \\
 &= \frac{6b^2 - (3b - 2a)^2}{32} \pi & [3b > 2a > b] \\
 &= \frac{a^2\pi}{4} & [b \geq 2a]
 \end{aligned}$$

BI (157)(18)

3.829

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{x^n - \sin^n x}{x^{n+2}} dx &= \frac{\pi}{2^n(n+1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} (n-2k)^{n+1} & \text{GW (333)(63)} \\
 2. \quad \int_0^\infty (1 - \cos^{2m-1} x) \frac{dx}{x^2} &= \int_0^\infty (1 - \cos^{2m} x) \frac{dx}{x^2} = \frac{m\pi}{2^{2m}} \binom{2m}{m} & \text{BI (158)(7, 8)}
 \end{aligned}$$

3.831

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin^{2n} ax - \sin^{2n} bx}{x} dx &= \frac{(2n-1)!!}{(2n)!!} \ln \frac{b}{a} & [ab > 0, \quad n = 1, 2, \dots] & \text{FI II 651} \\
 2. \quad \int_0^\infty \frac{\cos^{2n} ax - \cos^{2n} bx}{x} dx &= \left[1 - \frac{(2n-1)!!}{(2n)!!} \right] \ln \frac{b}{a} & [ab > 0, \quad n = 0, 1, \dots] & \text{FI II 651} \\
 3. \quad \int_0^\infty \frac{\cos^{2m+1} ax - \cos^{2m+1} bx}{x} dx &= \ln \frac{b}{a} & [ab > 0, \quad m = 0, 1, \dots] & \text{FI II} \\
 4. \quad \int_0^\infty \frac{\cos^m ax \cos max - \cos^m bx \cos mbx}{x} dx &= \left(1 - \frac{1}{2^m} \right) \ln \frac{b}{a} & [ab > 0, \quad m = 0, 1, \dots] & \text{LI (155)(8)}
 \end{aligned}$$

3.832

$$\begin{aligned}
 1. \quad \int_0^{\pi/2} x \cos^{p-1} x \sin ax \, dx &= \frac{\pi}{2^{p+1}} \Gamma(p) \frac{\psi\left(\frac{p+a+1}{2}\right) - \psi\left(\frac{p-a+1}{2}\right)}{\Gamma\left(\frac{p+a+1}{2}\right) \Gamma\left(\frac{p-a+1}{2}\right)} & [p > 0, \quad -(p+1) < a < p+1] & \text{BI (205)(6)} \\
 2.^3 \quad \int_0^\infty \sin^{2m+1} x \sin 2mx \frac{dx}{a^2 + x^2} &= \frac{(-1)^m \pi}{2^{2m+1} a} \left[(1 - e^{-2a})^{2m} - 1 \right] \sinh a & [a > 0, \quad m = 0, 1, \dots] & \text{BI (162)(17)} \\
 3. \quad \int_0^\infty \sin^{2m-1} x \sin[(2m-1)x] \frac{dx}{a^2 + x^2} &= \frac{(-1)^{m+1} \pi}{2^{2m} a} (1 - e^{-2a})^{2m-1} & [a > 0, \quad m = 1, 2, \dots] & \text{BI (162)(11)} \\
 4. \quad \int_0^\infty \sin^{2m-1} x \sin[(2m+1)x] \frac{dx}{a^2 + x^2} &= \frac{(-1)^{m-1} \pi}{2^{2m} a} e^{-2a} (1 - e^{-2a})^{2m-1} & [a > 0, \quad m = 1, 2, \dots] & \text{BI (162)(12)}
 \end{aligned}$$

$$5. \quad \int_0^\infty \sin^{2m+1} x \sin[3(2m+1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2a} e^{-3(2m+1)a} \sinh^{2m+1} a$$

$$[a > 0] \quad \text{BI (162)(18)}$$

$$6.^3 \quad \int_0^\infty \sin^{2m} x \sin[(2m-1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^a \left[(1 - e^{-2a})^{2m} - (1 + e^{-2a}) \right]$$

$$[a \geq 0, \quad m = 0, 1, \dots] \quad \text{BI (162)(13)}$$

$$7. \quad \int_0^\infty \sin^{2m} x \sin(2mx) \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1}} \left[(1 - e^{-2a})^{2m} - 1 \right]$$

$$[a > 0, \quad m = 0, 1, \dots] \quad \text{BI (162)(14)}$$

$$8. \quad \int_0^\infty \sin^{2m} x \sin[(2m+2)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^{-2a} (1 - e^{-2a})^{2m}$$

$$[a > 0, \quad m = 0, 1, \dots] \quad \text{BI (162)(15)}$$

$$9. \quad \int_0^\infty \sin^{2m} x \sin 4mx \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2} e^{-4ma} \sinh^{2m} a$$

$$[a > 0, \quad m = 1, 2, \dots] \quad \text{BI (162)(16)}$$

$$10. \quad \int_0^\infty \sin^{2m} x \cos x \frac{dx}{x^2} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$$

$$[m = 1, 2, \dots] \quad \text{GW (333)(15a)}$$

$$11. \quad \int_0^\infty \sin^{2m} x \cos[(2m-1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m} a} \left[(1 - e^{-2a})^{2m-1} - 1 \right] \sinh a$$

$$[a > 0, \quad m = 1, 2, \dots] \quad \text{BI (162)(25)}$$

$$12. \quad \int_0^\infty \sin^{2m} x \cos(2mx) \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} (1 - e^{-2a})^{2m}$$

$$[a > 0, \quad m = 0, 1, \dots] \quad \text{BI (162)(26)}$$

$$13. \quad \int_0^\infty \sin^{2m} x \cos[(2m+2)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} e^{-2a} (1 - e^{-2a})^{2m}$$

$$[a > 0, \quad m = 0, 1, \dots] \quad \text{BI (162)(27)}$$

$$14. \quad \int_0^\infty \sin^{2m} x \cos 4mx \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2a} e^{-4ma} \sinh^{2m} a$$

$$[a > 0, \quad m = 0, 1, \dots] \quad \text{BI (162)(28)}$$

$$15. \quad \int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x} = \frac{(2m-1)!!}{(2m+2)!!} \cdot \frac{\pi}{2}$$

$$[m = 0, 1, \dots] \quad \text{GW (333)(15)}$$

$$16.^3 \quad \int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x^3} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$$

$$[m = 1, 2, \dots] \quad \text{GW (333)(15b)}$$

$$17. \quad \int_0^\infty \sin^{2m-1} x \cos[(2m-1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m}} \left[(1 - e^{-2a})^{2m-1} - 1 \right]$$

$$[m = 1, 2, \dots, \quad a > 0] \quad \text{BI (162)(23)}$$

$$18.^3 \quad \int_0^\infty \sin^{2m+1} x \cos 2mx \frac{x dx}{a^2 + x^2} = \frac{(-1)^{m-1} \pi}{2^{2m+2}} \left\{ e^a \left[(1 - e^{-2a})^{2m+1} - 1 \right] - e^{-a} \right\}$$

$$[m = 0, 1, \dots, \quad a \geq 0] \quad \text{BI (162)(29)}$$

19.
$$\int_0^\infty \sin^{2m-1} x \cos[(2m+1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m}} e^{-2a} (1 - e^{-2a})^{2m-1}$$

$$[m = 1, 2, \dots, \quad a > 0] \quad \text{BI (162)(24)}$$
20.
$$\int_0^\infty \sin^{2m+1} x \cos[2(2m+1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^{m-1} \pi}{2} e^{-2(2m+1)a} \sinh^{2m+1} a$$

$$[m = 0, 1, \dots, \quad a > 0] \quad \text{BI (162)(30)}$$
21.
$$\int_0^\infty \cos^m x \sin mx \frac{x dx}{a^2 + x^2} = \frac{1}{2^{m+1}a} \sum_{k=1}^m \binom{m}{k} [e^{-2ka} \text{Ei}(2ka) - e^{2ka} \text{Ei}(-2ka)]$$

$$[a > 0] \quad \text{BI (162)(8)}$$
22.
$$\int_0^\infty \cos^n sx \sin nsx \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^{n+1}} [(1 + e^{-2as})^n - 1]$$

$$[s > 0, \quad \text{Re } a > 0, \quad n \geq 0] \quad \text{BI (163)(9)}$$
23.
$$\int_0^\infty \cos^n sx \sin nsx \frac{x dx}{a^2 - x^2} = \frac{\pi}{2} (2^{-n} - \cos^n as \cos nas)$$

$$[n = 0, 1, \dots] \quad \text{BI (166)(10)}$$
24.
$$\int_0^\infty \cos^{m-1} x \sin[(m+1)x] \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^m} e^{-2a} (1 + e^{-2a})^{m-1}$$

$$[a > 0, \quad m = 1, 2, \dots] \quad \text{BI (163)(6)}$$
25.
$$\int_0^\infty \cos^m x \sin[(m+1)x] \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^{m+1}} e^{-a} (1 + e^{-2a})^m$$

$$[m = 0, 1, \dots, \quad a > 0] \quad \text{BI (163)(10)}$$
- 26.³
$$\int_0^\infty \cos^m x \sin[(m-1)x] \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^m} \cosh a [(1 + e^{-2a})^{m-1} - 1]$$

$$[m = 0, 1, \dots, \quad a \geq 0] \quad \text{BI (163)(7)}$$
- 27.¹¹
$$\int_0^\infty \cos^m x \sin(3mx) \frac{x dx}{a^2 + x^2} = \frac{\pi}{2} e^{-3ma} \cosh^m a$$

$$[a > 0, \quad m = 1, 2, \dots] \quad \text{BI (163)(11)}$$
28.
$$\int_0^\infty \cos^n sx \cos nsx \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{n+1}a} (1 + e^{-2as})^n$$

$$[n = 0, 1, \dots] \quad \text{BI (163)(16)}$$
29.
$$\int_0^\infty \cos^n sx \cos nsx \frac{dx}{a^2 - x^2} = \frac{\pi}{2a} \cos^n as \sin nas$$

$$[n = 0, 1, \dots]$$
30.
$$\int_0^\infty \cos^{m-1} x \cos[(m+1)x] \frac{dx}{a^2 + x^2} = \frac{\pi}{2^m a} e^{-2a} (1 + e^{-2a})^{m-1}$$

$$[m = 1, 2, \dots, \quad a > 0] \quad \text{BI (163)(14)}$$
31.
$$\int_0^\infty \cos^m x \cos[(m-1)x] \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{m+1}a} e^a [(1 + e^{-2a})^m - (1 - e^{-2a})]$$

$$[m = 0, 1, \dots, \quad a > 0] \quad \text{BI (163)(15)}$$

$$32. \quad \int_0^\infty \cos^m x \cos[(m+1)x] \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{m+1}a} e^{-a} (1 + e^{-2a})^m$$

$$[m = 0, 1, \dots, \quad a > 0] \quad \text{BI (163)(17)}$$

$$33. \quad \int_0^\infty \sin^p x \cos x \frac{dx}{x^q} = \frac{p}{q-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{q-1}} dx - \frac{p+1}{q-1} \int_0^\infty \frac{\sin^{p+1} x}{x^{q-1}} dx \quad [p > q-1 > 0]$$

$$= \frac{p(p-1)}{(q-1)(q-2)} \int_0^\infty \sin^{p-2} x \cos x \frac{dx}{x^{q-2}}$$

$$- \frac{(p+1)^2}{(q-1)(q-2)} \int_0^\infty \sin^p x \cos x \frac{dx}{x^{q-2}} \quad [p > q-1 > 1]$$

$$\text{GW (333)(18)}$$

$$34. \quad \int_0^\infty \cos^{2m} x \cos 2nx \sin x \frac{dx}{x} = \int_0^\infty \cos^{2m-1} x \cos 2nx \sin \frac{dx}{x} = \frac{\pi}{2^{2m+1}} \binom{2m}{m+n}$$

$$\text{BI (152)(5, 6)}$$

$$35. \quad \int_0^\infty \cos^p ax \sin bx \cos x \frac{dx}{x} = \frac{\pi}{2} \quad [b > ap, \quad p > -1] \quad \text{BI (153)(12)}$$

$$36. \quad \int_0^\infty \cos^p ax \sin pax \cos x \frac{dx}{x} = \frac{\pi}{2^{p+1}} (2^p - 1) \quad [p > -1] \quad \text{BI (153)(2)}$$

$$37. \quad \int_0^\infty \frac{dx}{x^2} \left(\prod_{k=1}^n \cos^{p_k} a_k x \right) \sin bx \sin x = \frac{\pi}{2} \quad \left[b > \sum_{k=1}^n a_k p_k, \quad a_k > 0, \quad p_k > 0 \right]$$

$$\text{BI (157)(15)}$$

3.833

$$1.^{10} \quad \int_0^\infty \sin^{2m+1} x \cos^{2n} x \frac{dx}{x} = \int_0^\infty \sin^{2m+1} x \cos^{2n-1} x \frac{dx}{x} = \frac{(2m-1)!!(2n-1)!!}{2^{m+n+1}(m+n)!} \pi$$

$$= \frac{1}{2} B\left(m + \frac{1}{2}, n + \frac{1}{2}\right)$$

$$\text{BI (151)(24, 25)}$$

GW (333)(24)

$$2. \quad \int_0^\infty \sin^{2m+1} 2x \cos^{2n-1} 2x \cos^2 x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2m-1)!!(2n-1)!!}{(2m+2n)!!}$$

$$\text{LI (152)(4)}$$

3.834

$$1. \quad \int_0^\infty \frac{\sin^{2m+1} x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{(-1)^m \pi (1+a)^{4m}}{2^{2m+2} a^{2m+1}} \left\{ \left| \frac{1-a}{1+a} \right|^{2m-1} \right.$$

$$\left. - \sum_{k=0}^{2m} (-1)^k \binom{m - \frac{1}{2}}{k} \left(\frac{4a}{(1+a)^2} \right)^k \right\}$$

$$[|a| \neq 1] \quad \text{GW (333)(62a)}$$

$$\begin{aligned}
2. \quad & \int_0^\infty \frac{\sin^{2m+1} x \cos^n x}{(1 - 2a \cos x + a^2)^p} \cdot \frac{dx}{x} \\
&= \frac{n! \pi}{2^{n+1} (2m+n+1)! (1+a)^{2p}} \sum_{k=0}^n \frac{(-1)^k (2m+2n-2k+1)!! (2m+2k-1)!!}{k! (n-k)!} \\
&\quad \times F\left(m+n-k+\frac{3}{2}, p; 2m+n+2; \frac{4a}{(1+a)^2}\right) \\
&\quad [a \neq \pm 1]
\end{aligned}$$

GW (333)(62)

3.835

$$\begin{aligned}
1. \quad & \int_0^\infty \frac{\cos^{2m} x \cos 2mx \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{b^{2m-1}}{a(a+b)^{2m}} \quad [ab > 0] \quad \text{BI (182)(31)a} \\
2. \quad & \int_0^\infty \frac{\cos^{2m-1} x \cos 2mx \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \frac{b^{2m-1}}{(a+b)^{2m}} \quad [ab > 0] \quad \text{LI (182)(32)a}
\end{aligned}$$

3.836

$$\begin{aligned}
1. \quad & \int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin mx}{x} dx = \frac{\pi}{2} \quad [m \geq n] \quad \text{LI (159)(12)} \\
2.^{11} \quad & \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos mx dx = \frac{n\pi}{2^n} \sum_{k=0}^{\lfloor \frac{1}{2}(m+n) \rfloor} \frac{(-1)^k (n+m-2k)^{n-1}}{k! (n-k)!} \quad [0 \leq m < n] \\
&= 0 \quad [m \geq n \geq 2] \\
&= \frac{\pi}{4} \quad [m = n = 1]
\end{aligned}$$

GI(159)(14), ET I 20(11)

$$\begin{aligned}
3. \quad & \int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} \sin nx \cos x \frac{dx}{x} = \frac{\pi}{2} \quad [n \geq 1] \quad \text{BI (159)(20)} \\
4.^8 \quad & \int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin(ax)}{x} dx = \frac{\pi}{2} \left[1 - \frac{1}{2^{n-1} n!} \sum_{k=0}^{\lfloor \frac{1}{2} n(1+a) \rfloor} (-1)^k \binom{n}{k} (n+an-2k)^n \right] \\
&\quad [\text{all real } a, n \geq 1] \quad \text{ET I 20(11)}
\end{aligned}$$

$$\begin{aligned}
5.^{10} \quad & I_n(b) = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos bx dx = n (2^{n-1} n!)^{-1} \sum_{k=0}^{\lfloor r \rfloor} (-1)^k \binom{n}{k} (n-b-2k)^{n-1} \\
&\quad \text{where } 0 \leq b < n, n \geq 1, r = (n-b)/2, \text{ and } \lfloor r \rfloor \text{ is the largest integer contained in } r \\
&\quad \text{LO V 340(14)}
\end{aligned}$$

$$6.^{11} \quad \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos anx dx = 0 \quad [a \leq -1 \text{ or } a \geq 1, \quad n \geq 2; \quad \text{for } n = 1 \text{ see } \mathbf{3.741} \ 2]$$

3.837

$$\begin{aligned}
1. \quad & \int_0^{\pi/2} \frac{x^2 dx}{\sin^2 x} = \pi \ln 2 \quad \text{BI (206)(9)} \\
2. \quad & \int_0^{\pi/4} \frac{x^2 dx}{\sin^2 x} = -\frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 + \mathbf{G} = 0.8435118417 \dots \quad \text{BI (204)(10)}
\end{aligned}$$

3. $\int_0^{\pi/4} \frac{x^2 dx}{\cos^2 x} = \frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 - \mathbf{G}$ GW (333)(35a)
4. $\int_0^{\pi/4} \frac{x^{p+1}}{\sin^2 x} dx = -\left(\frac{\pi}{4}\right)^{p+1} + (p+1) \left(\frac{\pi}{4}\right)^p \left\{ \frac{1}{p} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)} \zeta(2k) \right\}$
 $[p > 0]$ LI (204)(14)
5. $\int_0^{\pi/2} \frac{x^2 \cos x}{\sin^2 x} dx = -\frac{\pi^2}{4} + 4\mathbf{G} = 1.1964612764 \dots$ BI (206)(7)
6. $\int_0^{\pi/2} \frac{x^3 \cos x}{\sin^3 x} dx = -\frac{\pi^3}{16} + \frac{3}{2} \pi \ln 2$ BI (206)(8)
7. $\int_0^{\infty} \frac{\cos 2nx}{\cos x} \sin^{2n} x \frac{dx}{x^m} = 0$ $\left[n > \frac{m-1}{2}, \quad m > 0 \right]$ BI (180)(16)
8. $\int_0^{\infty} \frac{\cos 2nx}{\cos x} \sin^{2n+1} x \frac{dx}{x^m} = 0$ $\left[n > \frac{m-2}{2}, \quad m > 0 \right]$ BI (180)(17)
9. $\int_0^1 \frac{x dx}{\cos ax \cos[a(1-x)]} = \frac{1}{a} \operatorname{cosec} a \cdot \ln \sec a$ $\left[a < \frac{\pi}{2} \right]$ BI (149)(20)
- 10.³ $\int_0^{\pi} \frac{x \sin(2n+1)x}{\sin x} dx = \frac{1}{2} \pi^2$ $[n = 0, 1, 2, \dots]$
- 11.³ $\int_0^{\pi} \frac{x \sin 2nx}{\sin x} dx = -4 \sum_{k=1}^n (2k-1)^{-2}$ $[n = 1, 2, 3, \dots]$

3.838

1. $\int_0^{\pi/2} \frac{x \cos^{p-1} x}{\sin^{p+1} x} dx = \frac{\pi}{2p} \sec \frac{\pi p}{2}$ $[p < 1]$ BI (206)(13)a
2. $\int_0^{\pi/4} \frac{x \sin^{p-1} x}{\cos^{p+1} x} dx = \frac{\pi}{4p} - \frac{1}{2p} \beta\left(\frac{p+1}{2}\right)$ $[p > -1]$ LI (204)(15)
3. $\int_0^{\pi/4} \frac{x \sin^{2m-1} x}{\cos^{2m+1} x} dx = \frac{\pi}{8m} (1 - \cos m\pi) + \frac{1}{2m} \sum_{k=0}^{m-1} \frac{(-1)^{k-1}}{2m-2k-1}$ BI (204)(17)
4. $\int_0^{\pi/4} \frac{x \sin^{2m} x}{\cos^{2m+2} x} dx = \frac{1}{2(2m+1)} \left[\frac{\pi}{2} + (-1)^{m-1} \ln 2 + \sum_{k=0}^{m-1} \frac{(-1)^{k-1}}{m-k} \right]$ BI (204)(16)

3.839

- 1.¹¹ $\int_0^{\pi/4} x \tan^2 x dx = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2} \ln 2$ BI (204)(3)
2. $\int_0^{\pi/4} x \tan^3 x dx = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8} \ln 2 - \frac{1}{2} \mathbf{G}$ BI (204)(7)
3. $\int_0^{\pi/4} \frac{x^2 \tan x}{\cos^2 x} dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{\pi^2}{16}$ (cf. **3.839** 1) BI (204)(13)

$$4. \quad \int_0^{\pi/4} \frac{x^2 \tan^2 x}{\cos^2 x} dx = \frac{1}{3} \left(1 - \frac{\pi}{4} \ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + \mathbf{G} \right) \quad (\text{cf. } \mathbf{3.839} \text{ 2}) \quad \text{BI (204)(12)}$$

$$5. \quad \int_0^{\pi/2} x \cos^p x \tan x dx = \frac{\pi}{2^{p+1}p} \cdot \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2} + 1\right) \right]^2} \quad [p > -1] \quad \text{BI (205)(3)}$$

$$6. \quad \int_0^{\pi/2} x \sin^p x \cot x dx = \frac{\pi}{2p} - \frac{2^{p-1}}{p} \mathbf{B} \left(\frac{p+1}{2}, \frac{p+1}{2} \right) \quad [p > -1] \quad \text{BI (206)(11)}$$

$$7. \quad \int_0^\infty \sin^{2n} x \tan x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!} \quad \text{GW (333)(16)}$$

$$8. \quad \int_0^\infty \cos^s rx \tan qx \frac{dx}{x} = \frac{\pi}{2} \quad [s > -1] \quad \text{BI (151)(26)}$$

$$9. \quad \int_0^\infty \frac{\cos[(2n-1)x]}{\cos x} \cdot \left(\frac{\sin x}{x} \right)^{2n} dx = (-1)^{n-1} \frac{2^{2n}-1}{(2n)!} \cdot 2^{2n-1} \pi |B_{2n}| \quad \text{BI (180)(15)}$$

$$10. \quad \int_0^\infty \tan^r px \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \sec \frac{r\pi}{2} \tanh^r pq \quad [r^2 < 1] \quad \text{BI (160)(19)}$$

3.84 Integrals containing $\sqrt{1 - k^2 \sin^2 x}$, $\sqrt{1 - k^2 \cos^2 x}$, and similar expressions

Notation: $k' = \sqrt{1 - k^2}$

3.841

$$1. \quad \int_0^\infty \sin x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = \mathbf{E}(k) \quad \text{BI (154)(8)}$$

$$2. \quad \int_0^\infty \sin x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = \mathbf{E}(k) \quad \text{BI (154)(20)}$$

$$3. \quad \int_0^\infty \tan x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = \mathbf{E}(k) \quad \text{BI (154)(9)}$$

$$4. \quad \int_0^\infty \tan x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = \mathbf{E}(k) \quad \text{BI (154)(21)}$$

3.842

$$\begin{aligned} 1.^{11} \quad \int_0^\infty \frac{\sin x}{\sqrt{1 + \sin^2 x}} \frac{dx}{x} &= \int_0^\infty \frac{\tan x}{\sqrt{1 + \sin^2 x}} \cdot \frac{dx}{x} \\ &= \int_0^\infty \frac{\sin x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} \mathbf{K} \left(\frac{1}{\sqrt{2}} \right) \approx 1.3110287771 \\ &\quad \text{BI (183)(4, 5, 9, 10)} \end{aligned}$$

$$2. \quad \int_u^{\pi/2} \frac{x \cos x dx}{\sqrt{\sin^2 x - \sin^2 u}} = \frac{\pi}{2} \ln(1 + \cos u) \quad \text{BI (226)(4)}$$

$$\begin{aligned}
 3. \quad \int_0^\infty \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} &= \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} \\
 &= \int_0^\infty \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \mathbf{K}(k)
 \end{aligned}$$

BI (183)(12, 13, 21, 22)

$$4. \quad \int_0^{\pi/2} \frac{x \sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} dx = \frac{1}{2k^2} [-\pi k' + 2 \mathbf{E}(k)] \quad \text{BI (211)(1)}$$

$$5. \quad \int_0^{\pi/2} \frac{x \sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} dx = \frac{1}{2k^2} [\pi - 2 \mathbf{E}(k)] \quad \text{BI (214)(1)}$$

$$6. \quad \int_0^\alpha \frac{x \sin x dx}{\cos^2 x \sqrt{\sin^2 \alpha - \sin^2 x}} = \frac{\pi \sin^2 \frac{\alpha}{2}}{\cos^2 \alpha} \quad \text{LO III 284}$$

$$7. \quad \int_0^\beta \frac{x \sin x dx}{(1 - \sin^2 \alpha \sin^2 x) \sqrt{\sin^2 \beta - \sin^2 x}} = \frac{\pi \ln \frac{\cos \alpha + \sqrt{1 - \sin^2 \alpha \sin^2 \beta}}{2 \cos \beta \cos^2 \frac{\alpha}{2}}}{2 \cos \alpha \sqrt{1 - \sin^2 \alpha \sin^2 \beta}} \quad \text{LO III 284}$$

3.843

$$1. \quad \int_0^\infty \tan x \sqrt{1 - k^2 \sin^2 2x} \frac{dx}{x} = \mathbf{E}(k) \quad \text{BI (154)(10)}$$

$$2. \quad \int_0^\infty \tan x \sqrt{1 - k^2 \cos^2 2x} \frac{dx}{x} = \mathbf{E}(k) \quad \text{BI (154)(22)}$$

$$3.^{11} \quad \int_0^\infty \frac{\tan x}{\sqrt{1 + \sin^2 2x}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1 + \cos^2 2x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} \mathbf{K} \left(\frac{1}{\sqrt{2}} \right) \approx 1.3110287771$$

BI (183)(6, 11)

$$4. \quad \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \sin^2 2x}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \cos^2 2x}} \frac{dx}{x} = \mathbf{K}(k) \quad \text{BI (183)(14, 23)}$$

3.844

$$1. \quad \int_0^\infty \frac{\sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BI (185)(20)}$$

$$2. \quad \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BI (185)(21)}$$

$$3. \quad \int_0^\infty \frac{\sin x \cos^3 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2) \mathbf{K}(k) - 2(1 + k^2) \mathbf{E}(k)] \quad \text{BI (185)(22)}$$

$$4. \quad \int_0^\infty \frac{\sin x \cos^4 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2) \mathbf{K}(k) - 2(1 + k^2) \mathbf{E}(k)] \quad \text{BI (185)(23)}$$

$$5. \quad \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)] \quad \text{BI (185)(24)}$$

$$6. \quad \int_0^\infty \frac{\sin^3 x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)] \quad \text{BI (185)(25)}$$

$$7. \quad \int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (184)(16)}$$

$$8. \quad \int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2+3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)] \quad \text{BI (184)(18)}$$

3.845

$$1.^{11} \quad \int_0^\infty \frac{\sin x \cos x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \frac{1}{2} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \right] \approx 0.5990701174 \quad \text{BI (185)(6)}$$

$$2.^{11} \quad \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \frac{1}{2} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \right] \approx 0.5990701174 \quad \text{BI (185)(7)}$$

$$3.^{11} \quad \int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right] \approx 0.7119586598 \quad \text{BU (184)(8)}$$

3.846

$$1. \quad \int_0^\infty \frac{\sin x \cos x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (185)(9)}$$

$$2. \quad \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (185)(10)}$$

$$3. \quad \int_0^\infty \frac{\sin x \cos^3 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2-3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)] \quad \text{BI (185)(11)}$$

$$4. \quad \int_0^\infty \frac{\sin x \cos^4 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2-3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)] \quad \text{BI (185)(12)}$$

$$5. \quad \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1+k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)] \quad \text{BI (185)(13)}$$

$$6. \quad \int_0^\infty \frac{\sin^3 x \cos^2 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1+k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)] \quad \text{BI (185)(14)}$$

$$7. \quad \int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BI (184)(9)}$$

$$8. \quad \int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2+k^2) \mathbf{K}(k) - 2(1+k^2) \mathbf{E}(k)] \quad \text{BI (184)(11)}$$

$$3.847^{11} \quad \int_0^\infty \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right] \approx 0.7119586598 \quad \text{BI (185)(3, 4)}$$

3.848

$$1. \quad \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BI (185)(15)}$$

$$2. \quad \int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (184)(12)}$$

$$3. \quad \int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2-3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)] \quad \text{BI (184)(13)}$$

$$4. \quad \int_0^\infty \frac{\sin^2 4x \tan x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{4}{3k^4} [(1+k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)] \quad \text{BI (184)(17)}$$

$$5. \quad \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (185)(26)}$$

$$6. \quad \int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1-k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BI (184)(19)}$$

$$7. \quad \int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1-k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2+k^2) \mathbf{K}(k) - 2(1+k^2) \mathbf{E}(k)] \quad \text{BI (184)(20)}$$

3.849

$$1.^{11} \quad \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1+\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{2\sqrt{2}} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right] \approx 0.1779896649 \quad \text{BI (185)(8)}$$

$$2.^{11} \quad \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \frac{\sqrt{2}}{8} \left[2\mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \right] \approx 0.1497675293 \quad \text{BI (185)(5)}$$

$$3.^{11} \quad \int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right] \approx 0.7119586598 \quad \text{BI (184)(7)}$$

3.85–3.88 Trigonometric functions of more complicated arguments combined with powers**3.851**

$$5. \quad \int_0^\infty \sin(ax^2) \cos(bx) \frac{dx}{x^2} = \frac{b\pi}{2} \left\{ S \left(\frac{b}{2\sqrt{a}} \right) - C \left(\frac{b}{2\sqrt{a}} \right) + \sqrt{a\pi} \sin \left(\frac{b^2}{4a} + \frac{\pi}{4} \right) \right\} \\ [a > 0, \quad b > 0], \quad (\text{cf. } \mathbf{3.691} \text{ 7}) \\ \text{ET I 23(3)a}$$

3.852

$$1. \quad \int_0^\infty \frac{\sin(ax^2)}{x^2} dx = \sqrt{\frac{a\pi}{2}} \quad [a \geq 0] \quad \text{BI (177)(10)a}$$

$$2. \quad \int_0^\infty \sin(ax^2) \cos(bx^2) \frac{dx}{x^2} = \frac{1}{2} \sqrt{\frac{\pi}{2}} (\sqrt{a+b} + \sqrt{a-b}) \quad [a > b > 0] \\ = \frac{1}{2} \sqrt{\pi a} \quad [b = a \geq 0] \\ = \frac{1}{2} \sqrt{\frac{\pi}{2}} (\sqrt{a+b} - \sqrt{b-a}) \\ [b > a > 0], \quad (\text{cf. } \mathbf{3.852} \text{ 1}) \quad \text{BI (177)(23)}$$

$$3. \quad \int_0^\infty \frac{\sin^2(a^2 x^2)}{x^4} dx = \frac{2\sqrt{\pi}}{3} a^3 \quad [a \geq 0] \quad \text{GW (333)(19e)}$$

$$4.^{10} \quad \int_0^\infty \frac{\sin^3(a^2 x^2)}{x^2} dx = \frac{a}{4} \sqrt{\frac{\pi}{2}} (3 - \sqrt{3}) \quad [\operatorname{Im} a^2 = 0] \quad \text{MC}$$

$$5. \quad \int_0^\infty (\sin^2 x - x^2 \cos x^2) \frac{dx}{x^4} = \frac{1}{3} \sqrt{\frac{\pi}{2}} \quad \text{BI (178)(8)}$$

$$6. \quad \int_0^\infty \left(\cos^2 x - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{1}{2} C \quad \text{BI (173)(22)}$$

3.853

$$1. \quad \int_0^\infty \frac{\sin(ax^2)}{\beta^2 + x^2} dx = \frac{\pi}{2\beta} \left[\sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) - \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) - \sin(a\beta^2) \right] \\ [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 219(33)a}$$

$$2. \quad \int_0^\infty \frac{\cos(ax^2)}{\beta^2 + x^2} dx = \frac{\pi}{2\beta} \left[\cos(a\beta^2) - \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) - \sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) \right] \\ [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 221(51)a}$$

$$3. \quad \int_0^\infty \frac{x^2 \sin(ax^2)}{\beta^2 + x^2} dx \\ = \frac{\beta\pi}{2} \left[\sin(a\beta^2) - \sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) + \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) \right] \\ - \frac{1}{2} \sqrt{\frac{\pi}{2a}} \\ [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 219(32)a}$$

$$4. \quad \int_0^\infty \frac{x^2 \cos(ax^2)}{\beta^2 + x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} - \frac{\beta\pi}{2} \left\{ \cos(a\beta^2) - \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) \right. \\ \left. - \sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) \right\} \\ [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 221(50)a}$$

3.854

$$1. \quad \int_0^\infty (\cos(ax^2) - \sin(ax^2)) \frac{dx}{x^4 + b^4} = \frac{\pi e^{-ab^2}}{2b^3 \sqrt{2}} \quad [a > 0, \quad b > 0] \\ \text{LI (178)(11)a, BI (168)(25)}$$

$$2. \quad \int_0^\infty (\cos(ax^2) + \sin(ax^2)) \frac{x^2 dx}{x^4 + b^4} = \frac{\pi e^{-ab^2}}{2b\sqrt{2}} \quad [a > 0, b > 0] \quad \text{LI (178)(12)}$$

$$3. \quad \int_0^\infty (\cos(ax^2) + \sin(ax^2)) \frac{x^2 dx}{(x^4 + b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b^3} \left(a + \frac{1}{2b^2} \right) \\ [a > 0, \quad b > 0] \quad \text{LI (178)(14)}$$

$$4. \quad \int_0^\infty (\cos(ax^2) - \sin(ax^2)) \frac{x^4 dx}{(x^4 + b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b} \left(\frac{1}{2b^2} - a \right) \\ [a > 0, \quad b > 0] \quad \text{BI (178)(15)}$$

3.855

$$1. \quad \int_0^\infty \frac{\sin(ax^2)}{\sqrt{\beta^2 + x^4}} dx = \frac{1}{2} \sqrt{\frac{a\pi}{2}} I_{\frac{1}{4}} \left(\frac{a\beta}{2} \right) K_{\frac{1}{4}} \left(\frac{a\beta}{2} \right) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 66(28)}$$

$$2. \quad \int_0^\infty \frac{\cos(ax^2)}{\sqrt{\beta^2 + x^4}} dx = \frac{1}{2} \sqrt{\frac{a\pi}{2}} I_{-\frac{1}{4}} \left(\frac{a\beta}{2} \right) K_{\frac{1}{4}} \left(\frac{a\beta}{2} \right) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 9(22)}$$

$$3. \quad \int_0^u \frac{\sin(a^2 x^2)}{\sqrt{u^4 - x^4}} dx = \frac{a}{4} \sqrt{\frac{\pi^3}{2}} \left[J_{\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) \right]^2 \quad [a > 0] \quad \text{ET I 66(29)}$$

$$4. \quad \int_u^\infty \frac{\sin(a^2 x^2)}{\sqrt{x^4 - u^4}} dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{2}} J_{\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) Y_{\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) \quad [a > 0] \quad \text{ET I 66(30)}$$

$$5. \quad \int_0^u \frac{\cos(a^2 x^2)}{\sqrt{u^4 - x^4}} dx = \frac{a}{4} \sqrt{\frac{\pi^3}{2}} \left[J_{-\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) \right]^2 \quad \text{ET I 9(23)}$$

$$6. \quad \int_u^\infty \frac{\cos(a^2 x^2)}{\sqrt{x^4 - u^4}} dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{2}} J_{-\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) Y_{-\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) \quad \text{ET I 10(24)}$$

3.856

$$1. \quad \int_0^\infty \frac{(\sqrt{\beta^4 + x^4} + x^2)^\nu}{\sqrt{\beta^4 + x^4}} \sin(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) K_{\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) \quad \left[\operatorname{Re} \nu < \frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 71(23)}$$

$$2. \quad \int_0^\infty \frac{(\sqrt{\beta^4 + x^4} + x^2)^\nu}{\sqrt{\beta^4 + x^4}} \cos(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) K_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) \quad \left[\operatorname{Re} \nu < \frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 12(16)}$$

$$3. \quad \int_0^\infty \frac{(\sqrt{\beta^4 + x^4} - x^2)^\nu}{\sqrt{\beta^4 + x^4}} \cos(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) K_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) \quad \left[\operatorname{Re} \nu > -\frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 12(17)}$$

$$4. \quad \int_0^\infty \frac{\sin(a^2 x^2) dx}{\sqrt{\beta^4 + x^4} \sqrt{x^2 + \sqrt{\beta^4 + x^4}}} = \frac{\sinh \frac{a^2 \beta^2}{2}}{\sqrt{2} \beta^2} K_0 \left(\frac{a^2 \beta^2}{2} \right) \quad [|\arg \beta| < \frac{\pi}{4}] \quad \text{ET I 66(32)}$$

$$5. \quad \int_0^\infty \frac{\cos(a^2 x^2) dx}{\sqrt{\beta^4 + x^4} \sqrt{(x^2 + \sqrt{\beta^4 + x^4})^3}} = \frac{\sinh \frac{a^2 \beta^2}{2}}{2\sqrt{2} \beta^4} K_1 \left(\frac{a^2 \beta^2}{2} \right) \quad [|\arg \beta| < \frac{\pi}{4}] \quad \text{ET I 10(27)}$$

$$6. \quad \int_0^\infty \frac{\sqrt{\sqrt{\beta^4 + x^4} + x^2}}{\sqrt{\beta^4 + x^4}} \sin(ax^2) dx = \frac{\pi}{2\sqrt{2}} e^{-\frac{a^2\beta^2}{2}} I_0\left(\frac{a^2\beta^2}{2}\right) \left[|\arg \beta| < \frac{\pi}{4}\right] \quad \text{ET I 67(33)}$$

3.857

$$1. \quad \int_0^\infty \frac{x^2}{R_1 R_2} \sqrt{\frac{R_2 - R_1}{R_2 + R_1}} \sin(ax^2) dx = \frac{1}{2\sqrt{b}} K_0(ac) \sin ab \left[R_1 = \sqrt{c^2 + (b - x^2)^2}, \quad R_2 = \sqrt{c^2 + (b + x^2)^2}, \quad a > 0, \quad c > 0 \right] \quad \text{ET I 67(34)}$$

$$2. \quad \int_0^\infty \frac{x^2}{R_1 R_2} \sqrt{\frac{R_2 + R_1}{R_2 - R_1}} \cos(ax^2) dx = \frac{1}{2\sqrt{b}} K_0(ac) \cos ab \left[R_1 = \sqrt{c^2 + (b - x^2)^2}, \quad R_2 = \sqrt{c^2 + (b + x^2)^2}, \quad a > 0, \quad c > 0 \right] \quad \text{ET I 10(26)}$$

3.858

$$1. \quad \int_u^\infty \frac{(x^2 + \sqrt{x^4 - u^4})^\nu + (x^2 - \sqrt{x^4 - u^4})^\nu}{\sqrt{x^4 - u^4}} \sin(a^2 x^2) dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{a}} u^{2\nu} \left[J_{\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) + J_{\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) \right] \left[\operatorname{Re} \nu < \frac{3}{2} \right] \quad \text{ET I 71(25)}$$

$$2. \quad \int_u^\infty \frac{(x^2 + \sqrt{x^4 - u^4})^\nu + (x^2 - \sqrt{x^4 - u^4})^\nu}{\sqrt{x^4 - u^4}} \cos(a^2 x^2) dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{a}} u^{2\nu} \left[J_{-\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{-\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) + J_{-\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{-\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) \right] \left[\operatorname{Re} \nu < \frac{3}{2} \right] \quad \text{ET I 13(26)}$$

$$3.859 \quad \int_0^\infty \left[\cos(x^{2n}) - \frac{1}{1 + x^{2n+1}} \right] \frac{dx}{x} = -\frac{1}{2n} C \quad \text{BI (173)(24)}$$

3.861

$$1. \quad \int_0^\infty \sin^{2n+1}(ax^2) \frac{dx}{x^{2m}} = \pm \frac{\sqrt{\pi} a^{m-\frac{1}{2}}}{2^{2n-m+\frac{1}{2}} (2m-1)!!} \sum_{k=1}^{n+1} (-1)^{k-1} \binom{2n+1}{n+k} (2k-1)^{m-\frac{1}{2}} \left[\begin{array}{l} \text{the + sign is taken when } m \equiv 0 \pmod{4} \text{ or } m \equiv 1 \pmod{4}, \\ \text{the - sign is taken when } m \equiv 2 \pmod{4} \text{ or } m \equiv 3 \pmod{4} \end{array} \right] \quad \text{BI (177)(19)a}$$

$$2. \quad \int_0^\infty \sin^{2n}(ax^2) \frac{dx}{x^{2m}} = \pm \frac{\sqrt{\pi} a^{m-\frac{1}{2}}}{2^{2n-2m+1} (2m-1)!!} \sum_{k=1}^n (-1)^k \binom{2n}{n+k} k^{m-\frac{1}{2}} \left[\begin{array}{l} \text{the + sign is taken when } m \equiv 0 \pmod{4} \text{ or } m \equiv 3 \pmod{4}, \\ \text{the - sign is taken when } m \equiv 2 \pmod{4} \text{ or } m \equiv 1 \pmod{4} \end{array} \right] \quad \text{BI (177)(18)a, LI (177)(18)}$$

$$3.862 \quad \int_0^\infty [\cos(ax^2\sqrt{n}) + \sin(ax^2\sqrt{n})] \left(\frac{\sin^2 x}{x^2} \right)^n dx = \frac{\sqrt{\pi}}{(2n-1)!!\sqrt{2}} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-2k+a\sqrt{n})^{n-\frac{1}{2}} \left[a > \sqrt{n} > 0 \right] \quad \text{BI (178)(9)}$$

3.863

1.
$$\int_0^\infty x^2 \cos(ax^4) \sin(2bx^2) dx = -\frac{\pi}{8} \sqrt{\frac{b^3}{a^3}} \left[\sin\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{\frac{3}{4}}\left(\frac{b^2}{2a}\right) \right]$$

[$a > 0, \quad b > 0$] ET I 25(22)
2.
$$\int_0^\infty x^2 \cos(ax^4) \cos(2bx^2) dx = -\frac{\pi}{8} \sqrt{\frac{b^3}{a^3}} \left[\sin\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{3}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right) \right]$$

[$a > 0, \quad b > 0$] ET I 25(23)

3.864

1.
$$\int_0^\infty \sin \frac{b}{x} \sin ax \frac{dx}{x} = \frac{\pi}{2} Y_0(2\sqrt{ab}) + K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0] \quad \text{WA 204(3)a}$$
2.
$$\int_0^\infty \cos \frac{b}{x} \cos ax \frac{dx}{x} = -\frac{\pi}{2} Y_0(2\sqrt{ab}) + K_0(2\sqrt{ab})$$

[$a > 0, \quad b > 0$]
WA 204(4)a, ET I 24 (12)

3.865

1.
$$\int_0^u \frac{(u^2 - x^2)^{\mu-1}}{x^{2\mu}} \sin \frac{a}{x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}} \Gamma(\mu) J_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$

[$a > 0, \quad u > 0, \quad 0 < \operatorname{Re} \mu < 1$]
ET II 189(30)
2.
$$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \sin \frac{a}{x} dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu} \Gamma(\mu) \sin \frac{a}{2u} J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right)$$

[$a > 0, \quad u > 0, \quad \operatorname{Re} \mu > 0$]
ET II 203(21)
3.
$$\int_0^u \frac{(u^2 - x^2)^{\mu-1}}{x^{2\mu}} \cos \frac{a}{x} dx = -\frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} \Gamma(\mu) u^{\mu-\frac{3}{2}} Y_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$

[$a > 0, \quad u > 0, \quad 0 < \operatorname{Re} \mu < 1$]
ET II 190(36)
4.
$$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \cos \frac{a}{x} dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu} \Gamma(\mu) \cos \frac{a}{2u} J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right)$$

[$a > 0, \quad u > 0, \quad \operatorname{Re} \mu > 0$]
ET II 204(26)

3.866

1.
$$\int_0^\infty x^{\mu-1} \sin \frac{b^2}{x} \sin(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a}\right)^\mu \operatorname{cosec} \frac{\mu\pi}{2} [J_\mu(2ab) - J_{-\mu}(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)]$$

[$a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1$]
ET I 322(42)
2.
$$\int_0^\infty x^{\mu-1} \sin \frac{b^2}{x} \cos(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a}\right)^\mu \sec \frac{\mu\pi}{2} [J_\mu(2ab) + J_{-\mu}(2ab) + I_\mu(2ab) - I_{-\mu}(2ab)]$$

[$a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1$]
ET I 322(43)

$$3. \quad \int_0^\infty x^{\mu-1} \cos \frac{b^2}{x} \cos(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a}\right)^\mu \operatorname{cosec} \frac{\mu\pi}{2} [J_{-\mu}(2ab) - J_\mu(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$$

ET I 322(44)

3.867

$$1. \quad \int_0^1 \frac{\cos ax - \cos \frac{a}{x}}{1-x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos ax - \cos \frac{a}{x}}{1-x^2} dx = \frac{\pi}{2} \sin a$$

$$[a > 0]$$

GW (334)(7a)

$$2. \quad \int_0^1 \frac{\cos ax + \cos \frac{a}{x}}{1+x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos ax + \cos \frac{a}{x}}{1+x^2} dx = \frac{\pi}{2} e^{-a}$$

$$[a > 0]$$

GW (334)(7b)

3.868

$$1. \quad \int_0^\infty \sin \left(a^2 x + \frac{b^2}{x}\right) \frac{dx}{x} = \pi J_0(2ab)$$

$$[a > 0, \quad b > 0]$$

GW (334)(11a), WA 200(16)

$$2. \quad \int_0^\infty \cos \left(a^2 x + \frac{b^2}{x}\right) \frac{dx}{x} = -\pi Y_0(2ab)$$

$$[a > 0, \quad b > 0]$$

GW (334)(11a)

$$3. \quad \int_0^\infty \sin \left(a^2 x - \frac{b^2}{x}\right) \frac{dx}{x} = 0$$

$$[a > 0, \quad b > 0]$$

GW (334)(11b)

$$4. \quad \int_0^\infty \cos \left(a^2 x - \frac{b^2}{x}\right) \frac{dx}{x} = 2 K_0(2ab)$$

$$[a > 0, \quad b > 0]$$

GW (334)(11b)

3.869

$$1. \quad \int_0^\infty \sin \left(ax - \frac{b}{x}\right) \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \exp \left(-\alpha\beta - \frac{b}{\beta}\right)$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0]$$

ET II 220(42)

$$2. \quad \int_0^\infty \cos \left(ax - \frac{b}{x}\right) \frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta} \exp \left(-a\beta - \frac{b}{\beta}\right)$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0]$$

ET II 222(58)

3.871

$$1. \quad \int_0^\infty x^{\mu-1} \sin \left[a \left(x + \frac{b^2}{x}\right)\right] dx = \pi b^\mu \left[J_\mu(2ab) \cos \frac{\mu\pi}{2} - Y_\mu(2ab) \sin \frac{\mu\pi}{2}\right]$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \mu < 1]$$

ET I 319(17)

$$2. \quad \int_0^\infty x^{\mu-1} \cos \left[a \left(x + \frac{b^2}{x}\right)\right] dx = -\pi b^\mu \left[J_\mu(2ab) \sin \frac{\mu\pi}{2} + Y_\mu(2ab) \cos \frac{\mu\pi}{2}\right]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$$

ET I 321(35)

$$3. \quad \int_0^\infty x^{\mu-1} \sin \left[a \left(x - \frac{b^2}{x}\right)\right] dx = 2b^\mu K_\mu(2ab) \sin \frac{\mu\pi}{2}$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$$

ET I 319(16)

$$4. \quad \int_0^\infty x^{\mu-1} \cos \left[a \left(x - \frac{b^2}{x} \right) \right] dx = 2b^\mu K_\mu(2ab) \cos \frac{\mu\pi}{2} \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$$

ET I 321(36)

3.872

$$1. \quad \int_0^1 \sin \left[a \left(x + \frac{1}{x} \right) \right] \sin \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1-x^2} \\ = \frac{1}{2} \int_0^\infty \sin \left[a \left(x + \frac{1}{x} \right) \right] \sin \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1-x^2} = -\frac{\pi}{4} \sin 2a$$

[a ≥ 0] BI (149)(15), GW (334)(8a)

$$2. \quad \int_0^1 \cos \left[a \left(x + \frac{1}{x} \right) \right] \cos \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1-x^2} \\ = \frac{1}{2} \int_0^\infty \cos \left[a \left(x + \frac{1}{x} \right) \right] \cos \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1+x^2} = \frac{\pi}{4} e^{-2a}$$

[a ≥ 0] GW (334)(8b)

3.873

$$1. \quad \int_0^\infty \sin \frac{a^2}{x^2} \cos b^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a} [\sin(2ab) + \cos(2ab) + e^{-2ab}]$$

[a > 0, \quad b > 0] ET I 24(15)

$$2. \quad \int_0^\infty \cos \frac{a^2}{x^2} \cos b^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a} [\cos(2ab) - \sin(2ab) + e^{-2ab}]$$

[a > 0, \quad b > 0] ET I 24(16)

3.874

$$1. \quad \int_0^\infty \sin \left(a^2 x^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \sin \left(2ab + \frac{\pi}{4} \right) \quad [a > 0, \quad b > 0]$$

BI (179)(6)a, GW(334)(10a)

$$2. \quad \int_0^\infty \cos \left(a^2 x^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \cos \left(2ab + \frac{\pi}{4} \right) \quad [a > 0, \quad b > 0]$$

GI (179)(8)a, GW(334)(10a)

$$3. \quad \int_0^\infty \sin \left(a^2 x^2 - \frac{b^2}{x^2} \right) \frac{dx}{x^2} = -\frac{\sqrt{\pi}}{2\sqrt{2}b} e^{-2ab} \quad [a \geq 0, \quad b > 0]$$

GW (335)(10b)

$$4. \quad \int_0^\infty \cos \left(a^2 x^2 - \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\sqrt{2}b} e^{-2ab} \quad [a \geq 0, \quad b > 0]$$

GW (334)(10b)

$$5. \quad \int_0^\infty \sin \left(ax - \frac{b}{x} \right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b} \quad [a > 0, \quad b > 0]$$

BI (179)(13)a

$$6. \quad \int_0^\infty \cos \left(ax - \frac{b}{x} \right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b} \quad [a > 0, \quad b > 0]$$

BI (179)(14)a

3.875

$$1. \quad \int_u^\infty \frac{x \sin(p\sqrt{x^2 - u^2})}{x^2 + a^2} \cos bx \, dx = \frac{\pi}{2} \exp(-p\sqrt{a^2 + u^2}) \cosh ab \quad [0 < b < p] \quad \text{ET I 27(39)}$$

$$2. \quad \int_u^\infty \frac{x \sin(p\sqrt{x^2 - u^2})}{a^2 + x^2 - u^2} \cos bx \, dx = \frac{\pi}{2} e^{-ap} \cos(b\sqrt{u^2 - a^2}) \quad [0 < b < p, \quad a > 0] \quad \text{ET I 27(38)}$$

$$3.^6 \quad \int_0^\infty \frac{\sin(p\sqrt{a^2 + x^2})}{(a^2 + x^2)^{3/2}} \cos bx \, dx = \frac{\pi p}{2a} e^{-ab} \quad [0 < p < b, \quad a > 0] \quad \text{ET I 26(29)}$$

3.876

$$1. \quad \int_0^\infty \frac{\sin(p\sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} \cos bx \, dx = \frac{\pi}{2} J_0(a\sqrt{p^2 - b^2}) \quad [0 < b < p] \\ = 0 \quad [b > p > 0] \\ [a > 0] \quad \text{ET I 26(30)}$$

$$2. \quad \int_0^\infty \frac{\cos(p\sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} \cos bx \, dx = -\frac{\pi}{2} Y_0(a\sqrt{p^2 - b^2}) \quad [0 < b < p] \\ = K_0(a\sqrt{b^2 - p^2}) \quad [b > p > 0] \\ [a > 0] \quad \text{ET I 26(34)}$$

$$3. \quad \int_0^\infty \frac{\cos(p\sqrt{x^2 + a^2})}{x^2 + c^2} \cos bx \, dx = \frac{\pi}{2c} e^{-bc} \cos(p\sqrt{a^2 - c^2}) \quad [c > 0, \quad b > p] \quad \text{ET I 26(33)}$$

$$4. \quad \int_0^\infty \frac{\sin(p\sqrt{x^2 + a^2})}{(x^2 + c^2)\sqrt{x^2 + a^2}} \cos bx \, dx = \frac{\pi}{2c} \frac{e^{-bc} \sin(p\sqrt{a^2 - c^2})}{\sqrt{a^2 - c^2}} \quad [c \neq a] \\ = \frac{\pi}{2} e^{-ba} \frac{p}{a} \quad [c = a] \\ [b > p, \quad c > 0] \quad \text{ET I 26(31)a}$$

$$5.^6 \quad \int_0^\infty \frac{\cos(p\sqrt{x^2 + a^2})}{x^2 + a^2} \cos bx \, dx = \frac{\pi}{2a} e^{-ab} \quad [b > p > 0; \quad a > 0] \quad \text{ET I 27(35)a}$$

$$6.^6 \quad \int_0^\infty \frac{x \cos(p\sqrt{x^2 + a^2})}{x^2 + a^2} \sin bx \, dx = \frac{\pi}{2} e^{-ab} \quad [a > 0, \quad b > p > 0] \quad \text{ET I 85(29)a}$$

$$7. \quad \int_0^u \frac{\cos(p\sqrt{u^2 - x^2})}{\sqrt{u^2 - x^2}} \cos bx \, dx = \frac{\pi}{2} J_0(u\sqrt{b^2 + p^2}) \quad \text{ET I 28(42)}$$

$$8. \quad \int_u^\infty \frac{\cos(p\sqrt{x^2 - u^2})}{\sqrt{x^2 - u^2}} \cos bx \, dx = K_0(u\sqrt{p^2 - b^2}) \quad [0 < b < |p|] \\ = -\frac{\pi}{2} Y_0(u\sqrt{b^2 - p^2}) \quad [b > |p|] \quad \text{ET I 28(43)}$$

3.877

$$1. \quad \int_0^u \frac{\sin(p\sqrt{u^2 - x^2})}{\sqrt[4]{(u^2 - x^2)^3}} \cos bx \, dx = \sqrt{\frac{\pi^3 p}{8}} J_{\frac{1}{4}} \left[\frac{u}{2} (\sqrt{b^2 + p^2} - b) \right] J_{\frac{1}{4}} \left[\frac{u}{2} (\sqrt{b^2 + p^2} + b) \right]$$

$$[b > 0, \quad p > 0] \quad \text{ET I 27(40)}$$

$$2. \quad \int_u^\infty \frac{\sin(p\sqrt{x^2 - u^2})}{\sqrt[4]{(x^2 - u^2)^3}} \cos bx \, dx = -\sqrt{\frac{\pi^3 p}{8}} J_{\frac{1}{4}} \left[\frac{u}{2} (b - \sqrt{b^2 - p^2}) \right] Z_{\frac{1}{4}} \left[\frac{u}{2} (b + \sqrt{b^2 - p^2}) \right]$$

$$[b > p > 0] \quad \text{ET I 27(41)}$$

$$3. \quad \int_0^u \frac{\cos(p\sqrt{u^2 - x^2})}{\sqrt[4]{(u^2 - x^2)^3}} \cos bx \, dx = \sqrt{\frac{\pi^3 p}{8}} J_{-\frac{1}{4}} \left[\frac{u}{2} (\sqrt{p^2 + b^2} - b) \right] J_{-\frac{1}{4}} \left[\frac{u}{2} (\sqrt{p^2 + b^2} + b) \right]$$

$$[u > 0, \quad p > 0] \quad \text{ET I 28(44)}$$

$$4. \quad \int_u^\infty \frac{\cos(p\sqrt{x^2 - u^2})}{\sqrt[4]{(x^2 - u^2)^3}} \cos bx \, dx = -\sqrt{\frac{\pi^3 p}{8}} J_{-\frac{1}{4}} \left[\frac{u}{2} (b - \sqrt{b^2 - p^2}) \right] Y_{\frac{1}{4}} \left[\frac{u}{2} (b + \sqrt{b^2 - p^2}) \right]$$

$$[b > p > 0] \quad \text{ET I 28(45)}$$

3.878

$$1. \quad \int_0^\infty \frac{\sin(p\sqrt{x^4 + a^4})}{\sqrt{x^4 + a^4}} \cos bx^2 \, dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3} b J_{-\frac{1}{4}} \left[\frac{a^2}{2} (p - \sqrt{p^2 - b^2}) \right] J_{\frac{1}{4}} \left[\frac{a^2}{2} (p + \sqrt{p^2 - b^2}) \right]$$

$$[p > b > 0] \quad \text{ET I 26(32)}$$

$$2. \quad \int_0^\infty \frac{\cos(p\sqrt{x^4 + a^4})}{\sqrt{x^4 + a^4}} \cos bx^2 \, dx$$

$$= -\frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3} b J_{-\frac{1}{4}} \left[\frac{a^2}{2} (p - \sqrt{p^2 - b^2}) \right] Y_{\frac{1}{4}} \left[\frac{a^2}{2} (p + \sqrt{p^2 - b^2}) \right]$$

$$[a > 0, \quad p > b > 0] \quad \text{ET I 27(36)}$$

$$3. \quad \int_0^u \frac{\cos(p\sqrt{u^4 - x^4})}{\sqrt{u^4 - x^4}} \cos bx^2 \, dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3} b J_{-\frac{1}{4}} \left[\frac{u^2}{2} (\sqrt{p^2 + b^2} - p) \right] J_{-\frac{1}{4}} \left[\frac{u^2}{2} (\sqrt{p^2 + b^2} + p) \right]$$

$$[p > 0, \quad b > 0] \quad \text{ET I 28(46)}$$

$$3.879 \quad \int_0^\infty \sin ax^p \frac{dx}{x} = \frac{\pi}{2p} \quad [a > 0, \quad p > 0] \quad \text{GW (334)(6)}$$

3.881

$$1. \quad \int_0^{\pi/2} x \sin(a \tan x) \, dx = \frac{\pi}{4} e^{-a} [C + \ln 2a - e^{2a} \text{Ei}(-2a)]$$

$$[a > 0] \quad \text{BI (205)(9)}$$

$$2. \quad \int_0^\infty \sin(a \tan x) \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a}) \quad [a > 0] \quad \text{BI (151)(6)}$$

$$3. \quad \int_0^\infty \sin(a \tan x) \cos x \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a}) \quad [a > 0] \quad \text{BI (151)(19)}$$

$$4. \quad \int_0^{\infty} \cos(a \tan x) \sin x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0] \quad \text{BI (151)(20)}$$

$$5. \quad \int_0^{\infty} \sin(a \tan x) \sin 2x \frac{dx}{x} = \frac{1+a}{2} \pi e^{-a} \quad [a > 0] \quad \text{BI (152)(11)}$$

$$6. \quad \int_0^{\infty} \cos(a \tan x) \sin^3 x \frac{dx}{x} = \frac{1-a}{4} \pi e^{-a} \quad [a > 0] \quad \text{BI (151)(23)}$$

$$7. \quad \int_0^{\infty} \sin(a \tan x) \tan \frac{x}{2} \cos^2 x \frac{dx}{x} = \frac{1+a}{4} \pi e^{-a} \quad [a > 0] \quad \text{BI (152)(13)}$$

$$8. \quad \int_0^{\pi/2} \cos(a \tan x) \frac{x dx}{\sin 2x} = -\frac{\pi}{4} \text{Ei}(-a) \quad [a > 0] \quad \text{BI (206)(15)}$$

$$9. \quad \int_0^{\pi/2} \sin(a \cot x) \frac{x dx}{\sin^2 x} = \frac{1-e^{-a}}{2a} \pi \quad [a > 0] \quad \text{LI (206)(14)}$$

$$10. \quad \int_0^{\pi/2} x \cos(a \tan x) \tan x dx = -\frac{\pi}{4} e^{-a} [C + \ln 2a + e^{2a} \text{Ei}(-2a)] \quad [a > 0] \quad \text{BI (205)(10)}$$

$$11. \quad \int_0^{\infty} \cos(a \tan x) \tan x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0] \quad \text{BI (151)(21)}$$

$$12. \quad \int_0^{\infty} \cos(a \tan x) \sin^2 x \tan x \frac{dx}{x} = \frac{1-a}{16} \pi e^{-a} \quad [a > 0] \quad \text{BI (152)(15)}$$

$$13. \quad \int_0^{\infty} \sin(a \tan x) \tan^2 x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0] \quad \text{BI (152)(9)}$$

$$14. \quad \int_0^{\infty} \cos(a \tan 2x) \tan x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0] \quad \text{BI (151)(22)}$$

$$15. \quad \int_0^{\infty} \sin(a \tan 2x) \cos^2 2x \tan x \frac{dx}{x} = \frac{1+a}{4} \pi e^{-a} \quad [a > 0] \quad \text{BI (152)(13)}$$

$$16. \quad \int_0^{\infty} \sin(a \tan 2x) \tan x \tan 2x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0] \quad \text{BI (152)(10)}$$

$$17. \quad \int_0^{\infty} \sin(a \tan 2x) \tan x \cot 2x \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a}) \quad [a > 0] \quad \text{BI (180)(6)}$$

3.882

$$1. \quad \int_0^{\infty} \sin(a \tan^2 x) \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\exp(-a \tanh b) - e^{-a}] \quad [a > 0, \quad b > 0] \quad \text{BI (160)(22)}$$

$$2. \quad \int_0^{\infty} \cos(a \tan^2 x) \cos x \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} [\cosh b \exp(-a \tanh b) - e^{-a} \sinh b] \quad [a > 0, \quad b > 0] \quad \text{BI (163)(3)}$$

$$3. \quad \int_0^{\infty} \cos(a \tan^2 x) \operatorname{cosec} 2x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2 \sinh 2b} \exp(-a \tanh b) \quad [a > 0, \quad b > 0] \quad \text{BI (191)(10)}$$

$$4. \quad \int_0^\infty \cos(a \tan^2 x) \tan x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2 \cosh b} [e^{-a} \cosh b - \exp(-a \tanh b) \sinh b]$$

$$[a > 0, \quad b > 0] \quad \text{BI (163)(4)}$$

$$5.^{11} \quad \int_0^\infty \cos(a \tan^2 x) \cot x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\coth b \exp(-a \tanh b) - e^{-a}]$$

$$[a > 0, \quad b > 0] \quad \text{BI (163)(5)}$$

$$6. \quad \int_0^\infty \cos(a \tan^2 x) \cot 2x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\coth 2b \exp(-a \tanh b) - e^{-a}]$$

$$[a > 0, \quad b > 0] \quad \text{BI (191)(11)}$$

3.883

$$1. \quad \int_0^1 \cos(a \ln x) \frac{dx}{(1+x)^2} = \frac{a\pi}{2 \sinh a\pi} \quad \text{BI (404)(4)}$$

$$2. \quad \int_0^1 x^{\mu-1} \sin(\beta \ln x) dx = -\frac{\beta}{\beta^2 + \mu^2} \quad [\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{ET I 319(19)}$$

$$3. \quad \int_0^1 x^{\mu-1} \cos(\beta \ln x) dx = \frac{\mu}{\beta^2 + \mu^2} \quad [\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{ET I 321(38)}$$

$$3.884^{11} \quad \int_{-\infty}^\infty \frac{\sin a \sqrt{|x|}}{x-b} \operatorname{sign} x dx = \pi [\exp(-a \sqrt{|-b|}) + \exp(-a \sqrt{|b|})]$$

$$[a > 0, \quad \operatorname{Im} b \neq 0] \quad \text{ET II 253(46)}$$

3.89–3.91 Trigonometric functions and exponentials**3.891**

$$1. \quad \int_0^{2\pi} e^{imx} \sin nx dx = 0 \quad [m \neq n; \text{ or } m = n = 0]$$

$$= \pi i \quad [m = n \neq 0]$$

$$2. \quad \int_0^{2\pi} e^{imx} \cos nx dx = 0 \quad [m \neq n]$$

$$= \pi \quad [m = n \neq 0]$$

$$= 2\pi \quad [m = n = 0]$$

3.892

$$1.^{11} \quad \int_0^\pi e^{i\beta x} \sin^{\nu-1} x dx = \frac{\pi e^{i\beta \frac{\pi}{2}}}{2^{\nu-1} \nu \operatorname{B}\left(\frac{\nu + \beta + 1}{2}, \frac{\nu - \beta + 1}{2}\right)}$$

$$[\operatorname{Re} \nu > -1] \quad \text{NH 158, EH I 12(29)}$$

$$2. \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^{\nu-1} x dx = \frac{\pi}{2^{\nu-1} \nu \operatorname{B}\left(\frac{\nu + \beta + 1}{2}, \frac{\nu - \beta + 1}{2}\right)}$$

$$[\operatorname{Re} \nu > -1] \quad \text{GW (335)(19)}$$

$$\begin{aligned}
3.^6 \quad \int_0^{\pi/2} e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x \, dx &= \frac{1}{2^{2\mu+2\nu+1}} \left\{ \exp \left[i\pi \left(\beta - \nu - \frac{1}{2} \right) \right] B(\beta - \mu - \nu, 2\nu + 1) \right. \\
&\quad \times F(-2\mu, \beta - \mu - \nu; 1 + \beta - \mu + \nu; -1) + \exp \left[i\pi \left(\mu + \frac{1}{2} \right) \right] \\
&\quad \times B(\beta - \mu - \nu, 2\mu + 1) F(-2\nu, \beta - \mu - \nu; 1 + \beta + \mu - \nu; -1) \left. \right\} \\
&\quad \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{EH I 80(6)}
\end{aligned}$$

$$4. \quad \int_0^{\pi} e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x \, dx = \frac{\pi \exp [i\pi(\beta - \nu)] F(-2\nu, \beta - \mu - \nu; 1 + \beta + \mu - \nu; -1)}{4^{\mu+\nu} (2\mu + 1) B(1 - \beta + \mu + \nu, 1 + \beta + \mu - \nu)}$$

EH I 80(8)

$$\begin{aligned}
5. \quad \int_0^{\pi/2} e^{i(\mu+\nu)x} \sin^{\mu-1} x \cos^{\nu-1} x \, dx &= e^{i\mu\frac{\pi}{2}} B(\mu, \nu) \\
&= \frac{1}{2^{\mu+\nu-1}} e^{i\mu\frac{\pi}{2}} \left\{ \frac{1}{\mu} F(1 - \nu, 1; \mu + 1; -1) + \frac{1}{\nu} F(1 - \mu, 1; \nu + 1; -1) \right\} \\
&\quad \left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right] \quad \text{EH I 80(7)}
\end{aligned}$$

3.893

$$1.^8 \quad \int_0^{\infty} e^{-px} \sin(qx + \lambda) \, dx = \frac{1}{p^2 + q^2} (q \cos \lambda + p \sin \lambda) \quad [\operatorname{Re} p > 0] \quad \text{BI (261)(3)}$$

$$2.^8 \quad \int_0^{\infty} e^{-px} \cos(qx + \lambda) \, dx = \frac{1}{p^2 + q^2} (p \cos \lambda - q \sin \lambda) \quad [\operatorname{Re} p > 0] \quad \text{BI (261)(4)}$$

$$3. \quad \int_0^{\infty} e^{-x \cos t} \cos(t - x \sin t) \, dx = 1 \quad \text{BI (261)(7)}$$

$$\begin{aligned}
4.^8 \quad \int_0^{\infty} \frac{e^{-\beta x} \sin ax}{\sin bx} \, dx &= \operatorname{Re} \left\{ \frac{1}{2bi} \left[\psi \left(\frac{a+b}{2b} - i \frac{\beta}{2b} \right) - \psi \left(\frac{b-a}{2b} - i \frac{\beta}{2b} \right) \right] \right\} \\
&\quad [\operatorname{Re} \beta > 0, \quad b \neq 0] \quad \text{GW (335)(15)}
\end{aligned}$$

$$5.^8 \quad \int_0^{\infty} \frac{e^{-2px} \sin[(2n+1)x]}{\sin x} \, dx = \frac{1}{2p} + \sum_{k=1}^n \frac{p}{p^2 + k^2} \quad [\operatorname{Re} p > 0] \quad \text{BI (267)(15)}$$

$$6.^8 \quad \int_0^{\infty} \frac{e^{-px} \sin 2nx}{\sin x} \, dx = 2p \sum_{k=0}^{n-1} \frac{1}{p^2 + (2k+1)^2} \quad [\operatorname{Re} p > 0] \quad \text{GW (335)(15c)}$$

$$\begin{aligned}
7. \quad \int_0^{\infty} e^{-px} \cos[(2n+1)x] \tan x \, dx &= \frac{2n+1}{p^2 + (2n+1)^2} + (-1)^n 2 \sum_{k=0}^{n-1} \frac{(-1)^k (2k+1)}{p^2 + (2k+1)^2} \\
&\quad [p > 0] \quad \text{LI (267)(16)}
\end{aligned}$$

$$\begin{aligned}
3.894 \quad \int_{-\pi}^{\pi} \left[\beta + \sqrt{\beta^2 - 1} \cos x \right]^{\nu} e^{inx} \, dx &= \frac{2\pi \Gamma(\nu+1) P_{\nu}^m(\beta)}{\Gamma(\nu+m+1)} \\
&\quad [\operatorname{Re} \beta > 0] \quad \text{ET I 157(15)}
\end{aligned}$$

3.895

$$\begin{aligned}
1. \quad \int_0^{\infty} e^{-\beta x} \sin^{2m} x \, dx &= \frac{(2m)!}{\beta(\beta^2 + 2^2)(\beta^2 + 4^2) \cdots [\beta^2 + (2m)^2]} \\
&\quad [\operatorname{Re} \beta > 0] \quad \text{FI II 615, WA 620a}
\end{aligned}$$

$$2.10 \quad \int_0^\pi e^{-px} \sin^{2m} x \, dx = \frac{(2m)! (1 - e^{-p\pi})}{p(p^2 + 2^2)(p^2 + 4^2) \cdots [p^2 + (2m)^2]} \quad \text{GW (335)(4a)}$$

$$3.10 \quad \int_0^{\pi/2} e^{-px} \sin^{2m} x \, dx = \frac{(2m)!}{p(p^2 + 2^2)(p^2 + 4^2) \cdots [p^2 + (2m)^2]} \times \left\{ 1 - e^{-\frac{p\pi}{2}} \left[1 + \frac{p^2}{2!} + \frac{p^2(p^2 + 2^2)}{4!} + \cdots + \frac{p^2(p^2 + 2^2) \cdots [p^2 + (2m-2)^2]}{(2m)!} \right] \right\} \quad \text{BI (270)(4)}$$

$$4. \quad \int_0^\infty e^{-\beta x} \sin^{2m+1} x \, dx = \frac{(2m+1)!}{(\beta^2 + 1^2)(\beta^2 + 3^2) \cdots [\beta^2 + (2m+1)^2]} \quad [\operatorname{Re} \beta > 0] \quad \text{FI II 615, WA 620a}$$

$$5.10 \quad \int_0^\pi e^{-px} \sin^{2m+1} x \, dx = \frac{(2m+1)! (1 + e^{-p\pi})}{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m+1)^2]} \quad \text{GW (335)(4b)}$$

$$6.8 \quad \int_0^{\pi/2} e^{-px} \sin^{2m+1} x \, dx = \frac{(2m+1)!}{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m+1)^2]} \times \left\{ 1 - pe^{-\frac{p\pi}{2}} \left[1 + \frac{p^2 + 1^2}{3!} + \cdots + \frac{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m-1)^2]}{(2m+1)!} \right] \right\} \quad \text{BI (270)(5)}$$

$$7. \quad \int_0^\infty e^{-px} \cos^{2m} x \, dx = \frac{(2m)!}{p(p^2 + 2^2) \cdots [p^2 + (2m)^2]} \times \left\{ 1 + \frac{p^2}{2!} + \frac{p^2(p^2 + 2^2)}{4!} + \cdots + \frac{p^2(p^2 + 2^2) \cdots [p^2 + (2m-2)^2]}{(2m)!} \right\} \quad [p > 0] \quad \text{BI (262)(3)}$$

$$8.10 \quad \int_0^{\pi/2} e^{-px} \cos^{2m} x \, dx = \frac{(2m)!}{p(p^2 + 2^2) \cdots [p^2 + (2m)^2]} \times \left\{ -e^{-p\frac{\pi}{2}} + 1 + \frac{p^2}{2!} + \frac{p^2(p^2 + 2^2)}{4!} + \cdots + \frac{p^2(p^2 + 2^2) \cdots [p^2 + (2m-2)^2]}{(2m)!} \right\} \quad \text{BI (270)(6)}$$

$$9.7 \quad \int_0^\infty e^{-px} \cos^{2m+1} x \, dx = \frac{(2m+1)!p}{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m+1)^2]} \times \left\{ 1 + \frac{p^2 + 1^2}{3!} + \frac{(p^2 + 1^2)(p^2 + 3^2)}{5!} + \cdots + \frac{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m-1)^2]}{(2m+1)!} \right\} \quad [p > 0] \quad \text{BI (262)(4)}$$

$$\begin{aligned}
 10.^{11} \quad & \int_0^{\pi/2} e^{-px} \cos^{2m+1} x \, dx \\
 &= \frac{(2m+1)!}{(p^2+1^2)(p^2+3^2)\cdots[p^2+(2m+1)^2]} \\
 &\quad \times \left\{ e^{-p\frac{\pi}{2}} + p \left[1 + \frac{p^2+1^2}{3!} + \cdots + \frac{(p^2+1)(p^2+3^2)\cdots[p^2+(2m-1)^2]}{(2m+1)!} \right] \right\} \\
 &\hspace{15em} \text{BI (270)(7)}
 \end{aligned}$$

$$\begin{aligned}
 11.^8 \quad & \int_0^\infty e^{-\beta x} \sin^n ax \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{2^{-n-2}}{a(n+1)} e^{\frac{1}{4}(1\mp 1+2n)\pi i} \\
 &\quad \times \left\{ \left(\frac{b+na+i\beta}{2a} \right)^{-1} \pm (-1)^n \left(\frac{b+na-i\beta}{2a} \right)^{-1} \right\} \\
 &\hspace{15em} [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0]
 \end{aligned}$$

$$12. \quad \int_0^\infty e^{-ax} \cos^2 mx \, dx = \frac{a^2 + 2m^2}{a(a^2 + 4m^2)} \quad \text{DW61 (861.06)}$$

$$13. \quad \int_0^\infty e^{-ax} \cos mx \cos nx \, dx = \frac{a(a^2 + m^2 + n^2)}{(a^2 + (m-n)^2)(a^2 + (m+n)^2)} \quad \text{DW61 (861.15)}$$

$$14. \quad \int_0^\infty e^{-ax} \sin mx \cos nx \, dx = \frac{m(a^2 + m^2 - n^2)}{(a^2 + (m-n)^2)(a^2 + (m+n)^2)} \quad \text{DW61 (861.14)}$$

$$15. \quad \int_0^\infty e^{-ax} \sin^2 mx \, dx = \frac{2m}{a(a^2 + 4m^2)} \quad [a > 0] \quad \text{DW61 (861.10)}$$

$$16. \quad \int_0^\infty e^{-ax} \sin mx \sin nx \, dx = \frac{2amn}{[a^2 + (m-n)^2][a^2 + (m+n)^2]} \quad \text{DW61 (861.13)}$$

3.896

$$1. \quad \int_{-\infty}^\infty e^{-q^2 x^2} \sin[p(x+\lambda)] \, dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \sin p\lambda \quad \text{BI (269)(2)}$$

$$2. \quad \int_{-\infty}^\infty e^{-q^2 x^2} \cos[p(x+\lambda)] \, dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda \quad \text{BI (269)(3)}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty e^{-ax^2} \sin bx \, dx = \frac{b}{2a} \exp\left(-\frac{b^2}{4a}\right) {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{b^2}{4a}\right) \\
 &= \frac{b}{2a} {}_1F_1\left(1; \frac{3}{2}; -\frac{b^2}{4a}\right) \quad \text{ET I 73(18)} \\
 &= \frac{b}{2a} \sum_{k=1}^\infty \frac{1}{(2k-1)!!} \left(-\frac{b^2}{2a}\right)^{k-1} \quad [a > 0] \quad \text{FI II 720}
 \end{aligned}$$

$$4. \quad \int_0^\infty e^{-\beta x^2} \cos bx \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{b^2}{4\beta}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{BI (263)(2)}$$

3.897

$$1.^8 \quad \int_0^\infty e^{-\beta x^2 - \gamma x} \sin bx \, dx = -\frac{i}{4} \sqrt{\frac{\pi}{\beta}} \left\{ \exp \frac{(\gamma - ib)^2}{4\beta} \left[1 - \Phi \left(\frac{\gamma - ib}{2\sqrt{\beta}} \right) \right] \right. \\ \left. - \exp \frac{(\gamma + ib)^2}{4\beta} \left[1 - \Phi \left(\frac{\gamma + ib}{2\sqrt{\beta}} \right) \right] \right\} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 74(27)}$$

$$2. \quad \int_0^\infty e^{-\beta x^2 - \gamma x} \cos bx \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ \exp \frac{(\gamma - ib)^2}{4\beta} \left[1 - \Phi \left(\frac{\gamma - ib}{2\sqrt{\beta}} \right) \right] \right. \\ \left. + \exp \frac{(\gamma + ib)^2}{4\beta} \left[1 - \Phi \left(\frac{\gamma + ib}{2\sqrt{\beta}} \right) \right] \right\} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 15(16)}$$

3.898

$$1. \quad \int_0^\infty e^{-\beta x^2} \sin ax \sin bx \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}} \right\} \quad [\operatorname{Re} \beta > 0] \quad \text{BI (263)(4)}$$

$$2. \quad \int_0^\infty e^{-\beta x^2} \cos ax \cos bx \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}} \right\} \quad [\operatorname{Re} \beta > 0] \quad \text{BI (263)(5)}$$

$$3.^8 \quad \int_0^\infty e^{-px^2} \sin^2 ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \left(1 - e^{-\frac{a^2}{p}} \right) \quad [\operatorname{Re} p > 0] \quad \text{BI (263)(6)}$$

3.899

$$1.^7 \quad \int_0^\infty \frac{e^{p^2 x^2} \sin[(2n+1)x]}{\sin x} \, dx = \frac{\sqrt{\pi}}{p} \left[\frac{1}{2} + \sum_{k=1}^n e^{-\left(\frac{k}{p}\right)^2} \right] \quad [p > 0] \quad \text{BI (267)(17)}$$

$$2. \quad \int_0^\infty \frac{e^{-p^2 x^2} \cos[(4n+1)x]}{\cos x} \, dx = \frac{\sqrt{\pi}}{p} \left[\frac{1}{2} + \sum_{k=0}^{2n} (-1)^k e^{-\left(\frac{k}{p}\right)^2} \right] \quad [p > 0] \quad \text{BI (267)(18)}$$

$$3. \quad \int_0^\infty \frac{e^{-px^2} \, dx}{1 - 2a \cos x + a^2} = \frac{\sqrt{\frac{\pi}{p}}}{1 - a^2} \left\{ \frac{1}{2} + \sum_{k=1}^\infty a^k \exp \left(-\frac{k^2}{4p} \right) \right\} \quad [a^2 < 1, \quad p > 0] \quad \text{EI (266)(1)} \\ = \frac{\sqrt{\frac{\pi}{p}}}{a^2 - 1} \left\{ \frac{1}{2} + \sum_{k=1}^\infty a^{-k} \exp \left(-\frac{k^2}{4p} \right) \right\} \quad [a^2 > 1, \quad p > 0] \quad \text{LI (266)(1)}$$

3.911

$$1. \quad \int_0^\infty \frac{\sin ax}{e^{\beta x} + 1} \, dx = \frac{1}{2a} - \frac{\pi}{2\beta \sinh \frac{a\pi}{\beta}} \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{BI (264)(1)}$$

$$2. \quad \int_0^\infty \frac{\sin ax}{e^{\beta x} - 1} \, dx = \frac{\pi}{2\beta} \coth \left(\frac{\pi a}{\beta} \right) - \frac{1}{2a} \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{BI (264)(2), WH}$$

$$3.^{11} \quad \int_0^\infty \frac{\sin ax}{e^x - 1} e^{x/2} \, dx = \frac{1}{2} \pi \tanh(a\pi) \quad [a > 0] \quad \text{ET I 73(13)}$$

$$4. \quad \int_0^\infty \frac{\sin ax}{1 - e^{-x}} e^{-nx} dx = \frac{\pi}{2} - \frac{1}{2a} + \frac{\pi}{e^{2\pi a} - 1} - \sum_{k=1}^{n-1} \frac{a}{a^2 + k^2} \quad [a > 0] \quad \text{BI (264)(8)}$$

$$5. \quad \int_0^\infty \frac{\sin ax}{e^{\beta x} - e^{\gamma x}} dx = \frac{1}{2i(\beta - \gamma)} \left[\psi \left(\frac{\beta + ia}{\beta - \gamma} \right) - \psi \left(\frac{\beta - ia}{\beta - \gamma} \right) \right] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{GW (335)(8)}$$

$$6. \quad \int_0^\infty \frac{\sin ax dx}{e^{\beta x} (e^{-x} - 1)} = \frac{i}{2} [\psi(\beta + ia) - \psi(\beta - ia)] \quad [\operatorname{Re} \beta > -1] \quad \text{ET 73(15)}$$

3.912

$$1. \quad \int_0^\infty e^{-\beta x} (1 - e^{-\gamma x})^{\nu-1} \sin ax dx = -\frac{i}{2\gamma} \left[\operatorname{B} \left(\nu, \frac{\beta - ia}{\gamma} \right) - \operatorname{B} \left(\nu, \frac{\beta + ia}{\gamma} \right) \right] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re} \nu > 0, \quad a > 0] \quad \text{ET I 73(17)}$$

$$2. \quad \int_0^\infty e^{-\beta x} (1 - e^{-\gamma x})^{\nu-1} \cos ax dx = \frac{1}{2\gamma} \left[\operatorname{B} \left(\nu, \frac{\beta - ia}{\gamma} \right) + \operatorname{B} \left(\nu, \frac{\beta + ia}{\gamma} \right) \right] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re} \nu > 0, \quad a > 0] \quad \text{ET I 15(10)}$$

3.913

$$1. \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^\nu x (\beta^2 e^{ix} + \nu^2 e^{-ix})^\mu dx = \frac{\pi {}_2F_1 \left(-\mu, \frac{\beta}{2} - \frac{\nu}{2} - \frac{\mu}{2}; 1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}; \frac{\beta^2}{\nu^2} \right)}{2^\nu (\nu + 1) \operatorname{B} \left(1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}, 1 - \frac{\beta}{2} + \frac{\nu}{2} + \frac{\mu}{2} \right)} \quad [\operatorname{Re} \nu > -1, \quad |\nu| > |\beta|] \quad \text{EH I 81(11)a}$$

$$2.^{11} \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-iux} \cos^\mu x (a^2 e^{ix} + b^2 e^{-ix})^\nu dx = \frac{\pi b^{2\nu} {}_2F_1 \left(-\nu, -\frac{u+\mu+\nu}{2}; 1 + \frac{\mu-\nu-u}{2}; \frac{a^2}{b^2} \right)}{2^\mu (\mu + 1) \operatorname{B} \left(1 - \frac{u+\nu-\mu}{2}, 1 + \frac{u+\mu+\nu}{2} \right)} \quad [\text{for } a^2 < b^2]$$

$$= \frac{\pi a^{2\nu} {}_2F_1 \left(-\nu, \frac{u-\mu-\nu}{2}; 1 + \frac{\mu-\nu+u}{2}; \frac{b^2}{a^2} \right)}{2^\mu (\mu + 1) \operatorname{B} \left(1 + \frac{u+\mu-\nu}{2}, 1 + \frac{\mu+\nu-u}{2} \right)} \quad [\text{for } b^2 < a^2]$$

$$[\operatorname{Re} \mu > -1] \quad \text{ET I 122(31)a}$$

3.914

$$1. \quad \int_0^\infty e^{-\beta \sqrt{\gamma^2 + x^2}} \cos bx dx = \frac{\beta \gamma}{\sqrt{\beta^2 + b^2}} K_1 \left(\gamma \sqrt{\beta^2 + b^2} \right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{ET I 16(26)}$$

$$2. \quad \int_0^\infty \sqrt{\gamma^2 + x^2} e^{-\beta \sqrt{\gamma^2 + x^2}} \cos bx dx = \frac{\beta^2 \gamma^2}{A^2} K_0(\gamma A) + \left(\frac{2\beta^2 \gamma}{A^3} - \frac{\gamma}{A} \right) K_1(\gamma A) \quad [A = \sqrt{\beta^2 + b^2}]$$

3.
$$\int_0^\infty (\gamma^2 + x^2) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx \, dx$$

$$= \left(-\frac{3\beta\gamma^2}{A^2} + \frac{4\beta^3\gamma^2}{A^4} \right) K_0(\gamma A) + \left(-\frac{6\beta\gamma}{A^3} + \frac{8\beta^3\gamma}{A^5} + \frac{\beta^3\gamma^3}{A^3} \right) K_1(\gamma A)$$

$$\left[A = \sqrt{\beta^2 + b^2} \right]$$
4.
$$\int_0^\infty \frac{e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \cos bx \, dx = K_0 \left(\gamma \sqrt{\beta^2 + b^2} \right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad b > 0]$$

ET I 16(27)
5.
$$\int_0^\infty \left(\frac{1}{\beta(\gamma^2 + x^2)^{3/2}} + \frac{1}{\gamma^2 + x^2} \right) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx \, dx = \frac{1}{\beta\gamma} \sqrt{\beta^2 + b^2} K_1 \left(\gamma \sqrt{\beta^2 + b^2} \right)$$

(6.726(4))
6.
$$\int_0^\infty x e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx \, dx = \frac{b\beta\gamma^2}{\beta^2 + b^2} K_2 \left(\gamma \sqrt{\beta^2 + b^2} \right)$$

ET I 175(35)
7.
$$\int_0^\infty x \sqrt{\gamma^2 + x^2} e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx \, dx$$

$$= \left(-\frac{b\gamma^2}{A^2} + \frac{4b\beta^2\gamma^2}{A^4} \right) K_0(\gamma A) + \left(-\frac{2b\gamma}{A^3} + \frac{8b\beta^2\gamma}{A^5} + \frac{b\beta^2\gamma^3}{A^3} \right) K_1(\gamma A)$$

$$\left[A = \sqrt{\beta^2 + b^2} \right]$$
8.
$$\int_0^\infty (\gamma^2 + x^2) e^{-\beta\sqrt{\gamma^2+x^2}} x \sin bx \, dx = \left(-\frac{12b\beta\gamma^2}{A^4} + \frac{24b\beta^3\gamma^2}{A^6} + \frac{b\beta^3\gamma^4}{A^4} \right) K_0(\gamma A)$$

$$+ \left(-\frac{24b\beta\gamma}{A^5} + \frac{48b\beta^3\gamma}{A^7} - \frac{3b\beta\gamma^3}{A^3} + \frac{8b\beta^3\gamma^3}{A^5} \right) K_1(\gamma A)$$

$$\left[A = \sqrt{\beta^2 + b^2} \right]$$
9.
$$\int_0^\infty \frac{x e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \sin bx \, dx = \frac{\gamma b}{\sqrt{\beta^2 + b^2}} K_1 \left(\gamma \sqrt{\beta^2 + b^2} \right)$$

ET I 75(36)
10.
$$\int_0^\infty \left(\frac{1}{\beta(\gamma^2 + x^2)^{3/2}} + \frac{1}{\gamma^2 + x^2} \right) e^{-\beta\sqrt{\gamma^2+x^2}} x \sin bx \, dx = \frac{b}{\beta} K_0 \left(\gamma \sqrt{\beta^2 + b^2} \right)$$

(6.726(3))

3.915

1.
$$\int_0^\pi e^{a \cos x} \sin x \, dx = \frac{2}{a} \sinh a$$

GW (337)(15c)
2.
$$\int_0^\pi e^{i\beta \cos x} \cos nx \, dx = i^n \pi J_n(\beta)$$

EH II 81(2)
- 3.³
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta \sin x} \cos^{2\nu} x \, dx = \sqrt{\pi} \left(\frac{2}{\beta} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) J_\nu(\beta) \quad [\operatorname{Re} \nu > -\frac{1}{2}]$$

EH II 81(6)
4.
$$\int_0^\pi e^{\pm i\beta \cos x} \sin^{2\nu} x \, dx = \sqrt{\pi} \left(\frac{2}{\beta} \right)^\nu \Gamma \left(\nu + \frac{1}{2} \right) I_\nu(\beta) \quad [\operatorname{Re} \nu > -\frac{1}{2}]$$

GW (337)(15b)

$$5. \quad \int_0^\pi e^{i\beta \cos x} \sin^{2\nu} x \, dx = \sqrt{\pi} \left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(\beta) \quad [\operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{WA 34(2), WA 60(6)}$$

3.916

$$1. \quad \int_0^{\pi/2} e^{-p^2 \tan x} \frac{\sin \frac{x}{2} \sqrt{\cos x}}{\sin 2x} \, dx = \left[C(p) - \frac{1}{2}\right]^2 + \left[S(p) - \frac{1}{2}\right]^2 \quad \text{NT 33(18)a}$$

$$2. \quad \int_0^{\pi/2} \frac{\exp(-p \tan x) \, dx}{\sin 2x + a \cos 2x + a} = -\frac{1}{2} e^{ap} \operatorname{Ei}(-ap) \quad [p > 0], \quad (\text{cf. } \mathbf{3552} \text{ 4 and 6})$$

BI (273)(11)

$$3. \quad \int_0^{\pi/2} \frac{\exp(-p \cot x) \, dx}{\sin 2x + a \cos 2x - a} = -\frac{1}{2} e^{-ap} \operatorname{Ei}(ap) \quad [p > 0], \quad (\text{cf. } \mathbf{3.552} \text{ 4 and 6})$$

BI (273)(12)

$$4. \quad \int_0^{\pi/2} \frac{\exp(-p \tan x) \sin 2x \, dx}{(1-a^2) - 2a^2 \cos 2x - (1+a^2) \cos^2 2x} = -\frac{1}{4} [e^{-ap} \operatorname{Ei}(ap) + e^{ap} \operatorname{Ei}(-ap)]$$

[p > 0] BI (273)(13)

$$5. \quad \int_0^{\pi/2} \frac{\exp(-p \cot x) \sin 2x \, dx}{(1-a^2) + 2a^2 \cos 2x - (1+a^2) \cos^2 2x} = -\frac{1}{4} [e^{-ap} \operatorname{Ei}(ap) + e^{ap} \operatorname{Ei}(-ap)]$$

[p > 0] BI (273)(14)

3.917

$$1. \quad \int_0^{\pi/2} e^{-2\beta \cot x} \cos^{\nu-1/2} x \sin^{-(\nu+1)} x \sin \left[\beta - \left(\nu - \frac{1}{2}\right) x\right] \, dx = \frac{\sqrt{\pi}}{2(2\beta)^\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(\beta)$$

[Re ν > -½] WA 186(7)

$$2. \quad \int_0^{\pi/2} e^{-2\beta \cot x} \cos^{\nu-1/2} x \sin^{-(\nu+1)} x \cos \left[\beta - \left(\nu - \frac{1}{2}\right) x\right] \, dx = \frac{\sqrt{\pi}}{2(2\beta)^\nu} \Gamma\left(\nu + \frac{1}{2}\right) Y_\nu(\beta)$$

[Re ν > -½] WA 186(8)

3.918

$$1. \quad \int_0^{\pi/2} \frac{\cos^\mu x}{\sin^{2\mu+2} x} e^{i\gamma(\beta-\mu x)-2\beta \cot x} \, dx = \frac{i\gamma}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) H_{\mu+\frac{1}{2}}^{(\varepsilon)}(\beta)$$

[ε = 1, 2, γ = (-1)^{ε+1}, Re β > 0, Re μ > -1] GW (337)(16)

$$2. \quad \int_0^{\pi/2} \frac{\cos^\mu x \sin(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \cot x} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(\beta)$$

[Re β > 0, Re μ > -1] WH

$$3. \quad \int_0^{\pi/2} \frac{\cos^\mu x \cos(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \cot x} \, dx = -\frac{1}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) Y_{\mu+\frac{1}{2}}(\beta)$$

[Re β > 0, Re μ > -1] GW (337)(17b)

3.919

1.
$$\int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \cdot \frac{dx}{\exp(2\pi \cot x) - 1} = (-1)^{n-1} \frac{2n-1}{4(2n+1)} \quad \text{BI (275)(6), LI (275)(6)}$$
2.
$$\int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \frac{dx}{\exp(\pi \cot x) - 1} = (-1)^{n-1} \frac{n}{2n+1} \quad \text{BI (275)(7), LI (275)(7)}$$

3.92 Trigonometric functions of more complicated arguments combined with exponentials

3.921⁶

1.
$$\int_0^\infty e^{-\gamma x} \cos ax^2 (\cos \gamma x - \sin \gamma x) dx = \sqrt{\frac{\pi}{8a}} \exp\left(-\frac{\gamma^2}{2a}\right) \quad [a > 0, \quad \operatorname{Re} \gamma \geq |\operatorname{Im} \gamma|] \quad \text{ET I 26(28)}$$
- 2.¹⁰
$$\int_0^{\pi/4} \prod_{n=1}^\infty \exp\left[-\frac{1}{n} \tan^{2n} x\right] = \frac{\pi}{2} - 1$$
- 3.¹⁰
$$\int_0^{\pi/2} \exp\left[-\sum_{n=1}^\infty \frac{1}{n} \sin^{2n} x\right] = \int_0^{\pi/2} \exp\left[-\sum_{n=1}^\infty \frac{1}{n} \cos^{2n} x\right] = \frac{\pi}{4}$$

3.922

1.
$$\begin{aligned} \int_0^\infty e^{-\beta x^2} \sin ax^2 dx &= \frac{1}{2} \int_{-\infty}^\infty e^{-\beta x^2} \sin ax^2 dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} - \beta}{\beta^2 + a^2}} \\ &= \frac{\sqrt{\pi}}{2^4 \sqrt{\beta^2 + a^2}} \sin\left(\frac{1}{2} \arctan \frac{a}{\beta}\right) \end{aligned} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{FI II 750, BI (263)(8)}$$
2.
$$\begin{aligned} \int_0^\infty e^{-\beta x^2} \cos ax^2 dx &= \frac{1}{2} \int_{-\infty}^\infty e^{-\beta x^2} \cos ax^2 dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} + \beta}{\beta^2 + a^2}} \\ &= \frac{\sqrt{\pi}}{2^4 \sqrt{\beta^2 + a^2}} \cos\left(\frac{1}{2} \arctan \frac{a}{\beta}\right) \end{aligned} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{FI II 750, BI (263)(9)}$$

[In formulas **3.922** 3 and 4, $a > 0$, $b > 0$, $\operatorname{Re} \beta > 0$, and

$$A = \frac{b^2}{4(a^2 + \beta^2)}, \quad B = \sqrt{\frac{1}{2} \left(\sqrt{\beta^2 + a^2} + \beta \right)}, \quad C = \sqrt{\frac{1}{2} \left(\sqrt{\beta^2 + a^2} - \beta \right)}.$$

If a is complex, then $\operatorname{Re} \beta > |\operatorname{Im} a|$.

3.
$$\begin{aligned} \int_0^\infty e^{-\beta x^2} \sin ax^2 \cos bx dx &= -\frac{1}{2} \sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} (B \sin Aa - C \cos Aa) \\ &= \frac{\sqrt{\pi}}{2^4 \sqrt{\beta^2 + a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2 + a^2)}\right) \sin\left\{\frac{1}{2} \arctan \frac{a}{\beta} - \frac{ab^2}{4(\beta^2 + a^2)}\right\} \end{aligned} \quad \text{LI (263)(10), GW (337)(5)}$$

$$\begin{aligned}
 4. \quad \int_0^\infty e^{-\beta x^2} \cos ax^2 \cos bx \, dx &= \frac{1}{2} \sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} (B \cos Aa + C \sin Aa) \\
 &= \frac{\sqrt{\pi}}{2 \sqrt[4]{\beta^2 + a^2}} \exp \left(-\frac{\beta b^2}{4(\beta^2 + a^2)} \right) \cos \left\{ \frac{1}{2} \arctan \frac{a}{\beta} - \frac{ab^2}{4(\beta^2 + a^2)} \right\} \\
 &\quad \text{LI (263)(11), GW (337)(5)}
 \end{aligned}$$

3.923

$$\begin{aligned}
 1. \quad \int_{-\infty}^\infty \exp [-(ax^2 + 2bx + c)] \sin (px^2 + 2qx + r) \, dx \\
 &= \frac{\sqrt{\pi}}{\sqrt[4]{a^2 + p^2}} \exp \frac{a(b^2 - ac) - (aq^2 - 2bpq + cp^2)}{a^2 + p^2} \\
 &\quad \times \sin \left\{ \frac{1}{2} \arctan \frac{p}{a} - \frac{p(q^2 - pr) - (b^2p - 2abq + a^2r)}{a^2 + p^2} \right\} \\
 &\quad [a > 0] \quad \text{GW (337)(3), BI (296)(6)} \\
 2. \quad \int_{-\infty}^\infty \exp [-(ax^2 + 2bx + c)] \cos (px^2 + 2qx + r) \, dx \\
 &= \frac{\sqrt{\pi}}{\sqrt[4]{a^2 + p^2}} \exp \frac{a(b^2 - ac) - (aq^2 - 2bpq + cp^2)}{a^2 + p^2} \\
 &\quad \times \cos \left\{ \frac{1}{2} \arctan \frac{p}{a} - \frac{p(q^2 - pr) - (b^2p - 2abq + a^2r)}{a^2 + p^2} \right\} \\
 &\quad [a > 0] \quad \text{GW (337)(3), BI (269)(7)}
 \end{aligned}$$

3.924

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-\beta x^4} \sin bx^2 \, dx &= \frac{\pi}{4} \sqrt{\frac{b}{2\beta}} \exp \left(-\frac{b^2}{8\beta} \right) I_{\frac{1}{4}} \left(\frac{b^2}{8\beta} \right) \\
 &\quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET 73(22)} \\
 2. \quad \int_0^\infty e^{-\beta x^4} \cos bx^2 \, dx &= \frac{\pi}{4} \sqrt{\frac{b}{2\beta}} \exp \left(-\frac{b^2}{8\beta} \right) I_{-\frac{1}{4}} \left(\frac{b^2}{8\beta} \right) \\
 &\quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET I 15(12)}
 \end{aligned}$$

3.925

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-\frac{p^2}{x^2}} \sin 2a^2 x^2 \, dx &= \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{p^2}{x^2}} \sin 2a^2 x^2 \, dx = \frac{\sqrt{\pi}}{4a} e^{-2ap} (\cos 2ap + \sin 2ap) \\
 &\quad [a > 0, \quad b > 0] \quad \text{BI (268)(12)} \\
 2. \quad \int_0^\infty e^{-\frac{p^2}{x^2}} \cos 2a^2 x^2 \, dx &= \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{p^2}{x^2}} \cos 2a^2 x^2 \, dx = \frac{\sqrt{\pi}}{4a} e^{-2ap} (\cos 2ap - \sin 2ap) \\
 &\quad [a > 0, \quad b > 0] \quad \text{BI (268)(13)}
 \end{aligned}$$

3.926 Notation:

$$u = \sqrt{\frac{\sqrt{a^2 + \beta^2} + \beta}{2}}, \quad v = \sqrt{\frac{\sqrt{a^2 + \beta^2} - \beta}{2}}$$

$$1. \quad \int_0^\infty e^{-(\beta x^2 + \frac{\gamma}{x^2})} \sin ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{a^2 + \beta^2}} e^{-2u\sqrt{\gamma}} [v \cos(2v\sqrt{\gamma}) + u \sin(2v\sqrt{\gamma})] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{BI (268)(14)}$$

$$2. \quad \int_0^\infty e^{-(\beta x^2 + \frac{\gamma}{x^2})} \cos ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{a^2 + \beta^2}} e^{-2u\sqrt{\gamma}} [u \cos(2v\sqrt{\gamma}) - v \sin(2v\sqrt{\gamma})] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{BI (268)(15)}$$

$$\mathbf{3.927} \quad \int_0^\infty e^{-\frac{p}{x}} \sin^2 \frac{a}{x} dx = a \arctan \frac{2a}{p} + \frac{p}{4} \ln \frac{p^2}{p^2 + 4a^2} \quad [a > 0, \quad p > 0] \quad \text{LI (268)(4)}$$

3.928

$$1. \quad \int_0^\infty \exp \left[- \left(p^2 x^2 + \frac{q^2}{x^2} \right) \right] \sin \left(a^2 x^2 + \frac{b^2}{x^2} \right) dx = \frac{\sqrt{\pi}}{2r} e^{-2rs \cos(A+B)} \sin \{ A + 2rs \sin(A+B) \} \\ \text{BI (268)(22)}$$

$$2. \quad \int_0^\infty \exp \left[- \left(p^2 x^2 + \frac{q^2}{x^2} \right) \right] \cos \left(a^2 x^2 + \frac{b^2}{x^2} \right) dx = \frac{\sqrt{\pi}}{2r} e^{-2rs \cos(A+B)} \cos \{ A + 2rs \sin(A+B) \} \\ \text{BI (268)(23)}$$

$$\mathbf{3.929} \quad \int_0^\infty \left[e^{-x} \cos(p\sqrt{x}) + p e^{-x^2} \sin px \right] dx = 1 \quad \text{LI (268)(3)}$$

Notation: For the formulas in **3.928**: $a^2 + p^2 > 0$, $r = \sqrt[4]{a^4 + p^4}$, $s = \sqrt[4]{b^4 + q^4}$, $A = \frac{1}{2} \arctan \frac{a^2}{p^2}$, and $B = \frac{1}{2} \arctan \frac{b^2}{q^2}$.

3.93 Trigonometric and exponential functions of trigonometric functions**3.931**

$$1. \quad \int_0^{\pi/2} e^{-p \cos x} \sin(p \sin x) dx = \operatorname{Ei}(-p) - \operatorname{ci}(p) \quad \text{NT 13(27)}$$

$$2. \quad \int_0^\pi e^{-p \cos x} \sin(p \sin x) dx = - \int_{-\pi}^0 e^{-p \cos x} \sin(p \sin x) dx = -2 \operatorname{shi}(p) \quad \text{GW (337)(11b)}$$

$$3. \quad \int_0^{\pi/2} e^{-p \cos x} \cos(p \sin x) dx = -\operatorname{si}(p) \quad \text{NT 13(26)}$$

$$4. \quad \int_0^{\pi/2} e^{-p \cos x} \cos(p \sin x) dx = \frac{1}{2} \int_0^{2\pi} e^{-p \cos x} \cos(p \sin x) dx = \pi \quad \text{GW (337)(11a)}$$

3.932

$$1. \quad \int_0^\pi e^{p \cos x} \sin(p \sin x) \sin mx dx = \frac{1}{2} \int_0^{2\pi} e^{p \cos x} \sin(p \sin x) \sin mx dx = \frac{\pi}{2} \cdot \frac{p^m}{m!} \\ \text{BI (277)(7), GW (337)(13a)}$$

$$2. \quad \int_0^\pi e^{p \cos x} \cos(p \sin x) \cos mx \, dx = \frac{1}{2} \int_0^{2\pi} e^{p \cos x} \cos(p \sin x) \cos mx \, dx = \frac{\pi}{2} \cdot \frac{p^m}{m!}$$

BI (277)(8), GW (337)(13b)

$$\mathbf{3.933} \quad \int_0^\pi e^{p \cos x} \sin(p \sin x) \operatorname{cosec} x \, dx = \pi \sinh p \quad \text{BI (278)(1)}$$

3.934

$$1. \quad \int_0^\pi e^{p \cos x} \sin(p \sin x) \tan \frac{x}{2} \, dx = \pi (1 - e^p) \quad \text{BI (271)(8)}$$

$$2. \quad \int_0^\pi e^{p \cos x} \sin(p \sin x) \cot \frac{x}{2} \, dx = \pi (e^p - 1) \quad \text{BI (272)(5)}$$

$$\mathbf{3.935} \quad \int_0^\pi e^{p \cos x} \cos(p \sin x) \frac{\sin 2nx}{\sin x} \, dx = \pi \sum_{k=0}^{n-1} \frac{p^{2k+1}}{(2k+1)!} \quad [p > 0] \quad \text{LI (278)(3)}$$

3.936

$$1. \quad \int_0^{2\pi} e^{p \cos x} \cos(p \sin x - mx) \, dx = 2 \int_0^\pi e^{p \cos x} \cos(p \sin x - mx) \, dx = \frac{2\pi p^m}{m!}$$

BI (277)(9), GW (337)(14a)

$$2. \quad \int_0^{2\pi} e^{p \sin x} \sin(p \cos x + mx) \, dx = \frac{2\pi p^m}{m!} \sin \frac{m\pi}{2} \quad [p > 0] \quad \text{GW (337)(14b)}$$

$$3. \quad \int_0^{2\pi} e^{p \sin x} \cos(p \cos x + mx) \, dx = \frac{2\pi p^m}{m!} \cos \frac{m\pi}{2} \quad [p > 0] \quad \text{GW (337)(14b)}$$

$$4. \quad \int_0^{2\pi} e^{\cos x} \sin(mx - \sin x) \, dx = 0 \quad \text{WH}$$

$$5. \quad \int_0^\pi e^{\beta \cos x} \cos(ax + \beta \sin x) \, dx = \beta^{-a} \sin(a\pi) \gamma(a, \beta) \quad \text{EH II 137(2)}$$

3.937 Notation: In formulas **3.937** 1 and 2, $(b-p)^2 + (a+q)^2 > 0$, $m = 0, 1, 2, \dots$, $A = p^2 - q^2 + a^2 - b^2$, $B = 2(pq + ab)$, $C = p^2 + q^2 - a^2 - b^2$, and $D = 2(ap + bq)$.

$$1.^{11} \quad \int_0^{2\pi} \exp(p \cos x + q \sin x) \sin(a \cos x + b \sin x - mx) \, dx$$

$$= i\pi [(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} \left\{ (A + iB)^{m/2} I_m(\sqrt{C - iD}) - (A - iB)^{m/2} I_m(\sqrt{C + iD}) \right\}$$

GW (337)(9b)

$$2. \quad \int_0^{2\pi} \exp(p \cos x + q \sin x) \cos(a \cos x + b \sin x - mx) \, dx$$

$$= \pi [(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} \left\{ (A + iB)^{\frac{m}{2}} I_m(\sqrt{C - iD}) + (A - iB)^{\frac{m}{2}} I_m(\sqrt{C + iD}) \right\}$$

GW (337)(9a)

$$3. \quad \int_0^{2\pi} \exp(p \cos x + q \sin x) \sin(q \cos x - p \sin x + mx) \, dx = \frac{2\pi}{m!} (p^2 + q^2)^{\frac{m}{2}} \sin \left(m \arctan \frac{q}{p} \right)$$

GW (337)(12)

$$4. \quad \int_0^{2\pi} \exp(p \cos x + q \sin x) \cos(q \cos x - p \sin x + mx) dx = \frac{2\pi}{m!} (p^2 + q^2)^{\frac{m}{2}} \cos\left(m \arctan \frac{q}{p}\right)$$

GW (337)(12)

3.938

$$1. \quad \int_0^\pi e^{r(\cos px + \cos qx)} \sin(r \sin px) \sin(r \sin qx) dx = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(pk+1)\Gamma(qk+1)} r^{(p+q)k}$$

BI (277)(14)

$$2. \quad \int_0^\pi e^{r(\cos px + \cos qx)} \cos(r \sin px) \cos(r \sin qx) dx = \frac{\pi}{2} \left(2 + \sum_{k=1}^{\infty} \frac{r^{(p+q)k}}{\Gamma(pk+1)\Gamma(qk+1)} \right)$$

BI (277)(15)

3.939

$$1. \quad \int_0^\pi e^{q \cos x} \frac{\sin rx}{1 - 2p^r \cos rx + p^{2r}} \sin(q \sin x) dx = \frac{\pi}{2pr} \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)}$$

[r > 0, 0 < p < 1] BI (278)(15)

$$2.^3 \quad \int_0^\pi e^{q \cos x} \frac{1 - p^r \cos rx}{1 - 2p^r \cos rx + p^{2r}} \cos(q \sin x) dx = \frac{\pi}{2} \left[2 + \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)} \right]$$

[r > 0, 0 < p < 1] BI (278)(16)

$$3. \quad \int_0^{\pi/2} \frac{e^{p \cos 2x} \cos(p \sin 2x) dx}{\cos^2 x + q^2 \sin^2 x} = \frac{\pi}{2q} \exp\left(p \frac{q-1}{q+1}\right)$$

BI (273)(8)

3.94–3.97 Combinations involving trigonometric functions, exponentials, and powers**3.941**

$$1. \quad \int_0^\infty e^{-px} \sin qx \frac{dx}{x} = \arctan \frac{q}{p} \quad [p > 0] \quad \text{BI (365)(1)}$$

$$2. \quad \int_0^\infty e^{-px} \cos qx \frac{dx}{x} = \infty \quad \text{BI (365)(2)}$$

3.942

$$1. \quad \int_0^\infty e^{-px} \cos px \frac{x dx}{b^4 + x^4} = \frac{\pi}{4b^2} \exp(-bp\sqrt{2}) \quad [p > 0, b > 0] \quad \text{BI (386)(6)a}$$

$$2. \quad \int_0^\infty e^{-px} \cos px \frac{x dx}{b^4 - x^4} = \frac{\pi}{4b^2} e^{-bp} \sin bp \quad [p > 0, b > 0] \quad \text{BI (386)(7)a}$$

3.943

$$\int_0^\infty e^{-\beta x} (1 - \cos ax) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2 + \beta^2}{\beta^2} \quad [\operatorname{Re} \beta > 0] \quad \text{BI (367)(6)}$$

3.944

$$1. \quad \int_0^u x^{\mu-1} e^{-\beta x} \sin \delta x dx = \frac{i}{2} (\beta + i\delta)^{-\mu} \gamma[\mu, (\beta + i\delta)u] - \frac{i}{2} (\beta - i\delta)^{-\mu} \gamma[\mu, (\beta - i\delta)u]$$

[Re μ > -1] ET I 318(8)

$$2. \quad \int_u^\infty x^{\mu-1} e^{-\beta x} \sin \delta x \, dx = \frac{i}{2} (\beta + i\delta)^{-\mu} \Gamma[\mu, (\beta + i\delta)u] - \frac{i}{2} (\beta - i\delta)^{-\mu} \Gamma[\mu, (\beta - i\delta)u]$$

$$[\operatorname{Re} \beta > |\operatorname{Im} \delta|] \quad \text{ET I 318(9)}$$

$$3. \quad \int_0^u x^{\mu-1} e^{-\beta x} \cos \delta x \, dx = \frac{1}{2} (\beta + i\delta)^{-\mu} \gamma[\mu, (\beta + i\delta)u] + \frac{1}{2} (\beta - i\delta)^{-\mu} \gamma[\mu, (\beta - i\delta)u]$$

$$[\operatorname{Re} \mu > 0] \quad \text{ET I 320(28)}$$

$$4. \quad \int_u^\infty x^{\mu-1} e^{-\beta x} \cos \delta x \, dx = \frac{1}{2} (\beta + i\delta)^{-\mu} \Gamma[\mu, (\beta + i\delta)u] + \frac{1}{2} (\beta - i\delta)^{-\mu} \Gamma[\mu, (\beta - i\delta)u]$$

$$[\operatorname{Re} \beta > |\operatorname{Im} \delta|] \quad \text{ET I 320(29)}$$

$$5.11 \quad \int_0^\infty x^{\mu-1} e^{-\beta x} \sin \delta x \, dx = \frac{\Gamma(\mu)}{(\beta^2 + \delta^2)^{\mu/2}} \sin\left(\mu \arctan \frac{\delta}{\beta}\right)$$

$$[\operatorname{Re} \mu > -1, \quad \operatorname{Re} \beta > |\operatorname{Im} \delta|]$$

$$\text{FI II 812, BI (361)(9)}$$

$$6. \quad \int_0^\infty x^{\mu-1} e^{-\beta x} \cos \delta x \, dx = \frac{\Gamma(\mu)}{(\delta^2 + \beta^2)^{\frac{\mu}{2}}} \cos\left(\mu \arctan \frac{\delta}{\beta}\right)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > |\operatorname{Im} \delta|]$$

$$\text{FI II 812, BI (361)(10)}$$

$$7. \quad \int_0^\infty x^{\mu-1} \exp(-ax \cos t) \sin(ax \sin t) \, dx = \Gamma(\mu) a^{-\mu} \sin(\mu t)$$

$$[\operatorname{Re} \mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2}]$$

$$\text{EH I 13(36)}$$

$$8. \quad \int_0^\infty x^{\mu-1} \exp(-ax \cos t) \cos(ax \sin t) \, dx = \Gamma(\mu) a^{-\mu} \cos(\mu t)$$

$$[\operatorname{Re} \mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2}]$$

$$\text{EH I 13(35)}$$

$$9. \quad \int_0^\infty x^{p-1} e^{-qx} \sin(qx \tan t) \, dx = \frac{1}{q^p} \Gamma(p) \cos^p t \sin pt \quad \left[|t| < \frac{\pi}{2}, \quad q > 0\right]$$

$$\text{LO V 288(16)}$$

$$10. \quad \int_0^\infty x^{p-1} e^{-qx} \cos(qx \tan t) \, dx = \frac{1}{q^p} \Gamma(p) \cos^p(t) \cos pt$$

$$\left[|t| < \frac{\pi}{2}, \quad q > 0\right] \quad \text{LO V 288(15)}$$

$$11. \quad \int_0^\infty x^n e^{-\beta x} \sin bx \, dx = n! \left(\frac{\beta}{\beta^2 + b^2}\right)^{n+1} \sum_{0 \leq 2k \leq n} (-1)^k \binom{n+1}{2k+1} \left(\frac{b}{\beta}\right)^{2k+1}$$

$$= (-1)^n \frac{\partial^n}{\partial \beta^n} \left(\frac{b}{b^2 + \beta^2}\right)$$

$$[\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{GW (336)(3), ET I 72(3)}$$

$$\begin{aligned}
 12. \quad \int_0^\infty x^n e^{-\beta x} \cos bx \, dx &= n! \left(\frac{\beta}{\beta^2 + b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n+1} (-1)^k \binom{n+1}{2k} \left(\frac{b}{\beta} \right)^{2k} \\
 &= (-1)^n \frac{\partial^n}{\partial \beta^n} \left(\frac{\beta}{b^2 + \beta^2} \right) \\
 &\quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{GW (336)(4), ET I 14(5)}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \int_0^\infty x^{n-1/2} e^{-\beta x} \sin bx \, dx &= (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \left(\frac{\sqrt{\sqrt{\beta^2 + b^2} - \beta}}{\sqrt{\beta^2 + b^2}} \right) \\
 &\quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET I 72(6)}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \int_0^\infty x^{n-1/2} e^{-\beta x} \cos bx \, dx &= (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \left(\frac{\sqrt{\sqrt{\beta^2 + b^2} + \beta}}{\sqrt{\beta^2 + b^2}} \right) \\
 &\quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET I 15(6)}
 \end{aligned}$$

3.945

$$\begin{aligned}
 1. \quad \int_0^\infty (e^{-\beta x} \sin ax - e^{-\gamma x} \sin bx) \frac{dx}{x^r} \\
 = \Gamma(1-r) \left\{ (b^2 + \gamma^2) \frac{r-1}{2} \sin \left[(r-1) \arctan \frac{b}{\gamma} \right] - (a^2 + \beta^2) \frac{r-1}{2} \sin \left[(r-1) \arctan \frac{a}{\beta} \right] \right\} \\
 \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad r < 2, \quad r \neq 1] \quad \text{BI (371)(6)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty (e^{-\beta x} \cos ax - e^{-\gamma x} \cos bx) \frac{dx}{x^r} \\
 = \Gamma(1-r) \left\{ (a^2 + \beta^2) \frac{r-1}{2} \cos \left[(r-1) \arctan \frac{a}{\beta} \right] - (b^2 + \gamma^2) \frac{r-1}{2} \cos \left[(r-1) \arctan \frac{b}{\gamma} \right] \right\} \\
 \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad r < 2, \quad r \neq 1] \quad \text{BI (371)(7)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty (ae^{-\beta x} \sin bx - be^{-\gamma x} \sin ax) \frac{dx}{x^2} &= ab \left[\frac{1}{2} \ln \frac{a^2 + \gamma^2}{b^2 + \beta^2} + \frac{\gamma}{a} \operatorname{arccot} \frac{\gamma}{a} - \frac{\beta}{b} \operatorname{arccot} \frac{\beta}{b} \right] \\
 &\quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{BI (368)(22)}
 \end{aligned}$$

3.946

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-px} \sin^{2m+1} ax \frac{dx}{x} &= \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \arctan \frac{(2m-2k+1)a}{p} \\
 &\quad [m = 0, 1, \dots, \quad p > 0] \quad \text{GW (336)(9a)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty e^{-px} \sin^{2m} ax \frac{dx}{x} &= \frac{(-1)^{m+1}}{2^{2m}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \ln [p^2 + (2m-2k)^2 a^2] - \frac{1}{2^{2m}} \binom{2m}{m} \ln p \\
 &\quad [m = 1, 2, \dots, \quad p > 0] \quad \text{GW (336)(9b)}
 \end{aligned}$$

3.947

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-\beta x} \sin \gamma x \sin ax \frac{dx}{x} &= \frac{1}{4} \ln \frac{\beta^2 + (a + \gamma)^2}{\beta^2 + (a - \gamma)^2} \\
 &\quad [\operatorname{Re} \beta > |\operatorname{Im} \gamma|, \quad a > 0] \quad \text{BI (365)(5)}
 \end{aligned}$$

$$2.^{11} \quad \int_0^\infty e^{-px} \sin ax \sin bx \frac{dx}{x^2} = \frac{|a+b|}{2} \arctan \left(\frac{|a+b|}{p} \right) - \frac{|a-b|}{2} \arctan \left(\frac{|a-b|}{p} \right) \\ + \frac{p}{4} \ln \left(\frac{p^2 + (a-b)^2}{p^2 + (a+b)^2} \right) \\ [p > 0, \quad \text{for } p = 0 \text{ see } \mathbf{3.741} \ 3] \quad \text{BI (368)(1), FI II 744}$$

$$3.^{11} \quad \int_0^\infty e^{-px} \sin ax \cos bx \frac{dx}{x} = \arctan \frac{a+b}{p} + \arctan \frac{a-b}{p} \\ [a \geq 0, \quad p > 0] \quad \text{GW (336)(10b)}$$

3.948

$$1.^{11} \quad \int_0^\infty e^{-\beta x} (\sin ax - \sin bx) \frac{dx}{x} = \arctan \frac{a}{\beta} - \arctan \frac{b}{\beta} \\ [\operatorname{Re} \beta > 0], \quad (\text{cf. } \mathbf{3.951} \ 2) \\ \text{BI (367)(7)}$$

$$2. \quad \int_0^\infty e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x} = \frac{1}{2} \ln \frac{b^2 + \beta^2}{a^2 + \beta^2} \\ [\operatorname{Re} \beta > 0], \quad (\text{cf. } \mathbf{3.951} \ 3) \\ \text{BI (367)(8), FI II 748a}$$

$$3. \quad \int_0^\infty e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x^2} = \frac{\beta}{2} \ln \frac{a^2 + \beta^2}{b^2 + \beta^2} + b \arctan \frac{b}{\beta} - a \arctan \frac{a}{\beta} \\ [\operatorname{Re} p > 0] \quad \text{BI (368)(20)}$$

$$4. \quad \int_0^\infty e^{-\beta x} (\sin^2 ax - \sin^2 bx) \frac{dx}{x^2} = a \arctan \frac{2a}{p} - b \arctan \frac{2b}{p} - \frac{p}{4} \ln \frac{p^2 + 4a^2}{p^2 + 4b^2} \\ [p > 0] \quad \text{BI (368)(25)}$$

$$5. \quad \int_0^\infty e^{-\beta x} (\cos^2 ax - \cos^2 bx) \frac{dx}{x^2} = -a \arctan \frac{2a}{p} + b \arctan \frac{2b}{p} + \frac{p}{4} \ln \frac{p^2 + 4a^2}{p^2 + 4b^2} \\ [p > 0] \quad \text{BI (368)(26)}$$

3.949

$$1. \quad \int_0^\infty e^{-px} \sin ax \sin bx \sin cx \frac{dx}{x} = -\frac{1}{4} \arctan \frac{a+b+c}{p} + \frac{1}{4} \arctan \frac{a+b-c}{p} + \frac{1}{4} \arctan \frac{a-b+c}{p} \\ + \frac{1}{4} \arctan \frac{-a+b+c}{p} \\ [p > 0] \quad \text{BI (365)(11)}$$

$$2.^8 \quad \int_0^\infty e^{-px} \sin^2 ax \sin bx \frac{dx}{x} = \frac{1}{2} \arctan \frac{b}{p} - \frac{1}{2} \left[\frac{1}{2} \arctan \frac{2pb}{p^2 + 4a^2 - b^2} + s \frac{\pi}{2} \right] \\ \left[s = \begin{cases} 1 & \text{for } p^2 + 4a^2 - b^2 < 0 \\ 0 & \text{for } p^2 + 4a^2 - b^2 \geq 0 \end{cases} \right] \\ \text{BI (365)(8)}$$

$$3. \quad \int_0^\infty e^{-px} \sin^2 ax \cos bx \frac{dx}{x} = \frac{1}{8} \ln \frac{[p^2 + (2a+b)^2] [p^2 + (2a-b)^2]}{(p^2 + b^2)^2} \\ [p > 0] \quad \text{BI (365)(9)}$$

$$4.^8 \quad \int_0^\infty e^{-px} \sin ax \cos^2 bx \frac{dx}{x} = \frac{1}{2} \arctan \frac{a}{p} + \frac{1}{2} \left[\frac{1}{2} \arctan \frac{2pa}{p^2 + 4b^2 - a^2} + s \frac{\pi}{2} \right]$$

$$\left[s = \begin{cases} 1 & \text{for } p^2 + 4b^2 - a^2 < 0 \\ 0 & \text{for } p^2 + 4b^2 - a^2 \geq 0 \end{cases} \right]$$

BI (365)(10)

$$5. \quad \int_0^\infty e^{-px} \sin^2 ax \sin bx \sin cx \frac{dx}{x} = \frac{1}{8} \ln \frac{p^2 + (b+c)^2}{p^2 + (b-c)^2}$$

$$+ \frac{1}{16} \ln \frac{[p^2 + (2a-b+c)^2] [p^2 + (2a+b-c)^2]}{[p^2 + (2a+b+c)^2] [p^2 + (2a-b-c)^2]}$$

[p > 0] BI (365)(15)

3.951

$$1. \quad \int_0^\infty (1 - e^{-x}) \cos x \frac{dx}{x} = \ln \sqrt{2}$$

FI II 745

$$2. \quad \int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \sin bx \, dx = \arctan \frac{(\beta - \gamma)b}{b^2 + \beta\gamma}$$

[Re β > 0, Re γ ≥ 0] BI (367)(3)

$$3. \quad \int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \cos bx \, dx = \frac{1}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2}$$

[Re β > 0, Re γ ≥ 0] BI (367)(4)

$$4.^{11} \quad \int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x^2} \sin bx \, dx = \frac{b}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2} + \beta \arctan \frac{b}{\beta} - \gamma \arctan \frac{b}{\gamma}$$

[Re β > 0, Re γ > 0] BI (368)(21)a

$$5. \quad \int_0^\infty \frac{x}{e^{\beta x} - 1} \cos bx \, dx = \frac{1}{2b^2} - \frac{\pi^2}{2\beta^2} \operatorname{cosech}^2 \frac{b\pi}{\beta}$$

[Re β > 0] ET I 15(18)

$$6. \quad \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \cos bx \, dx = \ln b - \frac{1}{2} [\psi(ib) + \psi(-ib)]$$

[b > 0] ET I 15(9)

$$7. \quad \int_0^\infty \frac{1 - \cos ax}{e^{2\pi x} - 1} \cdot \frac{dx}{x} = \frac{a}{4} + \frac{1}{2} \ln \frac{1 - e^{-a}}{a}$$

[a > 0] BI (387)(10)

$$8. \quad \int_0^\infty (e^{-\beta x} - e^{-\gamma x} \cos ax) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2 + \gamma^2}{\beta^2}$$

[Re β > 0, Re γ > 0] BI (367)(10)

$$9. \quad \int_0^\infty \frac{\cos px - e^{-px}}{b^4 + x^4} \frac{dx}{x} = \frac{\pi}{2b^4} \exp \left(-\frac{1}{2} bp\sqrt{2} \right) \sin \left(\frac{1}{2} bp\sqrt{2} \right)$$

[p > 0] BI (390)(6)

$$10. \quad \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = C$$

NT 65(8)

$$11. \quad \int_0^\infty \left(ae^{-px} - \frac{e^{-qx}}{x} \sin ax \right) \frac{dx}{x} = \frac{a}{2} \ln \frac{a^2 + q^2}{p^2} + q \arctan \frac{a}{q} - a$$

[p > 0, q > 0] BI (368)(24)

$$12. \quad \int_0^\infty \frac{x^{2m} \sin bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{2} \coth b\pi - \frac{1}{2b} \right] \quad [b > 0] \quad \text{GW (336)(15a)}$$

$$13. \quad \int_0^\infty \frac{x^{2m+1} \cos bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{2} \coth b\pi - \frac{1}{2b} \right] \quad [b > 0] \quad \text{GW (336)(15b)}$$

$$14. \quad \int_0^\infty \frac{x^{2m} \sin bx \, dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{4c} \tanh \frac{b\pi}{2c} - \sum_{k=1}^n \frac{b}{b^2 + (2k-1)^2 c^2} \right] \quad [b > 0] \quad \text{GW (336)(14a)}$$

$$15. \quad \int_0^\infty \frac{x^{2m+1} \cos bx \, dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{4c} \tanh \frac{b\pi}{2c} - \sum_{k=1}^n \frac{b}{b^2 + (2k-1)^2 c^2} \right] \quad [b > 0] \quad \text{GW (336)(14b)}$$

$$16. \quad \int_0^\infty \frac{x^{2m} \sin bx \, dx}{e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{4c} \coth \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right] \quad [b > 0, \quad c > 0] \quad \text{GW (336)(14c)}$$

$$17. \quad \int_0^\infty \frac{x^{2m+1} \cos bx \, dx}{e^{2n cx} - e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{4c} \coth \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right] \quad [b > 0, \quad c > 0] \quad \text{GW (336)(14d)}$$

$$18. \quad \int_0^\infty \frac{\cos ax - \cos bx}{e^{(2m+1)px} - e^{(2m-1)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{\cosh \frac{b\pi}{2p}}{\cosh \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^m \ln \frac{b^2 + (2k-1)^2 p^2}{a^2 + (2k-1)^2 p^2} \quad [p > 0] \quad \text{GW (336)(16a)}$$

$$19. \quad \int_0^\infty \frac{\cos ax - \cos bx}{e^{2mpx} - e^{(2m-2)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{a \sinh \frac{b\pi}{2p}}{b \sinh \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^{m-1} \ln \frac{b^2 + 4k^2 p^2}{a^2 + 4k^2 p^2} \quad [p > 0] \quad \text{GW (336)(16b)}$$

$$20. \quad \int_0^\infty \frac{\sin x \sin bx}{1 - e^x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{(b+1) \sinh[(b-1)\pi]}{(b-1) \sinh[(b+1)\pi]} \quad [b^2 \neq 1] \quad \text{LO V 305}$$

$$21. \quad \int_0^\infty \frac{\sin^2 ax}{1 - e^x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{2a\pi}{\sinh 2a\pi} \quad \text{LO V 306, BI (387)(5)}$$

3.952

$$1. \quad \int_0^\infty x e^{-p^2 x^2} \sin ax \, dx = \frac{a\sqrt{\pi}}{4p^3} \exp\left(-\frac{a^2}{4p^2}\right) \quad \text{BI (362)(1)}$$

$$2. \quad \int_0^\infty x e^{-p^2 x^2} \cos ax \, dx = \frac{1}{2p^2} - \frac{a}{4p^3} \sum_{k=0}^\infty \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1} \quad [a > 0] \quad \text{BI (362)(2)}$$

$$3. \quad \int_0^\infty x^2 e^{-p^2 x^2} \sin ax \, dx = \frac{a}{4p^4} + \frac{2p^2 - a^2}{8p^5} \sum_{k=0}^\infty \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1} \quad [a > 0] \quad \text{BI (362)(4)}$$

$$4. \quad \int_0^\infty x^2 e^{-p^2 x^2} \cos ax \, dx = \sqrt{\pi} \frac{2p^2 - a^2}{8p^5} \exp\left(-\frac{a^2}{4p^2}\right) \quad \text{BI (362)(5)}$$

$$5. \quad \int_0^\infty x^3 e^{-p^2 x^2} \sin ax \, dx = \sqrt{\pi} \frac{6ap^2 - a^3}{16p^7} \exp\left(-\frac{a^2}{4p^2}\right) \quad \text{BI (362)(6)}$$

$$6.^3 \quad \int_0^\infty e^{-p^2 x^2} \sin ax \frac{dx}{x} = \frac{a\sqrt{\pi}}{2p} \sum_{k=0}^\infty \frac{(-1)^k}{k!(2k+1)} \left(\frac{a}{2p}\right)^{2k} = \frac{\pi}{2} \Phi\left(\frac{a}{2p}\right) \quad \text{BI (365)(21)}$$

$$7. \quad \int_0^\infty x^{\mu-1} e^{-\beta x^2} \sin \gamma x \, dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}} \Gamma\left(\frac{1+\mu}{2}\right) {}_1F_1\left(1 - \frac{\mu}{2}; \frac{3}{2}; \frac{\gamma^2}{4\beta}\right) \quad [\operatorname{Re} \mu > -1] \quad \text{ET I 318(10)}$$

$$8.^{10} \quad \int_0^\infty x^{\mu-1} e^{-\beta x^2} \cos ax \, dx = \frac{1}{2} \beta^{-\mu/2} \Gamma\left(\frac{\mu}{2}\right) e^{-a^2/4\beta} {}_1F_1\left(-\frac{\mu}{2} + \frac{1}{2}; \frac{1}{2}; \frac{a^2}{4\beta}\right) \quad [\operatorname{Re} \mu > 0, \quad a > 0] \quad \text{ET I 320(30)}$$

$$9. \quad \begin{aligned} \int_0^\infty x^{2n} e^{-\beta^2 x^2} \cos ax \, dx &= (-1)^n \frac{\sqrt{\pi}}{2^{n+1} \beta^{2n+1}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n}\left(\frac{a}{\beta\sqrt{2}}\right) \\ &= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+1}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n}\left(\frac{a}{2\beta}\right) \end{aligned} \quad \left[|\arg \beta| < \frac{\pi}{4}, \quad a > 0\right] \quad \text{WH, ET I 15(13)}$$

$$10. \quad \begin{aligned} \int_0^\infty x^{2n+1} e^{-\beta^2 x^2} \sin ax \, dx &= (-1)^n \frac{\sqrt{\pi}}{2^{n+\frac{3}{2}} \beta^{2n+2}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n+1}\left(\frac{a}{\beta\sqrt{2}}\right) \\ &= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+2}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n+1}\left(\frac{a}{2\beta}\right) \end{aligned} \quad \left[|\arg \beta| < \frac{\pi}{4}, \quad a > 0\right] \quad \text{WH, ET I 74(23)}$$

3.953

$$1. \quad \begin{aligned} \int_0^\infty x^{\mu-1} e^{-\gamma x - \beta x^2} \sin ax \, dx \\ = -\frac{i}{2(2\beta)^{\frac{\mu}{2}}} \exp\left(\frac{\gamma^2 - a^2}{8\beta}\right) \Gamma(\mu) \left\{ \exp\left(-\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma - ia}{\sqrt{2\beta}}\right) - \exp\left(\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma + ia}{\sqrt{2\beta}}\right) \right\} \end{aligned} \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 318(11)}$$

$$2. \quad \begin{aligned} \int_0^\infty x^{\mu-1} e^{-\gamma x - \beta x^2} \cos ax \, dx \\ = \frac{1}{2(2\beta)^{\frac{\mu}{2}}} \exp\left(\frac{\gamma^2 - a^2}{8\beta}\right) \Gamma(\mu) \left\{ \exp\left(-\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma - ia}{\sqrt{2\beta}}\right) + \exp\left(\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma + ia}{\sqrt{2\beta}}\right) \right\} \end{aligned} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 16(18)}$$

$$3. \quad \int_0^\infty x e^{-\gamma x - \beta x^2} \sin ax \, dx = \frac{i\sqrt{\pi}}{8\sqrt{\beta^3}} \left\{ (\gamma - ia) \exp \left[-\frac{(\gamma - ia)^2}{4\beta} \right] \left[1 - \Phi \left(\frac{\gamma - ia}{2\sqrt{\beta}} \right) \right] \right. \\ \left. - (\gamma + ia) \exp \left[-\frac{(\gamma + ia)^2}{4\beta} \right] \left[1 - \Phi \left(\frac{\gamma + ia}{2\sqrt{\beta}} \right) \right] \right\} \\ [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 74(28)}$$

$$4. \quad \int_0^\infty x e^{-\gamma x - \beta x^2} \cos ax \, dx = -\frac{\sqrt{\pi}}{8\sqrt{\beta^3}} \left\{ (\gamma - ia) \exp \frac{(\gamma - ia)^2}{4\beta} \left[1 - \Phi \left(\frac{\gamma - ia}{2\sqrt{\beta}} \right) \right] \right. \\ \left. + (\gamma + ia) \exp \frac{(\gamma + ia)^2}{4\beta} \left[1 - \Phi \left(\frac{\gamma + ia}{2\sqrt{\beta}} \right) \right] \right\} + \frac{1}{2\beta} \\ [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 16(17)}$$

3.954

$$1.^{11} \quad \int_0^\infty e^{-\beta x^2} \sin ax \frac{x \, dx}{\gamma^2 + x^2} = -\frac{\pi}{4} e^{\beta \gamma^2} \left[2 \sinh a\gamma + e^{-\gamma a} \Phi \left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}} \right) - e^{\gamma a} \Phi \left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}} \right) \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad a > 0] \quad \text{ET I 74(26)a}$$

$$2.^{11} \quad \int_0^\infty e^{-\beta x^2} \cos ax \frac{dx}{\gamma^2 + x^2} = \frac{\pi}{4\gamma} e^{\beta \gamma^2} \left[2 \cosh a\gamma - e^{-\gamma a} \Phi \left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}} \right) - e^{\gamma a} \Phi \left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}} \right) \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad a > 0] \quad \text{ET I 15(15)}$$

$$3.955 \quad \int_0^\infty x^\nu e^{-\frac{x^2}{2}} \cos \left(\beta x - \nu \frac{\pi}{2} \right) dx = \sqrt{\frac{\pi}{2}} e^{-\frac{\beta^2}{4}} D_\nu(\beta) \quad [\operatorname{Re} \nu > -1] \quad \text{EH II 120(4)}$$

$$3.956 \quad \int_0^\infty e^{-x^2} (2x \cos x - \sin x) \sin x \frac{dx}{x^2} = \sqrt{\pi} \frac{e-1}{2e} \quad \text{BI (369)(19)}$$

3.957

$$1. \quad \int_0^\infty x^{\mu-1} \exp \left(\frac{-\beta^2}{4x} \right) \sin ax \, dx \\ = \frac{i}{2^\mu} \beta^\mu a^{-\frac{\mu}{2}} \left[\exp \left(-\frac{i}{4} \mu \pi \right) K_\mu \left(\beta e^{\frac{\pi i}{4}} \sqrt{a} \right) - \exp \left(\frac{i}{4} \mu \pi \right) K_\mu \left(\beta e^{-\pi i/4} \sqrt{a} \right) \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu < 1, \quad a > 0] \quad \text{ET I 318(12)}$$

$$2. \quad \int_0^\infty x^{\mu-1} \exp \left(\frac{-\beta^2}{4x} \right) \cos ax \, dx \\ = \frac{1}{2^\mu} \beta^\mu a^{-\frac{\mu}{2}} \left[\exp \left(-\frac{i}{4} \mu \pi \right) K_\mu \left(\beta e^{\pi i/4} \sqrt{a} \right) + \exp \left(\frac{i}{4} \mu \pi \right) K_\mu \left(\beta e^{-\pi i/4} \sqrt{a} \right) \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu < 1, \quad a > 0] \quad \text{ET I 320(32)a}$$

3.958

$$1. \quad \int_{-\infty}^\infty x^n e^{-(ax^2+bx+c)} \sin(px+q) \, dx = -\left(\frac{-1}{2a} \right)^n \sqrt{\frac{\pi}{a}} \exp \left(\frac{b^2-p^2}{4a} - c \right) \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!} a^k \\ \times \sum_{j=0}^{n-2k} \binom{n-2k}{j} b^{n-2k-j} p^j \sin \left(\frac{pb}{2a} - q + \frac{\pi}{2} j \right) \\ [a > 0] \quad \text{GW (37)(1b)}$$

$$\begin{aligned}
2. \quad \int_{-\infty}^{\infty} x^n e^{-(ax^2+bx+c)} \cos(px+q) dx &= \left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-p^2}{4a} - c\right) \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!} a^k \\
&\quad \times \sum_{j=0}^{n-2k} \binom{n-2k}{j} p^j \cos\left(\frac{pb}{2a} - q + \frac{\pi}{2}j\right) \\
&\quad [a > 0]
\end{aligned}$$

GW (337)(1a)

$$\begin{aligned}
3.959 \quad \int_0^{\infty} x e^{-p^2 x^2} \tan ax \, dx &= \frac{a\sqrt{\pi}}{p^3} \sum_{k=1}^{\infty} (-1)^k k \exp\left(-\frac{a^2 k^2}{p^2}\right) \\
&\quad [p > 0]
\end{aligned}$$

BI (362)(15)

$$\begin{aligned}
3.961 \quad 1. \quad \int_0^{\infty} \exp\left(-\beta\sqrt{\gamma^2+x^2}\right) \sin ax \frac{x \, dx}{\sqrt{\gamma^2+x^2}} &= \frac{a\gamma}{\sqrt{a^2+\beta^2}} K_1\left(\gamma\sqrt{a^2+\beta^2}\right) \\
&\quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad a > 0]
\end{aligned}$$

ET I 75(36)

$$\begin{aligned}
2. \quad \int_0^{\infty} \exp\left[-\beta\sqrt{\gamma^2+x^2}\right] \cos ax \frac{dx}{\sqrt{\gamma^2+x^2}} &= K_0\left(\gamma\sqrt{a^2+\beta^2}\right) \\
&\quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad a > 0]
\end{aligned}$$

ET I 17(27)

3.962

$$\begin{aligned}
1. \quad \int_0^{\infty} \frac{\sqrt{\sqrt{\gamma^2+x^2}-\gamma} \exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2}} \sin ax \, dx &= \sqrt{\frac{\pi}{2}} \frac{a \exp\left(-\gamma\sqrt{a^2+\beta^2}\right)}{\sqrt{\beta^2+a^2} \sqrt{\beta+\sqrt{a^2+\beta^2}}} \\
&\quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad a > 0]
\end{aligned}$$

ET I 75(38)

$$\begin{aligned}
2. \quad \int_0^{\infty} \frac{x \exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2} \sqrt{\sqrt{\gamma^2+x^2}-\gamma}} \cos ax \, dx &= \sqrt{\frac{\pi}{2}} \frac{\sqrt{\beta+\sqrt{a^2+\beta^2}}}{\sqrt{a^2+\beta^2}} \exp\left[-\gamma\sqrt{a^2+\beta^2}\right] \\
&\quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad a > 0]
\end{aligned}$$

ET I 17(29)

3.963

$$1. \quad \int_0^{\infty} e^{-\tan^2 x} \frac{\sin x}{\cos^2 x} \frac{dx}{x} = \frac{\sqrt{\pi}}{2}$$

BI (391)(1)

$$2. \quad \int_0^{\pi/2} e^{-p \tan x} \frac{x \, dx}{\cos^2 x} = \frac{1}{p} [\operatorname{ci}(p) \sin p - \cos p \operatorname{si}(p)] \quad [p > 0] \quad (\text{cf. } 3.339)$$

BI (396)(3)

$$3.^8 \quad \int_0^{\pi/2} x e^{-\tan^2 x} \sin 4x \frac{dx}{\cos^8 x} = -\frac{3}{2} \sqrt{\pi}$$

BI (396)(5)

$$4.^8 \quad \int_0^{\pi/2} x e^{-\tan^2 x} \sin^3 2x \frac{dx}{\cos^8 x} = 2\sqrt{\pi}$$

BI (396)(6)

3.964

$$1. \quad \int_0^{\pi/2} x e^{-p \tan x} \frac{p \sin x - \cos x}{\cos^3 x} dx = -\sin p \operatorname{si}(p) - \operatorname{ci}(p) \cos p \quad [p > 0] \quad \text{LI (396)(4)}$$

$$2. \quad \int_0^{\pi/2} x e^{-p \tan^2 x} \frac{p - \cos^2 x}{\cos^4 x \cot x} dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \quad [p > 0] \quad \text{BI (396)(7)}$$

$$3.^8 \quad \int_0^{\pi/2} x e^{-p \tan^2 x} \frac{p - 2 \cos^2 x}{\cos^6 x \cot x} dx = \frac{1 + 2p}{8p} \sqrt{\frac{\pi}{p}} \quad [p > 0] \quad \text{BI (396)(8)}$$

3.965

$$1. \quad \int_0^\infty x e^{-\beta x} \sin ax^2 \sin \beta x dx = \frac{\beta}{4} \sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}} \quad \left[|\arg \beta| < \frac{\pi}{4}, \quad a > 0 \right] \quad \text{ET I 84(17)}$$

$$2. \quad \int_0^\infty x e^{-\beta x} \cos ax^2 \cos \beta x dx = \frac{\beta}{4} \sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}} \quad [a > 0, \quad \operatorname{Re} \beta > |\operatorname{Im} \beta|] \quad \text{ET 26(27)}$$

3.966

$$1. \quad \int_0^\infty x e^{-px} \cos(2x^2 + px) dx = 0 \quad [p > 0] \quad \text{BI (361)(16)}$$

$$2. \quad \int_0^\infty x e^{-px} \cos(2x^2 - px) dx = \frac{p\sqrt{\pi}}{8} \exp\left(-\frac{1}{4}p^2\right) \quad [p > 0] \quad \text{BI (361)(17)}$$

$$3. \quad \int_0^\infty x^2 e^{-px} [\sin(2x^2 + px) + \cos(2x^2 + px)] dx = 0 \quad [p > 0] \quad \text{BI (361)(18)}$$

$$4. \quad \int_0^\infty x^2 e^{-px} [\sin(2x^2 - px) - \cos(2x^2 - px)] dx = \frac{\sqrt{\pi}}{16} (2 - p^2) \exp\left(-\frac{1}{4}p^2\right) \quad \text{BI (361)(19)}$$

$$5.^3 \quad \int_0^\infty x^{\mu-1} e^{-x} \cos(x + ax^2) dx = \frac{e^{\frac{1}{4a}} \Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \cos \frac{\mu\pi}{4} D_{-\mu}\left(\frac{1}{\sqrt{a}}\right) \quad [\operatorname{Re} \mu > 0, \quad a > 0] \quad \text{ET I 321(37)}$$

$$6.^6 \quad \int_0^\infty x^{\mu-1} e^{-x} \sin(x + ax^2) dx = \frac{e^{\frac{1}{4a}} \Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \sin \frac{\mu\pi}{4} D_{-\mu}\left(\frac{1}{\sqrt{a}}\right) \quad [\operatorname{Re} \mu > -1, \quad a > 0] \quad \text{ET I 319(18)}$$

3.967

$$1. \quad \int_0^\infty e^{-\frac{\beta^2}{x^2}} \sin a^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \sin(\sqrt{2}a\beta) \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 75(30)a, BI(369)(3)a}$$

$$2. \quad \int_0^\infty e^{-\frac{\beta^2}{x^2}} \cos a^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \cos(\sqrt{2}a\beta) \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (369)(4), ET I 16(20)}$$

$$3. \quad \int_0^\infty x^2 e^{-\beta x^2} \cos ax^2 dx = \frac{\sqrt{\pi}}{4\sqrt{(a^2 + \beta^2)^3}} \cos\left(\frac{3}{2} \arctan \frac{a}{\beta}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 14(3)a}$$

3.968

$$1. \quad \int_0^\infty e^{-\beta x^2} \sin ax^4 dx = -\frac{\pi}{8} \sqrt{\frac{\beta}{a}} \left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right) \cos\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8} + Y_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right) \sin\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8} \right] \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 75(34)}$$

$$2. \quad \int_0^\infty e^{-\beta x^2} \cos ax^4 dx = \frac{\pi}{8} \sqrt{\frac{\beta}{a}} \left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right) \sin\left(\frac{\beta^2}{8a} + \frac{\pi}{8}\right) - Y_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right) \cos\left(\frac{\beta^2}{8a} + \frac{\pi}{8}\right) + \frac{\pi}{8} \right] \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 16(24)}$$

3.969

$$1. \quad \int_0^\infty e^{-p^2 x^4 + q^2 x^2} [2px \cos(2pqx^3) + q \sin(2pqx^3)] dx = \frac{\sqrt{\pi}}{2} \quad \text{BI (363)(7)}$$

$$2. \quad \int_0^\infty e^{-p^2 x^4 + q^2 x^2} [2px \sin(2pqx^3) - q \cos(2pqx^3)] dx = 0 \quad \text{BI (363)(8)}$$

3.971 Notation: In formulas **3.971** 1 and 2, $p \geq 0$, $q \geq 0$, $r = \sqrt[4]{a^2 + p^2}$, $s = \sqrt[4]{b^2 + q^2}$, $A = \arctan \frac{a}{p}$, and $B = \arctan \frac{b}{q}$.

$$1. \quad \int_0^\infty \exp\left(-px^2 - \frac{q}{x^2}\right) \sin\left(ax^2 + \frac{b}{x^2}\right) \frac{dx}{x^2} = \frac{1}{2} \int_{-\infty}^\infty \exp\left(-px^2 - \frac{q}{x^2}\right) \sin\left(ax^2 + \frac{b}{x^2}\right) \frac{dx}{x^2} \\ = \frac{\sqrt{\pi}}{2s} \exp[-2rs \cos(A+B)] \sin[A + 2rs \sin(A+B)] \quad \text{BI (369)(16, 17)}$$

$$2. \quad \int_0^\infty \exp\left(-px^2 - \frac{q}{x^2}\right) \cos\left(ax^2 + \frac{b}{x^2}\right) \frac{dx}{x^2} = \frac{1}{2} \int_{-\infty}^\infty \exp\left(-px^2 - \frac{q}{x^2}\right) \cos\left(ax^2 + \frac{b}{x^2}\right) \frac{dx}{x^2} \\ = \frac{\sqrt{\pi}}{2s} \exp[-2rs \cos(A+B)] \cos[A + 2rs \sin(A+B)] \quad \text{BI (369)(15, 18)}$$

3.972

$$1. \quad \int_0^\infty \exp\left[-\beta \sqrt{\gamma^4 + x^4}\right] \sin ax^2 \frac{dx}{\sqrt{\gamma^4 + x^4}} \\ = \sqrt{\frac{a\pi}{8}} I_{1/4} \left[\frac{\gamma^2}{2} (\sqrt{\beta^2 + a^2} - \beta) \right] K_{1/4} \left[\frac{\gamma^2}{4} (\sqrt{\beta^2 + a^2} + \beta) \right] \quad [\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \frac{\pi}{4}, \quad a > 0] \quad \text{ET I 75(37)}$$

$$2. \quad \int_0^\infty \exp\left[-\beta \sqrt{\gamma^4 + x^4}\right] \cos ax^2 \frac{dx}{\sqrt{\gamma^4 + x^4}} \\ = \sqrt{\frac{a\pi}{8}} I_{-1/4} \left[\frac{\gamma^2}{2} (\sqrt{\beta^2 + a^2} - \beta) \right] K_{1/4} \left[\frac{\gamma^2}{4} (\sqrt{\beta^2 + a^2} + \beta) \right] \quad [\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \frac{\pi}{4}, \quad a > 0] \quad \text{ET I 17(28)}$$

3.973

$$1. \quad \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax) \frac{dx}{x} = \frac{\pi}{2} (e^p - 1) \quad [p > 0, \quad a > 0] \quad \text{WH, FI II 725}$$

$$2. \quad \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax + bx) \frac{x dx}{c^2 + x^2} = \frac{\pi}{2} \exp(-cb + pe^{-ac})$$

$$[a > 0, \quad b > 0, \quad c > 0, \quad p > 0] \quad \text{BI (372)(3)}$$

$$3. \quad \int_0^{\infty} \exp(p \cos ax) \cos(p \sin ax + bx) \frac{dx}{c^2 + x^2} = \frac{\pi}{2c} \exp(-cb + pe^{-ac})$$

$$[a > 0, \quad b > 0, \quad c > 0, \quad p > 0] \quad \text{BI (372)(4)}$$

$$4. \quad \int_0^{\infty} \exp(p \cos x) \sin(p \sin x + nx) \frac{dx}{x} = \frac{\pi}{2} e^p \quad [p > 0] \quad \text{BI (366)(2)}$$

$$5. \quad \int_0^{\infty} \exp(p \cos x) \sin(p \sin x) \cos nx \frac{dx}{x} = \frac{p^n}{n!} \cdot \frac{\pi}{4} + \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{p^k}{k!}$$

$$[p > 0] \quad \text{LI (366)(3)}$$

$$6. \quad \int_0^{\infty} \exp(p \cos x) \cos(p \sin x) \sin nx \frac{dx}{x} = \frac{\pi}{2} \sum_{k=0}^{n-1} \frac{p^k}{k!} + \frac{p^n}{n!} \frac{\pi}{4}$$

$$[p > 0] \quad \text{LI (366)(4)}$$

3.974

$$1. \quad \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax) \operatorname{cosec} ax \frac{dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab})]}{2b \sinh ab}$$

$$[a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(4)}$$

$$2. \quad \int_0^{\infty} [1 - \exp(p \cos ax) \cos(p \sin ax)] \operatorname{cosec} ax \frac{x dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab})]}{2 \sinh ab}$$

$$[a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(5)}$$

$$3. \quad \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax + ax) \operatorname{cosec} ax \frac{dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab} - ab)]}{2b \sinh ab}$$

$$[a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(6)}$$

$$4. \quad \int_0^{\infty} \exp(p \cos ax) \cos(p \sin ax + ax) \operatorname{cosec} ax \frac{x dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab} - ab)]}{2 \sinh ab}$$

$$[a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(7)}$$

$$5. \quad \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax) \frac{x dx}{b^2 - x^2} = \frac{\pi}{2} [1 - \exp(p \cos ab) \cos(p \sin ab)]$$

$$[p > 0, \quad a > 0] \quad \text{BI (378)(1)}$$

$$6. \quad \int_0^{\infty} \exp(p \cos ax) \cos(p \sin ax) \frac{dx}{b^2 - x^2} = \frac{\pi}{2b} \exp(p \cos ab) \sin(p \sin ab)$$

$$[a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (378)(2)}$$

$$7. \quad \int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \tan ax \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} \cdot \tanh ab [\exp(pe^{-ab}) - e^p] \\ [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (372)(14)}$$

$$8. \quad \int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \cot ax \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} \coth ab [e^p - \exp(pe^{-ab})] \\ [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (372)(15)}$$

$$9. \quad \int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \operatorname{cosec} ax \frac{dx}{b^2 - x^2} = \frac{\pi}{2b} \operatorname{cosec} ab [e^p - \exp(p \cos ab) \cos(p \sin ab)] \\ [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(12)}$$

$$10. \quad \int_0^\infty [1 - \exp(p \cos ax) \cos(p \sin ax)] \operatorname{cosec} ax \frac{x dx}{b^2 - x^2} = -\frac{\pi}{2} \exp(p \cos ab) \sin(p \sin ab) \operatorname{cosec} ab \\ [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(13)}$$

3.975

$$1. \quad \int_0^\infty \frac{\sin\left(\beta \arctan \frac{x}{\gamma}\right)}{(\gamma^2 + x^2)^{\frac{\beta}{2}}} \cdot \frac{dx}{e^{2\pi x} - 1} = \frac{1}{2} \zeta(\beta, \gamma) - \frac{1}{4\gamma^\beta} - \frac{\gamma^{1-\beta}}{2(\beta-1)} \\ [\operatorname{Re} \beta > 1, \quad \operatorname{Re} \gamma > 0] \quad \text{WH, ET I 26(7)}$$

$$2. \quad \int_0^\infty \frac{\sin(\beta \arctan x)}{(1+x^2)^{\frac{\beta}{2}}} \cdot \frac{dx}{e^{2\pi x} + 1} = \frac{1}{2(\beta-1)} - \frac{\zeta(\beta)}{2^\beta} \quad [\operatorname{Re} \beta > 1] \quad \text{EH I 33(13)}$$

$$3.976 \quad \int_0^\infty (1+x^2)^{\beta-\frac{1}{2}} e^{-px^2} \cos[2px + (2\beta-1) \arctan x] dx = \frac{e^{-p}}{2p^\beta} \sin \pi \beta \Gamma(\beta) \\ [\operatorname{Re} \beta > 0, \quad p > 0] \quad \text{WH}$$

3.98–3.99 Combinations of trigonometric and hyperbolic functions**3.981**

$$1. \quad \int_0^\infty \frac{\sin ax}{\sinh \beta x} dx = \frac{\pi}{2\beta} \tanh \frac{a\pi}{2\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (264)(16)}$$

$$2. \quad \int_0^\infty \frac{\sin ax}{\cosh \beta x} dx = -\frac{\pi}{2\beta} \tanh \frac{a\pi}{2\beta} - \frac{i}{2\beta} \left[\psi\left(\frac{\beta+ai}{4\beta}\right) - \psi\left(\frac{\beta-ai}{4\beta}\right) \right] \\ [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{GW (335)(12), ET I 88(1)}$$

$$3. \quad \int_0^\infty \frac{\cos ax}{\cosh \beta x} dx = \frac{\pi}{2\beta} \operatorname{sech} \frac{a\pi}{2\beta} \quad [\operatorname{Re} \beta > 0, \quad \text{all real } a] \quad \text{BI (264)(14)}$$

$$4. \quad \int_0^\infty \sin ax \frac{\sinh \beta x}{\sinh \gamma x} dx = \frac{\pi}{2\gamma} \frac{\sinh \frac{a\pi}{\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} + \frac{i}{2\gamma} \left[\psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right) - \psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right) \right] \\ [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{ET I 88(5)}$$

$$5. \quad \int_0^\infty \cos ax \frac{\sinh \beta x}{\sinh \gamma x} dx = \frac{\pi}{2\gamma} \frac{\sin \frac{\pi\beta}{\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma] \quad \text{BI (265)(7)}$$

$$6. \quad \int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh \gamma x} dx = \frac{\pi}{\gamma} \frac{\sin \frac{\beta\pi}{2\gamma} \sinh \frac{a\pi}{2\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{BI (265)(2)}$$

$$7. \quad \int_0^\infty \cos ax \frac{\sinh \beta x}{\cosh \gamma x} dx = \frac{1}{4\gamma} \left[\left\{ \psi \left(\frac{3\gamma - \beta + ia}{4\gamma} \right) + \psi \left(\frac{3\gamma - \beta - ia}{4\gamma} \right) - \psi \left(\frac{3\gamma + \beta - ia}{4\gamma} \right) \right\} \right. \\ \left. - \psi \left(\frac{3\gamma + \beta + ia}{4\gamma} \right) + \frac{2\pi \sin \frac{\pi\beta}{\gamma}}{\cos \frac{\pi\beta}{\gamma} + \cosh \frac{\pi a}{\gamma}} \right] \\ [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{ET I 31(13)}$$

$$8. \quad \int_0^\infty \sin ax \frac{\cosh \beta x}{\sinh \gamma x} dx = \frac{\pi}{2\gamma} \cdot \frac{\sinh \frac{\pi a}{\gamma}}{\cosh \frac{\pi a}{\gamma} + \cos \frac{\pi\beta}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{BI (265)(4)}$$

$$9. \quad \int_0^\infty \sin ax \frac{\cosh \beta x}{\cosh \gamma x} dx = \frac{i}{4\gamma} \left[\psi \left(\frac{3\gamma + \beta + ia}{4\gamma} \right) - \psi \left(\frac{3\gamma + \beta - ia}{4\gamma} \right) + \psi \left(\frac{3\gamma - \beta + ia}{4\gamma} \right) \right. \\ \left. - \psi \left(\frac{3\gamma - \beta - ia}{4\gamma} \right) - \frac{2\pi i \sinh \frac{\pi a}{\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \right] \\ [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{ET I 88(6)}$$

$$10. \quad \int_0^\infty \cos ax \frac{\cosh \beta x}{\cosh \gamma x} dx = \frac{\pi}{\gamma} \frac{\cos \frac{\beta\pi}{2\gamma} \cosh \frac{a\pi}{2\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad \text{all real } a] \quad \text{BI (265)(6)}$$

$$11.^{11} \quad \int_0^{\pi/2} \cos^{2m} x \cosh \beta x dx = \frac{(2m)! \sinh \frac{\pi\beta}{2}}{\beta (\beta^2 + 2^2) \dots [\beta^2 + (2m)^2]} \\ [\beta \neq 0] \quad \text{WA 620a}$$

$$12.^{11} \quad \int_0^{\pi/2} \cos^{2m+1} x \cosh \beta x dx = \frac{(2m+1)! \cosh \frac{\pi\beta}{2}}{(\beta^2 + 1^2)(\beta^2 + 3^2) \dots [\beta^2 + (2m+1)^2]} \quad \text{WA 620a}$$

3.982

$$1. \quad \int_0^\infty \frac{\cos ax}{\cosh^2 \beta x} dx = \frac{a\pi}{2\beta^2 \sinh \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (264)(16)}$$

$$2. \quad \int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh^2 \gamma x} dx = \frac{\pi \left(a \sin \frac{\beta\pi}{2\gamma} \cosh \frac{a\pi}{2\gamma} - \beta \cos \frac{\beta\pi}{2\gamma} \sinh \frac{a\pi}{2\gamma} \right)}{\gamma^2 \left(\cosh \frac{a\pi}{\gamma} - \cos \frac{\beta\pi}{\gamma} \right)} \\ [|\operatorname{Re} \beta| < 2 \operatorname{Re} \gamma, \quad a > 0] \quad \text{ET I 88(9)}$$

$$3.^{11} \quad \int_0^\infty \frac{\sin^2 x \cos ax}{\sinh^2 hx} dx = \frac{\pi}{4} \left\{ \frac{a+2}{1 - e^{-\pi(a+2)}} - \frac{2a}{1 - e^{-\pi a}} + \frac{a-2}{1 - e^{-\pi(a-2)}} \right\} = I(a) \\ \left[I(0) = \frac{1}{2} (\pi \coth \pi - 1), \quad I(\pm 2) = \frac{1}{4} + \frac{\pi}{2} (\coth 2\pi - \coth \pi) \right]$$

3.983

- 1.⁶
$$\int_0^\infty \frac{\cos ax \, dx}{b \cosh \beta x + c} = \frac{\pi \sin \left(\frac{a}{\beta} \operatorname{arccosh} \frac{c}{b} \right)}{\beta \sqrt{c^2 - b^2} \sinh \frac{a\pi}{\beta}} \quad [c > b > 0]$$

$$= \frac{\pi \sinh \left(\frac{a}{\beta} \arccos \frac{c}{b} \right)}{\beta \sqrt{b^2 - c^2} \sinh \frac{a\pi}{\beta}} \quad [b > |c| > 0]$$

[Re $\beta > 0$, $a > 0$] GW (335)(13a)
2.
$$\int_0^\infty \frac{\cos ax \, dx}{\cosh \beta x + \cos \gamma} = \frac{\pi}{\beta} \frac{\sinh \frac{a\gamma}{\beta}}{\sin \gamma \sinh \frac{a\pi}{\beta}} \quad [\pi \operatorname{Re} \beta < \operatorname{Im} \bar{\beta} \gamma, \quad a > 0] \quad \text{BI (267)(3)}$$
- 3.³
$$\int_0^\infty \frac{\cos ax \, dx}{\cosh x - \cosh b} = -\pi \coth a\pi \frac{\sin ab}{\sinh b} \quad [a > 0, \quad b > 0] \quad \text{ET I 30(8)}$$
4.
$$\int_0^\infty \frac{\cos ax \, dx}{1 + 2 \cosh \left(\sqrt{\frac{2}{3}} \pi x \right)} = \frac{\sqrt{\frac{\pi}{2}}}{1 + 2 \cosh \left(\sqrt{\frac{2}{3}} \pi a \right)} \quad [a > 0] \quad \text{ET I 30(9)}$$
5.
$$\int_0^\infty \frac{\sin ax \sinh \beta x}{\cosh \gamma x + \cos \delta} \, dx = \frac{\pi \left\{ \sin \left[\frac{\beta}{\gamma} (\pi - \delta) \right] \sinh \left[\frac{a}{\gamma} (\pi + \delta) \right] - \sin \left[\frac{\beta}{\gamma} (\pi + \delta) \right] \sinh \left[\frac{a}{\gamma} (\pi - \delta) \right] \right\}}{\gamma \sin \delta \left(\cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma} \right)}$$

[$\operatorname{Re} \gamma > |\operatorname{Re} \bar{\gamma} \delta|$, $|\operatorname{Re} \beta| < \operatorname{Re} \gamma$, $a > 0$] BI (267)(2)
6.
$$\int_0^\infty \frac{\cos ax \cosh \beta x}{\cosh \gamma x + \cos b} \, dx = \frac{\pi \left\{ \cos \left[\frac{\beta}{\gamma} (\pi - b) \right] \cosh \left[\frac{a}{\gamma} (\pi + b) \right] - \cos \left[\frac{\beta}{\gamma} (\pi + b) \right] \cosh \left[\frac{a}{\gamma} (\pi - b) \right] \right\}}{\gamma \sin b \left(\cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma} \right)}$$

[$|\operatorname{Re} \beta| < \operatorname{Re} \gamma$, $0 < b < \pi$, $a < 0$] BI (267)(6)
7.
$$\int_0^\infty \frac{\cos ax \, dx}{(\beta + \sqrt{\beta^2 - 1} \cosh x)^{\nu+1}} = \Gamma(\nu + 1 - ai) e^{a\pi} \frac{Q_\nu^{ai}(\beta)}{\Gamma(\nu + 1)}$$

[Re $\nu > -1$, $|\arg(\beta + 1)| < \pi$, $a > 0$] ET I 30(10)

3.984

- 1.⁶
$$\lim_{c \uparrow 1} \int_0^\infty \frac{\sin ax \sinh cx}{\cosh x + \cos b} \, dx = \pi \frac{\cosh ab}{\sinh a\pi} \quad [|b| \leq \pi, \quad a \text{ real}] \quad \text{BI (267)(1)}$$
- 2.⁶
$$\lim_{c \uparrow 1} \int_0^\infty \frac{\cos ax \cosh cx}{\cosh x + \cos b} \, dx = -\pi \cot b \frac{\sinh ab}{\sinh a\pi} \quad [0 < |b| < \pi, \quad a \text{ real}] \quad \text{BI (267)(5)}$$
- 3.⁸
$$\int_0^\infty \frac{\sin ax \sinh \frac{x}{2}}{\cosh x + \cos \beta} \, dx = \frac{\pi \sinh a\beta}{2 \sin \frac{\beta}{2} \cosh a\pi} \quad [\operatorname{Re} \beta < \pi, \quad a > 0] \quad \text{ET I 80(10)}$$
4.
$$\int_0^\infty \frac{\cos ax \cosh \frac{\beta}{2} x}{\cosh \beta x + \cosh \gamma} \, dx = \frac{\pi \cos \frac{a\gamma}{\beta}}{2\beta \cosh \frac{\gamma}{2} \cosh \frac{a\pi}{\beta}} \quad [\pi \operatorname{Re} \beta > |\operatorname{Im}(\bar{\beta} \gamma)|] \quad \text{ET I 31(16)}$$
5.
$$\int_0^\infty \frac{\sin ax \sinh \beta x}{\cosh 2\beta x + \cos 2ax} \, dx = \frac{a\pi}{4(a^2 + \beta^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{BI (267)(7)}$$

$$6. \quad \int_0^\infty \frac{\cos ax \cosh \beta x}{\cosh 2\beta x + \cos 2ax} dx = \frac{\beta\pi}{4(a^2 + \beta^2)} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (267)(8)}$$

$$7.^8 \quad \int_0^\infty \frac{\sinh^{2\mu-1} x \cosh^{2\nu-2\mu+1} x}{(\cosh^2 x - \beta \sinh^2 x)^e} dx = \frac{1}{2} \operatorname{B}(\mu, \nu - \mu) {}_2F_1(\varrho, \mu; \nu; \beta) \\ [\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{EH I 115(12)}$$

3.985

$$1. \quad \int_0^\infty \frac{\cos ax dx}{\cosh^\nu \beta x} = \frac{2^{\nu-2}}{\beta \Gamma(\nu)} \Gamma\left(\frac{\nu}{2} + \frac{ai}{2\beta}\right) \Gamma\left(\frac{\nu}{2} - \frac{ai}{2\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0, \quad a > 0] \\ \text{ET I 30(5)}$$

$$2. \quad \int_0^\infty \frac{\cos ax dx}{\cosh^{2n} \beta x} = \frac{4^{n-1} \pi a}{2(2n-1)! \beta^2 \sinh \frac{a\pi}{2\beta}} \prod_{k=1}^{n-1} \left(\frac{a^2}{4\beta^2} + k^2 \right) \\ = \frac{\pi a (a^2 + 2^2 \beta^2) (a^2 + 4^2 \beta^2) \cdots [a^2 + (2n-2)^2 \beta^2]}{2(2n-1)! \beta^{2n} \sinh \frac{a\pi}{2\beta}} \\ [n \geq 2, \quad a > 0] \quad \text{ET I 30(3)}$$

$$3. \quad \int_0^\infty \frac{\cos ax dx}{\cosh^{2n+1} \beta x} = \frac{\pi 2^{2n-1}}{(2n)! \beta \cosh \frac{a\pi}{2\beta}} \prod_{k=1}^n \left[\frac{a^2}{4\beta^2} + \left(\frac{2k-1}{2} \right)^2 \right] \\ = \frac{\pi (a^2 + \beta^2) (a^2 + 3^2 \beta^2) \cdots [a^2 + (2n-1)^2 \beta^2]}{2(2n)! \beta^{2n+1} \cosh \frac{a\pi}{2\beta}} \\ [\operatorname{Re} \beta > 0, \quad n = 0, 1, \dots, \text{ all real } a] \quad \text{ET I 30(4)}$$

3.986

$$1. \quad \int_0^\infty \frac{\sin \beta x \sin \gamma x}{\cosh \delta x} dx = \frac{\pi}{\delta} \cdot \frac{\sinh \frac{\beta\pi}{2\delta} \sinh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta}{\delta} \pi + \cosh \frac{\gamma}{\delta} \pi} \quad [|\operatorname{Im}(\beta + \gamma)| < \operatorname{Re} \delta] \quad \text{BI (264)(19)}$$

$$2. \quad \int_0^\infty \frac{\sin \alpha x \cos \beta x}{\sinh \gamma x} dx = \frac{\pi \sinh \frac{\pi\alpha}{\gamma}}{2\gamma \left(\cosh \frac{\alpha\pi}{\gamma} + \cosh \frac{\beta\pi}{\gamma} \right)} \quad [|\operatorname{Im}(\alpha + \beta)| < \operatorname{Re} \gamma] \quad \text{LI (264)(20)}$$

$$3. \quad \int_0^\infty \frac{\cos \beta x \cos \gamma x}{\cosh \delta x} dx = \frac{\pi}{\delta} \cdot \frac{\cosh \frac{\beta\pi}{2\delta} \cosh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta}{\delta} \pi + \cosh \frac{\gamma}{\delta} \pi} \quad [|\operatorname{Im}(\beta + \gamma)| < \operatorname{Re} \delta] \quad \text{BI (264)(21)}$$

$$4.^3 \quad \int_0^\infty \frac{\sin^2 \beta x}{\sinh^2 \pi x} dx = \frac{\beta}{\pi (e^{2\beta} - 1)} + \frac{\beta - 1}{2\pi} = \frac{\beta \coth \beta - 1}{2\pi} \\ [|\operatorname{Im} \beta| < \pi] \quad \text{EH I 44(3)}$$

3.987

$$1. \quad \int_0^\infty \sin ax (1 - \tanh \beta x) dx = \frac{1}{a} - \frac{\pi}{2\beta \sinh \frac{\alpha\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 88(4)a}$$

$$2. \quad \int_0^\infty \sin ax (\coth \beta x - 1) dx = \frac{\pi}{2\beta} \coth \frac{a\pi}{2\beta} - \frac{1}{a} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 88(3)}$$

3.988

$$1. \quad \int_0^{\pi/2} \frac{\cos ax \sinh(2b \cos x)}{\sqrt{\cos x}} dx = \frac{\pi}{2} \sqrt{\pi b} I_{\frac{\alpha}{2} + \frac{1}{4}}(b) I_{-\frac{\alpha}{2} + \frac{1}{4}}(b) \quad [a > 0] \quad \text{ET I 37(66)}$$

$$2. \quad \int_0^{\pi/2} \frac{\cos ax \cosh(2b \cos x)}{\sqrt{\cos x}} dx = \frac{\pi}{2} \sqrt{\pi b} I_{\frac{\alpha}{2} - \frac{1}{4}}(b) I_{-\frac{\alpha}{2} - \frac{1}{4}}(b) \quad [a > 0] \quad \text{ET I 37(67)}$$

$$3. \quad \int_0^\infty \frac{\cos ax \, dx}{\sqrt{\cosh x \cos b}} = \frac{\pi P_{-\frac{1}{2} + ia}(\cos b)}{\sqrt{2} \cosh a \pi} \quad [a > 0, \quad b > 0] \quad \text{ET I 30(7)}$$

3.989

$$1. \quad \int_0^\infty \frac{\sin \frac{a^2 x^2}{\pi} \sin bx}{\sinh ax} dx = \frac{\pi}{2a} \sin \frac{\pi b^2}{4a^2} \operatorname{cosech} \frac{\pi b}{2a} \quad [a > 0, \quad b > 0] \quad \text{ET I 93(44)}$$

$$2. \quad \int_0^\infty \frac{\cos \frac{a^2 x^2}{\pi} \sin bx}{\sinh ax} dx = \frac{\pi}{2a} \frac{\cosh \frac{\pi b}{a} - \cos \frac{\pi b^2}{4a^2}}{\sinh \frac{\pi b}{2a}} \quad [a > 0, \quad b > 0] \quad \text{ET I 93(45)}$$

$$3. \quad \int_0^\infty \frac{\sin \frac{x^2}{\pi} \cos ax}{\cosh x} dx = \frac{\pi}{2} \frac{\cos \frac{a^2 \pi}{4} - \frac{1}{\sqrt{2}}}{\cosh \frac{a\pi}{2}} \quad \text{ET I 36(54)}$$

$$4. \quad \int_0^\infty \frac{\cos \frac{x^2}{\pi} \cos ax}{\cosh x} dx = \frac{\pi}{2} \cdot \frac{\sin \frac{a^2 \pi}{4} + \frac{1}{\sqrt{2}}}{\cosh \frac{a\pi}{2}} \quad \text{ET I 36(55)}$$

$$5. \quad \int_0^\infty \frac{\sin(\pi a x^2) \cos bx}{\cosh \pi x} dx = - \sum_{k=0}^\infty \exp \left[- \left(k + \frac{1}{2} \right) b \right] \sin \left[\left(k + \frac{1}{2} \right)^2 \pi a \right] \\ + \frac{1}{\sqrt{a}} \sum_{k=0}^\infty \exp \left[- \frac{b \left(k + \frac{1}{2} \right)}{a} \right] \sin \left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k + \frac{1}{2} \right)^2 \pi}{a} \right] \quad [a > 0, \quad b > 0] \quad \text{ET I 36(56)}$$

$$6. \quad \int_0^\infty \frac{\cos(\pi a x^2) \cos bx}{\cosh \pi x} dx = \sum_{k=0}^\infty (-1)^k \exp \left[- \left(k + \frac{1}{2} \right) b \right] \cos \left[\left(k + \frac{1}{2} \right)^2 \pi a \right] \\ + \frac{1}{\sqrt{a}} \sum_{k=0}^\infty \exp \left[- \frac{b \left(k + \frac{1}{2} \right)}{a} \right] \cos \left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k + \frac{1}{2} \right)^2 \pi}{a} \right] \quad [a > 0, \quad b > 0] \quad \text{ET I 36(57)}$$

3.991

$$1. \quad \int_0^\infty \sin \pi x^2 \sin ax \coth \pi x \, dx = \frac{1}{2} \tanh \frac{a}{2} \sin \left(\frac{\pi}{4} + \frac{a^2}{4\pi} \right) \quad \text{ET I 93(42)}$$

$$2.^{11} \quad \int_0^\infty \cos \pi x^2 \sin ax \coth \pi x \, dx = \frac{1}{2} \tanh \frac{a}{2} \left[1 - \cos \left(\frac{\pi}{4} + \frac{a^2}{4\pi} \right) \right] \quad \text{ET I 93(43)}$$

3.992

$$1. \quad \int_0^\infty \frac{\sin \pi x^2 \cos ax}{1 + 2 \cosh \left(\frac{2}{\sqrt{3}} \pi x \right)} dx = -\sqrt{3} + \frac{\cos \left(\frac{\pi}{12} - \frac{a^2}{4\pi} \right)}{4 \cosh \frac{a}{\sqrt{3}} - 2} \quad \text{ET I 37(60)}$$

$$2. \quad \int_0^\infty \frac{\cos \pi x^2 \cos ax}{1 + 2 \cosh \left(\frac{2}{\sqrt{3}} \pi x \right)} dx = 1 - \frac{\sin \left(\frac{\pi}{12} - \frac{a^2}{4\pi} \right)}{4 \cosh \frac{a}{\sqrt{3}} - 2} \quad \text{ET I 37(61)}$$

$$\mathbf{3.993} \quad \int_0^\infty \frac{\sin^2 x + \cos x^2}{\cosh(\sqrt{\pi}x)} \cos ax \, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\sin^2 a + \cos a^2}{\cosh(\sqrt{\pi}a)} \quad \text{ET I 37(58)}$$

3.994

$$1. \quad \int_0^\infty \frac{\sin(2a \cosh x) \cos bx}{\sqrt{\cosh x}} dx = -\frac{\pi}{4} \sqrt{a\pi} \left[J_{\frac{1}{4} + \frac{ib}{2}}(a) Y_{\frac{1}{4} - \frac{ib}{2}}(a) + J_{\frac{1}{4} - \frac{ib}{2}}(a) Y_{\frac{1}{4} + \frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0] \quad \text{ET I 37(62)}$$

$$2. \quad \int_0^\infty \frac{\cos(2a \cosh x) \cos bx}{\sqrt{\cosh x}} dx = -\frac{\pi}{4} \sqrt{a\pi} \left[J_{-\frac{1}{4} + \frac{ib}{2}}(a) Y_{-\frac{1}{4} - \frac{ib}{2}}(a) + J_{-\frac{1}{4} - \frac{ib}{2}}(a) Y_{-\frac{1}{4} + \frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0] \quad \text{ET I 37(63)}$$

$$3. \quad \int_0^\infty \frac{\sin(2a \sinh x) \sin bx}{\sqrt{\sinh x}} dx = -\frac{i}{2} \sqrt{\pi a} \left[I_{\frac{1}{4} - \frac{ib}{2}}(a) K_{-\frac{1}{4} + \frac{ib}{2}}(a) - I_{\frac{1}{4} + \frac{ib}{2}}(a) K_{\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0] \quad \text{ET I 93(47)}$$

$$4. \quad \int_0^\infty \frac{\cos(2a \sinh x) \sin bx}{\sqrt{\sinh x}} dx = -\frac{i}{2} \sqrt{\pi a} \left[I_{-\frac{1}{4} - \frac{ib}{2}}(a) K_{-\frac{1}{4} + \frac{ib}{2}}(a) - I_{-\frac{1}{4} + \frac{ib}{2}}(a) K_{-\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0] \quad \text{ET I 93(48)}$$

$$5. \quad \int_0^\infty \frac{\sin(2a \sinh x) \cos bx}{\sqrt{\sinh x}} dx = \frac{\sqrt{\pi a}}{2} \left[I_{\frac{1}{4} - \frac{ib}{2}}(a) K_{\frac{1}{4} + \frac{ib}{2}}(a) + I_{\frac{1}{4} + \frac{ib}{2}}(a) K_{\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0] \quad \text{ET I 37(64)}$$

$$6. \quad \int_0^\infty \frac{\cos(2a \sinh x) \cos bx}{\sqrt{\sinh x}} dx = \frac{\sqrt{\pi a}}{2} \left[I_{-\frac{1}{4} - \frac{ib}{2}}(a) K_{-\frac{1}{4} + \frac{ib}{2}}(a) + I_{-\frac{1}{4} + \frac{ib}{2}}(a) K_{-\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0] \quad \text{ET I 37(65)}$$

$$7. \quad \int_0^\infty \sin(a \cosh x) \sin(a \sinh x) \frac{dx}{\sinh x} = \frac{\pi}{2} \sin a \quad [a > 0] \quad \text{BI (264)(22)}$$

3.995

$$1. \quad \int_0^{\pi/2} \frac{\sin(2a \cos^2 x) \cosh(a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} dx = \frac{\pi}{2bc} \sin \frac{2ac}{b+c} \\ [b > 0, \quad c > 0] \quad \text{BI (273)(9)}$$

$$2. \quad \int_0^{\pi/2} \frac{\cos(2a \cos^2 x) \cosh(a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} dx = \frac{\pi}{2bc} \cos \frac{2ac}{b+c} \\ [b > 0, \quad c > 0] \quad \text{BI (273)(10)}$$

3.996

$$1. \quad \int_0^\infty \sin(a \sinh x) \sinh \beta x \, dx = \sin \frac{\beta \pi}{2} K_\beta(a) \quad [|\operatorname{Re} \beta| < 1, \quad a > 0] \quad \text{EH II 82(26)}$$

$$2. \quad \int_0^\infty \cos(a \sinh x) \cosh \beta x \, dx = \cos \frac{\beta \pi}{2} K_\beta(a) \quad [|\operatorname{Re} \beta| < 1, \quad a > 0] \quad \text{WA 202(13)}$$

$$3. \quad \int_0^{\pi/2} \cos(a \sin x) \cosh(\beta \cos x) \, dx = \frac{\pi}{2} J_0\left(\sqrt{a^2 - \beta^2}\right) \quad \text{MO 40}$$

$$4. \quad \int_0^\infty \sin\left(a \cosh x - \frac{1}{2}\beta\pi\right) \cosh \beta x \, dx = \frac{\pi}{2} J_\beta(a) \quad [|\operatorname{Re} \beta| < 1, \quad a > 0] \quad \text{WA 199(12)}$$

$$5. \quad \int_0^\infty \cos\left(a \cosh x - \frac{1}{2}\beta\pi\right) \cosh \beta x \, dx = -\frac{\pi}{2} Y_\beta(a) \quad [|\operatorname{Re} \beta| < 1, \quad a > 0] \quad \text{WA 199(13)}$$

3.997

$$1. \quad \int_0^{\pi/2} \sin^\nu x \sinh(\beta \cos x) \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\nu+1}{2}\right) \mathbf{L}_{\frac{\nu}{2}}(\beta) \quad [\operatorname{Re} \nu > -1] \quad \text{EH II 38(53)}$$

$$2. \quad \int_0^\pi \sin^\nu x \cosh(\beta \cos x) \, dx = \sqrt{\pi} \left(\frac{2}{\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\nu+1}{2}\right) I_{\frac{\nu}{2}}(\beta) \quad [\operatorname{Re} \nu > -1] \quad \text{WH}$$

$$3. \quad \int_0^{\pi/2} \frac{dx}{\cosh(\tan x) \cos x \sqrt{\sin 2x}} = \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \quad \text{BI (276)(13)}$$

$$4. \quad \int_0^{\pi/2} \frac{\tan^q x}{\cosh(\tan x) + \cos \lambda \sin 2x} \frac{dx}{\sin \lambda} = \frac{\Gamma(q)}{\sin \lambda} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin k\lambda}{k^q} \quad [q > 0] \quad \text{BI (275)(20)}$$

4.11–4.12 Combinations involving trigonometric and hyperbolic functions and powers

4.111

$$1. \quad \int_0^\infty \frac{\sin ax}{\sinh \beta x} \cdot x^{2m} dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\tanh \frac{a\pi}{2\beta} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.981} \ 1) \quad \text{GW (336)(17a)}$$

$$2. \quad \int_0^\infty \frac{\cos ax}{\sinh \beta x} \cdot x^{2m+1} dx = (-1)^m \frac{\pi}{2\beta} \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\tanh \frac{a\pi}{2\beta} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.981} \ 1) \quad \text{GW (336)(17b)}$$

$$3. \quad \int_0^\infty \frac{\sin ax}{\cosh \beta x} \cdot x^{2m+1} dx = (-1)^{m+1} \frac{\pi}{2\beta} \cdot \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\frac{1}{\cosh \frac{a\pi}{2\beta}} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.981} \ 3) \quad \text{GW (336)(18b)}$$

$$4. \quad \int_0^\infty \frac{\cos ax}{\cosh \beta x} \cdot x^{2m} dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\frac{1}{\cosh \frac{a\pi}{2\beta}} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. } \mathbf{3.981} \ 3) \quad \text{GW (336)(18a)}$$

$$5. \quad \int_0^\infty x \frac{\sin 2ax}{\cosh \beta x} dx = \frac{\pi^2}{4\beta^2} \cdot \frac{\sinh \frac{a\pi}{\beta}}{\cosh^2 \frac{a\pi}{\beta}} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (364)(6)a}$$

$$6. \quad \int_0^\infty x \frac{\cos 2ax}{\sinh \beta x} dx = \frac{\pi^2}{4\beta^2} \cdot \frac{1}{\cosh^2 \frac{a\pi}{\beta}} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (364)(1)a}$$

$$7. \quad \int_0^\infty \frac{\sin ax}{\cosh \beta x} \frac{dx}{x} = 2 \arctan \left(\exp \frac{\pi a}{2\beta} \right) - \frac{\pi}{2} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (387)(1), ET I 89(13), LI (298)(17)}$$

4.112

$$1. \quad \int_0^\infty (x^2 + \beta^2) \frac{\cos ax}{\cosh \frac{\pi x}{2\beta}} dx = \frac{2\beta^3}{\cosh^3 a\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 32(19)}$$

$$2. \quad \int_0^\infty x (x^2 + 4\beta^2) \frac{\cos ax}{\sinh \frac{\pi x}{2\beta}} dx = \frac{6\beta^4}{\cosh^4 a\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 32(20)}$$

4.113

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{x^2 + \beta^2} &= -\frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{\beta \sin \pi \beta} \\
 &\quad + \frac{1}{2\beta^2} \left[{}_2F_1(1, -\beta; 1 - \beta; -e^{-a}) + {}_2F_1(1, \beta; 1 + \beta; -e^{-a}) \right] \\
 &= \frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{2\beta \sin \pi \beta} - \sum_{k=1}^\infty \frac{(-1)^k e^{-ak}}{k^2 - \beta^2} \\
 &\quad [\operatorname{Re} \beta > 0, \quad \beta \neq 0, 1, 2, \dots, \quad a > 0] \quad \text{ET I 90(18)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{x^2 + m^2} &= \frac{(-1)^m a e^{-ma}}{2m} + \frac{1}{2m} \sum_{k=1}^{m-1} \frac{(-1)^k e^{-ka}}{m - k} + \frac{(-1)^m e^{-ma}}{2m} \ln(1 + e^{-a}) \\
 &\quad + \frac{1}{2m!} \frac{d^{m-1}}{dz^{m-1}} \left[\frac{(1+z)^{m-1}}{z} \ln(1+z) \right]_{z=e^{-a}} \\
 &\quad [a > 0] \quad \text{ET I 89(17)}
 \end{aligned}$$

$$3. \quad \int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin ax}{\sinh \pi x} \frac{dx}{1+x^2} = -\frac{a}{2} \cosh a + \sinh a \ln \left(2 \cosh \frac{a}{2} \right) \quad \text{GW (336)(21b)}$$

$$\begin{aligned}
 4. \quad \int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} &= \frac{1}{2} \int_{-\infty}^\infty \frac{\sin ax}{\sinh \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{2} \sinh a - \cosh a \arctan(\sinh a) \\
 &\quad \text{GW (336)(21a)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{4} x} \cdot \frac{dx}{1+x^2} &= -\frac{\pi}{\sqrt{2}} e^{-a} + \frac{\sinh a}{\sqrt{2}} \ln \frac{2 \cosh a + \sqrt{2}}{2 \cosh a - \sqrt{2}} + \sqrt{2} \cosh a \arctan \frac{\sqrt{2}}{2 \sinh a} \\
 &\quad [a > 0] \quad \text{LI (389)(1)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_0^\infty \frac{\sin ax}{\cosh \frac{\pi}{4} x} \cdot \frac{x dx}{1+x^2} &= \frac{\pi}{\sqrt{2}} e^{-a} + \frac{\sinh a}{\sqrt{2}} \ln \frac{2 \cosh a + \sqrt{2}}{2 \cosh a - \sqrt{2}} - \sqrt{2} \cosh a \arctan \left(\frac{1}{\sqrt{2} \sinh a} \right) \\
 &\quad [a > 0] \quad \text{BI (388)(1)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \int_0^\infty \frac{\cos ax}{\sinh \pi x} \cdot \frac{x dx}{1+x^2} &= -\frac{1}{2} + \frac{a}{2} e^{-a} + \cosh a \ln(1 + e^{-a}) \\
 &\quad [a > 0] \quad \text{BI (389)(14), ET I 32(24)}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int_0^\infty \frac{\cos ax}{\sinh \frac{\pi}{2} x} \cdot \frac{x dx}{1+x^2} &= 2 \sinh a \arctan(e^{-a}) + \frac{\pi}{2} e^{-a} - 1 \\
 &\quad [a > 0] \quad \text{BI (389)(11)}
 \end{aligned}$$

$$\begin{aligned}
 9.^{11} \quad \int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{x^2 + \beta^2} &= \frac{\pi e^{-a\beta}}{2\beta \cos(\beta\pi)} - \sum_{k=0}^\infty \frac{(-1)^k e^{-(k+1/2)a}}{(k + \frac{1}{2})^2 - \beta^2} \\
 &\quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 32(26)}
 \end{aligned}$$

$$\begin{aligned}
 10.^{11} \quad \int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{x^2 + (m + \frac{1}{2})^2} &= \frac{(-1)^m e^{-a\beta} (a\beta + \frac{1}{2})}{2\beta^2} - \sum_{k=0}^\infty \frac{(-1)^k e^{-(k+1/2)a}}{(k + \frac{1}{2})^2 - \beta^2} \\
 &\quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 32(25)}
 \end{aligned}$$

$$11. \quad \int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{1+x^2} = 2 \cosh \frac{a}{2} - [e^a \arctan(e^{-\frac{a}{2}}) + e^{-a} \arctan(e^{\frac{a}{2}})]$$

[$a > 0$] ET I 32(21)

$$12. \quad \int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = ae^{-a} + \cosh a \ln(1 + e^{-2a}) \quad [a > 0]$$

BI (388)(6)

$$13. \quad \int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{4} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{\sqrt{2}} e^{-a} + \frac{2 \sinh a}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2} \sinh a}\right) - \frac{\cosh a}{\sqrt{2}} \ln \frac{2 \cosh a + \sqrt{2}}{2 \cosh a - \sqrt{2}}$$

[$a > 0$] BI (388)(5)

4.114

$$1. \quad \int_0^\infty \frac{\sin ax}{x} \frac{\sinh \beta x}{\sinh \gamma x} dx = \arctan\left(\tan \frac{\beta \pi}{2\gamma} \tanh \frac{a\pi}{2\gamma}\right) \quad [| \operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{BI (387)(6)a}$$

$$2. \quad \int_0^\infty \frac{\cos ax}{x} \frac{\sinh \beta x}{\cosh \gamma x} dx = \frac{1}{2} \ln \frac{\cosh \frac{a\pi}{2\gamma} + \sin \frac{\beta\pi}{2\gamma}}{\cosh \frac{a\pi}{2\gamma} - \sin \frac{\beta\pi}{2\gamma}} \quad [| \operatorname{Re} \beta| < \operatorname{Re} \gamma]$$

ET I 33(34)

4.115

$$1. \quad \int_0^\infty \frac{x \sin ax}{x^2 + b^2} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{\pi e^{-ab} \sin b\beta}{2 \sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{ke^{-ak} \sin k\beta}{k^2 - b^2}$$

[$0 < \operatorname{Re} \beta < \pi, \quad a > 0, \quad b > 0$] BI (389)(23)

$$2. \quad \int_0^\infty \frac{x \sin ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{1}{2} e^{-a} (a \sin \beta - \beta \cos \beta) - \frac{1}{2} \sinh a \sin \beta \ln [1 + 2e^{-a} \cos \beta + e^{-2a}]$$

$$+ \cosh a \cos \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$$

[$| \operatorname{Re} \beta| < \pi, \quad a > 0$] LI (389)(10)

$$3. \quad \int_0^\infty \frac{x \sin ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \frac{\pi}{2} x} dx$$

$$= \frac{\pi}{2} e^{-a} \sin \beta + \frac{1}{2} \cos \beta \sinh a \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta} - \sin \beta \cosh a \arctan\left(\frac{\cos \beta}{\sinh a}\right)$$

[$| \operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0$] BI (389)(8)

$$4. \quad \int_0^\infty \frac{\cos ax}{x^2 + b^2} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{\pi}{2b} \cdot \frac{e^{-ab} \sin b\beta}{\sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{e^{-ak} \sin k\beta}{k^2 - b^2}$$

[$0 < \operatorname{Re} \beta < \pi, \quad a > 0, \quad b > 0$] BI (389)(22)

$$5. \quad \int_0^\infty \frac{\cos ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{1}{2} e^{-a} (a \sin \beta - \beta \cos \beta) + \frac{1}{2} \cosh a \sin \beta \ln (1 + 2e^{-a} \cos \beta + e^{-2a})$$

$$- \sinh a \cos \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$$

[$| \operatorname{Re} \beta| < \pi, \quad a > 0, \quad b > 0$] BI (389)(20)a

$$6. \quad \int_0^\infty \frac{\cos ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \frac{\pi}{2} x} dx = \frac{\pi}{2} e^{-a} \sin \beta - \frac{1}{2} \cosh a \cos \beta \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta} + \sinh a \sin \beta \arctan \frac{\cos \beta}{\sinh a}$$

$$\left[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0, \quad b > 0 \right]$$

BI (389)(18)

$$7. \quad \int_0^\infty \frac{\sin ax}{x^2 + \frac{1}{4}} \cdot \frac{\sinh \beta x}{\cosh \pi x} dx = e^{-\frac{a}{2}} \left(a \sin \frac{\beta}{2} - \beta \cos \frac{\beta}{2} \right) - \sinh \frac{a}{2} \sin \frac{\beta}{2} \ln (1 + 2e^{-a} \cos \beta + e^{-2a})$$

$$+ \cosh \frac{a}{2} \cos \frac{\beta}{2} \arctan \frac{\sin \beta}{1 + e^{-a} \cos \beta}$$

$$[|\operatorname{Re} \beta| < \pi, \quad a > 0]$$

ET I 91(26)

$$8. \quad \int_0^\infty \frac{\sin ax}{x^2 + \beta^2} \cdot \frac{\cosh \gamma x}{\sinh \pi x} dx = \frac{1}{2\beta^2} - \frac{\pi}{2\beta} \cdot \frac{e^{-a\beta} \cos \beta \gamma}{\sin \beta \pi} + \sum_{k=1}^\infty (-1)^{k-1} \frac{e^{-ak} \cos k\gamma}{k^2 - \beta^2}$$

$$[0 \leq \operatorname{Re} \beta, \quad |\operatorname{Re} \gamma| < \pi, \quad a > 0]$$

BI (389)(21)

$$9. \quad \int_0^\infty \frac{\sin ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \pi x} dx = -\frac{1}{2} e^{-a} (a \cos \beta + \beta \sin \beta) + \frac{1}{2} \sinh a \cos \beta \ln (1 + 2e^{-a} \cos \beta + e^{-2a})$$

$$+ \cosh a \sin \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$$

$$[|\operatorname{Re} \beta| < \pi, \quad a > 0] \quad \text{ET I 91(25), LI (389)(9)}$$

$$10. \quad \int_0^\infty \frac{\sin ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \frac{\pi}{2} x} dx = -\frac{\pi}{2} e^{-a} \cos \beta + \frac{1}{2} \sinh a \sin \beta \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta} + \cosh a \cos \beta \arctan \frac{\cos \beta}{\sinh a}$$

$$[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0]$$

BI (389)(7)

$$11. \quad \int_0^\infty \frac{x \cos ax}{x^2 + b^2} \cdot \frac{\cosh \beta x}{\sinh \pi x} dx = \frac{\pi}{2} \cdot \frac{e^{-ab} \cos b\beta}{\sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{ke^{-ak} \cos k\beta}{k^2 - b^2}$$

$$[|\operatorname{Re} \beta| < \pi, \quad a > 0]$$

BI (389)(24)

$$12. \quad \int_0^\infty \frac{x \cos ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \pi x} dx = \frac{1}{2} e^{-a} (a \cos \beta + \beta \sin \beta)$$

$$- \frac{1}{2} + \frac{1}{2} \cosh a \cos \beta \ln [1 + 2e^{-a} \cos \beta + e^{-2a}]$$

$$+ \sinh a \sin \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$$

$$[|\operatorname{Re} \beta| < \pi, \quad a > 0]$$

BI (389)(19)

$$13. \quad \int_0^\infty \frac{x \cos ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \frac{\pi}{2} x} dx = -1 + \frac{\pi}{2} e^{-a} \cos \beta + \frac{1}{2} \cosh a \sin \beta \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta}$$

$$+ \sinh a \cos \beta \arctan \frac{\cos \beta}{\sinh a}$$

$$[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0]$$

BI (389)(17)

$$14. \quad \int_0^\infty \frac{\cos ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\cosh \frac{\pi}{2} x} dx = ae^{-a} \cos \beta + \beta e^{-a} \sin \beta + \sinh a \sin \beta \arctan \frac{e^{-2a} \sin 2\beta}{1 + e^{-2a} \cos 2\beta} \\ + \frac{1}{2} \cosh a \cos \beta \ln (1 + 2e^{-2a} \cos 2\beta + e^{-4a}) \\ \left[|\operatorname{Re} \beta| < \frac{\pi}{2}, a > 0 \right] \quad \text{ET I 34(37)}$$

4.116

$$1.^6 \quad \int_0^\infty x \cos 2ax \tanh x dx \quad \text{the integral is divergent} \quad \text{BI (364)(2)}$$

$$2. \quad \int_0^\infty \cos ax \tanh \beta x \frac{dx}{x} = \ln \coth \frac{a\pi}{4\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (387)(8)}$$

4.117

$$1. \quad \int_0^\infty \frac{\sin ax}{1 + x^2} \tanh \frac{\pi x}{2} dx = a \cosh a - \sinh a \ln (2 \sinh a) \\ [a > 0] \quad \text{BI (388)(3)}$$

$$2. \quad \int_0^\infty \frac{\sin ax}{1 + x^2} \tanh \frac{\pi x}{4} dx = -\frac{\pi}{2} e^a + \sinh a \ln \coth \frac{a}{2} + 2 \cosh a \arctan (e^a) \quad \text{BI (388)(4)}$$

$$3. \quad \int_0^\infty \frac{\sin ax}{1 + x^2} \coth \pi x dx = \frac{a}{2} e^{-a} - \sinh a \ln (1 - e^{-a}) \quad [a > 0] \quad \text{BI (389)(5)}$$

$$4. \quad \int_0^\infty \frac{\sin ax}{1 + x^2} \coth \frac{\pi}{2} x dx = \sinh a \ln \coth \frac{a}{2} \quad [a > 0] \quad \text{BI (389)(6)}$$

$$5. \quad \int_0^\infty \frac{x \cos ax}{1 + x^2} \tanh \frac{\pi}{2} x dx = -ae^{-a} - \cosh a \ln (1 - e^{-2a}) \\ [a > 0] \quad \text{BI (388)(7)}$$

$$6. \quad \int_0^\infty \frac{x \cos ax}{1 + x^2} \tanh \frac{\pi}{4} x dx - \frac{\pi}{2} e^a + \cosh a \ln \coth \frac{a}{2} + 2 \sinh a \arctan (e^a) \\ [a > 0] \quad \text{BI (388)(8)}$$

$$7. \quad \int_0^\infty \frac{x \cos ax}{1 + x^2} \coth \pi x dx = -\frac{a}{2} e^{-a} - \frac{1}{2} - \cosh a \ln (1 - e^{-a}) \quad \text{BI (389)(15)a, ET I 33(31)a}$$

$$8. \quad \int_0^\infty \frac{x \cos ax}{1 + x^2} \coth \frac{\pi}{2} x dx = -1 + \cosh a \ln \coth \frac{a}{2} \quad [a > 0] \quad \text{BI (389)(12)}$$

$$9. \quad \int_0^\infty \frac{x \cos ax}{1 + x^2} \coth \frac{\pi}{4} x dx = -2 + \frac{\pi}{2} e^{-a} + \cosh a \ln \coth \frac{a}{2} + 2 \sinh a \arctan (e^{-a}) \\ [a > 0] \quad \text{BI (389)(13)}$$

$$4.118^8 \quad \int_0^\infty \frac{x \sin ax}{\cosh^2 x} dx = \frac{\pi}{2} \frac{1}{\sinh \frac{1}{2} \pi a} \left(\frac{1}{2} \pi a \coth \frac{1}{2} \pi a - 1 \right) \quad \text{ET I 89(14)}$$

$$4.119 \quad \int_0^\infty \frac{1 - \cos px}{\sinh qx} \cdot \frac{dx}{x} = \ln \left(\cosh \frac{p\pi}{2q} \right) \quad \text{BI (387)(2)a}$$

4.121

1.
$$\int_0^\infty \frac{\sin ax - \sin bx}{\cosh \beta x} \cdot \frac{dx}{x} = 2 \arctan \frac{\exp \frac{a\pi}{2\beta} - \exp \frac{b\pi}{2\beta}}{1 + \exp \frac{(a+b)\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0] \quad \text{GW (336)(19b)}$$
2.
$$\int_0^\infty \frac{\cos ax - \cos bx}{\sinh \beta x} \cdot \frac{dx}{x} = \ln \frac{\cosh \frac{b\pi}{2\beta}}{\cosh \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0] \quad \text{GW (336)(19a)}$$

4.122

- 1.⁶
$$\int_0^\infty \frac{\cos \beta x \sin \gamma x}{\cosh \delta x} \cdot \frac{dx}{x} = \arctan \frac{\sinh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta\pi}{2\delta}} \quad [\operatorname{Re} \delta > |\operatorname{Im} \beta| + |\operatorname{Im} \gamma|] \quad \text{ET I 93(46)a}$$
2.
$$\int_0^\infty \sin^2 ax \frac{\cosh \beta x}{\sinh x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{\cosh 2a\pi + \cos \beta\pi}{1 + \cos \beta\pi} \quad [|\operatorname{Re} \beta| < 1] \quad \text{BI (387)(7)}$$

4.123

1.
$$\int_0^\infty \frac{\sin x}{\cosh ax + \cos x} \cdot \frac{x dx}{x^2 - \pi^2} = \arctan \frac{1}{a} - \frac{1}{a} \quad \text{BI (390)(1)}$$
2.
$$\int_0^\infty \frac{\sin x}{\cosh ax - \cos x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{a}{1 + a^2} - \arctan \frac{1}{a} \quad \text{BI (390)(2)}$$
3.
$$\int_0^\infty \frac{\sin 2x}{\cosh 2ax - \cos 2x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{1}{2a} \cdot \frac{1 + 2a^2}{1 + a^2} - \arctan \frac{1}{a} \quad \text{BI (390)(4)}$$
4.
$$\int_0^\infty \frac{\cosh ax \sin x}{\cosh 2ax - \cos 2x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{-1}{2a(1 + a^2)} \quad \text{LI (390)(3)}$$
5.
$$\int_0^\infty \frac{\cos ax}{\cosh \pi x + \cos \pi \beta} \cdot \frac{dx}{x^2 + \gamma^2} = \frac{\pi e^{-a\gamma}}{2\gamma (\cos \gamma\pi + \cos \beta\pi)} + \frac{1}{\sinh \beta\pi} \sum_{k=0}^{\infty} \left\{ \frac{e^{-(2k+1-\beta)a}}{\gamma^2 - (2k+1-\beta)^2} - \frac{e^{-(2k+1+\beta)a}}{\gamma^2 - (2k+1+\beta)^2} \right\} \quad [0 < \operatorname{Re} \beta < 1, \quad \operatorname{Re} \gamma > 0, \quad a > 0] \quad \text{ET I 33(27)}$$
6.
$$\int_0^\infty \frac{\sin ax \sinh bx}{\cos 2ax + \cosh 2bx} x^{p-1} dx = \frac{\Gamma(p)}{(a^2 + b^2)^{\frac{p}{2}}} \sin \left(p \arctan \frac{a}{b} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^p} \quad [p > 0] \quad \text{BI (364)(8)}$$
7.
$$\int_0^\infty \sin ax^2 \frac{\sin \frac{\pi x}{2} \sinh \frac{\pi x}{2}}{\cos \pi x + \cosh \pi x} \cdot x dx = \frac{1}{4} \left[\frac{\partial \vartheta_1(z | q)}{\partial z} \right]_{z=0, q=e^{-2a}} \quad [a > 0] \quad \text{ET I 93(49)}$$

4.124

$$1. \quad \int_0^1 \frac{\cos px \cosh(q\sqrt{1-x^2})}{\sqrt{1-x^2}} dx = \frac{\pi}{2} J_0(\sqrt{p^2 - q^2}) \quad \text{MO (40)}$$

$$2. \quad \int_u^\infty \cos ax \cosh \sqrt{\beta(u^2 - x^2)} \cdot \frac{dx}{\sqrt{u^2 - x^2}} = \frac{\pi}{2} J_0\left(\frac{u}{\sqrt{a^2 - \beta^2}}\right) \quad \text{ET I 34(38)}$$

4.125

$$1. \quad \int_0^\infty \sinh(a \sin x) \cos(a \cos x) \sin x \sin 2nx \frac{dx}{x} = \frac{(-1)^{n-1} a^{2n-1}}{(2n-1)!} \frac{\pi}{8} \left[1 + \frac{a^2}{2n(2n+1)}\right] \quad \text{LI (367)(14)}$$

$$2. \quad \int_0^\infty \cosh(a \sin x) \cos(a \cos x) \sin x \cos(2n-1)x \frac{dx}{x} = \frac{(-1)^{n-1} a^{2(n-1)}}{[2(n-1)]!} \frac{\pi}{8} \left[1 - \frac{a^2}{2n(2n-1)}\right] \quad \text{LI (367)(15)}$$

$$3. \quad \int_0^\infty \sinh(a \sin x) \cos(a \cos x) \cos x \cos 2nx \frac{dx}{x} = \frac{\pi}{2} \sum_{k=n+1}^\infty \frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^n a^{2n+1}}{(2n+1)!} \frac{3\pi}{8} + \frac{(-1)^{n-1} a^{2n-1}}{(2n-1)!} \frac{\pi}{8} \quad \text{LI (367)(21)}$$

4.126

$$1. \quad \int_0^\infty \sin(a \cos bx) \sinh(a \sin bx) \frac{x dx}{c^2 - x^2} = \frac{\pi}{2} [\cos(a \cos bc) \cosh(a \sin bc) - 1] \quad [b > 0] \quad \text{BI (381)(2)}$$

$$2. \quad \int_0^\infty \sin(a \cos bx) \cosh(a \sin bx) \frac{dx}{c^2 - x^2} = \frac{\pi}{2c} \cos(a \cos bc) \sinh(a \sin bc) \quad [b > 0, \quad c > 0] \quad \text{BI (381)(1)}$$

$$3. \quad \int_0^\infty \cos(a \cos bx) \sinh(a \sin bx) \frac{x dx}{c^2 - x^2} = \frac{\pi}{2} [a \cos bc - \sin(a \cos bc) \cosh(a \sin bc)] \quad [b > 0] \quad \text{BI (381)(4)}$$

$$4. \quad \int_0^\infty \cos(a \cos bx) \cosh(a \sin bx) \frac{dx}{c^2 - x^2} = -\frac{\pi}{2c} \sin(a \cos bc) \sinh(a \sin bc) \quad [b > 0] \quad \text{BI (381)(3)}$$

4.13 Combinations of trigonometric and hyperbolic functions and exponentials

4.131

$$1. \quad \int_0^\infty \sin ax \sinh^\nu \gamma x e^{-\beta x} dx = -\frac{i \Gamma(\nu+1)}{2^{\nu+2} \gamma} \left\{ \frac{\Gamma\left(\frac{\beta - \nu\gamma - ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta + \nu\gamma - ai}{2\gamma} + 1\right)} - \frac{\Gamma\left(\frac{\beta - \nu\gamma + ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta + \gamma\nu + ai}{2\gamma} + 1\right)} \right\} \quad [\operatorname{Re} \nu > -2, \quad \operatorname{Re} \gamma > 0, \quad |\operatorname{Re}(\gamma\nu)| < \operatorname{Re} \beta] \quad \text{ET I 91(30)a}$$

$$2. \quad \int_0^\infty \cos ax \sinh^\nu \gamma x e^{-\beta x} dx = \frac{\Gamma(\nu+1)}{2^{\nu+2}\gamma} \left\{ \frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu+ai}{2\gamma}+1\right)} \right\} \\ [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \gamma > 0, \quad |\operatorname{Re}(\gamma\nu)| < \operatorname{Re} \beta] \quad \text{ET I 34(40)a}$$

$$3. \quad \int_0^\infty e^{-\beta x} \frac{\sin ax}{\sinh \gamma x} dx = \sum_{k=1}^\infty \frac{2a}{a^2 + [\beta + (2k-1)\gamma]^2} \quad \text{BI (264)(9)a} \\ = \frac{1}{2\gamma i} \left[\psi\left(\frac{\beta + \gamma + ia}{2\gamma}\right) - \psi\left(\frac{\beta + \gamma - ia}{2\gamma}\right) \right] \quad [\operatorname{Re} \beta > |\operatorname{Re} \gamma|] \quad \text{ET I 91(28)}$$

$$4. \quad \int_0^\infty e^{-x} \frac{\sin ax}{\sinh x} dx = \frac{\pi}{2} \coth \frac{a\pi}{2} - \frac{1}{a} \quad \text{ET I 91(29)}$$

4.132

$$1. \quad \int_0^\infty \frac{\sin ax \sinh \beta x}{e^{\gamma x} - 1} dx = -\frac{a}{2(a^2 + \beta^2)} + \frac{\pi}{2\gamma} \cdot \frac{\sinh \frac{2\pi a}{\gamma}}{\cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma}} \\ + \frac{i}{2\gamma} \left[\psi\left(\frac{\beta}{\gamma} + i\frac{a}{\gamma} + 1\right) - \psi\left(\frac{\beta}{\gamma} - i\frac{a}{\gamma} + 1\right) \right] \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, a > 0] \quad \text{ET I 92(33)}$$

$$2. \quad \int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x} - 1} dx = -\frac{a}{2(a^2 + \beta^2)} + \frac{\pi}{2\gamma} \cdot \frac{\sinh \frac{2\pi a}{\gamma}}{\cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma}} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|] \quad \text{BI (265)(5)a, ET I 92(34)}$$

$$3. \quad \int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x} + 1} dx = \frac{a}{2(a^2 + \beta^2)} - \frac{\pi}{\gamma} \cdot \frac{\sinh \frac{a\pi}{\gamma} \cos \frac{\beta\pi}{\gamma}}{\cosh \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|] \quad \text{ET I 92(35)}$$

$$4. \quad \int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x} - 1} dx = \frac{\beta}{2(a^2 + \beta^2)} - \frac{\pi}{2\gamma} \cdot \frac{\sin \frac{2\pi \beta}{\gamma}}{\cosh \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|] \quad \text{LI (265)(8)}$$

$$5. \quad \int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x} + 1} dx = -\frac{\beta}{2(a^2 + \beta^2)} + \frac{\pi}{\gamma} \frac{\sin \frac{\pi \beta}{\gamma} \cosh \frac{\pi a}{\gamma}}{\cosh \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|] \quad \text{ET I 34(39)}$$

4.133

$$1.^{11} \quad \int_0^\infty \sin ax \sinh \beta x \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma} \exp[\gamma(\beta^2 - a^2)] \sin(2a\beta\gamma) \\ [\operatorname{Re} \gamma > 0] \quad \text{ET I 92(37)}$$

$$2.^{11} \quad \int_0^\infty \cos ax \cosh \beta x \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma} \exp[\gamma(\beta^2 - a^2)] \cos(2a\beta\gamma) \\ [\operatorname{Re} \gamma > 0] \quad \text{ET I 35(41)}$$

4.134

$$1. \quad \int_0^\infty e^{-\beta x^2} (\cosh x - \cos x) dx = \sqrt{\frac{\pi}{\beta}} \cosh \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$2. \quad \int_0^\infty e^{-\beta x^2} (\cosh x + \cos x) dx = \sqrt{\frac{\pi}{\beta}} \sinh \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

4.135

$$1. \quad \int_0^\infty \sin ax^2 \cosh 2\gamma x e^{-\beta x^2} dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2 + \beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2 + \beta^2}\right) \sin\left(\frac{a\gamma^2}{a^2 + \beta^2} + \frac{1}{2} \arctan \frac{a}{\beta}\right) \\ [\operatorname{Re} \beta > 0] \quad \text{LI (268)(7)}$$

$$2. \quad \int_0^\infty \cos ax^2 \cosh 2\gamma x e^{-\beta x^2} dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2 + \beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2 + \beta^2}\right) \cos\left(\frac{a\gamma^2}{a^2 + \beta^2} + \frac{1}{2} \arctan \frac{a}{\beta}\right) \\ [\operatorname{Re} \beta > 0] \quad \text{LI (268)(8)}$$

4.136

$$1. \quad \int_0^\infty (\sinh^2 x + \sin x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \cosh \frac{1}{8\beta} \\ [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$2. \quad \int_0^\infty (\sinh^2 x - \sin x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \sinh \frac{1}{8\beta} \\ [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$3. \quad \int_0^\infty (\cosh^2 x + \cos x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \cosh \frac{1}{8\beta} \\ [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$4. \quad \int_0^\infty (\cosh^2 x - \cos x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \sinh \frac{1}{8\beta} \\ [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

4.137

$$1. \quad \int_0^\infty \sin 2x^2 \sinh 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} + \frac{\pi}{4}\right) \\ [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

$$2. \quad \int_0^\infty \sin 2x^2 \cosh 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} - \frac{\pi}{4}\right) \\ [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

$$3. \quad \int_0^\infty \cos 2x^2 \sinh 2x^2 e^{-\beta x^4} dx = \frac{-\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} - \frac{\pi}{4}\right) \\ [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

$$4. \quad \int_0^\infty \cos 2x^2 \cosh 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128}\beta^2} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} + \frac{\pi}{4}\right)$$

[Re $\beta > 0$] MI 32

4.138

$$1. \quad \int_0^\infty (\sin^2 2x \cosh 2x^2 + \cos 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32}\beta^2} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right)$$

[Re $\beta > 0$] MI 32

$$2. \quad \int_0^\infty (\sin^2 2x \cosh 2x^2 - \cos 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32}\beta^2} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right)$$

[Re $\beta > 0$] MI 32

$$3. \quad \int_0^\infty (\cos^2 2x \cosh 2x^2 + \sin 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32}\beta^2} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right)$$

[Re $\beta > 0$] MI 32

$$4. \quad \int_0^\infty (\cos^2 2x \cosh 2x^2 - \sin 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32}\beta^2} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right)$$

[Re $\beta > 0$] MI 32

4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers

4.141

$$1. \quad \int_0^\infty x e^{-\beta x^2} \cosh x \sin x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{1}{2\beta} + \sin \frac{1}{2\beta} \right)$$

[Re $\beta > 0$] MI 32

$$2. \quad \int_0^\infty x e^{-\beta x^2} \sinh x \cos x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{1}{2\beta} - \sin \frac{1}{2\beta} \right)$$

[Re $\beta > 0$] MI 32

$$3. \quad \int_0^\infty x^2 e^{-\beta x^2} \cosh x \cos x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{1}{2\beta} - \frac{1}{\beta} \sin \frac{1}{2\beta} \right)$$

[Re $\beta > 0$] MI 32

$$4. \quad \int_0^\infty x^2 e^{-\beta x^2} \sinh x \sin x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\sin \frac{1}{2\beta} + \frac{1}{\beta} \cos \frac{1}{2\beta} \right)$$

[Re $\beta > 0$] MI 32

4.142

$$1. \quad \int_0^\infty x e^{-\beta x^2} (\sinh x + \sin x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \cosh \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$2. \quad \int_0^\infty x e^{-\beta x^2} (\sinh x - \sin x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \sinh \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$3. \quad \int_0^\infty x^2 e^{-\beta x^2} (\cosh x + \cos x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \left(\cosh \frac{1}{4\beta} + \frac{1}{2\beta} \sinh \frac{1}{4\beta} \right) \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$4. \quad \int_0^\infty x^2 e^{-\beta x^2} (\cosh x - \cos x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \left(\sinh \frac{1}{4\beta} + \frac{1}{2\beta} \cosh \frac{1}{4\beta} \right) \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

4.143

$$1. \quad \int_0^\infty x e^{-\beta x^2} (\cosh x \sin x + \sinh x \cos x) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \cos \frac{1}{2\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

$$2. \quad \int_0^\infty x e^{-\beta x^2} (\cosh x \sin x - \sinh x \cos x) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \sin \frac{1}{2\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

$$4.144 \quad \int_0^\infty e^{-x^2} \sinh x^2 \cos ax \frac{dx}{x^2} = \sqrt{\frac{\pi}{2}} e^{-\frac{a^2}{8}} - \frac{\pi a}{4} \left[1 - \Phi \left(\frac{a}{\sqrt{8}} \right) \right] \quad [a > 0] \quad \text{ET I 35(44)}$$

4.145

$$1. \quad \int_0^\infty x e^{-\beta x^2} \cosh (2ax \sin t) \sin (2ax \cos t) dx = \frac{a}{2} \sqrt{\frac{\pi}{\beta^3}} \exp \left(-\frac{a^2}{\beta} \cos 2t \right) \cos \left(t - \frac{a^2}{\beta} \sin 2t \right) \quad [\operatorname{Re} \beta > 0] \quad \text{BI (363)(5)}$$

$$2. \quad \int_0^\infty x e^{-\beta x^2} \sinh (2ax \sin t) \cos (2ax \cos t) dx = \frac{a}{2} \sqrt{\frac{\pi}{\beta^3}} \exp \left(-\frac{a^2}{\beta} \cos 2t \right) \sin \left(t - \frac{a^2}{\beta} \sin 2t \right) \quad [\operatorname{Re} \beta > 0] \quad \text{BI (363)(6)}$$

4.146¹⁰

$$1.^8 \quad \int_0^\infty e^{-\beta x^2} \sinh ax \sin bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \left(\frac{a^2 - b^2}{4\beta} \right) \sin \frac{ab}{2\beta} \quad [\operatorname{Re} \beta > 0]$$

$$2.^8 \quad \int_0^\infty e^{-\beta x^2} \cosh ax \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \left(\frac{a^2 - b^2}{4\beta} \right) \cos \frac{ab}{2\beta} \quad [\operatorname{Re} \beta > 0]$$

$$3. \quad \int_0^\infty x e^{-\beta x^2} \cosh ax \sin ax \, dx = \frac{a}{4\beta} \sqrt{\frac{\pi}{\beta}} \left(\cos \frac{a^2}{2\beta} + \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

$$4. \quad \int_0^\infty x e^{-\beta x^2} \sinh ax \cos ax \, dx = \frac{a}{4\beta} \sqrt{\frac{\pi}{\beta}} \left(\cos \frac{a^2}{2\beta} - \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

$$5.^8 \quad \int_0^\infty x^2 e^{-\beta x^2} \cosh ax \sin ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\sin \frac{a^2}{2\beta} + \frac{a^2}{\beta} \cos \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

$$6.^8 \quad \int_0^\infty x^2 e^{-\beta x^2} \cosh ax \cos ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{a^2}{2\beta} - \frac{a^2}{\beta} \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

4.2–4.4 Logarithmic Functions

4.21 Logarithmic functions

4.211

$$1. \quad \int_e^\infty \frac{dx}{\ln \frac{1}{x}} = -\infty \quad \text{BI (33)(9)}$$

$$2. \quad \int_0^u \frac{dx}{\ln x} = \operatorname{li} u \quad \text{FI III 653, FI II 606}$$

4.212

$$1.^7 \quad \int_0^1 \frac{dx}{a + \ln x} = e^{-a} \operatorname{Ei}(a) \quad [a > 0] \quad \text{BI (31)(4)}$$

$$2. \quad \int_0^1 \frac{dx}{a - \ln x} = -e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (31)(5)}$$

$$3.^7 \quad \int_0^1 \frac{dx}{(a + \ln x)^2} = -\frac{1}{a} + e^{-a} \operatorname{Ei}(a) \quad [a \geq 0] \quad \text{BI (31)(14)}$$

$$4. \quad \int_0^1 \frac{dx}{(a - \ln x)^2} = \frac{1}{a} + e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (31)(16)}$$

$$5.^8 \quad \int_0^1 \frac{\ln x \, dx}{(a + \ln x)^2} = 1 + (1 - a)e^{-a} \operatorname{Ei}(a) \quad [a \geq 0] \quad \text{BI (31)(15)}$$

$$6. \quad \int_0^1 \frac{\ln x \, dx}{(a - \ln x)^2} = 1 + (1 + a)e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (31)(17)}$$

$$7. \quad \int_1^e \frac{\ln x \, dx}{(1 + \ln x)^2} = \frac{e}{2} - 1 \quad \text{BI (33)(10)}$$

$$8.^7 \quad \int_0^1 \frac{dx}{(a + \ln x)^n} = \frac{1}{(n-1)!} e^{-a} \operatorname{Ei}(a) - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! a^{k-n} \quad [a \geq 0] \quad \text{BI (31)(22)}$$

$$9. \quad \int_0^1 \frac{dx}{(a - \ln x)^n} = \frac{(-1)^n}{(n-1)!} e^a \operatorname{Ei}(-a) + \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! (-a)^{k-n} \quad [a > 0, \quad n \text{ odd}] \quad \text{BI (31)(23)}$$

In integrals of the form $\int \frac{(\ln x)^m}{[a^n + (\ln x)^n]} dx$, it is convenient to make the substitution $x = e^{-t}$.

Results **4.212** 3, **4.212** 5, and **4.212** 8 [for $n > 1$] and **4.213** 6, **4.213** 8 below are divergent but may be considered to be valid if defined as follows:

$$\int_0^a \frac{f(z) dz}{(z - z_0)^n} = \frac{1}{(n-1)!} \left(\frac{d}{dz_0} \right)^{n-1} \left[\operatorname{PV} \int_0^a \frac{f(z)}{z - z_0} dz \right]$$

where $a > z_0 > 0, n = 1, 2, 3, \dots$ and PV indicates the Cauchy principal value.

4.213

$$1. \quad \int_0^1 \frac{dx}{a^2 + (\ln x)^2} = \frac{1}{a} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] \quad [a > 0] \quad \text{BI (31)(6)}$$

$$2.^7 \quad \int_0^1 \frac{dx}{a^2 - (\ln x)^2} = \frac{1}{2a} [e^{-a} \overline{\operatorname{Ei}}(a) - e^a \operatorname{Ei}(-a)] \quad [a > 0], \quad (\text{cf. } \mathbf{4.212} \text{ 1 and 2}) \quad \text{BI (31)(8)}$$

$$3. \quad \int_0^1 \frac{\ln x dx}{a^2 + (\ln x)^2} = \operatorname{ci}(a) \cos a + \operatorname{si}(a) \sin a \quad [a > 0] \quad \text{BI (31)(7)}$$

$$4.^7 \quad \int_0^1 \frac{\ln x dx}{a^2 - (\ln x)^2} = -\frac{1}{2} [e^{-a} \overline{\operatorname{Ei}}(a) + e^a \operatorname{Ei}(-a)] \quad [a > 0], \quad (\text{cf. } \mathbf{4.212} \text{ 1 and 2}) \quad \text{BI (31)(9)}$$

$$5. \quad \int_0^1 \frac{dx}{[a^2 + (\ln x)^2]^2} = \frac{1}{2a^3} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] - \frac{1}{2a^2} [\operatorname{ci}(a) \cos a + \operatorname{si}(a) \sin a] \quad [a > 0] \quad \text{LI (31)(18)}$$

$$6.^8 \quad \int_0^1 \frac{dx}{[a^2 - (\ln x)^2]^2} \quad \text{is divergent}$$

$$7. \quad \int_0^1 \frac{\ln x dx}{[a^2 + (\ln x)^2]^2} = \frac{1}{2a} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] - \frac{1}{2a^2} \quad [a > 0] \quad \text{BI (31)(19)}$$

$$8.^8 \quad \int_0^1 \frac{\ln x dx}{[a^2 - (\ln x)^2]^2} \quad \text{is divergent}$$

4.214

$$1. \quad \int_0^1 \frac{dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^3} [e^a \operatorname{Ei}(-a) - e^{-a} \overline{\operatorname{Ei}}(a) - 2 \operatorname{ci}(a) \sin a + 2 \operatorname{si}(a) \cos a] \\ [a > 0] \quad \text{BI (31)(10)}$$

$$2. \quad \int_0^1 \frac{\ln x \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^2} [e^a \operatorname{Ei}(-a) + e^{-a} \overline{\operatorname{Ei}}(a) - 2 \operatorname{ci}(a) \cos a - 2 \operatorname{si}(a) \sin a] \\ [a > 0] \quad \text{BI (31)(11)}$$

$$3. \quad \int_0^1 \frac{(\ln x)^2 \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a} [e^a \operatorname{Ei}(-a) - e^{-a} \overline{\operatorname{Ei}}(a) + 2 \operatorname{ci}(a) \sin a - 2 \operatorname{si}(a) \cos a] \\ [a > 0] \quad \text{BI (31)(12)}$$

$$4.^7 \quad \int_0^1 \frac{(\ln x)^3 \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4} [e^a \operatorname{Ei}(-a) + e^{-a} \overline{\operatorname{Ei}}(a) + 2 \operatorname{ci}(a) \cos a + 2 \operatorname{si}(a) \sin a] \\ [a > 0] \quad \text{BI (31)(13)}$$

4.215

$$1. \quad \int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} dx = \Gamma(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{FI II 778}$$

$$2. \quad \int_0^1 \frac{dx}{\left(\ln \frac{1}{x} \right)^\mu} = \frac{\pi}{\Gamma(\mu)} \operatorname{cosec} \mu\pi \quad [\operatorname{Re} \mu < 1] \quad \text{BI (31)(1)}$$

$$3. \quad \int_0^1 \sqrt{\ln \frac{1}{x}} \, dx = \frac{\sqrt{\pi}}{2} \quad \text{BI (32)(1)}$$

$$4. \quad \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \quad \text{BI (32)(3)}$$

4.216

$$1. \quad \int_0^{1/e} \frac{dx}{\sqrt{(\ln x)^2 - 1}} = K_0(1) \quad \text{GW (32)(2)}$$

$$2.^* \quad \int_0^{1/e} \frac{dx}{\sqrt{-\ln x - 1}} = \frac{\sqrt{\pi}}{e}$$

4.22 Logarithms of more complicated arguments**4.221**

$$1. \quad \int_0^1 \ln x \ln(1-x) \, dx = 2 - \frac{\pi^2}{6} \quad \text{BI (30)(7)}$$

$$2. \quad \int_0^1 \ln x \ln(1+x) \, dx = 2 - \frac{\pi^2}{12} - 2 \ln 2 \quad \text{BI (30)(8)}$$

$$3. \quad \int_0^1 \ln \frac{1-ax}{1-a} \frac{dx}{\ln x} = - \sum_{k=1}^{\infty} a^k \frac{\ln(1+k)}{k} \quad [a < 1] \quad \text{BI (31)(3)}$$

4.222

$$1. \quad \int_0^{\infty} \ln \frac{a^2 + x^2}{b^2 + x^2} dx = (a-b)\pi \quad [a > 0, \quad b > 0] \quad \text{GW (322)(20)}$$

$$2. \quad \int_0^{\infty} \ln x \ln \frac{a^2 + x^2}{b^2 + x^2} dx = \pi(b-a) + \pi \ln \frac{a^a}{b^b} \quad [a > 0, \quad b > 0] \quad \text{BI (33)(1)}$$

$$3. \quad \int_0^{\infty} \ln x \ln \left(1 + \frac{b^2}{x^2}\right) dx = \pi b (\ln b - 1) \quad [b > 0] \quad \text{BI (33)(2)}$$

$$4. \quad \int_0^{\infty} \ln(1 + a^2 x^2) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi \left[\frac{1+ab}{a} \ln(1+ab) - b \right] \\ [a > 0, \quad b > 0] \quad \text{BI (33)(3)}$$

$$5. \quad \int_0^{\infty} \ln(a^2 + x^2) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi [(a+b) \ln(a+b) - a \ln a - b] \\ [a > 0, \quad b > 0] \quad \text{BI (33)(4)}$$

$$6. \quad \int_0^{\infty} \ln \left(1 + \frac{a^2}{x^2}\right) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi [(a+b) \ln(a+b) - a \ln a - b \ln b] \\ [a > 0, \quad b > 0] \quad \text{BI (33)(5)}$$

$$7. \quad \int_0^{\infty} \ln \left(a^2 + \frac{1}{x^2}\right) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi \left[\frac{1+ab}{a} \ln(1+ab) - b \ln b \right] \\ [a > 0, \quad b > 0] \quad \text{BI (33)(7)}$$

$$8.* \quad \int_0^{\infty} \ln(1+ax) x^b e^{-x} dx = \sum_{m=0}^b \frac{b!}{(b-m)!} \left[\frac{(-1)^{b-m-1}}{a^{b-m}} e^{1/a} \text{Ei} \left(-\frac{1}{a}\right) + \sum_{k=1}^{b-m} \frac{(k-1)!}{(-a)^{b-m-k}} \right] \\ [b > 0, \quad \text{an integer}]$$

4.223

$$1. \quad \int_0^{\infty} \ln(1 + e^{-x}) dx = \frac{\pi^2}{12} \quad \text{BI (256)(10)}$$

$$2. \quad \int_0^{\infty} \ln(1 - e^{-x}) dx = -\frac{\pi^2}{6} \quad \text{BI (256)(11)}$$

$$3. \quad \int_0^{\infty} \ln(1 + 2e^{-x} \cos t + e^{-2x}) dx = \frac{\pi^2}{6} - \frac{t^2}{2} \quad [|t| < \pi] \quad \text{BI (256)(18)}$$

4.224

$$1. \quad \int_0^u \ln \sin x dx = L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right) \quad \text{LO III 186(15)}$$

$$2. \quad \int_0^{\pi/4} \ln \sin x dx = -\frac{\pi}{4} \ln 2 - \frac{1}{2} \mathbf{G} \quad \text{BI (285)(1)}$$

3. $\int_0^{\pi/2} \ln \sin x \, dx = \frac{1}{2} \int_0^{\pi} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$ FI II 629,643
4. $\int_0^u \ln \cos x \, dx = -L(u)$ LO III 184(10)
5. $\int_0^{\pi/4} \ln \cos x \, dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} \mathbf{G}$ BI (286)(1)
6. $\int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2$ BI 306(1)
7. $\int_0^{\pi/2} (\ln \sin x)^2 \, dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right]$ BI (305)(19)
8. $\int_0^{\pi/2} (\ln \cos x)^2 \, dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right]$ BI (306)(14)
- 9.⁸ $\int_0^{\pi} \ln(a + b \cos x) \, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2}$ [$a \geq |b| > 0$] GW (322)(15)
10. $\int_0^{\pi} \ln(1 \pm \sin x) \, dx = -\pi \ln 2 \pm 4 \mathbf{G}$ GW (322)(16a)
- 11.⁷ $\int_0^{\pi/2} \ln(1 + a \sin x) \, dx = \frac{\pi}{2} \ln \frac{a}{2} + 2 \mathbf{G} + 2 \sum_{k=1}^{\infty} \frac{b^k}{k} \sum_{n=1}^k \frac{(-1)^{n+1}}{2n-1}$ [$a > 0$] $b = \frac{1-a}{1+a}$
 $= -\frac{\pi}{2} \ln 2 + 2 \mathbf{G}$ [$a = 1$]
12. $\int_0^{\pi} \ln(1 + a \cos x) \, dx = \pi \ln \left(\frac{1 + \sqrt{1 - a^2}}{2} \right)$ [$a^2 \leq 1$] BI (330)(1)
- 12 (1) $\int_0^{\pi} \ln(1 + a \cos x)^2 \, dx = \begin{cases} 2\pi \ln \left(\frac{1 + \sqrt{1 - a^2}}{2} \right) & \text{for } a^2 \leq 1 \\ \frac{\pi}{2} \ln \frac{a^2}{4} & \text{for } a^2 \geq 1 \end{cases}$
13. $\int_0^{\pi/2} \ln(1 + 2a \sin x + a^2) \, dx = \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2}{(2k+1) \cdot (2k+1)!!} \left(\frac{2a}{1+a^2} \right)^{2k+1}$ [$a^2 \leq 1$] BI (308)(24)
- 14.¹¹ $\int_0^{n\pi} \ln(a^2 - 2ab \cos x + b^2) \, dx = 2n\pi \ln [\max(|a|, |b|)]$ [$ab > 0$] FI II 142, 163, 688
- 15.⁸ $\int_0^{n\pi} \ln(1 - 2a \cos x + a^2) \, dx = 0$ [$a^2 \leq 1$]
 $= n\pi \ln a^2$ [$a^2 \geq 1$]

4.225

1. $\int_0^{\pi/4} \ln(\cos x - \sin x) \, dx = -\frac{\pi}{8} \ln 2 - \frac{1}{2} \mathbf{G}$ GW (322)(9b)

$$2. \quad \int_0^{\pi/4} \ln(\cos x + \sin x) dx = \frac{1}{2} \int_0^{\pi/2} \ln(\cos x + \sin x) dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} G \quad \text{GW (322)(9a)}$$

$$3. \quad \int_0^{2\pi} \ln(1 + a \sin x + b \cos x) dx = 2\pi \ln \frac{1 + \sqrt{1 - a^2 - b^2}}{2} \quad [a^2 + b^2 < 1] \quad \text{BI (332)(2)}$$

$$4. \quad \int_0^{2\pi} \ln(1 + a^2 + b^2 + 2a \sin x + 2b \cos x) dx = 0 \quad [a^2 + b^2 \leq 1]$$

$$= 2\pi \ln(a^2 + b^2) \quad [a^2 + b^2 \geq 1] \quad \text{BI (322)(3)}$$

4.226

$$1. \quad \int_0^{\pi/2} \ln(a^2 - \sin^2 x)^2 dx = -2\pi \ln 2 \quad [a^2 \leq 1]$$

$$= 2\pi \ln \frac{a + \sqrt{a^2 - 1}}{2} = 2\pi (\operatorname{arccosh} a - \ln 2) \quad [a > 1] \quad \text{FI II 644, 687}$$

$$2. \quad \int_0^{\pi/2} \ln(1 + a \sin^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(1 + a \sin^2 x) dx = \int_0^{\pi/2} \ln(1 + a \cos^2 x) dx$$

$$= \frac{1}{2} \int_0^{\pi} \ln(1 + a \cos^2 x) dx = \pi \ln \frac{1 + \sqrt{1 + a}}{2} \quad [a \geq -1] \quad \text{BI (308)(15), GW(322)(12)}$$

$$3. \quad \int_0^u \ln(1 - \sin^2 \alpha \sin^2 x) dx = (\pi - 2\theta) \ln \cot \frac{\alpha}{2} + 2u \ln \left(\frac{1}{2} \sin \alpha \right) - \frac{\pi}{2} \ln 2$$

$$+ L(\theta + u) - L(\theta - u) + L\left(\frac{\pi}{2} - 2u\right) \quad \left[\cot \theta = \cos \alpha \tan u; \quad -\pi \leq \alpha \leq \pi, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \right] \quad \text{LO III 287}$$

$$4. \quad \int_0^{\pi/2} \ln[1 - \cos^2 x (\sin^2 \alpha - \sin^2 \beta \sin^2 x)] dx = \pi \ln \left[\frac{1}{2} \left(\cos^2 \frac{\alpha}{2} + \sqrt{\cos^4 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2}} \right) \right]$$

$$[\alpha > \beta > 0] \quad \text{LO III 283}$$

$$5. \quad \int_0^u \ln \left(1 - \frac{\sin^2 x}{\sin^2 \alpha} \right) dx = -u \ln \sin^2 \alpha - L\left(\frac{\pi}{2} - \alpha + u\right) + L\left(\frac{\pi}{2} - \alpha - u\right)$$

$$\left[-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad |\sin u| \leq |\sin \alpha| \right] \quad \text{LO III 287}$$

$$6. \quad \int_0^{\pi/2} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln \frac{a + b}{2}$$

$$[a > 0, \quad b > 0] \quad \text{GW (322)(13)}$$

$$7. \quad \int_0^{\pi/2} \ln \frac{1 + \sin t \cos^2 x}{1 - \sin t \cos^2 x} dx = \pi \ln \frac{1 + \sin \frac{t}{2}}{\cos \frac{t}{2}} = \pi \ln \cot \frac{\pi - t}{4}$$

$$\left[|t| < \frac{\pi}{2} \right] \quad \text{LO III 283}$$

4.227

$$1. \quad \int_0^u \ln \tan x dx = L(u) + L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right) \quad \text{LO III 186(16)}$$

$$2. \quad \int_0^{\pi/4} \ln \tan x dx = -\int_{\pi/4}^{\pi/2} \ln \tan x dx = -G \quad \text{BI (286)(11)}$$

$$3. \quad \int_0^{\pi/2} \ln(a \tan x) dx = \frac{\pi}{2} \ln a \quad [a > 0] \quad \text{BI (307)(2)}$$

$$4.^7 \quad \int_0^{\pi/4} (\ln \tan x)^n dx = n!(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}$$

$$= \frac{1}{2} \left(\frac{\pi}{2}\right)^{n+1} |E_n| \quad [n \text{ even}]$$

$$\text{BI (286)(21)}$$

$$5.^7 \quad \int_0^{\pi/2} (\ln \tan x)^{2n} dx = 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}| \quad \text{BI (307)(15)}$$

$$6. \quad \int_0^{\pi/2} (\ln \tan x)^{2n+1} dx = 0 \quad \text{BI (307)(14)}$$

$$7. \quad \int_0^{\pi/4} (\ln \tan x)^2 dx = \frac{\pi^3}{16} \quad \text{BI (286)(16)}$$

$$8. \quad \int_0^{\pi/4} (\ln \tan x)^4 dx = \frac{5}{64} \pi^5 \quad \text{BI (286)(19)}$$

$$9. \quad \int_0^{\pi/4} \ln(1 + \tan x) dx = \frac{\pi}{8} \ln 2 \quad \text{BI (287)(1)}$$

$$10. \quad \int_0^{\pi/2} \ln(1 + \tan x) dx = \frac{\pi}{4} \ln 2 + G \quad \text{BI (308)(9)}$$

$$11. \quad \int_0^{\pi/4} \ln(1 - \tan x) dx = \frac{\pi}{8} \ln 2 - G \quad \text{BI (287)(2)}$$

$$12.^{11} \quad \int_0^{\pi/2} (\ln(1 - \tan x))^2 dx = \frac{\pi}{2} \ln 2 - 2G \quad \text{BI (308)(10)}$$

$$13. \quad \int_0^{\pi/4} \ln(1 + \cot x) dx = \frac{\pi}{8} \ln 2 + G \quad \text{BI (287)(3)}$$

$$14. \quad \int_0^{\pi/4} \ln(\cot x - 1) dx = \frac{\pi}{8} \ln 2 \quad \text{BI (287)(4)}$$

$$15. \quad \int_0^{\pi/4} \ln(\tan x + \cot x) \, dx = \frac{1}{2} \int_0^{\pi/2} \ln(\tan x + \cot x) \, dx = \frac{\pi}{2} \ln 2 \quad \text{BI (287)(5), BI (308)(11)}$$

$$16.^{11} \quad \int_0^{\pi/4} (\ln(\cot x - \tan x))^2 \, dx = \frac{1}{2} \int_0^{\pi/2} (\ln(\cot x - \tan x))^2 \, dx = \frac{\pi}{2} \ln 2$$

BI (287)(6), BI (308)(12)

$$17. \quad \int_0^{\pi/2} \ln(a^2 + b^2 \tan^2 x) \, dx = \frac{1}{2} \int_0^{\pi} \ln(a^2 + b^2 \tan^2 x) \, dx = \pi \ln(a + b)$$

[$a > 0, \quad b > 0$] GW (322)(17)

4.228

$$1. \quad \int_0^{\pi/2} \ln(\sin t \sin x + \sqrt{1 - \cos^2 t \sin^2 x}) \, dx = \frac{\pi}{2} \ln 2 - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi - t}{2}\right) \quad \text{LO III 290}$$

$$2. \quad \int_0^u \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t}) \, dx = -\left(\frac{\pi}{2} - t - \varphi\right) \ln \cos t + \frac{1}{2} L(u + \varphi) - \frac{1}{2} L(u - \varphi) - L(\varphi)$$

[$\cos \varphi = \frac{\sin u}{\sin t} \quad 0 \leq u \leq t \leq \frac{\pi}{2}$] LO III 290

$$3. \quad \int_0^t \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t}) \, dx = -\left(\frac{\pi}{2} - t\right) \ln \cos t \quad \text{LO III 285}$$

$$4. \quad \int_0^u \ln \frac{\sin u + \sin t \cos x \sqrt{\sin^2 u - \sin^2 x}}{\sin u - \sin t \cos x \sqrt{\sin^2 u - \sin^2 x}} \, dx = \pi \ln \left[\tan \frac{t}{2} \sin u + \sqrt{\tan^2 \frac{t}{2} \sin^2 u + 1} \right]$$

[$t > 0, \quad u > 0$] LO III 283

$$5. \quad \int_0^{\pi/4} \sqrt{\ln \cot x} \, dx = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}} \quad \text{BI (297)(9)}$$

$$6. \quad \int_0^{\pi/4} \frac{dx}{\sqrt{\ln \cot x}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \quad \text{BI (304)(24)}$$

$$7. \quad \int_0^{\pi/4} \ln(\sqrt{\tan x} + \sqrt{\cot x}) \, dx = \frac{1}{2} \int_0^{\pi/2} \ln(\sqrt{\tan x} + \sqrt{\cot x}) \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} G$$

BI (287)(7), BI (308)(22)

$$8. \quad \int_0^{\pi/4} \ln^2(\sqrt{\cot x} - \sqrt{\tan x}) \, dx = \frac{1}{2} \int_0^{\pi/2} \ln^2(\sqrt{\cot x} - \sqrt{\tan x}) \, dx = \frac{\pi}{4} \ln 2 - G$$

BI (287)(8), BI (308)(23)

4.229

$$1. \quad \int_0^1 \ln\left(\ln \frac{1}{x}\right) \, dx = -C \quad \text{FI II 807}$$

$$2.^{11} \quad \text{PV} \int_0^1 \frac{dx}{\ln\left(\ln \frac{1}{x}\right)} = \text{PV} \int_0^{\infty} \frac{e^{-u}}{\ln u} \, du \approx -0.154479 \quad \text{BI (31)(2)}$$

3.
$$\int_0^1 \ln \left(\ln \frac{1}{x} \right) \frac{dx}{\sqrt{\ln \frac{1}{x}}} = - (C + 2 \ln 2) \sqrt{\pi} \quad \text{BI (32)(4)}$$
- 4.¹¹
$$\int_0^1 \ln \left(\ln \frac{1}{x} \right) \left(\ln \frac{1}{x} \right)^{\mu-1} dx = \psi(\mu) \Gamma(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (30)(10)}$$

 If the integrand contains $(\ln \ln \frac{1}{x})$, it is convenient to make the substitution $\ln \frac{1}{x} = u$ so that $x = e^{-u}$.
- 5.⁷
$$\int_0^1 \ln(a + \ln x) dx = \ln a - e^{-a} \operatorname{Ei}(a) \quad [a > 0] \quad \text{BI (30)(5)}$$
6.
$$\int_0^1 \ln(a - \ln x) dx = \ln a - e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (30)(6)}$$
7.
$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x dx = \frac{\pi}{2} \ln \left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sqrt{2\pi} \right) \quad \text{BI (308)(28)}$$

4.23 Combinations of logarithms and rational functions

4.231

1.
$$\int_0^1 \frac{\ln x}{1+x} dx = -\frac{\pi^2}{12} \quad \text{FI II 483a}$$
2.
$$\int_0^1 \frac{\ln x}{1-x} dx = -\frac{\pi^2}{6} \quad \text{FI II 714}$$
3.
$$\int_0^1 \frac{x \ln x}{1-x} dx = 1 - \frac{\pi^2}{6} \quad \text{BI (108)(7)}$$
4.
$$\int_0^1 \frac{1+x}{1-x} \ln x dx = 1 - \frac{\pi^2}{3} \quad \text{BI (108)(9)}$$
- 5.¹¹
$$\int_0^\infty \frac{\ln x dx}{(x+a)^2} = \frac{\ln a}{a} \quad [0 < a] \quad \text{BI (139)(1)}$$
6.
$$\int_0^1 \frac{\ln x}{(1+x)^2} dx = -\ln 2 \quad \text{BI (111)(1)}$$
- 7.⁷
$$\int_0^\infty \ln x \frac{dx}{(a^2 + b^2 x^2)^n} = \frac{\Gamma(n - \frac{1}{2}) \sqrt{\pi}}{4(n-1)! a^{2n-1} b} \left[2 \ln \frac{a}{2b} - C - \psi \left(n - \frac{1}{2} \right) \right] \quad [a > 0, \quad b > 0] \quad \text{LI (139)(3)}$$
8.
$$\int_0^\infty \frac{\ln x dx}{a^2 + b^2 x^2} = \frac{\pi}{2ab} \ln \frac{a}{b} \quad [ab > 0] \quad \text{BI (135)(6)}$$
9.
$$\int_0^\infty \frac{\ln px}{q^2 + x^2} dx = \frac{\pi}{2q} \ln pq \quad [p > 0, \quad q > 0] \quad \text{BI (135)(4)}$$
10.
$$\int_0^\infty \frac{\ln x dx}{a^2 - b^2 x^2} = -\frac{\pi^2}{4ab} \quad [ab > 0]$$

11. $\int_0^a \frac{\ln x \, dx}{x^2 + a^2} = \frac{\pi \ln a}{4a} - \frac{G}{a} \quad [a > 0] \quad \text{GW (324)(7b)}$
12. $\int_0^1 \frac{\ln x}{1+x^2} \, dx = -\int_1^\infty \frac{\ln x}{1+x^2} \, dx = -G \quad \text{FI II 482, 614}$
13. $\int_0^1 \frac{\ln x \, dx}{1-x^2} = -\frac{\pi^2}{8} \quad \text{BI (108)(11)}$
14. $\int_0^1 \frac{x \ln x}{1+x^2} \, dx = -\frac{\pi^2}{48} \quad \text{GW (324)(7b)}$
15. $\int_0^1 \frac{x \ln x}{1-x^2} \, dx = -\frac{\pi^2}{24}$
16. $\int_0^1 \ln x \frac{1-x^{2n+2}}{(1-x^2)^2} \, dx = -\frac{(n+1)\pi^2}{8} + \sum_{k=1}^n \frac{n-k+1}{(2k-1)^2} \quad \text{BI (111)(5)}$
17. $\int_0^1 \ln x \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = -\frac{(n+1)\pi^2}{12} - \sum_{k=1}^n (-1)^k \frac{n-k+1}{k^2} \quad \text{BI (111)(2)}$
18. $\int_0^1 \ln x \frac{1-x^{n+1}}{(1-x)^2} \, dx = -\frac{(n+1)\pi^2}{6} + \sum_{k=1}^n \frac{n-k+1}{k^2} \quad \text{BI (111)(3)}$
- 19.* $\int_0^1 \frac{x \ln x}{1+x} \, dx = -1 + \frac{\pi^2}{2}$
- 20.* $\int_0^1 \frac{(1-x) \ln x}{1+x} \, dx = 1 - \frac{\pi^2}{6}$

4.232

1. $\int_u^v \frac{\ln x \, dx}{(x+u)(x+v)} = \frac{\ln uv}{2(v-u)} \ln \frac{(u+v)^2}{4uv} \quad \text{BI (145)(32)}$
2. $\int_0^\infty \frac{\ln x \, dx}{(x+\beta)(x+\gamma)} = \frac{(\ln \beta)^2 - (\ln \gamma)^2}{2(\beta - \gamma)} \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi] \quad \text{ET II 218(24)}$
3. $\int_0^\infty \frac{\ln x}{x+a} \frac{dx}{x-1} = \frac{\pi^2 + (\ln a)^2}{2(a+1)} \quad [a > 0] \quad \text{BI (140)(10)}$

4.233

- 1.³ $\int_0^1 \frac{\ln x \, dx}{1+x+x^2} = \frac{2}{9} \left[\frac{2\pi^2}{3} - \psi' \left(\frac{1}{3} \right) \right] = -0.7813024129 \dots \quad \text{LI (113)(1)}$
- 2.³ $\int_0^1 \frac{\ln x \, dx}{1-x+x^2} = \frac{1}{3} \left[\frac{2\pi^2}{3} - \psi' \left(\frac{1}{3} \right) \right] = -1.17195361934 \dots \quad \text{LI (113)(2)}$
- 3.¹¹ $\int_0^1 \frac{x \ln x \, dx}{1+x+x^2} = -\frac{1}{9} \left[\frac{7\pi^2}{6} - \psi' \left(\frac{1}{3} \right) \right] = -0.15766014917 \dots \quad \text{LI (113)(2)}$
- 4.³ $\int_0^1 \frac{x \ln x \, dx}{1-x+x^2} = \frac{1}{6} \left[\frac{5\pi^2}{6} - \psi' \left(\frac{1}{3} \right) \right] = -0.3118211319 \dots \quad \text{LI (113)(4)}$

$$5. \quad \int_0^\infty \frac{\ln x \, dx}{x^2 + 2xa \cos t + a^2} = \frac{t \ln a}{a \sin t} \quad [a > 0, \quad 0 < t < \pi] \quad \text{GW (324)(13c)}$$

4.234

$$1.^{11} \quad \int_1^\infty \frac{\ln x \, dx}{(1+x^2)^2} = \frac{\mathbf{G}}{2} - \frac{\pi}{8} \quad \text{BI (144)(18)a}$$

$$2. \quad \int_0^1 \frac{x \ln x \, dx}{(1+x^2)^2} = -\frac{1}{4} \ln 2 \quad \text{BI (111)(4)}$$

$$3. \quad \int_0^\infty \frac{1+x^2}{(1-x^2)^2} \ln x \, dx = 0 \quad \text{BI (142)(2)a}$$

$$4. \quad \int_0^\infty \frac{1-x^2}{(1+x^2)^2} \ln x \, dx = -\frac{\pi}{2} \quad \text{BI (142)(1)a}$$

$$5. \quad \int_0^1 \frac{x^2 \ln x \, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16(2+\sqrt{2})} \quad \text{BI (112)(21)}$$

$$6. \quad \int_0^\infty \frac{\ln x \, dx}{(a^2 + b^2 x^2)(1+x^2)} = \frac{b\pi}{2a(b^2 - a^2)} \ln \frac{a}{b} \quad [ab > 0] \quad \text{BI (317)(16)a}$$

$$7. \quad \int_0^\infty \frac{\ln x}{x^2 + a^2} \cdot \frac{dx}{1 + b^2 x^2} = \frac{\pi}{2(1 - a^2 b^2)} \left(\frac{1}{a} \ln a + b \ln b \right) \\ [a > 0, \quad b > 0] \quad \text{LI (140)(12)}$$

$$8. \quad \int_0^\infty \frac{x^2 \ln x \, dx}{(a^2 + b^2 x^2)(1+x^2)} = \frac{a\pi}{2b(b^2 - a^2)} \ln \frac{b}{a} \quad [ab > 0] \quad \text{LI (140)(12), BI (317)(15)a}$$

4.235

$$1. \quad \int_0^\infty \ln x \frac{(1-x)x^{n-2}}{1-x^{2n}} \, dx = -\frac{\pi^2}{4n^2} \tan^2 \frac{\pi}{2n} \quad [n > 1] \quad \text{BI (135)(10)}$$

$$2. \quad \int_0^\infty \ln x \frac{(1-x^2)x^{m-1}}{1-x^{2n}} \, dx = -\frac{\pi^2 \sin\left(\frac{m+1}{n}\right) \pi \sin\left(\frac{\pi}{n}\right)}{4n^2 \sin^2\left(\frac{m\pi}{2n}\right) \sin^2\left(\frac{m+2}{2n}\pi\right)} \quad \text{LI (135)(12)}$$

$$3.^{11} \quad \int_0^\infty \ln x \frac{(1-x^2)x^{n-3}}{1-x^{2n}} \, dx = -\frac{\pi^2}{4n^2} \tan^2\left(\frac{\pi}{n}\right) \quad [n > 2] \quad \text{BI (135)(11)}$$

$$4. \quad \int_0^1 \ln x \frac{x^{m-1} + x^{n-m-1}}{1-x^n} \, dx = -\frac{\pi^2}{n^2 \sin^2\left(\frac{m}{n}\pi\right)} \quad [n > m] \quad \text{BI (108)(15)}$$

4.236

$$1. \quad \int_0^1 \left\{ \frac{1+(p-1)\ln x}{1-x} + \frac{x \ln x}{(1-x)^2} \right\} x^{p-1} \, dx = -1 + \psi'(p) \\ [p > 0] \quad \text{BI (111)(6)a, GW (326)(13)}$$

$$2. \quad \int_0^1 \left[\frac{1}{1-x} + \frac{x \ln x}{(1-x)^2} \right] \, dx = \frac{\pi^2}{6} - 1 \quad \text{GW (326)(13a)}$$

4.24 Combinations of logarithms and algebraic functions

4.241

$$1. \quad \int_0^1 \frac{x^{2n} \ln x}{\sqrt{1-x^2}} dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right) \quad \text{BI (118)(5)a}$$

$$2. \quad \int_0^1 \frac{x^{2n+1} \ln x}{\sqrt{1-x^2}} dx = \frac{(2n)!!}{(2n+1)!!} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right) \quad \text{BI (118)(5)a}$$

$$3. \quad \int_0^1 x^{2n} \sqrt{1-x^2} \ln x dx = \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{\pi}{2} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2 \right) \quad \text{LI (117)(4), GW (324)(53a)}$$

$$4. \quad \int_0^1 x^{2n+1} \sqrt{1-x^2} \ln x dx = \frac{(2n)!!}{(2n+3)!!} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} - \frac{1}{2n+3} \right) \quad \text{BI (117)(5), GW (324)(53b)}$$

$$5. \quad \int_0^1 \ln x \cdot \sqrt{(1-x^2)^{2n-1}} dx = -\frac{(2n-1)!!}{4 \cdot (2n)!!} \pi [\psi(n+1) + \mathbf{C} + \ln 4] \quad \text{BI (117)(3)}$$

$$6. \quad \int_0^{\sqrt{\frac{1}{2}}} \frac{\ln x dx}{\sqrt{1-x^2}} = -\frac{\pi}{4} \ln 2 - \frac{1}{2} \mathbf{G} \quad \text{BI (145)(1)}$$

$$7. \quad \int_0^1 \frac{\ln x dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \ln 2 \quad \text{FI II 614, 643}$$

$$8. \quad \int_1^\infty \frac{\ln x dx}{x^2 \sqrt{x^2-1}} = 1 - \ln 2 \quad \text{BI (144)(17)}$$

$$9. \quad \int_0^1 \sqrt{1-x^2} \ln x dx = -\frac{\pi}{8} - \frac{\pi}{4} \ln 2 \quad \text{BI (117)(1), GW (324)(53c)}$$

$$10. \quad \int_0^1 x \sqrt{1-x^2} \ln x dx = \frac{1}{3} \ln 2 - \frac{4}{9} \quad \text{BI (117)(2)}$$

$$11. \quad \int_0^1 \frac{\ln x dx}{\sqrt{x(1-x^2)}} = -\frac{\sqrt{2\pi}}{8} \left[\Gamma\left(\frac{1}{4}\right) \right]^2 \quad \text{GW (324)(54a)}$$

4.242

$$1. \quad \int_0^\infty \frac{\ln x dx}{\sqrt{(a^2+x^2)(x^2+b^2)}} = \frac{1}{2a} \mathbf{K} \left(\frac{\sqrt{a^2-b^2}}{a} \right) \ln ab \quad [a > b > 0] \quad \text{BY (800.04)}$$

$$2. \quad \int_0^b \frac{\ln x dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{2\sqrt{a^2+b^2}} \left[\mathbf{K} \left(\frac{b}{\sqrt{a^2+b^2}} \right) \ln ab - \frac{\pi}{2} \mathbf{K} \left(\frac{a}{\sqrt{a^2+b^2}} \right) \right] \quad [a > 0, \quad b > 0] \quad \text{BY (800.02)}$$

3.
$$\int_b^\infty \frac{\ln x \, dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{2\sqrt{a^2 + b^2}} \left[\mathbf{K} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \ln ab + \frac{\pi}{2} \mathbf{K} \left(\frac{b}{\sqrt{a^2 + b^2}} \right) \right]$$

[$a > 0, \quad b > 0$] BY (800.06)
4.
$$\int_0^b \frac{\ln x \, dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{2a} \left[\mathbf{K} \left(\frac{b}{a} \right) \ln ab - \frac{\pi}{2} \mathbf{K} \left(\frac{\sqrt{a^2 - b^2}}{a} \right) \right]$$

[$a > b > 0$] BY (800.01)
5.
$$\int_b^a \frac{\ln x \, dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{2a} \mathbf{K} \left(\frac{\sqrt{a^2 - b^2}}{a} \right) \ln ab$$
 BY (800.03)
6.
$$\int_a^\infty \frac{\ln x \, dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{2a} \left[\mathbf{K} \left(\frac{b}{a} \right) \ln ab + \frac{\pi}{2} \mathbf{K} \left(\frac{\sqrt{a^2 - b^2}}{a} \right) \right]$$

[$a > b > 0$] BY (800.05)
- 4.243**
$$\int_0^1 \frac{x \ln x}{\sqrt{1 - x^4}} \, dx = -\frac{\pi}{8} \ln 2$$
 GW (324)(56b)
- 4.244**
1.
$$\int_0^1 \frac{\ln x \, dx}{\sqrt[3]{x(1 - x^2)^2}} = -\frac{1}{8} \left[\Gamma \left(\frac{1}{3} \right) \right]^3$$
 GW (324)(54b)
2.
$$\int_0^1 \frac{\ln x \, dx}{\sqrt[3]{1 - x^3}} = -\frac{\pi}{3\sqrt{3}} \left(\ln 3 + \frac{\pi}{3\sqrt{3}} \right)$$
 BI (118)(7)
3.
$$\int_0^1 \frac{x \ln x \, dx}{\sqrt[3]{(1 - x^3)^2}} = \frac{\pi}{3\sqrt{3}} \left(\frac{\pi}{3\sqrt{3}} - \ln 3 \right)$$
 BI (118)(8)
- 4.245**
1.
$$\int_0^1 \frac{x^{4n+1} \ln x}{\sqrt{1 - x^4}} \, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{8} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right)$$
 GW (324)(56a)
2.
$$\int_0^1 \frac{x^{4n+3} \ln x}{\sqrt{1 - x^4}} \, dx = \frac{(2n)!!}{4 \cdot (2n+1)!!} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right)$$
 GW (324)(56c)
- 4.246**
$$\int_0^1 (1 - x^2)^{n-\frac{1}{2}} \ln x \, dx = -\frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{4} \left[2 \ln 2 + \sum_{k=1}^n \frac{1}{k} \right]$$
 GW (324)(55)
- 4.247**
- 1.⁶
$$\int_0^1 \frac{\ln x}{\sqrt[n]{1 - x^{2n}}} \, dx = -\frac{\pi \mathbf{B} \left(\frac{1}{2n}, \frac{1}{2n} \right)}{8n^2 \sin \frac{\pi}{2n}}$$
 [$n > 1$] GW (324)(54c)a
- 2.⁶
$$\int_0^1 \frac{\ln x \, dx}{\sqrt[n]{x^{n-1}(1 - x^2)}} = -\frac{\pi \mathbf{B} \left(\frac{1}{2n}, \frac{1}{2n} \right)}{8 \sin \frac{\pi}{2n}}$$
 GW (324)(54)

4.25 Combinations of logarithms and powers

4.251

1.
$$\int_0^\infty \frac{x^{\mu-1} \ln x}{\beta + x} dx = \frac{\pi \beta^{\mu-1}}{\sin \mu \pi} (\ln \beta - \pi \cot \mu \pi) \quad [|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < 1]$$
 BI (135)(1)
2.
$$\int_0^\infty \frac{x^{\mu-1} \ln x}{a - x} dx = \pi a^{\mu-1} \left(\cot \mu \pi \ln a - \frac{\pi}{\sin^2 \mu \pi} \right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1]$$
 ET I 314(5)
- 3.¹⁰
$$\int_0^1 \frac{x^{\mu-1} \ln x}{x + 1} dx = \beta'(\mu) \quad [\operatorname{Re} \mu > 0]$$
 GW (324)(6), ET I 314(3)
4.
$$\int_0^1 \frac{x^{\mu-1} \ln x}{1 - x} dx = -\psi'(\mu) = -\zeta(2, \mu) \quad [\operatorname{Re} \mu > 0]$$
 BI (108)(8)
- 5.¹¹
$$\int_0^1 \ln x \frac{x^{2n}}{1 + x} dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k^2}$$
 BI (108)(4)
- 6.¹¹
$$\int_0^1 \ln x \frac{x^{2n-1}}{1 + x} dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k^2}$$
 BI (108)(5)

4.252

1.
$$\int_0^\infty \frac{x^{\mu-1} \ln x}{(x + \beta)(x + \gamma)} dx = \frac{\pi}{(\gamma - \beta) \sin \mu \pi} [\beta^{\mu-1} \ln \beta - \gamma^{\mu-1} \ln \gamma - \pi \cot \mu \pi (\beta^{\mu-1} - \gamma^{\mu-1})]$$

 $[|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 2, \quad \mu \neq 1]$ BI (140)(9)a, ET 314(6)
2.
$$\int_0^\infty \frac{x^{\mu-1} \ln x dx}{(x + \beta)(x - 1)} = \frac{\pi}{(\beta + 1) \sin^2 \mu \pi} [\pi - \beta^{\mu-1} (\sin \mu \pi \ln \beta - \pi \cos \mu \pi)]$$

 $[|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < 2, \quad \mu \neq 1]$
BI (140)(11)
3.
$$\int_0^\infty \frac{x^{p-1} \ln x}{1 - x^2} dx = -\frac{\pi^2}{4} \operatorname{cosec}^2 \frac{p\pi}{2} \quad [0 < p < 2] \quad (\text{see also 4.254 2})$$
- 4.⁶
$$\int_0^\infty \frac{x^{\mu-1} \ln x}{(x + a)^2} dx = \frac{(1 - \mu)a^{\mu-2}\pi}{\sin \mu \pi} \left(\ln a - \pi \cot \mu \pi + \frac{1}{\mu - 1} \right)$$

 $[|\arg a| < \pi \quad 0 < \operatorname{Re} \mu < 2 \quad (\mu \neq 1)]$
GW (324)(13b)

4.253

- 1.⁸
$$\int_0^1 x^{\mu-1} (1 - x^r)^{\nu-1} \ln x dx = \frac{1}{r^2} B\left(\frac{\mu}{r}, \nu\right) \left[\psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r} + \nu\right) \right]$$

 $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad r > 0]$
GW (324)(3b)a, BI (107)(5)a
2.
$$\int_0^1 \frac{x^{p-1}}{(1 - x)^{p+1}} \ln x dx = -\frac{\pi}{p} \operatorname{cosec} p\pi \quad [0 < p < 1]$$
 bi (319)(10)a

3. $\int_u^\infty \frac{(x-u)^{\mu-1} \ln x \, dx}{x^\lambda} = u^{\mu-\lambda} \text{B}(\lambda-\mu, \mu) [\ln u + \psi(\lambda) - \psi(\lambda-\mu)]$
 $[0 < \text{Re } \mu < \text{Re } \lambda]$ ET II 203(18)
- 4.¹¹ $\int_0^\infty \ln x \left(\frac{x}{a^2 + x^2} \right)^p \frac{dx}{x} = \frac{\ln a}{2a^p} \text{B} \left(\frac{p}{2}, \frac{p}{2} \right)$
 $[a > 0, \quad p > 0]$ BI (140)(6)
5. $\int_1^\infty (x-1)^{p-1} \ln x \, dx = \frac{\pi}{p} \operatorname{cosec} \pi p$
 $[-1 < p < 0]$ BI (289)(12)a
- 6.⁷ $\int_0^\infty \ln x \frac{dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^\mu} (\ln a - \mathbf{C} - \psi(\mu))$
 $[\text{Re } \mu > 0, \quad a \neq 0, \quad |\arg a| < \pi]$
 NT 68(7)
- 7.⁷ $\int_0^\infty \ln x \frac{dx}{(a+x)^{n+\frac{1}{2}}} = \frac{2}{(2n-1)a^{n-\frac{1}{2}}} \left(\ln a + 2 \ln 2 - 2 \sum_{k=1}^{n-1} \frac{1}{2k-1} \right)$
 $[|\arg a| < \pi, \quad n = 1, 2, \dots]$ BI (142)(5)

4.254

1. $\int_0^1 \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{1}{q^2} \psi' \left(\frac{p}{q} \right)$
 $[p > 0, \quad q > 0]$ GW (324)(5)
2. $\int_0^\infty \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{\pi^2}{q^2 \sin^2 \frac{p\pi}{q}}$
 $[0 < p < q]$ BI (135)(8)
3. $\int_0^\infty \frac{\ln x}{x^q - 1} \frac{dx}{x^p} = \frac{\pi^2}{q^2 \sin^2 \frac{p-1}{q} \pi}$
 $[p < 1, \quad p+q > 1]$ BI (140)(2)
- 4.³ $\int_0^1 \frac{x^{p-1} \ln x}{1+x^q} \, dx = \frac{1}{q^2} \beta' \left(\frac{p}{q} \right)$
 $[p > 0, \quad q > 0]$ GW (324)(7)
5. $\int_0^\infty \frac{x^{p-1} \ln x}{1+x^q} \, dx = -\frac{\pi^2}{q^2} \frac{\cos \frac{p\pi}{q}}{\sin^2 \frac{p\pi}{q}}$
 $[0 < p < q]$ BI (135)(7)
6. $\int_0^1 \frac{x^{q-1} \ln x}{1-x^{2q}} \, dx = -\frac{\pi^2}{8q^2}$
 $[q > 0]$ BI (108)(12)

4.255

1. $\int_0^1 \ln x \frac{(1-x^2) x^{p-2}}{1+x^{2p}} \, dx = -\left(\frac{\pi}{2p} \right)^2 \frac{\sin \frac{\pi}{2p}}{\cos^2 \left(\frac{\pi}{2p} \right)}$
 $[p > 1]$ BI (108)(13)
2. $\int_0^1 \ln x \frac{(1+x^2) x^{p-2}}{1-x^{2p}} \, dx = -\left(\frac{\pi}{2p} \right)^2 \sec^2 \left(\frac{\pi}{2p} \right)$
 $[p > 1]$ BI (108)(14)
3. $\int_0^\infty \ln x \frac{1-x^p}{1-x^2} \, dx = \frac{\pi^2}{4} \tan^2 \left(\frac{p\pi}{2} \right)$
 $[p < 1]$ BI (140)(3)

$$4.256 \quad \int_0^1 \ln \frac{1}{x} \frac{x^{\mu-1} dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2} B\left(\frac{\mu}{n}, \frac{m}{n}\right) \left[\psi\left(\frac{\mu+m}{n}\right) - \psi\left(\frac{\mu}{n}\right) \right] \\ [Re \mu > 0] \quad \text{LI (118)(12)}$$

4.257

$$1. \quad \int_0^\infty \frac{x^\nu \ln \frac{x}{\beta} dx}{(x+\beta)(x+\gamma)} = \frac{\pi \left[\gamma^\nu \ln \frac{\gamma}{\beta} + \pi (\beta^\nu - \gamma^\nu) \cot \nu \pi \right]}{\sin \nu \pi (\gamma - \beta)} \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad |\operatorname{Re} \nu| < 1] \\ \text{ET II 219(30)}$$

$$2. \quad \int_0^\infty \ln \frac{x}{q} \left(\frac{x^p}{q^{2p} + x^{2p}} \right) \frac{dx}{x} = 0 \quad [q > 0] \quad \text{BI (140)(4)a}$$

$$3. \quad \int_0^\infty \ln \frac{x}{q} \left(\frac{x^p}{q^{2p} + x^{2p}} \right)^r \frac{dx}{q^2 + x^2} = 0 \quad [q > 0] \quad \text{BI (140)(4)a}$$

$$4. \quad \int_0^\infty \ln x \ln \frac{x}{a} \frac{dx}{(x-1)(x-a)} = \frac{\left[4\pi^2 + (\ln a)^2 \right] \ln a}{6(a-1)} \quad [a > 0] \quad (\text{for } a=1 \text{ see 4.261 5}) \\ \text{BI (141)(5)}$$

$$5. \quad \int_0^\infty \ln x \ln \frac{x}{a} \frac{x^p dx}{(x-1)(x-a)} = \frac{\pi^2 [(a^p + 1) \ln a - 2\pi (a^p - 1) \cot p\pi]}{(a-1) \sin^2 p\pi} \\ [p^2 < 1, \quad a > 0] \quad \text{BI (141)(6)}$$

4.26–4.27 Combinations involving powers of the logarithm and other powers

4.261

$$1.^7 \quad \int_0^1 (\ln x)^2 \frac{dx}{1+2x \cos t + x^2} = \frac{t(\pi^2 - t^2)}{6 \sin t} \quad [0 \leq t \leq \pi] \quad \text{BI (113)(7)}$$

$$2. \quad \int_0^1 \frac{(\ln x)^2 dx}{x^2 - x + 1} = \frac{1}{2} \int_0^\infty \frac{(\ln x)^2 dx}{x^2 - x + 1} = \frac{10\pi^3}{81\sqrt{3}} \quad \text{GW (324)(16c)}$$

$$3. \quad \int_0^1 \frac{(\ln x)^2 dx}{x^2 + x + 1} = \frac{1}{2} \int_0^\infty \frac{(\ln x)^2 dx}{x^2 + x + 1} = \frac{8\pi^3}{81\sqrt{3}} \quad \text{GW (324)(16b)}$$

$$4. \quad \int_0^\infty (\ln x)^2 \frac{dx}{(x-1)(x+a)} = \frac{\left[\pi^2 + (\ln a)^2 \right] \ln a}{3(1+a)} \quad [a > 0] \quad \text{BI (141)(1)}$$

$$5. \quad \int_0^\infty (\ln x)^2 \frac{dx}{(1-x)^2} = \frac{2}{3} \pi^2 \quad \text{BI (139)(4)}$$

$$6. \quad \int_0^1 (\ln x)^2 \frac{dx}{1+x^2} = \frac{\pi^3}{16} \quad \text{BI (109)(3)}$$

$$7. \quad \int_0^1 (\ln x)^2 \frac{1+x^2}{1+x^4} dx = \frac{1}{2} \int_0^\infty (\ln x)^2 \frac{1+x^2}{1+x^4} dx = \frac{3\sqrt{2}}{64} \pi^3 \quad \text{BI (109)(5), BI (135)(13)}$$

$$8.^{11} \quad \int_0^1 (\ln x)^2 \frac{1-x}{1-x^6} dx = \frac{8\sqrt{3}\pi^3 + 351\zeta(3)}{486}$$

9. $\int_0^1 (\ln x)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right]$ BI (118)(13)
10. $\int_0^\infty (\ln x)^2 \frac{x^{\mu-1}}{1+x} dx = \frac{\pi^3 (2 - \sin^2 \mu\pi)}{\sin^3 \mu\pi}$ ET I 315(10)
 $[0 < \operatorname{Re} \mu < 1]$
- 11.⁷ $\int_0^1 (\ln x)^2 \frac{x^n dx}{1+x} = 2 \sum_{k=n}^\infty \frac{(-1)^{n+k}}{(k+1)^3} = (-1)^n \left(\frac{3}{2} \zeta(3) + 2 \sum_{k=1}^n \frac{(-1)^k}{k^3} \right)$
 $[n = 0, 1, \dots]$ BI (109)(1)
- 12.⁷ $\int_0^1 (\ln x)^2 \frac{x^n dx}{1-x} = 2 \sum_{k=n}^\infty \frac{1}{(k+1)^3} = 2 \left(\zeta(3) - \sum_{k=1}^n \frac{1}{k^3} \right)$
 $[n = 0, 1, \dots]$ BI (109)(2)
- 13.¹¹ $\int_0^1 (\ln x)^2 \frac{x^{2n} dx}{1-x^2} = 2 \sum_{k=n}^\infty \frac{1}{(2k+1)^3} = \frac{7}{4} \zeta(3) - 2 \sum_{k=1}^n \frac{1}{(2k-1)^3}$
 $[n = 0, 1, \dots]$ BI (109)(4)
14. $\int_0^\infty (\ln x)^2 \frac{x^{p-1} dx}{x^2 + 2x \cos t + 1} = \frac{\pi \sin(1-p)t}{\sin t \sin p\pi} \{ \pi^2 - t^2 + 2\pi \cot p\pi [\pi \cot p\pi + t \cot(1-p)t] \}$
 $[0 < t < \pi, \quad 0 < p < 2, \quad p \neq 1]$ GW (324)(17)
15. $\int_0^1 (\ln x)^2 \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{2 \cdot (2n)!!} \pi \left\{ \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$ GW (324)(60a)
16. $\int_0^1 (\ln x)^2 \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!} \left\{ -\frac{\pi^2}{12} - \sum_{k=1}^{2n+1} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$
GW (324)(60b)
- 17.⁷ $\int_0^1 (\ln x)^2 x^{\mu-1} (1-x)^{\nu-1} dx = B(\mu, \nu) \left\{ [\psi(\mu) - \psi(\nu + \mu)]^2 + \psi'(\mu) - \psi'(\mu + \nu) \right\}$
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$ ET I 315(11)
18. $\int_0^1 (\ln x)^2 \frac{1-x^{n+1}}{(1-x)^2} dx = 2(n+1) \zeta(3) - 2 \sum_{k=1}^n \frac{n-k+1}{k^3}$ LI (111)(8)
19. $\int_0^1 (\ln x)^2 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} dx = \frac{3}{2} (n+1) \zeta(3) - 2 \sum_{k=1}^n (-1)^{k-1} \frac{n-k+1}{k^3}$ LI (111)(7)
- 20.⁷ $\int_0^1 (\ln x)^2 \frac{1-x^{2n+2}}{(1-x^2)^2} dx = \frac{7}{4} (n+1) \zeta(3) - 2 \sum_{k=1}^n \frac{n-k+1}{(2k-1)^3}$
 $[n = 0, 1, \dots]$ LI (111)(9)
21. $\int_0^1 (\ln x)^2 x^{p-1} (1-x^r)^{q-1} dx = \frac{1}{r^3} B\left(\frac{p}{r}, q\right) \left\{ \psi'\left(\frac{p}{r}\right) - \psi'\left(\frac{p}{r} + q\right) + \left[\psi\left(\frac{p}{r}\right) - \psi\left(\frac{p}{r} + q\right) \right]^2 \right\}$
 $[p > 0, \quad q > 0, \quad r > 0]$ GW (324)(8a)

4.262

$$1. \quad \int_0^1 (\ln x)^3 \frac{dx}{1+x} = -\frac{7}{120}\pi^4 \quad \text{BI (109)(9)}$$

$$2. \quad \int_0^1 (\ln x)^3 \frac{dx}{1-x} = -\frac{\pi^4}{15} \quad \text{BI (109)(11)}$$

$$3. \quad \int_0^\infty (\ln x)^3 \frac{dx}{(x+a)(x-1)} = \frac{[\pi^2 + (\ln a)^2]^2}{4(a+1)} \quad [a > 0] \quad \text{BI (141)(2)}$$

$$4. \quad \int_0^1 (\ln x)^3 \frac{x^n dx}{1+x} = (-1)^{n+1} \left[\frac{7\pi^4}{120} - 6 \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4} \right] \quad [n = 1, 2, \dots] \quad \text{BI (109)(10)}$$

$$5. \quad \int_0^1 (\ln x)^3 \frac{x^n dx}{1-x} = -\frac{\pi^4}{15} + 6 \sum_{k=0}^{n-1} \frac{1}{(k+1)^4} \quad [n = 1, 2, \dots] \quad \text{BI (109)(12)}$$

$$6. \quad \int_0^1 (\ln x)^3 \frac{x^{2n} dx}{1-x^2} = -\frac{\pi^4}{16} + 6 \sum_{k=0}^{n-1} \frac{1}{(2k+1)^4} \quad [n = 1, 2, \dots] \quad \text{BI (109)(14)}$$

$$7. \quad \int_0^1 (\ln x)^3 \frac{1-x^{n+1}}{(1-x)^2} dx = -\frac{(n+1)\pi^4}{15} + 6 \sum_{k=1}^n \frac{n-k+1}{k^4} \quad \text{BI (111)(11)}$$

$$8. \quad \int_0^1 (\ln x)^3 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} dx = -\frac{7(n+1)\pi^4}{120} + 6 \sum_{k=1}^n (-1)^{k-1} \frac{n-k+1}{k^4} \quad \text{BI (111)(10)}$$

$$9. \quad \int_0^1 (\ln x)^3 \frac{1-x^{2n+2}}{(1-x^2)^2} dx = -\frac{(n+1)\pi^4}{16} + 6 \sum_{k=1}^n \frac{n-k+1}{(2k-1)^4} \quad \text{BI (111)(12)}$$

4.263

$$1.^8 \quad \int_0^\infty (\ln x)^4 \frac{dx}{(x-1)(x+a)} = \frac{\ln a [\pi^2 + (\ln a)^2] [7\pi^2 + 3(\ln a)^2]}{15(1+a)} \quad [a > 0] \quad \text{BI (141)(3)}$$

$$2. \quad \int_0^1 (\ln x)^4 \frac{dx}{1+x^2} = \frac{5\pi^5}{64} \quad \text{BI (109)(17)}$$

$$3. \quad \int_0^1 (\ln x)^4 \frac{dx}{1+2x \cos t + x^2} = \frac{t(\pi^2 - t^2)(7\pi^2 - 3t^2)}{30 \sin t} \quad [|t| < \pi] \quad \text{BI (113)(8)}$$

4.264

$$1. \quad \int_0^1 (\ln x)^5 \frac{dx}{1+x} = -\frac{31\pi^6}{252} \quad \text{BI (109)(20)}$$

$$2. \quad \int_0^1 (\ln x)^5 \frac{dx}{1-x} = -\frac{8\pi^6}{63} \quad \text{BI (109)(21)}$$

3.
$$\int_0^\infty (\ln x)^5 \frac{dx}{(x-1)(x+a)} = \frac{[\pi^2 + (\ln a)^2]^2 [3\pi^2 + (\ln a)^2]}{6(1+a)} \quad [a > 0] \quad \text{BI (141)(4)}$$
- 4.265
$$\int_0^1 (\ln x)^6 \frac{dx}{1+x^2} = \frac{61\pi^7}{256} \quad \text{BI (109)(25)}$$
- 4.266
1.
$$\int_0^1 (\ln x)^7 \frac{dx}{1+x} = -\frac{127\pi^8}{240} \quad \text{BI (109)(28)}$$
2.
$$\int_0^1 (\ln x)^7 \frac{dx}{1-x} = -\frac{8\pi^8}{15} \quad \text{BI (109)(29)}$$
- 4.267
1.
$$\int_0^1 \frac{1-x}{1+x} \frac{dx}{\ln x} = \ln \frac{2}{\pi} \quad \text{BI (127)(3)}$$
2.
$$\int_0^1 \frac{(1-x)^2}{1+x^2} \frac{dx}{\ln x} = \ln \frac{\pi}{4} \quad \text{BI (128)(2)}$$
- 3.⁸
$$\int_0^1 \frac{(1-x)^2}{1+2x \cos \frac{m\pi}{n} + x^2} \cdot \frac{dx}{\ln x}$$

$$= \frac{1}{\sin(\frac{m\pi}{n})} \sum_{k=1}^{n-1} (-1)^k \sin\left(\frac{km\pi}{n}\right) \ln \frac{\{\Gamma(\frac{n+k+1}{2n})\}^2 \Gamma(\frac{k+2}{2n}) \Gamma(\frac{k}{2n})}{\{\Gamma(\frac{k+1}{2n})\}^2 \Gamma(\frac{n+k}{2n}) \Gamma(\frac{n+k+2}{2n})} \quad [m+n \text{ is odd}]$$

$$= \frac{1}{\sin(\frac{m\pi}{n})} \sum_{k=1}^{[\frac{1}{2}(n-1)]} (-1)^k \sin\left(\frac{km\pi}{n}\right) \ln \frac{\{\Gamma(\frac{n-k+1}{n})\}^2 \Gamma(\frac{k+2}{n}) \Gamma(\frac{k}{n})}{\{\Gamma(\frac{k+1}{n})\}^2 \Gamma(\frac{n-k}{n}) \Gamma(\frac{n-k+2}{n})} \quad [m+n \text{ is even}]$$

$$[m < n] \quad \text{BI (130)(3)}$$
4.
$$\int_0^1 \frac{1-x}{1+x} \cdot \frac{1}{1+x^2} \cdot \frac{dx}{\ln x} = -\frac{\ln 2}{2} \quad \text{BI (130)(16)}$$
5.
$$\int_0^1 \frac{1-x}{1+x} \cdot \frac{x^2}{1+x^2} \cdot \frac{dx}{\ln x} = \ln \frac{2\sqrt{2}}{\pi} \quad \text{BI (130)(17)}$$
- 6.¹¹
$$\int_0^1 (1-x)^p \frac{dx}{\ln x} = \sum_{k=1}^{\infty} (-1)^k \binom{p}{k} \ln(1+k) \quad [p \geq 1] \quad \text{BI (123)(2)}$$
7.
$$\int_0^1 \left(\frac{1-x^p}{1-x} - p \right) \frac{dx}{\ln x} = \ln \Gamma(p+1) \quad \text{GW (326)(10)}$$
8.
$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \ln \frac{p}{q} \quad [p > 0, \quad q > 0] \quad \text{FI II 647}$$
9.
$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} \cdot \frac{dx}{1+x} = \ln \frac{\Gamma(\frac{q}{2}) \Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}) \Gamma(\frac{q+1}{2})} \quad [p > 0, \quad q > 0] \quad \text{FI II 186}$$
10.
$$\int_0^1 \frac{x^{p-1} - x^{-p}}{(1+x) \ln x} dx = \frac{1}{2} \int_0^\infty \frac{x^{p-1} - x^{-p}}{(1+x) \ln x} dx = \ln \left(\tan \frac{p\pi}{2} \right) \quad [0 < p < 1] \quad \text{FI II 816}$$

$$11. \quad \int_0^1 (x^p - x^q) x^{r-1} \frac{dx}{\ln x} = \ln \frac{p+r}{r+q} \quad [r > 0, \quad p > 0, \quad q > 0] \quad \text{LI (123)(5)}$$

$$12. \quad \int_0^1 \frac{x^p - x^q}{(1-ax)^n} \frac{dx}{x \ln x} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k \ln \frac{p+k}{q+k} \quad [p > 0, \quad q > 0, \quad a^2 < 1] \quad \text{BI (130)(15)}$$

$$13. \quad \int_0^1 (x^p - 1)(x^q - 1) \frac{dx}{\ln x} = \ln \frac{p+q+1}{(p+1)(q+1)} \quad [p > -1, \quad q > -1, \quad p+q > -1] \\ \text{GW (324)(19b)}$$

$$14. \quad \int_0^1 \frac{x^p - x^q}{1+x} \cdot \frac{1+x^{2n+1}}{x \ln x} dx = \ln \frac{\Gamma\left(\frac{p}{2} + n + 1\right) \Gamma\left(\frac{q+1}{2} + n\right) \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2} + n + 1\right) \Gamma\left(\frac{p+1}{2} + n\right) \Gamma\left(\frac{q+1}{2}\right) \Gamma\left(\frac{p}{2}\right)} \\ [p > 0, \quad q > 0] \quad \text{BI (127)(7)}$$

$$15. \quad \int_0^1 \frac{x^p - x^q}{1-x} \cdot \frac{1-x^r}{\ln x} dx = \ln \frac{\Gamma(q+1) \Gamma(p+r+1)}{\Gamma(p+1) \Gamma(q+r+1)} \\ [p > -1, \quad q > -1, \quad p+r > -1, \quad q+r > -1] \quad \text{GW (324)(23)}$$

$$16. \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x} dx = \ln \frac{\Gamma\left(\frac{p+r}{2r}\right) \Gamma\left(\frac{q}{2r}\right)}{\Gamma\left(\frac{q+r}{2r}\right) \Gamma\left(\frac{p}{2r}\right)} \quad [p > 0, \quad q > 0, \quad r > 0] \quad \text{GW (324)(21)}$$

$$17. \quad \int_0^1 \frac{1-x^{2p-2q}}{1+x^{2p}} \frac{x^{q-1} dx}{\ln x} = \ln \tan \frac{q\pi}{4p} \quad [0 < q < p] \quad \text{BI (128)(6)}$$

$$18. \quad \int_0^{\infty} \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x} dx = \ln \left(\tan \frac{p\pi}{2r} \cot \frac{q\pi}{2r} \right) \quad [0 < p < r, \quad 0 < q < r] \\ \text{GW (324)(22), BI (143)(2)}$$

$$19. \quad \int_0^{\infty} \frac{x^{p-1} - x^{q-1}}{(1-x^r) \ln x} dx = \ln \left(\frac{\sin \frac{p\pi}{r}}{\sin \frac{q\pi}{r}} \right) \quad [0 < p < r, \quad 0 < q < r] \quad \text{BI (143)(4)}$$

$$20. \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x^{2n}} \cdot \frac{1-x^2}{\ln x} dx = \ln \frac{\Gamma\left(\frac{p+2}{2n}\right) \Gamma\left(\frac{q}{2n}\right)}{\Gamma\left(\frac{q+2}{2n}\right) \Gamma\left(\frac{p}{2n}\right)} \quad [p > 0, \quad q > 0] \quad \text{BI (128)(11)}$$

$$21. \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \frac{1+x^2}{\ln x} dx = \ln \frac{\Gamma\left(\frac{p+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{q+2}{4(2n+1)}\right) \Gamma\left(\frac{p+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{q}{4(2n+1)}\right)}{\Gamma\left(\frac{q+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{p+2}{4(2n+1)}\right) \Gamma\left(\frac{q+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{p}{4(2n+1)}\right)} \\ [p > 0, \quad q > 0] \quad \text{BI (128)(7)}$$

$$22. \quad \int_0^{\infty} \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \cdot \frac{1+x^2}{\ln x} dx = \ln \left\{ \tan \frac{p\pi}{4(2n+1)} \cdot \tan \frac{(p+2)\pi}{4(2n+1)} \cdot \cot \frac{q\pi}{4(2n+1)} \cdot \cot \frac{(q+2)\pi}{4(2n+1)} \right\} \\ [0 < p < 4n, \quad 0 < q < 4n] \quad \text{BI (143)(5)}$$

23.
$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1 - x^{2n}} \frac{1 - x^2}{\ln x} dx = \ln \frac{\sin \frac{p\pi}{2n} \cdot \sin \frac{(q+2)\pi}{2n}}{\sin \frac{q\pi}{2n} \cdot \sin \frac{(p+2)\pi}{2n}} \quad [0 < p < 2n, \quad 0 < q < 2n] \quad \text{BI (143)(6)}$$
24.
$$\int_0^1 (1 - x^p)(1 - x^q) \frac{x^{r-1} dx}{\ln x} = \ln \frac{(p+q+r)r}{(p+r)(q+r)} \quad [p > 0, \quad q > 0, \quad r > 0] \quad \text{BI (123)(8)}$$
25.
$$\int_0^1 (1 - x^p)(1 - x^q) \frac{x^{r-1} dx}{(1-x)\ln x} = \ln \frac{\Gamma(p+r)\Gamma(q+r)}{\Gamma(p+q+r)\Gamma(r)} \quad [r > 0, \quad r+p > 0, \quad r+q > 0, \quad r+p+q > 0] \quad \text{FI II 815a}$$
26.
$$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{dx}{\ln x} = \ln \frac{(p+q+1)(q+r+1)(r+p+1)}{(p+q+r+1)(p+1)(q+1)(r+1)} \quad [p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > -1] \quad \text{GW (324)(19c)}$$
27.
$$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{dx}{(1-x)\ln x} = \ln \frac{\Gamma(p+1)\Gamma(q+1)\Gamma(r+1)\Gamma(p+q+r+1)}{\Gamma(p+q+1)\Gamma(p+r+1)\Gamma(q+r+1)} \quad [p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > -1] \quad \text{FI II 815}$$
28.
$$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{x^{s-1} dx}{\ln x} = \ln \frac{(p+q+s)(p+r+s)(q+r+s)s}{(p+s)(q+s)(r+s)(p+q+r+s)} \quad [p > 0, \quad q > 0, \quad r > 0, \quad s > 0] \quad \text{BI (123)(10)}$$
29.
$$\int_0^1 (1 - x^p)(1 - x^q) \frac{x^{s-1} dx}{(1-x^r)\ln x} = \ln \frac{\Gamma(\frac{p+s}{r})\Gamma(\frac{q+s}{r})}{\Gamma(\frac{s}{r})\Gamma(\frac{p+q+s}{r})} \quad [p > 0, \quad q > 0, \quad r > 0, \quad s > 0] \quad \text{GW (324)(23a)}$$
30.
$$\begin{aligned} \int_0^\infty (1 - x^p)(1 - x^q) \frac{x^{s-1} dx}{(1 - x^{p+q+2s})\ln x} &= 2 \int_0^1 (1 - x^p)(1 - x^q) \frac{x^{s-1} dx}{(1 - x^{p+q+2s})\ln x} \\ &= 2 \ln \left(\sin \frac{s\pi}{p+q+2s} \operatorname{cosec} \frac{(p+s)\pi}{p+q+2s} \right) \quad [s > 0, \quad s+p > 0, \quad s+p+q > 0] \quad \text{GW (324)(23b)a} \end{aligned}$$
31.
$$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{x^{s-1} dx}{(1-x)\ln x} = \ln \frac{\Gamma(p+s)\Gamma(q+s)\Gamma(r+s)\Gamma(p+q+r+s)}{\Gamma(p+q+s)4\Gamma(p+r+s)\Gamma(q+r+s)\Gamma(s)} \quad [p > 0, \quad q > 0, \quad r > 0, \quad s > 0]^* \quad \text{BI (127)(11)}$$
32.
$$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{x^{s-1} dx}{(1-x^t)\ln x} = \ln \frac{\Gamma(\frac{p+s}{t})\Gamma(\frac{q+s}{t})\Gamma(\frac{r+s}{t})\Gamma(\frac{p+q+r+s}{t})}{\Gamma(\frac{p+q+s}{t})\Gamma(\frac{q+r+s}{t})\Gamma(\frac{p+r+s}{t})\Gamma(\frac{s}{t})} \quad [p > 0, \quad q > 0, \quad r > 0, \quad s > 0, \quad t > 0]^* \quad \text{GW (324)(23b)}$$

*In 4.267.31 the restrictions can be somewhat weakened by writing, for example, $s > 0, p+s > 0, q+s > 0, r+s > 0, p+q+s > 0, p+r+s > 0, q+r+s > 0, p+q+r+s > 0$, in **4.267** 31 and 32.

$$33. \int_0^1 \left\{ \frac{x^p - x^{p+q}}{1-x} - q \right\} \frac{dx}{\ln x} = \ln \frac{\Gamma(p+q+1)}{\Gamma(p+1)} \quad [p > -1, \quad p+q > -1] \quad \text{BI (127)(19)}$$

$$34. \int_0^1 \left\{ \frac{x^\mu - x}{x-1} - x(\mu-1) \right\} \frac{dx}{x \ln x} = \ln \Gamma(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{WH, BI (127)(18)}$$

$$35. \int_0^1 \left\{ 1-x - \frac{(1-x^p)(1-x^q)}{1-x} \right\} \frac{dx}{x \ln x} = -\ln \{B(p, q)\} \quad [p > 0, \quad q > 0] \quad \text{BI (130)(18)}$$

$$36. \int_0^1 \left\{ \frac{x^{p-1}}{1-x} - \frac{x^{pq-1}}{1-x^q} - \frac{1}{x(1-x)} + \frac{1}{x(1-x^q)} \right\} \frac{dx}{\ln x} = q \ln p \quad [p > 0] \quad \text{BI (130)(20)}$$

$$37. \int_0^1 \left\{ \frac{x^{q-1}}{1-x} - \frac{x^{pq-1}}{1-x^p} - \frac{p-1}{1-x^p} x^{p-1} - \frac{p-1}{2} x^{p-1} \right\} \frac{dx}{\ln x} = \frac{1-p}{2} \ln(2\pi) + \left(pq - \frac{1}{2} \right) \ln p \quad [p > 0, \quad q > 0] \quad \text{BI (130)(22)}$$

$$38. \int_0^1 \frac{(1-x^p)(1-x^q) - (1-x)^2}{x(1-x) \ln x} dx = \ln B(p, q) \quad [p > 0, \quad q > 0] \quad \text{GW (324)(24)}$$

$$39.^6 \int_0^1 (x^p - 1)^n \frac{dx}{\ln x} = \sum_{k=0}^n \binom{n}{n-k} (-1)^{n-k} \ln(pk+1) \quad [n > 0, \quad pn > -1] \quad \text{GW (324)(19d), BI (123)(12)a}$$

$$40.^6 \int_0^1 \frac{(1-x^p)^n}{1-x} \frac{dx}{\ln x} = \sum_{k=0}^n (-1)^{k-1} \ln \Gamma[(n-k)p+1] \quad [n > 1, \quad pn > -1] \quad \text{BI (127)(12)}$$

$$41. \int_0^1 (x^p - 1)^n x^{q-1} \frac{dx}{\ln x} = \sum_{k=0}^n (-1)^k \binom{n}{k} \ln[q + (n-k)p] \quad [n > 0, \quad q > 0, \quad pn > -q] \quad \text{BI (123)(12)}$$

$$42.^6 \int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(1-x) \ln x} = \sum_{k=0}^n (-1)^{k-1} \ln \Gamma[(n-k)p+q] \quad [n > 1, \quad q > 0, \quad pn > -q] \quad \text{BI (127)(13)}$$

$$43.^{10} \int_0^1 (x^p - 1)^n (x^q - 1)^m \frac{x^{r-1} dx}{\ln x} = \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{k=0}^m (-1)^k \binom{m}{k} \ln[r + (m-k)q + (n-j)p] \quad [n \geq 0, \quad m \geq 0, \quad n+m > 0, \quad r > 0, \quad pn+qm+r > 0] \quad \text{BI (123)(16)}$$

4.268

$$1. \int_0^1 \frac{(x^p - x^q)(1-x^r)}{(\ln x)^2} dx = (p+1) \ln(p+1) - (q+1) \ln(q+1) - (p+r+1) \ln(p+r+1) + (q+r+1) \ln(q+r+1) \quad [p > -1, \quad q > -1, \quad p+r > -1, \quad q+r > -1] \quad \text{GW (324)(26)}$$

$$2. \quad \int_0^1 (x^p - x^q)^2 \frac{dx}{(\ln x)^2} = (2p+1) \ln(2p+1) + (2q+1) \ln(2q+1) - 2(p+q+1) \ln(p+q+1) \\ [p > -\frac{1}{2}, \quad q > -\frac{1}{2}] \quad \text{GW (324)(26a)}$$

$$3. \quad \int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{dx}{(\ln x)^2} \\ = (p+q+1) \ln(p+q+1) + (q+r+1) \ln(q+r+1) + (p+r+1) \ln(p+r+1) \\ - (p+1) \ln(p+1) - (q+1) \ln(q+1) - (r+1) \ln(r+1) - (p+q+r) \ln(p+q+r) \\ [p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > 0] \\ \text{BI (124)(4)}$$

$$4. \quad \int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(\ln x)^2} = \frac{1}{2} \sum_{k=0}^n (-1)^k \binom{n}{k} (pk+q)^2 \ln(pk+q) \\ [q > 0, \quad p > -\frac{q}{n}] \quad \text{BI (124)(14)}$$

$$5. \quad \int_0^1 (1-x^p)^n (1-x^q)^m x^{r-1} \frac{dx}{(\ln x)^2} = \left(\sum_{j=0}^n (-1)^j \binom{n}{j} \right) \left(\sum_{k=0}^m (-1)^k \binom{m}{k} \right) \\ \times [(m-k)q + (n-j)p + r] \ln[(m-k)q + (n-j)p + r] \\ [r > 0, \quad mq+r > 0, \quad np+r > 0, \quad mq+np+r > 0] \quad \text{BI (124)(8)}$$

$$6. \quad \int_0^1 [(q-r)x^{p-1} + (r-p)x^{q-1} + (p-q)x^{r-1}] \frac{dx}{(\ln x)^2} \\ = (q-r)p \ln p + (r-p)q \ln q + (p-q)r \ln r \\ [p > 0, \quad q > 0, \quad r > 0] \quad \text{BI (124)(9)}$$

$$7. \quad \int_0^1 \left[\frac{x^{p-1}}{(p-q)(p-r)(p-s)} + \frac{x^{q-1}}{(q-p)(q-r)(q-s)} + \frac{x^{r-1}}{(r-p)(r-q)(r-s)} + \right. \\ \left. + \frac{x^{s-1}}{(s-p)(s-q)(s-r)} \right] \frac{dx}{(\ln x)^2} = \frac{1}{2} \left[\frac{p^2 \ln p}{(p-q)(p-r)(p-s)} + \frac{q^2 \ln q}{(q-p)(q-r)(q-s)} \right. \\ \left. + \frac{r^2 \ln r}{(r-p)(r-q)(r-s)} + \frac{s^2 \ln s}{(s-p)(s-q)(s-r)} \right] \\ [p > 0, \quad q > 0, \quad r > 0, \quad s > 0] \quad \text{BI (124)(16)}$$

4.269

$$1. \quad \int_0^1 \sqrt{\ln \frac{1}{x}} \frac{dx}{1+x^2} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}} \quad \text{BI (115)(33)}$$

$$2.^{11} \quad \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}} (1+x^2)} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \quad \text{BI (133)(2)}$$

$$3. \quad \int_0^1 \sqrt{\ln \frac{1}{x}} x^{p-1} dx = \frac{1}{2} \sqrt{\frac{\pi}{p^3}} \quad [p > 0] \quad \text{GW (324)(1c)}$$

$$4. \quad \int_0^1 \frac{x^{p-1}}{\sqrt{\ln \frac{1}{x}}} dx = \sqrt{\frac{\pi}{p}} \quad [p > 0] \quad \text{BI (133)(1)}$$

$$5. \quad \int_0^1 \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1} \sin nt}{1 - 2x \cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^n \frac{\sin kt}{\sqrt{k}} \quad [|t| < \pi] \quad \text{BI (133)(5)}$$

$$6. \quad \int_0^1 \frac{\cos t - x - x^{n-1} \cos nt + x^n \cos[(n-1)t]}{1 - 2x \cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^{n-1} \frac{\cos kt}{\sqrt{k}} \quad [|t| < \pi] \quad \text{BI (133)(6)}$$

$$7. \quad \int_u^v \frac{dx}{x \cdot \sqrt{\ln \frac{x}{u} \ln \frac{v}{x}}} = \pi \quad [uv > 0] \quad \text{BI (145)(37)}$$

4.271

$$1. \quad \int_0^1 (\ln x)^{2n} \frac{dx}{1+x} = \frac{2^{2n}-1}{2^{2n}} \cdot (2n)! \zeta(2n+1) \quad \text{BI (110)(1)}$$

$$2. \quad \int_0^1 (\ln x)^{2n-1} \frac{dx}{1+x} = \frac{1-2^{2n-1}}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (110)(2)}$$

$$3. \quad \int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x} = -\frac{1}{n} 2^{2n-2} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (110)(5), GW(324)(9a)}$$

$$4. \quad \int_0^1 (\ln x)^{p-1} \frac{dx}{1-x} = e^{i(p-1)\pi} \Gamma(p) \zeta(p) \quad [p > 1] \quad \text{GW (324)(9b)}$$

$$5. \quad \int_0^1 (\ln x)^n \frac{dx}{1+x^2} = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} \quad \text{BI (110)(11)}$$

$$6. \quad \int_0^1 (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{1}{2} \int_0^{\infty} (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{\pi^{2n+1}}{2^{2n+2}} |E_{2n}| \quad \text{GW (324)(10)a}$$

$$7. \quad \int_0^{\infty} \frac{(\ln x)^{2n+1}}{1+bx+x^2} dx = 0 \quad [|b| < 2] \quad \text{BI (135)(2)}$$

$$8. \quad \int_0^1 (\ln x)^{2n} \frac{dx}{1-x^2} = \frac{2^{2n+1}-1}{2^{2n+1}} \cdot (2n)! \zeta(2n+1) \quad [n = 1, 2, \dots] \quad \text{BI (110)(12)}$$

$$9. \quad \int_0^{\infty} (\ln x)^{2n} \frac{dx}{1-x^2} = 0 \quad \text{BI (312)(7)a}$$

$$10. \quad \int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1}{2} \int_0^{\infty} (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1-2^{2n}}{4n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (290)(17)a, BI(312)(6)a}$$

11. $\int_0^1 (\ln x)^{2n-1} \frac{x dx}{1-x^2} = -\frac{1}{4n} \pi^{2n} |B_{2n}|$ $[n = 1, 2, \dots]$ BI (290)(19)a
12. $\int_0^1 (\ln x)^{2n} \frac{1+x^2}{(1-x^2)^2} dx = \frac{2^{2n}-1}{2} \pi^{2n} |B_{2n}|$ $[n = 1, 2, \dots]$ BI (296)(17)a
13. $\int_0^1 (\ln x)^{2n+1} \frac{(\cos 2a\pi - x) dx}{1-2x \cos 2a\pi + x^2} = -(2n+1)! \sum_{k=1}^{\infty} \frac{\cos 2ak\pi}{k^{2n+2}}$
 $[a \text{ is not an integer}]$ LI (113)(10)
- 14.⁶ $\int_0^{\infty} (\ln x)^n \frac{x^{\nu-1} dx}{a^2 + 2ax \cos t + x^2} = -\pi \operatorname{cosec} t \frac{d^n}{d\nu^n} \left[a^{\nu-2} \frac{\sin(\nu-1)t}{\sin \nu\pi} \right]$
 $[a > 0, \quad 0 < \operatorname{Re} \nu < 2, \quad 0 < |t| < \pi]$ ET I 315(12)
15. $\int_0^1 (\ln x)^n \frac{x^{p-1}}{1-x^q} dx = -\frac{1}{q^{n+1}} \psi^{(n)} \left(\frac{p}{q} \right)$ $[p > 0, \quad q > 0]$ GW (324)(9)
- 16.³ $\int_0^1 (\ln x)^n \frac{x^{p-1}}{1+x^q} dx = \frac{1}{q^{n+1}} \beta^{(n)} \left(\frac{p}{q} \right)$ $[p > 0, \quad q > 0]$ GW (324)(10)

4.272

1. $\int_0^1 \frac{\left[\ln \left(\frac{1}{x} \right) \right]^{q-1} dx}{1+2x \cos t + x^2} = \operatorname{cosec} t \Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k^q}$ $[|t| < \pi, q < 1]$ LI (130)(1)
2. $\int_0^1 \left(\ln \frac{1}{x} \right)^{q-1} \frac{(1+x) dx}{1+2x \cos t + x^2} = \sec \frac{t}{2} \cdot \Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos \left[\left(k - \frac{1}{2} \right) t \right]}{k^q}$
 $[|t| < \pi, \quad q < \frac{1}{2}]$ LI (130)(5)
- 3.⁹ $\int_0^1 \left[\ln \left(\frac{1}{x} \right) \right]^{\mu} \frac{x^{\nu-1} dx}{1-2ax \cos t + x^2 a^2} = \frac{\Gamma(\mu+1)}{a \sin t} \sum_{k=1}^{\infty} \frac{a^k \sin kt}{(\nu+k-1)^{\mu+1}}$
 $[a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad -\pi < t < \pi]$ BI (140)(14)a
4. $\int_0^1 \left(\ln \frac{1}{x} \right)^{r-1} \frac{\cos \lambda - px}{1+p^2 x^2 - 2px \cos \lambda} x^{q-1} dx = \Gamma(r) \sum_{k=1}^{\infty} \frac{p^{k-1} \cos k\lambda}{(q+k-1)^r}$
 $[r > 0, \quad q > 0]$ BI (113)(11)
5. $\int_1^{\infty} (\ln x)^p \frac{dx}{x^2} = \Gamma(1+p)$ $[p > -1]$ BI (149)(1)
6. $\int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} x^{\nu-1} dx = \frac{1}{\nu^{\mu}} \Gamma(\mu)$ $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$ BI (107)(3)
7. $\int_0^1 \left(\ln \frac{1}{x} \right)^{n-\frac{1}{2}} x^{\nu-1} dx = \frac{(2n-1)!!}{(2\nu)^n} \sqrt{\frac{\pi}{\nu}}$ $[\operatorname{Re} \nu > 0]$ BI (107)(2)

$$8.^{11} \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1+x} dx = \Gamma\left(3 - \frac{1}{n}\right) \left(p^{\frac{1}{n}-3} - q^{\frac{1}{n}-3}\right) \quad [\operatorname{Re} \nu > 0] \quad \text{BI (110)(4)}$$

$$9. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1-x} dx = (n-1)! \zeta(n, \nu) \quad [\operatorname{Re} \nu > 0] \quad \text{BI (110)(7)}$$

$$10. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{\mu-1} (x-1)^n \left(a + \frac{nx}{x-1}\right) x^{a-1} dx = \Gamma(\mu) \sum_{k=0}^n \frac{(-1)^k n(n-1) \dots (n-k+1)}{(a+n-k)^{\mu-1} k!} \quad [\operatorname{Re} \mu > 0] \quad \text{LI (110)(10)}$$

$$11. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} \frac{1-x^m}{1-x} dx = (n-1)! \sum_{k=1}^m \frac{1}{k^n} \quad \text{LI (110)(9)}$$

$$12. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{\mu-1} \frac{x^{\nu-1} dx}{1-x^2} = \Gamma(\mu) \sum_{k=0}^{\infty} \frac{1}{(\nu+2k)^{\mu}} = \frac{1}{2^{\mu}} \Gamma(\mu) \zeta\left(\mu, \frac{\nu}{2}\right) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (110)(13)}$$

$$13. \quad \int_0^1 \frac{x^q - x^{-q}}{1-x^2} \left(\ln \frac{1}{x}\right)^p dx = \Gamma(p+1) \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k+q-1)^{p+1}} - \frac{1}{(2k-q-1)^{p+1}} \right\} \quad [p > -1, \quad q^2 < 1] \quad \text{LI (326)(12)a}$$

$$14. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{r-1} \frac{x^{p-1} dx}{(1+x^q)^s} = \Gamma(r) \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{(p+kq)^r} \quad [p > 0, \quad q > 0, \quad r > 0, \quad 0 < s < r+2] \quad \text{GW (324)(11)}$$

$$15. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^n (1+x^q)^m x^{p-1} dx = n! \sum_{k=0}^m \binom{m}{k} \frac{1}{(p+kq)^{n+1}} \quad [p > 0, \quad q > 0] \quad \text{BI (107)(6)}$$

$$16. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^n (1-x^q)^m x^{p-1} dx = n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(p+kq)^{n+1}} \quad [p > 0, \quad q > 0] \quad \text{BI (107)(7)}$$

$$17. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{p-1} \frac{x^{q-1} dx}{1-ax^q} = \frac{1}{aq^p} \Gamma(p) \sum_{k=1}^{\infty} \frac{a^k}{k^p} \quad [p > 0, \quad q > 0, \quad a < 1] \quad \text{LI (110)(8)}$$

$$18. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{2-\frac{1}{n}} (x^{p-1} - x^{q-1}) dx = \frac{n}{n-1} \Gamma\left(\frac{1}{n}\right) \left(q^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right) \quad [q > p > 0] \quad \text{BI (133)(4)}$$

$$19. \quad \int_0^1 \left(\ln \frac{1}{x}\right)^{2n-1} \frac{x^p - x^{-p}}{1-x^q} x^{q-1} dx = \frac{1}{p^{2n}} \sum_{k=n}^{\infty} \left(\frac{2p\pi}{q}\right)^k \frac{|B_{2k}|}{2k \cdot (2k-2n)!} \quad \left[p < \frac{q}{2}\right] \quad \text{LI (110)(16)}$$

$$4.273 \quad \int_u^v \left(\ln \frac{x}{u}\right)^{p-1} \left(\ln \frac{v}{x}\right)^{q-1} \frac{dx}{x} = B(p, q) \left(\ln \frac{v}{u}\right)^{p+q-1} \quad [p > 0, \quad q > 0, \quad uv > 0] \quad \text{BI (145)(36)}$$

$$4.274 \quad \int_0^1 \frac{e^{\frac{1}{x}} \sqrt[q]{x} dx}{x \sqrt{-(1 + \ln x)}} = \frac{\sqrt[q]{q\pi}}{\sqrt[q]{e}} \quad [q > 0] \quad \text{BI (145)(4)}$$

4.275

$$1. \quad \int_0^1 \left[\left(\ln \frac{1}{x}\right)^{q-1} - x^{p-1} (1-x)^{q-1} \right] dx = \frac{\Gamma(q)}{\Gamma(p+q)} [\Gamma(p+q) - \Gamma(p)]$$

$$[p > 0, \quad q > 0] \quad \text{BI (107)(8)}$$

$$2. \quad \int_0^1 \left[x - \left(\frac{1}{1 - \ln x} \right)^q \right] \frac{dx}{x \ln x} = -\psi(q)$$

$$[q > 0] \quad \text{BI (126)(5)}$$

4.28 Combinations of rational functions of $\ln x$ and powers

4.281

$$1. \quad \int_0^1 \left[\frac{1}{\ln x} + \frac{1}{1-x} \right] dx = C \quad \text{BI (127)(15)}$$

$$2. \quad \int_1^\infty \frac{dx}{x^2 (\ln p - \ln x)} = \frac{1}{p} \text{li}(p) \quad \text{LA 281(30)}$$

$$3. \quad \int_0^1 \frac{x^{p-1} dx}{q \pm \ln x} = \pm e^{\mp pq} \text{Ei}(\pm pq) \quad [p > 0, \quad q > 0] \quad \text{LI (144)(11,12)}$$

$$4. \quad \int_0^1 \left[\frac{1}{\ln x} + \frac{x^{\mu-1}}{1-x} \right] dx = -\psi(\mu) \quad [\text{Re } \mu > 0] \quad \text{WH}$$

$$5. \quad \int_0^1 \left[\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right] dx = \ln p - \psi(q) \quad [p > 0, \quad q > 0] \quad \text{BI (127)(17)}$$

$$6. \quad \int_0^1 \left[\frac{1}{1-x^2} + \frac{1}{2x \ln x} \right] \frac{dx}{\ln x} = \frac{\ln 2}{2} \quad \text{LI (130)(19)}$$

$$7. \quad \int_0^1 \left[q - \frac{1}{2} + \frac{(1-x)(1+q \ln x) + x \ln x}{(1-x)^2} x^{q-1} \right] \frac{dx}{\ln x} = \frac{1}{2} - q - \ln \Gamma(q) + \frac{\ln 2\pi}{2}$$

$$[q > 0] \quad \text{BI (128)(15)}$$

4.282

$$1. \quad \int_0^1 \frac{\ln x}{4\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x} = \frac{1}{4} - \frac{1}{2} C \quad \text{BI (129)(1)}$$

$$2. \quad \int_0^1 \frac{1}{a^2 + (\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{1}{2a} \beta\left(\frac{2a+\pi}{4\pi}\right) \quad \left[a > -\frac{\pi}{2}\right] \quad \text{BI (129)(9)}$$

$$3. \quad \int_0^1 \frac{1}{\pi^2 + (\ln x)^2} \frac{dx}{1+x^2} = \frac{4-\pi}{4\pi} \quad \text{BI (129)(6)}$$

$$4. \quad \int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x^2} = \frac{1}{2} \left(\frac{1}{2} - \ln 2 \right) \quad \text{BI (129)(10)}$$

$$5. \quad \int_0^1 \frac{\ln x}{a^2 + (\ln x)^2} \cdot \frac{x dx}{1-x^2} = \frac{1}{2} \left[\frac{\pi}{2a} + \ln \frac{\pi}{a} + \psi \left(\frac{a}{\pi} \right) \right] \quad [a > 0] \quad \text{BI (129)(14)}$$

$$6. \quad \int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{x dx}{1-x^2} = \frac{1}{2} \left(\frac{1}{2} - C \right) \quad \text{BI (129)(13)}$$

$$7. \quad \int_0^1 \frac{1}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{\ln 2}{4\pi} \quad \text{BI (129)(7)}$$

$$8. \quad \int_0^1 \frac{\ln x}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1-x^2} = \frac{2-\pi}{16} \quad \text{BI (129)(11)}$$

$$9.^{10} \quad \int_0^1 \frac{1}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{1}{8\pi\sqrt{2}} \left[\pi + 2 \ln(\sqrt{2}-1) \right] \quad \text{BI (129)(8)}$$

$$10. \quad \int_0^1 \frac{\ln x}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1-x^2} = -\frac{\pi}{32\sqrt{2}} + \frac{1}{16} + \frac{1}{16\sqrt{2}} \ln(\sqrt{2}-1) \quad \text{BI (129)(12)}$$

$$11. \quad \int_0^1 \frac{\ln x}{[a^2 + (\ln x)^2]^2} \frac{dx}{1-x} = -\frac{\pi^2}{a^4} \sum_{k=1}^{\infty} |B_{2k}| \left(\frac{2\pi}{a} \right)^{2k-2} \quad \text{BI (129)(4)}$$

$$12. \quad \int_0^1 \frac{\ln x}{[a^2 + (\ln x)^2]^2} \frac{x dx}{1-x^2} = -\frac{\pi^2}{4a^4} \sum_{k=1}^{\infty} |B_{2k}| \left(\frac{\pi}{a} \right)^{2k-2} \quad \text{BI (129)(16)}$$

$$13. \quad \int_0^1 \frac{x^p - x^{-p}}{x^2 - 1} \frac{dx}{q^2 + (\ln x)^2} = \frac{2\pi}{q} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kp\pi}{2q + k\pi} \quad [p^2 < 1] \quad \text{BI (132)(13)a}$$

4.283

$$1. \quad \int_0^1 \left(\frac{x-1}{\ln x} - x \right) \frac{dx}{\ln x} = \ln 2 - 1 \quad \text{BI (132)(17)a}$$

$$2. \quad \int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} - \frac{1}{2} \right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2} - 1 \quad \text{BI (127)(20)}$$

$$3. \quad \int_0^1 \left(\frac{1}{\ln x} + \frac{x}{1-x} + \frac{x}{2} \right) \frac{dx}{x \ln x} = \frac{\ln 2\pi}{2} \quad \text{BI (127)(23)}$$

$$4. \quad \int_0^1 \left[\frac{1}{(\ln x)^2} - \frac{x}{(1-x)^2} \right] dx = C - \frac{1}{2} \quad \text{GW (326)(8a)}$$

$$5. \quad \int_0^1 \left(\frac{1}{1-x^2} + \frac{1}{2\ln x} - \frac{1}{2} \right) \frac{dx}{\ln x} = \frac{\ln 2 - 1}{2} \quad \text{BI (128)(14)}$$

$$6. \quad \int_0^1 \left(\frac{1}{\ln x} + \frac{1}{2} \cdot \frac{1+x}{1-x} - \ln x \right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2} \quad \text{BI (127)(22)}$$

$$7. \quad \int_0^1 \left[\frac{1}{1-\ln x} - x \right] \frac{dx}{x \ln x} = -C \quad \text{GW (326)(11a)}$$

$$8. \quad \int_0^1 \left[\frac{x^q - 1}{x(\ln x)^2} - \frac{q}{\ln x} \right] dx = q \ln q - q \quad [q > 0] \quad \text{BI (126)(2)}$$

$$9. \quad \int_0^1 \left[x + \frac{1}{a \ln x - 1} \right] \frac{dx}{x \ln x} = \ln \frac{a}{q} + C \quad [a > 0, \quad q > 0] \quad \text{BI (126)(8)}$$

$$10. \quad \int_0^1 \left[\frac{1}{\ln x} + \frac{1+x}{2(1-x)} \right] \frac{x^{p-1}}{\ln x} dx = -\ln \Gamma(p) + \left(p - \frac{1}{2} \right) \ln p - p + \frac{\ln 2\pi}{2} \\ [p > 0] \quad \text{GW (326)(9)}$$

$$11. \quad \int_0^1 \left[p - 1 - \frac{1}{1-x} + \left(\frac{1}{2} - \frac{1}{\ln x} \right) x^{p-1} \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) \ln p + p - \frac{\ln 2\pi}{2} \\ [p > 0] \quad \text{BI (127)(25)}$$

$$12. \quad \int_0^1 \left[-\frac{1}{(\ln x)^2} + \frac{(p-2)x^p - (p-1)x^{p-1}}{(1-x)^2} \right] dx = -\psi(p) + p - \frac{3}{2} \\ [p > 0] \quad \text{GW (326)(8)}$$

$$13. \quad \int_0^1 \left[\left(p - \frac{1}{2} \right) x^3 + \frac{1}{2} \left(1 - \frac{1}{\ln x} \right) (x^{2p-1} - 1) \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) (\ln p - 1) \\ [p > 0] \quad \text{BI (132)(23)a}$$

$$14. \quad \int_0^1 \left[\left(q - \frac{1}{2} \right) \frac{x^{p-1} - x^{r-1}}{\ln x} + \frac{px^{pq-1}}{1-x^p} - \frac{rx^{rq-1}}{1-x^r} \right] \frac{dx}{\ln x} = (p-r) \left[\frac{1}{2} - q - \ln \Gamma(q) + \frac{\ln 2\pi}{2} \right] \\ [q > 0] \quad \text{BI (132)(13)}$$

4.284

$$1. \quad \int_0^1 \left[\frac{x^q - 1}{x (\ln x)^3} - \frac{q}{x (\ln x)^2} - \frac{q^2}{2 \ln x} \right] dx = \frac{q^2}{2} \ln q - \frac{3}{4} q^2 \\ [q > 0] \quad \text{BI (126)(3)}$$

$$2. \quad \int_0^1 \left[\frac{x^q - 1}{x (\ln x)^4} - \frac{q}{x (\ln x)^3} - \frac{q^2}{2x (\ln x)^2} - \frac{q^3}{6 \ln x} \right] dx = \frac{q^3}{6} \ln q - \frac{11}{36} q^3 \\ [q > 0] \quad \text{BI (126)(4)}$$

$$4.285 \quad \int_0^1 \frac{x^{p-1} dx}{(q + \ln x)^n} = \frac{p^{n-1}}{(n-1)!} e^{-pq} \text{Ei}(pq) - \frac{1}{(n-1)! q^{n-1}} \sum_{k=1}^{n-1} (n-k-1)! (pq)^{k-1} \\ [p > 0, \quad q < 0] \quad \text{BI (125)(21)}$$

In integrals of the form $\int \frac{x^a (\ln x)^n dx}{[b \pm (\ln x)^m]^l}$, we should make the substitution $x = e^t$ or $x = e^{-t}$ and then seek the resulting integrals in **3.351–3.356**.

4.29–4.32 Combinations of logarithmic functions of more complicated arguments and powers

4.291

$$1. \quad \int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12} \quad \text{FI II 483}$$

2. $\int_0^1 \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}$ FI II 714
3. $\int_0^{1/2} \frac{\ln(1-x)}{x} dx = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}$ BI (145)(2)
4. $\int_0^1 \ln\left(1 - \frac{x}{2}\right) \frac{dx}{x} = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}$ BI (114)(18)
5. $\int_0^1 \frac{\ln \frac{1+x}{2}}{1-x} dx = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}$ BI (115)(1)
6. $\int_0^1 \frac{\ln(1+x)}{1+x} dx = \frac{1}{2} (\ln 2)^2$ BI (114)(14)a
- 7.⁷ $\int_0^\infty \frac{\ln(1+ax)}{1+x^2} dx = \frac{\pi}{4} \ln(1+a^2) - \int_0^a \frac{\ln u du}{1+u^2}$ [a > 0] GI II (2209)
8. $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2$ FI II 157
9. $\int_0^\infty \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{4} \ln 2 + \mathbf{G}$ BI (136)(1)
10. $\int_0^1 \frac{\ln(1-x)}{1+x^2} dx = \frac{\pi}{8} \ln 2 - \mathbf{G}$ BI (114)(17)
11. $\int_1^\infty \frac{\ln(x-1)}{1+x^2} dx = \frac{\pi}{8} \ln 2$ BI (144)(4)
12. $\int_0^1 \frac{\ln(1+x)}{x(1+x)} dx = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2$ BI (144)(4)
13. $\int_0^\infty \frac{\ln(1+x)}{x(1+x)} dx = \frac{\pi^2}{6}$. BI (141)(9)a
14. $\int_0^1 \frac{\ln(1+x)}{(ax+b)^2} dx = \frac{1}{a(a-b)} \ln \frac{a+b}{b} + \frac{2 \ln 2}{b^2 - a^2}$ [a ≠ b, ab > 0]
 $\quad = \frac{1}{2a^2} (1 - \ln 2)$ [a = b]
LI (114)(5)a
15. $\int_0^\infty \frac{\ln(1+x)}{(ax+b)^2} dx = \frac{\ln \frac{a}{b}}{a(a-b)}$ [ab > 0] BI (139)(5)
16. $\int_0^1 \ln(a+x) \frac{dx}{a+x^2} = \frac{1}{2\sqrt{a}} \operatorname{arccot} \sqrt{a} \ln[(1+a)a]$ [a > 0] BI (114)(20)
17. $\int_0^\infty \ln(a+x) \frac{dx}{(b+x)^2} = \frac{a \ln a - b \ln b}{b(a-b)}$ [a > 0, b > 0, a ≠ b] LI (139)(6)
18. $\int_0^a \frac{\ln(1+ax)}{1+x^2} dx = \frac{1}{2} \arctan a \ln(1+a^2)$ GI II (2195)

$$19. \quad \int_0^1 \frac{\ln(1+ax)}{1+ax^2} dx = \frac{1}{2\sqrt{a}} \arctan \sqrt{a} \ln(1+a) \quad [a > 0] \quad \text{BI (114)(21)}$$

$$20. \quad \int_0^1 \frac{\ln(ax+b)}{(1+x)^2} dx = \frac{1}{a-b} \left[\frac{1}{2}(a+b) \ln(a+b) - b \ln b - a \ln 2 \right] \\ [a > 0, \quad b > 0, \quad a \neq b] \quad \text{BI (114)(22)}$$

$$21. \quad \int_0^\infty \frac{\ln(ax+b)}{(1+x)^2} dx = \frac{1}{a-b} [a \ln a - b \ln b] \quad [a > 0, \quad b > 0] \quad \text{BI (139)(8)}$$

$$22. \quad \int_0^\infty \ln(a+x) \frac{x dx}{(b^2+x^2)^2} = \frac{1}{2(a^2+b^2)} \left(\ln b + \frac{a\pi}{2b} + \frac{a^2}{b^2} \ln a \right) \\ [a > 0, \quad b > 0] \quad \text{BI (139)(9)}$$

$$23. \quad \int_0^1 \ln(1+x) \frac{1+x^2}{(1+x)^4} dx = -\frac{1}{3} \ln 2 + \frac{23}{72} \quad \text{LI (114)(12)}$$

$$24. \quad \int_0^1 \ln(1+x) \frac{1+x^2}{a^2+x^2} \cdot \frac{dx}{1+a^2x^2} = \frac{1}{2a(1+a^2)} \left[\frac{\pi}{2} \ln(1+a^2) - 2 \arctan a \cdot \ln a \right] \\ [a > 0] \quad \text{LI (114)(11)}$$

$$25. \quad \int_0^1 \ln(1+x) \frac{1-x^2}{(ax+b)^2} \frac{dx}{(bx+a)^2} = \frac{1}{a^2-b^2} \left\{ \frac{1}{a-b} \left[\frac{a+b}{ab} \ln(a+b) - \frac{1}{a} \ln b - \frac{1}{b} \ln a \right] \right. \\ \left. + \frac{4 \ln 2}{b^2-a^2} \right\} \\ [a > 0, \quad b > 0, \quad a^2 \neq b^2] \quad \text{LI (114)(13)}$$

$$26. \quad \int_0^\infty \ln(1+x) \frac{1-x^2}{(ax+b)^2} \cdot \frac{dx}{(bx+a)^2} = \frac{1}{ab(a^2-b^2)} \ln \frac{b}{a} \\ [a > 0, \quad b > 0] \quad \text{LI (139)(14)}$$

$$27. \quad \int_0^1 \ln(1+ax) \frac{1-x^2}{(1+x^2)^2} dx = \frac{1}{2} \frac{(1+a)^2}{1+a^2} \ln(1+a) - \frac{1}{2} \cdot \frac{a}{1+a^2} \ln 2 - \frac{\pi}{4} \cdot \frac{a^2}{1+a^2} \\ [a > -1] \quad \text{BI (114)(23)}$$

$$28. \quad \int_0^\infty \ln(a+x) \frac{b^2-x^2}{(b^2+x^2)^2} dx = \frac{1}{a^2+b^2} \left(a \ln \frac{b}{a} - \frac{b\pi}{2} \right) \\ [a > 0, \quad b > 0] \quad \text{BI (139)(11)}$$

$$29. \quad \int_0^\infty \ln^2(a-x) \frac{b^2-x^2}{(b^2+x^2)^2} dx = \frac{2}{a^2+b^2} \left(a \ln \frac{a}{b} - \frac{b\pi}{2} \right) \\ [a > 0, \quad b > 0] \quad \text{BI (139)(12)}$$

$$30. \quad \int_0^\infty \ln^2(a-x) \frac{x dx}{(b^2+x^2)^2} = \frac{1}{a^2+b^2} \left(\ln b - \frac{a\pi}{2b} + \frac{a^2}{b^2} \ln a \right) \\ [a > 0, \quad b > 0] \quad \text{BI (139)(10)}$$

4.292

$$1. \quad \int_0^1 \frac{\ln(1 \pm x)}{\sqrt{1-x^2}} dx = -\frac{\pi}{2} \ln 2 \pm 2\mathbf{G} \quad \text{GW (325)(20)}$$

$$2. \quad \int_0^1 \frac{x \ln(1 \pm x)}{\sqrt{1-x^2}} dx = -1 \pm \frac{\pi}{2} \quad \text{GW (325)(22c)}$$

$$3. \quad \int_{-a}^a \frac{\ln(1+bx)}{\sqrt{a^2-x^2}} dx = \pi \ln \frac{1+\sqrt{1-a^2b^2}}{2} \quad \left[0 \leq |b| \leq \frac{1}{a}\right] \\ \text{BI (145)(16, 17)a, GW (325)(21e)}$$

$$4. \quad \int_0^1 \frac{x \ln(1+ax)}{\sqrt{1-x^2}} dx = -1 + \frac{\pi}{2} \cdot \frac{1-\sqrt{1-a^2}}{a} + \frac{\sqrt{1-a^2}}{a} \arcsin a \quad [|a| \leq 1] \\ = -1 + \frac{\pi}{2a} + \frac{\sqrt{a^2-1}}{a} \ln(a + \sqrt{a^2-1}) \quad [a \geq 1] \\ \text{GW (325)(22)}$$

$$5. \quad \int_0^1 \frac{\ln(1+ax)}{x\sqrt{1-x^2}} dx = \frac{1}{2} \arcsin a (\pi - \arcsin a) = \frac{\pi^2}{8} - \frac{1}{2} (\arccos a)^2 \\ [|a| \leq 1] \quad \text{BI (120)(4), GW (325)(21a)}$$

4.293

$$1. \quad \int_0^1 x^{\mu-1} \ln(1+x) dx = \frac{1}{\mu} [\ln 2 - \beta(\mu+1)] \quad [\operatorname{Re} \mu > -1] \quad \text{BI (106)(4)a}$$

$$2.^6 \quad \int_1^\infty x^{\mu-1} \ln(1+x) dx = \frac{-1}{\mu} [\beta(-\mu) + \ln 2] \quad [\operatorname{Re} \mu < 0] \quad \text{ET I 315(17)}$$

$$3. \quad \int_0^\infty x^{\mu-1} \ln(1+x) dx = \frac{\pi}{\mu \sin \mu \pi} \quad [-1 < \operatorname{Re} \mu < 0] \quad \text{GW (325)(3)a}$$

$$4. \quad \int_0^1 x^{2n-1} \ln(1+x) dx = \frac{1}{2n} \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} \quad \text{GW (325)(2b)}$$

$$5. \quad \int_0^1 x^{2n} \ln(1+x) dx = \frac{1}{2n+1} \left[\ln 4 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right] \quad \text{GW (325)(2c)}$$

$$6.^{11} \quad \int_0^1 x^{n-\frac{1}{2}} \ln(1+x) dx = \frac{2 \ln 2}{2n+1} + \frac{(-1)^n \cdot 4}{2n+1} \left[\frac{\pi}{4} - \sum_{k=0}^n \frac{(-1)^k}{2k+1} \right] \quad \text{GW (325)(2f)}$$

$$7. \quad \int_0^\infty x^{\mu-1} \ln|1-x| dx = \frac{\pi}{\mu} \cot(\mu\pi) \quad [-1 < \operatorname{Re} \mu < 0] \\ \text{BI (134)(4), ET I 315(18)}$$

$$8. \quad \int_0^1 x^{\mu-1} \ln(1-x) dx = -\frac{1}{\mu} [\psi(\mu+1) - \psi(1)] = -\frac{1}{\mu} [\psi(\mu+1) + \mathbf{C}] \\ [\operatorname{Re} \mu > -1] \quad \text{ET I 316(19)}$$

$$9.^7 \quad \int_1^\infty x^{\mu-1} \ln(x-1) dx = \frac{1}{\mu} [\pi \cot(\mu\pi) + \psi(\mu+1) + \mathbf{C}] \\ [\operatorname{Re} \mu < 0] \quad \text{ET I 316(20)}$$

$$10. \quad \int_0^\infty x^{\mu-1} \ln(1+\gamma x) dx = \frac{\pi}{\mu \gamma^\mu \sin \mu \pi} \quad [-1 < \operatorname{Re} \mu < 0, \quad |\arg \gamma| < \pi] \quad \text{BI (134)(3)}$$

$$11.^{11} \quad \int_0^\infty \frac{x^{\mu-1} \ln(1+x)}{1+x} dx = -\frac{\pi}{\sin \mu \pi} [\mathbf{C} + \psi(1-\mu)] \quad [-1 < \operatorname{Re} \mu < 1] \quad \text{ET I 316(21)}$$

$$12. \quad \int_0^1 \frac{\ln(1+x)}{(1+x)^{\mu+1}} dx = -\frac{\ln 2}{2^\mu \mu} + \frac{2^\mu - 1}{2^\mu \mu^2} \quad \text{BI (114)(6)}$$

$$13. \quad \int_0^1 \frac{x^{\mu-1} \ln(1-x)}{(1-x)^{1-\nu}} dx = \mathbf{B}(\mu, \nu) [\psi(\nu) - \psi(\mu + \nu)] \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 316(122)}$$

$$14. \quad \int_0^\infty \frac{x^{\mu-1} \ln(\gamma + x)}{(\gamma + x)^\nu} dx = \gamma^{\mu-\nu} \mathbf{B}(\mu, \nu - \mu) [\psi(\nu) - \psi(\nu - \mu) + \ln \gamma] \quad [0 < \operatorname{Re} \mu < \operatorname{Re} \nu] \quad \text{ET I 316(23)}$$

4.294

$$1. \quad \int_0^1 \ln(1+x) \frac{(p-1)x^{p-1} - px^{-p}}{x} dx = 2 \ln 2 - \frac{\pi}{\sin p \pi} \quad [0 < p < 1] \quad \text{BI (114)(2)}$$

$$2. \quad \int_0^1 \ln(1+x) \frac{1+x^{2n+1}}{1+x} dx = 2 \ln 2 \sum_{k=0}^n \frac{1}{2k+1} - \sum_{j=1}^{2n+1} \frac{1}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k} \quad \text{BI (114)(7)}$$

$$3. \quad \int_0^1 \ln(1+x) \frac{1-x^{2n}}{1+x} dx = 2 \ln 2 \cdot \sum_{k=0}^{n-1} \frac{1}{2k+1} - \sum_{j=1}^{2n} \frac{1}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k} \quad \text{BI (114)(8)}$$

$$4. \quad \int_0^1 \ln(1+x) \frac{1-x^{2n}}{1-x} dx = 2 \ln 2 \cdot \sum_{k=0}^{n-1} \frac{1}{2k+1} + \sum_{i=1}^{2n} \frac{(-1)^j}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k} \quad \text{BI (114)(9)}$$

$$5. \quad \int_0^1 \ln(1+x) \frac{1-x^{2n+1}}{1-x} dx = 2 \ln 2 \sum_{k=0}^n \frac{1}{2k+1} + \sum_{j=1}^{2n+1} \frac{(-1)^j}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k} \quad \text{BI (114)(10)}$$

$$6. \quad \int_0^1 \ln(1-x) \frac{1-(-1)^n x^n}{1-x} dx = \sum_{j=1}^n \frac{(-1)^j}{j} \sum_{k=1}^j \frac{1}{k} \quad \text{BI (114)(15)}$$

$$7. \quad \int_0^1 \ln(1-x) \frac{1-x^n}{1-x} dx = -\sum_{j=1}^n \frac{1}{j} \sum_{k=1}^j \frac{1}{k} \quad \text{BI (114)(16)}$$

$$8. \quad \int_0^\infty \ln^2(1-x) x^p dx = \frac{2\pi}{p+1} \cot p\pi \quad [-2 < p < -1] \quad \text{BI (134)(13)a}$$

$$9. \quad \int_0^1 [\ln(1+x)]^n (1+x)^r dx = (-1)^{n-1} \frac{n!}{(r+1)^{n+1}} + 2^{r+1} \sum_{k=0}^n \frac{(-1)^k n! (\ln 2)^{n-k}}{(n-k)! (r+1)^{k+1}} \quad \text{LI (106)(34)a}$$

$$10. \quad \int_0^1 [\ln(1-x)]^n (1-x)^r dx = (-1)^n \frac{n!}{(r+1)^{n+1}} \quad [r > -1] \quad \text{BI (106)(35)a}$$

11. $\int_0^1 \left(\ln \frac{1}{1-x^2} \right)^n x^{2q-1} dx = \frac{n!}{2} \zeta(n+1, q+1) \quad [-1 < q < 0] \quad \text{BI (311)(15)a}$
12. $\int_0^1 (\ln x)^{2n} \ln(1-x^2) \frac{dx}{x} = -\frac{\pi^{2n+2}}{2(n+1)(2n+1)} |B_{2n+2}| \quad \text{BI (309)(5)a}$
- 13.⁶ $\int_0^1 \left[\ln \frac{1}{x} \right]^m \ln(1-x^2) dx = -\sum_{n=1}^{\infty} \frac{\Gamma(m+1)}{n(2n+1)^{m+1}} \quad [m+1 > 0, \quad n+1 > 0]$

4.295

1. $\int_0^{\infty} \ln(\mu x^2 + \beta) \frac{dx}{\gamma + x^2} = \frac{\pi}{\sqrt{\gamma}} \ln(\sqrt{\mu\gamma} + \sqrt{\beta}) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0, \quad |\arg \gamma| < \pi] \quad \text{ET II 218(27)}$
2. $\int_0^1 \ln(1+x^2) \frac{dx}{x^2} = \frac{\pi}{2} - \ln 2 \quad \text{GW (325)(2g)}$
3. $\int_0^{\infty} \ln(1+x^2) \frac{dx}{x^2} = \pi \quad \text{GW (325)(4c)}$
4. $\int_0^{\infty} \ln(1+x^2) \frac{dx}{(a+x)^2} = \frac{2a}{1+a^2} \left(\frac{\pi}{2a} + \ln a \right) \quad [a > 0] \quad \text{BI (319)(6)a}$
5. $\int_0^1 \ln(1+x^2) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 - \mathbf{G} \quad \text{BI (114)(24)}$
6. $\int_1^{\infty} \ln(1+x^2) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 + \mathbf{G} \quad \text{BI (114)(5)}$
7. $\int_0^{\infty} \ln(a^2 + b^2 x^2) \frac{dx}{c^2 + g^2 x^2} = \frac{\pi}{cg} \ln \frac{ag + bc}{g} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{BI (136)(11-14)a}$
8. $\int_0^{\infty} \ln(a^2 + b^2 x^2) \frac{dx}{c^2 - g^2 x^2} = -\frac{\pi}{cg} \arctan \frac{bc}{ag} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{BI (136)(15)a}$
9. $\int_0^{\infty} \frac{\ln(1+p^2 x^2) - \ln(1+q^2 x^2)}{x^2} dx = \pi(p-q) \quad [p > 0, \quad q > 0] \quad \text{FI II 645}$
10. $\int_0^1 \ln \frac{1+a^2 x^2}{1+a^2} \frac{dx}{1-x^2} = -(\arctan a)^2 \quad \text{BI (115)(2)}$
11. $\int_0^1 \ln(1-x^2) \frac{dx}{x} = -\frac{\pi^2}{12}$
12. $\int_0^{\infty} \ln^2(1-x^2) \frac{dx}{x^2} = 0 \quad \text{BI (142)(9)a}$
13. $\int_0^1 \ln(1-x^2) \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 - \mathbf{G} \quad \text{GW (325)(17)}$
14. $\int_1^{\infty} \ln(x^2-1) \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 + \mathbf{G} \quad \text{BI (144)(6)}$

$$15. \quad \int_0^\infty \ln^2(a^2 - x^2) \frac{dx}{b^2 + x^2} = \frac{\pi}{b} \ln(a^2 + b^2) \quad [b > 0] \quad \text{BI (136)(16)}$$

$$16. \quad \int_0^\infty \ln^2(a^2 - x^2) \frac{b^2 - x^2}{(b^2 + x^2)^2} dx = -\frac{2b\pi}{a^2 + b^2} \quad [b > 0] \quad \text{BI (136)(20)}$$

$$17. \quad \int_0^1 \ln(1 + x^2) \frac{dx}{x(1 + x^2)} = \frac{1}{2} \left[\frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2 \right] \quad \text{BI (114)(25)}$$

$$18. \quad \int_0^\infty \ln(1 + x^2) \frac{dx}{x(1 + x^2)} = \frac{\pi^2}{12} \quad \text{BI (141)(9)}$$

$$19. \quad \int_0^1 \ln(\cos^2 t + x^2 \sin^2 t) \frac{dx}{1 - x^2} = -t^2 \quad \text{BI (114)(27)a}$$

$$20. \quad \int_0^\infty \ln(a^2 + b^2 x^2) \frac{dx}{(c + gx)^2} = \frac{2 \ln b}{cg} + \frac{b^2}{a^2 g^2 + b^2 c^2} \left(\frac{a}{b} \pi + 2 \frac{c}{g} \ln \frac{c}{g} + 2 \frac{a^2 g}{b^2 c} \ln \frac{a}{b} \right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{BI (139)(16)a}$$

$$21. \quad \int_0^1 \ln(a^2 + b^2 x^2) \frac{dx}{(c + gx)^2} \\ = \frac{2}{c(c + g)} \ln a + \frac{b^2}{a^2 g^2 + b^2 c^2} \left[\frac{2a}{b} \operatorname{arccot} \frac{a}{b} + \frac{cb^2 - ga^2}{b^2(c + g)} \ln \frac{a^2 + b^2}{a^2} - 2 \frac{c}{g} \ln \frac{c + g}{c} \right] \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{BI (114)(28)a}$$

$$22.^{11} \quad \int_0^\infty \frac{\ln(1 + p^2 x^2)}{r^2 + q^2 x^2} dx = \int_0^\infty \frac{\ln(p^2 + x^2)}{q^2 + r^2 x^2} dx = \frac{\pi}{qr} \ln \frac{q + pr}{r} \\ [qr > 0, \quad p > 0] \quad \text{FI II 745a, BI (318)(1)a, BI (318)(4)a}$$

$$23. \quad \int_0^\infty \frac{\ln(1 + a^2 x^2)}{b^2 + c^2 x^2} \frac{dx}{d^2 + g^2 x^2} = \frac{\pi}{b^2 g^2 - c^2 d^2} \left[\frac{g}{d} \ln \left(1 + \frac{ad}{g} \right) - \frac{c}{b} \ln \left(1 + \frac{ab}{c} \right) \right] \\ [a > 0, \quad b > 0, \quad c > 0, \quad d > 0, \quad g > 0, \quad b^2 g^2 \neq c^2 d^2] \quad \text{BI (141)(10)}$$

$$24. \quad \int_0^\infty \frac{\ln(1 + a^2 x^2)}{b^2 + c^2 x^2} \frac{x^2 dx}{d^2 + g^2 x^2} = \frac{\pi}{b^2 g^2 - c^2 d^2} \left[\frac{b}{c} \ln \left(1 + \frac{ab}{c} \right) - \frac{d}{g} \ln \left(1 + \frac{ad}{g} \right) \right] \\ [a > 0, \quad b > 0, \quad c > 0, \quad d > 0, \quad g > 0, \quad b^2 g^2 \neq c^2 d^2] \quad \text{BI (141)(11)}$$

$$25. \quad \int_0^\infty \ln(a^2 + b^2 x^2) \frac{dx}{(c^2 + g^2 x^2)^2} = \frac{\pi}{2c^3 g} \left(\ln \frac{ag + bc}{g} - \frac{bc}{ag + bc} \right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{GW (325)(18a)}$$

$$26. \quad \int_0^\infty \ln(a^2 + b^2 x^2) \frac{x^2 dx}{(c^2 + g^2 x^2)^2} = \frac{\pi}{2cg^3} \left(\ln \frac{ag + bc}{g} + \frac{bc}{ag + bc} \right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{GW (325)(18b)}$$

- $$\begin{aligned}
27. \quad & \int_0^1 \ln(1+ax^2) \sqrt{1-x^2} \, dx = \frac{\pi}{2} \left\{ \ln \frac{1+\sqrt{1+a}}{2} + \frac{1}{2} \frac{1-\sqrt{1+a}}{1+\sqrt{1+a}} \right\} \\
& [a > 0] \quad \text{BI (117)(6)} \\
28. \quad & \int_0^1 \ln(1+a-ax^2) \sqrt{1-x^2} \, dx = \frac{\pi}{2} \left\{ \ln \frac{1+\sqrt{1+a}}{2} - \frac{1}{2} \frac{1-\sqrt{1+a}}{1+\sqrt{1+a}} \right\} \\
& [a > 0] \quad \text{BI (117)(7)} \\
29. \quad & \int_0^1 \ln(1-a^2x^2) \frac{dx}{\sqrt{1-x^2}} = \pi \ln \frac{1+\sqrt{1-a^2}}{2} \quad [a^2 < 1] \quad \text{BI (119)(1)} \\
30.^6 \quad & \int_0^1 \ln(1-a^2x^2) \frac{dx}{x\sqrt{1-x^2}} = -\left(\arccos|a| - \frac{\pi}{2}\right)^2 \quad \text{LI (120)(11)} \\
31. \quad & \int_0^1 \ln(1-x^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln \frac{k'}{k} \mathbf{K}(k) - \frac{\pi}{2} \mathbf{K}(k') \quad \text{BI (120)(12)} \\
32. \quad & \int_0^1 \ln(1 \pm kx^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln \frac{2 \pm 2k}{\sqrt{k}} \mathbf{K}(k) - \frac{\pi}{8} \mathbf{K}(k') \quad \text{BI (120)(8), BI (120)(14)} \\
33. \quad & \int_0^1 \frac{\ln(1-k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx = \ln k' \mathbf{K}(k) \quad \text{BI (119)(27)} \\
34. \quad & \int_0^1 \ln(1-k^2x^2) \sqrt{\frac{1-k^2x^2}{1-x^2}} \, dx = (2-k^2) \mathbf{K}(k) - (2-\ln k') \mathbf{E}(k) \quad \text{BI (119)(3)} \\
35. \quad & \int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}} \ln(1-k^2x^2) \, dx = \frac{1}{k^2} \left(1+k'^2 - k'^2 \ln k'\right) \mathbf{K}(k) - (2-\ln k') \mathbf{E}(k) \quad \text{BI (119)(7)} \\
36. \quad & \int_{-1}^1 \ln(1-x^2) \frac{dx}{(a+bx)\sqrt{1-x^2}} = \frac{2\pi}{\sqrt{a^2-b^2}} \ln \frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \\
& [a > 0, \quad b > 0, \quad a \neq b] \quad \text{BI (145)(15)} \\
37.^8 \quad & \int_0^1 \ln(1-x^2) (px^{p-1} - qx^{q-1}) \, dx = \psi\left(\frac{q}{2}+1\right) + \psi\left(\frac{p}{2}+1\right) \\
& [p > -2, \quad q > -2] \quad \text{BI (106)(15)} \\
38. \quad & \int_0^1 \ln(1+ax^2) \frac{dx}{\sqrt{1-x^2}} = \pi \ln \frac{1+\sqrt{1+a}}{2} \quad [a \geq -1] \quad \text{GW (325)(21b)} \\
39. \quad & \int_0^1 \ln(1+x^2) x^{\mu-1} \, dx = \frac{1}{\mu} \left[\ln 2 - \beta\left(\frac{\mu}{2}+1\right) \right] \quad [\operatorname{Re} \mu > -2] \quad \text{BI (106)(12)} \\
40. \quad & \int_0^\infty \ln(1+x^2) x^{\mu-1} \, dx = \frac{\pi}{\mu \sin \frac{\mu\pi}{2}} \quad [-2 < \operatorname{Re} \mu < 0]
\end{aligned}$$

BI (311)(4)a, ET I 315(15)

$$\begin{aligned}
 41. \quad \int_0^\infty \ln(1+x^2) \frac{x^{\mu-1} dx}{1+x} &= \frac{\pi}{\sin \mu\pi} \left\{ \ln 2 - (1-\mu) \sin \frac{\mu\pi}{2} \beta \left(\frac{1-\mu}{2} \right) - (2-\mu) \cos \frac{\mu\pi}{2} \beta \left(\frac{2-\mu}{2} \right) \right\} \\
 & \quad [-2 < \operatorname{Re} \mu < 1] \quad \text{ET I 316(25)}
 \end{aligned}$$

4.296

$$\begin{aligned}
 1. \quad \int_0^1 \ln(1+2x \cos t + x^2) \frac{dx}{x} &= \frac{\pi^2}{6} - \frac{t^2}{2} \quad \text{BI (114)(34)} \\
 2. \quad \int_{-\infty}^\infty \ln(a^2 - 2ax \cos t + x^2) \frac{dx}{1+x^2} &= \pi \ln(1+2a|\sin t| + a^2) \quad \text{BI (145)(28)} \\
 3. \quad \int_0^\infty \ln(1+2x \cos t + x^2) x^{\mu-1} dx &= \frac{2\pi \cos \mu t}{\mu \sin \mu\pi} \quad [|t| < \pi, \quad -1 < \operatorname{Re} \mu < 0] \quad \text{ET I 316(27)} \\
 4. \quad \int_0^\infty \ln \left(\frac{x^2 + 2ax \cos t + a^2}{x^2 - 2ax \cos t + a^2} \right) \frac{x dx}{x^2 + b^2} &= \frac{1}{2} \pi^2 - \pi t + \pi \arctan \frac{(a^2 - b^2) \cos t}{(a^2 + b^2) \sin t + 2ab} \\
 & \quad [a > 0, \quad b > 0, \quad 0 < t < \pi]
 \end{aligned}$$

4.297

$$\begin{aligned}
 1. \quad \int_0^1 \ln \frac{ax+b}{bx+a} \frac{dx}{(1+x)^2} &= \frac{1}{a-b} \left[(a+b) \ln \frac{a+b}{2} - a \ln a - b \ln b \right] \\
 & \quad [a > 0, \quad b > 0] \quad \text{BI (115)(16)} \\
 2. \quad \int_0^\infty \ln \frac{ax+b}{bx+a} \frac{dx}{(1+x)^2} &= 0 \quad [ab > 0] \quad \text{BI (139)(23)} \\
 3. \quad \int_0^1 \ln \frac{1-x}{x} \frac{dx}{1+x^2} &= \frac{\pi}{8} \ln 2 \quad \text{BI (115)(5)} \\
 4. \quad \int_0^1 \ln \frac{1+x}{1-x} \frac{dx}{1+x^2} &= \mathbf{G} \quad \text{BI (115)(17)} \\
 5.^{11} \quad \int_0^\infty \ln \left(\frac{1+x}{1-x} \right)^2 \frac{dx}{x(1+x^2)} &= \frac{\pi^2}{2} \quad \text{BI (141)(13)} \\
 6. \quad \int_u^v \ln \frac{v+x}{u+x} \frac{dx}{x} &= \frac{1}{2} \left(\ln \frac{v}{u} \right)^2 \quad [uv > 0] \quad \text{BI (145)(33)} \\
 7. \quad \int_0^\infty \frac{b \ln(1+ax) - a \ln(1+bx)}{x^2} dx &= ab \ln \frac{b}{a} \quad [a > 0, \quad b > 0] \quad \text{FI II 647} \\
 8. \quad \int_0^1 \ln \frac{1+ax}{1-ax} \frac{dx}{x\sqrt{1-x^2}} &= \pi \arcsin a \quad [|a| \leq 1] \quad \text{GW (325)(21c), BI (122)(2)} \\
 9. \quad \int_u^v \ln \left(\frac{1+ax}{1-ax} \right) \frac{dx}{\sqrt{(x^2-u^2)(v^2-x^2)}} &= \frac{\pi}{v} F \left(\arcsin av, \frac{u}{v} \right) \\
 & \quad [|av| < 1] \quad \text{BI (145)(35)} \\
 10.^8 \quad \text{PV} \int_0^1 \ln \left| \frac{a+y}{a-y} \right| \frac{dy}{y\sqrt{1-y^2}} &= \frac{\pi^2}{2} \quad [0 < a \leq 1]
 \end{aligned}$$

4.298

$$1. \quad \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1) \quad \text{BI (137)(1)}$$

$$2. \quad \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n}}{1+x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1) \quad \text{BI (137)(3)}$$

$$3. \quad \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1-x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1) \quad \text{BI (137)(2)}$$

$$4. \quad \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n}}{1-x} dx = -\frac{\ln 2}{2n} - \frac{1}{4n^2} + \frac{1}{2n} \beta(2n+1) \quad \text{BI (137)(4)}$$

$$5. \quad \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x^2} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1) \quad \text{BI (137)(10)}$$

$$6. \quad \int_0^1 \ln \frac{1+x^2}{x} x^{2n} dx = \frac{1}{2n+1} \left\{ (-1)^n \frac{\pi}{2} + \ln 2 - \frac{1}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \right\} \quad \text{BI (294)(8)}$$

$$7. \quad \int_0^1 \ln \frac{1+x^2}{x} x^{2n-1} dx = \frac{1}{2n} \left\{ (-1)^{n+1} \ln 2 + \ln 2 - \frac{1}{2n} + (-1)^{n+1} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right\} \quad \text{BI (294)(9)a}$$

$$8. \quad \int_0^1 \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 \quad \text{BI (115)(7)}$$

$$9. \quad \int_0^\infty \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \pi \ln 2 \quad \text{BI (137)(8)}$$

$$10. \quad \int_0^\infty \ln \frac{1+x^2}{x} \frac{dx}{1-x^2} = 0 \quad \text{BI (137)(9)}$$

$$11. \quad \int_0^1 \ln \frac{1-x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 \quad \text{BI (115)(9)}$$

$$12. \quad \int_1^\infty \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 \quad \text{BI (144)(8)}$$

$$13. \quad \int_0^1 \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 - \mathbf{G} \quad \text{BI (115)(18)}$$

$$14. \quad \int_1^\infty \ln \frac{1+x^2}{x-1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 + \mathbf{G} \quad \text{BI (144)(9)}$$

$$15. \quad \int_0^1 \ln \frac{1+x^2}{1-x} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 \quad \text{BI (115)(19)}$$

$$16. \quad \int_0^\infty \ln \frac{1+x^2}{x^2} \frac{x dx}{1+x^2} = \frac{\pi^2}{12} \quad \text{BI (138)(3)}$$

$$17. \quad \int_0^\infty \ln \frac{a^2+b^2x^2}{x^2} \frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg} \ln \frac{ag+bc}{c} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \\ \text{BI (138)(6, 7, 9, 10)a}$$

$$18. \quad \int_0^\infty \ln \frac{a^2+b^2x^2}{x^2} \frac{dx}{c^2-g^2x^2} = \frac{1}{cg} \arctan \frac{ag}{bc} \quad [a > 0, b > 0, c > 0, g > 0] \\ \text{BI (138)(8, 11)a}$$

$$19. \quad \int_0^\infty \ln \frac{1+x^2}{x^2} \frac{x^2 dx}{(1+x^2)^2} = \frac{\pi}{4} (\ln 4 - 1) \quad \text{BI (139)(21)}$$

$$20. \quad \int_0^1 \ln^2 \left(\frac{1-x^2}{x^2} \right) \sqrt{1-x^2} dx = \pi \quad \text{FI II 643a}$$

$$21. \quad \int_0^1 \ln \frac{1+2x \cos t + x^2}{(1+x)^2} \frac{dx}{x} = \frac{1}{2} \int_0^\infty \ln \frac{1+2x \cos t + x^2}{(1+x)^2} \frac{dx}{x} = -\frac{t^2}{2} \quad [|t| < \pi] \quad \text{BI (115)(23), BI (134)(15)}$$

$$22. \quad \int_0^\infty \ln \frac{1+2x \cos t + x^2}{(1+x)^2} x^{p-1} dx = -\frac{2\pi(1-\cos pt)}{p \sin p\pi} \quad [0 < |p| < 1, \quad |t| < \pi] \quad \text{BI (134)(17)}$$

$$23. \quad \int_0^1 \ln \frac{1+x^2 \sin t}{1-x^2 \sin t} \frac{dx}{\sqrt{1-x^2}} = \pi \ln \cot \left(\frac{\pi-t}{4} \right) \quad [|t| < \pi] \quad \text{GW (325)(21d)}$$

4.299

$$1. \quad \int_0^\infty \ln \frac{(x+1)(x+a^2)}{(x+a)^2} \frac{dx}{x} = (\ln a)^2 \quad [a > 0] \quad \text{BI (134)(14)}$$

$$2. \quad \int_0^1 \ln \frac{(1-ax)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{2\sqrt{a}} \arctan \sqrt{a} \ln(1+a) \quad [a > 0] \quad \text{BI (115)(25)}$$

$$3. \quad \int_0^1 \ln \frac{(1-a^2x^2)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}} \arctan \sqrt{a} \ln(1+a) \quad [a > 0] \quad \text{BI (115)(26)}$$

$$4. \quad \int_0^1 \ln \frac{(x+1)(x+a^2)}{(x+a)^2} x^{\mu-1} dx = \frac{\pi(a^\mu - 1)^2}{\mu \sin \mu\pi} \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (134)(16)}$$

4.311

$$1.^{11} \quad \int_0^\infty \frac{\ln(1+x^n)}{x^n} dx = \frac{\pi \operatorname{cosec}(\frac{\pi}{n})}{n-1} \quad n = 2, 3, \dots$$

$$2. \quad \int_0^\infty \ln(1+x^3) \frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}} \ln 3 \quad \text{LI (136)(8)}$$

$$3. \quad \int_0^\infty \ln(1+x^3) \frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9} \quad \text{LI (136)(6)}$$

$$4. \quad \int_0^\infty \ln(1+x^3) \frac{x dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 + \frac{\pi^2}{9} \quad \text{LI (136)(7)}$$

$$5. \quad \int_0^\infty \ln(1+x^3) \frac{1-x}{1+x^3} dx = -\frac{2}{9}\pi^2 \quad \text{BI (136)(9)}$$

$$6.^8 \quad \int_0^\infty \left| 1 - \frac{x^3}{a^3} \right| \frac{dx}{x^3} = -\frac{\pi\sqrt{3}}{6a^2}$$

4.312

$$1. \quad \int_0^\infty \ln \frac{1+x^3}{x^3} \frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 + \frac{\pi^2}{9} \quad \text{BI (138)(12)}$$

$$2. \quad \int_0^\infty \ln \frac{1+x^3}{x^3} \frac{x dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9} \quad \text{BI (138)(13)}$$

4.313

$$1. \quad \int_0^\infty \ln x \ln(1+a^2x^2) \frac{dx}{x^2} = \pi a(1 - \ln a) \quad [a > 0] \quad \text{BI (134)(18)}$$

$$2. \quad \int_0^\infty \ln(1+c^2x^2) \ln(a^2+b^2x^2) \frac{dx}{x^2} = 2\pi \left[\left(c + \frac{b}{a}\right) \ln(b+ac) - \frac{b}{a} \ln b - c \ln c \right] \\ [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (134)(20, 21)a}$$

$$3. \quad \int_0^\infty \ln(1+c^2x^2) \ln\left(a^2 + \frac{b^2}{x^2}\right) \frac{dx}{x^2} = 2\pi \left[\frac{a+bc}{b} \ln(a+bc) - \frac{a}{b} \ln a - c \right] \\ [a > 0, \quad a+bc > 0] \quad \text{BI (134)(22, 23)a}$$

$$4. \quad \int_0^\infty \ln x \ln \frac{1+a^2x^2}{1+b^2x^2} \frac{dx}{x^2} = \pi(a-b) + \pi \ln \frac{b^b}{a^a} \quad [a > 0, \quad b > 0] \quad \text{BI (134)(24)}$$

$$5. \quad \int_0^\infty \ln x \ln \frac{a^2+2bx+x^2}{a^2-2bx+x^2} \frac{dx}{x} = 2\pi \ln a \arcsin \frac{b}{a} \quad [a \geq |b|] \quad \text{BI (134)(25)}$$

$$6. \quad \int_0^\infty \ln(1+x) \frac{x \ln x - x - a}{(x+a)^2} \frac{dx}{x} = \frac{(\ln a)^2}{2(a-1)} \quad [a > 0] \quad \text{BI (141)(7)}$$

$$7. \quad \int_0^\infty \ln^2(1-x) \frac{x \ln x - x - a}{(x+a)^2} \frac{dx}{x} = \frac{\pi^2 + (\ln a)^2}{1+a} \quad [a > 0] \quad \text{LI (141)(8)}$$

4.314

$$1.^{11} \quad \int_0^1 \ln(1+ax) \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \sum_{k=1}^\infty (-1)^{k+1} \frac{a^k}{k} \ln \frac{p+k}{q+k} \\ [|a| < 1, \quad p > 0, \quad q > 0] \quad \text{BI (123)(18)}$$

$$2. \quad \int_0^\infty \left[\frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q} \right] \frac{dx}{x \ln(1+x)} = \ln \Gamma(q) \\ [q > 0] \quad \text{BI (143)(7)}$$

$$3. \quad \int_0^1 \frac{x \ln x + 1 - x}{x (\ln x)^2} \ln(1+x) dx = \ln \frac{4}{\pi} \quad \text{BI (126)(12)}$$

$$4. \quad \int_0^1 \frac{\ln(1-x^2) dx}{x (q^2 + (\ln x)^2)} = -\frac{\pi}{q} \ln \Gamma\left(\frac{q+\pi}{\pi}\right) + \frac{\pi}{2q} \ln 2q + \ln \frac{q}{\pi} - 1 \\ [q > 0] \quad \text{LI (327)(12)a}$$

4.315

$$1. \quad \int_0^1 \ln(1+x) (\ln x)^{n-1} \frac{dx}{x} = (-1)^{n-1} (n-1)! \left(1 - \frac{1}{2^n}\right) \zeta(n+1) \quad \text{BI (116)(3)}$$

$$2. \quad \int_0^1 \ln(1+x) (\ln x)^{2n} \frac{dx}{x} = \frac{2^{2n+1} - 1}{(2n+1)(2n+2)} \pi^{2n+2} |B_{2n+2}| \quad \text{BI (116)(1)}$$

$$3. \quad \int_0^1 \ln(1-x) (\ln x)^{n-1} \frac{dx}{x} = (-1)^n (n-1)! \zeta(n+1) \quad \text{BI (116)(4)}$$

$$4. \quad \int_0^1 \ln(1-x) (\ln x)^{2n} \frac{dx}{x} = -\frac{2^{2n}}{(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}| \quad \text{BI (116)(2)}$$

4.316

$$1. \quad \int_0^1 \ln(1-ax^r) \left(\ln \frac{1}{x}\right)^p \frac{dx}{x} = -\frac{1}{r^{p+1}} \Gamma(p+1) \sum_{k=1}^{\infty} \frac{a^k}{k^{p+2}} \quad [p > -1, \quad a < 1, \quad r > 0] \quad \text{BI (116)(7)}$$

$$2. \quad \int_0^1 \ln(1-2ax \cos t + a^2 x^2) \left(\ln \frac{1}{x}\right)^p \frac{dx}{x} = -2 \Gamma(p+1) \sum_{k=1}^{\infty} \frac{a^k \cos kt}{k^{p+2}} \quad \text{LI (116)(8)}$$

4.317

$$1. \quad \int_0^{\infty} \ln \frac{\sqrt{1+x^2} + a}{\sqrt{1+x^2} - a} \frac{dx}{\sqrt{1+x^2}} = \pi \arcsin a \quad [|a| < 1] \quad \text{BI (142)(11)}$$

$$2. \quad \int_0^1 \ln \frac{\sqrt{1-a^2 x^2} - x\sqrt{1-a^2}}{1-x} \frac{dx}{x} = \frac{1}{2} (\arcsin a)^2 \quad \text{BI (115)(32)}$$

$$3. \quad \int_0^1 \ln \frac{1 + \cos t \sqrt{1-x^2}}{1 - \cos t \sqrt{1-x^2}} \frac{dx}{x^2 + \tan^2 v} = \pi \cot t \frac{\cos \frac{v-t}{2}}{\sin \frac{v+t}{2}} \quad \text{BI (115)(30)}$$

$$4. \quad \int_0^1 \ln^2 \left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right) \frac{x dx}{1-x^2} = \frac{\pi^2}{2} \quad \text{BI (115)(31)}$$

$$5. \quad \int_0^1 \ln \left\{ \sqrt{1+kx} + \sqrt{1-kx} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{4} \ln(4k) \mathbf{K}(k) + \frac{\pi}{8} \mathbf{K}(k') \quad \text{BI (121)(8)}$$

$$6. \quad \int_0^1 \ln \left\{ \sqrt{1+kx} - \sqrt{1-kx} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{4} \ln(4k) \mathbf{K}(k) + \frac{3}{8} \pi \mathbf{K}(k') \quad \text{BI (121)(9)}$$

$$7. \quad \int_0^1 \ln \left\{ 1 + \sqrt{1-k^2 x^2} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{2} \ln k \mathbf{K}(k) + \frac{\pi}{4} \mathbf{K}(k') \quad \text{BI (121)(6)}$$

$$8. \quad \int_0^1 \ln \left\{ 1 - \sqrt{1-k^2 x^2} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{2} \ln k \mathbf{K}(k) - \frac{3}{4} \pi \mathbf{K}(k') \quad \text{BI (121)(7)}$$

$$9. \quad \int_0^1 \ln \frac{1+p\sqrt{1-x^2}}{1-p\sqrt{1-x^2}} \frac{dx}{1-x} = \pi \arcsin p \quad [p^2 < 1] \quad \text{BI (115)(29)}$$

$$10. \quad \int_0^1 \ln \frac{1 + q\sqrt{1 - k^2 x^2}}{1 - q\sqrt{1 - k^2 x^2}} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} = \pi F(\arcsin q, k')$$

$$[q^2 < 1] \quad \text{BI (122)(15)}$$

$$11.^{10} \quad \int_{-\infty}^{\infty} \ln \left| \frac{1 + 2\sqrt{1 + x^2}}{1 - 2\sqrt{1 + x^2}} \right| \frac{dx}{\sqrt{1 + x^2}} = \frac{\pi^2}{3}$$

4.318

$$1. \quad \int_0^1 \frac{\ln(1 - x^q)}{1 + (\ln x)^2} \frac{dx}{x} = \pi \left[\ln \Gamma\left(\frac{q}{2\pi} + 1\right) - \frac{\ln q}{2} + \frac{q}{2\pi} \left(\ln \frac{q}{2\pi} - 1 \right) \right]$$

$$[q > 0] \quad \text{BI (126)(11)}$$

$$2. \quad \int_0^{\infty} \ln(1 + x^r) \left[\frac{(p - r)x^p - (q - r)x^q}{\ln x} + \frac{x^q - x^p}{(\ln x)^2} \right] \frac{dx}{x^{r+1}} = r \ln \left(\tan \frac{q\pi}{2r} \cot \frac{p\pi}{2r} \right)$$

$$[p < r, \quad q < r] \quad \text{BI (143)(9)}$$

In integrals containing $\ln(a + bx^r)$, it is useful to make the substitution $x^r = t$ and then to seek the resulting integral in the tables. For example,

$$\int_0^{\infty} x^{p-1} \ln(1 + x^r) dx = \frac{1}{r} \int_0^{\infty} t^{\frac{p}{r}-1} \ln(1 + t) dt = \frac{\pi}{p \sin \frac{p\pi}{r}} \quad (\text{see 4.293 3})$$

4.319

$$1. \quad \int_0^{\infty} \ln(1 - e^{-2a\pi x}) \frac{dx}{1 + x^2} = -\pi \left[\frac{1}{2} \ln 2a\pi + a(\ln a - 1) - \ln \Gamma(a + 1) \right]$$

$$[a > 0] \quad \text{BI (354)(6)}$$

$$2. \quad \int_0^{\infty} \ln(1 + e^{-2a\pi x}) \frac{dx}{1 + x^2} = \pi \left[\ln \Gamma(2a) - \ln \Gamma(a) + a(1 - \ln a) - \left(2a - \frac{1}{2} \right) \ln 2 \right]$$

$$[a > 0] \quad \text{BI (354)(7)}$$

$$3. \quad \int_0^{\infty} \ln \frac{a + be^{-px}}{a + be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a + b} \ln \frac{p}{q} \quad \left[\frac{b}{a} > -1, \quad pq > 0 \right]$$

$$\text{FI II 635, BI (354)(1)}$$

4.321

$$1. \quad \int_{-\infty}^{\infty} x \ln \cosh x dx = 0 \quad \text{BI (358)(2)a}$$

$$2. \quad \int_{-\infty}^{\infty} \ln \cosh x \frac{dx}{1 - x^2} = 0 \quad \text{BI (138)(20)a}$$

4.322

$$1.^{11} \quad \int_0^{\pi} x \ln \sin x dx = \frac{1}{2} \int_0^{\pi} x \ln \cos^2 x dx = -\frac{\pi^2}{2} \ln 2 \quad \text{BI (432)(1, 2) FI II 643}$$

$$2. \quad \int_0^{\infty} \frac{\ln \sin^2 ax}{b^2 + x^2} dx = \frac{\pi}{b} \ln \frac{1 - e^{-2ab}}{2} \quad [a > 0, \quad b > 0] \quad \text{GW (338)(28b)}$$

3. $\int_0^\infty \frac{\ln \cos^2 ax}{b^2 + x^2} dx = \frac{\pi}{b} \ln \frac{1 + e^{-2ab}}{2}$ $[a > 0, \quad b > 0]$ GW (338)(28a)
4. $\int_0^\infty \frac{\ln \sin^2 ax}{b^2 - x^2} dx = -\frac{\pi^2}{2b} + a\pi$ $[a > 0, \quad b > 0]$ BI (418)(1)
- 5.¹¹ $\int_0^\infty \frac{\ln \cos^2 ax}{b^2 - x^2} dx = \infty$ BI (418)(2)
6. $\int_0^\infty \frac{\ln \cos^2 x}{x^2} dx = -\pi$ FI II 686
- 7.⁷ $\int_0^{\pi/4} \ln \sin xx^{\mu-1} dx = -\frac{1}{2\mu} \left(\frac{\pi}{4}\right)^\mu \left[\ln 2 + \frac{2}{\mu} - \sum_{k=1}^\infty \frac{\zeta(2k)}{4^{2k-1}(\mu + 2k)} \right]$
 $[\operatorname{Re} \mu > 0]$ LI (425)(1)
- 8.⁷ $\int_0^{\pi/2} \ln \sin xx^{\mu-1} dx = -\frac{1}{\mu} \left(\frac{\pi}{2}\right)^\mu \left[\frac{1}{\mu} - 2 \sum_{k=1}^\infty \frac{\zeta(2k)}{4^k(\mu + 2k)} \right]$
 $[\operatorname{Re} \mu > 0]$ LI (430)(1)
9. $\int_0^{\pi/2} \ln(1 - \cos x) x^{\mu-1} dx = -\frac{1}{\mu} \left(\frac{\pi}{2}\right)^\mu \left[\frac{2}{\mu} - \sum_{k=1}^\infty \frac{\zeta(2k)}{4^{2k-1}(\mu + 2k)} \right]$
 $[\operatorname{Re} \mu > 0]$ LI (430)(2)
10. $\int_0^\infty \ln(1 \pm 2p \cos \beta x + p^2) \frac{dx}{q^2 + x^2} = \frac{\pi}{q} \ln(1 \pm pe^{-\beta q}) \quad [p^2 < 1]$
 $= \frac{\pi}{q} \ln(p \pm e^{-\beta q}) \quad [p^2 > 1]$
 FI II 718a

4.323

- 1.¹¹ $\int_0^\pi x \ln \tan^2 x dx = 0$ BI (432)(3)
2. $\int_0^\infty \frac{\ln \tan^2 ax}{b^2 + x^2} dx = \frac{\pi}{b} \ln \tanh ab$ $[a > 0, \quad b > 0]$ GW (338)(28c)
3. $\int_0^\infty \ln \left(\frac{1 + \tan x}{1 - \tan x} \right)^2 \frac{dx}{x} = \frac{\pi^2}{2}$ GW (338)(26)

4.324

1. $\int_0^\infty \ln \left(\frac{1 + \sin x}{1 - \sin x} \right)^2 \frac{dx}{x} = \pi^2$ GW (338)(25)
2. $\int_0^\infty \ln \frac{1 + 2a \cos px + a^2}{1 + 2a \cos qx + a^2} \frac{dx}{x} = \ln(1 + a) \ln \frac{q^2}{p^2}$ $[-1 < a \leq 1]$
 $= \ln \left(1 + \frac{1}{a} \right) \ln \frac{q^2}{p^2}$ $[a < -1 \text{ or } a \geq 1]$
 GW (338)(27)

$$3. \quad \int_0^\infty \ln(a^2 \sin^2 px + b^2 \cos^2 px) \frac{dx}{c^2 + x^2} = \frac{\pi}{c} [\ln(a \sinh cp + b \cosh cp) - cp] \\ [a > 0, \quad b > 0, \quad c > 0, \quad p > 0] \\ \text{GW (338)(29)}$$

4.325

$$1.^3 \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x} = -C \ln 2 + \sum_{k=2}^\infty (-1)^k \frac{\ln k}{k} = -C \ln 2 + 0.159868905 \dots = -\frac{1}{2} (\ln 2)^2 \\ \text{GW (325)(25a)}$$

$$2. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{x + e^{i\lambda}} = \sum_{k=1}^\infty \frac{(-1)^k}{k} e^{-ik\lambda} (C + \ln k) \\ \text{GW (325)(26)}$$

$$3. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{(1+x)^2} = \int_1^\infty \ln \ln x \frac{dx}{(1+x)^2} = \frac{1}{2} \left[\psi \left(\frac{1}{2} \right) + \ln 2\pi \right] = \frac{1}{2} \left(\ln \frac{\pi}{2} - C \right) \\ \text{BI (147)(7)}$$

$$4. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x^2} = \int_1^\infty \ln \ln x \frac{dx}{1+x^2} = \frac{\pi}{2} \ln \frac{\sqrt{2\pi} \Gamma \left(\frac{3}{4} \right)}{\Gamma \left(\frac{1}{4} \right)} \\ \text{BI (148)(1)}$$

$$5. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x+x^2} = \int_1^\infty \ln \ln x \frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}} \ln \frac{\sqrt[3]{2\pi} \Gamma \left(\frac{2}{3} \right)}{\Gamma \left(\frac{1}{3} \right)} \\ \text{BI (148)(2)}$$

$$6. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1-x+x^2} = \int_1^\infty \ln \ln x \frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}} \left[\frac{5}{6} \ln 2\pi - \ln \Gamma \left(\frac{1}{6} \right) \right] \\ \text{BI (148)(5)}$$

$$7. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+2x \cos t + x^2} = \int_1^\infty \ln \ln x \frac{dx}{1+2x \cos t + x^2} = \frac{\pi}{2 \sin t} \ln \frac{(2\pi)^{t/\pi} \Gamma \left(\frac{1}{2} + \frac{t}{2\pi} \right)}{\Gamma \left(\frac{1}{2} - \frac{t}{2\pi} \right)} \\ \text{BI (147)(9)}$$

$$8. \quad \int_0^1 \ln \ln \frac{1}{x} x^{\mu-1} dx = -\frac{1}{\mu} (C + \ln \mu) \quad [\operatorname{Re} \mu > 0] \\ \text{BI (147)(1)}$$

$$9. \quad \int_1^\infty \ln \ln x \frac{x^{n-2} dx}{1+x^2+x^4+\dots+x^{2n-2}} \\ = \frac{\pi}{2n} \tan \frac{\pi}{2n} \ln 2\pi + \frac{\pi}{n} \sum_{k=1}^{n-1} (-1)^{k-1} \sin \frac{k\pi}{n} \ln \frac{\Gamma \left(\frac{n+k}{2n} \right)}{\Gamma \left(\frac{k}{2n} \right)} \quad [n \text{ is even}] \\ = \frac{\pi}{2n} \tan \frac{\pi}{2n} \ln \pi + \frac{\pi}{n} \sum_{k=1}^{n-1} \frac{n-1}{2} (-1)^{k-1} \sin \frac{k\pi}{n} \ln \frac{\Gamma \left(\frac{n-k}{n} \right)}{\Gamma \left(\frac{k}{n} \right)} \quad [n \text{ is odd}] \\ \text{BI (148)(4)}$$

$$\begin{aligned}
 10. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{(1+x^2) \sqrt{\ln \frac{1}{x}}} &= \int_1^\infty \ln \ln x \frac{dx}{(1+x^2) \sqrt{\ln x}} \\
 &= \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + 2 \ln 2 + C]
 \end{aligned}$$

BI (147)(4)

$$11. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{x^{\mu-1} dx}{\sqrt{\ln \frac{1}{x}}} = - (C + \ln 4\mu) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0]$$

BI (147)(3)

$$12. \quad \int_0^1 \ln \ln \left(\frac{1}{x} \right) \left(\ln \frac{1}{x} \right)^{\mu-1} x^{\nu-1} dx = \frac{1}{\nu^\mu} \Gamma(\mu) [\psi(\mu) - \ln(\nu)]$$

[$\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0$] BI (147)(2)

4.326

$$1. \quad \int_0^1 \ln(a - \ln x) x^{\mu-1} dx = \frac{1}{\mu} [\ln a - e^{a\mu} \operatorname{Ei}(-a\mu)] \quad [\operatorname{Re} \mu > 0, a > 0]$$

BI (107)(23)

$$2. \quad \int_0^{\frac{1}{e}} \ln \left(2 \ln \frac{1}{x} - 1 \right) \frac{x^{2\mu-1}}{\ln x} dx = -\frac{1}{2} [\operatorname{Ei}(-\mu)]^2 \quad [\operatorname{Re} \mu > 0]$$

BI (145)(5)

4.327

$$1. \quad \int_0^1 \ln [a^2 + (\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2 \Gamma \left(\frac{2a+3\pi}{4\pi} \right)}{\Gamma \left(\frac{2a+\pi}{4\pi} \right)} + \frac{\pi}{2} \ln \frac{\pi}{2}$$

[$a > -\frac{\pi}{2}$] BI (147)(10)

$$2. \quad \int_0^1 \ln [a^2 + 4(\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2 \Gamma \left(\frac{a+3\pi}{4\pi} \right)}{\Gamma \left(\frac{a+\pi}{4\pi} \right)} + \frac{\pi}{2} \ln \pi$$

[$a > -\pi$] BI (147)(16)a

$$3. \quad \int_0^\infty \ln [a^2 + (\ln x)^2] x^{\mu-1} dx = \frac{2}{\mu} [-\cos a\mu \operatorname{ci}(a\mu) - \sin a\mu \operatorname{si}(a\mu) + \ln a]$$

[$a > 0, \operatorname{Re} \mu > 0$] GW (325)(28)

If the integrand contains a logarithm whose argument also contains a logarithm, for example, if the integrand contains $\ln \ln \frac{1}{x}$, it is useful to make the substitution $\ln x = t$ and then seek the transformed integral in the tables.

4.33–4.34 Combinations of logarithms and exponentials**4.331**

$$1. \quad \int_0^\infty e^{-\mu x} \ln x dx = -\frac{1}{\mu} (C + \ln \mu) \quad [\operatorname{Re} \mu > 0]$$

BI (256)(2)

$$2. \quad \int_1^\infty e^{-\mu x} \ln x dx = -\frac{1}{\mu} \operatorname{Ei}(-\mu) \quad [\operatorname{Re} \mu > 0]$$

BI (260)(5)

$$3. \quad \int_0^1 e^{\mu x} \ln x \, dx = -\frac{1}{\mu} \int_0^1 \frac{e^{\mu x} - 1}{x} \, dx \quad [\mu \neq 0] \quad \text{GW (324)(81a)}$$

4.332

$$1. \quad \int_0^\infty \frac{\ln x \, dx}{e^x + e^{-x} - 1} = \frac{2\pi}{\sqrt{3}} \left[\frac{5}{6} \ln 2\pi - \ln \Gamma\left(\frac{1}{6}\right) \right] \quad (\text{cf. 4.325 6}) \quad \text{BI (257)(6)}$$

$$2. \quad \int_0^\infty \frac{\ln x \, dx}{e^x + e^{-x} + 1} = \frac{\pi}{\sqrt{3}} \ln \left[\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \sqrt{2\pi} \right] \quad (\text{cf. 4.325 5}) \quad \text{BI (257)(7)a, LI (260)(3)}$$

$$\mathbf{4.333} \quad \int_0^\infty e^{-\mu x^2} \ln x \, dx = -\frac{1}{4} (C + \ln 4\mu) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (256)(8), FI II 807a}$$

$$\mathbf{4.334} \quad \int_0^\infty \frac{\ln x \, dx}{e^{x^2} + 1 + e^{-x^2}} = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{k=1}^\infty (-1)^k \frac{C + \ln 4k}{\sqrt{k}} \sin \frac{k\pi}{3} \quad \text{BI (357)(13)}$$

4.335

$$1. \quad \int_0^\infty e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left[\frac{\pi^2}{6} + (C + \ln \mu)^2 \right] \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 149(13)}$$

$$2. \quad \int_0^\infty e^{-x^2} (\ln x)^2 \, dx = \frac{\sqrt{\pi}}{8} \left[(C + 2 \ln 2)^2 + \frac{\pi^2}{2} \right] \quad \text{FI II 808}$$

$$3.^7 \quad \int_0^\infty e^{-\mu x} (\ln x)^3 \, dx = -\frac{1}{\mu} \left[(C + \ln \mu)^3 + \frac{\pi^2}{2} (C + \ln \mu) - \psi''(1) \right] \quad \text{MI 26}$$

4.336

$$1.^7 \quad \text{PV} \int_0^\infty \frac{e^{-x}}{\ln x} \, dx = -0.154479567 \quad \text{BI (260)(9)}$$

$$2. \quad \int_0^\infty \frac{e^{-\mu x} \, dx}{\pi^2 + (\ln x)^2} = \nu'(\mu) - e^\mu \quad [\operatorname{Re} \mu > 0] \quad \text{MI 26}$$

4.337

$$1. \quad \int_0^\infty e^{-\mu x} \ln(\beta + x) \, dx = \frac{1}{\mu} [\ln \beta - e^{\mu\beta} \operatorname{Ei}(-\beta\mu)] \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0] \quad \text{BI (256)(3)}$$

$$2. \quad \int_0^\infty e^{-\mu x} \ln(1 + \beta x) \, dx = -\frac{1}{\mu} e^{\frac{\mu}{\beta}} \operatorname{Ei}\left(-\frac{\mu}{\beta}\right) \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0] \quad \text{ET I 148(4)}$$

$$3. \quad \int_0^\infty e^{-\mu x} \ln|a - x| \, dx = \frac{1}{\mu} [\ln a - e^{-a\mu} \operatorname{Ei}(a\mu)] \quad [a > 0, \operatorname{Re} \mu > 0] \quad \text{BI (256)(4)}$$

$$4.^7 \quad \int_0^\infty e^{-\mu x} \ln \left| \frac{\beta}{\beta - x} \right| \, dx = \frac{1}{\mu} [e^{-\beta\mu} \operatorname{Ei}(\beta\mu)] \quad [\operatorname{Re} \mu > 0] \quad \text{MI 26}$$

$$5.^* \quad \int_0^\infty \ln(1 + ax) x^\zeta e^{-x} \, dx = \sum_{\mu=0}^\zeta \frac{\zeta!}{(\zeta - \mu)!} \left[\frac{(-1)^{\zeta - \mu - 1}}{a^{\zeta - \mu}} e^{1/a} \operatorname{Ei}\left(-\frac{1}{a}\right) + \sum_{k=1}^{\zeta - \mu} (k - 1)! \left(-\frac{1}{a}\right)^{\zeta - \mu - k} \right]$$

4.338

$$1. \quad \int_0^\infty e^{-\mu x} \ln(\beta^2 + x^2) \, dx = \frac{2}{\mu} [\ln \beta - \operatorname{ci}(\beta\mu) \cos(\beta\mu) - \operatorname{si}(\beta\mu) \sin(\beta\mu)] \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \mu > 0] \quad \text{BI (256)(6)}$$

2. $\int_0^\infty e^{-\mu x} \ln^2(x^2 - \beta^2) dx = \frac{2}{\mu} [\ln^2 \beta - e^{\beta\mu} \text{Ei}(-\beta\mu) - e^{\beta\mu} \text{Ei}(\beta\mu)]$
 $[\text{Im } \beta > 0, \quad \text{Re } \mu > 0]$ BI (256)(5)
- 4.339 $\int_0^\infty e^{-\mu x} \ln \left| \frac{x+1}{x-1} \right| dx = \frac{1}{\mu} [e^{-\mu} (\ln 2\mu + \gamma) - e^\mu \text{Ei}(-2\mu)]$
 $[\text{Re } \mu > 0]$ MI 27
- 4.341 $\int_0^\infty e^{-\mu x} \ln \frac{\sqrt{x+ai} + \sqrt{x-ai}}{\sqrt{2a}} dx = \frac{\pi}{4\mu} [\mathbf{H}_0(a\mu) - Y_0(a\mu)]$
 $[a > 0, \quad \text{Re } \mu > 0]$ ET I 149(20)
- 4.342
1. $\int_0^\infty e^{-2nx} \ln(\sinh x) dx = \frac{1}{2n} \left[\frac{1}{n} + \ln 2 - 2\beta(2n+1) \right]$ BI (256)(17)
2. $\int_0^\infty e^{-\mu x} \ln(\cosh x) dx = \frac{1}{\mu} \left[\beta \left(\frac{\mu}{2} \right) - \frac{1}{\mu} \right]$ $[\text{Re } \mu > 0]$ ET I 165(32)
- 3.¹¹ $\int_0^\infty e^{-\mu x} [\ln(\sinh x) - \ln x] dx = \frac{1}{\mu} \left[\ln \frac{\mu}{2} - \frac{1}{\mu} - \psi \left(\frac{\mu}{2} \right) \right]$
 $[\text{Re } \mu > 0]$ ET I 165(33)
- 4.343 $\int_0^\pi e^{\mu \cos x} [\ln(2\mu \sin^2 x) + C] dx = -\pi K_0(\mu)$ WA 95(16)

4.35–4.36 Combinations of logarithms, exponentials, and powers

4.351

1. $\int_0^1 (1-x)e^{-x} \ln x dx = \frac{1-e}{e}$ BI (352)(1)
2. $\int_0^1 e^{\mu x} (\mu x^2 + 2x) \ln x dx = \frac{1}{\mu^2} [(1-\mu)e^\mu - 1]$ BI (352)(2)
3. $\int_1^\infty \frac{e^{-\mu x} \ln x}{1+x} dx = \frac{1}{2} e^\mu [\text{Ei}(-\mu)]^2$ $[\text{Re } \mu > 0]$ NT 32(10)

4.352

1. $\int_0^\infty x^{\nu-1} e^{-\mu x} \ln x dx = \frac{1}{\mu^\nu} \Gamma(\nu) [\psi(\nu) - \ln \mu]$ $[\text{Re } \mu > 0, \quad \text{Re } \nu > 0]$
BI (353)(3), ET I 315(10)a
2. $\int_0^\infty x^n e^{-\mu x} \ln x dx = \frac{n!}{\mu^{n+1}} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - C - \ln \mu \right]$
 $[\text{Re } \mu > 0]$ ET I 148(7)
3. $\int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x} \ln x dx = \sqrt{\pi} \frac{(2n-1)!!}{2^n \mu^{n+\frac{1}{2}}} \left[2 \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \right) - C - \ln 4\mu \right]$
 $[\text{Re } \mu > 0]$ ET I 148(10)

$$4. \quad \int_0^\infty x^{\mu-1} e^{-x} \ln x \, dx = \Gamma'(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{GW (324)(83a)}$$

4.353

$$1. \quad \int_0^\infty (x - \nu) x^{\nu-1} e^{-x} \ln x \, dx = \Gamma(\nu) \quad [\operatorname{Re} \nu > 0] \quad \text{GW (324)(84)}$$

$$2. \quad \int_0^\infty \left(\mu x - n - \frac{1}{2} \right) x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \frac{(2n-1)!!}{(2\mu)^n} \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (357)(2)}$$

$$3. \quad \int_0^1 (\mu x + n + 1) x^n e^{\mu x} \ln x \, dx = e^\mu \sum_{k=0}^n (-1)^{k-1} \frac{n!}{(n-k)! \mu^{k+1}} + (-1)^n \frac{n!}{\mu^{n+1}} \quad [\mu \neq 0] \quad \text{GW (324)(82)}$$

4.354

$$1.^6 \quad \int_0^\infty \frac{x^{\nu-1} \ln x}{e^x + 1} \, dx = \Gamma(\nu) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\nu} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 0]$$

$$= -\frac{1}{2} (\ln 2)^2 \quad [\text{for } \nu = 1] \quad \text{GW (324)(86a)}$$

$$2.^7 \quad \int_0^\infty \frac{x^{\nu-1} \ln x}{(e^x + 1)^2} \, dx = \Gamma(\nu) \sum_{k=2}^\infty \frac{(-1)^k (k-1)}{k^\nu} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 1] \quad \text{GW (324)(86b)}$$

$$3. \quad \int_0^\infty \frac{(x - \nu) e^x - \nu}{(e^x + 1)^2} x^{\nu-1} \ln x \, dx = \Gamma(\nu) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\nu} \quad [\operatorname{Re} \nu > 0] \quad \text{GW (324)(87a)}$$

$$4. \quad \int_0^\infty \frac{(x - 2n) e^x - 2n}{(e^x + 1)^2} x^{2n-1} \ln x \, dx = \frac{2^{2n-1} - 1}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{GW (324)(87b)}$$

$$5. \quad \int_0^\infty \frac{x^{\nu-1} \ln x}{(e^x + 1)^n} \, dx = (-1)^n \frac{\Gamma(\nu)}{(n-1)!} \sum_{k=n}^\infty \frac{(-1)^k (k-1)!}{(k-n)! k^\nu} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 0] \quad \text{GW (324)(86c)}$$

4.355

$$1. \quad \int_0^\infty x^2 e^{-\mu x^2} \ln x \, dx = \frac{1}{8\mu} (2 - \ln 4\mu - \mathbf{C}) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (357)(1)a}$$

$$2. \quad \int_0^\infty x (\mu x^2 - \nu x - 1) e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{4\mu} + \frac{\nu}{4\mu} \sqrt{\frac{\pi}{\mu}} \exp\left(\frac{\nu^2}{\mu}\right) \left[1 + \Phi\left(\frac{\nu}{\sqrt{\mu}}\right) \right] \quad [\operatorname{Re} \mu > 0] \quad \text{BI (358)(1)}$$

$$3. \quad \int_0^\infty (\mu x^2 - n) x^{2n-1} e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (353)(4)}$$

$$4. \quad \int_0^\infty (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n} \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (353)(5)}$$

4.356

$$1. \quad \int_0^\infty \exp \left[-\mu \left(\frac{x}{a} + \frac{a}{x} \right) \right] \ln x \frac{dx}{x} = 2 \ln a K_0(2\mu) \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{GW (324)(91)}$$

$$2. \quad \int_0^\infty \exp \left(-ax - \frac{b}{x} \right) \ln x [2ax^2 - (2n+1)x - 2b] x^{n-\frac{1}{2}} dx \\ = 2 \left(\frac{b}{a} \right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^\infty \frac{(n+k)!}{(n-k)!(2k)!! (2\sqrt{ab})^k} \quad [a > 0, \quad b > 0] \quad \text{BI (357)(4)}$$

$$3. \quad \int_0^\infty \exp \left(-ax - \frac{b}{x} \right) \ln x [2ax^2 + (2n-1)x - 2b] \frac{dx}{x^{n+\frac{3}{2}}} \\ = 2 \left(\frac{a}{b} \right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^\infty \frac{(n+k-1)!}{(n-k-1)!(2k)!! (2\sqrt{ab})^k} \quad [a > 0, \quad b > 0] \quad \text{BI (357)(11)}$$

For $n = \frac{1}{2}$:

$$4. \quad \int_0^\infty \exp \left(-ax - \frac{a}{x} \right) \ln x \frac{ax^2 - b}{x^2} dx = 2 K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0] \quad \text{GW (324)(92c)}$$

For $n = 0$:

$$5. \quad \int_0^\infty \exp \left(-ax - \frac{b}{x} \right) \ln x \frac{2ax^2 - x - 2b}{x\sqrt{x}} dx = 2\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \quad [a > 0, \quad b > 0] \quad \text{BI (357)(7), GW(324)(92a)}$$

For $n = -1$:

$$6. \quad \int_0^\infty \exp \left(-ax - \frac{b}{x} \right) \ln x \frac{2ax^2 - 3x - 2b}{\sqrt{x}} dx = \frac{1 + 2\sqrt{ab}}{a} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \quad [a > 0, \quad b > 0] \quad \text{LI (357)(6), GW (324)(92b)}$$

$$7.^9 \quad \int_0^\infty \exp \left(-ax - \frac{b}{x} \right) \ln x \left(a - \frac{b}{x^2} \right) dx = K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0]$$

$$\begin{aligned}
8.^9 \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 - (2n+1)x - 2b] x^{n-\frac{3}{2}} dx \\
= 4 \left(\frac{b}{a}\right)^{(2n+1)/4} K_{n+\frac{1}{2}}(2\sqrt{ab}) \\
= 2 \left(\frac{b}{a}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^n \frac{(n+k)!}{(n-k)!(2k)!! (2\sqrt{ab})^k} \\
[n = 0, 1, \dots, a > 0, \quad b > 0]
\end{aligned}$$

$$\begin{aligned}
9.^9 \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln [(ax^2 - b) \cos(\alpha \ln x) + \alpha x \sin(\alpha \ln x)] \frac{dx}{x^2} \\
= 2 \cos\left(\alpha \ln \sqrt{b/a}\right) K_{i\alpha}(2\sqrt{ab}) \\
[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]
\end{aligned}$$

$$\begin{aligned}
10.^9 \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x [(ax^2 - b) \sin(\alpha \ln x) - \alpha x \cos(\alpha \ln x)] \frac{dx}{x^2} \\
= 2 \sin\left(\alpha \ln \sqrt{b/a}\right) K_{i\alpha}(2\sqrt{ab}) \\
[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]
\end{aligned}$$

$$\begin{aligned}
11.^9 \quad q \int_0^\infty x^\alpha \ln x \left[a - \frac{\alpha}{x} - \frac{b}{x^2}\right] \exp\left(-ax - \frac{b}{x}\right) dx = 2 \left(\frac{b}{a}\right)^{\alpha/2} K_\alpha(2\sqrt{ab}) \\
[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]
\end{aligned}$$

4.357

$$1. \quad \int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{1+ax^2-x^4}{x^2} dx = -\frac{\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0] \quad \text{BI (357)(8)}$$

$$2. \quad \int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4+ax^2-1}{x^4} dx = \frac{\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0] \quad \text{BI (357)(9)}$$

$$3. \quad \int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4+3ax-1}{x^6} dx = \frac{(1+a)\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0] \quad \text{BI (357)(10)}$$

4.358

$$1.^6 \quad \int_1^\infty x^{\nu-1} e^{-\mu x} (\ln x)^m dx = \frac{\partial^m}{\partial \nu^m} \{\mu^{-\nu} \Gamma(\nu, \mu)\} \quad [m = 0, 1, \dots, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{MI 26}$$

$$2. \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^2 dx = \frac{\Gamma(\nu)}{\mu^\nu} \left\{ [\psi(\nu) - \ln \mu]^2 + \zeta(2, \nu) \right\} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{MI 26}$$

$$3.^9 \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^3 dx = \frac{\Gamma(\nu)}{\mu^\nu} \left\{ [\psi(\nu) - \ln \mu]^3 + 3\zeta(2, \nu) [\psi(\nu) - \ln \mu] - 2\zeta(3, \nu) \right\} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{MI 26}$$

$$4.7 \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^4 dx = \frac{\Gamma(\nu)}{\nu} \left\{ [\psi(\nu) - \ln \mu]^4 + 6 \zeta(2, \nu) [\psi(\nu) - \ln \mu]^2 \right. \\ \left. - 8 \zeta(3, \nu) [\psi(\nu) - \ln \mu] + 3 [\zeta(2, \nu)]^2 + 6 \zeta(4, \nu) \right\} \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$$

$$5.3 \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^n dx = \frac{\partial^n}{\partial \nu^n} \{ \mu^{-\nu} \Gamma(\nu) \} \quad [n = 0, 1, 2, \dots]$$

4.359

$$1. \quad \int_0^\infty e^{-\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \frac{1}{\mu} [\lambda(\mu, p-1) - \lambda(\mu, q-1)] \\ [\operatorname{Re} \mu > 0, \quad p > 0, \quad q > 0] \quad \text{MI 27}$$

$$2.11 \quad \int_0^1 e^{\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \sum_{k=0}^\infty \frac{\mu^k}{k!} \ln \frac{p+k}{q+k} \\ [p > 0, \quad q > 0] \quad \text{BI (352)(9)}$$

4.361

$$1. \quad \int_0^\infty \frac{(x+1)e^{-\mu x}}{\pi^2 + (\ln x)^2} dx = \nu'(\mu) - \nu''(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 27}$$

$$2. \quad \int_0^\infty \frac{e^{-\mu x} dx}{x [\pi^2 + (\ln x)^2]} = e^\mu - \nu(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 27}$$

4.362

$$1. \quad \int_0^1 x e^x \ln(1-x) dx = 1 - e \quad \text{BI (352)(5)a}$$

$$2. \quad \int_1^\infty e^{-\mu x} \ln(2x-1) \frac{dx}{x} = \frac{1}{2} \left[\operatorname{Ei} \left(-\frac{\mu}{2} \right) \right]^2 \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 148(8)}$$

4.363

$$1. \quad \int_0^\infty e^{-\mu x} \ln(a+x) \frac{\mu(x+a) \ln(x+a) - 2}{x+a} dx \\ = \frac{1}{4} \int_0^\infty e^{-\mu x} \ln^2(a-x) \frac{\mu(x-a) \ln^2(x-a) - 4}{x-a} dx = (\ln a)^2 \\ [\operatorname{Re} \mu > 0, \quad a > 0] \quad \text{BI (354)(4, 5)}$$

$$2. \quad \int_0^1 x(1-x)(2-x)e^{-(1-x)^2} \ln(1-x) dx = \frac{1-e}{4e} \quad \text{BI (352)(4)}$$

4.364

$$1. \quad \int_0^\infty e^{-\mu x} \ln[(x+a)(x+b)] \frac{dx}{x+a+b} = e^{(a+b)\mu} \{ \operatorname{Ei}(-a\mu) \operatorname{Ei}(-b\mu) - \ln(ab) \operatorname{Ei}[-(a+b)\mu] \} \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (354)(11)}$$

$$\begin{aligned}
2. \quad \int_0^\infty e^{-\mu x} \ln(x+a+b) \left(\frac{1}{x+a} + \frac{1}{x+b} \right) dx \\
= (1 + \ln a \ln b) \ln(a+b) + e^{-(a+b)\mu} \{ \text{Ei}(-\alpha\mu) \text{Ei}(-b\mu) \} \\
+ (1 - \ln(ab)) \text{Ei}[-(a+b)\mu] \\
[a > 0, \quad b > 0, \quad \text{Re } \mu > 0] \quad \text{BI (354)(12)}
\end{aligned}$$

$$\mathbf{4.365} \quad \int_0^\infty \left[e^{-x} - \frac{x}{(1+x)^{p+1} \ln(1+x)} \right] \frac{dx}{x} = \ln p \quad [p > 0] \quad \text{BI (354)(15)}$$

4.366

$$1. \quad \int_0^\infty e^{-\mu x} \ln \left(1 + \frac{x^2}{a^2} \right) \frac{dx}{x} = [\text{ci}(a\mu)]^2 + [\text{si}(a\mu)]^2 \quad [\text{Re } \mu > 0] \quad \text{NT 32(11)a}$$

$$2. \quad \int_0^\infty e^{-\mu x} \ln \left| 1 - \frac{x^2}{a^2} \right| \frac{dx}{x} = \text{Ei}(a\mu) \text{Ei}(-a\mu) \quad [\text{Re } \mu > 0] \quad \text{ME 18}$$

$$\begin{aligned}
3. \quad \int_0^\infty x e^{-\mu x^2} \ln \left| \frac{1+x^2}{1-x^2} \right| dx = \frac{1}{\mu} [\cosh \mu \sinh(i\mu) - \sinh \mu \cosh(i\mu)] \\
[\text{Re } \mu > 0]; \quad (\text{cf. } \mathbf{4.339}) \quad \text{MI 27}
\end{aligned}$$

$$\mathbf{4.367} \quad \int_0^\infty x e^{-\mu x^2} \ln \frac{x + \sqrt{x^2 + 2\beta}}{\sqrt{2\beta}} dx = \frac{e^{\beta\mu}}{4\mu} K_0(\beta\mu) \quad [|\arg \beta| < \pi, \quad \text{Re } \mu > 0] \quad \text{ET I 149(19)}$$

$$\begin{aligned}
\mathbf{4.368} \quad \int_0^{2u} e^{-\mu x^2} \ln \frac{x^2(4u^2 - x^2)}{u^4} \frac{dx}{\sqrt{4u^2 - x^2}} = \frac{\pi}{2} e^{-2u^2\mu} \left[\frac{\pi}{2} Y_0(2iu^2\mu) - (C - \ln 2) J_0(2iu^2\mu) \right] \\
[\text{Re } \mu > 0] \quad \text{ET I 149(21)a}
\end{aligned}$$

4.369

$$1. \quad \int_0^\infty x^{\nu-1} e^{-\mu x} [\psi(\nu) - \ln x] dx = \frac{\Gamma(\nu) \ln \mu}{\mu^\nu} \quad [\text{Re } \nu > 0] \quad \text{ET I 149(12)}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^n e^{-\mu x} \left\{ \left[\ln x - \frac{1}{2} \psi(n+1) \right]^2 - \frac{1}{2} \psi'(n+1) \right\} dx \\
= \frac{n!}{\mu^{n+1}} \left\{ \left[\ln \mu - \frac{1}{2} \psi(n+1) \right]^2 + \frac{1}{2} \psi'(n+1) \right\} \\
[\text{Re } \mu > 0] \quad \text{MI 26}
\end{aligned}$$

4.37 Combinations of logarithms and hyperbolic functions

4.371

$$1. \quad \int_0^\infty \frac{\ln x}{\cosh x} dx = \pi \ln \left[\frac{\sqrt{2\pi} \Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right] \quad \text{LI (260)(1)a}$$

$$\begin{aligned}
2. \quad \int_0^\infty \frac{\ln x dx}{\cosh x + \cos t} = \frac{\pi}{\sin t} \ln \frac{(2\pi)^{t/\pi} \Gamma\left(\frac{\pi+t}{2\pi}\right)}{\Gamma\left(\frac{\pi-t}{2\pi}\right)} \quad [t^2 < \pi^2] \quad \text{BI (257)(7)a}
\end{aligned}$$

$$3. \quad \int_0^\infty \frac{\ln x \, dx}{\cosh^2 x} = \psi\left(\frac{1}{2}\right) + \ln \pi = \ln \pi - 2 \ln 2 - C \quad \text{BI (257)(4)a}$$

4.372

$$1. \quad \int_1^\infty \ln x \frac{\sinh mx}{\sinh nx} dx = \frac{\pi}{2n} \tan \frac{m\pi}{2n} \ln 2\pi + \frac{\pi}{n} \sum_{k=1}^{n-1} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \quad [m+n \text{ is odd}]$$

$$= \frac{\pi}{2n} \tan \frac{m\pi}{2n} \ln \pi + \frac{\pi}{n} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \frac{\Gamma\left(\frac{n-k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \quad [m+n \text{ is even}]$$

BI (148)(3)a

$$2. \quad \int_1^\infty \ln x \frac{\cosh mx}{\cosh nx} dx$$

$$= \frac{\pi}{2n} \frac{\ln 2\pi}{\cos \frac{m\pi}{2n}} + \frac{\pi}{n} \sum_{k=1}^n (-1)^{k-1} \cos \frac{(2k-1)m\pi}{2n} \ln \frac{\Gamma\left(\frac{2n+2k-1}{4n}\right)}{\Gamma\left(\frac{2k-1}{4n}\right)} \quad [m+n \text{ is odd}]$$

$$= \frac{\pi}{2n} \frac{\ln \pi}{\cos \frac{m\pi}{2n}} + \frac{\pi}{n} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \cos \frac{(2k-1)m\pi}{2n} \ln \frac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)} \quad [m+n \text{ is even}]$$

BI (148)(6)a

4.373

$$1. \quad \int_0^\infty \frac{\ln(a^2 + x^2)}{\cosh bx} dx = \frac{\pi}{b} \left[2 \ln \frac{2\Gamma\left(\frac{2ab+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab+\pi}{4\pi}\right)} - \ln \frac{2b}{\pi} \right] \quad \left[b > 0, \quad a > -\frac{\pi}{2b} \right]. \quad \text{BI (258)(11)a}$$

$$2. \quad \int_0^\infty \ln(1+x^2) \frac{dx}{\cosh \frac{\pi x}{2}} = 2 \ln \frac{4}{\pi} \quad \text{BI (258)(1)a}$$

$$3. \quad \int_0^\infty \ln(a^2 + x^2) \frac{\sinh\left(\frac{2}{3}\pi x\right)}{\sinh \pi x} dx = 2 \sin \frac{\pi}{3} \ln \frac{6\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}{\Gamma\left(\frac{a+1}{6}\right)\Gamma\left(\frac{a+2}{6}\right)} \quad [a > -1]. \quad \text{BI (258)(12)}$$

$$4. \quad \int_0^\infty \ln(1+x^2) \frac{dx}{\sinh^2 ax} = \frac{2}{a} \left[\ln \frac{a}{\pi} + \frac{\pi}{2a} - \psi\left(\frac{\pi+a}{\pi}\right) \right] \quad [a > 0] \quad \text{BI (258)(5)}$$

$$5. \quad \int_0^\infty \ln(1+x^2) \frac{\cosh\left(\frac{\pi}{2}x\right)}{\sinh^2\left(\frac{\pi}{2}x\right)} dx = \frac{2\pi-4}{\pi} \quad \text{BI (258)(3)}$$

$$6. \quad \int_0^\infty \ln(1+x^2) \frac{\cosh\left(\frac{\pi}{4}x\right)}{\sinh^2\left(\frac{\pi}{4}x\right)} dx = 4\sqrt{2} - \frac{16}{\pi} + \frac{8\sqrt{2}}{\pi} \ln(\sqrt{2}+1) \quad \text{BI (258)(2)}$$

4.374

$$1. \quad \int_0^\infty \ln(\cos^2 t + e^{-2x} \sin^2 t) \frac{dx}{\sinh x} = -2t^2 \quad \text{BI (259)(10)a}$$

$$2. \quad \int_0^\infty \ln(a + be^{-2x}) \frac{dx}{\cosh^2 x} = \frac{2}{(b-a)} \left[\frac{a+b}{2} \ln(a+b) - a \ln a - b \ln 2 \right] \\ [a > 0, \quad a+b > 0] \quad \text{LI (259)(14)}$$

4.375

$$1.^{11} \quad \int_0^\infty \ln \cosh \frac{x}{2} \frac{dx}{\cosh x} = \mathbf{G} - \frac{\pi}{4} \ln 2 \quad \text{BI (259)(11)}$$

$$2. \quad \int_0^\infty \ln \coth x \frac{dx}{\cosh x} = \frac{\pi}{2} \ln 2 \quad \text{BI (259)(16)}$$

4.376

$$1. \quad \int_0^\infty \frac{\ln x}{\sqrt{x} \cosh x} dx = 2\sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^{k+1}}{\sqrt{2k+1}} \{ \ln(2k+1) + 2 \ln 2 + \mathbf{C} \} \quad \text{BI (147)(4)}$$

$$2. \quad \int_0^\infty \ln x \frac{(\mu+1) \cosh x - x \sinh x}{\cosh^2 x} x^\mu dx = 2 \Gamma(\mu+1) \sum_{k=0}^\infty \frac{(-1)^{k+1}}{(2k+1)^{\mu+1}} \\ [\operatorname{Re} \mu > -1] \quad \text{BI (356)(10)}$$

$$3. \quad \int_0^\infty \ln x \frac{(n+1) \cosh x - x \sinh x}{\cosh^2 x} x^n dx = \frac{(-1)^n}{2^n} \beta^{(n)} \left(\frac{1}{2} \right)$$

$$4. \quad \int_0^\infty \ln 2x \frac{n \sinh 2ax - ax}{\sinh^2 ax} x^{2n-1} dx = -\frac{1}{n} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \\ [n = 1, 2, \dots] \quad \text{BI (356)(9)a}$$

$$5. \quad \int_0^\infty \ln x \frac{ax \cosh ax - (2n+1) \sinh ax}{\sinh^2 ax} x^{2n} dx = 2 \frac{2^{2n+1} - 1}{(2a)^{2n+1}} (2n)! \zeta(2n+1) \quad \text{BI (356)(14)}$$

$$6. \quad \int_0^\infty \ln x \frac{ax \cosh ax - 2n \sinh ax}{\sinh^2 ax} x^{2n-1} dx = \frac{2^{2n-1} - 1}{2n} |B_{2n}| \left(\frac{\pi}{a} \right)^{2n} \\ [n = 1, 2, \dots, a > 0] \quad \text{BI (356)(15)}$$

$$7. \quad \int_0^\infty \ln \frac{(2n+1) \cosh ax - ax \sinh ax}{\cosh^2 ax} x^{2n} dx = -\left(\frac{\pi}{2a} \right)^{2n+1} |E_{2n}| \\ [a > 0] \quad \text{BI (356)(11)}$$

$$8.^6 \quad \int_0^\infty \ln x \frac{2ax \sinh ax - (2n+1) \cosh ax}{\cosh^3 ax} x^{2n} dx = \begin{cases} \frac{2}{a} (2^{2n-1} - 1) \left(\frac{\pi}{2a} \right)^{2n} |B_{2n}| & n = 1, 2, \dots \\ \frac{1}{a} & n = 0 \end{cases} \\ [a > 0] \quad \text{BI (356)(2)}$$

$$9.^6 \quad \int_0^\infty \ln x \frac{2ax \cosh ax - (2n+1) \sinh ax}{\sinh^3 ax} x^{2n} dx = \frac{1}{a} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \\ [a > 0, \quad n = 1, 2, \dots] \quad \text{BI (356)(6)a}$$

10.
$$\int_0^\infty \ln x \frac{x \sinh x - 6 \sinh^2 \left(\frac{x}{2}\right) - 6 \cos^2 \frac{t}{2}}{(\cosh x + \cos t)^2} x^2 dx = \frac{(\pi - t^2) t}{3 \sin t}$$

 $[0 < t < \pi]$ BI (356)(16)a
11.
$$\int_0^\infty \ln(1+x^2) \frac{\cosh \pi x + \pi x \sinh \pi x}{\cosh^2 \pi x} \frac{dx}{x^2} = 4 - \pi$$
 BI (356)(12)
12.
$$\int_0^\infty \ln(1+4x^2) \frac{\cosh \pi x + \pi x \sinh \pi x}{\cosh^2 \pi x} \frac{dx}{x^2} = 4 \ln 2$$
 BI (356)(13)
- 4.377
$$\int_0^\infty \ln 2x \frac{ax - n(1 - e^{-2ax})}{\sinh^2 ax} x^{2n-1} dx = \frac{1}{2n} \left(\frac{\pi}{a}\right)^{2n} |B_{2n}|$$

 $[n = 1, 2, \dots]$ LI (356)(8)a

4.38–4.41 Logarithms and trigonometric functions

4.381

1.
$$\int_0^1 \ln x \sin ax dx = -\frac{1}{a} [C + \ln a - \text{ci}(a)]$$
 $[a > 0]$ GW (338)(2a)
2.
$$\int_0^1 \ln x \cos ax dx = -\frac{1}{a} \left[\text{si}(a) + \frac{\pi}{2}\right]$$
 $[a > 0]$ BI (284)(2)
3.
$$\int_0^{2\pi} \ln x \sin nx dx = -\frac{1}{n} [C + \ln(2n\pi) - \text{ci}(2n\pi)]$$
 GW (338)(1a)
4.
$$\int_0^{2\pi} \ln x \cos nx dx = -\frac{1}{n} \left[\text{si}(2n\pi) + \frac{\pi}{2}\right]$$
 GW (338)(1b)

4.382

1.
$$\int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \sin bx dx = \frac{\pi}{b} \sin ab$$
 $[a < 0, \quad b > 0]$ ET I 77(11)
- 2.¹⁰
$$\int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \cos bx dx = \frac{2}{b} \left[\cos(ab) \left\{ \text{si}(ab) + \frac{\pi}{2} \right\} - \sin(ab) \text{ci}(ab) \right]$$

 $[a > 0, \quad b > 0]$ ET I 18(9)
3.
$$\int_0^\infty \ln \frac{a^2 + x^2}{b^2 + x^2} \cos cx dx = \frac{\pi}{c} (e^{-bc} - e^{-ac})$$
 $[a > 0, \quad b > 0, \quad c > 0]$
 FI III 648a, BI (337)(5)
4.
$$\int_0^\infty \ln \frac{x^2 + x + a^2}{x^2 - x + a^2} \sin bx dx = \frac{2\pi}{b} \exp \left(-b \sqrt{a^2 - \frac{1}{4}} \right) \sin \frac{b}{2}$$

 $[b > 0]$ ET I 77(12)
5.
$$\int_0^\infty \ln \frac{(x+\beta)^2 + \gamma^2}{(x-\beta)^2 + \gamma^2} \sin bx dx = \frac{2\pi}{b} e^{-\gamma b} \sin \beta b$$
 $[\text{Re } \gamma > 0, \quad |\text{Im } \beta| \leq \text{Re } \gamma, \quad b > 0]$
 ET I 77(13)

4.383

$$1. \quad \int_0^{\infty} \ln(1 + e^{-\beta x}) \cos bx \, dx = \frac{\beta}{2b^2} - \frac{\pi}{2b \sinh\left(\frac{\pi b}{\beta}\right)} \quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET I 18(13)}$$

$$2. \quad \int_0^{\infty} \ln(1 - e^{-\beta x}) \cos bx \, dx = \frac{\beta}{2b^2} - \frac{\pi}{2b} \coth\left(\frac{\pi b}{\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET I 18(14)}$$

4.384

$$1. \quad \int_0^1 \ln(\sin \pi x) \sin 2n\pi x \, dx = 0 \quad \text{GW (338)(3a)}$$

$$\begin{aligned} 2.^7 \quad \int_0^1 \ln(\sin \pi x) \sin(2n+1)\pi x \, dx &= 2 \int_0^{1/2} \ln(\sin \pi x) \sin(2n+1)\pi x \, dx \\ &= \frac{2}{(2n+1)\pi} \left[\ln 2 - \frac{1}{2n+1} - 2 \sum_{k=1}^n \frac{1}{2k-1} \right] \end{aligned}$$

GW (338)(3b)

$$\begin{aligned} 3.^6 \quad \int_0^1 \ln(\sin \pi x) \cos 2n\pi x \, dx &= 2 \int_0^{1/2} \ln(\sin \pi x) \cos 2n\pi x \, dx \\ &= -\ln 2 \quad [n = 0] \\ &= -\frac{1}{2n} \quad [n > 0] \end{aligned}$$

GW (338)(3c)

$$4. \quad \int_0^1 \ln(\sin \pi x) \cos(2n+1)\pi x \, dx = 0 \quad \text{GW (338)(3d)}$$

$$5. \quad \int_0^{\pi/2} \ln \sin x \sin x \, dx = \ln 2 - 1 \quad \text{BI (305)(4)}$$

$$6. \quad \int_0^{\pi/2} \ln \sin x \cos x \, dx = -1 \quad \text{BI (305)(5)}$$

$$7. \quad \int_0^{\pi/2} \ln \sin x \cos 2nx \, dx = \begin{cases} -\frac{\pi}{4n}, & \text{for } n > 0 \\ -\frac{\pi}{2} \ln 2, & \text{for } n = 0 \end{cases} \quad \text{LI (305)(6)}$$

$$8. \quad \int_0^{\pi} \ln \sin x \cos[2m(x-n)] \, dx = -\frac{\pi \cos 2mn}{2m} \quad \text{LI (330)(8)}$$

$$9. \quad \int_0^{\pi/2} \ln \sin x \sin^2 x \, dx = \frac{\pi}{8} (1 - \ln 4) \quad \text{BI (305)(7)}$$

$$10. \quad \int_0^{\pi/2} \ln \sin x \cos^2 x \, dx = -\frac{\pi}{8} (1 + \ln 4) \quad \text{BI (305)(8)}$$

$$11. \quad \int_0^{\pi/2} \ln \sin x \sin x \cos^2 x \, dx = \frac{1}{9} (\ln 8 - 4) \quad \text{BI (305)(9)}$$

$$12. \quad \int_0^{\pi/2} \ln \sin x \tan x \, dx = -\frac{\pi^2}{24} \quad \text{BI (305)(11)}$$

$$13. \quad \int_0^{\pi/2} \ln \sin 2x \sin x \, dx = \int_0^{\pi/2} \ln \sin 2x \cos x \, dx = 2(\ln 2 - 1) \quad \text{BI (305)(16, 17)}$$

$$14. \quad \int_0^{\pi} \frac{\ln(1 + p \cos x)}{\cos x} \, dx = \pi \arcsin p \quad [p^2 < 1] \quad \text{FI II 484}$$

$$15. \quad \int_0^{\pi} \ln \sin x \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^2}{2} \quad [a^2 < 1]$$

$$= \frac{\pi}{a^2 - 1} \ln \frac{a^2 - 1}{2a^2} \quad [a^2 > 1]$$

BI (331)(8)

$$16. \quad \int_0^{\pi} \ln \sin bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^{2b}}{2} \quad [a^2 < 1] \quad \text{BI (331)(10)}$$

$$17. \quad \int_0^{\pi} \ln \cos bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^{2b}}{2} \quad [a^2 < 1] \quad \text{BI (331)(11)}$$

$$18. \quad \int_0^{\pi/2} \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{1}{2} \int_0^{\pi} \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2}$$

$$= \frac{\pi}{2(1 - a^2)} \ln \frac{1 - a}{2} \quad [a^2 < 1]$$

$$= \frac{\pi}{2(a^2 - 1)} \ln \frac{a - 1}{2a} \quad [a^2 > 1]$$

BI (321)(1), BI (331)(13)

$$19. \quad \int_0^{\pi} \ln \sin bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^b}{2} \quad [a^2 < 1] \quad \text{BI (331)(18)}$$

$$20. \quad \int_0^{\pi} \ln \cos bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^b}{2} \quad [a^2 < 1] \quad \text{BI (331)(21)}$$

$$21. \quad \int_0^{\pi/2} \frac{\ln \cos x \, dx}{1 - 2p \cos 2x + p^2} = \frac{\pi}{2(1 - p^2)} \ln \frac{1 + p}{2} \quad [p^2 < 1]$$

$$= \frac{\pi}{2(p^2 - 1)} \ln \frac{p + 1}{2p} \quad [p^2 > 1]$$

BI (321)(8)

$$22. \quad \int_0^{\pi} \ln \sin x \frac{\cos x \, dx}{1 - 2a \cos x + a^2} = \frac{\pi}{2a} \frac{1 + a^2}{1 - a^2} \ln(1 - a^2) - \frac{a\pi \ln 2}{1 - a^2} \quad [a^2 < 1]$$

$$= \frac{\pi}{2a} \frac{a^2 + 1}{a^2 - 1} \ln \frac{a^2 - 1}{a^2} - \frac{\pi \ln 2}{a(a^2 - 1)} \quad [a^2 > 1]$$

LI (331)(9)

$$23. \quad \int_0^{\pi} \ln \sin bx \frac{\cos x \, dx}{1 - 2a \cos 2x + a^2} = \int_0^{\pi} \ln \cos bx \frac{\cos x \, dx}{1 - 2a \cos 2x + a^2} = 0$$

[0 < a < 1] BI (331)(19, 22)

$$24. \quad \int_0^{\pi} \ln \sin x \frac{\cos^2 x \, dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{4a} \frac{1 + a}{1 - a} \ln(1 - a) - \frac{\pi \ln 2}{2(1 - a)} \quad [0 < a < 1]$$

$$= \frac{\pi}{4a} \frac{a + 1}{a - 1} \ln \frac{a - 1}{a} - \frac{\pi \ln 2}{2a(a - 1)} \quad [a > 1]$$

BI (331)(16)

$$\begin{aligned}
 25. \quad \int_0^{\pi/2} \ln \sin x \frac{\cos 2x \, dx}{1 - 2a \cos 2x + a^2} &= \frac{1}{2} \int_0^{\pi} \ln \sin x \frac{\cos 2x \, dx}{1 - 2a \cos 2x + a^2} \\
 &= \frac{\pi}{2a(1-a^2)} \left\{ \frac{1+a^2}{2} \ln(1-a) - a^2 \ln 2 \right\} \quad [a^2 < 1] \\
 &= \frac{\pi}{2a(a^2-1)} \left\{ \frac{1+a^2}{2} \ln \frac{a-1}{a} - \ln 2 \right\} \quad [a^2 > 1] \\
 &\quad \text{BI (321)(2), BI (331)(15), LI (321)(2)}
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int_0^{\pi/2} \ln \cos x \frac{\cos 2x \, dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2a(1-a^2)} \left\{ \frac{1+a^2}{2} \ln(1+a) - a^2 \ln 2 \right\} \quad [a^2 < 1] \\
 &= \frac{\pi}{2a(a^2-1)} \left\{ \frac{1+a^2}{2} \ln \frac{1+a}{a} - \ln 2 \right\} \quad [a^2 > 1] \\
 &\quad \text{BI (321)(9)}
 \end{aligned}$$

4.385

$$1. \quad \int_0^{\pi} \ln \sin x \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} \ln \frac{\sqrt{a^2 - b^2}}{a + \sqrt{a^2 - b^2}} \quad [a > 0, \quad a > b] \quad \text{BI (331)(6)}$$

$$\begin{aligned}
 2. \quad \int_0^{\pi/2} \ln \sin x \frac{dx}{(a \sin x \pm b \cos x)^2} &= \int_0^{\pi/2} \ln \cos x \frac{dx}{(a \cos x \pm b \sin x)^2} \\
 &= \frac{1}{b(a^2 + b^2)} \left(\mp a \ln \frac{a}{b} - \frac{b\pi}{2} \right) \\
 &\quad [a > 0, \quad b > 0] \quad \text{BI (319)(1,6)a}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^{\pi/2} \frac{\ln \sin x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x} &= \int_0^{\pi/2} \frac{\ln \cos x \, dx}{b^2 \sin^2 x + a^2 \cos^2 x} = \frac{\pi}{2ab} \ln \frac{b}{a+b} \\
 &\quad [a > 0, \quad b > 0] \quad \text{BI (317)(4, 10)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_0^{\pi/2} \ln \sin x \frac{\sin 2x \, dx}{(a \sin^2 x + b \cos^2 x)^2} &= \int_0^{\pi/2} \ln \cos x \frac{\sin 2x \, dx}{(b \sin^2 x + a \cos^2 x)^2} \\
 &= \frac{1}{2b(b-a)} \ln \frac{a}{b} \\
 &\quad [a > 0, \quad b > 0] \quad \text{BI (319)(3, 7), LI (319)(3)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \int_0^{\pi/2} \ln \sin x \frac{a^2 \sin^2 x - b^2 \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} \, dx &= \int_0^{\pi/2} \ln \cos x \frac{a^2 \cos^2 x - b^2 \sin^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \, dx \\
 &= \frac{\pi}{2b(a+b)} \\
 &\quad [a > 0, \quad b > 0] \quad \text{LI (319)(2, 8)}
 \end{aligned}$$

4.386

$$1. \quad \int_0^{\pi/2} \ln \sin x \frac{\sin x}{\sqrt{1 + \sin^2 x}} \, dx = \int_0^{\pi/2} \frac{\cos x \ln \cos x}{\sqrt{1 + \cos^2 x}} \, dx = -\frac{\pi}{8} \ln 2 \quad \text{BI (322)(1, 6)}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^3 x \ln \sin x}{\sqrt{1 + \sin^2 x}} \, dx = \int_0^{\pi/2} \frac{\cos^3 x \ln \cos x}{\sqrt{1 + \cos^2 x}} \, dx = \frac{\ln 2 - 1}{4} \quad \text{BI (322)(2, 7)}$$

$$3. \quad \int_0^{\pi/2} \ln \sin x \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = -\frac{1}{2} \mathbf{K}(k) \ln k - \frac{\pi}{4} \mathbf{K}(k') \quad \text{BI (322)(3)}$$

$$4. \quad \int_0^{\pi/2} \frac{\ln \cos x \, dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{1}{2} \mathbf{K}(k) \ln \frac{k'}{k} - \frac{\pi}{4} \mathbf{K}(k') \quad \text{BI (322)(9)}$$

4.387

$$1. \quad \int_0^{\pi/2} \ln \sin x \sin^\mu x \cos^\nu x \, dx = \int_0^{\pi/2} \ln \cos x \cos^\mu x \sin^\nu x \, dx \\ = \frac{1}{4} \mathbf{B}\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) \left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu+\nu+2}{2}\right) \right] \\ [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1] \quad \text{GW (338)(6c)}$$

$$2. \quad \int_0^{\pi/2} \ln \sin x \sin^{\mu-1} x \, dx = \frac{\sqrt{\pi} \Gamma\left(\frac{\mu}{2}\right)}{4 \Gamma\left(\frac{\mu+1}{2}\right)} \left[\psi\left(\frac{\mu}{2}\right) - \psi\left(\frac{\mu+1}{2}\right) \right] \\ [\operatorname{Re} \mu > 0] \quad \text{GW (338)(6a)}$$

$$3. \quad \int_0^{\pi/2} \ln \sin x \cos^{\nu-1} x \, dx = \frac{\sqrt{\pi} \Gamma\left(\frac{\nu}{2}\right)}{4 \Gamma\left(\frac{\nu+1}{2}\right)} \left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu+1}{2}\right) \right] \\ [\operatorname{Re} \nu > 0] \quad \text{GW (338)(6b)}$$

$$4. \quad \int_0^{\pi/2} \ln \sin x \sin^{2n} x \, dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \left\{ \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} - \ln 2 \right\} \quad \text{FI II 811}$$

$$5. \quad \int_0^{\pi/2} \ln \sin x \sin^{2n+1} x \, dx = \frac{(2n)!!}{(2n+1)!!} \left\{ \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right\} \quad \text{BI (305)(13)}$$

$$6. \quad \int_0^{\pi/2} \ln \sin x \cos^{2n} x \, dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} \left[\sum_{k=1}^n \frac{1}{k} + \ln 4 \right] \\ = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} [\mathbf{C} + \psi(n+1) + \ln 4] \\ \text{BI (305)(14)}$$

$$7. \quad \int_0^{\pi/2} \ln \sin x \cos^{2n+1} x \, dx = -\frac{(2n)!!}{(2n+1)!!} \sum_{k=0}^n \frac{1}{2k+1} \\ = -\frac{(2n)!!}{2(2n+1)!!} \left[\psi\left(n + \frac{3}{2}\right) - \psi\left(\frac{1}{2}\right) \right] \\ \text{GW (338)(7b)}$$

$$8. \quad \int_0^{\pi/2} \ln \cos x \sin^{2n} x \, dx = -\frac{(2n-1)!!}{2^{n+1} \cdot n!} \frac{\pi}{2} \{ \mathbf{C} + 2 \ln 2 + \psi(n+1) \} \\ \text{BI (306)(8)}$$

$$9. \quad \int_0^{\pi/2} \ln \cos x \cos^{2n} x \, dx = -\frac{(2n-1)!!}{2^n n!} \frac{\pi}{2} \left(\ln 2 + \sum_{k=1}^{2n} \frac{(-1)^k}{k} \right) \quad \text{BI (306)(10)}$$

$$10. \quad \int_0^{\pi/2} \ln \cos x \cos^{2n} x \, dx = \frac{2^{n-1}(n-1)!}{(2n-1)!!} \left[\ln 2 + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k} \right] \quad \text{BI (306)(9)}$$

4.388

$$1. \quad \int_0^{\pi/4} \ln \sin x \frac{\sin^{2n} x}{\cos^{2n+2} x} \, dx = \frac{1}{2n+1} \left[\frac{1}{2} \ln 2 + (-1)^n \frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \right] \quad \text{BI (288)(1)}$$

$$2. \quad \int_0^{\pi/4} \ln \sin x \frac{\sin^{2n-1} x}{\cos^{2n+1} x} \, dx = \frac{1}{4n} \left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k} \right] \quad \text{LI (288)(2)}$$

$$3. \quad \int_0^{\pi/4} \ln \cos x \frac{\sin^{2n} x}{\cos^{2n+2} x} \, dx = \frac{1}{2n+1} \left[-\frac{1}{2} \ln 2 + (-1)^{n+1} \frac{\pi}{4} + \sum_{k=0}^n \frac{(-1)^{k-1}}{2n-2k+1} \right] \quad \text{BI (288)(10)}$$

$$4. \quad \int_0^{\pi/4} \ln \cos x \frac{\sin^{2n-1} x}{\cos^{2n+1} x} \, dx = \frac{1}{4n} \left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=0}^{n-1} \frac{(-1)^k}{n-k} \right] \quad \text{BI (288)(11)}$$

$$5. \quad \int_0^{\pi/2} \ln \sin x \frac{\sin^{p-1} x}{\cos^{p+1} x} \, dx = -\frac{\pi}{2p} \operatorname{cosec} \frac{p\pi}{2} \quad [0 < p < 2] \quad \text{BI (310)(4)}$$

$$6. \quad \int_0^{\pi/2} \ln \sin x \frac{dx}{\tan^{p-1} x \sin 2x} = \frac{1}{4} \frac{\pi}{p-1} \sec \frac{p\pi}{2} \quad [p^2 < 1] \quad \text{BI (310)(3)}$$

4.389

$$1. \quad \int_0^{\pi} \ln \sin x \sin^{2n} 2x \cos 2x \, dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4n+2} \quad \text{BI (330)(9)}$$

$$2. \quad \int_0^{\pi/4} \ln \sin x \cos^n 2x \sin 2x \, dx = -\frac{1}{4(n+1)} \{C + \psi(n+2) + \ln 2\} \quad \text{BI (285)(2)}$$

$$3. \quad \int_0^{\pi/4} \ln \cos x \cos^{\mu-1} 2x \tan 2x \, dx = \frac{1}{4(1-\mu)} \beta(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (286)(2)}$$

$$4. \quad \int_0^{\pi/2} \ln \sin x \sin^{\mu-1} x \cos x \, dx = \int_0^{\pi/2} \ln \cos x \cos^{\mu-1} x \sin x \, dx = -\frac{1}{\mu^2} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (306)(11)}$$

$$5.^3 \quad \int_{-\pi/2}^{\pi/2} \ln \cos x \cos^p x \cos px \, dx = \frac{\pi}{2^{p+1}} [C + \psi(p+1) - 2 \ln 2] \quad [p > -1]$$

$$6. \quad \int_0^{\pi/2} \ln \cos x \cos^{p-1} x \sin px \sin x \, dx = \frac{\pi}{2^{p+2}} \left[C + \psi(p) - \frac{1}{p} - 2 \ln 2 \right] \quad [p > 0] \quad \text{BI (306)(12)}$$

4.391

$$1. \quad \int_0^{\pi/4} (\ln \cos 2x)^n \cos^{p-1} 2x \tan x \, dx = \int_0^{\pi/4} (\ln \sin 2x)^n \sin^{p-1} 2x \tan \left(\frac{\pi}{4} - x \right) dx = \frac{1}{2} \beta^{(n)}(p) \quad [p > 0] \quad \text{BI (286)(10), BI (285)(18)}$$

$$2. \quad \int_0^{\pi/4} (\ln \sin 2x)^n \sin^{p-1} 2x \tan \left(\frac{\pi}{4} + x \right) dx = \frac{(-1)^n n!}{2} \zeta(n+1, p) \quad \text{BI (285)(17)}$$

$$3. \quad \int_0^{\pi/4} (\ln \cos 2x)^{2n-1} \tan x \, dx = \frac{1 - 2^{2n-1}}{4n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (286)(7)}$$

$$4. \quad \int_0^{\pi/4} (\ln \cos 2x)^{2n} \tan x \, dx = \frac{2^{2n} 1}{2^{2n+1}} (2n)! \zeta(2n+1) \quad \text{BI (286)(8)}$$

4.392

$$1. \quad \int_0^{\pi/4} \ln(\sin x \cos x) \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[(-1)^{n+1} \frac{\pi}{2} - \ln 2 + \frac{1}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{(-1)^{k-1}}{2n-2k-1} \right] \quad \text{BI (294)(8)}$$

$$2. \quad \int_0^{\pi/4} \ln(\sin x \cos x) \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{2n} \left[(-1)^n \ln 2 - \ln 2 + \frac{1}{2n} + (-1)^n \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right] \quad \text{BI (294)(9)}$$

4.393

$$1. \quad \int_0^{\pi/2} \ln \tan x \sin x \, dx = \ln 2 \quad \text{BI (307)(3)}$$

$$2. \quad \int_0^{\pi/2} \ln \tan x \cos x \, dx = -\ln 2 \quad \text{BI (307)(4)}$$

$$3. \quad \int_0^{\pi/2} \ln \tan x \sin^2 x \, dx = -\int_0^{\pi/2} \ln \tan x \cos^2 x \, dx = \frac{\pi}{4} \quad \text{BI (307)(5, 6)}$$

$$4. \quad \int_0^{\pi/4} \frac{\ln \tan x}{\cos 2x} dx = -\frac{\pi^2}{8} \quad \text{GW (338)(10b)a}$$

$$5. \quad \int_0^{\pi/2} \sin x \ln \cot \frac{x}{2} dx = \ln 2 \quad \text{LO III 290}$$

4.394

$$1. \quad \int_0^{\pi/2} \frac{\ln \tan x \, dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{2(1-a^2)} \ln \frac{1-a}{1+a} \quad [a^2 < 1]$$

$$= \frac{\pi}{2(a^2-1)} \ln \frac{a-1}{a+1} \quad [a^2 > 1] \quad \text{BI (321)(15)}$$

$$2. \quad \int_0^{\pi/2} \frac{\ln \tan x \cos 2x \, dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{4a} \frac{1+a^2}{1-a^2} \ln \frac{1-a}{1+a} \quad [a^2 < 1]$$

$$= \frac{\pi}{4a} \frac{a^2+1}{a^2-1} \ln \frac{a-1}{a+1} \quad [a^2 > 1] \quad \text{BI (321)(16)}$$

$$3. \quad \int_0^{\pi} \frac{\ln \tan bx \, dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^b}{1 + a^b} \quad [0 < a < 1, \quad b > 0] \quad \text{BI (331)(24)}$$

$$4. \quad \int_0^{\pi} \frac{\ln \tan bx \cos x \, dx}{1 - 2a \cos 2x + a^2} = 0 \quad [0 < a < 1] \quad \text{BI (331)(25)}$$

$$5. \quad \int_0^{\pi/4} \ln \tan x \frac{\cos 2x \, dx}{1 - a \sin 2x} = -\frac{\arcsin a}{4a} (\pi + \arcsin a) \quad [a^2 \leq 1] \quad \text{BI (291)(2,3)}$$

$$6. \quad \int_0^{\pi/4} \ln \tan x \frac{\cos 2x \, dx}{1 - a^2 \sin^2 2x} = -\frac{\pi}{4a} \arcsin a \quad [a^2 < 1] \quad \text{BI (291)(9)}$$

$$7. \quad \int_0^{\pi/4} \ln \tan x \frac{\cos 2x \, dx}{1 + a^2 \sin^2 2x} = -\frac{\pi}{4a} \operatorname{arcsinh} a = -\frac{\pi}{4a} \ln \left(a + \sqrt{1 + a^2} \right) \quad [a^2 < 1] \quad \text{BI (291)(10)}$$

$$8. \quad \int_0^u \frac{\sin x \ln \cot \frac{x}{2}}{1 - \cos^2 \alpha \sin^2 x} \, dx = \operatorname{cosec} 2\alpha \left\{ \frac{\pi}{2} \ln 2 + L(\varphi - \alpha) - L(\varphi + \alpha) - L\left(\frac{\pi}{2} - 2\alpha\right) \right\} \\ [\tan \varphi = \cot \alpha \cos u; \quad 0 < u < \pi] \quad \text{LO III 290}$$

$$9. \quad \int_0^{\pi/4} \frac{\ln \tan x \sin 2x \, dx}{1 - \cos^2 t \sin^2 2x} = \operatorname{cosec} 2t \left[L\left(\frac{\pi}{2} - t\right) - \left(\frac{\pi}{2} - t\right) \ln 2 \right] \quad \text{LO III 290a}$$

4.395

$$1. \quad \int_0^{\pi/2} \frac{\ln \tan x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = -\ln k' \, K(k) \quad \text{BI (322)(11)}$$

$$2. \quad \int_u^{\pi/4} \frac{\ln \tan x \sin 4x \, dx}{(\sin^2 u + \tan^2 v \sin^2 2x) \sqrt{\sin^2 2x - \sin^2 u}} = -\frac{\pi}{2} \frac{\cos^2 v}{\sin u \sin v} \ln \frac{\sin v + \sqrt{1 - \cos^2 u \cos^2 v}}{\sin u (1 + \sin v)} \\ \left[0 < u < \frac{\pi}{2}, \quad 0 < v < \frac{\pi}{2} \right] \quad \text{LO III 285a}$$

4.396

$$1. \quad \int_0^{\pi/2} \ln(a \tan x) \sin^{\mu-1} 2x \, dx = 2^{\mu-2} \ln a \frac{\left\{ \Gamma\left(\frac{a}{2}\right) \right\}^2}{\Gamma(a)} \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{LI (307)(8)}$$

$$2. \quad \int_0^{\pi/2} \ln \tan x \cos^{2(\mu-1)x} \, dx = -\frac{\sqrt{\pi}}{4} \frac{\Gamma\left(u - \frac{1}{2}\right)}{\Gamma(\mu)} \left[C + \psi\left(\frac{2\mu-1}{2}\right) + \ln 4 \right] \\ [\operatorname{Re} \mu > \frac{1}{2}] \quad \text{BI (307)(9)}$$

$$3. \quad \int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cot x \sin[(q+1)x] \, dx = -\frac{\pi}{2} [C + \psi(q+1)] \\ [q > -1] \quad \text{BI (307)(11)}$$

$$4. \quad \int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cos[(q+1)x] \, dx = -\frac{\pi}{2q} \quad [q > 0] \quad \text{BI (307)(10)}$$

$$5. \quad \int_0^{\pi/4} (\ln \tan x)^n \tan^p x \, dx = \frac{1}{2^{n+1}} B^{(n)} \left(\frac{p+1}{2} \right) \quad [p > -1] \quad \text{LI (286)(22)}$$

$$6. \quad \int_0^{\pi/2} (\ln \tan x)^{2n-1} \frac{dx}{\cos 2x} = \frac{1-2^{2n}}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (312)(6)}$$

$$7. \quad \int_0^{\pi/4} \ln \tan x \tan^{2n+1} x \, dx = \frac{(-1)^{n+1}}{4} \left[\frac{\pi^2}{12} + \sum_{k=1}^n \frac{(-1)^k}{k^2} \right] \quad \text{GW (338)(8a)}$$

4.397

$$1. \quad \int_0^{\pi/2} \ln(1 + p \sin x) \frac{dx}{\sin x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p^2) \quad [p^2 < 1] \quad \text{BI (313)(1)}$$

$$2. \quad \int_0^{\pi/2} \ln(1 + p \cos x) \frac{dx}{\cos x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p)^2 \quad [p^2 < 1] \quad \text{BI (313)(8)}$$

$$3. \quad \int_0^{\pi} \ln(1 + p \cos x) \frac{dx}{\cos x} = \pi \arcsin p \quad [p^2 < 1] \quad \text{BI (331)(1)}$$

$$4. \quad \int_0^{\pi/2} \frac{\cos x \ln(1 + \cos \alpha \cos x)}{1 - \cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2} - \alpha\right) - \alpha \ln \sin \alpha}{\sin \alpha \cos \alpha} \quad \left[0 < \alpha < \frac{\pi}{2}\right] \quad \text{LO III 291}$$

$$5. \quad \int_0^{\pi/2} \frac{\cos x \ln(1 - \cos \alpha \cos x)}{1 - \cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2} - \alpha\right) + (\pi - \alpha) \ln \sin \alpha}{\sin \alpha \cos \alpha} \quad \left[0 < \alpha < \frac{\pi}{2}\right] \quad \text{LO III 291}$$

$$6. \quad \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \cos nx \, dx$$

$$= \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \cos nx \, dx$$

$$= -\frac{\pi}{n} a^n \quad [a^2 < 1] \quad \text{BI (330)(11), BI (332)(5)}$$

$$= -\frac{\pi}{na^n} \quad [a^2 > 1] \quad \text{GW (338)(13a)}$$

$$7. \quad \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x \, dx = \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x \, dx$$

$$= \frac{\pi}{2} \left(\frac{a^{n+1}}{n+1} - \frac{a^{n-1}}{n-1} \right) \quad [a^2 > 1] \quad \text{BI (330)(10), BI (332)(4)}$$

$$8. \quad \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x \, dx = \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \cos nx \cos x \, dx$$

$$= -\frac{\pi}{2} \left(\frac{a^{n+1}}{n+1} + \frac{a^{n-1}}{n-1} \right)$$

$$9. \quad \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \cos(2n-1)x \, dx = 0 \quad [a^2 < 1] \quad \text{BI (330)(15)}$$

$$10. \quad \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \sin 2nx \sin x \, dx = 0 \quad [a^2 < 1] \quad \text{BI (330)(13)}$$

$$11. \quad \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \sin(2n-1)x \sin x \, dx = \frac{\pi}{2} \left(\frac{a^n}{n} - \frac{a^{n-1}}{n-1} \right) \\ [a^2 < 1] \quad \text{BI (330)(14)}$$

$$12. \quad \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \cos 2nx \cos x \, dx = 0 \quad [a^2 < 1] \quad \text{BI (330)(16)}$$

$$13. \quad \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \cos(2n-1)x \cos x \, dx = -\frac{\pi}{2} \left(\frac{a^n}{n} + \frac{a^{n-1}}{n-1} \right) \\ [a^2 < 1] \quad \text{BI (330)(17)}$$

$$14. \quad \int_0^{\pi/2} \ln(1 + 2a \cos 2x + a^2) \sin^2 x \, dx = -\frac{a\pi}{4} \quad [a^2 < 1] \\ = \frac{\pi \ln a^2}{4} - \frac{\pi}{4a} \quad [a^2 > 1] \quad \text{BI (309)(22), LI (309)(22)}$$

$$15. \quad \int_0^{\pi/2} \ln(1 + 2a \cos 2x + a^2) \cos^2 x \, dx = \frac{a\pi}{4} \quad [a^2 < 1] \\ = \frac{\pi \ln a^2}{4} + \frac{\pi}{4a} \quad [a^2 > 1] \quad \text{BI (309)(23), LI (309)(23)}$$

$$16. \quad \int_0^\pi \frac{\ln(1 - 2a \cos x + a^2)}{1 - 2b \cos x + b^2} \, dx = \frac{2\pi \ln(1 - ab)}{1 - b^2} \quad [a^2 \leq 1, \quad b^2 < 1] \quad \text{BI (331)(26)}$$

4.398

$$1. \quad \int_0^\pi \ln \frac{1 + 2a \cos x + a^2}{1 - 2a \cos x + a^2} \sin(2n+1)x \, dx = (-1)^n \frac{2\pi a^{2n+1}}{2n+1} \\ [a^2 < 1] \quad \text{BI (330)(18)}$$

$$2. \quad \int_0^{2\pi} \ln \frac{1 - 2a \cos x + a^2}{1 - 2a \cos nx + a^2} \cos mx \, dx = 2\pi \left(\frac{n}{m} a^{m/n} - \frac{a^m}{m} \right) \quad [a^2 \leq 1] \\ = 2\pi \left(\frac{n}{m} a^{-m/n} - \frac{a^{-m}}{m} \right) \quad [a^2 \geq 1] \quad \text{BI (332)(9)}$$

$$3. \quad \int_0^\pi \ln \frac{1 + 2a \cos 2x + a^2}{1 + 2a \cos 2nx + a^2} \cot x \, dx = 0 \quad \text{BI (331)(5), LI(331)(5)}$$

4.399

$$1. \quad \int_0^{\pi/2} \ln(1 + a \sin^2 x) \sin^2 x \, dx = \frac{\pi}{2} \left(\ln \frac{1 + \sqrt{1+a}}{2} - \frac{1}{2} \frac{1 - \sqrt{1+a}}{1 + \sqrt{1+a}} \right) \\ [a > -1] \quad \text{BI (309)(14)}$$

$$2. \quad \int_0^{\pi/2} \ln(1 + a \sin^2 x) \cos^2 x \, dx = \frac{\pi}{2} \left(\ln \frac{1 + \sqrt{1+a}}{2} + \frac{1}{2} \frac{1 - \sqrt{1+a}}{1 + \sqrt{1+a}} \right) \quad [a > -1] \quad \text{BI (309)(15)}$$

$$3. \quad \int_0^{\pi/2} \frac{\ln(1 - \cos^2 \beta \cos^2 x)}{1 - \cos^2 \alpha \cos^2 x} \, dx = -\frac{\pi}{\sin \alpha} \ln \frac{1 + \sin \alpha}{\sin \alpha + \sin \beta} \quad \left[0 < \beta < \frac{\pi}{2}, \quad 0 < \alpha < \frac{\pi}{2} \right] \quad \text{LO III 285}$$

4.411

$$1. \quad \int_0^{\pi} \ln \frac{1 + \sin x}{1 + \cos \lambda \sin x} \frac{dx}{\sin x} = \lambda^2 \quad [\lambda^2 < \pi^2] \quad \text{BI (331)(2)}$$

$$2. \quad \int_0^{\pi/2} \ln \frac{p + q \sin ax}{p - q \sin ax} \frac{dx}{\sin ax} = \int_0^{\pi/2} \ln \frac{p + q \cos ax}{p - q \cos ax} \frac{dx}{\cos ax} = \int_0^{\pi/2} \ln \frac{p + q \tan ax}{p - q \tan ax} \frac{dx}{\tan ax} = \pi \arcsin \frac{q}{p} \quad [p > q > 0] \quad \text{FI II 695a, BI (315)(5, 13,17)a}$$

$$3. \quad \int_0^{\pi/2} \frac{\cos x}{1 - \cos^2 \alpha \cos^2 x} \ln \frac{1 + \cos \beta \cos x}{1 - \cos \beta \cos x} \, dx = \frac{2\pi}{\sin 2\alpha} \ln \frac{\cos \frac{\alpha - \beta}{2}}{\sin \frac{\alpha + \beta}{2}} \quad \left[0 < \alpha \leq \beta < \frac{\pi}{2} \right] \quad \text{LO III 284}$$

4.412

$$1. \quad \int_0^{\pi/4} \ln \tan \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\sin 2x} = \pm \frac{\pi^2}{8} \quad \text{BI (293)(1)}$$

$$2. \quad \int_0^{\pi/4} \ln \tan \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\tan 2x} = \pm \frac{\pi^2}{16} \quad \text{BI (293)(2)}$$

$$3. \quad \int_0^{\pi/4} \ln \tan \left(\frac{\pi}{4} \pm x \right) (\ln \tan x)^{2n} \frac{dx}{\sin 2x} = \pm \frac{2^{2n+2} - 1}{4(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}| \quad \text{BI (294)(24)}$$

$$4. \quad \int_0^{\pi/4} \ln \tan \left(\frac{\pi}{4} \pm x \right) (\ln \tan x)^{2n-1} \frac{dx}{\sin 2x} = \pm \frac{1 - 2^{2n+1}}{2^{2n+2} n} (2n)! \zeta(2n+1) \quad \text{BI (294)(25)}$$

$$5. \quad \int_0^{\pi/4} \ln \tan \left(\frac{\pi}{4} \pm x \right) (\ln \sin 2x)^{n-1} \frac{dx}{\tan 2x} = \frac{(-1)^{n-1}}{2} (n-1)! \zeta(n+1) \quad \text{LI (294)(20)}$$

4.413

$$1. \quad \int_0^{\pi/2} \ln(p^2 + q^2 \tan^2 x) \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{ab} \ln \frac{ap + bq}{a} \quad [a > 0, \quad b > 0, \quad p > 0, \quad q > 0] \quad \text{BI (318)(1-4)a}$$

$$2. \quad \int_0^{\pi/2} \ln(1 + q^2 \tan^2 x) \frac{1}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} = \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{p^2 - r^2}{pr} \ln \left(1 + \frac{qr}{p} \right) + \frac{t^2 - s^2}{st} \ln \left(1 + \frac{qt}{s} \right) \right\} \quad [q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0] \quad \text{BI (320)(18)}$$

$$\begin{aligned}
3. \quad \int_0^{\pi/2} \ln(1 + q^2 \tan^2 x) \frac{\sin^2 x}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} \\
= \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{t}{s} \ln \left(1 + \frac{qr}{p} \right) - \frac{r}{p} \ln \left(1 + \frac{qt}{s} \right) \right\} \\
[q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0] \quad \text{BI (320)(20)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^{\pi/2} \ln(1 + q^2 \tan^2 x) \frac{\cos^2 x}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} \\
= \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{p}{r} \ln \left(1 + \frac{qr}{p} \right) - \frac{s}{t} \ln \left(1 + \frac{qt}{s} \right) \right\} \\
[q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0] \quad \text{BI (320)(21)}
\end{aligned}$$

$$5. \quad \int_0^{\pi} \frac{\ln \tan rx \, dx}{1 - 2p \cos x + p^2} = \frac{\pi}{1 - p^2} \ln \frac{1 - p^{2r}}{1 + p^{2r}} \quad [p^2 < 1] \quad \text{BI (331)(12)}$$

4.414

$$1. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \ln k' \mathbf{K}(k) \quad \text{BI (323)(1)}$$

$$\begin{aligned}
2. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \} \\
\text{BI (323)(3)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\cos^2 x}{dx} \sqrt{1 - k^2 \sin^2 x} = \frac{1}{k^2} \left[(1 + k'^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \right] \\
\text{BI (323)(6)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)^3}} = \frac{1}{k'^2} [(k^2 - 2) \mathbf{K}(k) + (2 + \ln k') \mathbf{E}(k)] \\
\text{BI (323)(9)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x}{dx} \sqrt{(1 - k^2 \sin^2 x)^3} \\
= \frac{1}{k^2 k'^2} \left[(2 + \ln k') \mathbf{E}(k) - (1 + k'^2 + k'^2 \ln k') \mathbf{K}(k) \right] \\
\text{BI (323)(10)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\cos^2 x \, dx}{\sqrt{(1 - k^2 \sin^2 x)^3}} = \frac{1}{k^2} \left[(1 + k'^2 + \ln k') \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k) \right] \\
\text{BI (323)(16)}
\end{aligned}$$

$$7. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} \, dx = (1 + k'^2) \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \quad \text{BI (324)(18)}$$

$$8. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \sin^2 x \sqrt{1 - k^2 \sin^2 x} dx = \frac{1}{9k^2} \left\{ \left(-2 + 11k^2 - 6k^4 + 3k'^2 \ln k' \right) \mathbf{K}(k) \right. \\ \left. + [2 - 10k^2 - 3(1 - 2k^2) \ln k'] \mathbf{E}(k) \right\} \quad \text{BI (324)(20)}$$

$$9. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \cos^2 x \sqrt{1 - k^2 \sin^2 x} dx = \frac{1}{9k^2} \left\{ \left(2 + 7k^2 - 3k^4 - 3k'^2 \ln k' \right) \mathbf{K}(k) \right. \\ \left. - [2 + 8k^2 - 3(1 + k^2) \ln k'] \mathbf{E}(k) \right\} \quad \text{BI (324)(21), LI (324)(21)}$$

$$10. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x dx}{\sqrt{(1 - k^2 \sin^2 x)^{2n+1}}} = \frac{2}{(2n-1)^2 k^2} \left\{ [1 + (2n-1) \ln k'] k'^{1-2n} - 1 \right\} \\ \text{BI (324)(17)}$$

4.415

$$1. \quad \int_0^\infty \ln x \sin ax^2 dx = -\frac{1}{4} \sqrt{\frac{\pi}{2a}} \left(\ln 4a + \mathbf{C} - \frac{\pi}{2} \right) \quad [a > 0] \quad \text{GW (338)(19)}$$

$$2. \quad \int_0^\infty \ln x \cos ax^2 dx = -\frac{1}{4} \sqrt{\frac{\pi}{2a}} \left(\ln 4a + \mathbf{C} - \frac{\pi}{2} \right) \quad [a > 0] \quad \text{GW (338)(19)}$$

4.416

$$1. \quad \int_0^{\pi/2} \frac{\cos x \ln \left(1 + \sqrt{\sin^2 \beta - \cos^2 \beta \tan^2 \alpha \sin^2 x} \right)}{1 - \sin^2 \alpha \cos^2 x} dx \\ = \operatorname{cosec} 2\alpha \{ (2\alpha + 2\gamma - \pi) \ln \cos \beta + 2L(\alpha) - 2L(\gamma) + L(\alpha + \gamma) - L(\alpha - \gamma) \} \\ \left[\cos \gamma = \frac{\sin \alpha}{\sin \beta}; \quad 0 < \alpha < \beta < \frac{\pi}{2} \right] \quad \text{LO III 291}$$

$$2. \quad \int_0^{\pi/2} \frac{\cos x \ln \left(1 - \sqrt{\sin^2 \beta - \cos^2 \beta \tan^2 \alpha \sin^2 x} \right)}{1 - \sin^2 \alpha \cos^2 x} dx \\ = \operatorname{cosec} 2\alpha \{ (\pi + 2\alpha - 2\gamma) \ln \cos \beta + 2L(\alpha) + 2L(\gamma) - L(\alpha + \gamma) + L(\alpha - \gamma) \} \\ \left[\cos \gamma = \frac{\sin \alpha}{\sin \beta}; \quad 0 < \alpha < \beta < \frac{\pi}{2} \right] \quad \text{LO III 291}$$

$$3. \quad \int_\beta^{\pi/2} \frac{\ln \left(\sin x + \sqrt{\sin^2 x - \sin^2 \beta} \right)}{1 - \cos^2 \alpha \cos^2 x} dx \\ = -\operatorname{cosec} \alpha \left\{ \arctan \left(\frac{\tan \beta}{\sin \alpha} \right) \ln \sin \beta + \frac{\pi}{2} \ln \frac{1 + \sin \alpha}{\sin \alpha + \sqrt{1 - \cos^2 \alpha \cos^2 \beta}} \right\} \\ \left[0 < \alpha < \pi, \quad 0 < \beta < \frac{\pi}{2} \right] \quad \text{LO III 285}$$

$$4.^7 \quad \int_0^{\pi/4} \ln \tan x (\ln \cos 2x)^{n-1} \tan 2x dx = \frac{1}{2} (-1)^n (n-1)! \left(1 - 2^{-(n+1)} \right) \zeta(n+1) \\ \text{BI (287)(20)}$$

4.42–4.43 Combinations of logarithms, trigonometric functions, and powers

4.421

$$1. \quad \int_0^{\infty} \ln x \sin ax \frac{dx}{x} = -\frac{\pi}{2} (C + \ln a) \quad [a > 0] \quad \text{FI II 810a}$$

$$2. \quad \int_0^{\infty} \ln ax \sin bx \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln(a\beta') - \frac{\pi}{4} \left[e^{b\beta'} \text{Ei}(-b\beta') + e^{-b\beta'} \text{Ei}(b\beta') \right] \\ [\beta' = \beta \operatorname{sign} \beta; \quad a > 0, \quad b > 0] \\ \text{ET I 76(5), NT 27(10)a}$$

$$3. \quad \int_0^{\infty} \ln ax \cos bx \frac{\beta' dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln(a\beta') + \frac{\pi}{4} \left[e^{b\beta'} \text{Ei}(-b\beta') - e^{-b\beta'} \text{Ei}(b\beta') \right] \\ [\beta' = \beta \operatorname{sign} \beta; \quad a > 0, \quad b > 0] \\ \text{ET I 17(3), NT 27(11)a}$$

$$4. \quad \int_0^{\infty} \ln ax \sin bx \frac{x dx}{x^2 - c^2} = \frac{\pi}{2} \{ -\operatorname{si}(bc) \sin bc + \cos bc [\ln ac - \operatorname{ci}(bc)] \} \\ [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (422)(5)}$$

$$5. \quad \int_0^{\infty} \ln ax \cos bx \frac{dx}{x^2 - c^2} = \frac{\pi}{2c} \{ \sin bc [\operatorname{ci}(bc) - \ln ac] - \cos bc \operatorname{si}(bc) \} \\ [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (422)(6)}$$

4.422

$$1. \quad \int_0^{\infty} \ln x \sin ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^{\mu}} \sin \frac{\mu\pi}{2} \left[\psi(\mu) - \ln a + \frac{\pi}{2} \cot \frac{\mu\pi}{2} \right] \\ [a > 0, \quad |\operatorname{Re} \mu| < 1] \quad \text{BI (411)(5)}$$

$$2. \quad \int_0^{\infty} \ln x \cos ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^{\mu}} \cos \frac{\mu\pi}{2} \left[\psi(\mu) - \ln a - \frac{\pi}{2} \tan \frac{\mu\pi}{2} \right] \\ [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{BI (411)(6)}$$

4.423

$$1. \quad \int_0^{\infty} \ln x \frac{\cos ax - \cos bx}{x} dx = \ln \frac{a}{b} \left(C + \frac{1}{2} \ln ab \right) \quad [a > 0, \quad b > 0] \quad \text{GW (338)(21a)}$$

$$2. \quad \int_0^{\infty} \ln x \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2} [(a-b)(C-1) + a \ln a - b \ln b] \\ [a > 0, \quad b > 0] \quad \text{GW (338)(21b)}$$

$$3. \quad \int_0^{\infty} \ln x \frac{\sin^2 ax}{x^2} dx = -\frac{a\pi}{2} (C + \ln 2a - 1) \quad [a > 0] \quad \text{GW (338)(20b)}$$

4.424

$$1. \quad \int_0^{\infty} (\ln x)^2 \sin ax \frac{dx}{x} = \frac{\pi}{2} C^2 + \frac{\pi^3}{24} + \pi C \ln a + \frac{\pi}{2} (\ln a)^2 \\ [a > 0] \quad \text{ET I 77(9), FI II 810a}$$

$$2.^6 \quad \int_0^\infty (\ln x)^2 \sin ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2} \left[\psi'(\mu) + \psi^2(\mu) + \pi \psi(\mu) \cot \frac{\mu\pi}{2} - 2 \psi(\mu) \ln a \right. \\ \left. - \pi \ln a \cot \frac{\mu\pi}{2} + (\ln a)^2 - \frac{1}{4} \pi^2 \right] \\ [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 77(10)}$$

4.425

$$1. \quad \int_0^\infty \ln(1+x) \cos ax \frac{dx}{x} = \frac{1}{2} \left\{ [\operatorname{si}(a)]^2 + [\operatorname{ci}(a)]^2 \right\} \quad [a > 0] \quad \text{ET I 18(8)}$$

$$2. \quad \int_0^\infty \ln^2 \left(\frac{b+x}{b-x} \right) \cos ax \frac{dx}{x} = -2\pi \operatorname{si}(ab) \quad [a \geq 0, \quad b > 0] \quad \text{ET I 18(11)}$$

$$3. \quad \int_0^\infty \ln(1+b^2x^2) \sin ax \frac{dx}{x} = -\pi \operatorname{Ei} \left(-\frac{a}{b} \right) \quad [a > 0, \quad b > 0] \\ \text{GW (338)(24), ET I 77(14)}$$

$$4. \quad \int_0^1 \ln(1-x^2) \cos(p \ln x) \frac{dx}{x} = \frac{1}{2p^2} + \frac{\pi}{2p} \coth \frac{p\pi}{2} \quad \text{LI (309)(1)a}$$

4.426

$$1.^{11} \quad \int_0^\infty x \ln \frac{b^2+x^2}{c^2+x^2} \sin ax dx = \frac{\pi}{a^2} [(1+ac)e^{-ac} - (1+ab)e^{-ab}] \\ [b \geq 0, \quad c \geq 0, \quad a > 0] \quad \text{GW (338)(23)}$$

$$2. \quad \int_0^\infty \ln \frac{b^2x^2+p^2}{c^2x^2+p^2} \sin ax \frac{dx}{x} = \pi \left[\operatorname{Ei} \left(-\frac{ap}{c} \right) - \operatorname{Ei} \left(-\frac{ap}{b} \right) \right] \\ [b > 0, \quad c > 0, \quad p > 0, \quad a > 0] \\ \text{ET I 77(15)}$$

$$4.427 \quad \int_0^\infty \ln \left(x + \sqrt{\beta^2 + x^2} \right) \frac{\sin ax}{\sqrt{\beta^2 + x^2}} dx = \frac{\pi}{2} K_0(a\beta) + \frac{\pi}{2} \ln(\beta) [I_0(a\beta) - \mathbf{L}(a\beta)] \\ [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 77(16)}$$

4.428

$$1. \quad \int_0^\infty \ln \cos^2 ax \frac{\cos bx}{x^2} dx = \pi b \ln 2 - a\pi \quad [a > 0, \quad b > 0] \quad \text{ET I 22(29)}$$

$$2. \quad \int_0^\infty \ln(4 \cos^2 ax) \frac{\cos bx}{x^2 + c^2} dx = \frac{\pi}{c} \cosh(bc) \ln(1 + e^{-2ac}) \\ \left[a < b < 2a < \frac{\pi}{c} \right] \quad \text{ET I 22(30)}$$

$$3. \quad \int_0^\infty \ln \cos^2 ax \frac{\sin bx}{x(1+x^2)} dx = \pi \ln(1 + e^{-2a}) \sinh b - \pi \ln 2 (1 - e^{-b}) \\ [a > 0, \quad b > 0] \quad \text{ET I 82(36)}$$

$$4. \quad \int_0^\infty \ln \cos^2 ax \frac{\cos bx}{x^2(1+x^2)} dx = -\pi \ln(1 + e^{-2a}) \cosh b + (b + e^{-b}) \pi \ln 2 - a\pi \\ [a > 0, \quad b > 0] \quad \text{ET I 22(31)}$$

$$4.429 \quad \int_0^1 \frac{(1+x)x}{\ln x} \sin(\ln x) dx = \frac{\pi}{4} \quad \text{BI (326)(2)a}$$

4.431

$$1. \quad \int_0^\infty \ln(2 \pm 2 \cos x) \frac{\sin bx}{x^2 + c^2} x \, dx = -\pi \sinh(bc) \ln(1 \pm e^{-c})$$

$$[b > 0, \quad c > 0] \quad \text{ET I 22(32)}$$

$$2. \quad \int_0^\infty \ln(2 \pm 2 \cos x) \frac{\cos bx}{x^2 + c^2} \, dx = \frac{\pi}{c} \cosh(bc) \ln(1 \pm e^{-c})$$

$$[b > 0, \quad c > 0] \quad \text{ET I 22(32)}$$

$$3. \quad \int_0^\infty \ln(1 + 2a \cos x + a^2) \frac{\sin bx}{x} \, dx = -\frac{\pi}{2} \sum_{k=1}^{[b]} \frac{(-a)^k}{k} [1 + \operatorname{sign}(b - k)]$$

$$[0 < a < 1, \quad b > 0] \quad \text{ET I 82(25)}$$

$$4. \quad \int_0^\infty \ln(1 - 2a \cos x + a^2) \frac{\cos bx}{x^2 + c^2} \, dx = \frac{\pi}{c} \ln(1 - ae^{-c}) \cosh(bc) + \frac{\pi}{c} \sum_{k=1}^{[b]} \frac{a^k}{k} \sinh[c(b - k)]$$

$$[|a| < 1, \quad b > 0, \quad c > 0] \quad \text{ET I 22(33)}$$

4.432

$$1. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \ln k' \mathbf{K}(k)$$

$$\text{BI ((412, 414))(4)}$$

$$2. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} x \, dx$$

$$= \frac{1}{k^2} \{ \pi k' (1 - \ln k') + (2 - k^2) \mathbf{K}(k) - (4 - \ln k') \mathbf{E}(k) \}$$

$$\text{BI (426)(3)}$$

$$3. \quad \int_0^{\pi/2} \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} x \, dx = \frac{1}{k^2} \{ -\pi - (2 - k^2) \mathbf{K}(k) + (4 - \ln k') \mathbf{E}(k) \}$$

$$\text{BI (426)(6)}$$

$$4. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \right\}$$

$$\text{BI (412)(5)}$$

$$5. \quad \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \}$$

$$\text{BI (414)(5)}$$

$$\begin{aligned}
6. \quad \int_0^\infty \ln(1 \pm k \sin^2 x) \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} &= \int_0^\infty \ln(1 \pm k \cos^2 x) \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} \\
&= \int_0^\infty \ln(1 \pm k \sin^2 x) \frac{\tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} \\
&= \int_0^\infty \ln(1 \pm k \cos^2 x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} \\
&= \int_0^\infty \ln(1 \pm k \sin^2 2x) \frac{\tan x}{\sqrt{1 - k^2 \sin^2 2x}} \frac{dx}{x} \\
&= \int_0^\infty \ln(1 \pm k^2 \cos^2 2x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 2x}} \frac{dx}{x} \\
&= \frac{1}{2} \ln \frac{2(1 \pm k)}{\sqrt{k}} \mathbf{K}(k) - \frac{\pi}{8} \mathbf{K}(k')
\end{aligned}$$

BI (413)(1-6), BI (415)(1-6)

$$7. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^3 x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \}$$

BI (412)(6)

$$8. \quad \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^3 x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \}$$

BI (414)(6)a

$$9. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \}$$

BI (412)(7)

$$10. \quad \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \}$$

BI (414)(7)

$$11. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \ln k' \mathbf{K}(k)$$

BI ((412, 414))(9)

$$12. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x \tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \}$$

BI (412)(8)

$$13. \quad \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^2 x \tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \}$$

BI (414)(8)

$$\begin{aligned}
14. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x} \\
&= \frac{1}{k'^2} \{ (k^2 - 2) \mathbf{K}(k) + (2 + \ln k') \mathbf{E}(k) \}
\end{aligned}$$

BI ((412, 414))(13)

$$15. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \sin^2 x)^3}} x \, dx = \frac{1}{k^2} \left\{ (1 + \ln k') \frac{\pi}{k'} - (2 + \ln k') \mathbf{K}(k) \right\}$$

BI (426)(9)

$$16. \quad \int_0^{\pi/2} \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \cos^2 x)^3}} x \, dx = \frac{1}{k^2} \{ -\pi + (2 + \ln k') \mathbf{K}(k) \}$$

BI (426)(15)

$$17. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^3 x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x}$$

$$= \frac{1}{k^2} \{ (2 - k^2 + \ln k') \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k) \}$$

BI (412)(14), BI(414)(15)

$$18. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^3 x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x}$$

$$= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x}$$

$$= \frac{1}{k^2 k'^2} \left\{ (2 + \ln k') \mathbf{E}(k) - (2 - k^2 + k'^2 \ln k') \mathbf{K}(k) \right\}$$

BI (412)(15), BI(414)(14)

$$19. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos^2 x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^2 x \tan x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x}$$

$$= \frac{1}{k^2} \{ (2 - k^2 + \ln k') \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k) \}$$

BI (412)(16), BI(414)(17)

$$20. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x \tan x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x}$$

$$= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos^2 x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x}$$

$$= \frac{1}{k^2 k'^2} \left\{ (2 + \ln k') \mathbf{E}(k) - (2 - k^2 + k'^2 \ln k') \mathbf{K}(k) \right\}$$

BI (412)(17), BI(414)(16)

$$21. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\tan x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\tan x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x}$$

$$= \frac{1}{k'^2} \{ (k^2 - 2) \mathbf{K}(k) + (2 + \ln k') \mathbf{E}(k) \}$$

BI ((412, 414))(18)

$$22. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} \sin x \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \sqrt{1 - k^2 \cos^2 x} \sin x \frac{dx}{x}$$

$$= (2 - k^2) \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k)$$

BI ((412, 414))(1)

$$\begin{aligned}
 23. \quad & \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} \sin x \cos x \cdot x \, dx \\
 &= \frac{1}{27k^2} \left\{ 3\pi k'^3 (1 - 3 \ln k') + (22k'^2 + 6k^4 - 3k'^2 \ln k') \mathbf{K}(k) \right\} - (2 - k^2) (14 - 6 \ln k') \mathbf{E}(k) \\
 & \qquad \qquad \qquad \text{BI (426)(1)}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & \int_0^{\pi/2} \ln(1 - k^2 \cos^2 x) \sqrt{1 - k^2 \cos^2 x} \sin x \cos x \cdot x \, dx \\
 &= \frac{1}{27k^2} \left\{ -3\pi - (22k'^2 + 6k^4 - 3k'^2 \ln k') \mathbf{K}(k) + (2 - k^2) (14 - 6 \ln k') \mathbf{E}(k) \right\} \\
 & \qquad \qquad \qquad \text{BI (426)(2)}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \int_0^\infty \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} \tan x \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \sqrt{1 - k^2 \cos^2 x} \tan x \frac{dx}{x} \\
 &= (2 - k^2) \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \\
 & \qquad \qquad \qquad ((412, 414))(2)
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & \int_0^\infty \ln(\sin^2 x + k' \cos^2 x) \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \int_0^\infty \ln(\sin^2 x + k' \cos^2 x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} \\
 &= \int_0^\infty \ln(\sin^2 2x + k' \cos^2 2x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 2x}} \frac{dx}{x} \\
 &= \frac{1}{2} \ln \left[\frac{2(\sqrt{k'})^3}{1 + k'} \right] \mathbf{K}(k) \\
 & \qquad \qquad \qquad \text{BI (415)(19-21)}
 \end{aligned}$$

4.44 Combinations of logarithms, trigonometric functions, and exponentials

4.441

$$\begin{aligned}
 1.^7 \quad & \int_0^\infty e^{-qx} \sin px \ln x \, dx = \frac{1}{p^2 + q^2} \left[q \arctan \frac{p}{q} - p \mathbf{C} - \frac{p}{c} \ln(p^2 - q^2) \right] \\
 & \qquad \qquad \qquad [q > 0, \quad p > 0] \qquad \qquad \qquad \text{BI (467)(1)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty e^{-qx} \cos px \ln x \, dx = -\frac{1}{p^2 + q^2} \left[\frac{q}{2} \ln(p^2 + q^2) + p \arctan \frac{p}{q} + q \mathbf{C} \right] \\
 & \qquad \qquad \qquad [q > 0] \qquad \qquad \qquad \text{BI (467)(2)}
 \end{aligned}$$

$$\begin{aligned}
 4.442 \quad & \int_0^{\pi/2} \frac{e^{-p \tan x} \ln \cos x \, dx}{\sin x \cos x} = -\frac{1}{2} [\text{ci}(p)]^2 + \frac{1}{2} [\text{si}(p)]^2 \qquad [\text{Re } p > 0] \\
 & \qquad \qquad \qquad \text{NT 32(11)}
 \end{aligned}$$

4.5 Inverse Trigonometric Functions

4.51 Inverse trigonometric functions

$$\begin{aligned}
 4.511 \quad & \int_0^\infty \text{arccot } px \text{arccot } qx \, dx = \frac{\pi}{2} \left\{ \frac{1}{p} \ln \left(1 + \frac{p}{q} \right) + \frac{1}{q} \ln \left(1 + \frac{q}{p} \right) \right\} \\
 & \qquad \qquad \qquad [p > 0, \quad q > 0] \qquad \qquad \qquad \text{BI (77)(8)}
 \end{aligned}$$

$$4.512 \quad \int_0^\pi \arctan(\cos x) \, dx = 0 \quad \text{BI (345)(1)}$$

4.52 Combinations of arcsines, arccosines, and powers

4.521

$$1. \quad \int_0^1 \frac{\arcsin x}{x} \, dx = \frac{\pi}{2} \ln 2 \quad \text{FI II 614, 623}$$

$$2. \quad \int_0^1 \frac{\arccos x}{1 \pm x} \, dx = \mp \frac{\pi}{2} \ln 2 + 2G \quad \text{BI (231)(7, 8)}$$

$$3. \quad \int_0^1 \arcsin x \frac{x}{1+qx^2} \, dx = \frac{\pi}{2q} \ln \frac{2\sqrt{1+q}}{1+\sqrt{1+q}} \quad [q > -1] \quad \text{BI (231)(1)}$$

$$4. \quad \int_0^1 \arcsin x \frac{x}{1-p^2x^2} \, dx = \frac{\pi}{2p^2} \ln \frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}} \quad [p^2 < 1] \quad \text{LI (231)(3)}$$

$$5. \quad \int_0^1 \arccos x \frac{dx}{\sin^2 \lambda - x^2} = 2 \operatorname{cosec} \lambda \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2} \quad \text{BI (231)(10)}$$

$$6. \quad \int_0^1 \arcsin x \frac{dx}{x(1+qx^2)} = \frac{\pi}{2} \ln \frac{1+\sqrt{1+q}}{\sqrt{1+q}} \quad [q > -1] \quad \text{BI (235)(10)}$$

$$7. \quad \int_0^1 \arcsin x \frac{x}{(1+qx^2)^2} \, dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \quad [q > -1] \quad \text{BI (234)(2)}$$

$$8. \quad \int_0^1 \arccos x \frac{x}{(1+qx^2)^2} \, dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \quad [q > -1] \quad \text{BI (234)(4)}$$

4.522

$$1. \quad \int_0^1 x \sqrt{1-k^2x^2} \arccos x \, dx = \frac{1}{9k^2} \left[\frac{3}{2}\pi + k'^2 \mathbf{K}(k) - 2(1+k'^2) \mathbf{E}(k) \right] \quad \text{BI (236)(9)}$$

$$2. \quad \int_0^1 x \sqrt{1-k^2x^2} \arcsin x \, dx = \frac{1}{9k^2} \left[-\frac{3}{2}\pi k'^3 - k'^2 \mathbf{K}(k) + 2(1+k'^2) \mathbf{E}(k) \right] \quad \text{BI (236)(1)}$$

$$3. \quad \int_0^1 x \sqrt{k'^2+k^2x^2} \arcsin x \, dx = \frac{1}{9k^2} \left[\frac{3}{2}\pi + k'^2 \mathbf{K}(k) - 2(1+k'^2) \mathbf{E}(k) \right] \quad \text{BI(236)(5)}$$

$$4. \quad \int_0^1 \frac{x \arcsin x}{\sqrt{1-k^2x^2}} \, dx = \frac{1}{k^2} \left[-\frac{\pi}{2}k' + \mathbf{E}(k) \right] \quad \text{BI (237)(1)}$$

$$5. \quad \int_0^1 \frac{x \arccos x}{\sqrt{1-k^2x^2}} \, dx = \frac{1}{k^2} \left[\frac{\pi}{2} - \mathbf{E}(k) \right] \quad \text{BI (240)(1)}$$

$$6. \quad \int_0^1 \frac{x \arcsin x}{\sqrt{k'^2+k^2x^2}} \, dx = \frac{1}{k^2} \left[\frac{\pi}{2} - \mathbf{E}(k) \right] \quad \text{BI (238)(1)}$$

$$7. \quad \int_0^1 \frac{x \arccos x}{\sqrt{k'^2+k^2x^2}} \, dx = \frac{1}{k^2} \left[-\frac{\pi}{2}k' + \mathbf{E}(k) \right] \quad \text{BI (241)(1)}$$

$$8. \quad \int_0^1 \frac{x \arcsin x \, dx}{(x^2 - \cos^2 \lambda) \sqrt{1-x^2}} = \frac{2}{\sin \lambda} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2} \quad \text{BI (243)(11)}$$

$$9. \quad \int_0^1 \frac{x \arcsin kx}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx = -\frac{\pi}{2k} \ln k' \quad \text{BI (239)(1)}$$

$$10. \quad \int_0^1 \frac{x \arccos kx}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx = \frac{\pi}{2k} \ln(1+k) \quad \text{BI (242)(1)}$$

4.523

$$1. \quad \int_0^1 x^{2n} \arcsin x \, dx = \frac{1}{2n+1} \left[\frac{\pi}{2} - \frac{2^n n!}{(2n+1)!!} \right] \quad \text{BI (229)(1)}$$

$$2. \quad \int_0^1 x^{2n-1} \arcsin x \, dx = \frac{\pi}{4n} \left[1 - \frac{(2n-1)!!}{2^n n!} \right] \quad \text{BI (229)(2)}$$

$$3. \quad \int_0^1 x^{2n} \arccos x \, dx = \frac{2^n n!}{(2n+1)(2n+1)!!} \quad \text{BI (229)(4)}$$

$$4. \quad \int_0^1 x^{2n-1} \arccos x \, dx = \frac{\pi}{4n} \frac{(2n-1)!!}{2^n n!} \quad \text{BI (229)(5)}$$

$$5. \quad \int_{-1}^1 (1-x^2)^n \arccos x \, dx = \pi \frac{2^n n!}{(2n+1)!!} \quad \text{BI (254)(2)}$$

$$6. \quad \int_{-1}^1 (1-x^2)^{n-\frac{1}{2}} \arccos x \, dx = \frac{\pi^2}{2} \frac{(2n-1)!!}{2^n n!} \quad \text{BI (254)(3)}$$

4.524

$$1. \quad \int_0^1 (\arcsin x)^2 \frac{dx}{x^2 \sqrt{1-x^2}} = \pi \ln 2 \quad \text{BI (243)(13)}$$

$$2. \quad \int_0^1 (\arccos x)^2 \frac{dx}{(\sqrt{1-x^2})^3} = \pi \ln 2 \quad \text{BI (244)(9)}$$

4.53–4.54 Combinations of arctangents, arccotangents, and powers**4.531**

$$1. \quad \int_0^1 \frac{\arctan x}{x} \, dx = \int_1^{\infty} \frac{\operatorname{arccot} x}{x} \, dx = \mathbf{G} \quad \text{FI II 482, BI (253)(8)}$$

$$2. \quad \int_0^{\infty} \frac{\operatorname{arccot} x}{1 \pm x} \, dx = \pm \frac{\pi}{4} \ln 2 + \mathbf{G} \quad \text{BI (248)(6, 7)}$$

$$3. \quad \int_0^1 \frac{\operatorname{arccot} x}{x(1+x)} \, dx = -\frac{\pi}{8} \ln 2 + \mathbf{G} \quad \text{BI (235)(11)}$$

$$4. \quad \int_0^{\infty} \frac{\arctan x}{1-x^2} \, dx = -\mathbf{G}. \quad \text{BI (248)(2)}$$

$$5. \quad \int_0^1 \arctan qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{(1+p)^2}{1+q^2} + \frac{q^2-p}{(1+p)(p^2+q^2)} \arctan q$$

[$p > -1$] BI (243)(7)

$$6. \quad \int_0^1 \operatorname{arccot} qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{1+q^2}{(1+p)^2} + \frac{p}{p^2+q^2} \arctan q + \frac{1}{1+p} \operatorname{arccot} q$$

[$p > -1$] BI (234)(10)

$$7. \quad \int_0^1 \frac{\arctan x}{x(1+x^2)} dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G}$$

BI (235)(12)

$$8. \quad \int_0^\infty \frac{x \arctan x}{1+x^4} dx = \frac{\pi^2}{16}$$

BI (248)(3)

$$9. \quad \int_0^\infty \frac{x \arctan x}{1-x^4} dx = -\frac{\pi}{8} \ln 2$$

BI (248)(4)

$$10.^{11} \quad \int_0^\infty \frac{x \operatorname{arccot} x}{1-x^4} dx = \frac{\pi}{8} \ln 2$$

BI (248)(12)

$$11. \quad \int_0^\infty \frac{\operatorname{arccot} x}{x\sqrt{1+x^2}} dx = \int_0^\infty \frac{\operatorname{arccot} x}{\sqrt{1+x^2}} dx = 2\mathbf{G}$$

BI (251)(3, 10)

$$12. \quad \int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(1+\sqrt{2})$$

FI II 694

$$13. \quad \int_0^1 \frac{x \arctan x dx}{\sqrt{(1+x^2)(1+k'^2x^2)}} = \frac{1}{k^2} \left[F\left(\frac{\pi}{4}, k\right) - \frac{\pi}{2\sqrt{2(1+k'^2)}} \right]$$

BI (294)(14)

4.532

$$1. \quad \int_0^1 x^p \arctan x dx = \frac{1}{2(p+1)} \left[\frac{\pi}{2} - \beta\left(\frac{p}{2}+1\right) \right]$$

[$p > -2$] BI (229)(7)

$$2. \quad \int_0^\infty x^p \arctan x dx = \frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2}$$

[$-1 > p > -2$] BI (246)(1)

$$3. \quad \int_0^1 x^p \operatorname{arccot} x dx = \frac{1}{2(p+1)} \left[\frac{\pi}{2} + \beta\left(\frac{p}{2}+1\right) \right]$$

[$p > -1$] BI (229)(8)

$$4. \quad \int_0^\infty x^p \operatorname{arccot} x dx = -\frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2}$$

[$-1 < p < 0$] BI (246)(2)

$$5. \quad \int_0^\infty \left(\frac{x^p}{1+x^{2p}} \right)^{2q} \arctan x \frac{dx}{x} = \frac{\sqrt{\pi^3}}{2^{2q+2}p} \frac{\Gamma(q)}{\Gamma(q+\frac{1}{2})}$$

[$q > 0$] BI (250)(10)

4.533

$$1. \quad \int_0^\infty (1-x \operatorname{arccot} x) dx = \frac{\pi}{4}$$

BI (246)(3)

$$2. \quad \int_0^1 \left(\frac{\pi}{4} - \arctan x \right) \frac{dx}{1-x} = -\frac{\pi}{8} \ln 2 + \mathbf{G}$$

BI (232)(2)

$$3. \quad \int_0^1 \left(\frac{\pi}{4} - \arctan x \right) \frac{1+x}{1-x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G} \quad \text{BI (235)(25)}$$

$$4. \quad \int_0^1 \left(x \operatorname{arccot} x - \frac{1}{x} \arctan x \right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \ln 2 \quad \text{BI (232)(1)}$$

$$\mathbf{4.534} \quad \int_0^\infty (\arctan x)^2 \frac{dx}{x^2 \sqrt{1+x^2}} = \int_0^\infty (\operatorname{arccot} x)^2 \frac{x dx}{\sqrt{1+x^2}} = -\frac{\pi^2}{4} + 4 \mathbf{G} \quad \text{BI (251)(9, 17)}$$

4.535

$$1. \quad \int_0^1 \frac{\arctan px}{1+p^2 x} dx = \frac{1}{2p^2} \arctan p \ln(1+p^2) \quad \text{BI (231)(19)}$$

$$2. \quad \int_0^1 \frac{\operatorname{arccot} px}{1+p^2 x} dx = \frac{1}{p^2} \left\{ \frac{\pi}{4} + \frac{1}{2} \operatorname{arccot} p \right\} \ln(1+p^2) \quad [p > 0] \quad \text{BI (231)(24)}$$

$$3. \quad \int_0^\infty \frac{\arctan qx}{(p+x)^2} dx = -\frac{q}{1+p^2 q^2} \left(\ln pq - \frac{\pi}{2} pq \right) \quad [p > 0, \quad q > 0] \quad \text{BI (249)(1)}$$

$$4. \quad \int_0^\infty \frac{\operatorname{arccot} qx}{(p+x)^2} dx = \frac{q}{1+p^2 q^2} \left(\ln pq + \frac{\pi}{2pq} \right) \quad [p > 0, \quad q > 0] \quad \text{BI (249)(8)}$$

$$5. \quad \int_0^\infty \frac{x \operatorname{arccot} px}{q^2 + x^2} dx = \frac{\pi}{2} \ln \frac{1+pq}{pq} \quad [p > 0, \quad q > 0] \quad \text{BI (248)(9)}$$

$$6. \quad \int_0^\infty \frac{x \operatorname{arccot} px dx}{x^2 - q^2} = \frac{\pi}{4} \ln \frac{1+p^2 q^2}{p^2 q^2} \quad [p > 0, \quad q > 0] \quad \text{BI (248)(10)}$$

$$7. \quad \int_0^\infty \frac{\arctan px}{x(1+x^2)} dx = \frac{\pi}{2} \ln(1+p) \quad [p \geq 0] \quad \text{FI II 745}$$

$$8. \quad \int_0^\infty \frac{\arctan px}{x(1-x^2)} dx = \frac{\pi}{4} \ln(1+p^2) \quad [p \geq 0] \quad \text{BI (250)(6)}$$

$$9. \quad \int_0^\infty \arctan qx \frac{dx}{x(p^2+x^2)} = \frac{\pi}{2p^2} \ln(1+pq) \quad [p > 0, \quad q \geq 0] \quad \text{BI (250)(3)}$$

$$10. \quad \int_0^\infty \arctan qx \frac{dx}{x(1-p^2 x^2)} = \frac{\pi}{4} \ln \frac{p^2+q^2}{p^2} \quad [p \geq 0] \quad \text{BI (250)(6)}$$

$$11. \quad \int_0^\infty \frac{x \arctan qx}{(p^2+x^2)^2} dx = \frac{\pi q}{4p(1+pq)} \quad [p > 0, \quad q \geq 0] \quad \text{BI (252)(12)a}$$

$$12. \quad \int_0^\infty \frac{x \operatorname{arccot} qx}{(p^2+x^2)^2} dx = \frac{\pi}{4p^2(1+pq)} \quad [p > 0, \quad q \geq 0] \quad \text{BI (252)(20)a}$$

$$13. \quad \int_0^1 \frac{\arctan qx}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(q + \sqrt{1+q^2}) \quad \text{BI (244)(11)}$$

$$14.^9 \quad \int_{-\infty}^\infty \frac{x \arctan(\alpha x) dx}{(x^2 + \beta^2)(x^2 + \gamma^2)} = \begin{cases} \frac{\pi}{\beta^2 - \gamma^2} \ln \left(\frac{1 + |\alpha\beta|}{1 + |\alpha\gamma|} \right) \operatorname{sign}(\alpha) & \text{for } \beta \neq \gamma \\ \frac{\pi\alpha}{2|\beta|(1 + |\alpha\beta|)} & \text{for } \beta = \gamma \end{cases}$$

for α, β, γ real

$$15.^9 \quad \int_{-\infty}^{\infty} \frac{x \arctan(\alpha/x) dx}{(x^2 + \beta^2)(x^2 + \gamma^2)} = \begin{cases} \frac{\pi}{\beta^2 - \gamma^2} \ln \left(\frac{1 + |\alpha/\gamma|}{1 + |\alpha/\beta|} \right) \operatorname{sign}(\alpha) & (\alpha, \beta, \gamma \text{ real}; \quad \beta \neq \gamma) \\ \frac{\pi\alpha}{2\beta^2(|\beta| + |\alpha|)} & (\beta = \gamma) \end{cases}$$

4.536

$$1. \quad \int_0^{\infty} \arctan qx \arcsin x \frac{dx}{x^2} = \frac{1}{2} q \pi \ln \frac{1 + \sqrt{1 + q^2}}{\sqrt{1 + q^2}} + \frac{\pi}{2} \ln \left(q + \sqrt{1 + q^2} \right) - \frac{\pi}{2} - \arctan q \quad \text{BI (230)(7)}$$

$$2. \quad \int_0^{\infty} \frac{\arctan px - \arctan qx}{x} dx = \frac{\pi}{2} \ln \frac{p}{q} \quad [p > 0, \quad q > 0] \quad \text{FI II 635}$$

$$3. \quad \int_0^{\infty} \frac{\arctan px \arctan qx}{x^2} dx = \frac{\pi}{2} \ln \frac{(p+q)^{p+q}}{p^p q^q} \quad [p > 0, \quad q > 0] \quad \text{FI II 745}$$

4.537

$$1.^8 \quad \int_0^1 \arctan \left(\sqrt{1 - x^2} \right) \frac{dx}{1 - x^2 \cos^2 \lambda} = \frac{\pi}{2 \cos \lambda} \ln \left[\cos \left(\frac{\pi - 4\lambda}{8} \right) \operatorname{cosec} \left(\frac{\pi + 4\lambda}{8} \right) \right] \quad \text{BI (245)(9)}$$

$$2. \quad \int_0^1 \arctan \left(p \sqrt{1 - x^2} \right) \frac{dx}{1 - x^2} = \frac{1}{2} \pi \ln \left(p + \sqrt{1 + p^2} \right) \quad [p > 0] \quad \text{BI (245)(10)}$$

$$3. \quad \int_0^1 \arctan \left(\tan \lambda \sqrt{1 - k^2 x^2} \right) \sqrt{\frac{1 - x^2}{1 - k^2 x^2}} dx = \frac{\pi}{2k^2} \left[E(\lambda, k) - k'^2 F(\gamma, k) \right] - \frac{\pi}{2k^2} \cot \gamma \left(1 - \sqrt{1 - k^2 \sin^2 \gamma} \right) \quad \text{BI (245)(12)}$$

$$4. \quad \int_0^1 \arctan \left(\tan \lambda \sqrt{1 - k^2 x^2} \right) \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} dx = \frac{\pi}{2} E(\lambda, k) - \frac{\pi}{2} \cot \lambda \left(1 - \sqrt{1 - k^2 \sin^2 \lambda} \right) \quad \text{BI (245)(11)}$$

$$5. \quad \int_0^1 \frac{\arctan \left(\tan \lambda \sqrt{1 - k^2 x^2} \right)}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} dx = \frac{\pi}{2} F(\lambda, k) \quad \text{BI (245)(13)}$$

4.538

$$1. \quad \int_0^{\infty} \arctan x^2 \frac{dx}{1 + x^2} = \int_0^{\infty} \arctan x^3 \frac{dx}{1 + x^2} = \int_0^{\infty} \operatorname{arccot} x^2 \frac{dx}{1 + x^2} = \int_0^{\infty} \operatorname{arccot} x^3 \frac{dx}{1 + x^2} = \frac{\pi^2}{8} \quad \begin{array}{l} \text{BI (252)(10, 11)} \\ \text{BI (252)(18, 19)} \end{array}$$

$$2. \quad \int_0^{\infty} \frac{1 - x^2}{x^2} \arctan x^2 dx = \frac{\pi}{2} (\sqrt{2} - 1) \quad \text{BI (244)(10)a}$$

$$4.539 \quad \int_0^{\infty} x^{s-1} \arctan (ae^{-x}) dx = 2^{-s-1} \Gamma(s) a \Phi \left(-a^2, s + 1, \frac{1}{2} \right) \quad \text{ET I 222(47)}$$

$$4.541 \quad \int_0^{\infty} \arctan \left(\frac{p \sin qx}{1 + p \cos qx} \right) \frac{x dx}{1 + x^2} = \frac{\pi}{2} \ln (1 + pe^{-q}) \quad [p > -e^q] \quad \text{BI (341)(14)a}$$

4.55 Combinations of inverse trigonometric functions and exponentials

4.551

$$1.^9 \quad \int_0^1 (\arcsin x) e^{-bx} dx = \frac{\pi}{2b} [I_0(b) - \mathbf{L}_0(b)] - \frac{\pi e^{-b}}{2b} \quad \text{ET I 160(1)}$$

$$2. \quad \int_0^1 x (\arcsin x) e^{-bx} dx = \frac{\pi}{2b^2} [\mathbf{L}_0(b) - I_0(b) + b \mathbf{L}_1(b) - b I_1(b)] + \frac{1}{b} \quad \text{ET I 161(2)}$$

$$3.^9 \quad \int_0^\infty \left(\arctan \frac{x}{a} \right) e^{-bx} dx = \frac{1}{b} [\text{ci}(ab) \sin(ab) - \text{si}(ab) \cos(ab)]$$

$$[\text{Re } b > 0] \quad \text{ET I 161(3)}$$

$$4.^9 \quad \int_0^\infty \left(\text{arccot} \frac{x}{a} \right) e^{-bx} dx = \frac{1}{b} \left[\frac{\pi}{2} - \text{ci}(ab) \sin(ab) + \text{si}(ab) \cos(ab) \right]$$

$$[\text{Re } b > 0] \quad \text{ET I 161(4)}$$

$$4.552 \quad \int_0^\infty \frac{\arctan \frac{x}{q}}{e^{2\pi x} - 1} dx = \frac{1}{2} \left[\ln \Gamma(q) - \left(q - \frac{1}{2} \right) \ln q + q - \frac{1}{2} \ln 2\pi \right]$$

$$[q > 0] \quad \text{WH}$$

$$4.553 \quad \int_0^\infty \left(\frac{2}{\pi} \text{arccot } x - e^{-px} \right) \frac{dx}{x} = C + \ln p \quad [p > 0] \quad \text{NT 66(12)}$$

4.56 A combination of the arctangent and a hyperbolic function

$$4.561 \quad \int_{-\infty}^\infty \frac{\arctan e^{-x}}{\cosh^{2q} px} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\Pi(x)}{\cosh^{2q} px} dx = \frac{\sqrt{\pi^3}}{4p} \frac{\Gamma(q)}{\Gamma(q + \frac{1}{2})}$$

$$[q > 0] \quad \text{LI (282)(10)}$$

4.57 Combinations of inverse and direct trigonometric functions

$$4.571 \quad \int_0^{\pi/2} \arcsin(k \sin x) \frac{\sin x dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{\pi}{2k} \ln k' \quad \text{BI (344)(2)}$$

$$4.572 \quad \int_0^\infty \left(\frac{2}{\pi} \text{arccot } x - \cos px \right) dx = C + \ln p \quad [p > 0] \quad \text{NT 66(12)}$$

4.573

$$1. \quad \int_0^\infty \text{arccot } qx \sin px dx = \frac{\pi}{2p} \left(1 - e^{-\frac{p}{q}} \right) \quad [p > 0, \quad q > 0] \quad \text{BI (347)(1)a}$$

$$2. \quad \int_0^\infty \text{arccot } qx \cos px dx = \frac{1}{2p} \left[e^{-\frac{p}{q}} \text{Ei} \left(\frac{p}{q} \right) - e^{\frac{p}{q}} \text{Ei} \left(-\frac{p}{q} \right) \right]$$

$$[p > 0, \quad q > 0] \quad \text{BI (347)(2)a}$$

$$3. \quad \int_0^\infty \text{arccot } rx \frac{\sin px dx}{1 \pm 2q \cos px + q^2} = \pm \frac{\pi}{2pq} \ln \frac{1 \pm q}{1 \pm qe^{-\frac{p}{r}}} \quad [p^2 < 1, \quad r > 0, \quad p > 0]$$

$$= \pm \frac{\pi}{2pq} \ln \frac{q \pm 1}{q \pm e^{-\frac{p}{r}}} \quad [q^2 > 1, \quad r > 0, \quad p > 0]$$

BI (347)(10)

$$4. \quad \int_0^\infty \operatorname{arccot} px \frac{\tan x \, dx}{q^2 \cos^2 x + r^2 \sin^2 x} = \frac{\pi}{2r^2} \ln \left(1 + \frac{r}{q} \tanh \frac{1}{p} \right) \\ [p > 0, \quad q > 0, \quad r > 0] \quad \text{BI (347)(9)}$$

4.574

$$1. \quad \int_0^\infty \arctan \left(\frac{2a}{x} \right) \sin(bx) \, dx = \frac{\pi}{b} e^{-ab} \sinh(ab) \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 87(8)}$$

$$2.^7 \quad \int_0^\infty \arctan \frac{a}{x} \cos(bx) \, dx = \frac{1}{2b} [e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab)] \\ [a > 0, \quad b > 0] \quad \text{ET I 29(7)}$$

$$3. \quad \int_0^\infty \arctan \left[\frac{2ax}{x^2 + c^2} \right] \sin(bx) \, dx = \frac{\pi}{b} e^{-b\sqrt{a^2 + c^2}} \sinh(ab) \\ [b > 0] \quad \text{ET I 87(9)}$$

$$4. \quad \int_0^\infty \arctan \left(\frac{2}{x^2} \right) \cos(bx) \, dx = \frac{\pi}{b} e^{-b} \sin b \quad [b > 0] \quad \text{ET I 29(8)}$$

4.575

$$1. \quad \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \sin nx \, dx = \frac{\pi}{2n} p^n \quad [p^2 < 1] \quad \text{BI (345)(4)}$$

$$2. \quad \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \sin nx \cos x \, dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} + \frac{p^{n-1}}{n-1} \right) \\ [p^2 < 1] \quad \text{BI (345)(5)}$$

$$3. \quad \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \cos nx \sin x \, dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} - \frac{p^{n-1}}{n-1} \right) \\ [p^2 < 1] \quad \text{BI (345)(6)}$$

4.576

$$1. \quad \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \frac{dx}{\sin x} = \frac{\pi}{2} \ln \frac{1+p}{1-p} \quad [p^2 < 1] \quad \text{BI(346)(1)}$$

$$2. \quad \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \frac{dx}{\tan x} = -\frac{\pi}{2} \ln(1 - p^2) \quad [p^2 < 1] \quad \text{BI(346)(3)}$$

4.577

$$1. \quad \int_0^{\pi/2} \arctan \left(\tan \lambda \sqrt{1 - k^2 \sin^2 x} \right) \frac{\sin^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} \\ = \frac{\pi}{2k^2} \left[F(\lambda, k) - E(\lambda, k) + \cot \lambda \left(1 - \sqrt{1 - k^2 \sin^2 \lambda} \right) \right] \\ \text{BI (344)(4)}$$

$$2. \quad \int_0^{\pi/2} \arctan \left(\tan \lambda \sqrt{1 - k^2 \sin^2 x} \right) \frac{\cos^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} \\ = \frac{\pi}{2k^2} \left[E(\lambda, k) - k'^2 F(\lambda, k) + \cot \lambda \left(\sqrt{1 - k^2 \sin^2 \lambda} - 1 \right) \right] \\ \text{BI (344)(5)}$$

4.58 A combination involving an inverse and a direct trigonometric function and a power

$$4.581^{10} \quad \int_0^\infty \arctan x \cos px \frac{dx}{x} = \int_0^\infty \arctan \frac{x}{p} \cos x \frac{dx}{x} = -\frac{\pi}{2} \operatorname{Ei}(-p) \\ [\operatorname{Re}(p) > 0] \quad \text{ET I 29(3), NT 25(13)}$$

4.59 Combinations of inverse trigonometric functions and logarithms

4.591

$$1. \quad \int_0^1 \arcsin x \ln x \, dx = 2 - \ln 2 - \frac{1}{2}\pi \quad \text{BI (339)(1)}$$

$$2. \quad \int_0^1 \arccos x \ln x \, dx = \ln 2 - 2 \quad \text{BI (339)(2)}$$

$$4.592 \quad \int_0^1 \arccos x \frac{dx}{\ln x} = -\sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{\ln(2k+2)}{2k+1} \quad \text{BI (339)(8)}$$

4.593

$$1. \quad \int_0^1 \arctan x \ln x \, dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{1}{48}\pi^2 \quad \text{BI (339)(3)}$$

$$2. \quad \int_0^1 \operatorname{arccot} x \ln x \, dx = -\frac{1}{48}\pi^2 - \frac{\pi}{4} - \frac{1}{2} \ln 2 \quad \text{BI (339)(4)}$$

$$4.594 \quad \int_0^1 \arctan x (\ln x)^{n-1} (\ln x + n) \, dx = \frac{n!}{(-2)^{n+1}} (2^{-n} - 1) \zeta(n+1) \quad \text{BI (339)(7)}$$

4.6 Multiple Integrals

4.60 Change of variables in multiple integrals

4.601

$$1. \quad \iint_{(\sigma)} f(x, y) \, dx \, dy = \iint_{(\sigma')} f[\varphi(u, v), \psi(u, v)] |\Delta| \, du \, dv$$

where $x = \varphi(u, v)$, $y = \psi(u, v)$, and $\Delta = \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \frac{\partial \varphi}{\partial v} \equiv \frac{D(\varphi, \psi)}{D(u, v)}$ is the Jacobian determinant of the functions φ and ψ .

$$2. \quad \iiint_{(V)} f(x, y, z) \, dx \, dy \, dz = \iiint_{(V')} f[\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w)] |\Delta| \, du \, dv \, dw$$

where $x = \varphi(u, v, w)$, $y = \psi(u, v, w)$, and $z = \chi(u, v, w)$ and where

$$\Delta = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} & \frac{\partial \varphi}{\partial w} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \\ \frac{\partial \chi}{\partial u} & \frac{\partial \chi}{\partial v} & \frac{\partial \chi}{\partial w} \end{vmatrix} \equiv \frac{D(\varphi, \psi, \chi)}{D(u, v, w)}$$

is the Jacobian determinant of the functions φ , ψ , and χ .

Here, we assume, both in (4.601 1) and in (4.601 2) that

- (a) the functions φ, ψ , and χ and also their first partial derivatives are continuous in the region of integration;
- (b) the Jacobian does not change sign in this region;
- (c) there exists a one-to-one correspondence between the old variables x, y, z and the new ones u, v, w in the region of integration;
- (d) when we change from the variables x, y, z to the variables u, v, w , the region V (resp. σ) is mapped into the region V' (resp. σ').

4.602 Transformation to polar coordinates:

$$x = r \cos \varphi, \quad y = r \sin \varphi; \quad \frac{D(x, y)}{D(r, \varphi)} = r$$

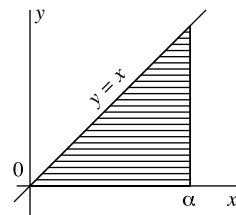
4.603 Transformation to spherical coordinates:

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta, \quad \frac{D(x, y, z)}{D(r, \theta, \varphi)} = r^2 \sin \theta$$

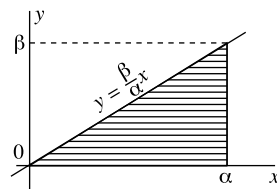
4.61 Change of the order of integration and change of variables

4.611

$$1. \quad \int_0^\alpha dx \int_0^x f(x, y) dy = \int_0^\alpha dy \int_y^\alpha f(x, y) dx$$

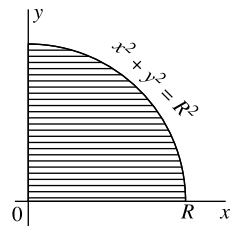


$$2. \quad \int_0^\alpha dx \int_0^{\frac{\beta}{\alpha}x} f(x, y) dy = \int_0^\beta dy \int_{\frac{\alpha}{\beta}y}^\alpha f(x, y) dx$$

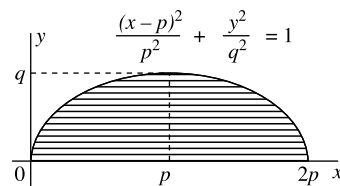


4.612

$$1. \quad \int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x, y) dy = \int_0^R dy \int_0^{\sqrt{R^2-y^2}} f(x, y) dx$$

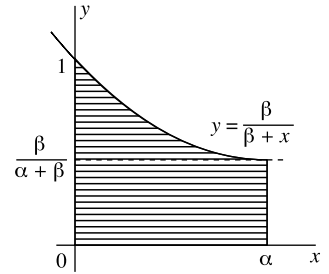


$$2. \quad \int_0^{2p} dx \int_0^{q/p\sqrt{2px-x^2}} f(x, y) dy = \int_0^q dy \int_p^{p[1+\sqrt{1-(y/q)^2}]} f(x, y) dx$$

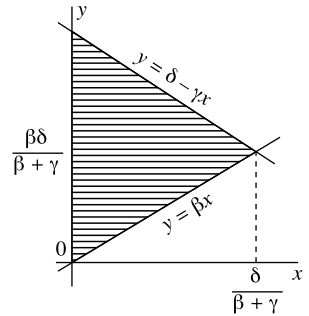


4.613

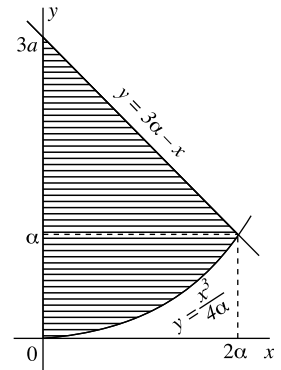
$$\begin{aligned}
 1. \quad \int_0^\alpha dx \int_0^{\beta/(\beta+x)} f(x, y) dy &= \int_0^{\beta/(\beta+\alpha)} dy \int_0^\alpha f(x, y) dx \\
 &+ \int_{\beta/(\beta+\alpha)}^1 dy \int_0^{\beta(1-y)/y} f(x, y) dx
 \end{aligned}$$



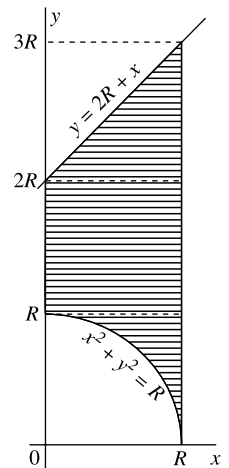
$$\begin{aligned}
 2. \quad \int_0^\alpha dx \int_{\beta x}^{\delta-\gamma x} f(x, y) dy &= \int_0^{\alpha\beta} dy \int_0^{y/\beta} f(x, y) dx \\
 &+ \int_{\alpha\beta}^\delta dy \int_0^{(\delta-y)/\gamma} f(x, y) dx \\
 \left[\alpha = \frac{\delta}{\beta+\gamma}, \quad a > 0, \quad \beta > 0, \quad \gamma > 0 \right]
 \end{aligned}$$



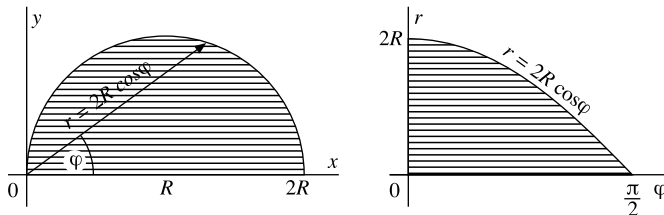
$$\begin{aligned}
 3. \quad \int_0^{2\alpha} dx \int_{x^2/4\alpha}^{3\alpha-x} f(x, y) dy &= \int_0^\alpha dy \int_0^{2\sqrt{\alpha y}} f(x, y) dx + \\
 &+ \int_\alpha^{3\alpha} dy \int_0^{3\alpha-y} f(x, y) dx
 \end{aligned}$$



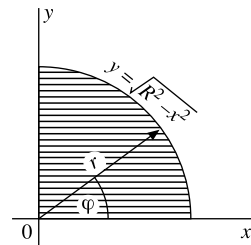
$$\begin{aligned}
 4. \quad \int_0^R dx \int_{\sqrt{R^2-x^2}}^{x+2R} f(x, y) dy &= \int_0^R dy \int_{\sqrt{R^2-y^2}}^R f(x, y) dx \\
 &+ \int_R^{2R} dy \int_0^R f(x, y) dx \\
 &+ \int_{2R}^{3R} dy \int_{y-2R}^R f(x, y) dx
 \end{aligned}$$



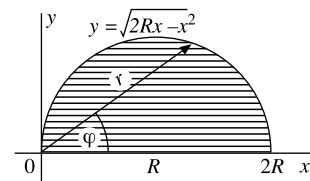
$$4.614 \quad \int_0^{\pi/2} d\varphi \int_0^{2R \cos \varphi} f(r, \varphi) dr = \int_0^{2R} dr \int_0^{\arccos \frac{r}{2R}} f(r, \varphi) d\varphi$$



$$4.615 \quad \int_0^R dx \int_0^{\sqrt{R^2 - x^2}} f(x, y) dy = \int_0^{\pi/2} d\varphi \int_0^R f(r \cos \varphi, r \sin \varphi) r dr$$



$$4.616 \quad \int_0^{2R} dx \int_0^{\sqrt{2Rx - x^2}} f(x, y) dy = \int_0^{\pi/2} d\varphi \int_0^{2R \cos \varphi} f(r \cos \varphi, r \sin \varphi) r dr$$



$$4.617 \quad \int_{\alpha}^{\beta} dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy = \int_0^{\beta} dx \int_0^{\varphi_2(x)} f(x, y) dy - \int_0^{\beta} dx \int_0^{\varphi_1(x)} f(x, y) dy - \int_0^{\alpha} dx \int_0^{\varphi_2(x)} f(x, y) dy + \int_0^{\alpha} dx \int_0^{\varphi_1(x)} f(x, y) dy$$

$$[\varphi_1(x) \leq \varphi_2(x) \text{ for } \alpha \leq x \leq \beta]$$

$$4.618 \quad \int_0^{\gamma} dx \int_0^{\varphi(x)} f(x, y) dy = \int_0^{\gamma} dx \int_0^1 f[x, z\varphi(x)] \varphi(x) dz \quad [y = z\varphi(x)]$$

$$= \gamma \int_0^1 dz \int_0^{\varphi(\gamma z)} f(\gamma z, y) dy \quad [x = \gamma z]$$

$$4.619 \quad \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} f(x, y) dy = \int_{x_0}^{x_1} dx \int_0^1 (y_1 - y_0) f[x, y_0 + (y_1 - y_0)t] dt$$

$$[y = y_0 + (y_1 - y_0)t]$$

4.62 Double and triple integrals with constant limits

4.620 General formulas

$$1. \quad \int_0^{\pi} d\omega \int_0^{\infty} f'(p \cosh x + q \cos \omega \sinh x) \sinh x dx = -\frac{\pi \operatorname{sign} p}{\sqrt{p^2 - q^2}} f\left(\operatorname{sign} p \sqrt{p^2 - q^2}\right)$$

$$\left[p^2 > q^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0\right] \quad \text{LO III 389}$$

2.
$$\int_0^{2\pi} d\omega \int_0^\infty f' [p \cosh x + (q \cos \omega + r \sin \omega) \sinh x] \sinh x \, dx$$

$$= -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f \left(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \right)$$

$$\left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 390}$$
3.
$$\int_0^\pi \int_0^\pi \frac{dx \, dy}{\sin x \sin^2 y} f' \left[\frac{p - q \cos x}{\sin x \sin y} + r \cot y \right] = -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f \left(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \right)$$

$$\left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 280}$$
4.
$$\int_{-\infty}^\infty dx \int_{-\infty}^\infty f' (p \cosh x \cosh y + q \sinh x \cosh y + r \sinh y) \cosh y \, dy$$

$$= -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f \left(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \right)$$

$$\left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 390}$$
5.
$$\int_0^\infty dx \int_0^\pi f (p \cosh x + q \cos \omega \sinh x) \sinh^2 x \sin \omega \, d\omega = 2 \int_0^\infty f \left(\operatorname{sign} p \sqrt{p^2 - q^2} \cosh x \right) \sinh^2 x \, dx$$

$$\left[\lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 391}$$
6.
$$\int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f [p \cosh x + (q \cos \omega + r \sin \omega) \sin \theta \sinh x] \sinh^2 x \sin \theta \, d\theta$$

$$= 4 \int_0^\infty f \left(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \cosh x \right) \sinh^2 x \, dx$$

$$\left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 390}$$
7.
$$\int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f \{p \cosh x + [(q \cos \omega + r \sin \omega) \sin \theta + s \cosh \theta] \sinh x\} \sinh^2 x \sin \theta \, d\theta$$

$$= 4\pi \int_0^\infty f \left(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2 - s^2} \cosh x \right) \sinh^2 x \, dx$$

$$\left[p^2 > q^2 + r^2 + s^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 391}$$

4.621

1.
$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} \, dx \, dy = \frac{\pi}{2\sqrt{1 - k^2}} \quad \text{LO I 252(90)}$$
2.
$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} \, dx \, dy = K(k) \quad \text{LO I 252(91)}$$
3.
$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin \alpha \sin y \, dx \, dy}{\sqrt{1 - \sin^2 \alpha \sin^2 x \sin^2 y}} = \frac{\pi \alpha}{2} \quad \text{LO I 253}$$

4.622

$$1. \quad \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx \, dy \, dz}{1 - \cos x \cos y \cos z} = 4\pi \mathbf{K}^2 \left(\frac{\sqrt{2}}{2} \right) \quad \text{MO 137}$$

$$2. \quad \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx \, dy \, dz}{3 - \cos y \cos z - \cos x \cos z - \cos x \cos y} = \sqrt{3}\pi \mathbf{K}^2 \left(\sin \frac{\pi}{12} \right) \quad \text{MO 137}$$

$$3. \quad \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx \, dy \, dz}{3 - \cos x - \cos y - \cos z} = 4\pi \left[18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6} \right] \mathbf{K}^2 \left[(2 - \sqrt{3}) (\sqrt{3} - \sqrt{2}) \right] \quad \text{MO 137}$$

$$4.623^3 \quad \int_0^\infty \int_0^\infty \varphi(a^2 x^2 + b^2 y^2) \, dx \, dy = \frac{\pi}{2ab} \int_0^\infty \varphi(x^2) x \, dx$$

$$4.624 \quad \int_0^\pi \int_0^{2\pi} f(\alpha \cos \theta + \beta \sin \theta \cos \psi + \gamma \sin \theta \sin \psi) \sin \theta \, d\theta \, d\psi \\ = 2\pi \int_0^\pi f(R \cos p) \sin p \, dp = 2\pi \int_{-1}^1 f(Rt) \, dt \\ \left[R = \sqrt{\alpha^2 + \beta^2 + \gamma^2} \right]$$

$$4.625^8 \quad p_l(a, b) = \int_0^a dx \int_0^b dy (x^2 + y^2 + 1)^{-3/2} P_l \left(1/\sqrt{x^2 + y^2 + 1} \right)$$

Then, for even and odd subscripts:

$$\bullet \quad p_{2l}(a, b) = \frac{1}{l(2l+1)2^{2l}} \frac{ab}{\sqrt{a^2 + b^2 + 1}} \sum_{k=0}^{l-1} \frac{(-1)^{l-k-1} 2^{2k} \binom{2l+2k}{l+k} \binom{l+k}{l-k-1}}{\binom{2k}{k} (2k+1)} \\ \times (2l+2k+1) \sum_{j=0}^k \frac{\binom{2j}{j}}{2^{2j}} \frac{1}{(a^2 + b^2 + 1)^j} \left(\frac{1}{(a^2 + 1)^{k-j+1}} + \frac{1}{(b^2 + 1)^{k-j+1}} \right) \\ \bullet \quad p_{2l+1}(a, b) = \frac{1}{2^{2l+1}(2l+1)} \sum_{k=0}^l \frac{(-1)^{l+k}}{2^{2k}} \binom{l}{k} \binom{l+k+1}{k} \binom{2l+2k+1}{l+k} \\ \times \left\{ \frac{1}{(b^2 + 1)^k} \frac{b}{\sqrt{b^2 + 1}} \arctan^{-1} \frac{a}{\sqrt{b^2 + 1}} + \frac{1}{(a^2 + 1)^k} \frac{a}{\sqrt{a^2 + 1}} \arctan^{-1} \frac{b}{\sqrt{a^2 + 1}} \right. \\ \left. + ab \sum_{j=1}^k \frac{2^{2j-1}}{j \binom{2j}{j}} \cdot \frac{1}{(a^2 + b^2 + 1)^j} \left(\frac{1}{(a^2 + 1)^{k-j+1}} + \frac{1}{(b^2 + 1)^{k-j+1}} \right) \right\}$$

4.63–4.64 Multiple integrals

$$4.631 \quad \int_p^x dt_{n-1} \int_p^{t_{n-1}} dt_{n-2} \dots \int_p^{t_1} f(t) \, dt = \frac{1}{(n-1)!} \int_p^x (x-t)^{n-1} f(t) \, dt,$$

where $f(t)$ is continuous on the interval $[p, q]$ and $p \leq x \leq q$.

4.632

$$1. \quad \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq h}} dx_1 dx_2 \cdots dx_n = \frac{h^n}{n!}$$

[the volume of an n -dimensional simplex] FI III 472

$$2. \quad \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2} dx_1 dx_2 \cdots dx_n = \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n \quad [\text{the volume of an } n\text{-dimensional sphere}]$$

FI III 473

$$4.633 \quad \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} \frac{dx_1 dx_2 \cdots dx_n}{\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}} = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \quad [n > 1]$$

[half-area of the surface of an $(n+1)$ -dimensional sphere $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$] FI III 474

$$4.634^8 \quad \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n$$

$$= \frac{q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n}}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} + 1\right)}$$

$$[\alpha_i > 0, \quad p_i > 0, \quad q_i > 0, \quad i = 1, 2, \dots, n] \quad \text{FI III 477}$$

4.635

$$1.^8 \quad \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \geq 1}} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n}\right]$$

$$\times x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n$$

$$= \frac{q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n}}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \int_1^\infty f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} - 1} dx$$

under the assumption that the integral on the right converges absolutely.

FI III 487

$$\begin{aligned}
2.^8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \leq 1}} f \left[\left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \right] \\
& \quad \times x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n \\
& = \frac{q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n}}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \int_0^1 f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} - 1} dx
\end{aligned}$$

under the assumptions that the one-dimensional integral on the right converges absolutely and that the numbers q_i , α_i , and p_i are positive. FI III 479

In particular,

$$\begin{aligned}
3. \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} e^{-q(x_1 + x_2 + \cdots + x_n)} dx_1 dx_2 \cdots dx_n \\
& = \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n)} \int_0^1 x^{p_1 + p_2 + \cdots + p_n - 1} e^{-qx} dx \\
& \quad [n > 0, \quad p_1 > 0, \quad p_2 > 0, \dots, p_n > 0] \\
4.^8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n} \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \cdots - x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n \\
& = \frac{\Gamma(1 - \mu)}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(1 - \mu + \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \\
& \quad [p_1 > 0, \quad p_2 > 0, \dots, p_n > 0, \quad \mu < 1] \quad \text{FI III 480}
\end{aligned}$$

4.636

$$\begin{aligned}
1.^8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n} \geq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n \\
& = \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n \left(\mu - \frac{p_1}{\alpha_1} - \frac{p_2}{\alpha_2} - \cdots - \frac{p_n}{\alpha_n}\right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \\
& \quad \left[p_1 > 0, \quad p_2 > 0, \dots, p_n > 0; \quad \mu > \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} \right] \quad \text{FI III 488}
\end{aligned}$$

$$\begin{aligned}
2.^8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n \\
&= \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n \left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} - \mu \right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \\
&\quad \left[\mu < \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} \right] \quad \text{FI III 480}
\end{aligned}$$

$$\begin{aligned}
3.^8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} \sqrt{\frac{1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \cdots - x_n^{\alpha_n}}{1 + x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n}}} dx_1 dx_2 \cdots dx_n \\
&= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{1}{\Gamma(m)} \left\{ \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \right\},
\end{aligned}$$

$$\text{where } m = \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}. \quad \text{FI III 480}$$

$$\begin{aligned}
4.637^8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, \quad x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq 1}} f(x_1 + x_2 + \cdots + x_n) \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n}{(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n + r)^{p_1 + p_2 + \cdots + p_n}} \\
&= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 p_2 + \cdots p_n)} \int_0^1 f(x) \frac{x^{p_1 p_2 + \cdots p_n - 1}}{(q_1 x + r)^{p_1} (q_2 x + r)^{p_2} \cdots (q_n x + r)^{p_n}} dx, \\
&\quad [q_1 \geq 0, \quad q_2 \geq 0, \dots, q_n \geq 0; \quad r > 0]
\end{aligned}$$

where $f(x)$ is continuous on the interval $(0, 1)$.

4.638

$$\begin{aligned}
1. \quad & \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} e^{-(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n)}}{(r_0 + r_1 x_1 + r_2 x_2 + \cdots + r_n x_n)^s} dx_1 dx_2 \cdots dx_n \\
&= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(s)} \int_0^\infty \frac{e^{r_0 x} x^{s-1} dx}{(q_1 r_1 x)^{p_1} (q_1 r_2 x)^{p_2} \cdots (q_n r_n x)^{p_n}}
\end{aligned}$$

where p_i, q_i, r_i , and s are positive. This result is also valid for $r_0 = 0$, provided $p_1 + p_2 + \cdots + p_n > s$.

$$\begin{aligned}
2. \quad & \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(r_0 + r_1 x_1 + r_2 x_2 + \cdots + r_n x_n)^s} dx_1 dx_2 \cdots dx_n \\
&= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n) \Gamma(sp_1 p_2 - \cdots - p_n)}{r_1^{p_1} r_2^{p_2} \cdots r_n^{p_n} r_0^{s-p_1-p_2-\cdots-p_n} \Gamma(s)} \\
&\quad [p_i > 0, \quad r_i > 0, \quad s > 0]
\end{aligned}$$

$$\begin{aligned}
3.^8 \quad & \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{[1 + (r_1 x_1)^{q_1} + (r_2 x_2)^{q_2} + \cdots + (r_n x_n)^{q_n}]^s} dx_1 dx_2 \cdots dx_n \\
&= \frac{\Gamma\left(\frac{p_1}{q_1}\right) \Gamma\left(\frac{p_2}{q_2}\right) \cdots \Gamma\left(\frac{p_n}{q_n}\right) \Gamma\left(s - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \cdots - \frac{p_n}{q_n}\right)}{q_1 q_2 \cdots q_n r_1^{p_1 q_1} r_2^{p_2 q_2} \cdots r_n^{p_n q_n} \Gamma(s)} \\
&\quad [p_i > 0, \quad q_i > 0, \quad r_i > 0, \quad s > 0]
\end{aligned}$$

4.639

$$\begin{aligned}
 1. \quad & \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} (p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^{2m} dx_1 dx_2 \cdots dx_n \\
 &= \frac{(2m-1)!!}{2^m} \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + m + 1\right)} (p_1^2 + p_2^2 + \cdots + p_n^2)^m
 \end{aligned}$$

FI III 482

$$2. \quad \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} (p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^{2m+1} dx_1 dx_2 \cdots dx_n = 0$$

FI III 483

4.641

$$\begin{aligned}
 1.^{11} \quad & \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} e^{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n} dx_1 dx_2 \cdots dx_n \\
 &= \sqrt{\pi^n} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma\left(\frac{n}{2} + k + 1\right)} \left(\frac{p_1^2 + p_2^2 + \cdots + p_n^2}{4}\right)^k
 \end{aligned}$$

FI III 483

$$2. \quad \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_{2n}^2 \leq 1} e^{p_1 x_1 p_2 x_2 + \cdots + p_{2n} x_{2n}} dx_1 dx_2 \cdots dx_{2n} = \frac{(2\pi)^n I_n \left(\sqrt{p_1^2 + p_2^2 + \cdots + p_{2n}^2}\right)}{(p_1^2 + p_2^2 + \cdots + p_{2n}^2)^{n/2}}$$

FI III 483a

$$4.642 \quad \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2} f\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\right) dx_1 dx_2 \cdots dx_n = \frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)} \int_0^R x^{n-1} f(x) dx,$$

FI III 485

where $f(x)$ is a function that is continuous on the interval $(0, R)$.

$$\begin{aligned}
 4.643 \quad & \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 x_2 \cdots x_n) (1-x_1)^{p_1-1} (1-x_2)^{p_2-1} \cdots (1-x_n)^{p_n-1} \\
 & \quad \times x_2^{p_1} x_3^{p_1+p_2} \cdots x_n^{p_1+p_2+\cdots+p_{n-1}} dx_1 dx_2 \cdots dx_n \\
 &= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n)} \int_0^1 f(x) (1-x)^{p_1+p_2+\cdots+p_n-1} dx
 \end{aligned}$$

under the assumption that the integral on the right converges absolutely.

FI III 488

$$\begin{aligned}
 4.644 \quad & \overbrace{\int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 1}}^{n-1} f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \frac{dx_1 dx_2 \cdots dx_{n-1}}{|x_n|} \\
 &= 2 \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 1} f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \frac{dx_1 dx_2 \cdots dx_{n-1}}{\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}} \\
 &= \frac{2\sqrt{\pi^{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi f\left(\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2} \cos x\right) \sin^{n-2} x dx
 \end{aligned}$$

[$n \geq 3$]

where $f(x)$ is continuous on the interval $\left\{-\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}, \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}\right\}$.

FI III 489

4.645 Suppose that two functions $f(x_1, x_2, \dots, x_n)$ and $g(x_1, x_2, \dots, x_n)$ are continuous in a closed, bounded region D and that the smallest and greatest values of the function g in D are m and M , respectively. Let $\varphi(u)$ denote a function that is continuous for $m \leq u \leq M$. We denote by $\psi(u)$ the integral

$$1. \quad \psi(u) = \int \int \cdots \int_{m \leq g(x_1, x_2, \dots, x_n) \leq u} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n,$$

over that portion of the region D on which the inequality $m \leq g(x_1, x_2, \dots, x_n) \leq u$ is satisfied. Then

$$2. \quad \int \int \cdots \int_{m \leq g(x_1, x_2, \dots, x_n) \leq M} f(x_1, x_2, \dots, x_n) \varphi[g(x_1, x_2, \dots, x_n)] dx_1 dx_2 \cdots dx_n \\ = (S) \int_m^M \varphi(u) d\psi(u) = (R) \int_m^M \varphi(u) \frac{d\psi(u)}{du} du$$

where the middle integral must be understood in the sense of Stieltjes. If the derivative $\frac{d\psi}{du}$ exists and is continuous, the Riemann integral on the right exists.

M may be $+\infty$ in formulas **4.645** 2, in which case $\int_m^{+\infty}$ should be understood to mean $\lim_{M \rightarrow +\infty} \int_m^M$.

$$\begin{aligned} \mathbf{4.646}^8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n)^r} dx_1 dx_2 \cdots dx_n \\ &= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n - r + 1) \Gamma(r)} \int_0^\infty \frac{x^{r-1} dx}{(1 + q_1 x)^{p_1} (1 + q_2 x)^{p_2} \cdots (1 + q_n x)^{p_n}} \\ &= [p_1 > 0, \quad p_2 > 0, \dots, p_n > 0, \quad q_1 > 0, \quad q_2 > 0, \dots, q_n > 0, \quad p_1 + p_2 + \cdots + p_n > r > 0] \end{aligned}$$

FI III 493

$$\begin{aligned} \mathbf{4.647} \quad & \int \int \cdots \int_{0 \leq x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} \exp \left\{ \frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} \right\} dx_1 dx_2 \cdots dx_n \\ &= \frac{2\sqrt{\pi^n}}{n(p_1^2 + p_2^2 + \cdots + p_n^2)^{\frac{n}{4} - \frac{1}{2}}} I_{\frac{n}{2}-1} \left(\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2} \right) \end{aligned}$$

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$$\begin{aligned} \mathbf{4.648}^8 \quad & \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp \left[- \left(x_1 + x_2 + \cdots + x_n + \frac{\lambda^{n+1}}{x_1 x_2 \cdots x_n} \right) \right] \\ & \quad \times x_1^{\frac{1}{n+1}-1} x_2^{\frac{2}{n+1}-1} \cdots x_n^{\frac{n}{n+1}-1} dx_1 dx_2 \cdots dx_n \\ &= \frac{1}{\sqrt{n+1}} (2\pi)^{\frac{n}{2}} e^{-(n+1)\lambda} \end{aligned}$$

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5 Indefinite Integrals of Special Functions

5.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1 - k^2}$ (cf. 8.1).

5.11 Complete elliptic integrals

5.111

$$1. \quad \int \mathbf{K}(k) k^{2p+3} dk = \frac{1}{(2p+3)^2} \left\{ 4(p+1)^2 \int \mathbf{K}(k) k^{2p+1} dk + k^{2p+2} \left[\mathbf{E}(k) - (2p+3) \mathbf{K}(k) k'^2 \right] \right\} \quad \text{BY (610.04)}$$

$$2. \quad \int \mathbf{E}(k) k^{2p+3} dk = \frac{1}{4p^2 + 16p + 15} \left\{ 4(p+1)^2 \int \mathbf{E}(k) k^{2p+1} dk - \mathbf{E}(k) k^{2p+2} \left[(2p+3) k'^2 - 2 \right] - k^{2p+2} k'^2 \mathbf{K}(k) \right\} \quad \text{BY (611.04)}$$

5.112

$$1. \quad \int \mathbf{K}(k) dk = \frac{\pi k}{2} \left[1 + \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(2j+1) 2^{4j} (j!)^4} \right] \quad \text{BY (610.00)}$$

$$2.^6 \quad \int \mathbf{E}(k) dk = \frac{\pi k}{2} \left[1 - \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(4j^2 - 1) 2^{4j} (j!)^4} \right] \quad \text{BY (611.00)}$$

$$3. \quad \int \mathbf{K}(k) k dk = \mathbf{E}(k) - k'^2 \mathbf{K}(k) \quad \text{BY (610.01)}$$

$$4. \quad \int \mathbf{E}(k) k dk = \frac{1}{3} \left[(1 + k^2) \mathbf{E}(k) - k'^2 \mathbf{K}(k) \right] \quad \text{BY (611.01)}$$

$$5. \quad \int \mathbf{K}(k) k^3 dk = \frac{1}{9} \left[(4 + k^2) \mathbf{E}(k) - k'^2 (4 + 3k^2) \mathbf{K}(k) \right] \quad \text{BY (610.02)}$$

$$6. \quad \int \mathbf{E}(k) k^3 dk = \frac{1}{45} \left[(4 + k^2 + 9k^4) \mathbf{E}(k) - k'^2 (4 + 3k^2) \mathbf{K}(k) \right] \quad \text{BY 611.02}$$

$$7. \quad \int \mathbf{K}(k) k^5 dk = \frac{1}{225} \left[(64 + 16k^2 + 9k^4) \mathbf{E}(k) - k'^2 (64 + 48k^2 + 45k^4) \mathbf{K}(k) \right] \quad \text{BY (610.03)}$$

$$8. \quad \int \mathbf{E}(k) k^5 dk = \frac{1}{1575} \left[(64 + 16k^2 + 9k^4 + 225k^6) \mathbf{E}(k) - k'^2 (64 + 48k^2 + 45k^4) \mathbf{K}(k) \right] \quad \text{BY (611.03)}$$

$$9. \quad \int \frac{\mathbf{K}(k)}{k^2} dk = -\frac{\mathbf{E}(k)}{k} \quad \text{BY (612.05)}$$

$$10. \quad \int \frac{\mathbf{E}(k)}{k^2} dk = \frac{1}{k} \left[k'^2 \mathbf{K}(k) - 2 \mathbf{E}(k) \right] \quad \text{BY (612.02)}$$

$$11. \quad \int \frac{\mathbf{E}(k)}{k'^2} dk = k \mathbf{K}(k) \quad \text{BY (612.01)}$$

$$12. \quad \int \frac{\mathbf{E}(k)}{k^4} dk = \frac{1}{9k^3} \left[2(k^2 - 2) \mathbf{E}(k) + k'^2 \mathbf{K}(k) \right] \quad \text{BY (612.03)}$$

$$13. \quad \int \frac{k \mathbf{E}(k)}{k'^2} dk = \mathbf{K}(k) - \mathbf{E}(k) \quad \text{BY (612.04)}$$

5.113

$$1. \quad \int [\mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k} = -\mathbf{E}(k) \quad \text{BY (612.06)}$$

$$2. \quad \int [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k} = 2 \mathbf{E}(k) - k'^2 \mathbf{K}(k) \quad \text{BY (612.09)}$$

$$3. \quad \int [(1 + k^2) \mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k} = -k'^2 \mathbf{K}(k) \quad \text{BY (612.12)}$$

$$4. \quad \int [\mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k^2} = \frac{1}{k} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BY (612.07)}$$

$$5. \quad \int [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k^2 k'^2} = \frac{1}{k} [\mathbf{K}(k) - \mathbf{E}(k)]$$

$$6. \quad \int [(1 + k^2) \mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k k'^4} = \frac{\mathbf{E}(k)}{k'^2} \quad \text{BY (612.13)}$$

$$5.114 \quad \int \frac{k \mathbf{K}(k) dk}{[\mathbf{E}(k) - k'^2 \mathbf{K}(k)]^2} = \frac{1}{k'^2 \mathbf{K}(k) - \mathbf{E}(k)} \quad \text{BY (612.11)}$$

5.115

$$1. \quad \int \Pi\left(\frac{\pi}{2}, r^2, k\right) k dk = (k^2 - r^2) \Pi\left(\frac{\pi}{2}, r^2, k\right) - \mathbf{K}(k) + \mathbf{E}(k) \quad \text{BY (612.14)}$$

$$2. \quad \int \left[\mathbf{K}(k) - \Pi\left(\frac{\pi}{2}, r^2, k\right) \right] k dk = k^2 \mathbf{K}(k) - (k^2 - r^2) \Pi\left(\frac{\pi}{2}, r^2, k\right) \quad \text{BY (612.15)}$$

$$3. \quad \int \left[\frac{\mathbf{E}(k)}{k'^2} + \Pi\left(\frac{\pi}{2}, r^2, k\right) \right] k dk = (k^2 - r^2) \Pi\left(\frac{\pi}{2}, r^2, k\right) \quad \text{BY (612.16)}$$

5.12 Elliptic integrals

$$5.121 \quad \int_0^x \frac{F(x, k) dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{[F(x, k)]^2}{2} \quad \left[0 < x \leq \frac{\pi}{2}\right] \quad \text{BY (630.01)}$$

$$5.122^{11} \quad \int_0^x E(x, k) \sqrt{1 - k^2 \sin^2 x} dx = \frac{[E(x, k)]^2}{2} \quad \text{BY (630.32)}$$

5.123

$$1. \quad \int_0^x F(x, k) \sin x dx = -\cos x F(x, k) + \frac{1}{k} \arcsin(k \sin x) \quad \text{BY (630.11)}$$

$$2. \quad \int_0^x F(x, k) \cos x dx = \sin x F(x, k) + \frac{1}{k} \operatorname{arccosh} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - \frac{1}{k} \operatorname{arccosh} \left(\frac{1}{k'} \right) \quad \text{BY (630.21)}$$

5.124

$$1. \quad \int_0^x E(x, k) \sin x dx = -\cos x E(x, k) + \frac{1}{2k} \left[k \sin x \sqrt{1 - k^2 \sin^2 x} + \arcsin(k \sin x) \right] \quad \text{BY (630.12)}$$

$$2. \quad \int_0^x E(x, k) \cos x dx = \sin x E(x, k) + \frac{1}{2k} \left[k \cos x \sqrt{1 - k^2 \sin^2 x} - k'^2 \operatorname{arccosh} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - k + k'^2 \operatorname{arccosh} \left(\frac{1}{k'} \right) \right] \quad \text{BY (630.22)}$$

$$3.* \quad \int_0^a \frac{x \mathbf{E}(x) dx}{(k'^2 + k^2 x^2)^2 \sqrt{a^2 - x^2}} = \frac{\pi}{4} \left(\frac{a \sqrt{1 - a^2}}{(k'^2 + k^2 a^2)^2} + \frac{a^2 E(\lambda, k)}{k'^2 (k'^2 + k^2 a^2)^{3/2}} + \frac{(1 - a^2) F(\lambda, k)}{(k'^2 + k^2 a^2)^{3/2}} \right) \\ \lambda = \arcsin \left(\frac{a}{\sqrt{k'^2 + k^2 a^2}} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < a < 1, \quad 0 < k < 1]$$

$$4.* \quad \int_0^a \frac{x \mathbf{E}(x) dx}{(k^2 - x^2)^2 \sqrt{a^2 - x^2}} = \frac{\pi}{4} \left(\frac{a \sqrt{1 - a^2}}{k^2 (k^2 - a^2)} + \frac{F(\phi, k)}{k^2 \sqrt{k^2 - a^2}} + \frac{a^2 E(\phi, k)}{k^2 (k^2 - a^2)^{3/2}} \right) \\ \phi = \arcsin \left(\frac{a}{k} \right) \quad [0 < a < k < 1]$$

$$5.* \quad \int_0^{\pi/2} \frac{E(x, k') \sin x \cos x dx}{(1 - k'^2 \cosh^2 v \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}} \\ = \frac{1}{k'^2 \sinh v \cosh v} \left\{ \mathbf{E}(k') \operatorname{arctanh} \left(\frac{\tanh v}{k} \right) - \frac{\pi \tanh v}{2} - \frac{\pi}{2} [F(\phi, k) - E(\phi, k)] \right\} \\ \phi = \arcsin \left(\frac{\tanh v}{k} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < \tanh v < k < 1]$$

$$\begin{aligned}
6.^* \quad & \int_0^{\pi/2} \frac{E(x, k) \sin x \cos x \, dx}{(1 - k^2 \cos^2 \psi \sin^2 x) \sqrt{1 - k^2 \sin^2 x}} \\
&= \frac{1}{k^2 \sin \psi \cos \psi} \left\{ \mathbf{E}(k) \arctan \left(\frac{\tan \psi}{k'} \right) - \frac{\pi}{2} E(\beta, k) + \frac{\pi}{2} \frac{\tan \psi}{\sqrt{1 - k^2 \cos^2 \psi}} \left(1 - \sqrt{1 - k^2 \cos^2 \psi} \right) \right\} \\
&\quad \beta = \arctan \left(\frac{\tan \psi}{k} \right) \quad k' = \sqrt{1 - k^2} \quad \left[0 < k < 1, \quad 0 < \psi < \frac{\pi}{2} \right]
\end{aligned}$$

$$\begin{aligned}
7.^* \quad & \int_0^{\pi/2} \frac{E(x, k') \sin x \cos x \, dx}{(1 + k'^2 \sinh^2 \mu \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}} \\
&= \frac{-1}{k'^2 \sinh \mu \cosh \mu} \left\{ \mathbf{E}(k') \operatorname{arctanh}(k \tanh \mu) - \frac{\pi}{2} \left[F(\phi, k) - E(\phi, k) + \tanh \mu \sqrt{1 + k'^2 \sinh^2 \mu} \right] \right. \\
&\quad \left. - \frac{\pi}{2} \coth \mu \left(1 - \sqrt{1 + k'^2 \sinh^2 \mu} \right) \right\} \\
&\quad \phi = \arcsin(\tanh \mu) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \tanh \mu < 1]
\end{aligned}$$

$$\begin{aligned}
8.^* \quad & \int_0^{\pi/2} \frac{F(x, k') \sin x \cos x \, dx}{(1 + k'^2 \sinh^2 \mu \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}} \\
&= \frac{-1}{k'^2 \sinh \mu \cosh \mu} \left[\mathbf{K}(k') \operatorname{arctanh}(k \tanh \mu) - \frac{\pi}{2} F(\phi, k) \right] \\
&\quad \phi = \arcsin(\tanh \mu) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \tanh \mu < 1]
\end{aligned}$$

$$\begin{aligned}
9.^* \quad & \int_0^{\pi/2} \frac{F(x, k') \sin x \cos x \, dx}{(1 - k'^2 \cosh^2 \nu \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}} \\
&= \frac{1}{k'^2 \sinh \nu \cosh \nu} \left[\mathbf{K}(k') \operatorname{arctanh} \left(\frac{\tanh \nu}{k} \right) - \frac{\pi}{2} F(\phi, k) \right] \\
&\quad \phi = \arcsin \left(\frac{\tanh \nu}{k} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \tanh \nu < 1]
\end{aligned}$$

$$\begin{aligned}
10.^* \quad & \int_0^{\pi/2} \frac{F(x, k) \sin x \cos x \, dx}{(1 - k^2 \cos^2 \psi \sin^2 x) \sqrt{1 - k^2 \sin^2 x}} \\
&= \frac{1}{k^2 \sin \psi \cos \psi} \left[\mathbf{K}(k') \operatorname{arctanh} \left(\frac{\tan \psi}{k'} \right) - \frac{\pi}{2} F(\beta, k) \right] \\
&\quad \beta = \arctan \left(\frac{\tan \psi}{k'} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \psi < 1]
\end{aligned}$$

$$\begin{aligned}
11.^* \quad & \int_a^b \ln \left(\frac{\epsilon + x}{\epsilon - x} \right) \frac{x^2 \, dx}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} = \frac{\pi}{\epsilon} \left(\epsilon^2 - \sqrt{(\epsilon^2 - a^2)(\epsilon^2 - b^2)} \right) + \pi \beta [F(\phi, k) - E(\phi, k)] \\
&\quad \phi = \arcsin \left(\frac{\beta}{\epsilon} \right) \quad k = \frac{a}{b} \quad [0 < a < b < \epsilon]
\end{aligned}$$

5.125

$$\begin{aligned}
 1. \quad \int_0^x \Pi(x, \alpha^2, k) \sin x \, dx \\
 &= -\cos x \Pi(x, \alpha^2, k) + \frac{1}{\sqrt{k^2 - \alpha^2}} \arctan \left[\sqrt{\frac{k^2 - \alpha^2}{1 - k^2 \sin^2 x}} \sin x \right] \quad [\alpha^2 < k^2] \\
 &= -\cos x \Pi(x, \alpha^2, k) + \frac{1}{\sqrt{\alpha^2 - k^2}} \operatorname{arctanh} \left[\sqrt{\frac{\alpha^2 - k^2}{1 - k^2 \sin^2 x}} \sin x \right] \quad [\alpha^2 > k^2]
 \end{aligned}$$

BY (630.13)

$$2. \quad \int_0^x \Pi(x, \alpha^2, k) \cos x \, dx = \sin x \Pi(x, \alpha^2, k) - f - f_0$$

where

$$\begin{aligned}
 f &= \frac{1}{2\sqrt{(1 - \alpha^2)(\alpha^2 - k^2)}} \arctan \left[\frac{2(1 - \alpha^2)(\alpha^2 - k^2) + (1 - \alpha^2 \sin^2 x)(2k^2 - \alpha^2 - \alpha^2 k^2)}{2\alpha^2 \sqrt{(1 - \alpha^2)(\alpha^2 - k^2)} \cos x \sqrt{1 - k^2 \sin^2 x}} \right] \\
 &\quad \text{for } (1 - \alpha^2)(\alpha^2 - k^2) > 0; \\
 &= \frac{1}{2\sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)}} \ln \left[\frac{2(\alpha^2 - 1)(\alpha^2 - k^2) + (1 - \alpha^2 \sin^2 x)(\alpha^2 + \alpha^2 k^2 - 2k^2)}{1 - \alpha^2 \sin^2 x} \right] \\
 &\quad + \frac{2\alpha^2 \sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)} \cos x \sqrt{1 - k^2 \sin^2 x}}{1 - \alpha^2 \sin^2 x} \Bigg] \\
 &\quad \text{for } (1 - \alpha^2)(\alpha^2 - k^2) < 0, \\
 f_0 &\text{ is the value of } f \text{ at } x = 0 \quad \text{BY (630.23)}
 \end{aligned}$$

Integration with respect to the modulus

$$5.126 \quad \int F(x, k) k \, dk = E(x, k) - k'^2 F(x, k) + \left(\sqrt{1 - k^2 \sin^2 x} - 1 \right) \cot x \quad \text{BY (613.01)}$$

$$5.127 \quad \int E(x, k) k \, dk = \frac{1}{3} \left[(1 + k^2) E(x, k) - k'^2 F(x, k) + \left(\sqrt{1 - k^2 \sin^2 x} - 1 \right) \cot x \right] \quad \text{BY (613.02)}$$

$$5.128 \quad \int \Pi(x, r^2, k) k \, dk = (k^2 - r^2) \Pi(x, r^2, k) - F(x, k) + E(x, k) + \left(\sqrt{1 - k^2 \sin^2 x} - 1 \right) \cot x$$

BY (613.03)

5.13 Jacobian elliptic functions

5.131

$$\begin{aligned}
 1. \quad \int \operatorname{sn}^m u \, du &= \frac{1}{m+1} \left[\operatorname{sn}^{m+1} u \operatorname{cn} u \operatorname{dn} u + (m+2)(1+k^2) \int \operatorname{sn}^{m+2} u \, du \right. \\
 &\quad \left. - (m+3)k^2 \int \operatorname{sn}^{m+4} u \, du \right]
 \end{aligned}$$

$$2. \quad \int \operatorname{cn}^m u \, du = \frac{1}{(m+1)k'^2} \left[-\operatorname{cn}^{m+1} u \operatorname{sn} u \operatorname{dn} u \right. \\ \left. + (m+2)(1-2k^2) \int \operatorname{cn}^{m+2} u \, du + (m+3)k^2 \int \operatorname{cn}^{m+4} u \, du \right] \quad \text{PE (568)}$$

$$3. \quad \int \operatorname{dn}^m u \, du = \frac{1}{(m+1)k'^2} \left[k^2 \operatorname{dn}^{m+1} u \operatorname{sn} u \operatorname{cn} u \right. \\ \left. + (m+2)(2-k^2) \int \operatorname{dn}^{m+2} u \, du - (m+3) \int \operatorname{dn}^{m+4} u \, du \right] \quad \text{PE (569)}$$

By using formulas **5.131**, we can reduce the integrals (for $m \neq 1$) $\int \operatorname{sn}^m u \, du$, $\int \operatorname{cn}^m u \, du$, and $\int \operatorname{dn}^m u \, du$ to the integrals **5.132**, **5.133** and **5.134**.

5.132

$$1. \quad \int \frac{du}{\operatorname{sn} u} = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u} \quad \text{H 87(164)} \\ = \ln \frac{\operatorname{dn} u - \operatorname{cn} u}{\operatorname{sn} u} \quad \text{SI 266(4)}$$

$$2. \quad \int \frac{du}{\operatorname{cn} u} = \frac{1}{k'} \ln \frac{k' \operatorname{sn} u + \operatorname{dn} u}{\operatorname{cn} u} \quad \text{SI 266(5)}$$

$$3. \quad \int \frac{du}{\operatorname{dn} u} = \frac{1}{k'} \arctan \frac{k' \operatorname{sn} u - \operatorname{cn} u}{k' \operatorname{sn} u + \operatorname{cn} u} \quad \text{H 88(166)} \\ = \frac{1}{k'} \arccos \frac{\operatorname{cn} u}{\operatorname{dn} u} \quad \text{JA} \\ = \frac{1}{k'} \ln \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u} \quad \text{SI 266(6)} \\ = \frac{1}{k'} \arcsin \frac{k' \operatorname{sn} u}{\operatorname{dn} u} \quad \text{JA}$$

5.133

$$1. \quad \int \operatorname{sn} u \, du = \frac{1}{k} \ln (\operatorname{dn} u - k \operatorname{cn} u) \quad \text{H 87(161)} \\ = \frac{1}{k} \operatorname{arccosh} \frac{\operatorname{dn} u - k^2 \operatorname{cn} u}{1 - k^2} \quad \text{JA} \\ = \frac{1}{k} \operatorname{arcsinh} \left(k \frac{\operatorname{dn} u - \operatorname{cn} u}{1 - k^2} \right); \quad \text{JA} \\ = -\frac{1}{k} \ln (\operatorname{dn} u + k \operatorname{cn} u) \quad \text{SI 365(1)}$$

$$2. \quad \int \operatorname{cn} u \, du = \frac{1}{k} \arccos (\operatorname{dn} u); \quad \text{H 87(162)} \\ = \frac{i}{k} \ln (\operatorname{dn} u - ik \operatorname{sn} u); \quad \text{SI 265(2)a, ZH 87(162)} \\ = \frac{1}{k} \arcsin (k \operatorname{sn} u) \quad \text{JA}$$

$$\begin{aligned}
3. \quad \int \operatorname{dn} u \, du &= \arcsin(\operatorname{sn} u); & \text{H 87(163)} \\
&= \operatorname{am} u = i \ln(\operatorname{cn} u - i \operatorname{sn} u) & \text{SI 266(3), ZH 87(163)}
\end{aligned}$$

5.134

$$\begin{aligned}
1. \quad \int \operatorname{sn}^2 u \, du &= \frac{1}{k^2} [u - E(\operatorname{am} u, k)] & \text{PE (564)} \\
2. \quad \int \operatorname{cn}^2 u \, du &= \frac{1}{k^2} [E(\operatorname{am} u, k) - k'^2 u] & \text{PE (565)} \\
3. \quad \int \operatorname{dn}^2 u \, du &= E(\operatorname{am} u, k) & \text{PE (566)}
\end{aligned}$$

5.135

$$\begin{aligned}
1. \quad \int \frac{\operatorname{sn} u}{\operatorname{cn} u} du &= \frac{1}{k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{cn} u} & \text{SI 266(7)} \\
&= \frac{1}{2k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'} & \text{H 88(167)} \\
2. \quad \int \frac{\operatorname{sn} u}{\operatorname{dn} u} du &= \frac{i}{kk'} \ln \frac{ik' - k \operatorname{cn} u}{\operatorname{dn} u} & \text{SI 266(8)} \\
&= \frac{1}{kk'} \operatorname{arccot} \frac{k \operatorname{cn} u}{k'} \\
3. \quad \int \frac{\operatorname{cn} u}{\operatorname{sn} u} du &= \ln \frac{1 - \operatorname{dn} u}{\operatorname{sn} u} & \text{SI 266(10)} \\
&= \frac{1}{2} \ln \frac{1 - \operatorname{dn} u}{1 + \operatorname{dn} u} & \text{H 88(168)} \\
4. \quad \int \frac{\operatorname{cn} u}{\operatorname{dn} u} du &= -\frac{1}{k} \ln \frac{1 - k \operatorname{sn} u}{\operatorname{dn} u} & \text{SI 266(9)} \\
&= \frac{1}{2k} \ln \frac{1 + k \operatorname{sn} u}{1 - k \operatorname{sn} u} \\
5. \quad \int \frac{\operatorname{dn} u}{\operatorname{cn} u} du &= \frac{1}{2} \ln \frac{1 + \operatorname{sn} u}{1 - \operatorname{sn} u} & \text{H 88(172)} \\
&= \ln \frac{1 + \operatorname{sn} u}{\operatorname{cn} u} & \text{JA} \\
6. \quad \int \frac{\operatorname{dn} u}{\operatorname{sn} u} du &= \frac{1}{2} \ln \frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u} & \text{H 87(170)}
\end{aligned}$$

5.136

$$\begin{aligned}
1. \quad \int \operatorname{sn} u \operatorname{cn} u \, du &= -\frac{1}{k^2} \operatorname{dn} u \\
2. \quad \int \operatorname{sn} u \operatorname{dn} u \, du &= -\operatorname{cn} u \\
3. \quad \int \operatorname{cn} u \operatorname{dn} u \, du &= \operatorname{sn} u
\end{aligned}$$

5.137

$$1. \quad \int \frac{\operatorname{sn} u}{\operatorname{cn}^2 u} du = \frac{1}{k'^2} \frac{\operatorname{dn} u}{\operatorname{cn} u} \quad \text{H 88(173)}$$

$$2. \quad \int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u} du = -\frac{1}{k'^2} \frac{\operatorname{cn} u}{\operatorname{dn} u} \quad \text{H 88(175)}$$

$$3. \quad \int \frac{\operatorname{cn} u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{dn} u}{\operatorname{sn} u} \quad \text{H 88(174)}$$

$$4. \quad \int \frac{\operatorname{cn} u}{\operatorname{dn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{dn} u} \quad \text{H 88(177)}$$

$$5. \quad \int \frac{\operatorname{dn} u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{cn} u}{\operatorname{sn} u} \quad \text{H 88(176)}$$

$$6. \quad \int \frac{\operatorname{dn} u}{\operatorname{cn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{cn} u} \quad \text{H 88(178)}$$

5.138

$$1. \quad \int \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} du = \ln \frac{\operatorname{sn} u}{\operatorname{dn} u} \quad \text{H 88(183)}$$

$$2. \quad \int \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} du = \frac{1}{k'^2} \ln \frac{\operatorname{dn} u}{\operatorname{cn} u} \quad \text{H 88(182)}$$

$$3. \quad \int \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} du = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u} \quad \text{H 88(184)}$$

5.139

$$1.^{11} \quad \int \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} du = \ln \operatorname{sn} u \quad \text{H 88(179)}$$

$$2. \quad \int \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} du = \ln \frac{1}{\operatorname{cn} u} \quad \text{H 88(180)}$$

$$3. \quad \int \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} du = -\frac{1}{k^2} \ln \operatorname{dn} u \quad \text{H 88(181)}$$

5.14 Weierstrass elliptic functions

The invariants g_1 and g_2 used below are defined in 8.161.

5.141

$$1. \quad \int \wp(u) du = -\zeta(u)$$

$$2. \quad \int \wp^2(u) du = \frac{1}{6} \wp'(u) + \frac{1}{12} g_2 u \quad \text{H 120(192)}$$

$$3. \quad \int \wp^3(u) du = \frac{1}{120} \wp'''(u) - \frac{3}{20} g_2 \zeta(u) + \frac{1}{10} g_3 u \quad \text{H 120(193)}$$

$$4.^8 \quad \int \frac{du}{\wp(u) - \wp(v)} = \frac{1}{\wp'(v)} \left[2u \zeta(v) + \ln \frac{\sigma(u-v)}{\sigma(u+v)} \right] \quad [\wp(v) \neq e_1, e_2, e_3] \quad (\text{see } \mathbf{8.162})$$

H 120(194)

$$5. \quad \int \frac{\alpha \wp(u) + \beta}{\gamma \wp(u) + \delta} du = \frac{\alpha u}{\gamma} + \frac{\alpha \delta - \beta \gamma}{\gamma^2 \wp'(v)} \left[\ln \frac{\sigma(u+v)}{\sigma(u-v)} - 2u \zeta(v) \right]$$

where $v = \wp^{-1} \left(\frac{-\delta}{\gamma} \right)$ H 120(195)

5.2 The Exponential Integral Function

5.21 The exponential integral function

$$\begin{aligned}
 5.211 \quad \int_x^\infty \text{Ei}(-\beta x) \text{Ei}(-\gamma x) dx &= \left(\frac{1}{\beta} + \frac{1}{\gamma} \right) \text{Ei}[-(\beta + \gamma)x] \\
 &\quad - x \text{Ei}(-\beta x) \text{Ei}(-\gamma x) - \frac{e^{-\beta x}}{\beta} \text{Ei}(-\gamma x) - \frac{e^{-\gamma x}}{\gamma} \text{Ei}(-\beta x) \\
 &\quad [\text{Re}(\beta + \gamma) > 0]
 \end{aligned}
 \tag{NT 53(2)}$$

5.22 Combinations of the exponential integral function and powers

5.221

$$\begin{aligned}
 1. \quad \int_x^\infty \frac{\text{Ei}[-a(x+b)]}{x^{n+1}} dx &= \left[\frac{1}{x^n} - \frac{(-1)^n}{b^n} \right] \frac{\text{Ei}[-a(x+b)]}{n} + \frac{e^{-ab}}{n} \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{b^{n-k}} \int_x^\infty \frac{e^{-ax}}{x^{k+1}} dx \\
 &\quad [a > 0, \quad b > 0]
 \end{aligned}
 \tag{NT 52(3)}$$

$$\begin{aligned}
 2. \quad \int_x^\infty \frac{\text{Ei}[-a(x+b)]}{x^2} dx &= \left(\frac{1}{x} + \frac{1}{b} \right) \text{Ei}[-a(x+b)] - \frac{e^{-ab} \text{Ei}(-ax)}{b} \\
 &\quad [a > 0, b > 0]
 \end{aligned}
 \tag{NT 52(4)}$$

$$3.* \quad \int x \text{Ei}(-ax) dx = \frac{x^2}{2} \text{Ei}(-ax) + \frac{1}{2a^2} e^{-ax} + \frac{x e^{-ax}}{2a} \quad [a > 0]$$

$$\begin{aligned}
 4.* \quad \int x^n \text{Ei}(-ax) dx &= \frac{x^{n+1}}{n+1} \text{Ei}(-ax) + \frac{n! e^{-ax}}{(n+1)a^{n+1}} \sum_{k=0}^{\infty} \frac{(ax)^k}{k!} \\
 &\quad [a > 0]
 \end{aligned}$$

$$\begin{aligned}
 5.* \quad \int x \text{Ei}(-ax) e^{-bx} dx &= \frac{1}{b^2} \text{Ei}[-(a+b)x] - \frac{1}{b^2} \text{Ei}(-ax) e^{-bx} - \frac{x}{b} \text{Ei}(-ax) e^{-bx} - \frac{1}{b(a+b)} e^{-(a+b)x} \\
 &\quad [a > 0, \quad b > 0]
 \end{aligned}$$

$$\begin{aligned}
 6.* \quad \int \text{Ei}^2(-ax) dx &= x \text{Ei}^2(-ax) + \frac{2}{a} [\text{Ei}(-ax) e^{-ax} - \text{Ei}(-2ax)] \\
 &\quad [a > 0]
 \end{aligned}$$

$$\begin{aligned}
 7.* \quad \int x \text{Ei}^2(-ax) dx &= \frac{x^2}{2} \text{Ei}^2(-ax) + \left(\frac{1}{a^2} + \frac{x}{a} \right) \text{Ei}(-ax) e^{-ax} - \frac{1}{a^2} \text{Ei}(-2ax) + \frac{1}{a^2} e^{-2ax} \\
 &\quad [a > 0]
 \end{aligned}$$

$$\begin{aligned}
 8.* \quad \int_0^u \text{Ei}(-ax) dx &= u \text{Ei}(-au) + \frac{e^{-au} - 1}{a} \\
 &\quad [a > 0]
 \end{aligned}$$

$$\begin{aligned}
 9.* \quad \int_0^\infty x \text{Ei}\left(-\frac{x}{a}\right) \text{Ei}\left(-\frac{x}{b}\right) dx &= \left(\frac{a^2 + b^2}{2} \right) \ln(a+b) - \frac{a^2}{2} \ln a - \frac{b^2}{2} \ln b - \frac{ab}{2} \\
 &\quad [a > 0, \quad b > 0]
 \end{aligned}$$

$$10.^* \quad \int_0^\infty x^2 \operatorname{Ei}\left(-\frac{x}{a}\right) \operatorname{Ei}\left(-\frac{x}{b}\right) dx = \frac{2}{3} \left[(a^3 + b^3) \ln(a+b) - a^3 \ln a - b^3 \ln b - \frac{ab}{a+b} (a^2 - ab + b^2) \right]$$

$$[a > 0, \quad b > 0]$$

5.23 Combinations of the exponential integral and the exponential

5.231

$$1. \quad \int_0^x e^x \operatorname{Ei}(-x) dx = -\ln x - C + e^x \operatorname{Ei}(-x) \quad \text{ET II 308(11)}$$

$$1. \quad \int_0^x e^{-\beta x} \operatorname{Ei}(-\alpha x) dx = -\frac{1}{\beta} \left\{ e^{-\beta x} \operatorname{Ei}(-\alpha x) + \ln \left(1 + \frac{\beta}{\alpha} \right) - \operatorname{Ei}[-(\alpha + \beta)x] \right\} \quad \text{ET II 308(12)}$$

5.3 The Sine Integral and the Cosine Integral

5.31

$$1. \quad \int \cos \alpha x \operatorname{ci}(\beta x) dx = \frac{\sin \alpha x \operatorname{ci}(\beta x)}{\alpha} - \frac{\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(1)}$$

$$2. \quad \int \sin \alpha x \operatorname{ci}(\beta x) dx = -\frac{\cos \alpha x \operatorname{ci}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x + \beta x) + \operatorname{ci}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(2)}$$

5.32

$$1. \quad \int \cos \alpha x \operatorname{si}(\beta x) dx = \frac{\sin \alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x + \beta x) - \operatorname{ci}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(3)}$$

$$2. \quad \int \sin \alpha x \operatorname{si}(\beta x) dx = -\frac{\cos \alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{si}(\alpha x + \beta x) - \operatorname{si}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(4)}$$

5.33

$$1. \quad \int \operatorname{ci}(\alpha x) \operatorname{ci}(\beta x) dx = x \operatorname{ci}(\alpha x) \operatorname{ci}(\beta x) + \frac{1}{2\alpha} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x))$$

$$+ \frac{1}{2\beta} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\beta x - \alpha x)) - \frac{1}{\alpha} \sin \alpha x \operatorname{ci}(\beta x) - \frac{1}{\beta} \sin \beta x \operatorname{ci}(\alpha x)$$

NT 53(5)

$$2. \quad \int \operatorname{si}(\alpha x) \operatorname{si}(\beta x) dx = x \operatorname{si}(\alpha x) \operatorname{si}(\beta x) - \frac{1}{2\beta} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x))$$

$$- \frac{1}{2\alpha} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\beta x + \alpha x)) + \frac{1}{\alpha} \cos \alpha x \operatorname{si}(\beta x) + \frac{1}{\beta} \cos \beta x \operatorname{si}(\alpha x)$$

NT 54(6)

$$3. \quad \int \operatorname{si}(\alpha x) \operatorname{ci}(\beta x) dx = x \operatorname{si}(\alpha x) \operatorname{ci}(\beta x) + \frac{1}{\alpha} \cos \alpha x \operatorname{ci}(\beta x)$$

$$- \frac{1}{\beta} \sin \beta x \operatorname{si}(\alpha x) - \left(\frac{1}{2\alpha} + \frac{1}{2\beta} \right) \operatorname{ci}(\alpha x + \beta x) - \left(\frac{1}{2\alpha} - \frac{1}{2\beta} \right) \operatorname{ci}(\alpha x - \beta x)$$

NT 54(10)

5.34

$$1. \quad \int_x^\infty \text{si}[a(x+b)] \frac{dx}{x^2} = \left(\frac{1}{x} + \frac{1}{b}\right) \text{si}[a(x+b)] - \frac{\cos ab \text{si}(ax) + \sin ab \text{ci}(ax)}{b} \\ [a > 0, \quad b > 0] \quad \text{NT 52(6)}$$

$$2. \quad \int_x^\infty \text{ci}[a(x+b)] \frac{dx}{x^2} = \left(\frac{1}{x} + \frac{1}{b}\right) \text{ci}[a(x+b)] + \frac{\sin ab \text{si}(ax) - \cos ab \text{ci}(ax)}{b} \\ [a > 0, \quad b > 0] \quad \text{NT 52(5)}$$

5.4 The Probability Integral and Fresnel Integrals

$$5.41^{11} \quad \int \Phi(\alpha x) dx = x \Phi(\alpha x) + \frac{e^{-\alpha^2 x^2}}{\alpha \sqrt{\pi}} \quad \text{NT 12(20)a}$$

$$5.42 \quad \int S(\alpha x) dx = x S(\alpha x) + \frac{\cos^2 \alpha x^2}{\alpha \sqrt{2\pi}} \quad \text{NT 12(22)a}$$

$$5.43 \quad \int C(\alpha x) dx = x C(\alpha x) - \frac{\sin^2 \alpha x^2}{\alpha \sqrt{2\pi}} \quad \text{NT 12(21)a}$$

5.5 Bessel Functions

Notation: Z and $\mathfrak{Z}$ denote any of $J, N, H^{(1)}, H^{(2)}$. In formulae 5.52–5.56, $Z_p(x)$ and $\mathfrak{Z}_p(x)$ are arbitrary Bessel functions of the first, second, or third kinds.

$$5.51 \quad \int J_p(x) dx = 2 \sum_{k=0}^{\infty} J_{p+2k+1}(x) \quad \text{JA, MO 30}$$

5.52

$$1. \quad \int x^{p+1} Z_p(x) dx = x^{p+1} Z_{p+1}(x) \quad \text{WA 132(1)}$$

$$2.^{11} \quad \int x^{-p} Z_{p+1}(x) dx = -x^{-p} Z_p(x) \quad \text{WA 132(2)}$$

$$5.53^{10} \quad \int \left[(\alpha^2 - \beta^2) x - \frac{p^2 - q^2}{x} \right] Z_p(\alpha x) \mathfrak{Z}_q(\beta x) dx \\ = \alpha x Z_{p+1}(\alpha x) \mathfrak{Z}_q(\beta x) - \beta x Z_p(\alpha x) \mathfrak{Z}_{q+1}(\beta x) - (p - q) Z_p(\alpha x) \mathfrak{Z}_q(\beta x) \\ = \beta x Z_p(\alpha x) \mathfrak{Z}_{q-1}(\beta x) - \alpha x Z_{p-1}(\alpha x) \mathfrak{Z}_q(\beta x) + (p - q) Z_p(\alpha x) \mathfrak{Z}_q(\beta x) \\ \text{JA, MO 30, WA 134(7)}$$

5.54

$$1.^{10} \quad \int x Z_p(\alpha x) \mathfrak{Z}_p(\beta x) dx = \frac{\alpha x Z_{p+1}(\alpha x) \mathfrak{Z}_p(\beta x) - \beta x Z_p(\alpha x) \mathfrak{Z}_{p+1}(\beta x)}{\alpha^2 - \beta^2} \\ = \frac{\beta x Z_p(\alpha x) \mathfrak{Z}_{p-1}(\beta x) - \alpha x Z_{p-1}(\alpha x) \mathfrak{Z}_p(\beta x)}{\alpha^2 - \beta^2} \quad \text{WA 134(8)}$$

$$2. \quad \int x [Z_p(\alpha x)]^2 dx = \frac{x^2}{2} \left\{ [Z_p(\alpha x)]^2 - Z_{p-1}(\alpha x) Z_{p+1}(\alpha x) \right\} \quad \text{WA 135(11)}$$

$$3.^* \quad \int x Z_p(ax) \mathfrak{Z}_p(ax) dx = \frac{x^4}{4} [2 Z_p(ax) \mathfrak{Z}_p(ax) - Z_{p-1}(ax) \mathfrak{Z}_{p+1}(ax) - Z_{p+1}(ax) \mathfrak{Z}_{p-1}(ax)]$$

$$\begin{aligned} 5.55^{10} \quad \int \frac{1}{x} Z_p(\alpha x) \mathfrak{Z}_q(\alpha x) dx &= \alpha x \frac{Z_p(\alpha x) \mathfrak{Z}_{q+1}(\alpha x) - Z_{p+1}(\alpha x) \mathfrak{Z}_q(\alpha x)}{p^2 - q^2} + \frac{Z_p(\alpha x) \mathfrak{Z}_q(\alpha x)}{p + q} \\ &= \alpha x \frac{Z_{p-1}(\alpha x) \mathfrak{Z}_q(\alpha x) - Z_p(\alpha x) \mathfrak{Z}_{q-1}(\alpha x)}{p^2 - q^2} - \frac{Z_p(\alpha x) \mathfrak{Z}_q(\alpha x)}{p + q} \end{aligned}$$

WA 135(13)

5.56

$$1. \quad \int Z_1(x) dx = -Z_0(x) \quad \text{JA}$$

$$2. \quad \int x Z_0(x) dx = x Z_1(x) \quad \text{JA}$$

6–7 Definite Integrals of Special Functions

6.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1 - k^2}$ (cf. 8.1).

6.11 Forms containing $F(x, k)$

$$\mathbf{6.111} \quad \int_0^{\pi/2} F(x, k) \cot x \, dx = \frac{\pi}{4} \mathbf{K}(k') + \frac{1}{2} \ln k \mathbf{K}(k) \quad \text{BI (350)(1)}$$

$$\mathbf{6.112} \quad 1. \quad \int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 + k \sin^2 x} \, dx = \frac{1}{4k} \mathbf{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \mathbf{K}(k') \quad \text{BI (350)(6)}$$

$$2. \quad \int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 - k \sin^2 x} \, dx = \frac{1}{4k} \mathbf{K}(k) \ln \frac{2}{(1-k)\sqrt{k}} - \frac{\pi}{16k} \mathbf{K}(k') \quad \text{BI (350)(7)}$$

$$3. \quad \int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x} \, dx = -\frac{1}{2k^2} \ln k' \mathbf{K}(k) \quad \text{BI (350)(2)a, BY(802.12)a}$$

6.113

$$1. \quad \int_0^{\pi/2} F(x, k') \frac{\sin x \cos x \, dx}{\cos^2 x + k \sin^2 x} = \frac{1}{4(1-k)} \ln \frac{2}{(1+k)\sqrt{k}} \mathbf{K}(k') \quad \text{BI (350)(5)}$$

$$2. \quad \int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \\ = -\frac{1}{k^2 \sin t \cos t} \left[\mathbf{K}(k) \arctan(k' \tan t) - \frac{\pi}{2} F(t, k) \right] \quad \text{BI (350)(12)}$$

$$\mathbf{6.114} \quad \int_u^v F(x, k) \frac{dx}{\sqrt{(\sin^2 x - \sin^2 u)(\sin^2 v - \sin^2 x)}} = \frac{1}{2 \cos u \sin v} \mathbf{K}(k) \mathbf{K} \left(\sqrt{1 - \tan^2 u \cot^2 v} \right) \\ [k^2 = 1 - \cot^2 u \cdot \cot^2 v] \quad \text{BI (351)(9)}$$

$$\mathbf{6.115} \quad \int_0^1 F(\arcsin x, k) \frac{x \, dx}{1 + kx^2} = \frac{1}{4k} \mathbf{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \mathbf{K}(k') \\ \text{(cf. 6.112 2)} \quad \text{BI (466)(1)}$$

This and similar formulas can be obtained from formulas **6.111–6.113** by means of the substitution $x = \arcsin t$.

6.12 Forms containing $E(x, k)$

$$\mathbf{6.121} \quad \int_0^{\pi/2} E(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x} dx = \frac{1}{2k^2} \left\{ (1 + k'^2) \mathbf{K}(k) - (2 + \ln k') E(k) \right\} \quad \text{BI (350)(4)}$$

$$\mathbf{6.122} \quad \int_0^{\pi/2} E(x, k) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \{ E(k) \mathbf{K}(k) - \ln k' \} \quad \text{BI (350)(10), BY (630.02)}$$

$$\begin{aligned} \mathbf{6.123} \quad \int_0^{\pi/2} E(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \\ = -\frac{1}{k^2 \sin t \cos t} \left[E(k) \arctan(k' \tan t) - \frac{\pi}{2} E(t, k) + \frac{\pi}{2} \cot t \left(1 - \sqrt{1 - k^2 \sin^2 t} \right) \right] \end{aligned} \quad \text{BI (350)(13)}$$

$$\begin{aligned} \mathbf{6.124} \quad \int_u^v E(x, k) \frac{dx}{\sqrt{(\sin^2 x - \sin^2 u)(\sin^2 v - \sin^2 x)}} = \frac{1}{2 \cos u \sin v} E(k) \mathbf{K} \left(\sqrt{1 - \frac{tg^2 u}{tg^2 v}} \right) \\ + \frac{k^2 \sin v}{2 \cos u} \mathbf{K} \left(\sqrt{1 - \frac{\sin^2 2u}{\sin^2 2v}} \right) \\ [k^2 = 1 - \cot^2 u \cot^2 v] \quad \text{BI (351)(10)} \end{aligned}$$

6.13 Integration of elliptic integrals with respect to the modulus

$$\mathbf{6.131} \quad \int_0^1 F(x, k) k dk = \frac{1 - \cos x}{\sin x} = \tan \frac{x}{2} \quad \text{BY (616.03)}$$

$$\mathbf{6.132} \quad \int_0^1 E(x, k) k dk = \frac{\sin^2 x + 1 - \cos x}{3 \sin x} \quad \text{BY (616.04)}$$

$$\mathbf{6.133} \quad \int_0^1 \Pi(x, r^2, k) k dk = \tan \frac{x}{2} - r \ln \sqrt{\frac{1 + r \sin x}{1 - r \sin x}} - r^2 \Pi(x, r^2, 0) \quad \text{BY (616.05)}$$

6.14–6.15 Complete elliptic integrals

6.141

$$1. \quad \int_0^1 \mathbf{K}(k) dk = 2G \quad \text{FI II 755}$$

$$2. \quad \int_0^1 \mathbf{K}(k') dk = \frac{\pi^2}{4} \quad \text{BY (615.03)}$$

$$\mathbf{6.142} \quad \int_0^1 \left(\mathbf{K}(k) - \frac{\pi}{2} \right) \frac{dk}{k} = \pi \ln 2 - 2G \quad \text{BY (615.05)}$$

$$\mathbf{6.143}^7 \quad \int_0^1 \mathbf{K}(k) \frac{dk}{k'} = \mathbf{K}^2 \left(\frac{\sqrt{2}}{2} \right) = \frac{1}{16\pi} \Gamma^4 \left(\frac{1}{4} \right) \quad \text{BY (615.08)}$$

$$\mathbf{6.144} \quad \int_0^1 \mathbf{K}(k) \frac{dk}{1+k} = \frac{\pi^2}{8} \quad \text{BY (615.09)}$$

$$\mathbf{6.145} \quad \int_0^1 \left(\mathbf{K}(k') - \ln \frac{4}{k} \right) \frac{dk}{k} = \frac{1}{12} \left[24 (\ln 2)^2 - \pi^2 \right] \quad \text{BY (615.13)}$$

$$\mathbf{6.146} \quad n^2 \int_0^1 k^n \mathbf{K}(k) dk = (n-1)^2 \int_0^1 k^{n-2} \mathbf{K}(k) dk + 1 \quad \text{BY (615.12)}$$

$$\mathbf{6.147} \quad n \int_0^1 k^n \mathbf{K}(k') dk = (n-1) \int_0^1 k^{n-2} \mathbf{E}(k) dk \quad [n > 1] \quad (\text{see } \mathbf{6.152}) \quad \text{BY (615.11)}$$

6.148

$$1. \quad \int_0^1 \mathbf{E}(k) dk = \frac{1}{2} + \mathbf{G} \quad \text{BY (615.02)}$$

$$2. \quad \int_0^1 \mathbf{E}(k') dk = \frac{\pi^2}{8} \quad \text{BY (615.04)}$$

$$3.* \quad \int_0^1 \frac{\mathbf{E}(k)}{1+k} dk = 1$$

6.149

$$1. \quad \int_0^1 \left(\mathbf{E}(k) - \frac{\pi}{2} \right) \frac{dk}{k} = \pi \ln 2 - 2\mathbf{G} + 1 - \frac{\pi}{2} \quad \text{BY (615.06)}$$

$$2. \quad \int_0^1 (\mathbf{E}(k') - 1) \frac{dk}{k} = 2 \ln 2 - 1 \quad \text{BY (615.07)}$$

$$3.* \quad \int_0^1 \frac{\mathbf{E}(k)}{1+k} dk = 1$$

$$4.* \quad \int_0^1 \frac{dx}{x^3} \left(\sqrt{a-x^2} \mathbf{K}(x) - \frac{\mathbf{E}(x)}{\sqrt{1-x^2}} + \frac{\pi}{4} x^2 \right) = -\frac{\pi}{4} \ln \left(\frac{4}{\sqrt{e}} \right)$$

$$\mathbf{6.151} \quad \int_0^1 \mathbf{E}(k) \frac{dk}{k'} = \frac{1}{8} \left[4 \mathbf{K}^2 \left(\frac{\sqrt{2}}{2} \right) + \frac{\pi^2}{\mathbf{K}^2 \left(\frac{\sqrt{2}}{2} \right)} \right] \quad \text{BY (615.10)}$$

$$\mathbf{6.152} \quad (n+2) \int_0^1 k^n \mathbf{E}(k') dk = (n+1) \int_0^1 k^n \mathbf{K}(k') dk \quad [n > 1] \quad (\text{see } \mathbf{6.147}) \quad \text{BY (615.14)}$$

$$\mathbf{6.153}^6 \quad \int_0^a \frac{\mathbf{K}(k) k dk}{k'^2 \sqrt{a^2 - k^2}} = \frac{\pi}{4} \frac{1}{\sqrt{1-a^2}} \ln \left(\frac{1+a}{1-a} \right) \quad [0 < a < 1] \quad \text{LO I 252}$$

$$\mathbf{6.154} \quad \int_0^{\pi/2} \frac{\mathbf{E}(p \sin x)}{1-p^2 \sin^2 x} \sin x dx = \frac{\pi}{2\sqrt{1-p^2}} \quad [p^2 > 1] \quad \text{FI II 489}$$

6.16 The theta function

6.161

$$1. \quad \int_0^\infty x^{s-1} \vartheta_2(0 | ix^2) dx = 2^s (1-2^{-s}) \pi^{-\frac{s}{2}} \Gamma\left(\frac{1}{2}s\right) \zeta(s) \quad [\text{Re } s > 2] \quad \text{ET I 339(20)}$$

$$2. \quad \int_0^\infty x^{s-1} [\vartheta_3(0 | ix^2) - 1] dx = \pi^{-\frac{s}{2}} \Gamma\left(\frac{1}{2}s\right) \zeta(s) \quad [\text{Re } s > 2] \quad \text{ET I 339(21)}$$

$$3. \quad \int_0^\infty x^{s-1} [1 - \vartheta_4(0 | ix^2)] dx = (1 - 2^{1-s}) \pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) \\ [\operatorname{Re} s > 2] \quad \text{ET I 339(22)}$$

$$4. \quad \int_0^\infty x^{s-1} [\vartheta_4(0 | ix^2) + \vartheta_2(0 | ix^2) - \vartheta_3(0 | ix^2)] dx = -(2^s - 1)(2^{1-s} - 1) \pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) \\ \text{ET I 339(24)}$$

6.162

$$1.^{11} \quad \int_0^\infty e^{-ax} \vartheta_4\left(\frac{b\pi}{2l} \left| \frac{i\pi x}{l^2} \right.\right) dx = \frac{l}{\sqrt{a}} \cosh(b\sqrt{a}) \operatorname{cosech}(l\sqrt{a}) \\ [\operatorname{Re} a > 0, \quad |b| \leq l] \quad \text{ET I 224(1)a}$$

$$2. \quad \int_0^\infty e^{-ax} \vartheta_1\left(\frac{b\pi}{2l} \left| \frac{i\pi x}{l^2} \right.\right) dx = -\frac{l}{\sqrt{a}} \sinh(b\sqrt{a}) \operatorname{sech}(l\sqrt{a}) \\ [\operatorname{Re} a > 0, \quad |b| \leq l] \quad \text{ET I 224(2)a}$$

$$3.^{11} \quad \int_0^\infty e^{-ax} \vartheta_2\left(\frac{(l+b)\pi}{2l} \left| \frac{i\pi x}{l^2} \right.\right) dx = -\frac{l}{\sqrt{a}} \sinh(b\sqrt{a}) \operatorname{sech}(l\sqrt{a}) \\ [\operatorname{Re} a > 0, \quad |b| \leq l] \quad \text{ET I 224(3)a}$$

$$4.^{11} \quad \int_0^\infty e^{-ax} \vartheta_3\left(\frac{(l+b)\pi}{2l} \left| \frac{i\pi x}{l^2} \right.\right) dx = \frac{l}{\sqrt{a}} \cosh(b\sqrt{a}) \operatorname{cosech}(l\sqrt{a}) \\ [\operatorname{Re} a > 0, \quad |b| \leq l] \quad \text{ET I 224(4)a}$$

6.163¹⁰

$$1. \quad \int_0^\infty e^{-(a-\mu)x} \vartheta_3(\pi\sqrt{\mu}x | i\pi x) dx = \frac{1}{2\sqrt{a}} [\coth(\sqrt{a} + \sqrt{\mu}) + \coth(\sqrt{a} - \sqrt{\mu})] \\ [\operatorname{Re} a > 0] \quad \text{ET I 224(7)a}$$

$$2.^{10} \quad \int_0^\infty \vartheta_3(i\pi kx | i\pi x) e^{-(k^2+l^2)x} dx = \frac{\sinh 2l}{l(\cosh 2l - \cos 2k)}$$

$$\mathbf{6.164}^{11} \quad \int_0^\infty [\vartheta_4(0 | ie^{2x}) + \vartheta_2(0 | ie^{2x}) - \vartheta_3(0 | ie^{2x})] e^{\frac{1}{2}x} \cos(ax) dx \\ = \frac{1}{2} \left(2^{\frac{1}{2}+ia} - 1\right) \left(1 - 2^{\frac{1}{2}-ia}\right) \pi^{-\frac{1}{4}-\frac{1}{2}ia} \Gamma\left(\frac{1}{4} + \frac{1}{2}ia\right) \zeta\left(\frac{1}{2} + ia\right) \\ [a > 0] \quad \text{ET I 61(11)}$$

$$\mathbf{6.165} \quad \int_0^\infty e^{\frac{1}{2}x} [\vartheta_3(0 | ie^{2x}) - 1] \cos(ax) dx \\ = \frac{2}{1+4a^2} \left\{1 + \left[\left(a^2 + \frac{1}{4}\right) \pi^{-\frac{1}{2}ia-\frac{1}{4}} \Gamma\left(\frac{1}{2}ia + \frac{1}{4}\right) \zeta\left(ia + \frac{1}{2}\right)\right]\right\} \\ [a > 0] \quad \text{ET I 61(12)}$$

6.17¹⁰ Generalized elliptic integrals

1. Set

$$\Omega_j(k) \equiv \int_0^\pi [1 - k^2 \cos \phi]^{-(j+\frac{1}{2})} d\phi,$$

$$\alpha_m(j) = \frac{\pi}{(64)^m} \frac{j!}{(2j)!} \frac{(4m+2j)!}{(2m+j)!} \left(\frac{1}{m!}\right)^2, \quad \lambda = \frac{\pi}{2} \sqrt{\frac{(2j+1)k^2}{1-k^2}},$$

then

$$\begin{aligned} \Omega_j(k) &= \sum_{m=0}^{\infty} \alpha_m(j) k^{4m} = \sqrt{\frac{\pi}{(2j+1)k^2}} (1-k^2)^{-j} \left[\operatorname{erf} \lambda + \frac{1}{2}(2j+1)^{-1} \left(1 + \frac{1}{2k^2}\right) \right. \\ &\quad \times \left\{ \operatorname{erf} \lambda - \left(\frac{2}{\sqrt{\pi}}\right) (\lambda e^{-\lambda^2}) \left(1 + \frac{2}{3}\lambda^2\right) \right\} - \frac{1}{12}(2j+1)^{-2} \left(16 + \frac{13}{k^2} + \frac{1}{k^4}\right) \\ &\quad \times \left. \left\{ \operatorname{erf} \lambda - \left(\frac{2}{\sqrt{\pi}}\right) (\lambda e^{-\lambda^2}) \left(1 + \frac{2}{3}\lambda^2 + \frac{4}{15}\lambda^4\right) \right\} + \dots \right] \end{aligned}$$

while for large λ

$$\begin{aligned} \lim_{j \rightarrow \infty} \Omega_j(k) &= \sqrt{\frac{\pi}{(2j+1)k^2}} (1-k^2)^{-j} \\ &\quad \times \left[1 + \frac{1}{2}(2j+1)^{-1} \left\{ 1 + \frac{1}{2k^2} \right\} - \frac{4}{3}(2j+1)^{-2} \left\{ 1 + \frac{13}{16k^2} + \frac{1}{16k^4} \right\} + \dots \right] \end{aligned}$$

2. Set

$$\begin{aligned} R_\mu(k, \alpha, \delta) &= \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\delta-2\alpha-1}(\theta/2) d\theta}{[1 - k^2 \cos \theta]^{\mu+\frac{1}{2}}}, \\ &\quad 0 < k < 1, \quad \operatorname{Re} \delta > \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -1/2, \\ M_\nu(\mu, \alpha, \delta) &= \frac{(-1)^\nu 2^\nu (\mu + \frac{1}{2})_\nu}{\nu!} \frac{\Gamma(\alpha) \Gamma(\delta - \alpha + \nu)}{\Gamma(\delta + \nu)}, \\ &\quad \text{with } (\lambda)_\nu = \Gamma(\lambda + \nu) / \Gamma(\lambda), \text{ and} \\ W_\nu(\mu, \alpha, \delta) &= \frac{2^\nu (\mu + \frac{1}{2})_\nu}{\nu!} \frac{\Gamma(\alpha + \nu) \Gamma(\delta - \alpha)}{\Gamma(\delta + \nu)}, \end{aligned}$$

then:

- for small k :

$$\begin{aligned} R_\mu(k, \alpha, \delta) &= (1-k^2)^{-(\mu+\frac{1}{2})} \sum_{\nu=0}^{\infty} [k^2 / (1-k^2)]^\nu M_\nu(\mu, \alpha, \delta) \\ &= (1+k^2)^{-(\mu+\frac{1}{2})} \sum_{\nu=0}^{\infty} [k^2 / (1+k^2)]^\nu W_\nu(\mu, \alpha, \delta), \end{aligned}$$

- for k^2 close to 1:

$$\begin{aligned}
 R_\mu(k, \alpha, \delta) &= [\Gamma(\delta - \alpha) \Gamma(\mu + \alpha - \delta + \tfrac{1}{2}) \Gamma(\mu + \tfrac{1}{2})] (2k^2)^{\alpha - \delta} (1 - k^2)^{\delta - \alpha - \mu - \frac{1}{2}} \\
 &\quad \times \left\{ \Gamma(\delta - \alpha - \mu - \tfrac{1}{2}) \Gamma(\alpha) \left[\Gamma(\delta - \mu - \tfrac{1}{2}) (2k^2)^{\mu + \frac{1}{2}} \right] \right\} \\
 &\quad \left[\operatorname{Re}(\mu + \alpha - \delta + \tfrac{1}{2}) \text{ not an integer} \right] \\
 &= \left[2^{\mu + \frac{1}{2}} k^{2\mu + 1} \Gamma(\mu + \tfrac{1}{2}) \Gamma(1 - \alpha) \right] \\
 &\quad \times \sum_{n=0}^{\infty} [\Gamma(\delta - \alpha + n) \Gamma(1 - \alpha + n) \Gamma(\alpha - \delta + \mu - n + \tfrac{1}{2}) n!] [2k^2 / (1 - k^2)]^{\alpha - \delta + \mu - n + \frac{1}{2}} \\
 &\quad \left[\alpha - \delta + \mu + \tfrac{1}{2} = m, \text{ with } m \text{ a non-negative integer} \right]
 \end{aligned}$$

6.2–6.3 The Exponential Integral Function and Functions Generated by It

6.21 The logarithm integral

$$6.211 \quad \int_0^1 \operatorname{li}(x) dx = -\ln 2 \quad \text{BI (79)(5)}$$

6.212

$$1. \quad \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) x dx = 0 \quad \text{BI (255)(1)}$$

$$2. \quad \int_0^1 \operatorname{li}(x) x^{p-1} dx = -\frac{1}{p} \ln(p+1) \quad [p > -1] \quad \text{BI (255)(2)}$$

$$3. \quad \int_0^1 \operatorname{li}(x) \frac{dx}{x^{q+1}} = \frac{1}{q} \ln(1-q) \quad [q < 1] \quad \text{BI (255)(3)}$$

$$4. \quad \int_1^\infty \operatorname{li}(x) \frac{dx}{x^{q+1}} = -\frac{1}{q} \ln(q-1) \quad [q > 1] \quad \text{BI (255)(4)}$$

6.213

$$1. \quad \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x) dx = \frac{1}{1+a^2} \left(a \ln a - \frac{\pi}{2} \right) \quad [a > 0] \quad \text{BI (475)(1)}$$

$$2. \quad \int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x) dx = -\frac{1}{1+a^2} \left(\frac{\pi}{2} + a \ln a \right) \quad [a > 0] \quad \text{BI (475)(9)}$$

$$3. \quad \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x) dx = -\frac{1}{1+a^2} \left(\ln a + \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (475)(2)}$$

$$4. \quad \int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x) dx = \frac{1}{1+a^2} \left(\ln a - \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (475)(10)}$$

$$5. \quad \int_0^1 \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x} = \frac{\ln(1+a^2)}{2a} \quad [a > 0] \quad \text{BI(479)(1), ET I 98(20)a}$$

$$6. \quad \int_0^1 \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x} = -\frac{\arctan a}{a} \quad \text{BI (479)(2)}$$

$$7. \quad \int_0^1 \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left(a \ln a + \frac{\pi}{2} \right) \quad [a > 0] \quad \text{BI (479)(3)}$$

$$8. \quad \int_1^\infty \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left(\frac{\pi}{2} - a \ln a \right) \quad [a > 0] \quad \text{BI (479)(13)}$$

$$9. \quad \int_0^1 \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left(\ln a - \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (479)(4)}$$

$$10. \quad \int_1^\infty \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x^2} = -\frac{1}{1+a^2} \left(\ln a + \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (479)(14)}$$

$$11. \quad \int_0^1 \operatorname{li}(x) \sin(a \ln x) x^{p-1} dx = \frac{1}{a^2 + p^2} \left\{ \frac{a}{2} \ln [(1+p)^2 + a^2] - p \arctan \frac{a}{1+p} \right\} \\ [p > 0] \quad \text{BI (477)(1)}$$

$$12. \quad \int_0^1 \operatorname{li}(x) \cos(a \ln x) x^{p-1} dx = -\frac{1}{a^2 + p^2} \left\{ a \arctan \frac{a}{1+p} + \frac{p}{2} \ln [(1+p)^2 + a^2] \right\} \\ [p > 0] \quad \text{BI (477)(2)}$$

6.214

$$1. \quad \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \left(\ln \frac{1}{x}\right)^{p-1} dx = -\pi \cot p\pi \cdot \Gamma(p) \quad [0 < p < 1] \quad \text{BI (340)(1)}$$

$$2. \quad \int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) (\ln x)^{p-1} dx = -\frac{\pi}{\sin p\pi} \Gamma(p) \quad [0 < p < 1] \quad \text{BI (340)(9)}$$

6.215

$$1. \quad \int_0^1 \operatorname{li}(x) \frac{x^{p-1}}{\sqrt{\ln\left(\frac{1}{x}\right)}} dx = -2\sqrt{\frac{\pi}{p}} \operatorname{arcsinh} \sqrt{p} = -2\sqrt{\frac{\pi}{p}} \ln(\sqrt{p} + \sqrt{p+1}) \\ [p > 0] \quad \text{BI (444)(3)}$$

$$2. \quad \int_0^1 \operatorname{li}(x) \frac{dx}{x^{p+1} \sqrt{\ln\left(\frac{1}{x}\right)}} = -2\sqrt{\frac{\pi}{p}} \operatorname{arcsin} \sqrt{p} \quad [1 > p > 0] \quad \text{BI (444)(4)}$$

6.216

$$1. \quad \int_0^1 \operatorname{li}(x) \left[\ln\left(\frac{1}{x}\right) \right]^{p-1} \frac{ax}{x} = -\frac{1}{p} \Gamma(p) \quad [0 < p \leq 1] \quad \text{BI (444)(1)}$$

$$2. \quad \int_0^1 \operatorname{li}(x) \left[\ln\left(\frac{1}{x}\right) \right]^{p-1} \frac{dx}{x^2} = -\frac{\pi \Gamma(p)}{\sin p\pi} \quad [0 < p < 1] \quad \text{BI (444)(2)}$$

6.22–6.23 The exponential integral function

$$6.221 \quad \int_0^p \operatorname{Ei}(\alpha x) dx = p \operatorname{Ei}(\alpha p) + \frac{1 - e^{\alpha p}}{\alpha} \quad \text{NT 11(7)}$$

$$6.222 \quad \int_0^\infty \operatorname{Ei}(-px) \operatorname{Ei}(-qx) dx = \left(\frac{1}{p} + \frac{1}{q} \right) \ln(p+q) - \frac{\ln q}{p} - \frac{\ln p}{q} \\ [p > 0, \quad q > 0] \quad \text{FI II 653, NT 53(3)}$$

$$6.223 \quad \int_0^\infty \operatorname{Ei}(-\beta x) x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu \beta^\mu} \quad [\operatorname{Re} \beta \geq 0, \quad \operatorname{Re} \mu > 0] \\ \text{NT 55(7), ET I 325(10)}$$

6.224

$$1. \quad \int_0^\infty \operatorname{Ei}(-\beta x) e^{-\mu x} dx = -\frac{1}{\mu} \ln \left(1 + \frac{\mu}{\beta} \right) \quad [\operatorname{Re}(\beta + \mu) \geq 0, \quad \mu > 0] \\ = -1/\beta \quad [\mu = 0] \\ \text{FI II 652, NT 48(8)}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(ax) e^{-\mu x} dx = -\frac{1}{\mu} \ln \left(\frac{\mu}{a} - 1 \right) \quad [a > 0, \quad \operatorname{Re} \mu > 0, \quad \mu > a] \\ \text{ET I 178(23)a, BI (283)(3)}$$

6.225

$$1. \quad \int_0^\infty \operatorname{Ei}(-x^2) e^{-\mu x^2} dx = -\sqrt{\frac{\pi}{\mu}} \operatorname{arcsinh} \sqrt{\mu} = -\sqrt{\frac{\pi}{\mu}} \ln \left(\sqrt{\mu} + \sqrt{1+\mu} \right) \\ [\operatorname{Re} \mu > 0] \quad \text{BI (283)(5), ET I 178(25)a}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(-x^2) e^{px^2} dx = -\sqrt{\frac{\pi}{p}} \operatorname{arcsin} \sqrt{p} \quad [1 > p > 0] \quad \text{NT 59(9)a}$$

6.226

$$1. \quad \int_0^\infty \operatorname{Ei} \left(-\frac{1}{4x} \right) e^{-\mu x} dx = -\frac{2}{\mu} K_0(\sqrt{\mu}) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$2. \quad \int_0^\infty \operatorname{Ei} \left(\frac{a^2}{4x} \right) e^{-\mu x} dx = -\frac{2}{\mu} K_0(a\sqrt{\mu}) \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$3. \quad \int_0^\infty \operatorname{Ei} \left(-\frac{1}{4x^2} \right) e^{-\mu x^2} dx = \sqrt{\frac{\pi}{\mu}} \operatorname{Ei}(-\sqrt{\mu}) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$4. \quad \int_0^\infty \operatorname{Ei} \left(-\frac{1}{4x^2} \right) e^{-\mu x^2 + \frac{1}{4x^2}} dx = \sqrt{\frac{\pi}{\mu}} [\cos \sqrt{\mu} \operatorname{ci} \sqrt{\mu} - \sin \sqrt{\mu} \operatorname{si} \sqrt{\mu}] \\ [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

6.227

$$1. \quad \int_0^\infty \operatorname{Ei}(-x) e^{-\mu x} x dx = \frac{1}{\mu(\mu+1)} - \frac{1}{\mu^2} \ln(1+\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$2. \quad \int_0^\infty \left[\frac{e^{-ax} \operatorname{Ei}(ax)}{x-b} - \frac{e^{ax} \operatorname{Ei}(-ax)}{x+b} \right] dx = 0 \quad [a > 0, \quad b < 0]$$

$$= \pi^2 e^{-ab} \quad [a > 0, \quad b > 0]$$

ET II 253(1)a

6.228

$$1. \quad \int_0^\infty \operatorname{Ei}(-x) e^x x^{\nu-1} dx = -\frac{\pi \Gamma(\nu)}{\sin \nu \pi} \quad [0 < \operatorname{Re} \nu < 1] \quad \text{ET II 308(13)}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(-\beta x) e^{-\mu x} x^{\nu-1} dx = -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^\nu} {}_2F_1 \left(1, \nu; \nu+1; \frac{\mu}{\beta+\mu} \right)$$

$$[|\arg \beta| < \pi, \quad \operatorname{Re}(\beta+\mu) > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 308(14)}$$

$$6.229 \quad \int_0^\infty \operatorname{Ei} \left(-\frac{1}{4x^2} \right) \exp \left(-\mu x^2 + \frac{1}{4x^2} \right) \frac{dx}{x^2} = 2\sqrt{\pi} (\cos \sqrt{\mu} \operatorname{si} \sqrt{\mu} - \sin \sqrt{\mu} \operatorname{ci} \sqrt{\mu})$$

$$[\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$6.231 \quad \int_{-\ln a}^\infty [\operatorname{Ei}(-a) - \operatorname{Ei}(-e^{-x})] e^{-\mu x} dx = \frac{1}{\mu} \gamma(\mu, a) \quad [a < 1, \quad \operatorname{Re} \mu > 0] \quad \text{MI 34}$$

6.232

$$1. \quad \int_0^\infty \operatorname{Ei}(-ax) \sin bx dx = -\frac{\ln \left(1 + \frac{b^2}{a^2} \right)}{2b} \quad [a > 0, \quad b > 0] \quad \text{BI (473)(1)a}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(-ax) \cos bx dx = -\frac{1}{b} \arctan \frac{b}{a} \quad [a > 0, \quad b > 0] \quad \text{BI (473)(2)a}$$

6.233

$$1. \quad \int_0^\infty \operatorname{Ei}(-x) e^{-\mu x} \sin \beta x dx = -\frac{1}{\beta^2 + \mu^2} \left\{ \frac{\beta}{2} \ln [(1+\mu)^2 + \beta^2] - \mu \arctan \frac{\beta}{1+\mu} \right\}$$

$$[\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{BI (473)(7)a}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(-x) e^{-\mu x} \cos \beta x dx = -\frac{1}{\beta^2 + \mu^2} \left\{ \frac{\mu}{2} \ln [(1+\mu)^2 + \beta^2] + \beta \arctan \frac{\beta}{1+\mu} \right\}$$

$$[\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{BI (473)(8)a}$$

$$6.234 \quad \int_0^\infty \operatorname{Ei}(-x) \ln x dx = C + 1 \quad \text{NT 56(10)}$$

6.24–6.26 The sine integral and cosine integral functions**6.241**

$$1. \quad \int_0^\infty \operatorname{si}(px) \operatorname{si}(qx) dx = \frac{\pi}{2p} \quad [p \geq q] \quad \text{BI II 653, NT 54(8)}$$

$$2. \quad \int_0^\infty \operatorname{ci}(px) \operatorname{ci}(qx) dx = \frac{\pi}{2p} \quad [p \geq q] \quad \text{FI II 653, NT 54(7)}$$

$$\begin{aligned}
 3. \quad \int_0^\infty \operatorname{si}(px) \operatorname{ci}(qx) dx &= \frac{1}{4q} \ln \left(\frac{p+q}{p-q} \right)^2 + \frac{1}{4p} \ln \frac{(p^2 - q^2)^2}{q^4} & [p \neq q] \\
 &= \frac{1}{q} \ln 2 & [p = q]
 \end{aligned}$$

FI II 653, NT 54(10, 12)

$$6.242 \quad \int_0^\infty \frac{\operatorname{ci}(ax)}{\beta + x} dx = -\frac{1}{2} \left\{ [\operatorname{si}(a\beta)]^2 + [\operatorname{ci}(a\beta)]^2 \right\} \quad [a > 0, \quad |\arg \beta| < \pi] \quad \text{ET II 224(1)}$$

6.243

$$1. \quad \int_{-\infty}^\infty \frac{\operatorname{si}(a|x|)}{x-b} \operatorname{sign} x dx = \pi \operatorname{ci}(a|b|) \quad [a > 0, \quad b > 0] \quad \text{ET II 253(3)}$$

$$2. \quad \int_{-\infty}^\infty \frac{\operatorname{ci}(a|x|)}{x-b} dx = -\pi \operatorname{sign} b \cdot \operatorname{si}(a|b|) \quad [a > 0] \quad \text{ET II 253(2)}$$

6.244

$$1.^8 \quad \int_0^\infty \operatorname{si}(px) \frac{x dx}{q^2 + x^2} = \frac{\pi}{2} \operatorname{Ei}(-pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(6)}$$

$$2.^8 \quad \int_0^\infty \operatorname{si}(px) \frac{x dx}{q^2 - x^2} = -\frac{\pi}{2} \operatorname{ci}(pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(6)}$$

6.245

$$1. \quad \int_0^\infty \operatorname{ci}(px) \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \operatorname{Ei}(-pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(7)}$$

$$2. \quad \int_0^\infty \operatorname{ci}(px) \frac{dx}{q^2 - x^2} = \frac{\pi}{2q} \operatorname{si}(pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(8)}$$

6.246

$$1. \quad \int_0^\infty \operatorname{si}(ax) x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu a^\mu} \sin \frac{\mu\pi}{2} \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1]$$

NT 56(9), ET I 325(12)a

$$2. \quad \int_0^\infty \operatorname{ci}(ax) x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu a^\mu} \cos \frac{\mu\pi}{2} \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1]$$

NT 56(8), ET I 325(13)a

6.247

$$1. \quad \int_0^\infty \operatorname{si}(\beta x) e^{-\mu x} dx = -\frac{1}{\mu} \arctan \frac{\mu}{\beta} \quad [\operatorname{Re} \mu > 0] \quad \text{NT 49(12), ET I 177(18)}$$

$$2. \quad \int_0^\infty \operatorname{ci}(\beta x) e^{-\mu x} dx = -\frac{1}{\mu} \ln \sqrt{1 + \frac{\mu^2}{\beta^2}} \quad [\operatorname{Re} \mu > 0] \quad \text{NT 49(11), ET I 178(19)a}$$

6.248

$$1.^8 \quad \int_0^\infty \operatorname{si}(x) e^{-\mu x^2} x dx = \frac{\pi}{4\mu} \left[\Phi \left(\frac{1}{2\sqrt{\mu}} \right) - 1 \right] \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

2. $\int_0^\infty \text{ci}(x) e^{-\mu x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu}} \text{Ei} \left(-\frac{1}{4\mu} \right)$ $[\text{Re } \mu > 0]$ MI 34
- 6.249** $\int_0^\infty \left[\text{si}(x^2) + \frac{\pi}{2} \right] e^{-\mu x} dx = \frac{\pi}{\mu} \left\{ \left[S \left(\frac{\mu^2}{4} \right) - \frac{1}{2} \right]^2 + \left[C \left(\frac{\mu^2}{4} \right) - \frac{1}{2} \right]^2 \right\}$
 $[\text{Re } \mu > 0]$ ME 26
- 6.251**
1. $\int_0^\infty \text{si} \left(\frac{1}{x} \right) e^{-\mu x} dx = \frac{2}{\mu} \text{kei} (2\sqrt{\mu})$ $[\text{Re } \mu > 0]$ MI 34
2. $\int_0^\infty \text{ci} \left(\frac{1}{x} \right) e^{-\mu x} dx = -\frac{2}{\mu} \text{ker} (2\sqrt{\mu})$ $[\text{Re } \mu > 0]$ MI 34
- 6.252**
1. $\int_0^\infty \sin px \text{si}(qx) dx = -\frac{\pi}{2p}$ $[p^2 > q^2]$
 $= -\frac{\pi}{4p}$ $[p^2 = q^2]$
 $= 0$ $[p^2 < q^2]$
 FI II 652, NT 50(8)
- 2.⁶ $\int_0^\infty \cos px \text{si}(qx) dx = -\frac{1}{4p} \ln \left(\frac{p+q}{p-q} \right)^2$ $[p \neq 0, \quad p^2 \neq q^2]$
 $= \frac{1}{q}$ $[p = 0]$
 FI II 652, NT 50(10)
3. $\int_0^\infty \sin px \text{ci}(qx) dx = -\frac{1}{4p} \ln \left(\frac{p^2}{q^2} - 1 \right)^2$ $[p \neq 0, \quad p^2 \neq q^2]$
 $= 0$ $[p = 0]$
 FI II 652, NT 50(9)
4. $\int_0^\infty \cos px \text{ci}(qx) dx = -\frac{\pi}{2p}$ $[p^2 > q^2]$
 $= -\frac{\pi}{4p}$ $[p^2 = q^2]$
 $= 0$ $[p^2 < q^2]$
 FI II 654, NT 50(7)
- 6.253** $\int_0^\infty \frac{\text{si}(ax) \sin bx}{1 - 2r \cos x + r^2} dx = -\frac{\pi (r^m + r^{m+1})}{4b(1-r)(1-r^2)}$ $[b = a - m]$
 $= -\frac{\pi (2 + 2r - r^m - r^{m+1})}{4b(1-r)(1-r^2)}$ $[b = a + m]$
 $= -\frac{\pi r^{m+1}}{2b(1-r)(1-r^2)}$ $[a - m - 1 < b < a - m]$
 $= -\frac{\pi (1 + r - r^{m+1})}{2b(1-r)(1-r^2)}$ $[a + m < b < a + m + 1]$

6.254

$$1.^* \quad \int_0^\infty \text{ci}(x) \sin^2 x \frac{dx}{x} = \frac{1}{2} \left[L_2 \left(\frac{1}{2} \right) - L_2 \left(-\frac{1}{2} \right) \right]$$

where $L_2(x)$ is the Euler dilogarithm defined as $L_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$ and this in turn can be expressed as $L_2(z) = \Phi(z, 2, 1)$ in terms of the Lerch function defined in 9.550, with z real.

$$2.^{11} \quad \int_0^\infty \left[\text{si}(ax) + \frac{\pi}{2} \right] \cos bx \cdot \frac{dx}{x} = \frac{\pi}{2} \ln \frac{a}{b} H(a-b) \\ [a > 0, \quad b > 0, \quad H(x) \text{ is the Heaviside step function}] \quad \text{ET I 41(11)}$$

6.255

$$1. \quad \int_{-\infty}^\infty [\cos ax \text{ci}(a|x|) + \sin(a|x|) \text{si}(a|x|)] \frac{dx}{x-b} = -\pi [\text{sign } b \cos ab \text{si}(a|b|) - \sin ab \text{ci}(a|b|)] \\ [a > 0] \quad \text{ET II 253(4)}$$

$$2. \quad \int_{-\infty}^\infty [\sin ax \text{ci}(a|x|) - \text{sign } x \cos ax \text{si}(a|x|)] \frac{dx}{x-b} = -\pi [\sin(a|b|) \text{si}(a|b|) + \cos ab \text{ci}(a|b|)] \\ [a > 0] \quad \text{ET II 253(5)}$$

6.256

$$1. \quad \int_0^\infty [\text{si}^2(x) + \text{ci}^2(x)] \cos ax \, dx = \frac{\pi}{a} \ln(1+a) \quad [a > 0]$$

$$2.^* \quad \int_0^\infty [\text{si}(x) \cos x - \text{ci}(x) \sin x]^2 \, dx = \frac{\pi}{2}$$

$$3.^* \quad \int_0^\infty \text{si}^2(x) \cos(ax) \, dx = \frac{\pi}{2a} \log(1+a) \quad [0 \leq a \leq 2]$$

$$4.^* \quad \int_0^\infty \text{ci}^2(x) \cos(ax) \, dx = \frac{\pi}{2a} \log(1+a) \quad [0 \leq a \leq 2]$$

$$6.257 \quad \int_0^\infty \text{si} \left(\frac{a}{x} \right) \sin bx \, dx = -\frac{\pi}{2b} J_0(2\sqrt{ab}) \quad [b > 0] \quad \text{ET I 42(18)}$$

6.258

$$1. \quad \int_0^\infty \left[\text{si}(ax) + \frac{\pi}{2} \right] \sin bx \frac{dx}{x^2 + c^2} \\ = \frac{\pi}{4c} \{ e^{-bc} [\text{Ei}(bc) - \text{Ei}(-ac)] + e^{bc} [\text{Ei}(-ac) - \text{Ei}(-bc)] \} \quad [0 < b \leq a, \quad c > 0] \\ = \frac{\pi}{4c} e^{-bc} [\text{Ei}(ac) - \text{Ei}(-ac)] \quad [0 < a \leq b, \quad c > 0] \\ \text{BI (460)(1)}$$

$$2. \quad \int_0^\infty \left[\text{si}(ax) + \frac{\pi}{2} \right] \cos bx \frac{x \, dx}{x^2 + c^2} \\ = -\frac{\pi}{4} \{ e^{-bc} [\text{Ei}(bc) - \text{Ei}(-ac)] + e^{bc} [\text{Ei}(-bc) - \text{Ei}(-ac)] \} \quad [0 < b \leq a, \quad c > 0] \\ = \frac{\pi}{4} e^{-bc} [\text{Ei}(-ac) - \text{Ei}(ac)] \quad [0 < a \leq b, \quad c > 0] \\ \text{BI (460)(2, 5)}$$

6.259

$$\begin{aligned}
 1. \quad \int_0^\infty \text{si}(ax) \sin bx \frac{dx}{x^2 + c^2} &= \frac{\pi}{2c} \text{Ei}(-ac) \sinh(bc) & [0 < b \leq a, \quad c > 0] \\
 &= \frac{\pi}{4c} e^{-cb} [\text{Ei}(-bc) + \text{Ei}(bc) - \text{Ei}(-ac) - \text{Ei}(ac)] \\
 &\quad + \frac{\pi}{2c} \text{Ei}(-bc) \sinh(bc) & [0 < a \leq b, \quad c > 0] \\
 &\hspace{15em} \text{ET I 96(8)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \text{ci}(ax) \sin bx \frac{x dx}{x^2 + c^2} &= -\frac{\pi}{2} \sinh(bc) \text{Ei}(-ac) & [0 < b \leq a, \quad c > 0] \\
 &= -\frac{\pi}{2} \sinh(bc) \text{Ei}(-bc) + \frac{\pi}{4} e^{-bc} [\text{Ei}(-bc) + \text{Ei}(bc) \\
 &\quad - \text{Ei}(-ac) - \text{Ei}(ac)] & [0 < a \leq b, \quad c > 0] \\
 &\hspace{15em} \text{BI (460)(3)a, ET I 97(15)a}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty \text{ci}(ax) \cos bx \frac{dx}{x^2 + c^2} &= \frac{\pi}{2c} \cosh bc \text{Ei}(-ac) & [0 < b \leq a, \quad c > 0] \\
 &= \frac{\pi}{4c} \{e^{-bc} [\text{Ei}(ac) + \text{Ei}(-ac) - \text{Ei}(bc)] + e^{bc} \text{Ei}(-bc)\} & [0 < a \leq b, \quad c > 0] \\
 &\hspace{15em} \text{BI (460)(4), ET I 41(15)}
 \end{aligned}$$

$$\begin{aligned}
 4.* \quad \int_0^\infty [\text{ci}(x) \sin x - \text{Si}(x) \cos x] \sin x \frac{x dx}{a^2 + x^2} &= \frac{1}{8} [\text{Ei}(a)e^{-a} - \text{Ei}(-a)e^a]^2 \\
 &\hspace{15em} [a \text{ real}]
 \end{aligned}$$

$$\begin{aligned}
 5.* \quad \int_0^\infty [\text{ci}(x) \sin x - \text{Si}(x) \cos x]^2 \frac{x dx}{a^2 + x^2} &= \frac{\pi^3 e^{-|a|}}{8a} \sinh(a) - \frac{\pi}{8|a|} [\text{Ei}(a)e^{-a} - \text{Ei}(-a)e^a]^2 \\
 &\hspace{15em} [a \text{ real}]
 \end{aligned}$$

6.261

$$\begin{aligned}
 1. \quad \int_0^\infty \text{si}(bx) \cos ax e^{-px} dx &= -\frac{1}{2(a^2 + p^2)} \left[\frac{a}{2} \ln \frac{p^2 + (a+b)^2}{p^2 + (a-b)^2} + p \arctan \frac{2bp}{b^2 - a^2 - p^2} \right] \\
 &\hspace{15em} [a > 0, \quad b > 0, \quad p > 0] \quad \text{ET I 40(8)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \text{si}(\beta x) \cos ax e^{-\mu x} dx &= -\frac{\arctan \frac{\mu + ai}{\beta}}{2(\mu + ai)} - \frac{\arctan \frac{\mu - ai}{\beta}}{2(\mu - ai)} \\
 &\hspace{15em} [a > 0, \quad \text{Re } \mu > |\text{Im } \beta|] \quad \text{ET I 40(9)}
 \end{aligned}$$

6.262

$$\begin{aligned}
 1. \quad \int_0^\infty \text{ci}(bx) \sin ax e^{-\mu x} dx &= \frac{1}{2(a^2 + \mu^2)} \left\{ \mu \arctan \frac{2a\mu}{\mu^2 + b^2 - a^2} - \frac{a}{2} \ln \frac{(\mu^2 + b^2 - a^2)^2 + 4a^2\mu^2}{b^4} \right\} \\
 &\hspace{15em} [a > 0, \quad b > 0, \quad \text{Re } \mu > 0] \\
 &\hspace{15em} \text{ET I 98(16)a}
 \end{aligned}$$

$$2. \quad \int_0^\infty \text{ci}(bx) \cos ax e^{-px} dx = \frac{-1}{2(a^2 + p^2)} \left\{ \frac{p}{2} \ln \left[\frac{(b^2 + p^2 - a^2)^2 + 4a^2 p^2}{b^4} \right] + a \arctan \frac{2ap}{b^2 + p^2 - a^2} \right\}$$

[$a > 0, \quad b > 0, \quad \text{Re } p > 0$] ET I 41(16)

$$3. \quad \int_0^\infty \text{ci}(\beta x) \cos ax e^{-\mu x} dx = \frac{-\ln \left[1 + \frac{(\mu + ai)^2}{\beta^2} \right]}{4(\mu + ai)} - \frac{\ln \left[1 + \frac{(\mu - ai)^2}{\beta^2} \right]}{4(\mu - ai)}$$

[$a > 0, \quad \text{Re } \mu > |\text{Im } \beta|$] ET I 41(17)

6.263

$$1. \quad \int_0^\infty [\text{ci}(x) \cos x + \text{si}(x) \sin x] e^{-\mu x} dx = \frac{-\frac{\pi}{2} - \mu \ln \mu}{1 + \mu^2} \quad [\text{Re } \mu > 0] \quad \text{ME 26a, ET I 178(21)a}$$

$$2. \quad \int_0^\infty [\text{si}(x) \cos x - \text{ci}(x) \sin x] e^{-\mu x} dx = \frac{-\frac{\pi}{2} \mu + \ln \mu}{1 + \mu^2} \quad [\text{Re } \mu > 0] \quad \text{ME 26a, ET I 178(20)a}$$

$$3. \quad \int_0^\infty [\sin x - x \text{ci}(x)] e^{-\mu x} dx = \frac{\ln(1 + \mu^2)}{2\mu^2} \quad [\text{Re } \mu > 0] \quad \text{ME 26}$$

6.264

$$1. \quad \int_0^\infty \text{si}(x) \ln x dx = \mathbf{C} + 1 \quad \text{NT 46(10)}$$

$$2. \quad \int_0^\infty \text{ci}(x) \ln x dx = \frac{\pi}{2} \quad \text{NT 56(11)}$$

6.27 The hyperbolic sine integral and hyperbolic cosine integral functions**6.271**

$$1. \quad \int_0^\infty \text{shi}(x) e^{-\mu x} dx = \frac{1}{2\mu} \ln \frac{\mu + 1}{\mu - 1} = \frac{1}{\mu} \text{arccoth } \mu \quad [\text{Re } \mu > 1] \quad \text{MI 34}$$

$$2.^{11} \quad \int_0^\infty \text{chi}(x) e^{-\mu x} dx = -\frac{1}{2\mu} \ln(\mu^2 - 1) \quad [\text{Re } \mu > 1] \quad \text{MI 34}$$

$$6.272^{11} \quad \int_0^\infty \text{chi}(x) e^{-px^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \text{Ei} \left(\frac{1}{4p} \right) \quad [p > 0] \quad \text{MI 35}$$

6.273

$$1.^{11} \quad \int_0^\infty [\cosh x \text{shi}(x) - \sinh x \text{chi}(x)] e^{-\mu x} dx = \frac{\ln \mu}{\mu^2 - 1} \quad [\text{Re } \mu > 0] \quad \text{MI 35}$$

$$2.^{11} \quad \int_0^\infty [\cosh x \text{chi}(x) + \sinh x \text{shi}(x)] e^{-\mu x} dx = \frac{\mu \ln \mu}{1 - \mu^2} \quad [\text{Re } \mu > 2] \quad \text{MI 35}$$

$$6.274^{11} \quad \int_0^\infty [\cosh x \operatorname{shi}(x) - \sinh x \operatorname{chi}(x)] e^{-\mu x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu}} e^{\frac{1}{4\mu}} \operatorname{Ei} \left(-\frac{1}{4\mu} \right) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 35}$$

$$6.275 \quad \int_0^\infty [x \operatorname{chi}(x) - \sinh x] e^{-\mu x} dx = -\frac{\ln(\mu^2 - 1)}{2\mu^2} \quad [\operatorname{Re} \mu > 1] \quad \text{MI 35}$$

$$6.276 \quad \int_0^\infty [\cosh x \operatorname{chi}(x) + \sinh x \operatorname{shi}(x)] e^{-\mu x^2} x dx = \frac{1}{8} \sqrt{\frac{\pi}{\mu^3}} \exp \left(\frac{1}{4\mu} \right) \operatorname{Ei} \left(-\frac{1}{4\mu} \right) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 35}$$

6.277

$$1. \quad \int_0^\infty [\operatorname{chi}(x) + \operatorname{ci}(x)] e^{-\mu x} dx = -\frac{\ln(\mu^4 - 1)}{2\mu} \quad [\operatorname{Re} \mu > 1] \quad \text{MI 34}$$

$$2. \quad \int_0^\infty [\operatorname{chi}(x) - \operatorname{ci}(x)] e^{-\mu x} dx = \frac{1}{2\mu} \ln \frac{\mu^2 + 1}{\mu^2 - 1} \quad [\operatorname{Re} \mu > 1] \quad \text{MI 35}$$

6.28–6.31 The probability integral

6.281

$$1.^6 \quad \int_0^\infty [1 - \Phi(px)] x^{2q-1} dx = \frac{\Gamma(q + \frac{1}{2})}{2\sqrt{\pi}qp^{2q}} \quad [\operatorname{Re} q > 0, \quad \operatorname{Re} p > 0] \quad \text{NT 56(12), ET II 306(1)a}$$

$$2.^6 \quad \int_0^\infty \left[1 - \Phi \left(at^\alpha \pm \frac{b}{t^\alpha} \right) \right] dt = \frac{2b}{\sqrt{\pi}} \left(\frac{b}{a} \right)^{\frac{1-\alpha}{2\alpha}} \left[K_{\frac{1+\alpha}{2\alpha}}(2ab) \pm K_{\frac{1-\alpha}{2\alpha}}(2ab) \right] e^{\pm 2ab} \quad [a > 0, \quad b > 0, \quad \alpha \neq 0]$$

6.282

$$1. \quad \int_0^\infty \Phi(qt) e^{-pt} dt = \frac{1}{p} \left[1 - \Phi \left(\frac{p}{2q} \right) \right] \exp \left(\frac{p^2}{4q^2} \right) \quad [\operatorname{Re} p > 0, \quad |\arg q| < \frac{\pi}{4}] \quad \text{MO 175, EH II 148(11)}$$

$$2. \quad \int_0^\infty \left[\Phi \left(x + \frac{1}{2} \right) - \Phi \left(\frac{1}{2} \right) \right] e^{-\mu x + \frac{1}{4}} dx = \frac{1}{(\mu + 1)(\mu + 2)} \exp \frac{(\mu + 1)^2}{4} \left[1 - \Phi \left(\frac{\mu + 1}{2} \right) \right] \quad \text{ME 27}$$

6.283

$$1. \quad \int_0^\infty e^{\beta x} [1 - \Phi(\sqrt{\alpha x})] dx = \frac{1}{\beta} \left[\frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} - 1 \right] \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta < \operatorname{Re} \alpha] \quad \text{ET II 307(5)}$$

$$2. \quad \int_0^\infty \Phi(\sqrt{qt}) e^{-pt} dt = \frac{\sqrt{q}}{p} \frac{1}{\sqrt{p+q}} \quad [\operatorname{Re} p > 0, \quad \operatorname{Re}(q+p) > 0] \quad \text{EH II 148(12)}$$

$$6.284 \quad \int_0^\infty \left[1 - \Phi \left(\frac{q}{2\sqrt{x}} \right) \right] e^{-px} dx = \frac{1}{p} e^{-q\sqrt{p}} \quad [\operatorname{Re} p > 0, \quad |\arg q| < \frac{\pi}{4}] \quad \text{EF 147(235), EH II 148(13)}$$

6.285

$$1. \quad \int_0^\infty [1 - \Phi(x)] e^{-\mu^2 x^2} dx = \frac{\arctan \mu}{\sqrt{\pi} \mu} \quad [\operatorname{Re} \mu > 0] \quad \text{MI 37}$$

$$2. \quad \int_0^\infty \Phi(iat) e^{-a^2 t^2 - st} dt = \frac{-1}{2ai\sqrt{\pi}} \exp\left(\frac{s^2}{4a^2}\right) \operatorname{Ei}\left(-\frac{s^2}{4a^2}\right) \\ \left[\operatorname{Re} s > 0, \quad |\arg a| < \frac{\pi}{4}\right] \quad \text{EH II 148(14)a}$$

6.286

$$1. \quad \int_0^\infty [1 - \Phi(\beta x)] e^{\mu^2 x^2} x^{\nu-1} dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi} \nu \beta^\nu} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2} + 1; \frac{\mu^2}{\beta^2}\right) \\ [\operatorname{Re}^2 \beta > \operatorname{Re} \mu^2, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 306(2)}$$

$$2. \quad \int_0^\infty \left[1 - \Phi\left(\frac{\sqrt{2}x}{2}\right)\right] e^{\frac{x^2}{2}} x^{\nu-1} dx = 2^{\frac{\nu}{2}-1} \sec \frac{\nu\pi}{2} \Gamma\left(\frac{\nu}{2}\right) \\ [0 < \operatorname{Re} \nu < 1] \quad \text{ET I 325(9)}$$

6.287

$$1. \quad \int_0^\infty \Phi(\beta x) e^{-\mu x^2} x dx = \frac{\beta}{2\mu\sqrt{\mu + \beta^2}} \quad [\operatorname{Re} \mu > -\operatorname{Re} \beta^2, \quad \operatorname{Re} \mu > 0] \\ \text{ME 27a, ET I 176(4)}$$

$$2. \quad \int_0^\infty [1 - \Phi(\beta x)] e^{-\mu x^2} x dx = \frac{1}{2\mu} \left(1 - \frac{\beta}{\sqrt{\mu + \beta^2}}\right) \quad [\operatorname{Re} \mu > -\operatorname{Re} \beta^2, \quad \operatorname{Re} \mu > 0] \\ \text{NT 49(14), ET I 177(9)}$$

$$3.* \quad I = \int_{-\infty}^\infty \frac{r}{\sigma^2} \exp\left(\frac{r}{\sigma^2}\right) Q(rA) Q(rB) dr = \frac{1}{4} - \frac{1}{2\pi} \left[\alpha \arctan\left(\frac{A}{\alpha B}\right) + \beta \arctan\left(\frac{B}{\beta A}\right)\right] \quad B \neq A \\ = \frac{1}{4} - \frac{1}{\pi} \alpha \arctan \frac{1}{\alpha} \quad B = A \\ Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt = \frac{1}{2} \left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right], \quad \alpha = \sqrt{\frac{\sigma^2 A^2}{1 + \sigma^2 A^2}}, \quad \beta = \sqrt{\frac{\sigma^2 B^2}{1 + \sigma^2 B^2}},$$

$$6.288 \quad \int_0^\infty \Phi(iax) e^{-\mu x^2} x dx = \frac{ai}{2\mu\sqrt{\mu - a^2}} \quad [a > 0, \quad \operatorname{Re} \mu > \operatorname{Re} a^2] \quad \text{MI 37a}$$

6.289

$$1. \quad \int_0^\infty \Phi(\beta x) e^{(\beta^2 - \mu^2)x^2} x dx = \frac{\beta}{2\mu(\mu^2 - \beta^2)} \quad \left[\operatorname{Re}^2 \mu > \operatorname{Re} \beta^2, \quad |\arg \mu| < \frac{\pi}{4}\right] \\ \text{ET I 176(5)}$$

$$2. \quad \int_0^\infty [1 - \Phi(\beta x)] e^{(\beta^2 - \mu^2)x^2} x dx = \frac{1}{2\mu(\mu + \beta)} \quad \left[\operatorname{Re}^2 \mu > \operatorname{Re} \beta^2, \quad \arg \mu < \frac{\pi}{4}\right] \\ \text{ET I 177(10)}$$

$$3. \quad \int_0^\infty \Phi(\sqrt{b-ax}) e^{-(a+\mu)x^2} x dx = \frac{\sqrt{b-a}}{2(\mu+a)\sqrt{\mu+b}} \quad [\operatorname{Re} \mu > -a > 0, \quad b > a] \quad \text{ME 27}$$

$$6.291 \quad \int_0^\infty \Phi(ix) e^{-(\mu x + x^2)} x dx = \frac{i}{\sqrt{\pi}} \left[\frac{1}{\mu} + \frac{\mu}{4} \operatorname{Ei} \left(-\frac{\mu^2}{4} \right) \right] \quad [\operatorname{Re} \mu > 0] \quad \text{MI 37}$$

$$6.292 \quad \int_0^\infty [1 - \Phi(x)] e^{-\mu^2 x^2} x^2 dx = \frac{1}{2\sqrt{\pi}} \left\{ \frac{\arctan \mu}{\mu^3} - \frac{1}{\mu^2(\mu^2 + 1)} \right\} \quad \left[|\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$6.293 \quad \int_0^\infty \Phi(x) e^{-\mu x^2} \frac{dx}{x} = \frac{1}{2} \ln \frac{\sqrt{\mu+1} + 1}{\sqrt{\mu+1} - 1} = \operatorname{arccoth} \sqrt{\mu+1} \quad [\operatorname{Re} \mu > 0] \quad \text{MI 37a}$$

6.294

$$1. \quad \int_0^\infty \left[1 - \Phi \left(\frac{\beta}{x} \right) \right] e^{-\mu^2 x^2} x dx = \frac{1}{2\mu^2} \exp(-2\beta\mu) \quad \left[|\arg \beta| < \frac{\pi}{4}, \quad |\arg \mu| < \frac{\pi}{4} \right] \quad \text{ET I 177(11)}$$

$$2. \quad \int_0^\infty \left[1 - \Phi \left(\frac{1}{x} \right) \right] e^{-\mu^2 x^2} \frac{dx}{x} = -\operatorname{Ei}(-2\mu) \quad \left[|\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

6.295

$$1. \quad \int_0^\infty \left[1 - \Phi \left(\frac{1}{x} \right) \right] \exp \left(-\mu^2 x^2 + \frac{1}{x^2} \right) dx = \frac{1}{\sqrt{\pi}\mu} [\sin 2\mu \operatorname{ci}(2\mu) - \cos 2\mu \operatorname{si}(2\mu)] \quad \left[|\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$2. \quad \int_0^\infty \left[1 - \Phi \left(\frac{1}{x} \right) \right] \exp \left(-\mu^2 x^2 + \frac{1}{x^2} \right) x dx = \frac{\pi}{2\mu} [\mathbf{H}_1(2\mu) - Y_1(2\mu)] - \frac{1}{\mu^2} \quad \left[|\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$3. \quad \int_0^\infty \left[1 - \Phi \left(\frac{1}{x} \right) \right] \exp \left(-\mu^2 x^2 + \frac{1}{x^2} \right) \frac{dx}{x} = \frac{\pi}{2} [\mathbf{H}_0(2\mu) - Y_0(2\mu)] \quad \left[|\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$6.296 \quad \int_0^\infty \left\{ (x^2 + a^2) \left[1 - \Phi \left(\frac{a}{\sqrt{2}x} \right) \right] - \sqrt{\frac{2}{\pi}} ax \cdot e^{-\frac{a^2}{2x^2}} \right\} e^{-\mu^2 x^2} x dx = \frac{1}{2\mu^4} e^{-a\mu\sqrt{2}} \quad \left[|\arg \mu| < \frac{\pi}{4}, \quad a > 0 \right] \quad \text{MI 38a}$$

6.297

$$1. \quad \int_0^\infty \left[1 - \Phi \left(\gamma x + \frac{\beta}{x} \right) \right] e^{(\gamma^2 - \mu)x^2} x dx = \frac{1}{2\sqrt{\mu}(\sqrt{\mu} + \gamma)} \exp[-2(\beta\gamma + \beta\sqrt{\mu})] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 177(12)a}$$

$$2. \quad \int_0^\infty \left[1 - \Phi \left(\frac{b + 2ax^2}{2x} \right) \right] \exp[-(\mu^2 - a^2)x^2 + ab] x dx = \frac{e^{-b\mu}}{2\mu(\mu + a)} \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0] \quad \text{MI 38}$$

$$3. \quad \int_0^\infty \left\{ \left[1 - \Phi \left(\frac{b - 2ax^2}{2x} \right) \right] e^{-ab} + \left[1 - \Phi \left(\frac{b + 2ax^2}{2x} \right) \right] e^{ab} \right\} e^{-\mu x^2} x dx = \frac{1}{\mu} \exp \left(-b\sqrt{a^2 + \mu} \right) \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0] \quad \text{MI 38}$$

$$6.298 \quad \int_0^\infty \left\{ 2 \cosh ab - e^{-ab} \Phi \left(\frac{b - 2ax^2}{2x} \right) - e^{ab} \Phi \left(\frac{b + 2ax^2}{2x} \right) \right\} e^{-(\mu - a^2)x^2} x dx = \frac{1}{\mu - a^2} \exp(-b\sqrt{\mu}) \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0] \quad \text{MI 38}$$

$$6.299 \quad \int_0^\infty \cosh(2\nu t) \exp \left[(a \cosh t)^2 \right] [1 - \Phi(a \cosh t)] dt = \frac{1}{2 \cos(\nu\pi)} \exp \left(\frac{1}{2} a^2 \right) K_\nu(a^2) \\ [\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET II 308(10)}$$

$$6.311 \quad \int_0^\infty [1 - \Phi(ax)] \sin bx dx = \frac{1}{b} \left(1 - e^{-\frac{b^2}{4a^2}} \right) \quad [a > 0, \quad b > 0] \quad \text{ET I 96(4)}$$

$$6.312 \quad \int_0^\infty \Phi(ax) \sin bx^2 dx = \frac{1}{4\sqrt{2\pi b}} \left(\ln \frac{b + a^2 + a\sqrt{2b}}{b + a^2 - a\sqrt{2b}} + 2 \arctan \frac{a\sqrt{2b}}{b - a^2} \right) \\ [a > 0, \quad b > 0] \quad \text{ET I 96(3)}$$

6.313

$$1. \quad \int_0^\infty \sin(\beta x) [1 - \Phi(\sqrt{\alpha x})] dx = \frac{1}{\beta} - \left(\frac{\frac{\alpha}{2}}{\alpha^2 + \beta^2} \right)^{\frac{1}{2}} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha \right]^{-\frac{1}{2}} \\ [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{ET II 307(6)}$$

$$2. \quad \int_0^\infty \cos(\beta x) [1 - \Phi(\sqrt{\alpha x})] dx = \left(\frac{\frac{\alpha}{2}}{\alpha^2 + \beta^2} \right)^{\frac{1}{2}} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^{-\frac{1}{2}} \\ [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{ET II 307(7)}$$

6.314

$$1. \quad \int_0^\infty \sin(bx) \left[1 - \Phi \left(\sqrt{\frac{a}{x}} \right) \right] dx = b^{-1} \exp \left[-(2ab)^{\frac{1}{2}} \right] \cos \left[(2ab)^{\frac{1}{2}} \right] \\ [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 307(8)}$$

$$2. \quad \int_0^\infty \cos(bx) \left[1 - \Phi \left(\sqrt{\frac{a}{x}} \right) \right] dx = -b^{-1} \exp \left[-(2ab)^{\frac{1}{2}} \right] \sin \left[(2ab)^{\frac{1}{2}} \right] \\ [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 307(9)}$$

6.315

$$1. \quad \int_0^\infty x^{\nu-1} \sin(\beta x) [1 - \Phi(\alpha x)] dx = \frac{\Gamma \left(1 + \frac{1}{2}\nu \right) \beta}{\sqrt{\pi}(\nu+1)\alpha^{\nu+1}} {}_2F_2 \left(\frac{\nu+1}{2}, \frac{\nu}{2} + 1; \frac{3}{2}, \frac{\nu+3}{2}; -\frac{\beta^2}{4\alpha^2} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 307(3)}$$

$$2. \quad \int_0^\infty x^{\nu-1} \cos(\beta x) [1 - \Phi(\alpha x)] dx = \frac{\Gamma \left(\frac{1}{2} + \frac{1}{2}\nu \right)}{\sqrt{\pi}\nu\alpha^\nu} {}_2F_2 \left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}, \frac{\nu}{2} + 1; -\frac{\beta^2}{4\alpha^2} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 307(4)}$$

3.
$$\int_0^\infty [1 - \Phi(ax)] \cos bx \cdot x \, dx = \frac{1}{2a^2} \exp\left(-\frac{b^2}{4a^2}\right) - \frac{1}{b^2} \left[1 - \exp\left(-\frac{b^2}{4a^2}\right)\right]$$

 $[a > 0, \quad b > 0]$ ET I 40(5)
4.
$$\int_0^\infty [\Phi(ax) - \Phi(bx)] \cos px \frac{dx}{x} = \frac{1}{2} \left[\operatorname{Ei}\left(-\frac{p^2}{4b^2}\right) - \operatorname{Ei}\left(-\frac{p^2}{4a^2}\right) \right]$$

 $[a > 0, \quad b > 0, \quad p > 0]$ ET I 40(6)
5.
$$\int_0^\infty x^{-\frac{1}{2}} \Phi(a\sqrt{x}) \sin bx \, dx = \frac{1}{2\sqrt{2\pi}b} \left\{ \ln \left[\frac{b + a\sqrt{2b} + a^2}{b - a\sqrt{2b} + a^2} \right] + 2 \arctan \left[\frac{a\sqrt{2b}}{b - a^2} \right] \right\}$$

 $[a > 0, \quad b > 0]$ ET I 96(3)
- 6.316**
$$\int_0^\infty e^{\frac{1}{2}x^2} \left[1 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right] \sin bx \, dx = \sqrt{\frac{\pi}{2}} e^{\frac{b^2}{2}} \left[1 - \Phi\left(\frac{b}{\sqrt{2}}\right)\right]$$

 $[b > 0]$ ET I 96(5)
- 6.317**⁶
$$\int_0^\infty e^{-a^2x^2} \Phi(iax) \sin bx \, dx = \frac{i}{a} \frac{\sqrt{\pi}}{2} e^{-\frac{b^2}{4a^2}}$$

 $[b > 0]$ ET I 96(2)
- 6.318**
$$\int_0^\infty [1 - \Phi(x)] \operatorname{si}(2px) \, dx = \frac{2}{\pi p} (1 - e^{-p^2}) - \frac{2}{\sqrt{\pi}} (1 - \Phi(p))$$

 $[p > 0]$ NT 61(13)a

6.32 Fresnel integrals

6.321

1.
$$\int_0^\infty \left[\frac{1}{2} - S(px) \right] x^{2q-1} \, dx = \frac{\sqrt{2} \Gamma\left(q + \frac{1}{2}\right) \sin \frac{2q+1}{4} \pi}{4\sqrt{\pi} q p^{2q}}$$

 $[0 < \operatorname{Re} q < \frac{3}{2}, \quad p > 0]$ NT 56(14)a
2.
$$\int_0^\infty \left[\frac{1}{2} - C(px) \right] x^{2q-1} \, dx = \frac{\sqrt{2} \Gamma\left(q + \frac{1}{2}\right) \cos \frac{2q+1}{4} \pi}{4\sqrt{\pi} q p^{2q}}$$

 $[0 < \operatorname{Re} q < \frac{3}{2}, \quad p > 0]$ NT 56(13)a

6.322

1.
$$\int_0^\infty S(t) e^{-pt} \, dt = \frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[\frac{1}{2} - C\left(\frac{p}{2}\right) \right] + \sin \frac{p^2}{4} \left[\frac{1}{2} - S\left(\frac{p}{2}\right) \right] \right\}$$
 MO 173a
2.
$$\int_0^\infty C(t) e^{-pt} \, dt = \frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[\frac{1}{2} - S\left(\frac{p}{2}\right) \right] - \sin \frac{p^2}{4} \left[\frac{1}{2} - C\left(\frac{p}{2}\right) \right] \right\}$$
 MO 172a

6.323

1.
$$\int_0^\infty S(\sqrt{t}) e^{-pt} \, dx = \frac{(\sqrt{p^2+1} - p)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}$$
 EF 122(58)a

$$2. \quad \int_0^\infty C(\sqrt{t}) e^{-pt} dt = \frac{(\sqrt{p^2+1}+p)^{\frac{1}{2}}}{2p\sqrt{p^2+1}} \quad \text{EF 122(58)a}$$

6.324

$$1. \quad \int_0^\infty \left[\frac{1}{2} - S(x) \right] \sin 2px dx = \frac{1 + \sin p^2 - \cos p^2}{4p} \quad [p > 0] \quad \text{NT 61(12)a}$$

$$2. \quad \int_0^\infty \left[\frac{1}{2} - C(x) \right] \sin 2px dx = \frac{1 - \sin p^2 - \cos p^2}{4p} \quad [p > 0] \quad \text{NT 61(11)a}$$

6.325

$$1. \quad \int_0^\infty S(x) \sin b^2 x^2 dx = \frac{\sqrt{\pi}}{b} 2^{-\frac{5}{2}} \quad [0 < b^2 < 1] \\ = 0 \quad [b^2 > 1] \quad \text{ET I 98(21)a}$$

$$2. \quad \int_0^\infty C(x) \cos b^2 x^2 dx = \frac{\sqrt{\pi}}{b} 2^{-\frac{5}{2}} \quad [0 < b^2 < 1] \\ = 0 \quad [b^2 > 1] \quad \text{ET I 42(22)}$$

6.326

$$1. \quad \int_0^\infty \left[\frac{1}{2} - S(x) \right] \text{si}(2px) dx = \left(\frac{\pi}{8} \right)^{1/2} (S(p) + C(p) - 1) - \frac{1 + \sin p^2 - \cos p^2}{4p} \\ [p > 0] \quad \text{NT 61(15)a}$$

$$2. \quad \int_0^\infty \left[\frac{1}{2} - C(x) \right] \text{si}(2px) dx = \left(\frac{\pi}{8} \right)^{1/2} (S(p) - C(p)) - \frac{1 - \sin p^2 - \cos p^2}{4p} \\ [p > 0] \quad \text{NT 61(14)a}$$

6.4 The Gamma Function and Functions Generated by It**6.41 The gamma function**

$$6.411^{11} \quad \int_{-\infty}^\infty \Gamma(\alpha+x) \Gamma(\beta-x) dx = -i\pi 2^{1-\alpha-\beta} \Gamma(\alpha+\beta) \\ [\text{Re}(\alpha+\beta) < 1 \text{ and either } \text{Im } \alpha < 0 < \text{Im } \beta \text{ or } \text{Im } \beta < 0 < \text{Im } \alpha] \quad \text{ET II 297(1)}$$

$$= i\pi 2^{1-\alpha-\beta} \Gamma(\alpha+\beta) \\ [\text{Re}(\alpha+\beta) < 1, \quad \text{Im } \alpha < 0, \quad \text{Im } \beta < 0] \quad \text{ET II 297(2)}$$

$$= 0 \\ [\text{Re}(\alpha+\beta) < 1, \quad \text{Im } \alpha > 0, \quad \text{Im } \beta > 0] \quad \text{ET II 297(3)}$$

$$\begin{aligned}
 6.412 \quad \int_{-\infty}^{i\infty} \Gamma(\alpha+s) \Gamma(\beta+s) \Gamma(\gamma-s) \Gamma(\delta-s) ds &= 2\pi i \frac{\Gamma(\alpha+\gamma) \Gamma(\alpha+\delta) \Gamma(\beta+\gamma) \Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} \\
 &\quad [\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re} \gamma, \operatorname{Re} \delta > 0] \\
 &\quad \text{ET II 302(32)}
 \end{aligned}$$

6.413

$$\begin{aligned}
 1. \quad \int_0^\infty |\Gamma(a+ix) \Gamma(b+ix)|^2 dx &= \frac{\sqrt{\pi} \Gamma(a) \Gamma(a+\frac{1}{2}) \Gamma(b) \Gamma(b+\frac{1}{2}) \Gamma(a+b)}{2 \Gamma(a+b+\frac{1}{2})} \\
 &\quad [a > 0, \quad b > 0] \quad \text{ET II 302(27)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \left| \frac{\Gamma(a+ix)}{\Gamma(b+ix)} \right|^2 dx &= \frac{\sqrt{\pi} \Gamma(a) \Gamma(a+\frac{1}{2}) \Gamma(b-a-\frac{1}{2})}{2 \Gamma(b) \Gamma(b-\frac{1}{2}) \Gamma(b-a)} \\
 &\quad [0 < a < b - \frac{1}{2}] \quad \text{ET II 302(28)}
 \end{aligned}$$

6.414

$$\begin{aligned}
 1. \quad \int_{-\infty}^\infty \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} dx &= 0 \quad [\operatorname{Im} \alpha \neq 0, \quad \operatorname{Re}(\alpha-\beta) < -1] \\
 &\quad \text{ET II 297(4)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha+x) \Gamma(\beta-x)} &= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \quad [\operatorname{Re}(\alpha+\beta) > 1] \\
 &\quad \text{ET II 297(5)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_{-\infty}^\infty \frac{\Gamma(\gamma+x) \Gamma(\delta+x)}{\Gamma(\alpha+x) \Gamma(\beta+x)} dx &= 0 \\
 &\quad [\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im} \gamma, \operatorname{Im} \delta > 0] \quad \text{ET II 299(18)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_{-\infty}^\infty \frac{\Gamma(\gamma+x) \Gamma(\delta+x)}{\Gamma(\alpha+x) \Gamma(\beta+x)} dx &= \frac{\pm 2\pi^2 i \Gamma(\alpha+\beta-\gamma-\delta-1)}{\sin[\pi(\gamma-\delta)] \Gamma(\alpha-\gamma) \Gamma(\alpha-\delta) \Gamma(\beta-\gamma) \Gamma(\beta-\delta)} \\
 &\quad [\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im} \gamma < 0, \quad \operatorname{Im} \delta < 0. \text{ In the numerator, we take the plus sign if } \\
 &\quad \operatorname{Im} \gamma > \operatorname{Im} \delta \text{ and the minus sign if } \operatorname{Im} \gamma < \operatorname{Im} \delta.] \quad \text{ET II 300(19)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \int_{-\infty}^\infty \frac{\Gamma(\alpha-\beta-\gamma+x+1) dx}{\Gamma(\alpha+x) \Gamma(\beta-x) \Gamma(\gamma+x)} &= \frac{\pi \exp(\pm \frac{1}{2} \pi (\delta-\gamma) i)}{\Gamma(\beta+\gamma-1) \Gamma(\frac{1}{2}(\alpha+\beta)) \Gamma(\frac{1}{2}(\gamma-\delta+1))} \\
 &\quad [\operatorname{Re}(\beta+\gamma) > 1, \quad \delta = \alpha - \beta - \gamma + 1, \quad \operatorname{Im} \delta \neq 0. \text{ The sign is plus in the argument if the } \\
 &\quad \text{exponential for } \operatorname{Im} \delta > 0 \text{ and minus for } \operatorname{Im} \delta < 0.] \quad \text{ET II 300(20)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)} &= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1) \Gamma(\beta+\gamma-1) \Gamma(\gamma+\delta-1) \Gamma(\delta+\alpha-1)} \\
 &\quad [\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3] \quad \text{ET II 300(21)}
 \end{aligned}$$

6.415

$$\begin{aligned}
 1. \quad \int_{-\infty}^\infty \frac{R(x) dx}{\Gamma(\alpha+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)} &= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1) \Gamma(\beta+\gamma-1) \Gamma(\gamma+\delta-1) \Gamma(\delta+\alpha-1)} \int_0^1 R(t) dt \\
 &\quad [\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3, \quad R(x+1) = R(x)] \quad \text{ET II 301(24)}
 \end{aligned}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{R(x) dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\int_0^1 R(t) \cos \left[\frac{1}{2}\pi(2t+\alpha-\beta) \right] dt}{\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$

$$[\alpha+\delta=\beta+\gamma, \quad \operatorname{Re}(\alpha+\beta+\gamma+\delta) > 2, \quad R(x+1) = -R(x)] \quad \text{ET II 301(25)}$$

6.42 Combinations of the gamma function, the exponential, and powers

6.421

$$1. \quad \int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x) \exp[2(\pi n + \theta)xi] dx = 2\pi i \Gamma(\alpha+\beta)(2\cos\theta)^{-\alpha-\beta} \exp[(\beta-\alpha)i\theta]$$

$$\times [\eta_n(\beta) \exp(2n\pi\beta i) - \eta_n(-\alpha) \exp(-2n\pi\alpha i)]$$

$$\left[\operatorname{Re}(\alpha+\beta) < 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n \text{ an integer}, \quad \eta_n(\xi) = \begin{cases} 0 & \text{if } (\frac{1}{2}-n) \operatorname{Im} \xi > 0 \\ \operatorname{sign}(\frac{1}{2}-n) & \text{if } (\frac{1}{2}-n) \operatorname{Im} \xi < 0 \end{cases} \right]$$

ET II 298(7)

$$2. \quad \int_{-\infty}^{\infty} \frac{e^{\pi icx} dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+kx)\Gamma(\delta-kx)} = 0$$

$$[\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 2, \quad c \text{ and } k \text{ are real}, \quad |c| > |k| + 1] \quad \text{ET II 301(26)}$$

$$3. \quad \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta)xi] dx$$

$$= 2\pi i \operatorname{sign}\left(n + \frac{1}{2}\right) \frac{(2\cos\theta)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \exp[-(2\pi n + \pi - \theta)\alpha i + \theta i(\beta-1)]$$

$$\left[\operatorname{Re}(\beta-\alpha) > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n \text{ is an integer}, \quad \left(n + \frac{1}{2}\right) \operatorname{Im} \alpha < 0 \right] \quad \text{ET II 298(8)}$$

$$4. \quad \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta)xi] dx = 0$$

$$\left[\operatorname{Re}(\beta-\alpha) > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n \text{ is an integer}, \quad \left(n + \frac{1}{2}\right) \operatorname{Im} \alpha > 0 \right] \quad \text{ET II 297(6)}$$

6.422

$$1. \quad \int_{-i\infty}^{i\infty} \Gamma(s-k-\lambda) \Gamma\left(\lambda+\mu-s+\frac{1}{2}\right) \Gamma\left(\lambda-\mu-s+\frac{1}{2}\right) z^s ds$$

$$= 2\pi i \Gamma\left(\frac{1}{2}-k-\mu\right) \Gamma\left(\frac{1}{2}-k+\mu\right) z^\lambda e^{\frac{z}{2}} W_{k,\mu}(z)$$

$$\left[\operatorname{Re}(k+\lambda) < 0, \quad \operatorname{Re} \lambda > |\operatorname{Re} \mu| - \frac{1}{2}, \quad |\arg z| < \frac{3}{2}\pi \right] \quad \text{ET II 302(29)}$$

$$2. \quad \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\alpha+s) \Gamma(-s) \Gamma(1-c-s) x^s ds = 2\pi i \Gamma(\alpha) \Gamma(\alpha-c+1) \Psi(\alpha, c; x)$$

$$\left[-\operatorname{Re} \alpha < \gamma < \min(0, 1 - \operatorname{Re} c), \quad -\frac{3}{2}\pi < \arg x < \frac{3}{2}\pi \right] \quad \text{EH I 256(5)}$$

$$3. \quad \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s) \Gamma(\beta+s) t^s ds = 2\pi i \Gamma(\beta) (1+t)^{-\beta} \quad [0 > \gamma > \operatorname{Re}(1-\beta), \quad |\arg t| < \pi]$$

EH I 256, BU 75

$$4. \quad \int_{-\infty i}^{\infty i} \Gamma\left(\frac{t-p}{2}\right) \Gamma(-t) (\sqrt{2})^{t-p-2} z^t dt = 2\pi i e^{\frac{1}{4}z^2} \Gamma(-p) D_p(z) \\ [|\arg z| < \frac{3}{4}\pi, \quad p \text{ is not a positive integer}] \quad \text{WH}$$

$$5. \quad \int_{-i\infty}^{i\infty} \Gamma(s) \Gamma\left(\frac{1}{2}\nu + \frac{1}{4} - s\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{4} - s\right) \left(\frac{z^2}{2}\right)^s ds \\ = 2\pi i \cdot 2^{\frac{1}{4}-\frac{1}{2}\nu} z^{-\frac{1}{2}} e^{\frac{3}{4}z^2} \Gamma\left(\frac{1}{2}\nu + \frac{1}{4}\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{4}\right) D_\nu(z) \\ [|\arg z| < \frac{3}{4}\pi, \quad \nu \neq \frac{1}{2}, \quad -\frac{1}{2}, \quad -\frac{3}{2}, \dots] \quad \text{EH II 120}$$

$$6.^3 \quad \int_{c-i\infty}^{c+i\infty} \left(\frac{1}{2}x\right)^{-s} \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}s\right) [\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}s)]^{-1} ds = 4\pi i J_\nu(x) \\ [x > 0, -\operatorname{Re} \nu < c < 1] \quad \text{EH II 21(34)}$$

$$7. \quad \int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s) \Gamma(-s) \left(-\frac{1}{2}iz\right)^{\nu+2s} ds = -2\pi^2 e^{\frac{1}{2}i\nu\pi} H_\nu^{(1)}(z) \\ [|\arg(-iz)| < \frac{\pi}{2}, \quad 0 < \operatorname{Re} \nu < c] \\ \text{EH II 83(34)}$$

$$8. \quad \int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s) \Gamma(-s) \left(\frac{1}{2}iz\right)^{\nu+2s} ds = 2\pi^2 e^{-\frac{1}{2}i\nu\pi} H_\nu^{(2)}(z) \\ [|\arg(iz)| < \frac{\pi}{2}, \quad 0 < \operatorname{Re} \nu < c] \\ \text{EH II 83(35)}$$

$$9. \quad \int_{-i\infty}^{i\infty} \Gamma(-s) \frac{\left(\frac{1}{2}x\right)^{\nu+2s}}{\Gamma(\nu+s+1)} ds = 2\pi i J_\nu(x) \quad [x > 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH II 83(36)}$$

$$10. \quad \int_{-i\infty}^{i\infty} \Gamma(-s) \Gamma(-2\nu-s) \Gamma\left(\nu+s+\frac{1}{2}\right) (-2iz)^s ds = -\pi^{\frac{5}{2}} e^{-i(z-\nu\pi)} \sec(\nu\pi) (2z)^{-\nu} H_\nu^{(1)}(z) \\ [|\arg(-iz)| < \frac{3}{2}\pi, \quad 2\nu \neq \pm 1, \quad \pm 3, \dots] \\ \text{EH II 83(37)}$$

$$11. \quad \int_{-i\infty}^{i\infty} \Gamma(-s) \Gamma(-2\nu-s) \Gamma\left(\nu+s+\frac{1}{2}\right) (2iz)^s ds = \pi^{\frac{5}{2}} e^{i(z-\nu\pi)} \sec(\nu\pi) (2z)^{-\nu} H_\nu^{(2)}(z) \\ [|\arg(iz)| < \frac{3}{2}\pi, \quad 2\nu \neq \pm 1, \quad \pm 3, \dots] \\ \text{EH II 84(38)}$$

$$12. \quad \int_{-i\infty}^{i\infty} \Gamma(s) \Gamma\left(\frac{1}{2}-s-\nu\right) \Gamma\left(\frac{1}{2}-s+\nu\right) (2z)^s ds = 2^{\frac{3}{2}} \pi^{\frac{3}{2}} iz^{\frac{1}{2}} e^z \sec(\nu\pi) K_\nu(z) \\ [|\arg z| < \frac{3}{2}\pi, \quad 2\nu \neq \pm 1, \quad \pm 3, \dots] \\ \text{EH II 84(39)}$$

$$13. \quad \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-s)}{s\Gamma(1+s)} x^{2s} ds = 4\pi \int_{2x}^{\infty} \frac{J_0(t)}{t} dt \quad [x > 0] \quad \text{MO 41}$$

$$14. \quad \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s ds = 2\pi i \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha, \beta; \gamma; z)$$

[For $\arg(-z) < \pi$, the path of integration must separate the poles of the integrand at the points $s = 0, 1, 2, 3, \dots$ from the poles $s = -\alpha - n$ and $s = -\beta - n$ (for $n = 0, 1, 2, \dots$).]

$$15. \quad \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s ds = \frac{2\pi i \Gamma(\alpha)}{\Gamma(\gamma)} {}_1F_1(\alpha; \gamma; z) \\ \left[-\frac{\pi}{2} < \arg(-z) < \frac{\pi}{2}, \quad 0 > \delta > -\operatorname{Re} \alpha, \quad \gamma \neq 0, 1, 2, \dots \right] \quad \text{EH I 62(15), EH I 256(4)}$$

$$16. \quad \int_{-i\infty}^{i\infty} \left[\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)} \right]^2 z^s ds = 2\pi i z^{\frac{1}{2}} \left[2\pi^{-1} K_0\left(4z^{\frac{1}{4}}\right) - Y_0\left(4z^{\frac{1}{4}}\right) \right] \\ [z > 0] \quad \text{ET II 303(33)}$$

$$17. \quad \int_{-i\infty}^{i\infty} \frac{\Gamma(\lambda+\mu-s+\frac{1}{2})\Gamma(\lambda-\mu-s+\frac{1}{2})}{\Gamma(\lambda-k-s+1)} z^s ds = 2\pi i z^\lambda e^{-\frac{z}{2}} W_{k,\mu}(z) \\ \left[\operatorname{Re} \lambda > |\operatorname{Re} \mu| - \frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right] \quad \text{ET II 302(30)}$$

$$18. \quad \int_{-i\infty}^{i\infty} \frac{\Gamma(k-\lambda+s)\Gamma(\lambda+\mu-s+\frac{1}{2})}{\Gamma(\mu-\lambda+s+\frac{1}{2})} z^s ds = 2\pi i \frac{\Gamma(k+\mu+\frac{1}{2})}{\Gamma(2\mu+1)} z^\lambda e^{-\frac{z}{2}} M_{k,\mu}(z) \\ \left[\operatorname{Re}(k-\lambda) > 0, \quad \operatorname{Re}(\lambda+\mu) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right] \quad \text{ET II 302(31)}$$

$$19. \quad \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=n+1}^p \Gamma(a_j-s)} z^s ds = 2\pi i G_{mn}^{pq} \left(z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ \left[p+q < 2(m+n); \quad |\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi; \right. \\ \left. \operatorname{Re} a_k < 1, \quad k=1, \dots, n; \quad \operatorname{Re} b_j > 0, \quad j=1, \dots, m \right] \quad \text{ET II 303(34)}$$

6.423

$$1. \quad \int_0^\infty e^{-\alpha x} \frac{dx}{\Gamma(1+x)} = \nu(e^{-\alpha}) \quad \text{MI 39, EH III 222(16)}$$

$$2. \quad \int_0^\infty e^{-\alpha x} \frac{dx}{\Gamma(x+\beta+1)} = e^{\beta\alpha} \nu(e^{-\alpha}, \beta) \quad \text{MI 39, EH III 222(16)}$$

$$3. \quad \int_0^\infty e^{-\alpha x} \frac{x^m}{\Gamma(x+1)} dx = \mu(e^{-\alpha}, m) \Gamma(m+1) \quad [\operatorname{Re} m > -1] \quad \text{MI 39, EH III 222(17)}$$

$$4. \quad \int_0^\infty e^{-\alpha x} \frac{x^m}{\Gamma(x+n+1)} dx = e^{n\alpha} \mu(e^{-\alpha}, m, n) \Gamma(m+1) \quad \text{MI 39, EH III 222(17)}$$

$$6.424 \quad \int_{-\infty}^\infty \frac{R(x) \exp[(2\pi n + \theta)xi] dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{\left[2 \cos\left(\frac{\theta}{2}\right)\right]^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \exp\left[\frac{1}{2}\theta(\beta-\alpha)i\right] \int_0^1 R(t) \exp(2\pi nti) dt$$

[Re($\alpha + \beta$) > 1, $-\pi < \theta < \pi$, n is an integer, $R(x+1) = R(x)$] ET II 299(16)

6.43 Combinations of the gamma function and trigonometric functions

6.431

$$1. \quad \int_{-\infty}^\infty \frac{\sin rx dx}{\Gamma(p+x)\Gamma(q-x)} = \frac{\left(2 \cos \frac{r}{2}\right)^{p+q-2} \sin \frac{r(q-p)}{2}}{\Gamma(p+q-1)} \quad [|r| < \pi]$$

$= 0 \quad [|r| > \pi]$

[r is real; $\text{Re}(p+q) > 1$] MO 10a, ET II 298(9, 10)

$$2. \quad \int_{-\infty}^\infty \frac{\cos rx dx}{\Gamma(p+x)\Gamma(q-x)} = \frac{\left(2 \cos \frac{r}{2}\right)^{p+q-2} \cos \frac{r(q-p)}{2}}{\Gamma(p+q-1)} \quad [|r| < \pi]$$

$= 0 \quad [|r| > \pi]$

[r is real; $\text{Re}(p+q) > 1$] MO 10a, ET II 299(13, 14)

$$6.432 \quad \int_{-\infty}^\infty \frac{\sin(m\pi x)}{\sin(\pi x)} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = 0 \quad [m \text{ is an even integer}]$$

$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \quad [m \text{ is an odd integer}]$

[Re($\alpha + \beta$) > 1] ET II 298(11, 12)

6.433

$$1. \quad \int_{-\infty}^\infty \frac{\sin \pi x dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\sin\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$

[$\alpha + \delta = \beta + \gamma$, $\text{Re}(\alpha + \beta + \gamma + \delta) > 2$] ET II 300(22)

$$2. \quad \int_{-\infty}^\infty \frac{\cos \pi x dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\cos\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$

[$\alpha + \delta = \beta + \gamma$, $\text{Re}(\alpha + \beta + \gamma + \delta) > 2$] ET II 301(23)

6.44 The logarithm of the gamma function*

6.441

$$1. \quad \int_p^{p+1} \ln \Gamma(x) dx = \frac{1}{2} \ln 2\pi + p \ln p - p \quad \text{FI II 784}$$

$$2. \quad \int_0^1 \ln \Gamma(x) dx = \int_0^1 \ln \Gamma(1-x) dx = \frac{1}{2} \ln 2\pi \quad \text{FI II 783}$$

$$3. \quad \int_0^1 \ln \Gamma(x+q) dx = \frac{1}{2} \ln 2\pi + q \ln q - q \quad [q \geq 0] \quad \text{NH 89(17), ET II 304(40)}$$

$$4. \quad \int_0^z \ln \Gamma(x+1) dx = \frac{z}{2} \ln 2\pi - \frac{z(z+1)}{2} + z \ln \Gamma(z+1) - \ln G(z+1),$$

where $G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z(z+1)}{2} - \frac{Cz^2}{2}\right) \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k}\right) \right\}$ WH

$$5. \quad \int_0^n \ln \Gamma(\alpha+x) dx = \sum_{k=0}^{n-1} (a+k) \ln(a+k) - na + \frac{1}{2} n \ln(2\pi) - \frac{1}{2} n(n-1)$$

$[a \geq 0; \quad n = 1, 2, \dots] \quad \text{ET II 304(41)}$

$$6.442 \quad \int_0^1 \exp(2\pi n x i) \ln \Gamma(a+x) dx = (2\pi n i)^{-1} [\ln a - \exp(-2\pi n a i) \text{Ei}(2\pi n a i)]$$

$[a > 0; \quad n = \pm 1, \pm 2, \dots] \quad \text{ET II 304(38)}$

6.443

$$1. \quad \int_0^1 \ln \Gamma(x) \sin 2\pi n x dx = \frac{1}{2\pi n} [\ln(2\pi n) + C] \quad \text{NH 203(5), ET II 304(42)}$$

$$2. \quad \int_0^1 \ln \Gamma(x) \sin(2n+1)\pi x dx = \frac{1}{(2n+1)\pi} \left[\ln\left(\frac{\pi}{2}\right) + 2 \left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1}\right) + \frac{1}{2n+1} \right]$$

ET II 305(43)

$$3. \quad \int_0^1 \ln \Gamma(x) \cos 2\pi n x dx = \frac{1}{4n} \quad \text{NH 203(6), ET II 305(44)}$$

$$4.^8 \quad \int_0^1 \ln \Gamma(x) \cos(2n+1)\pi x dx = \frac{2}{\pi^2} \left[\frac{1}{(2n+1)^2} (C + \ln 2\pi) + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - (2n+1)^2} \right] \quad \text{NH 203(6)}$$

$$5. \quad \int_0^1 \sin(2\pi n x) \ln \Gamma(a+x) dx = -(2\pi n)^{-1} [\ln a + \cos(2\pi n a) \text{ci}(2\pi n a) - \sin(2\pi n a) \text{si}(2\pi n a)]$$

$[a > 0; \quad n = 1, 2, \dots] \quad \text{ET II 304(36)}$

$$6. \quad \int_0^1 \cos(2\pi n x) \ln \Gamma(a+x) dx = -(2\pi n)^{-1} [\sin(2\pi n a) \text{ci}(2\pi n a) + \cos(2\pi n a) \text{si}(2\pi n a)]$$

$[a > 0; \quad n = 1, 2, \dots] \quad \text{ET II 304(37)}$

*Here, we are violating our usual order of presentation of the formulas in order to make it easier to examine the integrals involving the gamma function.

6.45 The incomplete gamma function

6.451

$$1. \quad \int_0^\infty e^{-\alpha x} \gamma(\beta, x) dx = \frac{1}{\alpha} \Gamma(\beta) (1 + \alpha)^{-\beta} \quad [\beta > 0] \quad \text{MI 39}$$

$$2. \quad \int_0^\infty e^{-\alpha x} \Gamma(\beta, x) dx = \frac{1}{\alpha} \Gamma(\beta) \left[1 - \frac{1}{(\alpha + 1)^\beta} \right] \quad [\beta > 0] \quad \text{MI 39}$$

6.452

$$1. \quad \int_0^\infty e^{-\mu x} \gamma\left(\nu, \frac{x^2}{8a^2}\right) dx = \frac{1}{\mu} 2^{-\nu-1} \Gamma(2\nu) e^{(a\mu)^2} D_{-2\nu}(2a\mu) \quad \left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > 0 \right] \quad \text{ET I 179(36)}$$

$$2. \quad \int_0^\infty e^{-\mu x} \gamma\left(\frac{1}{4}, \frac{x^2}{8a^2}\right) dx = \frac{2^{\frac{3}{4}} \sqrt{a}}{\sqrt{\mu}} e^{(a\mu)^2} K_{\frac{1}{4}}(a^2 \mu^2) \quad \left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \mu > 0 \right] \quad \text{ET I 179(35)}$$

$$6.453 \quad \int_0^\infty e^{-\mu x} \Gamma\left(\nu, \frac{a}{x}\right) dx = 2a^{\frac{1}{2}\nu} \mu^{\frac{1}{2}\nu-1} K_\nu(2\sqrt{\mu a}) \quad \left[|\arg a| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0 \right] \quad \text{ET I 179(32)}$$

$$6.454 \quad \int_0^\infty e^{-\beta x} \gamma(\nu, \alpha \sqrt{x}) dx = 2^{-\frac{1}{2}\nu} \alpha^\nu \beta^{-\frac{1}{2}\nu-1} \Gamma(\nu) \exp\left(\frac{\alpha^2}{8\beta}\right) D_{-\nu}\left(\frac{\alpha}{\sqrt{2\beta}}\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 309(19), MI 39a}$$

6.455

$$1. \quad \int_0^\infty x^{\mu-1} e^{-\beta x} \Gamma(\nu, \alpha x) dx = \frac{\alpha^\nu \Gamma(\mu + \nu)}{\mu(\alpha + \beta)^{\mu+\nu}} {}_2F_1\left(1, \mu + \nu; \mu + 1; \frac{\beta}{\alpha + \beta}\right) \quad [\operatorname{Re}(\alpha + \beta) > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(\mu + \nu) > 0] \quad \text{ET II 309(16)}$$

$$2. \quad \int_0^\infty x^{\mu-1} e^{-\beta x} \gamma(\nu, \alpha x) dx = \frac{\alpha^\nu \Gamma(\mu + \nu)}{\nu(\alpha + \beta)^{\mu+\nu}} {}_2F_1\left(1, \mu + \nu; \nu + 1; \frac{\alpha}{\alpha + \beta}\right) \quad [\operatorname{Re}(\alpha + \beta) > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\mu + \nu) > 0] \quad \text{ET II 308(15)}$$

6.456

$$1. \quad \int_0^\infty e^{-\alpha x} (4x)^{\nu-\frac{1}{2}} \gamma\left(\nu, \frac{1}{4x}\right) dx = \sqrt{\pi} \frac{\gamma(2\nu, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}} \quad \text{MI 39a}$$

$$2. \quad \int_0^\infty e^{-\alpha x} (4x)^{\nu-\frac{1}{2}} \Gamma\left(\nu, \frac{1}{4x}\right) dx = \frac{\sqrt{\pi} \Gamma(2\nu, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}} \quad \text{MI 39a}$$

6.457

$$1. \quad \int_0^\infty e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \gamma\left(\nu + 1, \frac{1}{4x}\right) dx = \sqrt{\pi} \frac{\gamma(2\nu + 1, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}} \quad \text{MI 39}$$

$$2. \quad \int_0^\infty e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \Gamma\left(\nu + 1, \frac{1}{4x}\right) dx = \sqrt{\pi} \frac{\Gamma(2\nu + 1, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}} \quad \text{MI 39}$$

$$\begin{aligned}
 6.458 \quad \int_0^\infty x^{1-2\nu} \exp(\alpha x^2) \sin(bx) \Gamma(\nu, \alpha x^2) dx &= \pi^{\frac{1}{2}} 2^{-\nu} \alpha^{\nu-1} \Gamma\left(\frac{3}{2} - \nu\right) \exp\left(\frac{b^2}{8\alpha}\right) D_{2\nu-2} \left[\frac{b}{(2\alpha)^{\frac{1}{2}}} \right] \\
 &\quad \left[|\arg \alpha| < \frac{3\pi}{2}, \quad 0 < \operatorname{Re} \nu < 1 \right] \\
 &\quad \text{ET II 309(18)}
 \end{aligned}$$

6.46–6.47 The function $\psi(x)$

$$\begin{aligned}
 6.461 \quad \int_1^x \psi(t) dt &= \ln \Gamma(x) \\
 6.462 \quad \int_0^1 \psi(\alpha + x) dx &= \ln \alpha \quad [\alpha > 0] \quad \text{ET II 305(1)} \\
 6.463 \quad \int_0^\infty x^{-\alpha} [C + \psi(1+x)] dx &= -\pi \operatorname{cosec}(\pi\alpha) \zeta(\alpha) \quad [1 < \operatorname{Re} \alpha < 2] \quad \text{ET II 305(6)} \\
 6.464 \quad \int_0^1 e^{2\pi n x i} \psi(\alpha + x) dx &= e^{-2\pi n \alpha i} \operatorname{Ei}(2\pi n \alpha i) \quad [\alpha > 0; \quad n = \pm i, \pm 2, \dots] \quad \text{ET II 305(2)} \\
 6.465 \quad 1.^8 \int_0^1 \psi(x) \sin \pi x dx &= -\frac{2}{\pi} \left[C + \ln 2\pi + 2 \sum_{k=2}^\infty \frac{\ln k}{4k^2 - 1} \right] \\
 &\quad \text{(see 6.443 4)} \quad \text{NH 204} \\
 2. \quad \int_0^1 \psi(x) \sin(2\pi n x) dx &= -\frac{1}{2} \pi \quad [n = 1, 2, \dots] \quad \text{ET II 305(3)} \\
 6.466 \quad \int_0^\infty [\psi(\alpha + ix) - \psi(\alpha - ix)] \sin xy dx &= i\pi e^{-\alpha y} (1 - e^{-y})^{-1} \\
 &\quad [\alpha > 0, \quad y > 0] \quad \text{ET I 96(1)} \\
 6.467 \quad 1. \quad \int_0^1 \sin(2\pi n x) \psi(\alpha + x) dx &= \sin(2\pi n \alpha) \operatorname{ci}(2\pi n \alpha) + \cos(2\pi n \alpha) \operatorname{si}(2\pi n \alpha) \\
 &\quad [\alpha \geq 0; \quad n = 1, 2, \dots] \quad \text{ET II 305(4)} \\
 2. \quad \int_0^1 \cos(2\pi n x) \psi(\alpha + x) dx &= \sin(2\pi n \alpha) \operatorname{si}(2\pi n \alpha) - \cos(2\pi n \alpha) \operatorname{ci}(2\pi n \alpha) \\
 &\quad [\alpha > 0; \quad n = 1, 2, \dots] \quad \text{ET II 305(5)} \\
 6.468 \quad \int_0^1 \psi(x) \sin^2 \pi x dx &= -\frac{1}{2} [C + \ln(2\pi)] \quad \text{NH 204} \\
 6.469 \quad 1. \quad \int_0^1 \psi(x) \sin \pi x \cos \pi x dx &= -\frac{\pi}{4} \quad \text{NH 204} \\
 2.^8 \quad \int_0^1 \psi(x) \sin \pi x \sin(n\pi x) dx &= \frac{n}{1 - n^2} \quad [n \text{ is even}] \\
 &= \frac{1}{2} \ln \frac{n-1}{n+1} \quad [n > 1 \text{ is odd}] \\
 &\quad \text{NH 204(8)a}
 \end{aligned}$$

6.471

$$1. \quad \int_0^\infty x^{-\alpha} [\ln x - \psi(1+x)] dx = \pi \operatorname{cosec}(\pi\alpha) \zeta(\alpha) \quad [0 < \operatorname{Re} \alpha < 1] \quad \text{ET II 306(7)}$$

$$2. \quad \int_0^\infty x^{-\alpha} [\ln(1+x) - \psi(1+x)] dx = \pi \operatorname{cosec}(\pi\alpha) [\zeta(\alpha) - (\alpha-1)^{-1}]$$

$$[0 < \operatorname{Re} \alpha < 1] \quad \text{ET II 306(8)}$$

$$3. \quad \int_0^\infty [\psi(x+1) - \ln x] \cos(2\pi xy) dx = \frac{1}{2} [\psi(y+1) - \ln y] \quad \text{ET II 306(12)}$$

6.472

$$1. \quad \int_0^\infty x^{-\alpha} [(1+x)^{-1} - \psi'(1+x)] dx = -\pi\alpha \operatorname{cosec}(\pi\alpha) [\zeta(1+\alpha) - \alpha^{-1}]$$

$$[|\operatorname{Re} \alpha| < 1] \quad \text{ET II 306(9)}$$

$$2. \quad \int_0^\infty x^{-\alpha} [x^{-1} - \psi'(1+x)] dx = -\pi\alpha \operatorname{cosec}(\pi\alpha) \zeta(1+\alpha)$$

$$[-2 < \operatorname{Re} \alpha < 0] \quad \text{ET II 306(10)}$$

$$\mathbf{6.473} \quad \int_0^\infty x^{-\alpha} \psi^{(n)}(1+x) dx = (-1)^{n-1} \frac{\pi \Gamma(\alpha+n)}{\Gamma(\alpha) \sin \pi\alpha} \zeta(\alpha+n)$$

$$[n = 1, 2, \dots; \quad 0 < \operatorname{Re} \alpha < 1] \quad \text{ET II 306(11)}$$

6.5–6.7 Bessel Functions**6.51 Bessel functions****6.511**

$$1. \quad \int_0^\infty J_\nu(bx) dx = \frac{1}{b} \quad [\operatorname{Re} \nu > -1, \quad b > 0] \quad \text{ET II 22(3)}$$

$$2. \quad \int_0^\infty Y_\nu(bx) dx = -\frac{1}{b} \tan\left(\frac{\nu\pi}{2}\right) \quad [|\operatorname{Re} \nu| < 1, \quad b > 0]$$

WA 432(7), ET II 96(1)

$$3. \quad \int_0^a J_\nu(x) dx = 2 \sum_{k=0}^\infty J_{\nu+2k+1}(a) \quad [\operatorname{Re} \nu > -1] \quad \text{ET II 333(1)}$$

$$4. \quad \int_0^a J_{\frac{1}{2}}(t) dt = 2 S(\sqrt{a}) \quad \text{WA 599(4)}$$

$$5. \quad \int_0^a J_{-\frac{1}{2}}(t) dt = 2 C(\sqrt{a}) \quad \text{WA 599(3)}$$

$$6. \quad \int_0^a J_0(x) dx = a J_0(a) + \frac{\pi a}{2} [J_1(a) \mathbf{H}_0(a) - J_0(a) \mathbf{H}_1(a)]$$

$$[a > 0] \quad \text{ET II 7(2)}$$

$$7. \quad \int_0^a J_1(x) dx = 1 - J_0(a) \quad [a > 0] \quad \text{ET II 18(1)}$$

$$8. \quad \int_a^\infty J_0(x) dx = 1 - a J_0(a) + \frac{\pi a}{2} [J_0(a) \mathbf{H}_1(a) - J_1(a) \mathbf{H}_0(a)]$$

$$[a > 0] \quad \text{ET II 7(3)}$$

$$9. \quad \int_a^\infty J_1(x) dx = J_0(a) \quad [a > 0] \quad \text{ET II 18(2)}$$

$$10. \quad \int_a^b Y_\nu(x) dx = 2 \sum_{n=0}^\infty [Y_{\nu+2n+1}(b) - Y_{\nu+2n+1}(a)] \quad \text{ET II 339(46)}$$

$$11. \quad \int_0^a I_\nu(x) dx = 2 \sum_{n=0}^\infty (-1)^n I_{\nu+2n+1}(a) \quad [\operatorname{Re} \nu > -1] \quad \text{ET II 364(1)}$$

$$12.^* \quad \int_0^\infty K_0(ax) dx = \frac{\pi}{2a} \quad [a > 0]$$

$$13.^* \quad \int_0^\infty K_0^2(ax) dx = \frac{\pi^2}{4a} \quad [a > 0]$$

6.512

$$1.^{11} \quad \int_0^\infty J_\mu(ax) J_\nu(bx) dx = b^\nu a^{-\nu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1) \Gamma\left(\frac{\mu-\nu+1}{2}\right)} F\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right)$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re}(\mu+\nu) > -1, \quad b < a.]$$

For $a > b$, the positions of μ and ν should be reversed.]

ET II 48(6)

$$2.^7 \quad \int_0^\infty J_{\nu+n}(\alpha t) J_{\nu-n-1}(\beta t) dt = \frac{\beta^{\nu-n-1} \Gamma(\nu)}{\alpha^{\nu-n} n! \Gamma(\nu-n)} F\left(\nu, -n; \nu-n; \frac{\beta^2}{\alpha^2}\right) \quad [0 < \beta < \alpha]$$

$$= (-1)^n \frac{1}{2\alpha} \quad [0 < \beta = \alpha]$$

$$= 0 \quad [0 < \alpha < \beta]$$

[$\operatorname{Re}(\nu) > 0$] MO 50

$$3.^8 \quad \int_0^\infty J_\nu(\alpha x) J_{\nu-1}(\beta x) dx = \frac{\beta^{\nu-1}}{\alpha^\nu} \quad [\beta < \alpha]$$

$$= \frac{1}{2\beta} \quad [\beta = \alpha]$$

$$= 0 \quad [\beta > \alpha]$$

$$[\operatorname{Re} \nu > 0] \quad \text{WA 444(8), KU (40)a}$$

$$4. \quad \int_0^\infty J_{\nu+2n+1}(ax) J_\nu(bx) dx = b^\nu a^{-\nu-1} P_n^{(\nu,0)}\left(1 - \frac{2b^2}{a^2}\right) \quad [\operatorname{Re} \nu > -1-n, \quad 0 < b < a]$$

$$= 0 \quad [\operatorname{Re} \nu > -1-n, \quad 0 < a < b]$$

ET II 47(5)

$$5. \quad \int_0^\infty J_{\nu+n}(ax) Y_{\nu-n}(ax) dx = (-1)^{n+1} \frac{1}{2a} \quad [\operatorname{Re} \nu > -\tfrac{1}{2}, \quad a > 0, \quad n = 0, 1, 2, \dots] \\ \text{ET II 347(57)}$$

$$6. \quad \int_0^\infty J_1(bx) Y_0(ax) dx = -\frac{b^{-1}}{\pi} \ln \left(1 - \frac{b^2}{a^2} \right) \quad [0 < b < a] \quad \text{ET II 21(31)}$$

$$7. \quad \int_0^a J_\nu(x) J_{\nu+1}(x) dx = \sum_{n=0}^\infty [J_{\nu+n+1}(a)]^2 \quad [\operatorname{Re} \nu > -1] \quad \text{ET II 338(37)}$$

$$8.^9 \quad \int_0^\infty k J_n(ka) J_n(kb) dk = \frac{1}{a} \delta(b-a) \quad [n = 0, 1, \dots] \quad \text{JAC 110}$$

$$9.^* \quad \int_0^\infty K_0(ax) J_1(bx) dx = \frac{1}{2b} \ln \left(1 + \frac{b^2}{a^2} \right) \quad [a > 0, \quad b > 0]$$

$$10.^* \quad \int_0^\infty K_0(ax) I_1(bx) dx = -\frac{1}{2b} \ln \left(1 - \frac{b^2}{a^2} \right) \quad [a > 0, \quad b > 0]$$

6.513

$$1. \quad \int_0^\infty [J_\mu(ax)]^2 J_\nu(bx) dx = a^{2\mu} b^{-2\mu-1} \frac{\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{[\Gamma(\mu+1)]^2 \Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \\ \times \left[F\left(\frac{1-\nu+2\mu}{2}, \frac{1+\nu+2\mu}{2}; \mu+1; \frac{1-\sqrt{1-\frac{4a^2}{b^2}}}{2}\right) \right]^2 \\ [\operatorname{Re} \nu + \operatorname{Re} 2\mu > -1, \quad 0 < 2a < b] \quad \text{ET II 52(33)}$$

$$2. \quad \int_0^\infty [J_\mu(ax)]^2 K_\nu(bx) dx = \frac{b^{-1}}{2} \Gamma\left(\frac{2\mu+\nu+1}{2}\right) \Gamma\left(\frac{2\mu-\nu+1}{2}\right) \left[P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left(\sqrt{1+\frac{4a^2}{b^2}} \right) \right]^2 \\ [2 \operatorname{Re} \mu > |\operatorname{Re} \nu| - 1, \quad \operatorname{Re} b > 2|\operatorname{Im} a|] \\ \text{ET II 138(18)}$$

$$3. \quad \int_0^\infty I_\mu(ax) K_\mu(ax) J_\nu(bx) dx = \frac{e^{\mu\pi i} \Gamma\left(\frac{\nu+2\mu+1}{2}\right)}{b \Gamma\left(\frac{\nu-2\mu+1}{2}\right)} P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left(\sqrt{1+\frac{4a^2}{b^2}} \right) Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left(\sqrt{1+\frac{4a^2}{b^2}} \right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu+2\mu) > -1] \quad \text{ET II 65(20)}$$

$$4. \quad \int_0^\infty J_\mu(ax) J_{-\mu}(ax) K_\nu(bx) dx = \frac{\pi}{2b} \sec\left(\frac{\nu\pi}{2}\right) P_{\frac{1}{2}\nu-\frac{1}{2}}^\mu \left(\sqrt{1+\frac{4a^2}{b^2}} \right) P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left(\sqrt{1+\frac{4a^2}{b^2}} \right) \\ [|\operatorname{Re} \nu| < 1, \quad \operatorname{Re} b > 2|\operatorname{Im} a|] \\ \text{ET II 138(21)}$$

$$5. \quad \int_0^\infty [K_\mu(ax)]^2 J_\nu(bx) dx = \frac{e^{2\mu\pi i} \Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{b \Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \left[Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left(\sqrt{1 + \frac{4a^2}{b^2}} \right) \right]^2$$

[Re $a > 0$, $b > 0$, Re $(\frac{1}{2}\nu \pm \mu) > -\frac{1}{2}$] ET II 66(28)

$$6. \quad \int_0^z J_\mu(x) J_\nu(z-x) dx = 2 \sum_{k=0}^\infty (-1)^k J_{\mu+\nu+2k+1}(z) \quad [\text{Re } \mu > -1, \text{ Re } \nu > -1] \quad \text{WA 414(2)}$$

$$7. \quad \int_0^z J_\mu(x) J_{-\mu}(z-x) dx = \sin z \quad [-1 < \text{Re } \mu < 1] \quad \text{WA 415(4)}$$

$$8. \quad \int_0^z J_\mu(x) J_{1-\mu}(z-x) dx = J_0(z) - \cos(z) \quad [-1 < \text{Re } \mu < 2] \quad \text{WA 415(4)}$$

$$9.^* \quad \int_0^\infty J_0^2(ax) J_1(bx) dx = \frac{1}{b} \quad [b > 2a > 0]$$

$$= \frac{2}{\pi b} \arcsin\left(\frac{b}{2a}\right) \quad [2a > b > 0]$$

6.514

$$1. \quad \int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu(bx) dx = b^{-1} J_{2\nu}(2\sqrt{ab}) \quad [a > 0, b > 0, \text{ Re } \nu > -\frac{1}{2}]$$

ET II 57(9)

$$2. \quad \int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu(bx) dx = b^{-1} \left[Y_{2\nu}(2\sqrt{ab}) + \frac{2}{\pi} K_{2\nu}(\sqrt{2ab}) \right]$$

[$a > 0$, $b > 0$, $-\frac{1}{2} < \text{Re } \nu < \frac{3}{2}$] ET II 110(12)

$$3. \quad \int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu(bx) dx = b^{-1} e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu} \left[2e^{\frac{1}{4}i\pi} \sqrt{ab} \right] + b^{-1} e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu} \left[2e^{-\frac{1}{4}i\pi} \sqrt{ab} \right]$$

[$a > 0$, Re $b > 0$, $|\text{Re } \nu| < \frac{5}{2}$] ET II 141(31)

$$4. \quad \int_0^\infty Y_\nu\left(\frac{a}{x}\right) J_\nu(bx) dx = -\frac{2b^{-1}}{\pi} \left[K_{2\nu}(2\sqrt{ab}) - \frac{\pi}{2} Y_{2\nu}(2\sqrt{ab}) \right]$$

[$a > 0$, $b > 0$, $|\text{Re } \nu| < \frac{1}{2}$] ET II 62(37)a

$$5. \quad \int_0^\infty Y_\nu\left(\frac{a}{x}\right) Y_\nu(bx) dx = -b^{-1} J_{2\nu}(2\sqrt{ab}) \quad [a > 0, b > 0, |\text{Re } \nu| < \frac{1}{2}]$$

ET II 110(14)

$$6. \quad \int_0^\infty Y_\nu\left(\frac{a}{x}\right) K_\nu(bx) dx = -b^{-1} e^{\frac{1}{2}\nu\pi i} K_{2\nu} \left(2e^{\frac{1}{4}\pi i} \sqrt{ab} \right) - b^{-1} e^{-\frac{1}{2}\nu\pi i} K_{2\nu} \left(2e^{-\frac{1}{4}\pi i} \sqrt{ab} \right)$$

[$a > 0$, Re $b > 0$, $|\text{Re } \nu| < \frac{5}{2}$] ET II 143(37)

$$7. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu(bx) dx = -2b^{-1} \left[\sin\left(\frac{3\nu\pi}{2}\right) \ker_{2\nu}(2\sqrt{ab}) + \cos\left(\frac{3\nu\pi}{2}\right) \operatorname{kei}_{2\nu}(2\sqrt{ab}) \right] \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right] \quad \text{ET II 113(28)}$$

$$8. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu(bx) dx = \pi b^{-1} K_{2\nu}(2\sqrt{ab}) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 146(54)}$$

6.515

$$1. \quad \int_0^\infty J_\mu\left(\frac{a}{x}\right) Y_\mu\left(\frac{a}{x}\right) K_0(bx) dx = -2b^{-1} J_{2\mu}(2\sqrt{ab}) K_{2\mu}(2\sqrt{ab}) \\ [a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 143(42)}$$

$$2. \quad \int_0^\infty \left[K_\mu\left(\frac{a}{x}\right) \right]^2 K_0(bx) dx = 2\pi b^{-1} K_{2\mu}(2e^{\frac{1}{4}\pi i} \sqrt{ab}) K_{2\mu}(2e^{-\frac{1}{4}\pi i} \sqrt{ab}) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 147(59)}$$

$$3. \quad \int_0^\infty H_\mu^{(1)}\left(\frac{a^2}{x}\right) H_\mu^{(2)}\left(\frac{a^2}{x}\right) J_0(bx) dx = 16\pi^{-2} b^{-1} \cos \mu\pi K_{2\mu}(2e^{\pi i/4} a\sqrt{b}) K_{2\mu}(2e^{-\pi i/4} a\sqrt{b}) \\ \left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad |\operatorname{Re} \mu| < \frac{1}{4} \right] \\ \text{ET II 17(36)}$$

6.516

$$1. \quad \int_0^\infty J_{2\nu}(a\sqrt{x}) J_\nu(bx) dx = b^{-1} J_\nu\left(\frac{a^2}{4b}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\ \text{ET II 58(16)}$$

$$2. \quad \int_0^\infty J_{2\nu}(a\sqrt{x}) Y_\nu(bx) dx = -b^{-1} \mathbf{H}_\nu\left(\frac{a^2}{4b}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\ \text{ET II 111(18)}$$

$$3. \quad \int_0^\infty J_{2\nu}(a\sqrt{x}) K_\nu(bx) dx = \frac{\pi}{2} b^{-1} \left[I_\nu\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right) \right] \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 144(45)}$$

$$4.^{10} \quad \int_0^\infty Y_{2\nu}(a\sqrt{x}) J_\nu(bx) dx = \frac{1}{b} J_\nu\left(\frac{a^2}{4b}\right) \cot(2\pi\nu) - \frac{1}{2b} J_{-\nu}\left(\frac{a^2}{4b}\right) \operatorname{cosec}(2\pi\nu) \\ - \frac{2^{3\nu-3} a^{2-2\nu} b^{\nu-2}}{\pi^{3/2}} \Gamma\left(\nu - \frac{1}{2}\right) {}_1F_2\left(1; \frac{3}{2}, \frac{3}{2} - \nu; \frac{a^4}{64b^2}\right) \\ [a > 0, \quad b > 0] \quad \text{MC}$$

$$5. \quad \int_0^\infty Y_{2\nu}(a\sqrt{x}) Y_\nu(bx) dx \\ = \frac{b^{-1}}{2} \left[\sec(\nu\pi) J_{-\nu}\left(\frac{a^2}{4b}\right) + \operatorname{cosec}(\nu\pi) \mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) - 2 \cot(2\nu\pi) \mathbf{H}_\nu\left(\frac{a^2}{4b}\right) \right] \\ [a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 111(19)}$$

$$\begin{aligned}
 6. \quad \int_0^\infty Y_{2\nu}(a\sqrt{x}) K_\nu(bx) dx &= \frac{\pi b^{-1}}{2} \left[\operatorname{cosec}(2\nu\pi) \mathbf{L}_{-\nu} \left(\frac{a^2}{4b} \right) - \cot(2\nu\pi) \mathbf{L}_\nu \left(\frac{a^2}{4b} \right) \right. \\
 &\quad \left. - \tan(\nu\pi) I_\nu \left(\frac{a^2}{4b} \right) - \frac{\sec(\nu\pi)}{\pi} K_\nu \left(\frac{a^2}{4b} \right) \right] \\
 &\quad [\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 144(46)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \int_0^\infty K_{2\nu}(a\sqrt{x}) J_\nu(bx) dx &= \frac{1}{4} \pi b^{-1} \sec(\nu\pi) \left[\mathbf{H}_{-\nu} \left(\frac{a^2}{4b} \right) - Y_{-\nu} \left(\frac{a^2}{4b} \right) \right] \\
 &\quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
 &\quad \text{ET II 70(22)}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int_0^\infty K_{2\nu}(a\sqrt{x}) Y_\nu(bx) dx \\
 &= -\frac{1}{4} \pi b^{-1} \left[\sec(\nu\pi) J_{-\nu} \left(\frac{a^2}{4b} \right) - \operatorname{cosec}(\nu\pi) \mathbf{H}_{-\nu} \left(\frac{a^2}{4b} \right) + 2 \operatorname{cosec}(2\nu\pi) \mathbf{H}_\nu \left(\frac{a^2}{4b} \right) \right] \\
 &\quad [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 114(34)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_0^\infty K_{2\nu}(a\sqrt{x}) K_\nu(bx) dx &= \frac{\pi b^{-1}}{4 \cos(\nu\pi)} \left\{ K_\nu \left(\frac{a^2}{4b} \right) + \frac{\pi}{2 \sin(\nu\pi)} \left[\mathbf{L}_{-\nu} \left(\frac{a^2}{4b} \right) - \mathbf{L}_\nu \left(\frac{a^2}{4b} \right) \right] \right\} \\
 &\quad [\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 147(63)}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \int_0^\infty I_{2\nu}(a\sqrt{x}) K_\nu(bx) dx &= \frac{\pi b^{-1}}{2} \left[I_\nu \left(\frac{a^2}{4b} \right) + \mathbf{L}_\nu \left(\frac{a^2}{4b} \right) \right] \\
 &\quad [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 147(60)}
 \end{aligned}$$

$$6.517 \quad \int_0^z J_0(\sqrt{z^2 - x^2}) dx = \sin z \quad \text{MO 48}$$

$$6.518 \quad \int_0^\infty K_{2\nu}(2z \sinh x) dx = \frac{\pi^2}{8 \cos \nu\pi} (J_\nu^2(z) + N_\nu^2(z)) \quad [\operatorname{Re} z > 0, \quad -\tfrac{1}{2} < \operatorname{Re} \nu < \tfrac{1}{2}] \quad \text{MO 45}$$

6.519

$$1. \quad \int_0^{\pi/2} J_{2\nu}(2z \cos x) dx = \frac{\pi}{2} J_\nu^2(z) \quad [\operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{WH}$$

$$2. \quad \int_0^{\pi/2} J_{2\nu}(2z \sin x) dx = \frac{\pi}{2} J_\nu^2(z) \quad [\operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{WA 42(1)a}$$

6.52 Bessel functions combined with x and x^2

6.521

$$\begin{aligned}
 1. \quad \int_0^1 x J_\nu(\alpha x) J_\nu(\beta x) dx &= \frac{\beta J_{\nu-1}(\beta) J_\nu(\alpha) - \alpha J_{\nu-1}(\alpha) J_\nu(\beta)}{\alpha^2 - \beta^2} \quad [\alpha \neq \beta, \quad \nu > -1] \\
 &= \frac{\alpha J_\nu(\beta) J'_\nu(\alpha) - \beta J_\nu(\alpha) J'_\nu(\beta)}{\beta^2 - \alpha^2} \quad [\alpha \neq \beta, \quad \nu > -1]
 \end{aligned}$$

WH

$$2.^{10} \quad \int_0^\infty x K_\nu(ax) J_\nu(bx) dx = \frac{b^\nu}{a^\nu (b^2 + a^2)} \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 63(2)}$$

$$3. \quad \int_0^\infty x K_\nu(ax) K_\nu(bx) dx = \frac{\pi(ab)^{-\nu} (a^{2\nu} - b^{2\nu})}{2 \sin(\nu\pi) (a^2 - b^2)} \quad [|\operatorname{Re} \nu| < 1, \quad \operatorname{Re}(a + b) > 0] \\ \text{ET II 145(48)}$$

$$4. \quad \int_0^a x J_\nu(\lambda x) K_\nu(\mu x) dx = (\mu^2 + \lambda^2)^{-1} \left[\left(\frac{\lambda}{\mu} \right)^\nu + \lambda a J_{\nu+1}(\lambda a) K_\nu(\mu a) - \mu a J_\nu(\lambda a) K_{\nu+1}(\mu a) \right] \\ [\operatorname{Re} \nu > -1] \quad \text{ET II 367(26)}$$

$$5.* \quad \int_0^\infty x K_1(ax) = \frac{\pi}{2a^2} \quad [a > 0]$$

$$6.* \quad \int_0^\infty x K_0^2(ax) = \frac{1}{2a^2} \quad [a > 0]$$

$$7.* \quad \int_0^\infty x K_1(ax) J_1(bx) = \frac{b}{a(a^2 + b^2)} \quad [a > 0, \quad b > 0]$$

$$8.* \quad \int_0^\infty x K_0(ax) I_0(bx) = \frac{1}{a^2 - b^2} \quad [a > b > 0]$$

$$9.* \quad \int_0^\infty x K_1(ax) I_1(bx) = \frac{b}{a(a^2 - b^2)} \quad [a > b > 0]$$

$$10.* \quad \int_0^\infty x^2 K_0(ax) = \frac{\pi}{2a^3} \quad [a > 0]$$

$$11.* \quad \int_0^\infty x^2 K_1(ax) = \frac{2}{a^3} \quad [a > 0]$$

$$12.* \quad \int_0^\infty x^2 K_0(ax) J_1(bx) = \frac{2b}{(a^2 + b^2)^2} \quad [a > 0, \quad b > 0]$$

$$13.* \quad \int_0^\infty x^2 K_1(ax) J_0(bx) = \frac{2a}{(a^2 + b^2)^2} \quad [a > b > 0]$$

$$14.* \quad \int_0^\infty x^2 K_0(ax) I_1(bx) = \frac{2b}{(a^2 - b^2)^2} \quad [a > b > 0]$$

$$15.* \quad \int_0^\infty x^2 K_1(ax) I_0(bx) = \frac{2a}{(a^2 - b^2)^2} \quad [a > b > 0]$$

6.522 Notation: $\ell_1 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \ell_2 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$

$$1.^8 \quad \int_0^\infty x [J_\mu(ax)]^2 K_\nu(bx) dx = \Gamma(\mu + \tfrac{1}{2}\nu + 1) \Gamma(\mu - \tfrac{1}{2}\nu + 1) b^{-2} \\ \times (1 + 4a^2 b^{-2})^{-\frac{1}{2}} P_{\frac{1}{2}\nu}^{-\mu} \left[(1 + 4a^2 b^{-2})^{\frac{1}{2}} \right] P_{-\frac{1}{2}\nu}^{-\mu} \left[(1 + 4a^2 b^{-2})^{\frac{1}{2}} \right] \\ [\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2\operatorname{Re} \mu > |\operatorname{Re} \nu| - 2] \quad \text{ET II 138(19)}$$

$$\begin{aligned}
2. \quad \int_0^\infty x [K_\mu(ax)]^2 J_\nu(bx) dx &= \frac{2e^{2\mu\pi i} \Gamma\left(1 + \frac{1}{2}\nu + \mu\right)}{b(4a^2 + b^2)^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\nu - \mu\right)} \\
&\quad \times Q_{\frac{1}{2}\nu}^{-\mu}\left(\sqrt{(1 + 4a^2b^{-2})}\right) Q_{\frac{1}{2}\nu-1}^{-\mu}\left(\sqrt{(1 + 4a^2b^{-2})}\right) \\
&\quad [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\left(\frac{1}{2}\nu \pm \mu\right) > -1] \quad \text{ET II 66(27)a}
\end{aligned}$$

$$\begin{aligned}
3.^{11} \quad \int_0^\infty x K_0(ax) J_\nu(bx) J_\nu(cx) dx &= r_1^{-1} r_2^{-1} (r_2 - r_1)^\nu (r_2 - r_1)^{-\nu} = \frac{\ell_1^\nu}{\ell_2^\nu (\ell_2^2 - \ell_1^2)}, \\
&\quad \left[r_1 = \sqrt{a^2 + (b - c)^2}, \quad r_2 = \sqrt{a^2 + (b + c)^2}, \quad c > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} a > |\operatorname{Im} b| \right] \\
&\quad \text{ET II 63(6)}
\end{aligned}$$

$$\begin{aligned}
4.^{10} \quad \int_0^\infty x I_0(ax) K_0(bx) J_0(cx) dx &= (a^4 + b^4 + c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2)^{-\frac{1}{2}} \\
&\quad [\operatorname{Re} b > \operatorname{Re} a, \quad c > 0] \quad \text{ET II 16(27)}
\end{aligned}$$

alternatively, with a and c interchanged

$$\int_0^\infty x I_0(cx) K_0(bx) J_0(ax) dx = \frac{1}{\ell_2^2 - \ell_1^2} \quad [\operatorname{Re} b > \operatorname{Re} c, \quad a > 0]$$

$$\begin{aligned}
5.^{10} \quad \int_0^\infty x J_0(ax) K_0(bx) J_0(cx) dx &= (a^4 + b^4 + c^4 - 2a^2c^2 + 2a^2b^2 + 2b^2c^2)^{-\frac{1}{2}} \\
&\quad [\operatorname{Re} b > |\operatorname{Im} a|, \quad c > 0] \quad \text{ET II 15(25)}
\end{aligned}$$

alternatively, with a and b interchanged

$$\int_0^\infty x J_0(bx) K_0(ax) J_0(cx) dx = \frac{1}{\ell_2^2 - \ell_1^2} \quad [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

$$\begin{aligned}
6. \quad \int_0^\infty x J_0(ax) Y_0(ax) J_0(bx) dx &= 0 \quad [0 < b < 2a] \\
&= -2\pi^{-1} b^{-1} [b^2 - 4a^2]^{-\frac{1}{2}} \quad [0 < 2a < b < \infty] \\
&\quad \text{ET II 15(21)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x J_\mu(ax) J_{\mu+1}(ax) K_\nu(bx) dx &= \Gamma\left(\mu + \frac{3+\nu}{2}\right) \Gamma\left(\mu + \frac{3-\nu}{2}\right) b^{-2} (1 + 4a^2b^{-2})^{-\frac{1}{2}} \\
&\quad \times P_{-\mu}^{\frac{1}{2}\nu-\frac{1}{2}}\left[\sqrt{1 + 4a^2b^{-2}}\right] P_{-\mu-1}^{\frac{1}{2}\nu-\frac{1}{2}}\left[\sqrt{1 + 4a^2b^{-2}}\right] \\
&\quad [\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2\operatorname{Re} \mu > |\operatorname{Re} \nu| - 3] \quad \text{ET II 138(20)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_0^\infty x K_{\mu-\frac{1}{2}}(ax) K_{\mu+\frac{1}{2}}(ax) J_\nu(bx) dx \\
&= -\frac{2e^{2\mu\pi i} \Gamma\left(\frac{1}{2}\nu + \mu + 1\right)}{b \Gamma\left(\frac{1}{2}\nu - \mu\right) (b^2 + 4a^2)^{\frac{1}{2}}} Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu+\frac{1}{2}}\left[(1 + 4a^2b^{-2})^{\frac{1}{2}}\right] Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}\left[(1 + 4a^2b^{-2})^{\frac{1}{2}}\right] \\
&\quad [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1, \quad |\operatorname{Re} \mu| < 1 + \frac{1}{2} \operatorname{Re} \nu] \quad \text{ET II 67(29)a}
\end{aligned}$$

$$\begin{aligned}
9.^8 \quad \int_0^\infty x I_{\frac{1}{2}\nu}(ax) K_{\frac{1}{2}\nu}(ax) J_\nu(bx) dx &= b^{-1} (b^2 + 4a^2)^{-\frac{1}{2}} \\
&\quad [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1] \\
&\quad \text{ET II 65(16)}
\end{aligned}$$

$$\begin{aligned}
 10. \quad \int_0^\infty x J_{\frac{1}{2}\nu}(ax) Y_{\frac{1}{2}\nu}(ax) J_\nu(bx) dx \\
 &= 0 \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 0 < b < 2a] \\
 &= -2\pi^{-1} b^{-1} (b^2 - 4a^2)^{-\frac{1}{2}} \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 2a < b < \infty] \\
 &\quad \text{ET II 55(48)}
 \end{aligned}$$

$$\begin{aligned}
 11.^8 \quad \int_0^\infty x J_{\frac{1}{2}(\nu+n)}(ax) J_{\frac{1}{2}(\nu-n)}(ax) J_\nu(bx) dx \\
 &= 2\pi^{-1} b^{-1} (4a^2 - b^2)^{-\frac{1}{2}} T_n \left(\frac{b}{2a} \right) \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 0 < b < 2a] \\
 &= 0 \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 2a < b] \\
 &\quad \text{ET II 52(32)}
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \int_0^\infty x I_{\frac{1}{2}(\nu-\mu)}(ax) K_{\frac{1}{2}(\nu+\mu)}(ax) J_\nu(bx) dx = 2^{-\mu} a^{-\mu} b^{-1} (b^2 + 4a^2)^{-\frac{1}{2}} \left[b + (b^2 + 4a^2)^{\frac{1}{2}} \right]^\mu \\
 [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - \mu) > -2] \quad \text{ET II 66(23)}
 \end{aligned}$$

$$\begin{aligned}
 13.^8 \quad \int_0^\infty x J_\mu(xa \sin \varphi) K_{\nu-\mu}(ax \cos \varphi \cos \psi) J_\nu(xa \sin \psi) dx = \frac{(\sin \varphi)^\mu (\sin \psi)^\nu (\cos \varphi)^{\nu-\mu} (\cos \psi)^{\mu-\nu}}{a^2 (1 - \sin^2 \varphi \sin^2 \psi)} \\
 \left[a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1 \right] \quad \text{ET II 64(10)}
 \end{aligned}$$

$$\begin{aligned}
 14.^8 \quad \int_0^\infty x J_\mu(xa \sin \varphi \cos \psi) J_{\nu-\mu}(ax) J_\nu(xa \cos \varphi \sin \psi) dx \\
 = -2\pi^{-1} a^{-2} \sin(\mu\pi) (\sin \varphi)^\mu (\sin \psi)^\nu (\cos \varphi)^{-\nu} (\cos \psi)^{-\mu} [\cos(\varphi + \psi) \cos(\varphi - \psi)]^{-1} \\
 \left[a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -1 \right] \quad \text{ET II 54(39)}
 \end{aligned}$$

$$\begin{aligned}
 15.^{10} \quad \int_0^\infty x^{\nu+1} J_\nu(bx) K_\nu(ax) J_\nu(cx) dx = \frac{2^{3\nu} (abc)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\ell_2^2 - \ell_1^2)^{2\nu+1}} \\
 [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]
 \end{aligned}$$

$$\begin{aligned}
 16.^{10} \quad \int_0^\infty x^{\nu+1} I_\nu(cx) K_\nu(bx) J_\nu(ax) dx = \frac{2^{3\nu} (abc)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\ell_2^2 - \ell_1^2)^{2\nu+1}} \\
 [\operatorname{Re} b > |\operatorname{Im} a| + |\operatorname{Im} c|]
 \end{aligned}$$

$$\begin{aligned}
 17.^{11} \quad \int_0^\infty t^{\nu-\mu-\rho+1} J_\mu(ct) J_\nu(bt) K_\rho(at) dt \\
 = \frac{2^{1+\nu-\mu-\rho}}{c^\mu b^\nu a^\rho \Gamma(\mu - \nu + \rho)} \int_0^{\ell_1} \frac{x^{1+2\nu-2\rho} [(\ell_1^2 - x^2) (\ell_2^2 - x^2)]^{\mu-\nu+\rho-1}}{(b^2 - x^2)^{\mu-\nu}} dx \\
 \ell_1 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \quad \ell_2 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right] \\
 [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]
 \end{aligned}$$

$$\begin{aligned}
 18.^{11} \quad & \int_0^\infty t^{\mu-\nu+\rho+1} J_\mu(ct) J_\nu(bt) K_\rho(at) dt \\
 &= \frac{2^{1+\mu-\nu+\rho} a^\rho}{c^\mu b^\nu \Gamma(\nu-\mu-\rho)} \int_0^{\ell_1} \frac{x^{1+2\mu+2\rho} [(\ell_1^2 - x^2)(\ell_2^2 - x^2)]^{\nu-\mu-\rho-1}}{(c^2 - x^2)^{\nu-\mu}} dx \\
 &\ell_1 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \quad \ell_2 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right] \\
 &\quad [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]
 \end{aligned}$$

$$\begin{aligned}
 6.523 \quad & \int_0^\infty x [2\pi^{-1} K_0(ax) - Y_0(ax)] K_0(bx) dx = 2\pi^{-1} \left[(a^2 + b^2)^{-1} + (b^2 - a^2)^{-1} \right] \ln \frac{b}{a} \\
 &\quad [\operatorname{Re} b > |\operatorname{Im} a|, \quad \operatorname{Re}(a+b) > 0] \\
 &\quad \text{ET II 145(50)}
 \end{aligned}$$

6.524

$$\begin{aligned}
 1. \quad & \int_0^\infty x J_\nu^2(ax) J_\nu(bx) Y_\nu(bx) dx = 0 \quad [0 < a < b, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
 &= -(2\pi ab)^{-1} \quad [0 < b < a, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
 &\quad \text{ET II 352(14)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x [J_0(ax) K_0(bx)]^2 dx = \frac{\pi}{8ab} - \frac{1}{4ab} \arcsin \left(\frac{b^2 - a^2}{b^2 + a^2} \right) \\
 &\quad [a > 0, \quad b > 0] \quad \text{ET II 373(9)}
 \end{aligned}$$

$$6.525 \quad \text{Notation: } \ell_1 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \ell_2 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$$

$$\begin{aligned}
 1.^{10} \quad & \int_0^\infty x^2 J_1(ax) K_0(bx) J_0(cx) dx = 2a (a^2 + b^2 - c^2) \left[(a^2 + b^2 + c^2)^2 - 4a^2 c^2 \right]^{-\frac{3}{2}} \\
 &\quad [c > 0, \quad \operatorname{Re} b \geq |\operatorname{Im} a|, \quad \operatorname{Re} a > 0] \\
 &\quad \text{ET II 15(26)}
 \end{aligned}$$

alternatively, with a and b interchanged

$$\int_0^\infty x^2 J_1(bx) K_0(ax) J_0(cx) dx = \frac{2b (a^2 + b^2 - c^2)}{(\ell_2^2 - \ell_1^2)^3} \quad [\operatorname{Re} a > |\operatorname{Im} b|, \quad \operatorname{Re} b > 0, \quad c > 0]$$

$$\begin{aligned}
 2.^{10} \quad & \int_0^\infty x^2 I_0(ax) K_1(bx) J_0(cx) dx = 2b (b^2 + c^2 - a^2) \left[(a^2 + b^2 + c^2)^2 - 4a^2 b^2 \right]^{-\frac{3}{2}} \\
 &\quad [\operatorname{Re} b > |\operatorname{Re} a|, \quad c > 0] \quad \text{ET II 16(28)}
 \end{aligned}$$

$$3.^{10} \quad \int_0^\infty x^2 I_0(cx) K_0(bx) J_0(ax) dx = \frac{2b (a^2 + b^2 - c^2)}{(\ell_2^2 - \ell_1^2)^3} \quad [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

6.526

$$\begin{aligned}
 1. \quad & \int_0^\infty x J_{\frac{1}{2}\nu}(ax^2) J_\nu(bx) dx = (2a)^{-1} J_{\frac{1}{2}\nu} \left(\frac{b^2}{4a} \right) \\
 &\quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 56(1)}
 \end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x J_{\frac{1}{2}\nu}(ax^2) Y_\nu(bx) dx \\
= (4a)^{-1} \left[Y_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \tan\left(\frac{\nu\pi}{2}\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \sec\left(\frac{\nu\pi}{2}\right) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\
[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 109(9)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x J_{\frac{1}{2}\nu}(ax^2) K_\nu(bx) dx = \frac{\pi}{8a \cos\left(\frac{\nu\pi}{2}\right)} \left[\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - Y_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\
[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1] \\
\text{ET II 140(27)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x Y_{\frac{1}{2}\nu}(ax^2) J_\nu(bx) dx = -(2a)^{-1} \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \quad [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1] \\
\text{ET II 61(35)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x Y_{\frac{1}{2}\nu}(ax^2) K_\nu(bx) dx \\
= \frac{\pi}{4a \sin(\nu\pi)} \left[\cos\left(\frac{\nu\pi}{2}\right) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \sin\left(\frac{\nu\pi}{2}\right) J_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\
[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 141(28)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x K_{\frac{1}{2}\nu}(ax^2) J_\nu(bx) dx = \frac{\pi}{4a} \left[I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\
[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \\
\text{ET II 68(9)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x K_{\frac{1}{2}\nu}(ax^2) Y_\nu(bx) dx = \frac{\pi}{4a} \left[\operatorname{cosec}(\nu\pi) \mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \cot(\nu\pi) \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right. \\
\left. - \tan\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \frac{1}{\pi} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\
[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 112(25)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_0^\infty x K_{\frac{1}{2}\nu}(ax^2) K_\nu(bx) dx \\
= \frac{\pi}{8a} \left\{ \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \pi \operatorname{cosec}(\nu\pi) \left[\mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \right\} \\
[\operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 146(52)}
\end{aligned}$$

6.527

$$1. \quad \int_0^\infty x^2 J_{2\nu}(2ax) J_{\nu-\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu+\frac{1}{2}}(a^2) \quad [a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 355(33)}$$

$$2. \quad \int_0^\infty x^2 J_{2\nu}(2ax) J_{\nu+\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu-\frac{1}{2}}(a^2) \quad [a > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 355(35)}$$

$$3. \quad \int_0^\infty x^2 J_{2\nu}(2ax) Y_{\nu+\frac{1}{2}}(x^2) dx = -\frac{1}{2} a \mathbf{H}_{\nu-\frac{1}{2}}(a^2) \quad [a > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 355(36)}$$

$$6.528 \quad \int_0^\infty x K_{\frac{1}{4}\nu} \left(\frac{x^2}{4} \right) I_{\frac{1}{4}\nu} \left(\frac{x^2}{4} \right) J_\nu(bx) dx = K_{\frac{1}{4}\nu} \left(\frac{x^2}{4} \right) I_{\frac{1}{4}\nu} \left(\frac{b^2}{4} \right) \\ [b > 0, \quad \nu > -1] \quad \text{MO 183a}$$

6.529

$$1. \quad \int_0^\infty x J_\nu(2\sqrt{ax}) K_\nu(2\sqrt{ax}) J_\nu(bx) dx = \frac{1}{2} b^{-2} e^{-\frac{2a}{b}} \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 70(23)}$$

$$2. \quad \int_0^a x J_\lambda(2a) I_\lambda(2x) J_\mu \left(2\sqrt{a^2 - x^2} \right) I_\mu \left(2\sqrt{a^2 - x^2} \right) dx \\ = \frac{a^{2\lambda+2\mu+2}}{2\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\lambda+\mu+2)} \\ \times {}_1F_4 \left(\frac{\lambda+\mu+1}{2}; \lambda+1, \mu+1, \lambda+\mu+1, \frac{\lambda+\mu+3}{2}; -a^4 \right) \\ [\operatorname{Re} \lambda > -1, \quad \operatorname{Re} \mu > -1] \quad \text{ET II 376(31)}$$

6.53–6.54 Combinations of Bessel functions and rational functions

6.531

$$1.^{10} \quad \int_0^\infty \frac{Y_\nu(bx)}{x+a} dx \\ = -\pi J_\nu(ab) \cot(\pi\nu) \operatorname{cosec}(\pi\nu) - \pi J_{-\nu}(ab) \operatorname{cosec}^2(\pi\nu) + \frac{1}{\nu} \cot \frac{\pi\nu}{2} {}_1F_2 \left(1; \frac{2-\nu}{2}, \frac{2+\nu}{2}; -\frac{a^2b^2}{4} \right) \\ + \frac{ab}{\nu^2-1} {}_1F_2 \left(1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; -\frac{a^2b^2}{4} \right) \tan \frac{\pi\nu}{2} \\ [\operatorname{Re} \nu < 1, \quad \arg a \neq \pi, \quad b > 0] \quad \text{MC}$$

$$2. \quad \int_0^\infty \frac{Y_\nu(bx)}{x-a} dx = \pi \left\{ \cot(\nu\pi) [Y_\nu(ab) + \mathbf{E}_\nu(ab)] + \mathbf{J}_\nu(ab) + 2 [\cot(\nu\pi)]^2 [\mathbf{J}_\nu(ab) - J_\nu(ab)] \right\} \\ [b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < 1] \\ \text{ET II 98(9)}$$

$$3. \quad \int_0^\infty \frac{K_\nu(bx)}{x+a} dx = \frac{\pi^2}{2} [\operatorname{cosec}(\nu\pi)]^2 \left[I_\nu(ab) + I_{-\nu}(ab) - e^{-\frac{1}{2}i\nu\pi} \mathbf{J}_\nu(iab) - e^{\frac{1}{2}i\nu\pi} \mathbf{J}_{-\nu}(iab) \right] \\ [\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad |\operatorname{Re} \nu| < 1] \\ \text{ET II 128(5)}$$

6.532

$$1.^{11} \quad \int_0^\infty \frac{J_\nu(x)}{x^2+a^2} dx = \frac{i}{a} \left[S_{0,\nu}(ia) - e^{-i\nu\pi/2} K_\nu(a) \right] = \frac{1}{a} \left[i s_{0,\nu}(ia) + \frac{\pi}{2} \sec \left(\frac{\nu\pi}{2} \right) I_\nu(a) \right] \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1]$$

$$2. \quad \int_0^\infty \frac{Y_\nu(x)}{x^2 + a^2} dx = \frac{1}{\cos \frac{\nu\pi}{2}} \left[-\frac{\pi}{2a} \tan\left(\frac{\nu\pi}{2}\right) I_\nu(ab) - \frac{1}{a} K_\nu(ab) + \frac{b \sin\left(\frac{\nu\pi}{2}\right)}{1 - \nu^2} {}_1F_2\left(1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{a^2 b^2}{4}\right) \right] \\ [b > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 99(13)}$$

$$3. \quad \int_0^\infty \frac{Y_\nu(bx)}{x^2 - a^2} dx = \frac{\pi}{2a} \left\{ J_\nu(ab) + \tan\left(\frac{\nu\pi}{2}\right) \left\{ \tan\left(\frac{\nu\pi}{2}\right) [\mathbf{J}_\nu(ab) - J_\nu(ab)] - \mathbf{E}_\nu(ab) - Y_\nu(ab) \right\} \right\} \\ [b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 101(21)}$$

$$4. \quad \int_0^\infty \frac{x J_0(ax)}{x^2 + k^2} dx = K_0(ak) \quad [a > 0, \quad \operatorname{Re} k > 0] \quad \text{WA 466(5)}$$

$$5. \quad \int_0^\infty \frac{Y_0(ax)}{x^2 + k^2} dx = -\frac{K_0(ak)}{k} \quad [a > 0, \quad \operatorname{Re} k > 0] \quad \text{WA 466(6)}$$

$$6. \quad \int_0^\infty \frac{J_0(ax)}{x^2 + k^2} dx = \frac{\pi}{2k} [I_0(ak) - \mathbf{L}_0(ak)] \quad [a > 0, \quad \operatorname{Re} k > 0] \quad \text{WA 467(7)}$$

6.533

$$1. \quad \int_0^z J_p(x) J_q(z-x) \frac{dx}{x} = \frac{J_{p+q}(z)}{p} \quad [\operatorname{Re} p > 0, \quad \operatorname{Re} q > -1] \quad \text{WA 415(3)}$$

$$2. \quad \int_0^z \frac{J_p(x)}{x} \frac{J_q(z-x)}{z-x} dx = \left(\frac{1}{p} + \frac{1}{q} \right) \frac{J_{p+q}(z)}{z} \quad [\operatorname{Re} p > 0, \quad \operatorname{Re} q > 0] \quad \text{WA 415(5)}$$

$$3.^{11} \quad \int_0^\infty [J_0(ax) - 1] J_1(bx) \frac{dx}{x^2} = -\frac{b}{4} \left[1 + 2 \ln \frac{a}{b} \right] \quad [0 < b < a] \\ = -\frac{a^2}{4b} \quad [0 < a < b] \quad \text{ET II 21(28)a}$$

$$3b \quad \int_0^\infty [J_0(ax) - 1] J_1(bx) \frac{dx}{x} = \begin{cases} \frac{b}{2a} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{b^2}{a^2}\right) - 1 & [0 < b < a] \\ \frac{2}{\pi} \mathbf{E}\left(\frac{b^2}{a^2}\right) - 1 & [0 < a < b] \end{cases}$$

$$4. \quad \int_0^\infty [1 - J_0(ax)] J_0(bx) \frac{dx}{x} = 0 \quad [0 < a < b] \\ = \ln \frac{a}{b} \quad [0 < b < a] \quad \text{ET II 14(16)}$$

$$\mathbf{6.534} \quad \int_0^\infty \frac{x^3 J_0(x)}{x^4 - a^4} dx = \frac{1}{2} K_0(a) - \frac{1}{4} \pi Y_0(a) \quad [a > 0] \quad \text{ET II 340(5)}$$

$$\mathbf{6.535} \quad \int_0^\infty \frac{x}{x^2 + a^2} [J_\nu(x)]^2 dx = I_\nu(a) K_\nu(a) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 342(26)}$$

$$\mathbf{6.536} \quad \int_0^\infty \frac{x^3 J_0(bx)}{x^4 + a^4} dx = \ker(ab) \quad [b > 0, \quad |\arg a| < \frac{1}{4}\pi] \quad \text{ET II 8(9), MO 46a}$$

$$6.537 \quad \int_0^\infty \frac{x^2 J_0(bx)}{x^4 + a^4} dx = -\frac{1}{a^2} \operatorname{kei}(ab) \quad \left[b > 0, \quad |\arg a| < \frac{\pi}{4} \right] \quad \text{MO 46a}$$

6.538

$$1. \quad \int_0^\infty J_1(ax) J_1(bx) \frac{dx}{x^2} = \frac{a+b}{\pi} \left[E \left(\frac{2i\sqrt{ab}}{|b-a|} \right) - K \left(\frac{2i\sqrt{ab}}{|b-a|} \right) \right] \\ [a > 0, \quad b > 0] \quad \text{ET II 21(30)}$$

$$2.^8 \quad \int_0^\infty x^{-1} J_{\nu+2n+1}(x) J_{\nu+2m+1}(x) dx = 0 \quad [m \neq n \text{ with } m, n \text{ integers, } \nu > -1] \\ = (4n + 2\nu + 2)^{-1} \quad [m = n, \quad \nu > -1]$$

EH II 64

6.539

$$1. \quad \int_a^b \frac{dx}{x [J_\nu(x)]^2} = \frac{\pi}{2} \left[\frac{Y_\nu(b)}{J_\nu(b)} - \frac{Y_\nu(a)}{J_\nu(a)} \right] \quad [J_\nu(x) \neq 0 \text{ for } x \in [a, b]] \quad \text{ET II 338(41)}$$

$$2. \quad \int_a^b \frac{dx}{x [Y_\nu(x)]^2} = \frac{\pi}{2} \left[\frac{J_\nu(a)}{Y_\nu(a)} - \frac{J_\nu(b)}{Y_\nu(b)} \right] \quad [Y_\nu(x) \neq 0 \text{ for } x \in [a, b]] \\ \text{ET II 339(49)}$$

$$3. \quad \int_a^b \frac{dx}{x J_\nu(x) Y_\nu(x)} = \frac{\pi}{2} \ln \left[\frac{J_\nu(a) Y_\nu(b)}{J_\nu(b) Y_\nu(a)} \right] \quad \text{ET II 339(50)}$$

6.541

$$1. \quad \int_0^\infty x J_\nu(ax) J_\nu(bx) \frac{dx}{x^2 + c^2} = I_\nu(bc) K_\nu(ac) \quad [0 < b < a, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > -1] \\ = I_\nu(ac) K_\nu(bc) \quad [0 < a < b, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 49(10)}$$

$$2.^8 \quad \int_0^\infty x^{1-2n} J_\nu(ax) J_\nu(bx) \frac{dx}{x^2 + c^2} \\ = \left(-\frac{1}{c^2} \right)^n \left[I_\nu(bc) K_\nu(ac) - \frac{1}{2} \left(\frac{b}{a} \right)^\nu \frac{\pi}{\sin(\pi\nu)} \sum_{p=0}^{n-1} \frac{(a^2 c^2/4)^p}{p! \Gamma(1-\nu+p)} \sum_{k=0}^{n-1-p} \frac{(b^2 c^2/4)^k}{k! \Gamma(1-\nu+k)} \right] \\ [0 < b < a] \\ = \left(-\frac{1}{c^2} \right)^n \left[I_\nu(bc) K_\nu(ac) - \frac{1}{2\nu} \left(\frac{b}{a} \right)^\nu \sum_{p=0}^{n-1} \frac{(a^2 c^2/4)^p}{p! (1-\nu)_p} \sum_{k=0}^{n-1-p} \frac{(b^2 c^2/4)^k}{k! (1+\nu)_k} \right] \\ [n = 1, 2, \dots, \quad \operatorname{Re} \nu > n-1, \quad \operatorname{Re} c > 0, \quad 0 < b < a]$$

- 3.⁸
$$\int_0^\infty \frac{x^{\alpha-1}}{(x^2+z^2)^\rho} J_\mu(cx) J_\nu(cx) dx = \frac{1}{2} \left(\frac{c}{2}\right)^{2\rho-\alpha} \times \Gamma \left[\begin{matrix} (\mu+\nu+\alpha)/2 - \rho, 1+2\rho-\alpha \\ (\mu-\nu-\alpha)/2 + \rho + 1, (\mu+\nu-\alpha)/2 + \rho + 1, (\nu-\mu-\alpha)/2 + \rho + 1 \end{matrix} \right] \times {}_3F_4 \left(\begin{matrix} \frac{1-\alpha}{2} + \rho, 1 - \frac{\alpha}{2} + \rho, \rho; \rho + 1 - \frac{\mu+\nu+\alpha}{2}, \rho + 1 + \frac{\mu-\nu-\alpha}{2}, \rho + 1 + \frac{\mu+\nu-\alpha}{2}, \rho + 1 + \frac{\nu-\mu-\alpha}{2}; c^2 z^2 \end{matrix} \right) + \frac{z^{\alpha-2\rho}}{2} \left(\frac{cz}{2}\right)^{\mu+\nu},$$
- $$\Gamma \left[\begin{matrix} \rho - (\alpha + \mu + \nu)/2, (\alpha + \mu + \nu)/2 \\ \rho, \mu + 1, \nu + 1 \end{matrix} \right] {}_3F_4 \left(\begin{matrix} \frac{1+\mu+\nu}{2}, 1 + \frac{\mu+\nu}{2}, \frac{\alpha + \mu + \nu}{2}; 1 - \rho + \frac{\alpha + \mu + \nu}{2}, \mu + 1, \nu + 1, \mu + \nu + 1; c^2 z^2 \end{matrix} \right)$$
- $$\left[\Gamma \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right] = \frac{\Gamma(a_1) \dots \Gamma(a_p)}{\Gamma(b_1) \dots \Gamma(b_q)}, \quad c > 0, \quad \operatorname{Re} z > 0, \quad \operatorname{Re}(\alpha + \mu + \nu) > 0; \quad \operatorname{Re}(\alpha - 2\rho) > 1 \right]$$
- 6.542
$$\int_0^\infty \frac{J_\nu(ax) Y_\nu(bx) - J_\nu(bx) Y_\nu(ax)}{x \left\{ [J_\nu(bx)]^2 + [Y_\nu(bx)]^2 \right\}} dx = -\frac{\pi}{2} \left(\frac{b}{a}\right)^\nu \quad [0 < b < a] \quad \text{ET II 352(16)}$$
- 6.543
$$\int_0^\infty J_\mu(bx) \left\{ \cos \left[\frac{1}{2}(\nu - \mu)\pi \right] J_\nu(ax) - \sin \left[\frac{1}{2}(\nu - \mu)\pi \right] Y_\nu(ax) \right\} \frac{x dx}{x^2 + r^2} = I_\mu(br) K_\nu(ar)$$

$$[\operatorname{Re} r > 0, \quad a \geq b > 0, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| - 2]$$
- 6.544
1.
$$\int_0^\infty J_\nu \left(\frac{a}{x}\right) Y_\nu \left(\frac{x}{b}\right) \frac{dx}{x^2} = -\frac{1}{a} \left[\frac{2}{\pi} K_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - Y_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \right]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{EI II 357(47)}$$
 2.
$$\int_0^\infty J_\nu \left(\frac{a}{x}\right) J_\nu \left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{1}{a} J_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}}\right)$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 57(10)}$$
 3.
$$\int_0^\infty J_\nu \left(\frac{a}{x}\right) K_\nu \left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{1}{a} e^{\frac{1}{2}i\nu\pi} K_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}} e^{\frac{1}{4}i\pi}\right) + \frac{1}{a} e^{-\frac{1}{2}i\nu\pi} K_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}} e^{-\frac{1}{4}i\pi}\right)$$

$$[\operatorname{Re} b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 142(32)}$$
 4.
$$\int_0^\infty Y_\nu \left(\frac{a}{x}\right) J_\nu \left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{2}{a\pi} \left[K_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}}\right) + \frac{\pi}{2} Y_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \right]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 62(38)}$$
 5.
$$\int_0^\infty Y_\nu \left(\frac{a}{x}\right) K_\nu \left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{4}{a} \left[e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}} e^{\frac{1}{4}i\pi}\right) + e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu} \left(\frac{2\sqrt{a}}{\sqrt{b}} e^{-\frac{1}{4}i\pi}\right) \right]$$

$$[\operatorname{Re} b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 143(38)}$$

$$6. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{i}{a} \left[e^{\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{\frac{1}{4}\pi i} \frac{2\sqrt{a}}{\sqrt{b}}\right) - e^{-\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{-\frac{1}{4}\pi i} \frac{2\sqrt{a}}{\sqrt{b}}\right) \right] \\ [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2}] \quad \text{ET II 70(19)}$$

$$7. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{2}{a} \left[\sin\left(\frac{3}{2}\pi\nu\right) \operatorname{kei}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - \cos\left(\frac{3}{2}\pi\nu\right) \operatorname{ker}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \right] \\ [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2}] \quad \text{ET II 113(29)}$$

$$8. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{\pi}{a} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 146(55)}$$

6.55 Combinations of Bessel functions and algebraic functions

6.551¹⁰

$$1. \quad \int_0^1 x^{1/2} J_\nu(xy) dx = \sqrt{2} y^{-3/2} \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}\nu\right)} \\ + y^{-1/2} \left[\left(\nu - \frac{1}{2}\right) J_\nu(y) S_{-1/2, \nu-1}(y) - J_{\nu-1}(y) S_{1/2, \nu}(y) \right] \\ [y > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET II 21(1)}$$

$$2. \quad \int_1^\infty x^{1/2} J_\nu(xy) dx = y^{-1/2} [J_{\nu-1}(y) S_{1/2, \nu}(y) + \left(\frac{1}{2} - \nu\right) J_\nu(y) S_{-1/2, \nu-1}(y)] \\ [y > 0] \quad \text{ET II 22(2)}$$

6.552

$$1. \quad \int_0^\infty J_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = I_{\nu/2}\left(\frac{1}{2}ay\right) K_{\nu/2}\left(\frac{1}{2}ay\right) \quad [\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 23(11), WA 477(3), MO 44}$$

$$2. \quad \int_0^\infty Y_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = -\frac{1}{\pi} \sec\left(\frac{1}{2}\nu\pi\right) K_{\nu/2}\left(\frac{1}{2}ay\right) [K_{\nu/2}\left(\frac{1}{2}ay\right) + \pi \sin\left(\frac{1}{2}\nu\pi\right) I_{\nu/2}\left(\frac{1}{2}ay\right)] \\ [y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 100(18)}$$

$$3. \quad \int_0^\infty K_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = \frac{\pi^2}{8} \sec\left(\frac{1}{2}\nu\pi\right) \left\{ [J_{\nu/2}\left(\frac{1}{2}ay\right)]^2 + [Y_{\nu/2}\left(\frac{1}{2}ay\right)]^2 \right\} \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 128(6)}$$

$$4. \quad \int_0^1 J_\nu(xy) \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2} [J_{\nu/2}\left(\frac{1}{2}y\right)]^2 \quad [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 24(22)a}$$

$$5. \quad \int_0^1 Y_0(xy) \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2} J_0\left(\frac{1}{2}y\right) Y_0\left(\frac{1}{2}y\right) \quad [y > 0] \quad \text{ET II 102(26)a}$$

$$6. \quad \int_1^\infty J_\nu(xy) \frac{dx}{(x^2 - 1)^{1/2}} = -\frac{\pi}{2} J_{\nu/2}\left(\frac{1}{2}y\right) Y_{\nu/2}\left(\frac{1}{2}y\right) \quad [y > 0] \quad \text{ET II 24(23)a}$$

$$7. \quad \int_1^\infty Y_\nu(xy) \frac{dx}{(x^2-1)^{1/2}} = \frac{\pi}{4} \left\{ [J_{\nu/2}(\tfrac{1}{2}y)]^2 - [Y_{\nu/2}(\tfrac{1}{2}y)]^2 \right\} \quad [y > 0] \quad \text{ET II 102(27)}$$

$$6.553 \quad \int_0^\infty x^{-1/2} I_\nu(x) K_\nu(x) K_\mu(2x) dx = \frac{\Gamma(\frac{1}{4} + \frac{1}{2}\mu) \Gamma(\frac{1}{4} - \frac{1}{2}\mu) \Gamma(\frac{1}{4} + \nu + \frac{1}{2}\mu) \Gamma(\frac{1}{4} + \nu - \frac{1}{2}\mu)}{4 \Gamma(\frac{3}{4} + \nu + \frac{1}{2}\mu) \Gamma(\frac{3}{4} + \nu - \frac{1}{2}\mu)} \quad [|\operatorname{Re} \mu| < \tfrac{1}{2}, \quad 2 \operatorname{Re} \nu > |\operatorname{Re} \mu| - \tfrac{1}{2}] \quad \text{ET II 372(2)}$$

6.554

$$1. \quad \int_0^\infty x J_0(xy) \frac{dx}{(a^2 + x^2)^{1/2}} = y^{-1} e^{-ay} \quad [y > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 7(4)}$$

$$2. \quad \int_0^1 x J_0(xy) \frac{dx}{(1-x^2)^{1/2}} = y^{-1} \sin y \quad [y > 0] \quad \text{ET II 7(5)a}$$

$$3. \quad \int_1^\infty x J_0(xy) \frac{dx}{(x^2-1)^{1/2}} = y^{-1} \cos y \quad [y > 0] \quad \text{ET II 7(6)a}$$

$$4. \quad \int_0^\infty x J_0(xy) \frac{dx}{(x^2 + a^2)^{3/2}} = a^{-1} e^{-ay} \quad [y > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 7(7)a}$$

$$5.^{11} \quad \int_0^\infty \frac{x^{\nu+1} J_\nu(ax)}{(x^4 + 4k^4)^{\nu+1/2}} dx = \frac{(\frac{1}{2}a)^\nu \sqrt{\pi}}{(2k)^{2\nu} \Gamma(\nu + \frac{1}{2})} J_\nu(ak) K_\nu(ak) \quad [a > 0, \quad |\arg k| > \frac{\pi}{4}, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{WA 473(1)}$$

$$6.555 \quad \int_0^\infty x^{1/2} J_{2\nu-1}(ax^{1/2}) Y_\nu(xy) dx = -\frac{a}{2y^2} \mathbf{H}_{\nu-1}\left(\frac{a^2}{4y}\right) \quad [a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 111(17)}$$

$$6.556 \quad \int_0^\infty J_\nu[a(x^2+1)^{1/2}] \frac{dx}{\sqrt{x^2+1}} = -\frac{\pi}{2} J_{\nu/2}\left(\frac{a}{2}\right) Y_{\nu/2}\left(\frac{a}{2}\right) \quad [\operatorname{Re} \nu > -1, \quad a > 0] \quad \text{MO 46}$$

6.56–6.58 Combinations of Bessel functions and powers**6.561**

$$1. \quad \int_0^1 x^\nu J_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) [J_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) J_{\nu-1}(a)] \quad [\operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 333(2)a}$$

$$2. \quad \int_0^1 x^\nu Y_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) [Y_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) Y_{\nu-1}(a)] \quad [\operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 338(43)a}$$

$$3. \quad \int_0^1 x^\nu I_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) [I_\nu(a) \mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a) I_{\nu-1}(a)] \quad [\operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 364(2)a}$$

4.
$$\int_0^1 x^\nu K_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) [K_\nu(a) \mathbf{L}_{\nu-1}(a) + \mathbf{L}_\nu(a) K_{\nu-1}(a)]$$

$$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 367(21)a}$$
5.
$$\int_0^1 x^{\nu+1} J_\nu(ax) dx = a^{-1} J_{\nu+1}(a)$$

$$[\operatorname{Re} \nu > -1] \quad \text{ET II 333(3)a}$$
6.
$$\int_0^1 x^{\nu+1} Y_\nu(ax) dx = a^{-1} Y_{\nu+1}(a) + 2^{\nu+1} a^{-\nu-2} \pi^{-1} \Gamma(\nu + 1)$$

$$[\operatorname{Re} \nu > -1] \quad \text{ET II 339(44)a}$$
7.
$$\int_0^1 x^{\nu+1} I_\nu(ax) dx = a^{-1} I_{\nu+1}(a)$$

$$[\operatorname{Re} \nu > -1] \quad \text{ET II 365(3)a}$$
8.
$$\int_0^1 x^{\nu+1} K_\nu(ax) dx = 2^\nu a^{-\nu-2} \Gamma(\nu + 1) - a^{-1} K_{\nu+1}(a)$$

$$[\operatorname{Re} \nu > -1] \quad \text{ET II 367(22)a}$$
9.
$$\int_0^1 x^{1-\nu} J_\nu(ax) dx = \frac{a^{\nu-2}}{2^{\nu-1} \Gamma(\nu)} - a^{-1} J_{\nu-1}(a)$$

$$\text{ET II 333(4)a}$$
10.
$$\int_0^1 x^{1-\nu} Y_\nu(ax) dx = \frac{a^{\nu-2} \cot(\nu\pi)}{2^{\nu-1} \Gamma(\nu)} - a^{-1} Y_{\nu-1}(a)$$

$$[\operatorname{Re} \nu < 1] \quad \text{ET II 339(45)a}$$
11.
$$\int_0^1 x^{1-\nu} I_\nu(ax) dx = a^{-1} I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1} \Gamma(\nu)}$$

$$\text{ET II 365(4)a}$$
12.
$$\int_0^1 x^{1-\nu} K_\nu(ax) dx = 2^{-\nu} a^{\nu-2} \Gamma(1 - \nu) - a^{-1} K_{\nu-1}(a)$$

$$[\operatorname{Re} \nu < 1] \quad \text{ET II 367(23)a}$$
- 13.⁷
$$\int_0^1 x^\mu J_\nu(ax) dx = \frac{2^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{a^{\mu+1} \Gamma\left(\frac{\nu-\mu+1}{2}\right)} + a^{-\mu} \{(\mu + \nu - 1) J_\nu(a) S_{\mu-1, \nu-1}(a) - J_{\nu-1}(a) S_{\mu, \nu}(a)\}$$

$$[a > 0, \quad \operatorname{Re}(\mu + \nu) > -1] \quad \text{ET II 22(8)a}$$
14.
$$\int_0^\infty x^\mu J_\nu(ax) dx = 2^\mu a^{-\mu-1} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu\right)}$$

$$[-\operatorname{Re} \nu - 1 < \operatorname{Re} \mu < \frac{1}{2}, \quad a > 0]$$

$$\text{EH II 49(19)}$$
15.
$$\int_0^\infty x^\mu Y_\nu(ax) dx = 2^\mu \cot\left[\frac{1}{2}(\nu + 1 - \mu)\pi\right] a^{-\mu-1} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu\right)}$$

$$[|\operatorname{Re} \nu| - 1 < \mu < \frac{1}{2}, \quad a > 0]$$

$$\text{ET II 97(3)a}$$
16.
$$\int_0^\infty x^\mu K_\nu(ax) dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1 + \mu + \nu}{2}\right) \Gamma\left(\frac{1 + \mu - \nu}{2}\right)$$

$$[\operatorname{Re}(\mu + 1 \pm \nu) > 0, \quad \operatorname{Re} a > 0]$$

$$\text{EH II 51(27)}$$

$$17. \quad \int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}} dx = \frac{\Gamma\left(\frac{1}{2}q + \frac{1}{2}\right)}{2^{\nu-q} a^{q-\nu+1} \Gamma\left(\nu - \frac{1}{2}q + \frac{1}{2}\right)} \quad \left[-1 < \operatorname{Re} q < \operatorname{Re} \nu - \frac{1}{2}\right]$$

WA 428(1), KU 144(5)

$$18. \quad \int_0^\infty \frac{Y_\nu(x)}{x^{\nu-\mu}} dx = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \nu\right) \sin\left(\frac{1}{2}\mu - \nu\right) \pi}{2^{\nu-\mu} \pi}$$

[$|\operatorname{Re} \nu| < \operatorname{Re}(1 + \mu - \nu) < \frac{3}{2}$]
WA 430(5)

$$19. \quad \int_0^1 x^{2m+n+1/2} K_{n+1/2}(\alpha x) dx = \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} \frac{\gamma(2m+n-k+1, \alpha)}{\alpha^{2m+n+3/2} 2^k}$$

STR

6.562

$$1. \quad \int_0^\infty x^\mu Y_\nu(bx) \frac{dx}{x+a} = (2a)^\mu \pi^{-1} \left\{ \sin\left[\frac{1}{2}\pi(\mu - \nu)\right] \Gamma\left[\frac{1}{2}(\mu + \nu + 1)\right] \Gamma\left[\frac{1}{2}(1 + \mu - \nu)\right] S_{-\mu, \nu}(ab) \right. \\ \left. - 2 \cos\left[\frac{1}{2}\pi(\mu - \nu)\right] \Gamma\left(1 + \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(1 + \frac{1}{2}\mu - \frac{1}{2}\nu\right) S_{-\mu-1, \nu}(ab) \right\}$$

[$b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\mu \pm \nu) > -1, \quad \operatorname{Re} \mu < \frac{3}{2}$] ET II 98(8)

$$2. \quad \int_0^\infty \frac{x^\nu J_\nu(ax)}{x+k} dx = \frac{\pi k^\nu}{2 \cos \nu \pi} [\mathbf{H}_{-\nu}(ak) - Y_{-\nu}(ak)] \quad \left[-\frac{1}{2} < \operatorname{Re} \nu < \frac{3}{2}, \quad a > 0, \quad |\arg k| < \pi\right]$$

WA 479(7)

$$3. \quad \int_0^\infty x^\mu K_\nu(bx) \frac{dx}{x+a} \\ = 2^{\mu-2} \Gamma\left[\frac{1}{2}(\mu + \nu)\right] \Gamma\left[\frac{1}{2}(\mu - \nu)\right] b^{-\mu} {}_1F_2\left(1; 1 - \frac{\mu + \nu}{2}, 1 - \frac{\mu - \nu}{2}; \frac{a^2 b^2}{4}\right) \\ - 2^{\mu-3} \Gamma\left[\frac{1}{2}(\mu - \nu - 1)\right] \Gamma\left[\frac{1}{2}(\mu + \nu - 1)\right] ab^{1-\mu} {}_1F_2\left(1; \frac{3 - \mu - \nu}{2}, \frac{3 - \mu + \nu}{2}; \frac{a^2 b^2}{4}\right) \\ - \pi a^\mu \operatorname{cosec}[\pi(\mu - \nu)] \{K_\nu(ab) + \pi \cos(\mu\pi) \operatorname{cosec}[\pi(\nu + \mu)] I_\nu(ab)\}$$

[$\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| - 1$] ET II 127(4)

$$6.563 \quad \int_0^\infty x^{\varrho-1} J_\nu(bx) \frac{dx}{(x+a)^{1+\mu}} = \frac{\pi a^{\varrho-\mu-1}}{\sin[(\varrho + \nu - \mu)\pi] \Gamma(\mu + 1)} \\ \times \left\{ \sum_{m=0}^\infty \frac{(-1)^m \left(\frac{1}{2}ab\right)^{\nu+2m} \Gamma(\varrho + \nu + 2m)}{m! \Gamma(\nu + m + 1) \Gamma(\varrho + \nu - \mu + 2m)} \right. \\ \left. - \sum_{m=0}^\infty \frac{\left(\frac{1}{2}ab\right)^{\mu+1-\varrho+m} \Gamma(\mu + m + 1) \sin\left[\frac{1}{2}(\varrho + \nu - \mu - m)\pi\right]}{m! \Gamma\left[\frac{1}{2}(\mu + \nu - \varrho + m + 3)\right] \Gamma\left[\frac{1}{2}(\mu - \nu - \varrho + m + 3)\right]} \right\}$$

[$b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\varrho + \nu) > 0, \quad \operatorname{Re}(\varrho - \mu) < \frac{5}{2}$] ET II 23(10), WA 479

6.564

$$1. \quad \int_0^\infty x^{\nu+1} J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} = \sqrt{\frac{2}{\pi b}} a^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}(ab) \quad [\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}]$$

ET II 23(15)

$$2. \quad \int_0^\infty x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} = \sqrt{\frac{\pi}{2b}} a^{\frac{1}{2}-\nu} \left[I_{\nu-\frac{1}{2}}(ab) - \mathbf{L}_{\nu-\frac{1}{2}}(ab) \right] \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\ \text{ET II 23(16)}$$

6.565

$$1. \quad \int_0^\infty x^{-\nu} (x^2 + a^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = 2^\nu a^{-2\nu} b^\nu \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} I_\nu\left(\frac{ab}{2}\right) K_\nu\left(\frac{ab}{2}\right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\ \text{WA 477(4), ET II 23(17)}$$

$$2. \quad \int_0^\infty x^{\nu+1} (x^2 + a^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = \frac{\sqrt{\pi} b^{\nu-1}}{2^\nu e^{ab} \Gamma(\nu + \frac{1}{2})} \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\ \text{ET II 24(18)}$$

$$3. \quad \int_0^\infty x^{\nu+1} (x^2 + a^2)^{-\nu-\frac{3}{2}} J_\nu(bx) dx = \frac{b^\nu \sqrt{\pi}}{2^{\nu+1} a e^{ab} \Gamma(\nu + \frac{3}{2})} \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 24(19)}$$

$$4. \quad \int_0^\infty \frac{J_\nu(bx) x^{\nu+1}}{(x^2 + a^2)^{\mu+1}} dx = \frac{a^{\nu-\mu} b^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ab) \\ [-1 < \operatorname{Re} \nu < \operatorname{Re}(2\mu + \tfrac{3}{2}), \quad a > 0, \quad b > 0] \quad \text{MO 43}$$

$$5. \quad \int_0^\infty x^{\nu+1} (x^2 + a^2)^\mu Y_\nu(bx) dx = 2^{\nu-1} \pi^{-1} a^{2\mu+2} (1+\mu)^{-1} \Gamma(\nu) b^{-\nu} \\ \times {}_1F_2\left(1; 1-\nu, 2+\mu; \frac{a^2 b^2}{4}\right) - 2^\mu a^{\mu+\nu+1} [\sin(\nu\pi)]^{-1} \\ \times \Gamma(\mu+1) b^{-1-\mu} [I_{\mu+\nu+1}(ab) - 2 \cos(\mu\pi) K_{\mu+\nu+1}(ab)] \\ [b > 0, \quad \operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < -2 \operatorname{Re} \mu] \quad \text{ET II 100(19)}$$

$$6.^{10} \quad \int_0^\infty x^{1-\nu} (x^2 + a^2)^\mu Y_\nu(bx) dx = \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu} \pi}{\Gamma(-\mu)} I_{-1-\mu+\nu}(ab) \cot[\pi(\mu-\nu)] \operatorname{cosec}(\pi\mu) \\ - \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu} \pi}{\Gamma(-\mu)} I_{1+\mu-\nu}(ab) \operatorname{cosec}[\pi(\mu-\nu)] \operatorname{cosec}(\pi\nu) \\ + \frac{2^{-1-\nu} a^{2+2\mu} b^\nu}{(1+\mu)\pi} \cos(\pi\nu) \Gamma(-\mu) {}_1F_2\left(1; 2+\mu, 1+\nu; \frac{a^2 b^2}{4}\right) \\ [\operatorname{Re} \nu < 1, \quad \operatorname{Re}(\nu-2\mu) > -3, \quad \arg a^2 \neq \pi, \quad b > 0] \quad \text{MC}$$

$$7. \quad \int_0^\infty x^{1+\nu} (x^2 + a^2)^\mu K_\nu(bx) dx = 2^\nu \Gamma(\nu+1) a^{\nu+\mu+1} b^{-1-\mu} S_{\mu-\nu, \mu+\nu+1}(ab) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 128(8)}$$

$$\begin{aligned}
8.11 \quad \int_0^\infty \frac{x^{\varrho-1} J_\nu(ax)}{(x^2 + k^2)^{\mu+1}} dx = & \frac{a^\nu k^{\varrho+\nu-2\mu-2} \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\nu\right) \Gamma\left(\mu + 1 - \frac{1}{2}\varrho - \frac{1}{2}\nu\right)}{2^{\nu+1} \Gamma(\mu+1) \Gamma(\nu+1)} \\
& \times {}_1F_2\left(\frac{\varrho+\nu}{2}; \frac{\varrho+\nu}{2} - \mu, \nu+1; \frac{a^2 k^2}{4}\right) \\
& + \frac{a^{2\mu+2-\varrho} \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\varrho - \mu - 1\right)}{2^{2\mu+3-\varrho} \Gamma\left(\mu+2 + \frac{1}{2}\nu - \frac{1}{2}\varrho\right)} \\
& \times {}_1F_2\left(\mu+1; \mu+2 + \frac{\nu-\varrho}{2}, \mu+2 - \frac{\nu+\varrho}{2}; \frac{a^2 k^2}{4}\right) \\
& [a > 0, \quad -\operatorname{Re} \nu < \operatorname{Re} \varrho < 2\operatorname{Re} \mu + \frac{7}{2}, \quad \operatorname{Re} k > 0] \quad \text{WA 477(1)}
\end{aligned}$$

6.566

$$\begin{aligned}
1. \quad \int_0^\infty x^\mu Y_\nu(bx) \frac{dx}{x^2 + a^2} = & 2^{\mu-2} \pi^{-1} b^{1-\mu} \\
& \times \cos\left[\frac{\pi}{2}(\mu - \nu + 1)\right] \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2}\right) \\
& \times {}_1F_2\left(1; 2 - \frac{\mu+1+\nu}{2}, 2 - \frac{\mu+1-\nu}{2}; \frac{a^2 b^2}{4}\right) \\
& - \frac{1}{2} \pi a^{\mu-1} \operatorname{cosec}\left[\frac{\pi}{2}(\mu + \nu + 1)\right] \cot\left[\frac{\pi}{2}(\mu - \nu + 1)\right] I_\nu(ab) \\
& - a^{\mu-1} \operatorname{cosec}\left[\frac{\pi}{2}(\mu - \nu + 1)\right] K_\nu(ab) \\
& [b > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| - 1 < \operatorname{Re} \mu < \frac{5}{2}] \quad \text{ET II 100(17)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{\nu+1} J_\nu(ax) \frac{dx}{x^2 + b^2} = & b^\nu K_\nu(ab) \\
& [a > 0, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{3}{2}] \\
& \text{EH II 96(58)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^\nu K_\nu(ax) \frac{dx}{x^2 + b^2} = & \frac{\pi^2 b^{\nu-1}}{4 \cos \nu \pi} [\mathbf{H}_{-\nu}(ab) - Y_{-\nu}(ab)] \\
& [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\
& \text{WA 468(9)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{-\nu} K_\nu(ax) \frac{dx}{x^2 + b^2} = & \frac{\pi^2}{4b^{\nu+1} \cos \nu \pi} [\mathbf{H}_\nu(ab) - Y_\nu(ab)] \\
& [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu < \frac{1}{2}] \\
& \text{WA 468(10)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{-\nu} J_\nu(ax) \frac{dx}{x^2 + b^2} = & \frac{\pi}{2b^{\nu+1}} [I_\nu(ab) - \mathbf{L}_\nu(ab)] \\
& [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{5}{2}] \\
& \text{WA 468(11)}
\end{aligned}$$

6.567

$$\begin{aligned}
1. \quad \int_0^1 x^{\nu+1} (1-x^2)^\mu J_\nu(bx) dx = & 2^\mu \Gamma(\mu+1) b^{-(\mu+1)} J_{\nu+\mu+1}(b) \\
& [b > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1] \\
& \text{ET II 26(33)a}
\end{aligned}$$

2.
$$\int_0^1 x^{\nu+1} (1-x^2)^\mu Y_\nu(bx) dx$$

$$= b^{-(\mu+1)} [2^\mu \Gamma(\mu+1) Y_{\mu+\nu+1}(b) + 2^{\nu+1} \pi^{-1} \Gamma(\nu+1) S_{\mu-\nu, \mu+\nu+1}(b)]$$

$$[b > 0, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 103(35)a}$$
3.
$$\int_0^1 x^{1-\nu} (1-x^2)^\mu J_\nu(bx) dx = \frac{2^{1-\nu} S_{\nu+\mu, \mu-\nu+1}(b)}{b^{\mu+1} \Gamma(\nu)} \quad [b > 0, \quad \operatorname{Re} \mu > -1] \quad \text{ET II 25(31)a}$$
4.
$$\int_0^1 x^{1-\nu} (1-x^2)^\mu Y_\nu(bx) dx = b^{-(\mu+1)} \left[2^{1-\nu} \pi^{-1} \cos(\nu\pi) \Gamma(1-\nu) \right.$$

$$\left. \times S_{\mu+\nu, \mu-\nu+1}(b) - 2^\mu \operatorname{cosec}(\nu\pi) \Gamma(\mu+1) J_{\mu-\nu+1}(b) \right]$$

$$[b > 0, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 104(37)a}$$
5.
$$\int_0^1 x^{1-\nu} (1-x^2)^\mu K_\nu(bx) dx = 2^{-\nu-2} b^\nu (\mu+1)^{-1} \Gamma(-\nu) {}_1F_2 \left(1; \nu+1, \mu+2; \frac{b^2}{4} \right)$$

$$+ \pi 2^{\mu-1} b^{-(\mu+1)} \operatorname{cosec}(\nu\pi) \Gamma(\mu+1) I_{\mu-\nu+1}(b)$$

$$[\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 129(12)a}$$
6.
$$\int_0^1 x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \mathbf{H}_{\nu-\frac{1}{2}}(b) \quad [b > 0] \quad \text{ET II 24(24)a}$$
7.
$$\int_0^1 x^{1+\nu} Y_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \operatorname{cosec}(\nu\pi) \left[\cos(\nu\pi) J_{\nu+\frac{1}{2}}(b) - \mathbf{H}_{-\nu-\frac{1}{2}}(b) \right]$$

$$[b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 102(28)a}$$
8.
$$\int_0^1 x^{1-\nu} Y_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \left\{ \cot(\nu\pi) \left[\mathbf{H}_{\nu-\frac{1}{2}}(b) - Y_{\nu-\frac{1}{2}}(b) \right] - J_{\nu-\frac{1}{2}}(b) \right\}$$

$$[b > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 102(30)a}$$
9.
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} J_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[J_\nu\left(\frac{b}{2}\right) \right]^2$$

$$[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 24(25)a}$$
10.
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} Y_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_\nu\left(\frac{b}{2}\right) Y_\nu\left(\frac{b}{2}\right)$$

$$[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 102(31)a}$$
11.
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} K_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) I_\nu\left(\frac{b}{2}\right) K_\nu\left(\frac{b}{2}\right)$$

$$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 129(10)a}$$
12.
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} I_\nu(bx) dx = 2^{-\nu-1} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[I_\nu\left(\frac{b}{2}\right) \right]^2 \quad \text{ET II 365(5)a}$$
13.
$$\int_0^1 x^{\nu+1} (1-x^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = 2^{-\nu} \frac{b^{\nu-1}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - \nu\right) \sin b$$

$$[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 25(27)a}$$

14. $\int_1^\infty x^\nu (x^2 - 1)^{\nu - \frac{1}{2}} Y_\nu(bx) dx = 2^{\nu-2} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[J_\nu\left(\frac{b}{2}\right) J_{-\nu}\left(\frac{b}{2}\right) - Y_\nu\left(\frac{b}{2}\right) Y_{-\nu}\left(\frac{b}{2}\right) \right]$
 $[|\operatorname{Re} \nu| < \frac{1}{2}, \quad b > 0] \quad \text{ET II 103(32)a}$
15. $\int_1^\infty x^\nu (x^2 - 1)^{\nu - \frac{1}{2}} K_\nu(bx) dx = \frac{2^{\nu-1}}{\sqrt{\pi}} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[K_\nu\left(\frac{b}{2}\right) \right]^2$
 $[\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 129(11)a}$
16. $\int_1^\infty x^{-\nu} (x^2 - 1)^{-\nu - \frac{1}{2}} J_\nu(bx) dx = -2^{-\nu-1} \sqrt{\pi} b^\nu \Gamma\left(\frac{1}{2} - \nu\right) J_\nu\left(\frac{b}{2}\right) Y_\nu\left(\frac{b}{2}\right)$
 $[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 25(26)a}$
- 17.⁸ $\int_1^\infty x^{-\nu+1} (x^2 - 1)^{\nu - \frac{1}{2}} J_\nu(bx) dx = \frac{2^\nu}{\sqrt{\pi}} b^{-\nu-1} \Gamma\left(\frac{1}{2} + \nu\right) \cos b$
 $[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 25(28)}$

6.568

1. $\int_0^\infty x^\nu Y_\nu(bx) \frac{dx}{x^2 - a^2} = \frac{\pi}{2} a^{\nu-1} J_\nu(ab)$
 $[a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{2}]$
 ET II 101(22)
2. $\int_0^\infty x^\mu Y_\nu(bx) \frac{dx}{x^2 - a^2} = \frac{\pi}{2} a^{\mu-1} J_\nu(ab) + 2^\mu \pi^{-1} a^{\mu-1} \cos \left[\frac{\pi}{2} (\mu - \nu + 1) \right]$
 $\times \Gamma\left(\frac{\mu - \nu + 1}{2}\right) \Gamma\left(\frac{\mu + \nu + 1}{2}\right) S_{-\mu, \nu}(ab)$
 $[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| - 1 < \operatorname{Re} \mu < \frac{5}{2}] \quad \text{ET II (101)(25)}$

6.569 $\int_0^1 x^\lambda (1-x)^{\mu-1} J_\nu(ax) dx$
 $= \frac{\Gamma(\mu) \Gamma(1 + \lambda + \nu) 2^{-\nu} a^\nu}{\Gamma(\nu + 1) \Gamma(1 + \lambda + \mu + \nu)}$
 $\times {}_2F_3\left(\frac{\lambda + 1 + \nu}{2}, \frac{\lambda + 2 + \nu}{2}; \nu + 1, \frac{\lambda + 1 + \mu + \nu}{2}, \frac{\lambda + 2 + \mu + \nu}{2}; -\frac{a^2}{4}\right)$
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\lambda + \nu) > -1] \quad \text{ET II 193(56)a}$

6.571

1. $\int_0^\infty \left[(x^2 + a^2)^{\frac{1}{2}} \pm x \right]^\mu J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} = a^\mu I_{\frac{1}{2}(\nu \mp \mu)}\left(\frac{ab}{2}\right) K_{\frac{1}{2}(\nu \pm \mu)}\left(\frac{ab}{2}\right)$
 $[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{3}{2}] \quad \text{ET II 26(38)}$
2. $\int_0^\infty \left[(x^2 + a^2)^{\frac{1}{2}} - x \right]^\mu Y_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}}$
 $= a^\mu \left[\cot(\nu\pi) I_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right) K_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right) - \operatorname{cosec}(\nu\pi) I_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right) K_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right) \right]$
 $[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \mu > -\frac{3}{2}, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 104(40)}$

$$\begin{aligned}
3. \quad & \int_0^\infty \left[(x^2 + a^2)^{\frac{1}{2}} + x \right]^\mu K_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} \\
&= \frac{\pi^2}{4} a^\mu \operatorname{cosec}(\nu\pi) \left[J_{\frac{1}{2}(\nu-\mu)} \left(\frac{ab}{2} \right) Y_{-\frac{1}{2}(\nu+\mu)} \left(\frac{ab}{2} \right) - Y_{\frac{1}{2}(\nu-\mu)} \left(\frac{ab}{2} \right) J_{-\frac{1}{2}(\nu+\mu)} \left(\frac{ab}{2} \right) \right] \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 130(15)}
\end{aligned}$$

6.572

$$\begin{aligned}
1. \quad & \int_0^\infty x^{-\mu} \left[(x^2 + a^2)^{\frac{1}{2}} + a \right]^\mu J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} = \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}{ab \Gamma(\nu+1)} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \\
&\quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\nu - \mu) > -1] \quad \text{ET II 26(40)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{-\mu} \left[(x^2 + a^2)^{\frac{1}{2}} + a \right]^\mu K_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} \\
&= \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right) \Gamma\left(\frac{1-\nu-\mu}{2}\right)}{2ab} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(iab) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(-iab) \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \mu + |\operatorname{Re} \nu| < 1] \quad \text{ET II 130(18), BU 87(6a)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{-\mu} \left[(x^2 + a^2)^{\frac{1}{2}} - a \right]^\mu Y_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} \\
&= -\frac{1}{ab} W_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \left\{ \frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma(\nu+1)} \tan\left(\frac{\nu-\mu}{2}\pi\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \right. \\
&\quad \left. + \sec\left(\frac{\nu-\mu}{2}\pi\right) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \right\} \\
&\quad [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} + \frac{1}{2} \operatorname{Re} \mu] \quad \text{ET II 105(42)}
\end{aligned}$$

6.573

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\nu-M+1} J_\nu(bx) \prod_{i=1}^k J_{\mu_i}(a_i x) dx = 0 \quad M = \sum_{i=1}^k \mu_i \\
&\quad \left[a_i > 0, \quad \sum_{i=1}^k a_i < b < \infty, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{1}{2}k - \frac{1}{2} \right] \quad \text{ET II 54(42)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\nu-M-1} J_\nu(bx) \prod_{i=1}^k J_{\mu_i}(a_i x) dx = 2^{\nu-M-1} b^{-\nu} \Gamma(\nu) \prod_{i=1}^k \frac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}, \quad M = \sum_{i=1}^k \mu_i \\
&\quad \left[a_i > 0, \quad \sum_{i=1}^k a_i < b < \infty, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{1}{2}k + \frac{3}{2} \right] \quad \text{WA 460(16)a, ET II 54(43)}
\end{aligned}$$

6.574

$$1.^8 \quad \int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda} dt = \frac{\alpha^\nu \Gamma\left(\frac{\nu + \mu - \lambda + 1}{2}\right)}{2^\lambda \beta^{\nu-\lambda+1} \Gamma\left(\frac{-\nu + \mu + \lambda + 1}{2}\right) \Gamma(\nu + 1)} \\ \times F\left(\frac{\nu + \mu - \lambda + 1}{2}, \frac{\nu - \mu - \lambda + 1}{2}; \nu + 1; \frac{\alpha^2}{\beta^2}\right) \\ [\operatorname{Re}(\nu + \mu - \lambda + 1) > 0, \quad \operatorname{Re} \lambda > -1, \quad 0 < \alpha < \beta] \quad \text{WA 439(2)a, MO 49}$$

If we reverse the positions of ν and μ and at the same time reverse the positions of α and β , the function on the right-hand side of this equation will change. Thus, the right-hand side represents a function of $\frac{\alpha}{\beta}$ that is not analytic at $\frac{\alpha}{\beta} = 1$.

For $\alpha = \beta$, we have the following equation:

$$2. \quad \int_0^\infty J_\nu(\alpha t) J_\mu(\alpha t) t^{-\lambda} dt = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\nu + \mu - \lambda + 1}{2}\right)}{2^\lambda \Gamma\left(\frac{-\nu + \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu + \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right)} \\ [\operatorname{Re}(\nu + \mu + 1) > \operatorname{Re} \lambda > 0, \quad \alpha > 0] \\ \text{MO 49, WA 441(2)a}$$

If $\mu - \nu + \lambda + 1$ (or $\nu - \mu + \lambda + 1$) is a negative integer, the right-hand side of equation **6.574** 1 (or **6.574** 3) vanishes. The cases in which the hypergeometric function F in **6.574** 3 (or **6.574** 1) can be reduced to an elementary function are then especially important.

$$3.^* \quad \int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda} dt = \frac{\beta^\nu \Gamma\left(\frac{\mu + \nu - \lambda + 1}{2}\right)}{2^\lambda \alpha^{\mu-\lambda+1} \Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right) \Gamma(\nu + 1)} \\ \times F\left(\frac{\nu + \mu - \lambda + 1}{2}, \frac{-\nu + \mu - \lambda + 1}{2}; \mu + 1; \frac{\beta^2}{\alpha^2}\right) \\ [\operatorname{Re}(\nu + \mu - \lambda + 1) > 0, \quad \operatorname{Re} \lambda > -1, \quad 0 < \beta < \alpha] \quad \text{MO 50, WA 440(3)a}$$

If $\mu - \nu + \lambda + 1$ (or $\nu - \mu + \lambda + 1$) is a negative integer, the right-hand side of equation **6.754** 1 (or **6.574** 3) vanishes. The cases in which the hypergeometric function F in **6.754** 3 (or **6.574** 1) can be reduced to an elementary function are then especially important.

6.575

$$1.^{11} \quad \int_0^\infty J_{\nu+1}(\alpha t) J_\mu(\beta t) t^{\mu-\nu} dt = 0 \quad [\alpha < \beta] \\ = \frac{(\alpha^2 - \beta^2)^{\nu-\mu} \beta^\mu}{2^{\nu-\mu} \alpha^{\nu+1} \Gamma(\nu - \mu + 1)} \quad [\alpha \geq \beta] \\ [\operatorname{Re}(\nu + 1) > \operatorname{Re} \mu > -1] \quad \text{MO 51}$$

$$2. \quad \int_0^\infty \frac{J_\nu(x) J_\mu(x)}{x^{\nu+\mu}} dx = \frac{\sqrt{\pi} \Gamma(\nu + \mu)}{2^{\nu+\mu} \Gamma\left(\nu + \mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\mu + \frac{1}{2}\right)} \\ [\operatorname{Re}(\nu + \mu) > 0] \quad \text{KU 147(17), WA 434(1)}$$

6.576

$$1. \quad \int_0^\infty x^{\mu-\nu+1} J_\mu(x) K_\nu(x) dx = \frac{1}{2} \Gamma(\mu - \nu + 1) \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re}(\mu - \nu) > -1]$$

ET II 370(47)

$$2.^{11} \quad \int_0^\infty x^{-\lambda} J_\nu(ax) J_\nu(bx) dx = \frac{a^\nu b^\nu \Gamma\left(\nu + \frac{1-\lambda}{2}\right)}{2^\lambda (a+b)^{2\nu-\lambda+1} \Gamma(\nu+1) \Gamma\left(\frac{1+\lambda}{2}\right)} \\ \times F\left(\nu + \frac{1-\lambda}{2}, \nu + \frac{1}{2}; 2\nu+1; \frac{4ab}{(a+b)^2}\right) \\ [a > 0, \quad b > 0, \quad 2\operatorname{Re} \nu + 1 > \operatorname{Re} \lambda > -1] \quad \text{ET II 47(4)}$$

$$3. \quad \int_0^\infty x^{-\lambda} K_\mu(ax) J_\nu(bx) dx = \frac{b^\nu \Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right) \Gamma\left(\frac{\nu-\lambda-\mu+1}{2}\right)}{2^{\lambda+1} a^{\nu-\lambda+1} \Gamma(1+\nu)} \\ \times F\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; -\frac{b^2}{a^2}\right) \\ [\operatorname{Re}(a \pm ib) > 0, \quad \operatorname{Re}(\nu - \lambda + 1) > |\operatorname{Re} \mu|] \quad \text{EH II 52(31), ET II 63(4), WA 449(1)}$$

$$4. \quad \int_0^\infty x^{-\lambda} K_\mu(ax) K_\nu(bx) dx = \frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \\ \times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \\ \times F\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}; 1-\lambda; 1 - \frac{b^2}{a^2}\right) \\ [\operatorname{Re} a + b > 0, \quad \operatorname{Re} \lambda < 1 - |\operatorname{Re} \mu| - |\operatorname{Re} \nu|] \quad \text{ET II 145(49), EH II 93(36)}$$

$$5. \quad \int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx) dx = \frac{b^\nu \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{2^{\lambda+1} \Gamma(\nu+1) a^{-\lambda+\nu+1}} \\ \times F\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu; \nu+1; \frac{b^2}{a^2}\right) \\ [\operatorname{Re}(\nu + 1 - \lambda \pm \mu) > 0, \quad a > b] \quad \text{EH II 93(35)}$$

$$6. \quad \int_0^\infty x^{-\lambda} Y_\mu(ax) J_\nu(bx) dx = \frac{2}{\pi} \sin \frac{\pi(\nu - \mu - \lambda)}{2} \int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx) dx \\ [a > b, \quad \operatorname{Re} \lambda > -1, \quad \operatorname{Re}(\nu - \lambda + 1 \pm \mu) > 0] \quad (\text{see 6.576 5}) \quad \text{EH II 93(37)}$$

$$7.^8 \quad \int_0^\infty x^{\mu+\nu+1} J_\mu(ax) K_\nu(bx) dx = 2^{\mu+\nu} a^\mu b^\nu \frac{\Gamma(\mu + \nu + 1)}{(a^2 + b^2)^{\mu+\nu+1}} \\ [\operatorname{Re} \mu > |\operatorname{Re} \nu| - 1, \quad \operatorname{Re} b > |\operatorname{Im} a|] \\ \text{ET 137(16), EH II 93(36)}$$

6.577

$$1.^8 \quad \int_0^\infty x^{\nu-\mu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\nu-\mu+2n} I_\mu(ac) K_\nu(bc) \\ [a > 0, \quad b > a, \quad \operatorname{Re} c > 0, \quad 2 + \operatorname{Re} \mu - 2n > \operatorname{Re} \nu > -1 - n, \quad n \geq 0 \text{ an integer}] \quad \text{ET II 49(13)}$$

$$2.^8 \quad \int_0^\infty x^{\mu-\nu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\mu-\nu+2n} I_\nu(bc) K_\mu(ac) \\ [b > 0, \quad a > b, \quad \operatorname{Re} \nu - 2n + 2 > \operatorname{Re} \mu > -n - 1, \quad n \geq 0 \text{ an integer}] \quad \text{ET II 49(15)}$$

6.578

$$1. \quad \int_0^\infty x^{\varrho-1} J_\lambda(ax) J_\mu(bx) J_\nu(cx) dx = \frac{2^{\varrho-1} a^\lambda b^\mu c^{-\lambda-\mu-\varrho} \Gamma\left(\frac{\lambda+\mu+\nu+\varrho}{2}\right)}{\Gamma(\lambda+1) \Gamma(\mu+1) \Gamma\left(1 - \frac{\lambda+\mu-\nu+\varrho}{2}\right)} \\ \times F_4\left(\frac{\lambda+\mu-\nu+\varrho}{2}, \frac{\lambda+\mu+\nu+\varrho}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right) \\ \left[\operatorname{Re}(\lambda+\mu+\nu+\varrho) > 0, \quad \operatorname{Re} \varrho < \frac{5}{2}, \quad a > 0, \quad b > 0, \quad c > 0, \quad c > a+b\right] \quad \text{ET II 351(9)}$$

$$2. \quad \int_0^\infty x^{\varrho-1} J_\lambda(ax) J_\mu(bx) K_\nu(cx) dx \\ = \frac{2^{\varrho-2} a^\lambda b^\mu c^{-\varrho-\lambda-\mu}}{\Gamma(\lambda+1) \Gamma(\mu+1)} \Gamma\left(\frac{\varrho+\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{\varrho+\lambda+\mu+\nu}{2}\right) \\ \times F_4\left(\frac{\varrho+\lambda+\mu-\nu}{2}, \frac{\varrho+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}\right) \\ [\operatorname{Re}(\varrho+\lambda+\mu) > |\operatorname{Re} \nu|, \quad \operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|] \quad \text{ET II 373(8)}$$

$$3. \quad \int_0^\infty x^{\lambda-\mu-\nu+1} J_\nu(ax) J_\mu(bx) J_\lambda(cx) dx = 0 \\ [\operatorname{Re} \lambda > -1, \quad \operatorname{Re}(\lambda - \mu - \nu) < \frac{1}{2}, \quad c > b > 0, \quad 0 < a < c - b] \quad \text{ET II 53(36)}$$

$$4. \quad \int_0^\infty x^{\lambda-\mu-\nu-1} J_\nu(ax) J_\mu(bx) J_\lambda(cx) dx = \frac{2^{\lambda-\mu-\nu-1} a^\nu b^\mu \Gamma(\lambda)}{c^\lambda \Gamma(\mu+1) \Gamma(\nu+1)} \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda - \mu - \nu) < \frac{5}{2}, \quad c > b > 0, \quad 0 < a < c - b] \quad \text{ET II 53(37)}$$

$$5. \quad \int_0^\infty x^{1+\mu} Y_\mu(ax) J_\nu(bx) J_\nu(cx) dx = 0 \quad [0 < b < c, \quad 0 < a < c - b] \\ \text{ET II 352(13)}$$

$$6.^{11} \quad \int_0^\infty x^{\mu+1} K_\mu(ax) J_\nu(bx) J_\nu(cx) dx = \frac{1}{\sqrt{2\pi}} a^\mu b^{-\mu-1} c^{-\mu-1} e^{-(\mu+\frac{1}{2})\pi i} (u^2 - 1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(u) \\ [2bcu = a^2 + b^2 + c^2, \quad \operatorname{Re} a > |\operatorname{Im} b| + |\operatorname{Im} c|, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1] \\ \text{WA 452(2), ET II 64(12)}$$

$$7.^{11} \quad \int_0^\infty x^{\mu+1} I_\nu(ax) K_\mu(bx) J_\nu(cx) dx = \frac{1}{\sqrt{2\pi}} a^{-\mu-1} b^\mu c^{-\mu-1} e^{-(\mu-\frac{1}{2}\nu+\frac{1}{4})\pi i} (v^2 + 1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(iv), \\ 2acv = b^2 - a^2 + c^2 \quad [\operatorname{Re} b > |\operatorname{Re} a| + |\operatorname{Im} c|; \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1] \quad \text{ET II 66(22)}$$

- 8.¹¹
$$\begin{aligned} \int_0^\infty x^{1-\mu} J_\mu(ax) J_\nu(bx) J_\nu(cx) dx \\ &= \sqrt{\frac{2}{\pi^3}} a^{-\mu} (bc)^{\mu-1} (\sinh u)^{\mu-\frac{1}{2}} \sin[(\mu-\nu)\pi] e^{(\mu-\frac{1}{2})\pi i} Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh u) \quad [a > b+c] \\ &= \frac{1}{\sqrt{2\pi}} a^{-\mu} (bc)^{\mu-1} (\sin v)^{\mu-\frac{1}{2}} P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cos v) \quad [|b-c| < a < b+c] \\ &= 0 \quad [0 < a < |b-c|] \\ [2bc \cosh u = a^2 - b^2 - c^2, \quad 2bc \cos v = b^2 + c^2 - a^2, \quad b > 0, \quad c > 0; \quad \operatorname{Re} \nu > -1, \operatorname{Re} \mu > -\frac{1}{2}] \end{aligned}$$
9.
$$\begin{aligned} \int_0^\infty J_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1-\nu} dx &= 0 \quad [0 < c \leq |a-b| \text{ or } c \geq a+b] \\ &= \frac{2^{\nu-1} \Delta^{2\nu-1}}{(abc)^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \quad [|a-b| < c < a+b] \\ \Delta &= \frac{1}{4} \sqrt{[c^2 - (a-b)^2][(a+b)^2 - c^2]}, \quad [a > 0, \quad b > 0, \quad c > 0; \quad \operatorname{Re} \nu > -\frac{1}{2}] \end{aligned}$$
- ($\Delta > 0$ is equal to the area of a triangle whose sides are a , b , and c .)
- 10.¹¹
$$\begin{aligned} \int_0^\infty x^{\nu+1} K_\mu(ax) K_\mu(bx) J_\nu(cx) dx &= \frac{\sqrt{\pi} c^\nu \Gamma(\nu + \mu + 1) \Gamma(\nu - \mu + 1)}{2^{3/2} (ab)^{\nu+1} (u^2 - 1)^{\frac{1}{2}\nu + \frac{1}{4}}} P_{\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(u) \\ [2abu = a^2 + b^2 + c^2, \quad \operatorname{Re}(a+b) > |\operatorname{Im} c|, \quad \operatorname{Re}(\nu \pm \mu) > -1, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 67(30)} \end{aligned}$$
- 11.¹¹
$$\begin{aligned} \int_0^\infty x^{\nu+1} K_\mu(ax) I_\mu(bx) J_\nu(cx) dx &= \frac{(ab)^{-\nu-1} c^\nu e^{-(\nu+\frac{1}{2})\pi i} Q_{\mu-\frac{1}{2}}^{\nu+\frac{1}{2}}(u)}{\sqrt{2\pi} (u^2 - 1)^{\frac{1}{2}\nu + \frac{1}{4}}} \quad 2abu = a^2 + b^2 + c^2 \\ [\operatorname{Re} a > |\operatorname{Re} b| + |\operatorname{Im} c|; \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1] \quad \text{ET II 66(24)} \end{aligned}$$
- 12.⁸
$$\begin{aligned} \int_0^\infty x^{\nu+1} [J_\nu(ax)]^2 Y_\nu(bx) dx &= 0 \quad [0 < b < 2a, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ &= \frac{2^{3\nu+1} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \nu)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \quad [0 < 2a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \end{aligned}$$

ET II 109(3)
13.
$$\begin{aligned} \int_0^\infty x^{\nu+1} J_\nu(ax) Y_\nu(ax) J_\nu(bx) dx \\ &= 0 \quad [a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}, \quad 0 < b < 2a] \\ &= -\frac{2^{3\nu+1} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \nu)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \quad [a > 0, \quad 2a < b < \infty, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \end{aligned}$$

ET II 55(49)
14.
$$\begin{aligned} \int_0^\infty x^{\nu+1} J_\mu(xa \sin \psi) J_\nu(xa \sin \varphi) K_\mu(xa \cos \varphi \cos \psi) dx \\ &= \frac{2^\nu \Gamma(\mu + \nu + 1) (\sin \varphi)^\nu (\cos \frac{\alpha}{2})^{2\nu+1}}{a^{\nu+2} (\cos \psi)^{2\nu+2}} P_\nu^{-\mu}(\cos \alpha) \\ [\tan \frac{1}{2} \alpha = \tan \psi \cos \varphi, \quad a > 0, \quad \frac{\pi}{2} > \varphi > 0, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1] \end{aligned}$$

ET II 64(11)

$$15. \quad \int_0^\infty x^{\nu+1} J_\nu(ax) K_\nu(bx) J_\nu(cx) dx = \frac{2^{3\nu} (abc)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} \left[(a^2 + b^2 + c^2)^2 - 4a^2 c^2\right]^{\nu + \frac{1}{2}}} \\ \left[\operatorname{Re} b > |\operatorname{Im} a|, \quad c > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}\right] \\ \text{ET II 63(8)}$$

$$16.^8 \quad \int_0^\infty x^{\nu+1} I_\nu(ax) K_\nu(bx) J_\nu(cx) dx = \frac{2^{3\nu} (abc)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} \left[(b^2 - a^2 + c^2)^2 + 4a^2 c^2\right]^{\nu + \frac{1}{2}}} \\ \left[\operatorname{Re} b > |\operatorname{Re} a| + |\operatorname{Im} c|; \quad \operatorname{Re} \nu > -\frac{1}{2}\right] \\ \text{ET II 65(18)}$$

6.579

$$1. \quad \int_0^\infty x^{2\nu+1} J_\nu(ax) Y_\nu(ax) J_\nu(bx) Y_\nu(bx) dx \\ = \frac{a^{2\nu} \Gamma(3\nu + 1)}{2\pi b^{4\nu+2} \Gamma\left(\frac{1}{2} - \nu\right) \Gamma\left(2\nu + \frac{3}{2}\right)} F\left(\nu + \frac{1}{2}, 3\nu + 1; 2\nu + \frac{3}{2}; \frac{a^2}{b^2}\right) \\ \left[0 < a < b, \quad -\frac{1}{3} < \operatorname{Re} \nu < \frac{1}{2}\right] \quad \text{EH II 94(45), ET II 352(15)}$$

$$2. \quad \int_0^\infty x^{2\nu+1} J_\nu(ax) K_\nu(ax) J_\nu(bx) K_\nu(bx) dx \\ = \frac{2^{\nu-3} a^{2\nu} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{3\nu+1}{2}\right)}{\sqrt{\pi} b^{4\nu+2} \Gamma(\nu + 1)} F\left(\nu + \frac{1}{2}, \frac{3\nu+1}{2}; 2\nu + 1; 1 - \frac{a^4}{b^4}\right) \\ \left[0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{3}\right] \quad \text{ET II 373(10)}$$

$$3. \quad \int_0^\infty x^{1-2\nu} [J_\nu(ax)]^4 dx = \frac{\Gamma(\nu) \Gamma(2\nu)}{2\pi \left[\Gamma\left(\nu + \frac{1}{2}\right)\right]^2 \Gamma(3\nu)} \quad \left[\operatorname{Re} \nu > 0\right] \quad \text{ET II 342(25)}$$

$$4. \quad \int_0^\infty x^{1-2\nu} [J_\nu(ax)]^2 [J_\nu(bx)]^2 dx = \frac{a^{2\nu-1} \Gamma(\nu)}{2\pi b \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(2\nu + \frac{1}{2}\right)} F\left(\nu, \frac{1}{2} - \nu; 2\nu + \frac{1}{2}; \frac{a^2}{b^2}\right) \\ \text{ET II 351(10)}$$

6.581

$$1. \quad \int_0^a x^{\lambda-1} J_\mu(x) J_\nu(a-x) dx = 2^\lambda \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\lambda + \mu + m) \Gamma(\lambda + m)}{m! \Gamma(\lambda) \Gamma(\mu + m + 1)} J_{\lambda+\mu+\nu+2m}(a) \\ \left[\operatorname{Re}(\lambda + \mu) > 0, \quad \operatorname{Re} \nu > -1\right] \\ \text{ET II 354(25)}$$

$$2.^8 \quad \int_0^a x^{\lambda-1} (a-x)^{-1} J_\mu(x) J_\nu(a-x) dx \\ = \frac{2^\lambda}{a\nu} \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\lambda + \mu + m) \Gamma(\lambda + m)}{m! \Gamma(\lambda) \Gamma(\mu + m + 1)} (\lambda + \mu + \nu + 2m) J_{\lambda+\mu+\nu+2m}(a) \\ \left[\operatorname{Re}(\lambda + \mu) > 0, \quad \operatorname{Re} \nu > 0\right] \quad \text{ET II 354(27)}$$

$$3. \quad \int_0^a x^\mu (a-x)^\nu J_\mu(x) J_\nu(a-x) dx = \frac{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{2\pi} \Gamma(\mu + \nu + 1)} a^{\mu+\nu+\frac{1}{2}} J_{\mu+\nu+\frac{1}{2}}(a) \\ \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}\right] \\ \text{ET II 354(28), EH II 46(6)}$$

$$4. \quad \int_0^a x^\mu (a-x)^{\nu+1} J_\mu(x) J_\nu(a-x) dx = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\nu + \frac{3}{2})}{\sqrt{2\pi} \Gamma(\mu + \nu + 2)} a^{\mu+\nu+\frac{3}{2}} J_{\mu+\nu+\frac{1}{2}}(a) \\ [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{ET II 354(29)}$$

$$5. \quad \int_0^a x^\mu (a-x)^{-\mu-1} J_\mu(x) J_\nu(a-x) dx = \frac{2^\mu \Gamma(\mu + \frac{1}{2}) \Gamma(\nu - \mu)}{\sqrt{\pi} \Gamma(\mu + \nu + 1)} a^\mu J_\nu(a) \\ [\operatorname{Re} \nu > \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{ET II 355(30)}$$

$$6.582 \quad \int_0^\infty x^{\mu-1} |x-b|^{-\mu} K_\mu(|x-b|) K_\nu(x) dx = \frac{1}{\sqrt{\pi}} (2b)^{-\mu} \Gamma(\frac{1}{2} - \mu) \Gamma(\mu + \nu) \Gamma(\mu - \nu) K_\nu(b) \\ [b > 0, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu|] \quad \text{ET II 374(14)}$$

$$6.583 \quad \int_0^\infty x^{\mu-1} (x+b)^{-\mu} K_\mu(x+b) K_\nu(x) dx = \frac{\sqrt{\pi} \Gamma(\mu + \nu) \Gamma(\mu - \nu)}{2^\mu b^\mu \Gamma(\mu + \frac{1}{2})} K_\nu(b) \\ [|\arg b| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu|] \quad \text{ET II 374(15)}$$

6.584

$$1.^8 \quad \int_0^\infty \frac{x^{\varrho-1} [H_\nu^{(1)}(ax) - e^{\varrho\pi i} H_\nu^{(1)}(axe^{\pi i})]}{(x^2 - r^2)^{m+1}} dx = \frac{\pi i}{2^m m!} \left(\frac{d}{r dr} \right)^m [r^{\varrho-2} H_\nu^{(1)}(ar)] \\ [m = 0, 1, 2, \dots, \quad \operatorname{Im} r > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \operatorname{Re} \varrho < 2m + \frac{7}{2}] \quad \text{WA 465}$$

$$2.^8 \quad \int_0^\infty \left[\cos \frac{1}{2}(\varrho - \nu)\pi J_\nu(ax) + \sin \frac{1}{2}(\varrho - \nu)\pi Y_\nu(ax) \right] \frac{x^{\varrho-1}}{(x^2 + k^2)^{m+1}} dx \\ = \frac{(-1)^{m+1}}{2^m \cdot m!} \left(\frac{d}{k dk} \right)^m [k^{\varrho-2} K_\nu(ak)] \\ [m = 0, 1, 2, \dots, \quad \operatorname{Re} k > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \operatorname{Re} \varrho < 2m + \frac{7}{2}] \quad \text{WA 466(2)}$$

$$3. \quad \int_0^\infty \{ \cos \nu\pi J_\nu(ax) - \sin \nu\pi Y_\nu(ax) \} \frac{x^{1-\nu} dx}{(x^2 + k^2)^{m+1}} = \frac{a^m K_{\nu+m}(ak)}{2^m \cdot m! k^{\nu+m}} \\ [m = 0, 1, 2, \dots, \quad \operatorname{Re} k > 0, \quad a > 0, \quad -2m - \frac{3}{2} < \operatorname{Re} \nu < 1] \quad \text{WA 466(3)}$$

$$4. \quad \int_0^\infty \{ \cos [(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu)\pi] J_\nu(ax) + \sin [(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu)\pi] Y_\nu(ax) \} \frac{x^{\varrho-1}}{(x^2 + k^2)^{\mu+1}} dx \\ = \frac{\pi k^{\varrho-2\mu-2}}{2 \sin \nu\pi \cdot \Gamma(\mu + 1)} \left[\frac{(\frac{1}{2}ak)^\nu \Gamma(\frac{1}{2}\varrho + \frac{1}{2}\nu)}{\Gamma(\nu + 1) \Gamma(\frac{1}{2}\varrho + \frac{1}{2}\nu - \mu)} {}_1F_2 \left(\frac{\varrho + \nu}{2}; \frac{\varrho + \nu}{2} - \mu, \nu + 1; \frac{a^2 k^2}{4} \right) \right. \\ \left. - \frac{(\frac{1}{2}ak)^{-\nu} \Gamma(\frac{1}{2}\varrho - \frac{1}{2}\nu)}{\Gamma(1 - \nu) \Gamma(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu)} {}_1F_2 \left(\frac{\varrho - \nu}{2}; \frac{\varrho - \nu}{2} - \mu, 1 - \nu; \frac{a^2 k^2}{4} \right) \right] \\ [a > 0, \quad \operatorname{Re} k > 0, \quad |\operatorname{Re} \nu| < \operatorname{Re} \varrho < 2 \operatorname{Re} \mu + \frac{7}{2}] \quad \text{WA 407(1)}$$

$$\begin{aligned}
5.8 \quad \int_0^\infty \left[\prod_{j=1}^n J_{\mu_j}(b_n x) \right] & \left\{ \cos \left[\frac{1}{2} \left(\varrho + \sum_j \mu_j - \nu \right) \pi \right] J_\nu(ax) \right. \\
& + \sin \left[\frac{1}{2} \left(\varrho + \sum_j \mu_j - \nu \right) \pi \right] Y_\nu(ax) \left. \right\} \frac{x^{\varrho-1}}{x^2 + k^2} dx \\
& = - \left[\prod_{j=1}^n I_{\mu_j}(b_n k) \right] K_\nu(ak) k^{\varrho-2} \\
& \left[\operatorname{Re} k > 0, \quad a > \sum_j |\operatorname{Re} b_j|, \quad \operatorname{Re} \left(\varrho + \sum_j \mu_j \right) > |\operatorname{Re} \nu| \right] \quad \text{WA 472(9)}
\end{aligned}$$

6.59 Combinations of powers and Bessel functions of more complicated arguments

6.591

1.
$$\int_0^\infty x^{2\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} J_{1+2\nu}(\sqrt{2ab}) K_{1+2\nu}(\sqrt{2ab})$$

 $[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1]$
ET II 142(35)
2.
$$\int_0^\infty x^{2\nu+\frac{1}{2}} Y_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} Y_{2\nu+1}(\sqrt{2ab}) K_{2\nu+1}(\sqrt{2ab})$$

 $[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1]$
ET II 143(41)
3.
$$\int_0^\infty x^{2\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} K_{2\nu+1}(e^{\frac{1}{4}i\pi}\sqrt{2ab}) K_{2\nu+1}(e^{-\frac{1}{4}i\pi}\sqrt{2ab})$$

 $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$
ET II 146(56)
4.
$$\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} K_{2\nu-1}(\sqrt{2ab})$$

 $\times \left[\sin(\nu\pi) J_{2\nu-1}(\sqrt{2ab}) + \cos(\nu\pi) Y_{2\nu-1}(\sqrt{2ab}) \right]$
 $[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu < 1]$
ET II 142(34)
5.
$$\int_0^\infty x^{-2\nu+\frac{1}{2}} Y_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = -\sqrt{\frac{\pi}{2}} b^{\nu-1} a^{\frac{1}{2}-\nu} \sec(\nu\pi) K_{2\nu-1}(\sqrt{2ab})$$

 $\times \left[J_{2\nu-1}(\sqrt{2ab}) - J_{1-2\nu}(\sqrt{2ab}) \right]$
 $[a > 0, \quad \operatorname{Re} \nu < 1]$
ET II 143(40)
6.
$$\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\frac{1}{2}-\nu}\left(\frac{a}{x}\right) J_\nu(bx) dx$$

 $= -\frac{1}{2} i \operatorname{cosec}(2\nu\pi) b^{\nu-1} a^{\frac{1}{2}-\nu} [e^{2\nu\pi i} J_{1-2\nu}(u) J_{2\nu-1}(v) - e^{-2\nu\pi i} J_{2\nu-1}(u) J_{1-2\nu}(v)]$
 $\left[u = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}, \quad v = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}, \quad a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 3 \right]$
ET II 58(12)

$$7. \quad \int_0^\infty x^{-2\nu+\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) Y_\nu(bx) dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} Y_{2\nu-1}\left(\sqrt{2ab}\right) K_{2\nu-1}\left(\sqrt{2ab}\right) \\ [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > \frac{1}{6}] \quad \text{ET II 113(30)}$$

$$8. \quad \int_0^\infty x^{\varrho-1} J_\mu(ax) J_\nu\left(\frac{b}{x}\right) dx = \frac{a^{\nu-\varrho} b^\nu \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\varrho - \frac{1}{2}\nu\right)}{2^{2\nu-\varrho+1} \Gamma(\nu+1) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\varrho + 1\right)} \\ \times {}_0F_3\left(\nu+1, \frac{\nu-\mu-\varrho}{2}+1, \frac{\nu+\mu-\varrho}{2}+1; \frac{a^2 b^2}{16}\right) \\ + \frac{a^\mu b^{\mu+\varrho} \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2}\varrho\right)}{2^{2\mu+\varrho+1} \Gamma(\mu+1) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\varrho + 1\right)} \\ \times {}_0F_3\left(\mu+1, \frac{\mu-\nu+\varrho}{2}+1, \frac{\nu+\mu+\varrho}{2}+1; \frac{a^2 b^2}{16}\right) \\ [a > 0, \quad b > 0, \quad -\operatorname{Re}\left(\mu + \frac{3}{2}\right) < \operatorname{Re} \varrho < \operatorname{Re}\left(\nu + \frac{3}{2}\right)] \quad \text{WA 480(1)}$$

6.592

$$1. \quad \int_0^\infty x^\lambda (1-x)^{\mu-1} Y_\nu(a\sqrt{x}) dx = 2^{-\nu} a^\nu \cot(\nu\pi) \frac{\Gamma(\mu) \Gamma\left(\lambda+1+\frac{1}{2}\nu\right)}{\Gamma(1+\nu) \Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)} \\ \times {}_1F_2\left(\lambda+1+\frac{1}{2}\nu; 1+\nu, \lambda+1+\mu+\frac{1}{2}\nu; -\frac{a^2}{4}\right) \\ - 2^\nu a^{-\nu} \operatorname{cosec}(\nu\pi) \frac{\Gamma(\mu) \Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma(1-\nu) \Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)} \\ \times {}_1F_2\left(\lambda-\frac{1}{2}\nu+1; 1-\nu, \lambda+1+\mu-\frac{1}{2}\nu; -\frac{a^2}{4}\right) \\ [\operatorname{Re} \lambda > -1 + \frac{1}{2}|\operatorname{Re} \nu|, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 197(76)a}$$

$$2.^{10} \quad \int_0^1 x^\lambda (1-x)^{\mu-1} K_\nu(a\sqrt{x}) dx \\ = 2^{-\nu-1} a^{-\nu} \frac{\Gamma(\nu) \Gamma(\mu) \Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)} {}_1F_2\left(\lambda+1-\frac{1}{2}\nu; 1-\nu, \lambda+1+\mu-\frac{1}{2}\nu; \frac{a^2}{4}\right) \\ + 2^{-1-\nu} a^\nu \frac{\Gamma(-\nu) \Gamma\left(\lambda+1+\frac{1}{2}\nu\right) \Gamma(\mu)}{\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)} {}_1F_2\left(\lambda+1+\frac{1}{2}\nu; 1+\nu, \lambda+1+\mu+\frac{1}{2}\nu; \frac{a^2}{4}\right) \\ = \frac{2^{\nu-1}}{a^\nu} \Gamma(\mu) G_{13}^{21}\left(\frac{a^2}{4} \left| \begin{matrix} \frac{\nu}{2} - \lambda \\ \nu, 0, \frac{\nu}{2} - \lambda - \mu \end{matrix} \right. \right)$$

OB 159 (3.16)

$$[\operatorname{Re} \lambda > -1 + \frac{1}{2}|\operatorname{Re} \nu|, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 198(87)a}$$

- 3.¹¹ $\int_1^\infty x^\lambda (x-1)^{\mu-1} J_\nu(a\sqrt{x}) dx = 2^{2\lambda} a^{-2\lambda} G_{13}^{20} \left(\frac{a^2}{4} \middle| \begin{matrix} 0 \\ -\mu, \lambda + \frac{1}{2}\nu, \lambda - \frac{1}{2}\nu \end{matrix} \right) \Gamma(\mu)$
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{3}{4} - \operatorname{Re} \lambda]$ ET II 205(36)a
4. $\int_1^\infty x^\lambda (x-1)^{\mu-1} K_\nu(a\sqrt{x}) dx = \Gamma(\mu) 2^{2\lambda-1} a^{-2\lambda} G_{13}^{30} \left(\frac{a^2}{4} \middle| \begin{matrix} 0 \\ -\mu, \frac{1}{2}\nu + \lambda, -\frac{1}{2}\nu + \lambda \end{matrix} \right)$
 $[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0]$ ET II 209(60)a
5. $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} J_\nu(a\sqrt{x}) dx = \pi \left[J_{\frac{1}{2}\nu} \left(\frac{1}{2}a \right) \right]^2$ $[\operatorname{Re} \nu > -1]$ ET II 194(59)a
6. $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} I_\nu(a\sqrt{x}) dx = \pi \left[I_{\frac{1}{2}\nu} \left(\frac{1}{2}a \right) \right]^2$ $[\operatorname{Re} \nu > -1]$ ET II 197(79)
7. $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} K_\nu(a\sqrt{x}) dx = \frac{1}{2} \pi \sec \left(\frac{1}{2} \nu \pi \right) \left[I_{\frac{\nu}{2}} \left(\frac{a}{2} \right) + I_{-\frac{\nu}{2}} \left(\frac{a}{2} \right) \right] K_{\frac{\nu}{2}} \left(\frac{a}{2} \right)$
 $[|\operatorname{Re} \nu| < 1]$ ET II 198(85)a
8. $\int_1^\infty x^{-\frac{1}{2}} (x-1)^{-\frac{1}{2}} K_\nu(a\sqrt{x}) dx = \left[K_{\frac{\nu}{2}} \left(\frac{a}{2} \right) \right]^2$ $[\operatorname{Re} a > 0]$ ET II 208(56)a
9. $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} Y_\nu(a\sqrt{x}) dx = \pi \left\{ \cot(\nu\pi) \left[J_{\frac{\nu}{2}} \left(\frac{a}{2} \right) \right]^2 - \operatorname{cosec}(\nu\pi) \left[J_{-\frac{\nu}{2}} \left(\frac{a}{2} \right) \right]^2 \right\}$
 $[|\operatorname{Re} \nu| < 1]$ ET II 195(68)a
10. $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} J_\nu(a\sqrt{x}) dx = \Gamma(\mu) 2^\mu a^{-\mu} J_{\nu-\mu}(a)$
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}]$ ET II 205(34)a
11. $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} J_{-\nu}(a\sqrt{x}) dx = \Gamma(\mu) 2^\mu a^{-\mu} [\cos(\nu\pi) J_{\nu-\mu}(a) - \sin(\nu\pi) Y_{\nu-\mu}(a)]$
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}]$ ET II 205(35)a
12. $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} K_\nu(a\sqrt{x}) dx = \Gamma(\mu) 2^\mu a^{-\mu} K_{\nu-\mu}(a)$
 $[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0]$ ET II 209(59)a
13. $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} Y_\nu(a\sqrt{x}) dx = 2^\mu a^{-\mu} Y_{\nu-\mu}(a) \Gamma(\mu)$
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}]$ ET II 206(40)a
14. $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} H_\nu^{(1)}(a\sqrt{x}) dx = 2^\mu a^{-\mu} H_{\nu-\mu}^{(1)}(a) \Gamma(\mu)$
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Im} a > 0]$ ET II 206(45)a
15. $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} H_\nu^{(2)}(a\sqrt{x}) dx = 2^\mu a^{-\mu} H_{\nu-\mu}^{(2)}(a) \Gamma(\mu)$
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Im} a < 0]$ ET II 207(48)a

$$16. \quad \int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1} J_\nu(a\sqrt{x}) dx = \frac{2^{2-\nu}a^{-\mu}}{\Gamma(\nu)} s_{\mu+\nu-1,\mu-\nu}(a) \quad [\operatorname{Re} \mu > 0] \quad \text{ET II 194(64)a}$$

$$17. \quad \int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1} Y_\nu(a\sqrt{x}) dx = \frac{2^{2-\nu}a^{-\mu} \cot(\nu\pi)}{\Gamma(\nu)} s_{\mu+\nu-1,\mu-\nu}(a) \\ - 2^\mu a^{-\mu} \operatorname{cosec}(\nu\pi) J_{\mu-\nu}(a) \Gamma(\mu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 196(75)a}$$

6.593

$$1. \quad \int_0^\infty \sqrt{x} J_{2\nu-1}(a\sqrt{x}) J_\nu(bx) dx = \frac{1}{2} ab^{-2} J_{\nu-1}\left(\frac{a^2}{4b}\right) \quad [b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 58(15)}$$

$$2. \quad \int_0^\infty \sqrt{x} J_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx = \frac{\pi a}{4b^2} \left[I_{\nu-1}\left(\frac{a^2}{4b}\right) - \mathbf{L}_{\nu-1}\left(\frac{a^2}{4b}\right) \right] \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 144(44)}$$

6.594

$$1. \quad \int_0^\infty x^\nu I_{2\nu-1}(a\sqrt{x}) J_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx = \sqrt{\pi} 2^{-\nu} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 148(65)}$$

$$2. \quad \int_0^\infty x^\nu I_{2\nu-1}(a\sqrt{x}) Y_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx \\ = \sqrt{\pi} 2^{-\nu-1} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \\ \times \left[\mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) + \cos(\nu\pi) J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) + \sin(\nu\pi) Y_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) \right] \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 148(66)}$$

$$3. \quad \int_0^\infty x^\nu J_{2\nu-1}(a\sqrt{x}) K_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx \\ = \pi^2 2^{-\nu-2} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \left[\mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) - Y_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) \right] \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 148(67)}$$

6.595

$$1. \quad \int_0^\infty x^{\nu+1} J_\nu(cx) \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i) dx = 0 \quad z_i = \sqrt{x^2 + b_i^2} \\ \left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c; \quad \operatorname{Re} \left(\frac{1}{2}n + \sum_{i=1}^n \mu_i - \frac{1}{2} \right) > \operatorname{Re} \nu > -1 \right] \\ \text{EH II 52(33), ET II 60(26)}$$

$$2. \quad \int_0^\infty x^{\nu-1} J_\nu(cx) \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i) dx = 2^{\nu-1} \Gamma(\nu) c^{-\nu} \prod_{i=1}^n [b_i^{-\mu_i} J_{\mu_i}(a_i b_i)] \quad z_i = \sqrt{x^2 + b_i^2} \\ \left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c, \quad \operatorname{Re} \left(\frac{1}{2}n + \sum_{i=1}^n \mu_i + \frac{3}{2} \right) > \operatorname{Re} \nu > 0 \right] \\ \text{EH II 52(34), ET II 60(27)}$$

6.596

1.
$$\int_0^\infty J_\nu \left(\alpha \sqrt{x^2 + z^2} \right) \frac{x^{2\mu+1}}{\sqrt{(x^2 + z^2)^\nu}} dx = \frac{2^\mu \Gamma(\mu + 1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} J_{\nu-\mu-1}(\alpha z)$$

$$\left[\alpha > 0, \quad \operatorname{Re} \left(\frac{1}{2} \nu - \frac{1}{4} \right) > \operatorname{Re} \mu > -1 \right]$$

WA 457(5)
2.
$$\int_0^\infty \frac{J_\nu (\alpha \sqrt{t^2 + 1})}{\sqrt{t^2 + 1}} dt = -\frac{\pi}{2} J_{\frac{\nu}{2}} \left(\frac{\alpha}{2} \right) Y_{\frac{\nu}{2}} \left(\frac{\alpha}{2} \right) \quad [\operatorname{Re} \nu > -1, \quad \alpha > 0]$$

MO 46
3.
$$\int_0^\infty K_\nu \left(\alpha \sqrt{x^2 + z^2} \right) \frac{x^{2\mu+1}}{\sqrt{(x^2 + z^2)^\nu}} dx = \frac{2^\mu \Gamma(\mu + 1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} K_{\nu-\mu-1}(\alpha z)$$

$$[\alpha > 0, \quad \operatorname{Re} \mu > -1]$$

WA 457(6)
- 4.⁸
$$\int_0^\infty J_\nu(\beta x) \frac{J_{\mu-1} \{ \alpha \sqrt{x^2 + z^2} \}}{(x^2 + z^2)^{\frac{1}{2}\mu + \frac{1}{2}}} x^{\nu+1} dx = \frac{\alpha^{\mu-1} z^\nu}{2^{\mu-1} \Gamma(\mu)} K_\nu(\beta z)$$

$$[\alpha < \beta, \quad \operatorname{Re}(\mu + 2) > \operatorname{Re} \nu > -1]$$

ET II 59(19)
- 5.⁸
$$\int_0^\infty J_\nu(\beta x) \frac{J_\mu \{ \alpha \sqrt{x^2 + z^2} \}}{\sqrt{(x^2 + z^2)^\mu}} x^{\nu-1} dx = \frac{2^{\nu-1} \Gamma(\nu)}{\beta^\nu} \frac{J_\mu(\alpha z)}{z^\mu}$$

$$[\operatorname{Re}(\mu + 2) > \operatorname{Re} \nu > 0, \quad \beta > \alpha > 0]$$

WA 459(12)
- 6.⁶
$$\int_0^\infty J_\nu(\beta x) \frac{J_\mu (\alpha \sqrt{x^2 + z^2})}{\sqrt{(x^2 + z^2)^\mu}} x^{\nu+1} dx$$

$$= 0 \quad [0 < \alpha < \beta]$$

$$= \frac{\beta^\nu}{\alpha^\mu} \left(\frac{\sqrt{\alpha^2 - \beta^2}}{z} \right)^{\mu-\nu-1} J_{\mu-\nu-1} \left\{ z \sqrt{\alpha^2 - \beta^2} \right\} \quad [\alpha > \beta > 0]$$

$$[\operatorname{Re} \mu > \operatorname{Re} \nu > -1]$$

WA 415(1)
- 7.⁸
$$\int_0^\infty J_\nu(\beta x) \frac{K_\mu (\alpha \sqrt{x^2 + z^2})}{\sqrt{(x^2 + z^2)^\mu}} x^{\nu+1} dx = \frac{\beta^\nu}{\alpha^\mu} \left(\frac{\sqrt{\alpha^2 + \beta^2}}{z} \right)^{\mu-\nu-1} K_{\mu-\nu-1} \left(z \sqrt{\alpha^2 + \beta^2} \right)$$

$$\left[\alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1, \quad |\arg z| < \frac{\pi}{2} \right] \quad \text{KU 151(31), WA 416(2)}$$
- 8.⁸
$$\int_0^\infty J_\nu(ux) K_\mu \left(v \sqrt{x^2 - y^2} \right) (x^2 - y^2)^{-\frac{\mu}{2}} x^{\nu+1} dx = \frac{\pi}{2} \exp \left[-i\pi \left(\mu - \nu - \frac{1}{2} \right) \right] \cdot \frac{u^\nu}{v^\mu}$$

$$\cdot \left[\frac{\sqrt{u^2 + v^2}}{y} \right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)} \left(y \sqrt{u^2 + v^2} \right)$$

$$\left[\operatorname{Re} \mu < 1, \quad \operatorname{Re} \nu > -1, u > 0, \quad v > 0, y > 0; \quad (x^2 - y^2)^{\frac{1}{2}\alpha} = e^{\frac{1}{2}\alpha\pi i} (y^2 - n^2)^{\frac{1}{2}\alpha} \text{ if } x < y \right]$$

$$\begin{aligned}
 9.^8 \quad \int_0^\infty J_\nu(ux) H_\mu^{(2)}\left(v\sqrt{x^2+y^2}\right) (x^2+y^2)^{-\frac{\mu}{2}} x^{\nu+1} dx \\
 = \frac{u^\nu}{v^\mu} \left[\frac{\sqrt{v^2-u^2}}{y} \right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)}\left(y\sqrt{v^2-u^2}\right) \\
 [u < v] \\
 \left[\operatorname{Re} \mu > \operatorname{Re} \nu > -1, \quad u > 0, \quad v > 0, \quad y > 0; , \quad \arg \sqrt{v^2-u^2} = 0, \text{ for } v > u \right. \\
 \left. \arg(v^2-u^2)^\sigma = -\pi\sigma \text{ for } v < u, \text{ where } \sigma = \frac{1}{2} \text{ or } \sigma = \frac{\mu-\nu-1}{2} \right]
 \end{aligned}$$

MO 43

$$\begin{aligned}
 10.^8 \quad \int_0^\infty J_\nu(\beta x) J_\mu\left(\alpha\sqrt{x^2+z^2}\right) J_\mu\left(\gamma\sqrt{x^2+z^2}\right) \frac{x^{\nu-1}}{(x^2+z^2)^\mu} dx = \frac{2^{\nu-1} \Gamma(\nu)}{\beta^\nu} \frac{J_\mu(\alpha z)}{z^\mu} \frac{J_\mu(\gamma z)}{z^\mu} \\
 [\alpha > 0; \quad \beta > \alpha + \gamma; \quad \gamma > 0, \quad \operatorname{Re}(2\mu + \frac{5}{2}) > \operatorname{Re} \nu > 0] \quad \text{WA 459(14)}
 \end{aligned}$$

$$\begin{aligned}
 11.^8 \quad \int_0^\infty J_\nu(\beta t) t^{\nu-1} \prod_{k=1}^n J_\mu\left(\alpha_k \sqrt{t^2+x^2}\right) \sqrt{(t^2+x^2)^{-n\mu}} dt = 2^{\nu-1} \beta^{-\nu} \Gamma(\nu) \prod_{k=1}^n [x^{-\mu} J_\mu(\alpha_k x)] \\
 \left[x > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \dots, \alpha_n > 0, \quad \beta > \prod_{k=1}^n \alpha_k; \quad \operatorname{Re}\left(n\mu + \frac{1}{2}n + \frac{1}{2}\right) > \operatorname{Re} \nu > 0 \right]
 \end{aligned}$$

MO 43

$$12.^8 \quad \int_0^\infty \frac{J_\mu^2(\sqrt{a^2+x^2})}{(a^2+x^2)^\nu} x^{2\nu-2} dx = \frac{\Gamma(\nu - \frac{1}{2})}{2a^{\nu+1}\sqrt{\pi}} \mathbf{H}_\nu(2a) \quad [\operatorname{Re} \nu > \frac{1}{2}] \quad \text{WA 457(8)}$$

$$\begin{aligned}
 6.597 \quad \int_0^\infty t^{\nu+1} J_\mu\left[b(t^2+y^2)^{\frac{1}{2}}\right] (t^2+y^2)^{-\frac{1}{2}\mu} (t^2+\beta^2)^{-1} J_\nu(at) dt \\
 = \beta^\nu J_\mu\left[b(y^2-\beta^2)^{\frac{1}{2}}\right] (y^2-\beta^2)^{-\frac{1}{2}\mu} K_\nu(a\beta) \\
 [a \geq b, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < 2 + \operatorname{Re} \mu] \quad \text{EH II 95(56)}
 \end{aligned}$$

$$\begin{aligned}
 6.598 \quad \int_0^1 x^{\frac{\mu}{2}} (1-x)^{\frac{\nu}{2}} J_\mu(a\sqrt{x}) J_\nu(b\sqrt{1-x}) dx = 2a^\mu b^\nu (a^2+b^2)^{-\frac{1}{2}(\nu+\mu+1)} J_{\nu+\mu+1}\left(\sqrt{a^2+b^2}\right) \\
 [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1] \quad \text{EH II 46a}
 \end{aligned}$$

6.61 Combinations of Bessel functions and exponentials

6.611

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-\alpha x} J_\nu(\beta x) dx = \frac{\beta^{-\nu} [\sqrt{\alpha^2 + \beta^2} - \alpha]^\nu}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re}(\alpha \pm i\beta) > 0] \\
 \text{EH II 49(18), WA 422(8)}
 \end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty e^{-\alpha x} Y_\nu(\beta x) dx &= (\alpha^2 + \beta^2)^{-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \\
&\times \left\{ \beta^\nu \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^{-\nu} \cos(\nu\pi) - \beta^{-\nu} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^\nu \right\} \\
&\quad [\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{MO 179, ET II 105(1)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty e^{-\alpha x} K_\nu(\beta x) dx &= \frac{\pi}{\beta \sin(\nu\pi)} \frac{\sin(\nu\theta)}{\sin \theta} \\
&\quad \left[\cos \theta = \frac{\alpha}{\beta}; \quad \theta \rightarrow \frac{\pi}{2} \quad \text{for } \beta \rightarrow \infty \right] \\
&\quad \text{ET II 131(22)} \\
&= \frac{\pi \operatorname{cosec}(\nu\pi)}{2\sqrt{\alpha^2 - \beta^2}} \left[\beta^{-\nu} \left(\alpha + \sqrt{\alpha^2 - \beta^2} \right)^\nu - \beta^\nu \left(\sqrt{\alpha^2 - \beta^2} + \alpha \right)^{-\nu} \right] \\
&\quad [|\operatorname{Re} \nu| < 1, \quad \operatorname{Re}(\alpha + \beta) > 0] \\
&\quad \text{ET I 197(24), MO 180}
\end{aligned}$$

$$\begin{aligned}
4.8 \quad \int_0^\infty e^{-\alpha x} I_\nu(\beta x) dx &= \frac{\beta^{-\nu} \left[\alpha - \sqrt{\alpha^2 - \beta^2} \right]^\nu}{\sqrt{\alpha^2 - \beta^2}} \\
&\quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > |\operatorname{Re} \beta|] \\
&\quad \text{MO 180, ET I 195(1)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty e^{-\alpha x} H_\nu^{(1,2)}(\beta x) dx &= \frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^\nu}{\beta^\nu \sqrt{\alpha^2 + \beta^2}} \left\{ 1 \pm \frac{i}{\sin(\nu\pi)} \left[\cos(\nu\pi) - \frac{(\alpha + \sqrt{\alpha^2 + \beta^2})^{2\nu}}{b^{2\nu}} \right] \right\} \\
&\quad [-1 < \operatorname{Re} \nu < 1; \text{ a plus sign corresponds to the function } H_\nu^{(1)}, \text{ a minus sign to the function } H_\nu^{(2)}.] \\
&\quad \text{MO 180, ET I 188(54, 55)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty e^{-\alpha x} H_0^{(1)}(\beta x) dx &= \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ 1 - \frac{2i}{\pi} \ln \left[\frac{\alpha}{\beta} + \sqrt{1 + \left(\frac{\alpha}{\beta} \right)^2} \right] \right\} \\
&\quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{MO 180, ET I 188(53)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty e^{-\alpha x} H_0^{(2)}(\beta x) dx &= \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ 1 + \frac{2i}{\pi} \ln \left[\frac{\alpha}{\beta} + \sqrt{1 + \left(\frac{\alpha}{\beta} \right)^2} \right] \right\} \\
&\quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{MO 180, ET I 188(53)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_0^\infty e^{-\alpha x} Y_0(\beta x) dx &= \frac{-2}{\pi \sqrt{\alpha^2 + \beta^2}} \ln \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{\beta} \\
&\quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{MO 47, ET I 187(44)}
\end{aligned}$$

$$\begin{aligned}
9.11 \quad \int_0^\infty e^{-\alpha x} K_0(\beta x) dx &= \frac{\arccos \frac{\alpha}{\beta}}{\sqrt{\beta^2 - \alpha^2}} \\
&\quad [\operatorname{Re}(\alpha + \beta) > 0] \quad \text{WA 424, ET II 131(22)} \\
&= \frac{1}{\sqrt{\alpha^2 - \beta^2}} \ln \left(\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - 1} \right) \\
&\quad [\operatorname{Re}(\alpha + \beta) > 0]
\end{aligned}$$

$$10.^{10} \int_a^b \alpha d\alpha \int_0^\infty dk J_1(k\alpha) e^{-k|\beta|} = \int_a^b \left(1 - \frac{|\beta|}{\sqrt{\alpha^2 + \beta^2}}\right) d\alpha$$

(see **3.241** 6)

6.612

$$1. \quad \int_0^\infty e^{-2\alpha x} J_0(x) Y_0(x) dx = \frac{\mathbf{K} \left[\alpha (\alpha^2 + 1)^{-\frac{1}{2}} \right]}{\pi (\alpha^2 + 1)^{\frac{1}{2}}} \quad [\operatorname{Re} \alpha > 0] \quad \text{ET II 347(58)}$$

$$2. \quad \int_0^\infty e^{-2\alpha x} I_0(x) K_0(x) dx = \frac{1}{2} \mathbf{K} \left[(1 - \alpha^2)^{\frac{1}{2}} \right] \quad [0 < \alpha < 1]$$

$$= \frac{1}{2\alpha} \mathbf{K} \left[\left(1 - \frac{1}{\alpha^2}\right)^{\frac{1}{2}} \right] \quad [1 < \alpha < \infty]$$

ET II 370(48)

$$3. \quad \int_0^\infty e^{-\alpha x} J_\nu(\beta x) J_\nu(\gamma x) dx = \frac{1}{\pi \sqrt{\gamma \beta}} Q_{\nu-\frac{1}{2}} \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2\beta\gamma} \right)$$

[$\operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 0$, $\gamma > 0$, $\operatorname{Re} \nu > -\frac{1}{2}$] WA 426(2), ET II 50(17)

$$4. \quad \int_0^\infty e^{-\alpha x} [J_0(\beta x)]^2 dx = \frac{2}{\pi \sqrt{\alpha^2 + 4\beta^2}} \mathbf{K} \left(\frac{2\beta}{\sqrt{\alpha^2 + 4\beta^2}} \right) \quad \text{MO 178}$$

$$5. \quad \int_0^\infty e^{-2\alpha x} J_1^2(\beta x) dx = \frac{(2\alpha^2 + \beta^2) \mathbf{K} \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) - 2(\alpha^2 + \beta^2) \mathbf{E} \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right)}{\pi \beta^2 \sqrt{\alpha^2 + \beta^2}} \quad \text{WA 428(3)}$$

$$6. \quad \int_0^\infty e^{-3x} I_l(x) I_m(x) I_n(x) dx = r_1 g + \frac{r_2}{\pi^2 g} + r_3$$

where

$$g = \frac{\sqrt{3}-1}{96\pi^3} \Gamma^2 \left(\frac{1}{24} \right) \Gamma^2 \left(\frac{11}{24} \right)$$

and

(lmn)	r_1	r_2	r_3	(lmn)	r_1	r_2	r_3
000	1	0	0	432	$525/32$	$-4617/112$	0
100	1	0	$-1/3$	433	$-595/72$	$8809/420$	0
110	$5/12$	$-1/2$	0	440	$6025/36$	$-620161/1470$	0
111	$-1/8$	$3/4$	0	441	$-29175/224$	$131379/400$	0
200	$10/3$	2	-2	442	$2975/48$	$-31231/200$	0
210	$3/8$	$-9/4$	$1/3$	443	$-539/32$	$119271/2800$	0
211	$-2/3$	2	0	444	$77/8$	$-186003/7700$	0
220	$73/36$	$-29/6$	0	500	$9287/12$	$3005/2$	$-2077/3$
221	$-15/16$	$21/8$	0	510	$-189029/180$	$-138331/50$	348
222	$5/8$	$-27/20$	0	511	$275/4$	$5751/10$	-150
300	$35/2$	21	-13	520	$2897/16$	$-15123/20$	$-229/3$
310	$-79/36$	$-85/6$	4	521	$-937/12$	$27059/30$	24
311	$-11/4$	$21/2$	$-2/3$	522	$509/8$	$-4209/28$	0
320	$319/48$	$-119/8$	$-1/3$	530	$3589/18$	$-1993883/3075$	0
321	$-125/36$	$269/30$	0	531	$-1329/8$	$297981/700$	$-4/3$
322	$35/16$	$-213/40$	0	532	$2555/36$	$-187777/1050$	0
330	$50/3$	$-1046/25$	0	533	$-2233/48$	$164399/1400$	0
331	$-35/3$	$148/5$	0	540	$18471/32$	$-28493109/19600$	$-1/3$
332	$35/9$	$-1012/105$	0	541	$-1390/3$	$286274/245$	0
333	$-35/16$	$1587/280$	0	542	$7777/32$	$-1715589/2800$	0
400	$994/9$	$542/3$	-92	543	$-5621/72$	$4550057/23100$	0
410	$-515/16$	$-879/8$	$115/3$	544	$1155/32$	$-560001/6160$	0
411	$-9/2$	$357/5$	-12	550	$197045/108$	$-101441689/22050$	0
420	$12907/120$	$-13903/10$	-6	551	$-12023/8$	$18569853/4900$	0
421	$-229/16$	$1251/40$	1	552	$1683/2$	$-5718309/2695$	0
422	$35/3$	$-1024/35$	0	553	$-5159/16$	$2504541/3080$	0
430	$2641/48$	$-28049/200$	$1/3$	554	$24563/312$	$-1527851/77000$	0
431	$-1505/36$	$118051/1050$	0	555	$-9251/208$	$12099711/107800$	0

$$6.613^{11} \int_0^\infty e^{-xz} J_{\nu+\frac{1}{2}} \left(\frac{x^2}{2} \right) dx = \frac{\Gamma(\nu+1)}{\sqrt{\pi}} D_{-\nu-1} \left(ze^{\frac{\pi}{4}i} \right) D_{-\nu-1} \left(ze^{-\frac{\pi}{4}i} \right) \quad [\operatorname{Re} \nu > -1] \quad \text{MO 122}$$

6.614

$$1. \int_0^\infty e^{-\alpha x} J_\nu (\beta \sqrt{x}) dx = \frac{\beta}{4} \sqrt{\frac{\pi}{\alpha^3}} \exp \left(-\frac{\beta^2}{8\alpha} \right) \left[I_{\frac{1}{2}(\nu-1)} \left(\frac{\beta^2}{8\alpha} \right) - I_{\frac{1}{2}(\nu+1)} \left(\frac{\beta^2}{8\alpha} \right) \right] \\ = \frac{1}{\alpha} e^{-\beta^2/4\alpha} \quad [\nu = 0]$$

MO 178

$$2. \int_0^\infty e^{-\alpha x} Y_{2\nu} (2\sqrt{\beta x}) dx = \frac{e^{-\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \left\{ \cot(\nu\pi) \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{\frac{1}{2},\nu} \left(\frac{\beta}{\alpha} \right) - \operatorname{cosec}(\nu\pi) W_{\frac{1}{2},\nu} \left(\frac{\beta}{\alpha} \right) \right\} \\ [\operatorname{Re} \alpha > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 188(50)a}$$

$$3. \int_0^\infty e^{-\alpha x} I_{2\nu} (2\sqrt{\beta x}) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{-\frac{1}{2},\nu} \left(\frac{\beta}{\alpha} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 197(20)a}$$

$$4. \quad \int_0^\infty e^{-\alpha x} K_{2\nu} \left(2\sqrt{\beta x} \right) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{2\sqrt{\alpha\beta}} \Gamma(\nu+1) \Gamma(1-\nu) W_{-\frac{1}{2}, \nu} \left(\frac{\beta}{\alpha} \right) \\ [\operatorname{Re} \alpha > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 199(37)a}$$

$$5. \quad \int_0^\infty e^{-\alpha x} K_1 \left(\beta\sqrt{x} \right) dx = \frac{\beta}{8} \sqrt{\frac{\pi}{\alpha^3}} \exp \left(\frac{\beta^2}{8\alpha} \right) \left[K_1 \left(\frac{\beta^2}{8\alpha} \right) - K_0 \left(\frac{\beta^2}{8\alpha} \right) \right] \quad \text{MO 181}$$

$$6.615 \quad \int_0^\infty e^{-\alpha x} J_\nu \left(2\beta\sqrt{x} \right) J_\nu \left(2\gamma\sqrt{x} \right) dx = \frac{1}{\alpha} I_\nu \left(\frac{2\beta\gamma}{\alpha} \right) \exp \left(-\frac{\beta^2 + \gamma^2}{\alpha} \right) \quad [\operatorname{Re} \nu > -1] \\ \text{MO 178}$$

6.616

$$1. \quad \int_0^\infty e^{-\alpha x} J_0 \left(\beta\sqrt{x^2 + 2\gamma x} \right) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \exp \left[\gamma \left(\alpha - \sqrt{\alpha^2 + \beta^2} \right) \right] \quad \text{MO 179}$$

$$2. \quad \int_1^\infty e^{-\alpha x} J_0 \left(\beta\sqrt{x^2 - 1} \right) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \exp \left(-\sqrt{\alpha^2 + \beta^2} \right) \quad \text{MO 179}$$

$$3. \quad \int_{-\infty}^\infty e^{itx} H_0^{(1)} \left(r\sqrt{\alpha^2 - t^2} \right) dt = -2i \frac{e^{i\alpha\sqrt{r^2 + x^2}}}{\sqrt{r^2 + x^2}} \\ \left[0 \leq \arg \sqrt{\alpha^2 - t^2} < \pi, \quad 0 \leq \arg \alpha < \pi; \quad r \text{ and } x \text{ are real} \right] \quad \text{MO 49}$$

$$4. \quad \int_{-\infty}^\infty e^{-itx} H_0^{(2)} \left(r\sqrt{\alpha^2 - t^2} \right) dt = 2i \frac{e^{-i\alpha\sqrt{r^2 + x^2}}}{\sqrt{r^2 + x^2}} \\ \left[-\pi < \arg \sqrt{\alpha^2 - t^2} \leq 0, \quad -\pi < \arg \alpha \leq 0, \quad r \text{ and } x \text{ are real} \right] \quad \text{MO 49}$$

$$5.^3 \quad \int_{-1}^1 e^{-ax} I_0 \left(b\sqrt{1-x^2} \right) dx = 2 \left(a^2 + b^2 \right)^{-1/2} \sinh \sqrt{a^2 + b^2} \\ [a > 0, \quad b > 0]$$

$$6.^8 \quad \int_0^\infty e^{-xy} J_0 \left[y\sqrt{1-x^2} \right] / (\alpha + y) dy = \sum_{n=0}^\infty n! \frac{P_n(x)}{\alpha^{n+1}}$$

6.617

$$1. \quad \int_0^\infty K_{q-p} (2z \sinh x) e^{(p+q)x} dx = \frac{\pi^2}{4 \sin[(p-q)\pi]} [J_p(z) Y_q(z) - J_q(z) Y_p(z)] \\ [\operatorname{Re} z > 0, \quad -1 < \operatorname{Re}(p-q) < 1] \quad \text{MO 44}$$

$$2. \quad \int_0^\infty K_0 (2z \sinh x) e^{-2px} dx = -\frac{\pi}{4} \left\{ J_p(z) \frac{\partial Y_p(z)}{\partial p} - Y_p(z) \frac{\partial J_p(z)}{\partial p} \right\} \\ [\operatorname{Re} z > 0] \quad \text{MO 44}$$

6.618

$$1. \quad \int_0^\infty e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp \left(-\frac{\beta^2}{8\alpha} \right) I_{\frac{1}{2}\nu} \left(\frac{\beta^2}{8\alpha} \right) \quad [\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1] \\ \text{WA 432(5), ET II 29(8)}$$

2.
$$\int_0^\infty e^{-\alpha x^2} Y_\nu(\beta x) dx = -\frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[\tan \frac{\nu\pi}{2} I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \right]$$

[Re $\alpha > 0$, $\beta > 0$, $|\operatorname{Re} \nu| < 1$]
WA 432(6), ET II 106(3)
3.
$$\int_0^\infty e^{-\alpha x^2} K_\nu(\beta x) dx = \frac{1}{4} \sec\left(\frac{\nu\pi}{2}\right) \frac{\sqrt{\pi}}{\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)$$

[Re $\alpha > 0$, $|\operatorname{Re} \nu| < 1$]
EH II 51(28), ET II 132(24)
4.
$$\int_0^\infty e^{-\alpha x^2} I_\nu(\beta x) dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \quad [\operatorname{Re} \nu > -1, \operatorname{Re} \alpha > 0] \quad \text{EH II 92(27)}$$
5.
$$\int_0^\infty e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\beta x) dx$$
$$= 2^{-\nu-\mu-1} \alpha^{-\frac{\nu+\mu+1}{2}} \beta^{\nu+\mu} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\Gamma(\nu+1)}$$
$$\times {}_3F_3\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \mu+1, \nu+1, \nu+\mu+1; -\frac{\beta^2}{\alpha}\right)$$

[Re($\nu+\mu$) > -1 , $\operatorname{Re} \alpha > 0$] EH II 50(21)a

6.62–6.63 Combinations of Bessel functions, exponentials, and powers

6.621 Notation:

$$\ell_1 = \frac{1}{2} \left[\sqrt{(a+\rho)^2 + z^2} - \sqrt{(a-\rho)^2 + z^2} \right], \quad \ell_2 = \frac{1}{2} \left[\sqrt{(a+\rho)^2 + z^2} + \sqrt{(a-\rho)^2 + z^2} \right]$$

1.
$$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\mu-1} dx$$
$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$$

WA 421(2)

$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} \left(1 + \frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}-\mu} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2} + 1; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$$

WA 421(3)

$$= \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{(\alpha^2 + \beta^2)^{\nu+\mu}} \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{\beta^2}{\alpha^2 + \beta^2}\right)$$

[Re($\nu+\mu$) > 0 , $\operatorname{Re}(\alpha + i\beta) > 0$, $\operatorname{Re}(\alpha - i\beta) > 0$]
WA 421(3)

$$= (\alpha^2 + \beta^2)^{-\frac{1}{2}\mu} \Gamma(\nu+\mu) P_{\mu-1}^{-\nu} \left[\alpha (\alpha^2 + \beta^2)^{-\frac{1}{2}} \right]$$

[$\alpha > 0$, $\beta > 0$, $\operatorname{Re}(\nu+\mu) > 0$]
ET II 29(6)

$$\begin{aligned}
2. \quad & \int_0^\infty e^{-\alpha x} Y_\nu(\beta x) x^{\mu-1} dx \\
&= \cot \nu \pi \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu + \mu)}{\sqrt{(\alpha^2 + \beta^2)^{\nu+\mu}} \Gamma(\nu + 1)} F\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu + 1}{2}; \nu + 1; \frac{\beta^2}{\alpha^2 + \beta^2}\right) \\
&\quad - \operatorname{cosec} \nu \pi \frac{\left(\frac{\beta}{2}\right)^{-\nu} \Gamma(\mu - \nu)}{\sqrt{(\alpha^2 + \beta^2)^{\mu-\nu}} \Gamma(1 - \nu)} F\left(\frac{\mu - \nu}{2}, \frac{1 - \nu - \mu}{2}; 1 - \nu; \frac{\beta^2}{\alpha^2 + \beta^2}\right) \\
&\quad [\operatorname{Re} \mu \geq |\operatorname{Re} \nu|, \quad \operatorname{Re}(\alpha \pm i\beta) > 0] \\
&\quad \text{WA 421(4)} \\
&= -\frac{2}{\pi} \Gamma(\nu + \mu) (\beta^2 + \alpha^2)^{-\frac{1}{2}\mu} Q_{\mu-1}^{-\nu} \left[\alpha (\alpha^2 + \beta^2)^{-\frac{1}{2}} \right] \\
&\quad [\alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu|] \\
&\quad \text{ET II 105(2)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\mu-1} e^{-\alpha x} K_\nu(\beta x) dx = \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha + \beta)^{\mu+\nu}} \frac{\Gamma(\mu + \nu) \Gamma(\mu - \nu)}{\Gamma(\mu + \frac{1}{2})} F\left(\mu + \nu, \nu + \frac{1}{2}; \mu + \frac{1}{2}; \frac{\alpha - \beta}{\alpha + \beta}\right) \\
&\quad [\operatorname{Re} \mu > |\operatorname{Re} \nu|, \quad \operatorname{Re}(\alpha + \beta) > 0] \\
&\quad \text{ET II 131(23)a, EH II 50(26)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{m+1} e^{-\alpha x} J_\nu(\beta x) dx = (-1)^{m+1} \beta^{-\nu} \frac{d^{m+1}}{d\alpha^{m+1}} \left[\frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^\nu}{\sqrt{\alpha^2 + \beta^2}} \right] \\
&\quad [\beta > 0, \quad \operatorname{Re} \nu > -m - 2] \quad \text{ET II 28(3)}
\end{aligned}$$

$$\begin{aligned}
5.^{10} \quad & \int_0^\infty e^{-zx} J_1(ax) J_{1/2}(\rho x) x^{-3/2} dx \\
&= \frac{1}{a} \sqrt{\frac{2}{\pi \rho}} \left\{ \frac{\ell_1}{2} \sqrt{a^2 - \ell_1^2} + \frac{a^2}{2} \arcsin\left(\frac{\ell_1}{2}\right) + z \left[\sqrt{\rho^2 - \ell_1^2} - \rho \right] \right\} \\
&\quad [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]
\end{aligned}$$

$$\begin{aligned}
6.^{10} \quad & \int_0^\infty e^{-zx} J_1(ax) J_{1/2}(\rho x) x^{-1/2} dx = \frac{1}{a} \sqrt{\frac{2}{\pi \rho}} \left[\rho - \sqrt{\rho^2 - \ell_1^2} \right] \\
&\quad [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]
\end{aligned}$$

$$\begin{aligned}
7.^{10} \quad & \int_0^\infty e^{-zx} J_1(ax) J_{1/2}(\rho x) x^{1/2} dx = \frac{1}{a} \sqrt{\frac{2}{\pi \rho}} \frac{\ell_1 \sqrt{a^2 - \ell_1^2}}{\ell_2^2 - \ell_1^2} \\
&\quad [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]
\end{aligned}$$

$$\begin{aligned}
8.^{10} \quad & \int_0^\infty e^{-zx} J_1(ax) J_{3/2}(\rho x) x^{1/2} dx = \sqrt{\frac{2}{\pi}} \frac{\ell_1^2 \sqrt{\rho^2 - \ell_1^2}}{\rho^{3/2} a (\ell_2^2 - \ell_1^2)} \\
&\quad [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]
\end{aligned}$$

$$\begin{aligned}
9.^{10} \quad & \int_0^\infty e^{-zx} J_1(ax) J_{3/2}(\rho x) x^{-3/2} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{3/2} a} \left[a^2 \arcsin\left(\frac{\ell_1}{a}\right) - \ell_1 \sqrt{a^2 - \ell_1^2} \right] \\
&\quad [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]
\end{aligned}$$

$$10.^{10} \quad \int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-1/2} dx = \frac{1}{\sqrt{2\pi}} \frac{z}{\rho^{5/2} a} \left[\ell_1 \sqrt{a^2 - \ell_1^2} + \frac{2a^2 \ell_1}{\sqrt{a^2 - \ell_1^2}} - 3a^2 \arcsin\left(\frac{\ell_1}{a}\right) \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$11.^{10} \quad \int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-3/2} dx \\ = \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{5/2} a} \left[\frac{\ell_1}{\sqrt{a^2 - \ell_1^2}} \left(\frac{7a^2}{8} - a^2 z^2 - \frac{\ell_1^4}{4} - \frac{5a^2 \ell_1^2}{8} \right) \right. \\ \left. - \frac{1}{2} (\ell_1^2 + \ell_2^2) \ell_1 \sqrt{a^2 - \ell_1^2} + \arcsin\left(\frac{\ell_1}{a}\right) \left(\frac{3}{2} a^2 z^2 + \frac{1}{2} a^2 \rho^2 - \frac{3a^4}{8} \right) \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$12.^{10} \quad \int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-5/2} dx \\ = \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{5/2} a} \left\{ \frac{2 \left[\rho^{5/2} - (\rho^2 - \ell_1^2)^{5/2} \right]}{15} + z a^2 \arcsin\left(\frac{\ell_1}{a}\right) \left[\frac{3a^2}{8} - \frac{\rho^2}{2} - \frac{z^2}{2} \right] \right. \\ \left. + z \ell_1 \sqrt{a^2 - \ell_1^2} \left[\frac{\rho^2}{2} - \frac{3a^2}{8} + \frac{z^2}{6} - \frac{\ell_1^2}{4} \right] + \frac{z^3 a^2 \ell_1}{3 \sqrt{a^2 - \ell_1^2}} \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$13.^{10} \quad \int_0^\infty e^{-zx} J_2(ax) J_{3/2}(\rho x) x^{1/2} dx = \sqrt{\frac{2}{\pi}} a^2 \rho^{3/2} \frac{\sqrt{\ell_2^2 - \rho^2}}{(\ell_2^2 - \ell_1^2) \ell_2^4} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$14.^{10} \quad \int_0^\infty e^{-zx} J_2(ax) J_{3/2}(\rho x) x^{-1/2} dx = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{a^2} \left[\frac{2}{3} - \frac{\sqrt{\rho^2 - \ell_1^2}}{\rho} + \frac{(\rho^2 - \ell_1^2)^{3/2}}{3\rho^3} \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$15.^{10} \quad \int_0^\infty e^{-zx} J_3(ax) J_{1/2}(\rho x) x^{-1/2} dx \\ = \sqrt{\frac{2}{\pi}} \frac{1}{3a^3} \left\{ \rho [3a^2 - 4\rho^2 + 12z^2] - \sqrt{\rho^2 - \ell_1^2} \{ 12\ell_2^2 - 16\rho^2 + 4\ell_1^2 - 3a^2 \} \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$16.^{10} \quad \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{1/2} dx \\ = \sqrt{\frac{2}{\pi}} \rho^{3/2} \left\{ \frac{4}{a^3} \left[\frac{2}{3} - \frac{\sqrt{\rho^2 - \ell_1^2}}{\rho} + \frac{(\rho^2 - \ell_1^2)^{3/2}}{3\rho^2} \right] - \frac{a \sqrt{\ell_2^2 - a^2}}{(\ell_2^2 - \ell_1^2) \ell_2^3} \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$17.^{10} \quad \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-1/2} dx = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left[\sqrt{\ell_2^2 - \rho^2} \left(\frac{4\rho^2 (2\rho^2 - \ell_1^2) - \ell_1^4}{\rho^4} \right) - 8z \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$18.^{10} \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-3/2} dx \\ = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left\{ a^2 - \frac{4}{5}\rho^2 + 4z^2 - \sqrt{\rho^2 - \ell_1^2} \left[\frac{4\ell_2^2}{\rho} - \frac{24\rho}{5} + \frac{8\ell_1^2}{5\rho} - \frac{a^2}{\rho} + \frac{\ell_1^4}{5\rho^3} \right] \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$19.^{10} \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-5/2} dx \\ = -\sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left\{ \left(a^2 - \frac{4}{5}\rho^2 \right) z + \frac{4z^3}{3} \right. \\ \left. + \sqrt{\ell_2^2 - \rho^2} \left[a^2 + \frac{32}{15}\rho^2 - \frac{12}{5}\ell_1^2 - \frac{4}{3}\ell_2^2 + \frac{2\ell_1^4}{5\rho^2} + \frac{a^4\ell_1^2}{16\rho^4} + \frac{a^2\ell_1^2}{24\rho^4} + \frac{\ell_1^6}{30\rho^4} \right] \right. \\ \left. - \frac{a^6}{16\rho^3} \arcsin\left(\frac{\rho}{\ell_2}\right) \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

6.622

$$1. \int_0^\infty (J_0(x) - e^{-\alpha x}) \frac{dx}{x} = \ln 2\alpha \quad [\alpha > 0] \quad \text{NT 66(13)}$$

$$2. \int_0^\infty \frac{e^{i(u+x)}}{u+x} J_0(x) dx = \frac{\pi}{2} i H_0^{(1)}(u) \quad \text{MO 44}$$

$$3.^8 \int_0^\infty e^{-x \cosh \alpha} I_\nu(x) x^{\mu-1} dx = \sqrt{\frac{2}{\pi}} e^{-(\mu-\frac{1}{2})\pi i} \frac{Q_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(\cosh \alpha)}{\sinh^{\mu-\frac{1}{2}} \alpha} \\ [\operatorname{Re}(\mu + \nu) > 0, \quad \operatorname{Re}(\cosh \alpha) > 1] \quad \text{WA 388(6)a}$$

6.623

$$1. \int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^\nu dx = \frac{(2\beta)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\alpha^2 + \beta^2)^{\nu+\frac{1}{2}}} \quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|] \\ \text{WA 422(5)}$$

$$2. \int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\nu+1} dx = \frac{2\alpha(2\beta)^\nu \Gamma(\nu + \frac{3}{2})}{\sqrt{\pi} (\alpha^2 + \beta^2)^{\nu+\frac{3}{2}}} \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|] \\ \text{WA 422(6)}$$

$$3. \int_0^\infty e^{-\alpha x} J_\nu(\beta x) \frac{dx}{x} = \frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^\nu}{\nu \beta^\nu} \\ [\operatorname{Re} \nu > 0; \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad (\text{cf. 6.611 1}) \quad \text{WA 422(7)}$$

6.624

$$1. \int_0^\infty x e^{-\alpha x} K_0(\beta x) dx = \frac{1}{\alpha^2 - \beta^2} \left\{ \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \ln \left[\frac{\alpha}{\beta} + \sqrt{\left(\frac{\alpha}{\beta}\right)^2 - 1} \right] - 1 \right\} \quad \text{MO 181}$$

$$2. \quad \int_0^\infty \sqrt{x} e^{-\alpha x} K_{\pm \frac{1}{2}}(\beta x) dx = \sqrt{\frac{\pi}{2\beta}} \frac{1}{\alpha + \beta} \quad \text{MO 181}$$

$$3. \quad \int_0^\infty e^{-tz(z^2-1)^{-1/2}} K_\mu(t) t^\nu dt = \frac{\Gamma(\nu - \mu + 1)}{(z^2 - 1)^{-\frac{1}{2}(\nu+1)}} e^{i\mu\pi} Q_\nu^\mu(z) \\ [\operatorname{Re}(\nu \pm \mu) > -1] \quad \text{EH II 57(7)}$$

$$4. \quad \int_0^\infty e^{-tz(z^2-1)^{-1/2}} I_{-\mu}(t) t^\nu dt = \frac{\Gamma(-\nu - \mu)}{(z^2 - 1)^{\frac{1}{2}\nu}} P_\nu^\mu(z) \quad [\operatorname{Re}(\nu + \mu) < 0] \quad \text{EH II 57(8)}$$

$$5. \quad \int_0^\infty e^{-tz(z^2-1)^{-1/2}} I_\mu(t) t^\nu dt = \frac{\Gamma(\nu + \mu + 1)}{(z^2 - 1)^{-\frac{1}{2}(\nu+1)}} P_\nu^{-\mu}(z) \\ [\operatorname{Re}(\nu + \mu) > -1] \quad \text{EH II 57(9)}$$

$$6. \quad \int_0^\infty e^{-t \cos \theta} J_\mu(t \sin \theta) t^\nu dt = \Gamma(\nu + \mu + 1) P_\nu^{-\mu}(\cos \theta) \\ [\operatorname{Re}(\nu + \mu) > -1, \quad 0 \leq \theta < \frac{1}{2}\pi] \quad \text{EH II 57(10)}$$

$$7. \quad \int_0^\infty \frac{J_\nu(bx) x^\nu}{e^{\pi x} - 1} dx = \frac{(2b)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} \sum_{n=1}^\infty \frac{1}{(n^2 \pi^2 + b^2)^{\nu + \frac{1}{2}}} \\ [\operatorname{Re} \nu > 0, \quad |\operatorname{Im} b| < \pi] \quad \text{WA 423(9)}$$

6.625

$$1. \quad \int_0^1 x^{\lambda-\nu-1} (1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x) dx = \frac{2^{-\nu} \alpha^\nu \Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda + \mu) \Gamma(\nu + 1)} {}_2F_2 \left(\lambda, \nu + \frac{1}{2}; \lambda + \mu, 2\nu + 1; \pm 2i\alpha \right) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 194(58)a}$$

$$2. \quad \int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x) dx = \frac{(2\alpha)^\nu \Gamma(\mu) \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu + 2\nu + 1)} {}_1F_1 \left(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2i\alpha \right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 194(57)a}$$

$$3. \quad \int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm \alpha x} J_\nu(\alpha x) dx = \frac{(2\alpha)^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\mu)}{\sqrt{\pi} \Gamma(\mu + 2\nu + 1)} {}_1F_1 \left(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2\alpha \right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{BU 9(16a), ET II 197(77)a}$$

$$4. \quad \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} e^{\pm \alpha x} I_\nu(\alpha x) dx = \frac{(\frac{1}{2}\alpha)^\nu \Gamma(\lambda + \nu) \Gamma(\mu)}{\Gamma(\nu + 1) \Gamma(\lambda + \mu + \nu)} \\ \times {}_2F_2 \left(\nu + \frac{1}{2}, \lambda + \nu; 2\nu + 1, \mu + \lambda + \nu; \pm 2\alpha \right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\lambda + \nu) > 0] \quad \text{ET II 197(78)a}$$

$$5. \quad \int_0^1 x^{\mu-\kappa} (1-x)^{2\kappa-1} I_{\mu-\kappa} \left(\frac{1}{2} xz \right) e^{-\frac{1}{2} xz} dx = \frac{\Gamma(2\kappa)}{\sqrt{\pi} \Gamma(1 + 2\mu)} e^{\frac{\pi}{2}} z^{-\kappa - \frac{1}{2}} M_{\kappa, u}(z) \\ [\operatorname{Re}(\kappa - \frac{1}{2} - \mu) < 0, \quad \operatorname{Re} \kappa > 0] \quad \text{BU 129(14a)}$$

$$6. \quad \int_1^\infty x^{-\lambda} (x-1)^{\mu-1} e^{-\alpha x} I_\nu(\alpha x) dx = \frac{(2\alpha)^\lambda \Gamma(\mu)}{\sqrt{\pi}} G_{23}^{21} \left(2\alpha \left| \begin{matrix} \frac{1}{2} - \lambda, 0 \\ -\mu, \nu - \lambda, -\nu - \lambda \end{matrix} \right. \right) \\ [0 < \operatorname{Re} \mu < \tfrac{1}{2} + \operatorname{Re} \lambda, \quad \operatorname{Re} \alpha > 0] \quad \text{ET II 207(50)a}$$

$$7. \quad \int_1^\infty x^{-\lambda} (x-1)^{\mu-1} e^{-\alpha x} K_\nu(\alpha x) dx = \Gamma(\mu) \sqrt{\pi} (2\alpha)^\lambda G_{23}^{30} \left(2\alpha \left| \begin{matrix} 0, \frac{1}{2} - \lambda \\ -\mu, \nu - \lambda, -\nu - \lambda \end{matrix} \right. \right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \alpha > 0] \quad \text{ET II 208(55)a}$$

$$8. \quad \int_1^\infty x^{-\nu} (x-1)^{\mu-1} e^{-\alpha x} I_\nu(\alpha x) dx = \frac{(2\alpha)^{\nu-\mu} \Gamma(\tfrac{1}{2} - \mu + \nu) \Gamma(\mu)}{\sqrt{\pi} \Gamma(1 - \mu + 2\nu)} \\ \times {}_1F_1 \left(\tfrac{1}{2} - \mu + \nu; 1 - \mu + 2\nu; -2\alpha \right) \\ [0 < \operatorname{Re} \mu < \tfrac{1}{2} + \operatorname{Re} \nu, \quad \operatorname{Re} \alpha > 0] \quad \text{ET II 207(49)a}$$

$$9. \quad \int_1^\infty x^{-\nu} (x-1)^{\mu-1} e^{-\alpha x} K_\nu(\alpha x) dx = \sqrt{\pi} \Gamma(\mu) (2\alpha)^{-\frac{1}{2}\mu - \frac{1}{2}} e^{-\alpha} W_{-\frac{1}{2}\mu, \nu - \frac{1}{2}\mu}(2\alpha) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \alpha > 0] \quad \text{ET II 208(53)a}$$

$$10. \quad \int_1^\infty x^{-\mu - \frac{1}{2}(x-1)\mu-1} e^{-\alpha x} K_\nu(\alpha x) dx = \sqrt{\pi} \Gamma(\mu) (2\alpha)^{-\frac{1}{2}} e^{-\alpha} W_{-\mu, \nu}(2\alpha) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \alpha > 0] \quad \text{ET II 207(51)a}$$

$$11.^3 \quad \int_{-1}^1 (1-x^2)^{-1/2} x e^{-ax} I_1(b\sqrt{1-x^2}) dx = \frac{2}{b} \left\{ \sinh a - a (a^2 + b^2)^{-1/2} \sinh \sqrt{a^2 + b^2} \right\} \\ [a > 0, \quad b > 0]$$

6.626

$$1.^{11} \quad \int_0^\infty x^{\lambda-1} e^{-\alpha x} J_\mu(\beta x) J_\nu(\gamma x) dx = \frac{\beta^\mu \gamma^\nu}{\Gamma(\nu+1)} 2^{-\nu-\mu} \alpha^{-\lambda-\mu-\nu} \sum_{m=0}^\infty \frac{\Gamma(\lambda + \mu + \nu + 2m)}{m! \Gamma(\mu + m + 1)} \\ \times F \left(-m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2} \right) \left(-\frac{\beta^2}{4\alpha^2} \right)^m \\ [\operatorname{Re}(\lambda + \mu + \nu) > 0, \quad \operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 1] \quad \text{EH II 48(15)}$$

$$2. \quad \int_0^\infty e^{-2\alpha x} J_\nu(\beta x) J_\mu(\beta x) x^{\nu+\mu} dx = \frac{\Gamma(\nu + \mu + \tfrac{1}{2}) \beta^{\nu+\mu}}{\sqrt{\pi^3}} \\ \times \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu+\mu} \varphi \cos(\nu - \mu) \varphi}{(\alpha^2 + \beta^2 \cos^2 \varphi)^{\nu+\mu} \sqrt{\alpha^2 + \beta^2 \cos^2 \varphi}} d\varphi \\ [\operatorname{Re} \alpha > |\operatorname{Im} \beta|, \quad \operatorname{Re}(\nu + \mu) > -\tfrac{1}{2}] \quad \text{WA 427(1)}$$

$$3. \quad \int_0^\infty e^{-2\alpha x} J_0(\beta x) J_1(\beta x) x dx = \frac{\mathbf{K} \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) - \mathbf{E} \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right)}{2\pi\beta\sqrt{\alpha^2 + \beta^2}} \quad \text{WA 427(2)}$$

$$4. \quad \int_0^\infty e^{-2\alpha x} I_0(\beta x) I_1(\beta x) x dx = \frac{1}{2\pi\beta} \left\{ \frac{\alpha}{\alpha^2 - \beta^2} \mathbf{E} \left(\frac{\beta}{\alpha} \right) - \frac{1}{\alpha} \mathbf{K} \left(\frac{\beta}{\alpha} \right) \right\} \\ [\operatorname{Re} \alpha > \operatorname{Re} \beta] \quad \text{WA 428(5)}$$

$$\begin{aligned}
5.10 \quad \int_0^\infty x^{\nu-\mu+2n} e^{-zx} J_\mu(\alpha x) J_\nu(\rho x) dx &= \frac{1}{\sqrt{\pi}} \left(\frac{a}{2}\right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a}\right)^\nu \\
&\times \frac{1}{\Gamma(\mu-\nu-n+\frac{1}{2})} \sum_{q=0}^\infty \frac{\Gamma(\nu+n+q+\frac{1}{2}) (\nu-\mu+n+\frac{1}{2})_q}{q! \Gamma(\nu+q+\frac{1}{2})} \\
&\times a^{-2q} \int_0^{\ell_1/\rho} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2}\right)^q \\
\text{where } \ell_1 &= \frac{1}{2} \left[\sqrt{(a+\rho)^2 + z^2} - \sqrt{(a-\rho)^2 + z^2} \right] \quad [\mu > \nu + 2n, \quad n = 0, 1, \dots, \quad \nu > -\frac{1}{2}]
\end{aligned}$$

$$6.627 \quad \int_0^\infty \frac{x^{-1/2}}{x+a} e^{-x} K_\nu(x) dx = \frac{\pi e^a K_\nu(a)}{\sqrt{a} \cos(\nu\pi)} \quad [|\arg a| < \pi, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 368(29)}$$

6.628

$$\begin{aligned}
1. \quad \int_0^\infty e^{-x \cos \beta} J_{-\nu}(x \sin \beta) x^\mu dx &= \Gamma(\mu - \nu + 1) P_\mu^\nu(\cos \beta) \\
&[0 < \beta < \tfrac{\pi}{2}, \quad \operatorname{Re}(\mu - \nu) > -1] \\
&\text{WA 424(3), WH}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty e^{-x \cos \beta} Y_\nu(x \sin \beta) x^\mu dx &= -\frac{\sin \mu \pi}{\sin(\mu + \nu) \pi} \frac{\Gamma(\mu - \nu + 1)}{\pi} \\
&\times \left[Q_\mu^\nu(\cos \beta + 0 \cdot i) e^{\frac{1}{2}\nu\pi i} + Q_\mu^\nu(\cos \beta - 0 \cdot i) e^{-\frac{1}{2}\nu\pi i} \right] \\
&[\operatorname{Re}(\mu + \nu) > -1, \quad 0 < \beta < \tfrac{\pi}{2}] \quad \text{WA 424(4)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^1 e^{\frac{xu}{2}} (1-x)^{2\nu-1} x^{\mu-\nu} J_{\mu-\nu}\left(\frac{ixu}{2}\right) dx &= 2^{2(\nu-\mu)} e^{\frac{\pi}{2}(\mu-\nu)i} \frac{B(2\nu, 2\mu-2\nu+1)}{\Gamma(\mu-\nu+1)} \frac{e^{\frac{u}{2}}}{u^{\nu+\frac{1}{2}}} M_{\nu,\mu}(u) \\
&\text{MO 118a}
\end{aligned}$$

$$\begin{aligned}
4.8 \quad \int_0^\infty e^{-x \cosh \alpha} I_\nu(x \sinh \alpha) x^\mu dx &= \Gamma(\nu + \mu + 1) P_\mu^{-\nu}(\cosh \alpha) \\
&[\operatorname{Re}(\mu + \nu) > -1, \quad |\operatorname{Im} \alpha| < \tfrac{1}{2}\pi] \\
&\text{WA 423(1)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty e^{-x \cosh \alpha} K_\nu(x \sinh \alpha) x^\mu dx &= \frac{\sin \mu \pi}{\sin(\nu + \mu) \pi} \Gamma(\mu - \nu + 1) Q_\mu^\nu(\cosh \alpha) \\
&[\operatorname{Re}(\mu + 1) > |\operatorname{Re} \nu|] \quad \text{WA 423(2)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty e^{-x \cosh \alpha} I_\nu(x) x^{\mu-1} dx &= \frac{\cos \nu \pi}{\sin(\mu + \nu) \pi} \frac{Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh \alpha)}{\sqrt{\frac{\pi}{2}} (\sinh \alpha)^{\mu-\frac{1}{2}}} \\
&[\operatorname{Re}(\mu + \nu) > 0, \quad \operatorname{Re}(\cosh \alpha) > 1] \\
&\text{WA 424(6)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty e^{-x \cosh \alpha} K_\nu(x) x^{\mu-1} dx &= \sqrt{\frac{\pi}{2}} \Gamma(\mu - \nu) \Gamma(\mu + \nu) \frac{P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh \alpha)}{(\sinh \alpha)^{\mu-\frac{1}{2}}} \\
&[\operatorname{Re} \mu > |\operatorname{Re} \nu|, \quad \operatorname{Re}(\cosh \alpha) > -1] \\
&\text{WA 424(7)}
\end{aligned}$$

$$\begin{aligned}
6.629^8 \int_0^\infty x^{-1/2} e^{-x\alpha \cos \varphi \cos \psi} J_\mu(\alpha x \sin \varphi) J_\nu(\alpha x \sin \psi) dx \\
= \Gamma\left(\mu + \nu + \frac{1}{2}\right) \alpha^{-\frac{1}{2}} P_{\nu-\frac{1}{2}}^{-\mu}(\cos \varphi) P_{\mu-\frac{1}{2}}^{-\nu}(\cos \psi) \\
\left[\alpha > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re}(\mu + \nu) > -\frac{1}{2} \right] \quad \text{ET II 50(19)}
\end{aligned}$$

6.631

$$\begin{aligned}
1. \quad \int_0^\infty x^\mu e^{-\alpha x^2} J_\nu(\beta x) dx &= \frac{\beta^\nu \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\right)}{2^{\nu+1} \alpha^{\frac{1}{2}(\mu+\nu+1)} \Gamma(\nu+1)} {}_1F_1\left(\frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{4\alpha}\right) \\
&\quad \text{BU 8(15)} \\
&= \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\right)}{\beta \alpha^{\frac{1}{2}} \Gamma(\nu+1)} \exp\left(-\frac{\beta^2}{8\alpha}\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) \\
&\quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re}(\mu + \nu) > -1] \\
&\quad \text{EH II 50(22), ET II 30(14), BU 14(13b)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^\mu e^{-\alpha x^2} Y_\nu(\beta x) dx \\
= -\alpha^{-\frac{1}{2}\mu} \beta^{-1} \sec\left(\frac{\nu-\mu}{2}\pi\right) \exp\left(-\frac{\beta^2}{8\alpha}\right) \\
\times \left\{ \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{\Gamma(1+\nu)} \sin\left(\frac{\nu-\mu}{2}\pi\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) + W_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) \right\} \\
[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| - 1, \quad \beta > 0] \quad \text{ET II 106(4)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^\mu e^{-\alpha x^2} K_\nu(\beta x) dx &= \frac{1}{2} \alpha^{-\frac{1}{2}\mu} \beta^{-1} \Gamma\left(\frac{1+\nu+\mu}{2}\right) \Gamma\left(\frac{1-\nu+\mu}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{-\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) \\
&\quad [\operatorname{Re} \mu > |\operatorname{Re} \nu| - 1] \quad \text{ET II 132(25)}
\end{aligned}$$

$$\begin{aligned}
4.^{11} \quad \int_0^\infty x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x) dx &= \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left(-\frac{\beta^2}{4\alpha}\right) \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1] \\
&\quad \text{WA 431(4), ET II 29(10)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{\nu-1} e^{-\alpha x^2} J_\nu(\beta x) dx &= 2^{\nu-1} \beta^{-\nu} \left[1 - \gamma\left(\nu, \frac{\beta^2}{4\alpha}\right) \right] \\
&\quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 30(11)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x^{\nu+1} e^{\pm i\alpha x^2} J_\nu(\beta x) dx &= \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left[\pm i\left(\frac{\nu+1}{2}\pi - \frac{\beta^2}{4\alpha}\right)\right] \\
&\quad [\alpha > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}, \quad \beta > 0] \\
&\quad \text{ET II 30(12)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x e^{-\alpha x^2} J_\nu(\beta x) dx &= \frac{\sqrt{\pi}\beta}{8\alpha^{\frac{3}{2}}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[I_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) \right] \\
&\quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 29(9)}
\end{aligned}$$

$$8. \quad \int_0^1 x^{n+1} e^{-\alpha x^2} I_n(2\alpha x) dx = \frac{1}{4\alpha} \left[e^\alpha - e^{-\alpha} \sum_{r=-n}^n I_r(2\alpha) \right] \\ [n = 0, 1, \dots] \quad \text{ET II 365(8)a}$$

$$9. \quad \int_1^\infty x^{1-n} e^{-\alpha x^2} I_n(2\alpha x) dx = \frac{1}{4\alpha} \left[e^\alpha - e^{-\alpha} \sum_{r=1-n}^{n-1} I_r(2\alpha) \right] \\ [n = 1, 2, \dots] \quad \text{ET II 367(20)a}$$

$$10. \quad \int_0^\infty e^{-x^2} x^{2n+\mu+1} J_\mu(2x\sqrt{z}) dx = \frac{n!}{2} e^{-z} z^{\frac{1}{2}\mu} L_n^\mu(z) \quad [n = 0, 1, \dots; \quad n + \operatorname{Re} \mu > -1] \\ \text{BU 135(5)}$$

$$\mathbf{6.632} \quad \int_0^\infty x^{-\frac{1}{2}} \exp \left[- (x^2 + a^2 - 2ax \cos \varphi)^{\frac{1}{2}} \right] [x^2 + a^2 - 2ax \cos \varphi]^{-\frac{1}{2}} K_\nu(x) dx \\ = \pi a^{-\frac{1}{2}} \sec(\nu\pi) P_{\nu-\frac{1}{2}}(-\cos \varphi) K_\nu(a) \\ [|\arg a| + |\operatorname{Re} \varphi| < \pi, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 368(32)}$$

6.633

$$1. \quad \int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x) dx = \frac{\beta^\mu \gamma^\nu \alpha^{-\frac{\mu+\nu+\lambda+2}{2}}}{2^{\nu+\mu+1} \Gamma(\nu+1)} \sum_{m=0}^\infty \frac{\Gamma(m + \frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\lambda + 1)}{m! \Gamma(m + \mu + 1)} \left(-\frac{\beta^2}{4\alpha} \right)^m \\ \times F \left(-m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2} \right) \\ [\operatorname{Re} \alpha > 0, \operatorname{Re}(\mu + \nu + \lambda) > -2, \beta > 0, \quad \gamma > 0] \quad \text{EH II 49(20)a, ET II 51(24)a}$$

$$2. \quad \int_0^\infty e^{-\varrho^2 x^2} J_p(\alpha x) J_p(\beta x) x dx = \frac{1}{2\varrho^2} \exp \left(-\frac{\alpha^2 + \beta^2}{4\varrho^2} \right) I_p \left(\frac{\alpha\beta}{2\varrho^2} \right) \\ \left[\operatorname{Re} p > -1, \quad |\arg \varrho| < \frac{\pi}{4}, \quad \alpha > 0, \quad \beta > 0 \right] \quad \text{KU 146(16)a, WA 433(1)}$$

$$3. \quad \int_0^\infty x^{2\nu+1} e^{-\alpha x^2} J_\nu(x) Y_\nu(x) dx = -\frac{1}{2\sqrt{\pi}} \alpha^{-\frac{3}{2}\nu-\frac{1}{2}} \exp \left(-\frac{1}{2\alpha} \right) W_{\frac{1}{2}\nu, \frac{1}{2}\nu} \left(\frac{1}{\alpha} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 347(59)}$$

$$4. \quad \int_0^\infty x e^{-\alpha x^2} I_\nu(\beta x) J_\nu(\gamma x) dx = \frac{1}{2\alpha} \exp \left(\frac{\beta^2 - \gamma^2}{4\alpha} \right) J_\nu \left(\frac{\beta\gamma}{2\alpha} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 63(1)}$$

$$5. \quad \int_0^\infty x^{\lambda-1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\beta x) dx \\ = 2^{-\nu-\mu-1} \alpha^{-\frac{1}{2}(\nu+\lambda+\mu)} \beta^{\nu+\mu} \frac{\Gamma(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu)}{\Gamma(\mu+1) \Gamma(\nu+1)} \\ \times {}_3F_3 \left[\frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2}, \frac{\nu}{2} + \frac{\mu}{2} + 1, \frac{\nu+\mu+\lambda}{2}; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha} \right] \\ [\operatorname{Re}(\nu + \lambda + \mu) > 0, \quad \operatorname{Re} \alpha > 0] \quad \text{WA 434, EH II 50(21)}$$

$$6.634 \quad \int_0^\infty x e^{-\frac{x^2}{2a}} [I_\nu(x) + I_{-\nu}(x)] K_\nu(x) dx = a e^a K_\nu(a) \quad [\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1]$$

ET II 371(49)

6.635

$$1. \quad \int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} J_\nu(\beta x) dx = 2 J_\nu(\sqrt{2\alpha\beta}) K_\nu(\sqrt{2\alpha\beta})$$

[$\operatorname{Re} \alpha > 0, \quad \beta > 0$] ET II 30(15)

$$2. \quad \int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} Y_\nu(\beta x) dx = 2 Y_\nu(\sqrt{2\alpha\beta}) K_\nu(\sqrt{2\alpha\beta})$$

[$\operatorname{Re} \alpha > 0, \quad \beta > 0$] ET II 106(5)

$$3. \quad \int_0^\infty x^{-1} e^{-\frac{\alpha}{x} - \beta x} J_\nu(\gamma x) dx = 2 J_\nu \left\{ \sqrt{2\alpha} \left[\sqrt{\beta^2 + \gamma^2} - \beta \right]^{\frac{1}{2}} \right\} K_\nu \left\{ \sqrt{2\alpha} \left[\sqrt{\beta^2 + \gamma^2} + \beta \right]^{\frac{1}{2}} \right\}$$

[$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma > 0$] ET II 30(16)

$$6.636 \quad \int_0^\infty x^{-\frac{1}{2}} e^{-\alpha\sqrt{x}} J_\nu(\beta x) dx = \frac{\sqrt{2}}{\sqrt{\pi}\beta} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}} \left(2^{-\frac{1}{2}} \alpha e^{\frac{1}{4}\pi i} \beta^{-\frac{1}{2}}\right) D_{-\nu-\frac{1}{2}} \left(2^{-\frac{1}{2}} \alpha e^{-\frac{1}{4}\pi i} \beta^{-\frac{1}{2}}\right)$$

[$\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$] ET II 30(17)

6.637

$$1. \quad \int_0^\infty (\beta^2 + x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2 + x^2)^{\frac{1}{2}}\right] J_\nu(\gamma x) dx$$

$$= I_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} - \alpha \right] \right\} K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\}$$

[$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma > 0, \quad \operatorname{Re} \nu > -1$] ET II 31(20)

$$2. \quad \int_0^\infty (\beta^2 + x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2 + x^2)^{\frac{1}{2}}\right] Y_\nu(\gamma x) dx$$

$$= -\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\}$$

$$\times \left(\frac{1}{\pi} K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\} + \sin\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} - \alpha \right] \right\} \right)$$

[$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma > 0, \quad |\operatorname{Re} \nu| < 1$] ET II 106(6)

$$3. \quad \int_0^\infty (x^2 + \beta^2)^{-\frac{1}{2}} \exp\left[-\alpha(x^2 + \beta^2)^{\frac{1}{2}}\right] K_\nu(\gamma x) dx$$

$$= \frac{1}{2} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu} \left(\frac{1}{2}\beta \left[\alpha + (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] \right) K_{\frac{1}{2}\nu} \left(\frac{1}{2}\beta \left[\alpha - (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] \right)$$

[$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\gamma + \beta) > 0, \quad |\operatorname{Re} \nu| < 1$] ET II 132(26)

6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and powers

$$6.641 \quad \int_0^\infty \sqrt{x} e^{-\alpha x} J_{\pm\frac{1}{4}}(x^2) dx = \frac{\sqrt{\pi\alpha}}{4} \left[\mathbf{H}_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right) - Y_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right) \right]$$

MI 42

6.642

$$1.^{10} \quad \int_0^\infty x^{-1} e^{-\alpha x} Y_\nu \left(\frac{2}{x} \right) dx = 2 K_\nu (2\sqrt{\alpha}) Y_\nu (2\sqrt{\alpha})$$

[Re $\alpha > 0$] MC

$$2. \quad \int_0^\infty x^{-1} e^{-\alpha x} H_\nu^{(1,2)} \left(\frac{2}{x} \right) dx = H_\nu^{(1,2)} (\sqrt{\alpha}) K_\nu (\sqrt{\alpha})$$

MI 44, EH II 91(26)

6.643

$$1. \quad \int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} J_{2\nu} (2\beta\sqrt{x}) dx = \frac{\Gamma(\mu+\nu+\frac{1}{2})}{\beta\Gamma(2\nu+1)} e^{-\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{\mu,\nu} \left(\frac{\beta^2}{\alpha} \right)$$

[Re $(\mu+\nu+\frac{1}{2}) > 0$] BU 14(13a), MI 42a

$$2. \quad \int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} I_{2\nu} (2\beta\sqrt{x}) dx = \frac{\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(2\nu+1)} \beta^{-1} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{-\mu,\nu} \left(\frac{\beta^2}{\alpha} \right)$$

[Re $(\mu+\nu+\frac{1}{2}) > 0$] MI 45

$$3. \quad \int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} K_{2\nu} (2\beta\sqrt{x}) dx = \frac{\Gamma(\mu+\nu+\frac{1}{2}) \Gamma(\mu-\nu+\frac{1}{2})}{2\beta} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} W_{-\mu,\nu} \left(\frac{\beta^2}{\alpha} \right)$$

[Re $(\mu+\nu+\frac{1}{2}) > 0$], (cf. **6.631** 3) MI 47a

$$4. \quad \int_0^\infty x^{n+\frac{1}{2}\nu} e^{-\alpha x} J_\nu (2\beta\sqrt{x}) dx = n! \beta^\nu e^{-\frac{\beta^2}{\alpha}} \alpha^{-n-\nu-1} L_n^\nu \left(\frac{\beta^2}{\alpha} \right)$$

[$n+\nu > -1$] MO 178a

$$5. \quad \int_0^\infty x^{-\frac{1}{2}} e^{-\alpha x} Y_{2\nu} (\beta\sqrt{x}) dx = -\sqrt{\frac{\pi}{\alpha}} \frac{\exp\left(-\frac{\beta^2}{8\alpha}\right)}{\cos(\nu\pi)} \left[\sin(\nu\pi) I_\nu \left(\frac{\beta^2}{8\alpha} \right) + \frac{1}{\pi} K_\nu \left(\frac{\beta^2}{8\alpha} \right) \right]$$

[|Re ν | $< \frac{1}{2}$] MI 44

$$6. \quad \int_0^\infty x^{\frac{1}{2}m} e^{-\alpha x} K_m (2\sqrt{x}) dx = \frac{\Gamma(m+1)}{2\alpha} \left(\frac{1}{\alpha} \right)^{\frac{1}{2}m-\frac{1}{2}} e^{\frac{1}{2\alpha}} W_{-\frac{1}{2}(m+1), -\frac{1}{2}m} \left(\frac{1}{\alpha} \right)$$

MI 48a

$$\mathbf{6.644} \quad \int_0^\infty e^{-\beta x} J_{2\nu} (2a\sqrt{x}) J_\nu (bx) dx = \exp \left(-\frac{a^2\beta}{\beta^2+b^2} \right) J_\nu \left(\frac{a^2b}{\beta^2+b^2} \right) \frac{1}{\sqrt{\beta^2+b^2}}$$

[Re $\beta > 0$, $b > 0$, Re $\nu > -\frac{1}{2}$] ET II 58(17)

6.645

$$1. \quad \int_1^\infty (x^2-1)^{-\frac{1}{2}} e^{-\alpha x} J_\nu (\beta\sqrt{x^2-1}) dx = I_{\frac{1}{2}\nu} \left[\frac{1}{2} (\sqrt{\alpha^2+\beta^2}-\alpha) \right] K_{\frac{1}{2}\nu} \left[\frac{1}{2} (\sqrt{\alpha^2+\beta^2}+\alpha) \right]$$

MO 179a

$$2. \quad \int_1^\infty (x^2-1)^{\frac{1}{2}\nu} e^{-\alpha x} J_\nu (\beta\sqrt{x^2-1}) dx = \sqrt{\frac{2}{\pi}} \beta^\nu (\alpha^2+\beta^2)^{-\frac{1}{2}\nu-\frac{1}{4}} K_{\nu+\frac{1}{2}} (\sqrt{\alpha^2+\beta^2})$$

MO 179a

$$3.^3 \quad \int_{-1}^1 (1-x^2)^{-1/2} e^{-ax} I_1(b\sqrt{1-x^2}) dx = \frac{2}{b} (\cosh \sqrt{a^2+b^2} - \cosh a)$$

$$[a > 0, \quad b > 0]$$

6.646

$$1. \quad \int_1^\infty \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x} J_\nu(\beta\sqrt{x^2-1}) dx = \frac{\exp(-\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}} \left(\frac{\beta}{\alpha+\sqrt{\alpha^2+\beta^2}}\right)^\nu$$

$$[\operatorname{Re} \nu > -1] \quad \text{EF 89(52), MO 179}$$

$$2. \quad \int_1^\infty \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x} I_\nu(\beta\sqrt{x^2-1}) dx = \frac{\exp(-\sqrt{\alpha^2-\beta^2})}{\sqrt{\alpha^2-\beta^2}} \left(\frac{\beta}{\alpha+\sqrt{\alpha^2-\beta^2}}\right)^\nu$$

$$[\operatorname{Re} \nu > -1, \quad \alpha > \beta] \quad \text{MO 180}$$

$$3.^7 \quad \int_b^\infty e^{-pt} \left(\frac{t-b}{t+b}\right)^{\nu/2} K_\nu[a(t^2-b^2)^{1/2}] dt = \frac{\Gamma(\nu+1)}{2sa^\nu} [x^\nu e^{-bx} \Gamma(-\nu, bx) - y^\nu e^{bs} \Gamma(-\nu, by)]$$

$$\text{where } x = p-s, \quad y = p+s, \quad s = (p^2-a^2)^{1/2} \quad [\operatorname{Re}(p+a) > 0, \quad |\operatorname{Re}(\nu)| < 1].$$

ME 39a

6.647

$$1. \quad \int_0^\infty x^{-\lambda-\frac{1}{2}} (\beta+x)^{\lambda-\frac{1}{2}} e^{-\alpha x} K_{2\mu}[\sqrt{x(\beta+x)}] dx$$

$$= \frac{1}{\beta} e^{\frac{1}{2}\alpha\beta} \Gamma\left(\frac{1}{2}-\lambda+\mu\right) \Gamma\left(\frac{1}{2}-\lambda-\mu\right) W_{\lambda,\mu}(z_1) W_{\lambda,\mu}(z_2)$$

$$z_1 = \frac{1}{2}\beta(\alpha+\sqrt{\alpha^2-1}), \quad z_2 = \frac{1}{2}\beta(\alpha-\sqrt{\alpha^2-1})$$

$$[|\arg \beta| < \pi, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \lambda + |\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{ET II 377(37)}$$

$$2. \quad \int_0^\infty (\alpha+x)^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-x \cosh t} K_\nu[\sqrt{x(\alpha+x)}] dx$$

$$= \frac{1}{2} \sec\left(\frac{\nu\pi}{2}\right) e^{\frac{1}{2}\alpha \cosh t} K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^t\right) K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^{-t}\right)$$

$$[-1 < \operatorname{Re} \nu < 1] \quad \text{ET II 377(36)}$$

$$3.^{11} \quad \int_0^\alpha x^{\lambda-\frac{1}{2}} (\alpha-x)^{-\lambda-\frac{1}{2}} e^{-x \sinh t} I_{2\mu}[\sqrt{x(\alpha-x)}] dx$$

$$= e^{-(\alpha/2) \sinh t} \frac{2\Gamma\left(\frac{1}{2}+\lambda+\mu\right) \Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{\alpha [\Gamma(2\mu+1)]^2} M_{\lambda,\mu}\left(\frac{1}{2}\alpha e^t\right) M_{-\lambda,\mu}\left(\frac{1}{2}\alpha e^{-t}\right)$$

$$[\operatorname{Re} \mu > |\operatorname{Re} \lambda| - \frac{1}{2}] \quad \text{ET II 377(32)}$$

$$6.648 \quad \int_{-\infty}^\infty e^{qx} \left(\frac{\alpha+\beta e^x}{\alpha e^x+\beta}\right)^\nu K_{2\nu}[(\alpha^2+\beta^2+2\alpha\beta \cosh x)^{\frac{1}{2}}] dx = 2 K_{\nu+\varrho}(\alpha) K_{\nu-\varrho}(\beta)$$

$$[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 379(45)}$$

6.649

$$1. \quad \int_0^\infty K_{\mu-\nu}(2z \sinh x) e^{(\nu+\mu)x} dx = \frac{\pi^2}{4 \sin[(\nu-\mu)\pi]} [J_\nu(z) Y_\mu(z) - J_\mu(z) Y_\nu(z)]$$

[Re $z > 0$, $-1 < \operatorname{Re}(\nu - \mu) < 1$] MO 44

$$2. \quad \int_0^\infty J_{\nu+\mu}(2x \sinh t) e^{(\nu-\mu)t} dt = K_\nu(x) I_\mu(x)$$

[Re $(\nu - \mu) < \frac{3}{2}$, $\operatorname{Re}(\nu + \mu) > -1$, $x > 0$] EH II 97(68)

$$3. \quad \int_0^\infty Y_{\nu-\mu}(2x \sinh t) e^{-(\nu+\mu)t} dt = \frac{1}{\sin[\pi(\mu-\nu)]} \{I_\mu(x) K_\nu(x) - \cos[(\nu-\mu)\pi] I_\nu(x) K_\mu(x)\}$$

[|Re $(\nu - \mu)$ | < 1, $\operatorname{Re}(\nu + \mu) > -\frac{1}{2}$, $x > 0$] EH II 97(73)

$$4. \quad \int_0^\infty K_0(2z \sinh x) e^{-2\nu x} dx = -\frac{\pi}{4} \left\{ J_\nu(z) \frac{\partial Y_\nu(z)}{\partial \nu} - Y_\nu(z) \frac{\partial J_\nu(z)}{\partial \nu} \right\}$$

6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers

6.651

$$1. \quad \int_0^\infty x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} I_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx$$

$$= \frac{1}{\sqrt{2\pi}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{21} \left(\frac{\beta^2}{2\alpha^2} \middle| \begin{matrix} 1-\mu, 1+\mu \\ h, \frac{1}{2}, k \end{matrix} \right)$$

$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu$, $k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu$

[|arg α | < $\frac{\pi}{4}$, $\beta > 0$, $-\frac{3}{2} - \operatorname{Re}(2\mu + \nu) < \operatorname{Re} \lambda < 0$] ET II 68(8)

$$2. \quad \int_0^\infty x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} K_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx$$

$$= \sqrt{\frac{\pi}{2}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{12} \left(\frac{\beta^2}{2\alpha^2} \middle| \begin{matrix} 1-\mu, 1+\mu \\ h, \frac{1}{2}, k \end{matrix} \right)$$

$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu$, $k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu$

[|arg α | < $\frac{\pi}{4}$, $\operatorname{Re}(\lambda + \nu \pm 2\mu) > -\frac{3}{2}$] ET II 69(15)

$$3. \quad \int_0^\infty x^{2\mu-\nu+1} e^{-\frac{1}{4}\alpha x^2} I_\mu\left(\frac{1}{4}\alpha x^2\right) J_\nu(\beta x) dx$$

$$= 2^{\mu-\nu+\frac{1}{2}} (\pi\alpha)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} + \mu\right) \frac{\beta^{\nu-2\mu-1}}{\Gamma\left(\frac{1}{2} - \mu + \nu\right)} {}_1F_1\left(\frac{1}{2} + \mu; \frac{1}{2} - \mu + \nu; -\frac{\beta^2}{2\alpha}\right)$$

[Re $\alpha > 0$, $\beta > 0$, $\operatorname{Re} \nu > 2 \operatorname{Re} \mu + \frac{1}{2} > -\frac{1}{2}$] ET II 68(6)

$$\begin{aligned}
4. \quad \int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{4}\alpha^2 x^2} K_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx \\
= \sqrt{\pi} 2^\mu \alpha^{-2\mu-2\nu-2} \beta^\nu \frac{\Gamma(1+2\mu+\nu)}{\Gamma(\mu+\nu+\frac{3}{2})} {}_1F_1\left(1+2\mu+\nu; \mu+\nu+\frac{3}{2}; -\frac{\beta^2}{2\alpha^2}\right) \\
[\arg \alpha| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(2\mu+\nu) > -1, \quad \beta > 0] \quad \text{ET II 69(13)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{2}\alpha x^2} I_\mu\left(\frac{1}{2}\alpha x^2\right) K_\nu(\beta x) dx \\
= \frac{2^{\mu-\frac{1}{2}}}{\sqrt{\pi}} \beta^{-\mu-\frac{3}{2}} \alpha^{-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{4}} \Gamma(2\mu+\nu+1) \Gamma\left(\mu+\frac{1}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{k,m}\left(\frac{\beta^2}{4\alpha}\right) \\
2k = -3\mu - \nu - \frac{1}{2}, \quad 2m = \mu + \nu + \frac{1}{2} \\
[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1] \quad \text{ET II 146(53)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x e^{-\frac{1}{4}\alpha x^2} J_{\frac{1}{2}\nu}\left(\frac{1}{4}\beta x^2\right) J_\nu(\gamma x) dx = 2(\alpha^2 + \beta^2)^{-\frac{1}{2}} \exp\left(-\frac{\alpha\gamma^2}{\alpha^2 + \beta^2}\right) J_{\frac{1}{2}\nu}\left(\frac{\beta\gamma^2}{\alpha^2 + \beta^2}\right) \\
[\gamma > 0, \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|, \quad \operatorname{Re} \nu > -1] \\
\text{ET II 56(2)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x e^{-\frac{1}{4}\alpha x^2} I_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha x^2\right) J_\nu(\beta x) dx = \left(\frac{1}{2}\pi\alpha\right)^{-\frac{1}{2}} \beta^{-1} \exp\left(-\frac{\beta^2}{2\alpha}\right) \\
[\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1] \\
\text{ET II 67(3)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_0^\infty x^{1-\nu} e^{-\frac{1}{4}\alpha^2 x^2} I_\nu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx = \sqrt{\frac{2}{\pi}} \frac{\beta^{\nu-1}}{\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu}\left(\frac{\beta}{\alpha}\right) \\
[\arg \alpha| < \frac{1}{4}\pi, \quad \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\
\text{ET II 67(1)}
\end{aligned}$$

$$\begin{aligned}
9. \quad \int_0^\infty x^{-\nu-1} e^{-\frac{1}{4}\alpha^2 x^2} I_{\nu+1}\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx = \sqrt{\frac{2}{\pi}} \beta^\nu \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu-3}\left(\frac{\beta}{\alpha}\right) \\
[\arg \alpha| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1, \quad \beta > 0] \\
\text{ET II 67(2)}
\end{aligned}$$

$$\begin{aligned}
6.652 \quad \int_0^\infty x^{2\nu} e^{-\left(\frac{x^2}{8} + \alpha x\right)} I_\nu\left(\frac{x^2}{8}\right) dx = \frac{\Gamma(4\nu+1)}{2^{4\nu} \Gamma(\nu+1)} \frac{e^{\frac{\alpha^2}{2}}}{\alpha^{\nu+1}} W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}(\alpha^2) \\
[\operatorname{Re}(\nu + \frac{1}{4}) > 0] \quad \text{MI 45}
\end{aligned}$$

6.653

$$\begin{aligned}
1. \quad \int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}(a^2 + b^2)\right] I_\nu\left(\frac{ab}{x}\right) \frac{dx}{x} = 2 I_\nu(a) K_\nu(b) \quad [0 < a < b] \\
= 2 K_\nu(a) I_\nu(b) \quad [0 < b < a] \\
[\operatorname{Re} \nu > -1] \quad \text{WA 482(2)a, EH II 53(37), WA 482(3)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}(z^2 + w^2)\right] K_\nu\left(\frac{zw}{x}\right) \frac{dx}{x} = 2 K_\nu(z) K_\nu(w) \\
[|\arg z| < \pi, \quad |\arg w| < \pi, \quad \arg(z+w) < \frac{1}{4}\pi] \quad \text{WA 483(1), EH II 53(36)}
\end{aligned}$$

$$6.654 \quad \int_0^\infty x^{-\frac{1}{2}} e^{-\frac{\beta^2}{8x} - \alpha x} K_\nu \left(\frac{\beta^2}{8x} \right) dx = \sqrt{4\pi} \alpha^{-\frac{1}{2}} K_{2\nu} (\beta \sqrt{\alpha}) \quad \text{ME 39}$$

$$6.655 \quad \int_0^\infty x (\beta^2 + x^2)^{-\frac{1}{2}} \exp \left(-\frac{\alpha^2 \beta}{\beta^2 + x^2} \right) J_\nu \left(\frac{\alpha^2 x}{\beta^2 + x^2} \right) J_\nu(\gamma x) dx = \gamma^{-1} e^{-\beta \gamma} J_{2\nu}(2\alpha \sqrt{\gamma})$$

$$[\operatorname{Re} \beta > 0, \quad \gamma > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}]$$

ET II 58(14)

6.656

$$1. \quad \int_0^\infty e^{-(\xi-z) \cosh t} J_{2\nu} \left[2(z\xi)^{\frac{1}{2}} \sinh t \right] dt = I_\nu(z) K_\nu(\xi)$$

$$[\operatorname{Re} \nu > -\tfrac{1}{2}, \quad \operatorname{Re}(\xi - z) > 0]$$

EH II 98(78)

$$2. \quad \int_0^\infty e^{-(\xi+z) \cosh t} K_{2\nu} \left[2(z\xi)^{\frac{1}{2}} \sinh t \right] dt = \frac{1}{2} K_\nu(z) K_\nu(\xi) \sec(\nu\pi)$$

$$\left[|\operatorname{Re} \nu| < \tfrac{1}{2}, \quad \operatorname{Re} \left(z^{\frac{1}{2}} + \xi^{\frac{1}{2}} \right)^2 \geq 0 \right]$$

EH II 98(79)

6.66 Combinations of Bessel, hyperbolic, and exponential functions

Bessel and hyperbolic functions

6.661

$$1. \quad \int_0^\infty \sinh(ax) K_\nu(bx) dx = \frac{\pi \operatorname{cosec} \left(\frac{\nu\pi}{2} \right) \sin \left[\nu \arcsin \left(\frac{a}{b} \right) \right]}{2 \sqrt{b^2 - a^2}}$$

$$[\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re} \nu| < 2]$$

ET II 133(32)

$$2. \quad \int_0^\infty \cosh(ax) K_\nu(bx) dx = \frac{\pi \cos \left[\nu \arcsin \left(\frac{a}{b} \right) \right]}{2 \sqrt{b^2 - a^2} \cos \left(\frac{\nu\pi}{2} \right)}$$

$$[\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re} \nu| < 1]$$

ET II 134(33)

6.662 Notation:

$$\ell_1 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \quad \ell_2 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$$

$$1.^{10} \quad \int_0^\infty \cosh(\beta x) K_0(\alpha x) J_0(\gamma x) dx = \frac{\mathbf{K}(k)}{\sqrt{u+v}}$$

$$u = \frac{1}{2} \left\{ \sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2 \beta^2} \right\} + \alpha^2 - \beta^2 - \gamma^2$$

$$v = \frac{1}{2} \left\{ \sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2 \beta^2} \right\} - \alpha^2 + \beta^2 + \gamma^2$$

$$k^2 = v(u+v)^{-1} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0]$$

ET II 15(23)

alternatively, with $a = \gamma$, $b = \beta$, $c = \alpha$,

$$\int_0^\infty \cosh(bx) K_0(cx) J_0(ax) dx = \frac{\mathbf{K}(k)}{\sqrt{\ell_2^2 - \ell_1^2}}$$

$$k^2 = \frac{\ell_2^2 - c^2}{\ell_2^2 - \ell_1^2}, \quad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

$$\begin{aligned} 2.10 \quad \int_0^\infty \sinh(\beta x) K_1(\alpha x) J_0(\gamma x) dx &= a^{-1} \left[u \mathbf{E}(k) - \mathbf{K}(k) \mathbf{E}(u) + \frac{\mathbf{K}(k) \operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} \right] \\ \operatorname{cn}^2 u &= 2\gamma^2 \left\{ \left[(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2 \beta^2 \right]^{\frac{1}{2}} - \alpha^2 + \beta^2 + \gamma^2 \right\}^{-1} \\ k^2 &= \frac{1}{2} \left\{ 1 - (\alpha^2 - \beta^2 - \gamma^2) \left[(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2 \beta^2 \right]^{-\frac{1}{2}} \right\} \\ &[\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0] \\ &\text{ET II 15(24)} \end{aligned}$$

alternatively, with $a = \gamma$, $b = \beta$, $c = \alpha$,

$$\begin{aligned} \int_0^\infty \sinh(bx) K_1(cx) J_0(ax) dx &= c^{-1} \left[u \mathbf{E}(k) - \mathbf{K}(k) \mathbf{E}(u) + \frac{\mathbf{K}(k) \operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} \right] \\ \operatorname{cn}^2 u &= \frac{a^2}{\ell_2^2 - c^2}, \quad k^2 = \frac{\ell_2^2 - c^2}{\ell_2^2 - \ell_1^2} \quad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0] \end{aligned}$$

6.663

1. $\int_0^\infty K_{\nu \pm \mu} (2z \cosh t) \cosh [(\mu \mp \nu) t] dt = \frac{1}{2} K_\mu(z) K_\nu(z)$
 $[\operatorname{Re} z > 0] \quad \text{WA 484(1), EH II 54(39)}$
2. $\int_0^\infty Y_{\mu+\nu} (2z \cosh t) \cosh[(\mu - \nu)t] dt = \frac{\pi}{4} [J_\mu(z) J_\nu(z) - Y_\mu(z) Y_\nu(z)]$
 $[z > 0] \quad \text{EH II 96(64)}$
3. $\int_0^\infty J_{\mu+\nu} (2z \cosh t) \cosh[(\mu - \nu)t] dt = -\frac{\pi}{4} [J_\mu(z) Y_\nu(z) + J_\nu(z) Y_\mu(z)]$
 $[z > 0] \quad \text{EH II 97(65)}$
4. $\int_0^\infty J_{\mu+\nu} (2z \sinh t) \cosh[(\mu - \nu)t] dt = \frac{1}{2} [I_\nu(z) K_\mu(z) + I_\mu(z) K_\nu(z)]$
 $[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \frac{3}{2}, \quad z > 0] \quad \text{EH II 97(71)}$
5. $\int_0^\infty J_{\mu+\nu} (2z \sinh t) \sinh[(\mu - \nu)t] dt = \frac{1}{2} [I_\nu(z) K_\mu(z) - I_\mu(z) K_\nu(z)]$
 $[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \frac{3}{2}, \quad z > 0] \quad \text{EH II 97(72)}$

6.664

1. $\int_0^\infty J_0 (2z \sinh t) \sinh(2\nu t) dt = \frac{\sin(\nu\pi)}{\pi} [K_\nu(z)]^2 \quad [|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0] \quad \text{EH II 97(69)}$

2. $\int_0^\infty Y_0(2z \sinh t) \cosh(2\nu t) dt = -\frac{\cos(\nu\pi)}{\pi} [K_\nu(z)]^2 \quad [|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0] \quad \text{EH II 97(70)}$
 3. $\int_0^\infty Y_0(2z \sinh t) \sinh(2\nu t) dt = \frac{1}{\pi} \left[I_\nu(z) \frac{\partial K_\nu(z)}{\partial \nu} - K_\nu(z) \frac{\partial I_\nu(z)}{\partial \nu} \right] - \frac{1}{\pi} \cos(\nu\pi) [K_\nu(z)]^2$
 $[|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0] \quad \text{EH II 97(75)}$
 4. $\int_0^\infty K_0(2z \sinh t) \cosh 2\nu t dt = \frac{\pi^2}{8} \{J_\nu^2(z) + N_\nu^2(z)\} \quad [\operatorname{Re} z > 0] \quad \text{MO 44}$
 5. $\int_0^\infty K_{2\mu}(z \sinh 2t) \coth^{2\nu} t dt = \frac{1}{4z} \Gamma\left(\frac{1}{2} + \mu - \nu\right) \Gamma\left(\frac{1}{2} - \mu - \nu\right) W_{\nu,\mu}(iz) W_{\nu,\mu}(-iz)$
 $[|\arg z| \leq \frac{\pi}{2}, \quad |\operatorname{Re} \mu| + \operatorname{Re} \nu < \frac{1}{2}] \quad \text{MO 119}$
 6. $\int_0^\infty \cosh(2\mu x) K_{2\nu}(2a \cosh x) dx = \frac{1}{2} K_{\mu+\nu}(a) K_{\mu-\nu}(a)$
 $[\operatorname{Re} a > 0] \quad \text{ET II 378(42)}$
- 6.665** $\int_0^\infty \operatorname{sech} x \cosh(2\lambda x) I_{2\mu}(a \operatorname{sech} x) dx = \frac{\Gamma(\frac{1}{2} + \lambda + \mu) \Gamma(\frac{1}{2} - \lambda + \mu)}{2a [\Gamma(2\mu + 1)]^2} M_{\lambda,\mu}(a) M_{-\lambda,\mu}(a)$
 $[|\operatorname{Re} \lambda| - \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 378(43)}$

Bessel, hyperbolic, and algebraic functions

- 6.666** $\int_0^\infty x^{\nu+1} \sinh(\alpha x) \operatorname{cosech}(\pi x) J_\nu(\beta x) dx = \frac{2}{\pi} \sum_{n=1}^\infty (-1)^{n-1} n^{\nu+1} \sin(n\alpha) K_\nu(n\beta)$
 $[|\operatorname{Re} \alpha| < \pi, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 41(3), WA 469(12)}$
- 6.667**
- 1.³ $\int_0^a \frac{\cosh(\sqrt{a^2 - x^2}) \sinh t I_{2\nu}(x)}{\sqrt{a^2 - x^2}} dx = \frac{\pi}{2} I_\nu\left(\frac{1}{2}ae^t\right) I_\nu\left(\frac{1}{2}ae^{-t}\right)$
 $[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 365(10)}$
 2. $\int_0^a \frac{\cosh(\sqrt{a^2 - x^2} \sinh t) K_{2\nu}(x)}{\sqrt{a^2 - x^2}} dx = \frac{\pi^2}{4} \operatorname{cosec}(\nu\pi) [I_{-\nu}(ae^t) I_{-\nu}(ae^{-t}) - I_\nu(ae^t) I_\nu(ae^{-t})]$
 $[|\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 367(25)}$

Exponential, hyperbolic, and Bessel functions

6.668 Notation:

$$\ell_1 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \quad \ell_2 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$$

$$1.^{10} \quad \int_0^\infty e^{-\alpha x} \sinh(\beta x) J_0(\gamma x) dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^{\frac{1}{2}} (r_2 + r_1)^{-\frac{1}{2}} \\ r_1 = \sqrt{\gamma^2 + (\beta - \alpha)^2}, \quad r_2 = \sqrt{\gamma^2 + (\beta + \alpha)^2}, \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0] \quad \text{ET II 12(52)}$$

alternatively, with $a = \gamma$, $b = \beta$, $c = \alpha$,

$$\int_0^\infty e^{-cx} \sinh(bx) J_0(ax) dx = \frac{\ell_1}{\ell_2^2 - \ell_1^2} \\ [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

$$2.^{10} \quad \int_0^\infty e^{-\alpha x} \cosh(\beta x) J_0(\gamma x) dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^{\frac{1}{2}} (r_2 + r_1)^{-\frac{1}{2}} \\ r_1 = \sqrt{\gamma^2 + (\beta - \alpha)^2}, \quad r_2 = \sqrt{\gamma^2 + (\beta + \alpha)^2}, \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0] \quad \text{ET II 12(54)}$$

alternatively, with $a = \gamma$, $b = \beta$, $c = \alpha$,

$$\int_0^\infty e^{-cx} \cosh(bx) J_0(ax) dx = \frac{\ell_2}{\ell_2^2 - \ell_1^2} \\ [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

6.669

$$1. \quad \int_0^\infty \left[\coth \left(\frac{1}{2} x \right) \right]^{2\lambda} e^{-\beta \cosh x} J_{2\mu}(\alpha \sinh x) dx = \frac{\Gamma(\frac{1}{2} - \lambda + \mu)}{\alpha \Gamma(2\mu + 1)} M_{-\lambda, \mu} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} - \beta \right] \\ \times W_{\lambda, \mu} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \beta \right] \\ [\operatorname{Re} \beta > |\operatorname{Re} \alpha|, \quad \operatorname{Re}(\mu - \lambda) > -\frac{1}{2}] \quad \text{BU 86(5b)a, ET II 363(34)}$$

$$2. \quad \int_0^\infty \left[\coth \left(\frac{1}{2} x \right) \right]^{2\lambda} e^{-\beta \cosh x} Y_{2\mu}(\alpha \sinh x) dx \\ = -\frac{\sec[(\mu + \lambda)\pi]}{\alpha} W_{\lambda, \mu}(\sqrt{\alpha^2 + \beta^2} + \beta) W_{-\lambda, \mu}(\sqrt{\alpha^2 + \beta^2} - \beta) \\ - \frac{\tan[(\mu + \lambda)\pi] \Gamma(\frac{1}{2} - \lambda + \mu)}{\alpha \Gamma(2\mu + 1)} W_{\lambda, \mu}(\sqrt{\alpha^2 + \beta^2} + \beta) M_{-\lambda, \mu}(\sqrt{\alpha^2 + \beta^2} - \beta) \\ [\operatorname{Re} \beta > |\operatorname{Re} \alpha|, \quad \operatorname{Re} \lambda < \frac{1}{2} - |\operatorname{Re} \mu|] \quad \text{ET II 363(35)}$$

$$3. \quad \int_0^\infty e^{-\frac{1}{2}(a_1 a_2)t \cosh x} \left[\coth \left(\frac{1}{2} x \right) \right]^{2\nu} K_{2\mu}(t\sqrt{a_1 a_2} \sinh x) dx \\ = \frac{\Gamma(\frac{1}{2} + \mu - \nu) \Gamma(\frac{1}{2} - \mu - \nu)}{2t\sqrt{a_1 a_2}} W_{\nu, \mu}(a_1 t) W_{\nu, \mu}(a_2 t) \\ \left[\operatorname{Re} \nu < \operatorname{Re} \frac{1 \pm 2\mu}{2}, \quad \operatorname{Re} \left[t(\sqrt{a_1} + \sqrt{a_2})^2 \right] > 0 \right] \quad \text{BU 85(4a)}$$

$$4. \quad \int_0^\infty e^{-\frac{1}{2}(a_1 a_2)t \cosh x} \left[\coth \left(\frac{x}{2} \right) \right]^{2\nu} I_{2\mu}(t\sqrt{a_1 a_2} \sinh x) dx = \frac{\Gamma(\frac{1}{2} + \mu - \nu)}{t\sqrt{a_1 a_2} \Gamma(1 + 2\mu)} W_{\nu, \mu}(a_1 t) M_{\nu, \mu}(a_2 t) \\ [\operatorname{Re}(\frac{1}{2} + \mu - \nu) > 0, \quad \operatorname{Re} \mu > 0, \quad a_1 > a_2] \quad \text{BU 86(5c)}$$

$$5. \quad \int_{-\infty}^\infty e^{2\nu s - \frac{x-y}{2} \tanh s} I_{2\mu} \left(\frac{\sqrt{xy}}{\cosh s} \right) \frac{ds}{\cosh s} = \frac{\Gamma(\frac{1}{2} + \mu + \nu) \Gamma(\frac{1}{2} + \mu - \nu)}{\sqrt{xy} [\Gamma(1 + 2\mu)]^2} M_{\nu, \mu}(x) M_{-\nu, \mu}(y) \\ [\operatorname{Re}(\pm \nu + \frac{1}{2} + \mu) > 0] \quad \text{BU 83(3a)a}$$

$$6. \quad \int_{-\infty}^{\infty} e^{2\nu s - \frac{x+y}{2} \tanh s} J_{2\mu} \left(\frac{\sqrt{xy}}{\cosh s} \right) \frac{ds}{\cosh s} = \frac{\Gamma\left(\frac{1}{2} + \mu + \nu\right) \Gamma\left(\frac{1}{2} + \mu - \nu\right)}{\sqrt{xy} [\Gamma(1 + 2\mu)]^2} M_{\nu, \mu}(x) M_{\nu, \mu}(y) \\ [\operatorname{Re}(\mp\nu + \tfrac{1}{2} + \mu) > 0] \quad \text{BU 84(3b)a}$$

6.67–6.68 Combinations of Bessel and trigonometric functions

6.671

$$1. \quad \int_0^{\infty} J_{\nu}(\alpha x) \sin \beta x \, dx = \frac{\sin\left(\nu \arcsin \frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2 - \beta^2}} \quad [\beta < \alpha] \\ = \infty \text{ or } 0 \quad [\beta = \alpha] \\ = \frac{\alpha^{\nu} \cos \frac{\nu\pi}{2}}{\sqrt{\beta^2 - \alpha^2} \left(\beta + \sqrt{\beta^2 - \alpha^2}\right)^{\nu}} \quad [\beta > \alpha] \\ [\operatorname{Re} \nu > -2] \quad \text{WA 444(4)}$$

$$2. \quad \int_0^{\infty} J_{\nu}(\alpha x) \cos \beta x \, dx = \frac{\cos\left(\nu \arcsin \frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2 - \beta^2}} \quad [\beta < \alpha] \\ = \infty \text{ or } 0 \quad [\beta = \alpha] \\ = \frac{-\alpha^{\nu} \sin \frac{\nu\pi}{2}}{\sqrt{\beta^2 - \alpha^2} \left(\beta + \sqrt{\beta^2 - \alpha^2}\right)^{\nu}} \quad [\beta > \alpha] \\ [\operatorname{Re} \nu > -1] \quad \text{WA 444(5)}$$

$$3. \quad \int_0^{\infty} Y_{\nu}(ax) \sin(bx) \, dx \\ = \cot\left(\frac{\nu\pi}{2}\right) (a^2 - b^2)^{-\frac{1}{2}} \sin\left[\nu \arcsin\left(\frac{b}{a}\right)\right] \quad [0 < b < a, |\operatorname{Re} \nu| < 2] \\ = \frac{1}{2} \operatorname{cosec}\left(\frac{\nu\pi}{2}\right) (b^2 - a^2)^{-\frac{1}{2}} \\ \times \left\{ a^{-\nu} \cos(\nu\pi) \left[b - (b^2 - a^2)^{\frac{1}{2}}\right]^{\nu} - a^{\nu} \left[b - (b^2 - a^2)^{\frac{1}{2}}\right]^{-\nu} \right\} \quad [0 < a < b, |\operatorname{Re} \nu| < 2] \\ \text{ET I 103(33)}$$

$$4. \quad \int_0^{\infty} Y_{\nu}(ax) \cos(bx) \, dx \\ = \frac{\tan\left(\frac{\nu\pi}{2}\right)}{(a^2 - b^2)^{\frac{1}{2}}} \cos\left[\nu \arcsin\left(\frac{b}{a}\right)\right] \quad [0 < b < a, |\operatorname{Re} \nu| < 1] \\ = -\sin\left(\frac{\nu\pi}{2}\right) (b^2 - a^2)^{-\frac{1}{2}} \left\{ a^{-\nu} \left[b - (b^2 - a^2)^{\frac{1}{2}}\right]^{\nu} + \cot(\nu\pi) \right. \\ \left. + a^{\nu} \left[b - (b^2 - a^2)^{\frac{1}{2}}\right]^{-\nu} \operatorname{cosec}(\nu\pi) \right\} \quad [0 < a < b, |\operatorname{Re} \nu| < 1] \\ \text{ET I 47(29)}$$

$$\begin{aligned}
5. \quad \int_0^\infty K_\nu(ax) \sin(bx) dx &= \frac{1}{4} \pi a^{-\nu} \operatorname{cosec} \left(\frac{\nu\pi}{2} \right) (a^2 + b^2)^{-\frac{1}{2}} \left\{ \left[(b^2 + a^2)^{\frac{1}{2}} + b \right]^\nu - \left[(b^2 + a^2)^{\frac{1}{2}} - b \right]^\nu \right\} \\
&\quad [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < 2, \quad \nu \neq 0] \quad \text{ET I 105(48)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty K_\nu(ax) \cos(bx) dx &= \frac{\pi}{4} (b^2 + a^2)^{-\frac{1}{2}} \sec \left(\frac{\nu\pi}{2} \right) \left\{ a^{-\nu} \left[b + (b^2 + a^2)^{\frac{1}{2}} \right]^\nu + a^\nu \left[b + (b^2 + a^2)^{\frac{1}{2}} \right]^{-\nu} \right\} \\
&\quad [\operatorname{Re} a > 0, b > 0, |\operatorname{Re} \nu| < 1] \quad \text{ET I 49(40)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty J_0(ax) \sin(bx) dx &= 0 & [0 < b < a] \\
&= \frac{1}{\sqrt{b^2 - a^2}} & [0 < a < b]
\end{aligned}$$

ET I 99(1)

$$\begin{aligned}
8. \quad \int_0^\infty J_0(ax) \cos(bx) dx &= \frac{1}{\sqrt{a^2 - b^2}} & [0 < b < a] \\
&= \infty & [a = b] \\
&= 0 & [0 < a < b]
\end{aligned}$$

ET I 43(1)

$$\begin{aligned}
9. \quad \int_0^\infty J_{2n+1}(ax) \sin(bx) dx &= (-1)^n \frac{1}{\sqrt{a^2 - b^2}} T_{2n+1} \left(\frac{b}{a} \right) & [0 < b < a] \\
&= 0 & [0 < a < b]
\end{aligned}$$

ET I 99(2)

$$\begin{aligned}
10. \quad \int_0^\infty J_{2n}(ax) \cos(bx) dx &= (-1)^n \frac{1}{\sqrt{a^2 - b^2}} T_{2n} \left(\frac{b}{a} \right) & [0 < b < a] \\
&= 0 & [0 < a < b]
\end{aligned}$$

ET I 43(2)

$$\begin{aligned}
11. \quad \int_0^\infty Y_0(ax) \sin(bx) dx &= \frac{2 \arcsin \left(\frac{b}{a} \right)}{\pi \sqrt{a^2 - b^2}} & [0 < b < a] \\
&= \frac{2}{\pi \sqrt{b^2 - a^2}} \ln \left[\frac{b}{a} - \sqrt{\frac{b^2}{a^2} - 1} \right] & [0 < a < b]
\end{aligned}$$

ET I 103(31)

$$\begin{aligned}
12. \quad \int_0^\infty Y_0(ax) \cos(bx) dx &= 0 & [0 < b < a] \\
&= -\frac{1}{\sqrt{b^2 - a^2}} & [0 < a < b]
\end{aligned}$$

ET I 47(28)

$$13. \quad \int_0^\infty K_0(\beta x) \sin \alpha x \, dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \ln \left(\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} + 1} \right) \quad [\alpha > 0, \quad \beta > 0] \quad \text{WA 425(11)a, MO 48}$$

$$14.^8 \quad \int_0^\infty K_0(\beta x) \cos \alpha x \, dx = \frac{\pi}{2\sqrt{\alpha^2 + \beta^2}} \quad [\alpha > 0] \quad \text{WA 425(10)a, MO 48}$$

6.672

$$1. \quad \int_0^\infty J_\nu(ax) J_\nu(bx) \sin(cx) \, dx$$

$$= 0 \quad [\operatorname{Re} \nu > -1, \quad 0 < c < b - a, \quad 0 < a < b]$$

$$= \frac{1}{2\sqrt{ab}} P_{\nu-\frac{1}{2}} \left(\frac{b^2 + a^2 - c^2}{2ab} \right) \quad [\operatorname{Re} \nu > -1, \quad b - a < c < b + a, \quad 0 < a < b]$$

$$= -\frac{\cos(\nu\pi)}{\pi\sqrt{ab}} Q_{\nu-\frac{1}{2}} \left(-\frac{b^2 + a^2 - c^2}{2ab} \right) \quad [\operatorname{Re} \nu > -1, \quad b + a < c, \quad 0 < a < b]$$

ET I 102(27)

$$2. \quad \int_0^\infty J_\nu(x) J_{-\nu}(x) \cos(bx) \, dx = \frac{1}{2} P_{\nu-\frac{1}{2}} \left(\frac{1}{2} b^2 - 1 \right) \quad [0 < b < 2]$$

$$= 0 \quad [2 < b]$$

ET I 46(21)

$$3. \quad \int_0^\infty K_\nu(ax) K_\nu(bx) \cos(cx) \, dx = \frac{\pi^2}{4\sqrt{ab}} \sec(\nu\pi) P_{\nu-\frac{1}{2}} \left[(a^2 + b^2 + c^2) (2ab)^{-1} \right]$$

$$[\operatorname{Re}(a+b) > 0, \quad c > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}]$$

ET I 50(51)

$$4. \quad \int_0^\infty K_\nu(ax) I_\nu(bx) \cos(cx) \, dx = \frac{1}{2\sqrt{ab}} Q_{\nu-\frac{1}{2}} \left(\frac{a^2 + b^2 + c^2}{2ab} \right)$$

$$[\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}]$$

ET I 49(47)

$$5. \quad \int_0^\infty \sin(2ax) [J_\nu(x)]^2 \, dx = \frac{1}{2} P_{\nu-\frac{1}{2}} (1 - 2a^2) \quad [0 < a < 1, \quad \operatorname{Re} \nu > -1]$$

$$= \frac{1}{\pi} \cos(\nu\pi) Q_{\nu-\frac{1}{2}} (2a^2 - 1) \quad [a > 1, \quad \operatorname{Re} \nu > -1]$$

ET II 343(30)

$$6. \quad \int_0^\infty \cos(2ax) [J_\nu(x)]^2 \, dx = \frac{1}{\pi} Q_{\nu-\frac{1}{2}} (1 - 2a^2) \quad [0 < a < 1, \quad \operatorname{Re} \nu > -\tfrac{1}{2}]$$

$$= -\frac{1}{\pi} \sin(\nu\pi) Q_{\nu-\frac{1}{2}} (2a^2 - 1) \quad [a > 1, \quad \operatorname{Re} \nu > -\tfrac{1}{2}]$$

ET II 344(32)

$$7. \quad \int_0^\infty \sin(2ax) J_0(x) Y_0(x) \, dx = 0 \quad [0 < a < 1]$$

$$= -\frac{\mathbf{K} \left[(1 - a^{-2})^{\frac{1}{2}} \right]}{\pi a} \quad [a > 1]$$

ET II 348(60)

$$8. \quad \int_0^\infty K_0(ax) I_0(bx) \cos(cx) dx = \frac{1}{\sqrt{c^2 + (a+b)^2}} \mathbf{K} \left\{ \frac{2\sqrt{ab}}{\sqrt{c^2 + (a+b)^2}} \right\} \\ [\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0] \quad \text{ET I 49(46)}$$

$$9. \quad \int_0^\infty \cos(2ax) J_0(x) Y_0(x) dx = -\frac{1}{\pi} \mathbf{K}(a) \quad [0 < a < 1] \\ = -\frac{1}{\pi a} \mathbf{K}\left(\frac{1}{a}\right) \quad [a > 1] \\ \text{ET II 348(61)}$$

$$10. \quad \int_0^\infty \cos(2ax) [Y_0(x)]^2 dx = \frac{1}{\pi} \mathbf{K}\left(\sqrt{1-a^2}\right) \quad [0 < a < 1] \\ = \frac{2}{\pi a} \mathbf{K}\left(\sqrt{1-\frac{1}{a^2}}\right) \quad [a > 1] \\ \text{ET II 348(62)}$$

6.673

$$1. \quad \int_0^\infty \left[J_\nu(ax) \cos\left(\frac{\nu\pi}{2}\right) - Y_\nu(ax) \sin\left(\frac{\nu\pi}{2}\right) \right] \sin(bx) dx \\ = 0 \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 2] \\ = \frac{1}{2a^\nu \sqrt{b^2 - a^2}} \left\{ \left[b + (b^2 - a^2)^{\frac{1}{2}} \right]^\nu + \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu \right\} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < 2] \\ \text{ET I 104(39)}$$

$$2. \quad \int_0^\infty \left[Y_\nu(ax) \cos\left(\frac{\nu\pi}{2}\right) + J_\nu(ax) \sin\left(\frac{\nu\pi}{2}\right) \right] \cos(bx) dx \\ = 0 \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 1] \\ = -\frac{1}{2a^\nu \sqrt{b^2 - a^2}} \left\{ \left[b + (b^2 - a^2)^{\frac{1}{2}} \right]^\nu + \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu \right\} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < 1] \\ \text{ET I 48(32)}$$

$$3.* \quad \int_0^{\pi/2} [\cos x I_0(a \cos x) + I_1(a \cos x)] dx = \frac{e^a - 1}{a}$$

6.674

$$1. \quad \int_0^a \sin(a-x) J_\nu(x) dx = a J_{\nu+1}(a) - 2\nu \sum_{n=0}^\infty (-1)^n J_{\nu+2n+2}(a) \\ [\operatorname{Re} \nu > -1] \quad \text{ET II 334(12)}$$

$$2. \quad \int_0^a \cos(a-x) J_\nu(x) dx = a J_\nu(a) - 2\nu \sum_{n=0}^\infty (-1)^n J_{\nu+2n+1}(a) \\ [\operatorname{Re} \nu > -1] \quad \text{ET II 336(23)}$$

$$3. \quad \int_0^a \sin(a-x) J_{2n}(x) dx = a J_{2n+1}(a) + (-1)^n 2n \left[\cos a - J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a) \right] \\ [n = 0, 1, 2, \dots] \quad \text{ET II 334(10)}$$

$$4. \quad \int_0^a \cos(a-x) J_{2n}(x) dx = a J_{2n}(a) - (-1)^n 2n \left[\sin a - 2 \sum_{m=0}^{n-1} (-1)^m J_{2m+1}(a) \right] \\ [n = 0, 1, 2, \dots] \quad \text{ET II 335(21)}$$

$$5. \quad \int_0^a \sin(a-x) J_{2n+1}(x) dx = a J_{2n+2}(a) + (-1)^n (2n+1) \left[\sin a - 2 \sum_{m=0}^n (-1)^m J_{2m+1}(a) \right] \\ [n = 0, 1, 2, \dots] \quad \text{ET II 334(11)}$$

$$6. \quad \int_0^a \cos(a-x) J_{2n+1}(x) dx = a J_{2n+1}(a) + (-1)^n (2n+1) \left[\cos a - J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a) \right] \\ [n = 0, 1, 2, \dots] \quad \text{ET II 336(22)}$$

$$7. \quad \int_0^z \sin(z-x) J_0(x) dx = z J_1(z) \quad \text{WA 415(2)}$$

$$8. \quad \int_0^z \cos(z-x) J_0(x) dx = z J_0(z) \quad \text{WA 415(1)}$$

6.675

$$1. \quad \int_0^\infty J_\nu(a\sqrt{x}) \sin(bx) dx = \frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) - \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -4] \quad \text{ET I 110(23)}$$

$$2. \quad \int_0^\infty J_\nu(a\sqrt{x}) \cos(bx) dx \\ = -\frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) + \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET I 53(22)a}$$

$$3. \quad \int_0^\infty J_0(a\sqrt{x}) \sin(bx) dx = \frac{1}{b} \cos\left(\frac{a^2}{4b}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 110(22)}$$

$$4. \quad \int_0^\infty J_0(a\sqrt{x}) \cos(bx) dx = \frac{1}{b} \sin\left(\frac{a^2}{4b}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 53(21)}$$

6.676

$$1. \quad \int_0^\infty J_\nu(a\sqrt{x}) J_\nu(b\sqrt{x}) \sin(cx) dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \cos\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET I 111(29)a}$$

$$2. \quad \int_0^\infty J_\nu(a\sqrt{x}) J_\nu(b\sqrt{x}) \cos(cx) dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \sin\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 54(27)}$$

$$3. \quad \int_0^\infty J_0(a\sqrt{x}) K_0(a\sqrt{x}) \sin(bx) dx = \frac{1}{2b} K_0\left(\frac{a^2}{2b}\right) \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 111(31)}$$

$$4. \quad \int_0^\infty J_0(\sqrt{ax}) K_0(\sqrt{ax}) \cos(bx) dx = \frac{\pi}{4b} \left[I_0\left(\frac{a}{2b}\right) - L_0\left(\frac{a}{2b}\right) \right] \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 54(29)}$$

$$5. \quad \int_0^\infty K_0(\sqrt{ax}) Y_0(\sqrt{ax}) \cos(bx) dx = -\frac{1}{2b} K_0\left(\frac{a}{2b}\right) \quad [\operatorname{Re} \sqrt{a} > 0, \quad b > 0] \quad \text{ET I 54(30)}$$

$$6. \quad \int_0^\infty K_0\left(\sqrt{ax}e^{\frac{1}{4}\pi i}\right) K_0\left(\sqrt{ax}e^{-\frac{1}{4}\pi i}\right) \cos(bx) dx = \frac{\pi^2}{8b} \left[\mathbf{H}_0\left(\frac{a}{2b}\right) - Y_0\left(\frac{a}{2b}\right) \right] \quad [\operatorname{Re} a > 0, b > 0] \quad \text{ET I 54(31)}$$

6.677

$$1. \quad \int_a^\infty J_0\left(b\sqrt{x^2 - a^2}\right) \sin(cx) dx = 0 \quad [0 < c < b]$$

$$= \frac{\cos(a\sqrt{c^2 - b^2})}{\sqrt{c^2 - b^2}} \quad [0 < b < c] \quad \text{ET I 113(47)}$$

$$2. \quad \int_a^\infty J_0\left(b\sqrt{x^2 - a^2}\right) \cos(cx) dx = \frac{\exp(-a\sqrt{b^2 - c^2})}{\sqrt{b^2 - c^2}} \quad [0 < c < b]$$

$$= \frac{-\sin(a\sqrt{c^2 - b^2})}{\sqrt{c^2 - b^2}} \quad [0 < b < c] \quad \text{ET I 57(48)a}$$

$$3.^6 \quad \int_0^\infty J_0\left(\alpha\sqrt{x^2 + z^2}\right) \cos \beta x dx = \frac{\cos z\sqrt{\alpha^2 - \beta^2}}{\sqrt{\alpha^2 - \beta^2}} \quad [0 < \beta < \alpha, \quad z > 0]$$

$$= 0 \quad [0 < \alpha < \beta, \quad z > 0] \quad \text{MO 47a}$$

$$4. \quad \int_0^\infty Y_0\left(\alpha\sqrt{x^2 + z^2}\right) \cos \beta x dx = \frac{1}{\sqrt{\alpha^2 - \beta^2}} \sin\left(z\sqrt{\alpha^2 - \beta^2}\right) \quad [0 < \beta < \alpha, \quad z > 0]$$

$$= -\frac{1}{\sqrt{\beta^2 - \alpha^2}} \exp\left(-z\sqrt{\beta^2 - \alpha^2}\right) \quad [0 < \alpha < \beta, \quad z > 0] \quad \text{MO 47a}$$

$$5. \quad \int_0^\infty K_0\left[\alpha\sqrt{x^2 + \beta^2}\right] \cos(\gamma x) dx = \frac{\pi}{2\sqrt{\alpha^2 + \gamma^2}} \exp\left(-\beta\sqrt{\alpha^2 + \gamma^2}\right) \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma > 0]$$

$$\quad \text{ET I 56(43)}$$

$$6. \quad \int_0^a J_0\left(b\sqrt{a^2 - x^2}\right) \cos(cx) dx = \frac{\sin(a\sqrt{b^2 + c^2})}{\sqrt{b^2 + c^2}} \quad [b > 0] \quad \text{MO 48a, ET I 57(47)}$$

$$7. \quad \int_0^\infty J_0\left(b\sqrt{x^2 - a^2}\right) \cos(cx) dx = \frac{\cosh(a\sqrt{b^2 - c^2})}{\sqrt{b^2 - c^2}} \quad [0 < c < b, \quad a > 0]$$

$$= 0 \quad [0 < b < c, \quad a > 0] \quad \text{ET I 57(49)}$$

8.
$$\int_0^\infty H_0^{(1)}(\alpha\sqrt{\beta^2 - x^2}) \cos(\gamma x) dx = -i \frac{\exp(i\beta\sqrt{\alpha^2 + \gamma^2})}{\sqrt{\alpha^2 + \gamma^2}} \left[\pi > \arg \sqrt{\beta^2 - x^2} \geq 0, \quad \alpha > 0, \quad \gamma > 0 \right] \quad \text{ET I 59(59)}$$
9.
$$\int_0^\infty H_0^{(2)}(\alpha\sqrt{\beta^2 - x^2}) \cos(\gamma x) dx = \frac{i \exp(-i\beta\sqrt{\alpha^2 + \gamma^2})}{\sqrt{\alpha^2 + \gamma^2}} \left[-\pi < \arg \sqrt{\beta^2 - x^2} \leq 0, \quad \alpha > 0, \quad \gamma > 0 \right] \quad \text{ET I 58(58)}$$
- 6.678**
$$\int_0^\infty \left[K_0(2\sqrt{x}) + \frac{\pi}{2} Y_0(2\sqrt{x}) \right] \sin(bx) dx = \frac{\pi}{2b} \sin\left(\frac{1}{b}\right) \quad [b > 0] \quad \text{ET I 111(34)}$$
- 6.679**
1.
$$\int_0^\infty J_{2\nu} \left[2b \sinh\left(\frac{x}{2}\right) \right] \sin(bx) dx = -i [I_{\nu-ib}(a) K_{\nu+ib}(a) - I_{\nu+ib}(a) K_{\nu-ib}(a)] \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 115(59)}$$
2.
$$\int_0^\infty J_{2\nu} \left[2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = I_{\nu-ib}(a) K_{\nu+ib}(a) + I_{\nu+ib}(a) K_{\nu-ib}(a) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 59(64)}$$
3.
$$\int_0^\infty J_{2\nu} \left[2a \cosh\left(\frac{x}{2}\right) \right] \cos(bx) dx = -\frac{\pi}{2} [J_{\nu+ib}(a) Y_{\nu-ib}(a) + J_{\nu-ib}(a) Y_{\nu+ib}(a)] \quad \text{ET I 59(63)}$$
4.
$$\int_0^\infty J_0 \left[2a \sinh\left(\frac{x}{2}\right) \right] \sin(bx) dx = \frac{2}{\pi} \sinh(\pi b) [K_{ib}(a)]^2 \quad [a > 0, \quad b > 0] \quad \text{ET I 115(58)}$$
5.
$$\int_0^\infty J_0 \left[2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = [I_{ib}(a) + I_{-ib}(a)] K_{ib}(a) \quad [a > 0, \quad b > 0] \quad \text{ET I 59(62)}$$
6.
$$\int_0^\infty Y_0 \left[2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = -\frac{2}{\pi} \cosh(\pi b) [K_{ib}(a)]^2 \quad [a > 0, \quad b > 0] \quad \text{ET I 59(65)}$$
7.
$$\int_0^\infty K_0 \left[2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = \frac{\pi^2}{4} \left\{ [J_{ib}(a)]^2 + [Y_{ib}(a)]^2 \right\} \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 59(66)}$$

6.681

1.
$$\int_0^{\frac{\pi}{2}} \cos(2\mu x) J_{2\nu}(2a \cos x) dx = \frac{\pi}{2} J_{\nu+\mu}(a) J_{\nu-\mu}(a) \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 361(23)}$$
2.
$$\int_0^{\frac{\pi}{2}} \cos(2\mu x) Y_{2\nu}(2a \cos x) dx = \frac{\pi}{2} [\cot(2\nu\pi) J_{\nu+\mu}(a) J_{\nu-\mu}(a) - \operatorname{cosec}(2\nu\pi) J_{\mu-\nu}(a) J_{-\mu-\nu}(a)] \quad [|\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 361(24)}$$

3. $\int_0^{\frac{\pi}{2}} \cos(2\mu x) I_{2\nu}(2a \cos x) dx = \frac{\pi}{2} I_{\nu-\mu}(a) I_{\nu+\mu}(a) \quad [\operatorname{Re} \nu > -\frac{1}{2}]$ ET I 59(61)
4. $\int_0^{\frac{\pi}{2}} \cos(\nu x) K_\nu(2a \cos x) dx = \frac{\pi}{2} I_0(a) K_\nu(a) \quad [\operatorname{Re} \nu < 1]$ WA 484(3)
5. $\int_0^\pi J_0(2z \cos x) \cos 2nx dx = (-1)^n \pi J_n^2(z).$ MO 45
6. $\int_0^\pi J_0(2z \sin x) \cos 2nx dx = \pi J_n^2(z).$ WA 43(3), MO 45
7. $\int_0^{\frac{\pi}{2}} \cos(2n\pi) Y_0(2a \sin x) dx = \frac{\pi}{2} J_n(a) Y_n(a) \quad [n = 0, 1, 2, \dots]$ ET II 360(16)
8. $\int_0^\pi \sin(2\mu x) J_{2\nu}(2a \sin x) dx = \pi \sin(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a)$
 $[\operatorname{Re} \nu > -1]$ ET II 360(13)
9. $\int_0^\pi \cos(2\mu x) J_{2\nu}(2a \sin x) dx = \pi \cos(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a)$
 $[\operatorname{Re} \nu > -\frac{1}{2}]$ ET II 360(14)
10. $\int_0^{\frac{\pi}{2}} J_{\nu+\mu}(2z \cos x) \cos[(\nu - \mu)x] dx = \frac{\pi}{2} J_\nu(z) J_\mu(z) \quad [\operatorname{Re}(\nu + \mu) > -1]$ MO 42
11. $\int_0^{\frac{\pi}{2}} \cos[(\mu - \nu)x] I_{\mu+\nu}(2a \cos x) dx = \frac{\pi}{2} I_\mu(a) I_\nu(a) \quad [\operatorname{Re}(\mu + \nu) > -1]$
 WA 484(2), ET II 378(39)
12. $\int_0^{\frac{\pi}{2}} \cos[(\mu - \nu)x] K_{\mu+\nu}(2a \cos x) dx = \frac{\pi^2}{4} \operatorname{cosec}[(\mu + \nu)\pi] [I_{-\mu}(a) I_{-\nu}(a) - I_\mu(a) I_\nu(a)]$
 $[|\operatorname{Re}(\mu + \nu)| < 1]$ ET II 378(40)
- 13.⁸ $\int_0^{\frac{\pi}{2}} K_{\nu-m}(2a \cos x) \cos[(m + \nu)x] dx = (-1)^m \frac{\pi}{2} I_m(a) K_\nu(a)$
 $[|\operatorname{Re}(\nu - m)| < 1]$ WA 485(4)

6.682

- 1.⁷ $\int_0^{\frac{\pi}{2}} J_{\nu-\frac{1}{2}}(x \sin t) \sin^{\nu+\frac{1}{2}} t dt = \sqrt{\frac{\pi}{2x}} J_\nu(x)$
 $[\nu \text{ may be zero, a natural number, one half, or a natural number plus one half; } x > 0]$ MO 42a
2. $\int_0^{\frac{\pi}{2}} J_\nu(z \sin x) \sin^\nu x \cos^{2\nu} x dx = 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) z^{-\nu} J_\nu^2\left(\frac{z}{2}\right)$
 $[\operatorname{Re} \nu > -\frac{1}{2}]$ MO 42a

6.683

1.
$$\int_0^{\frac{\pi}{2}} J_\nu(z \sin x) I_\mu(z \cos x) \tan^{\nu+1} x \, dx = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{\mu-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2} + 1\right)} J_\mu(z)$$

[Re $\nu > \text{Re } \mu > -1$] WA 407(4)
2.
$$\int_0^{\frac{\pi}{2}} J_\nu(z_1 \sin x) J_\mu(z_2 \cos x) \sin^{\nu+1} x \cos^{\mu+1} x \, dx = \frac{z_1^\nu z_2^\mu J_{\nu+\mu+1}\left(\sqrt{z_1^2 + z_2^2}\right)}{\sqrt{(z_1^2 + z_2^2)^{\nu+\mu+1}}}$$

[Re $\nu > -1$, Re $\mu > -1$] WA 410(1)
3.
$$\int_0^{\frac{\pi}{2}} J_\nu(z \cos^2 x) J_\mu(z \sin^2 x) \sin x \cos x \, dx = \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k J_{\nu+\mu+2k+1}(z)$$

[Re $\nu > -1$, Re $\mu > -1$] (see also 6.513 6) WA 414(1)
4.
$$\int_0^{\frac{\pi}{2}} J_\mu(z \sin \theta) (\sin \theta)^{1-\mu} (\cos \theta)^{2\nu+1} \, d\theta = \frac{s_{\mu+\nu, \nu-\mu+1}(z)}{2^{\mu-1} z^{\nu+1} \Gamma(\mu)}$$

[Re $\nu > -1$] WA 407(2)
5.
$$\int_0^{\frac{\pi}{2}} J_\mu(z \sin \theta) (\sin \theta)^{1-\mu} \, d\theta = \frac{\mathbf{H}_{\mu-\frac{1}{2}}(z)}{\sqrt{\frac{2z}{\pi}}}$$

WA 407(3)
6.
$$\int_0^{\frac{\pi}{2}} J_\mu(a \sin \theta) (\sin \theta)^{\mu+1} (\cos \theta)^{2\varrho+1} \, d\theta = 2^\varrho \Gamma(\varrho+1) a^{-\varrho-1} J_{\varrho+\mu+1}(a)$$

[Re $\varrho > -1$, Re $\mu > -1$] WA 406(1), EH II 46(5)
7.
$$\begin{aligned} \int_0^{\frac{\pi}{2}} J_\nu(2z \sin \theta) (\sin \theta)^\nu (\cos \theta)^{2\nu} \, d\theta \\ = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m z^{\nu+2m} \Gamma\left(\nu+m+\frac{1}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right)}{m! \Gamma(\nu+m+1) \Gamma(2\nu+m+1)} \\ = \frac{1}{2} z^{-\nu} \sqrt{\pi} \Gamma\left(\nu+\frac{1}{2}\right) [J_\nu(z)]^2 \end{aligned}$$

[Re $\nu > -\frac{1}{2}$] EH II 47(10)
8.
$$\int_0^{\frac{\pi}{2}} J_\nu(z \sin \theta) (\sin \theta)^{\nu+1} (\cos \theta)^{-2\nu} \, d\theta = 2^{-\nu} \frac{z^{\nu-1}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - \nu\right) \sin z$$

[$-1 < \text{Re } \nu < \frac{1}{2}$] EH II 68(39)
9.
$$\int_0^{\frac{\pi}{2}} J_\nu(z \sin^2 \theta) J_\nu(z \cos^2 \theta) (\sin \theta)^{2\nu+1} (\cos \theta)^{2\nu+1} \, d\theta = \frac{\Gamma\left(\frac{1}{2} + \nu\right) J_{2\nu+\frac{1}{2}}(z)}{2^{2\nu+\frac{3}{2}} \Gamma(\nu+1) \sqrt{z}}$$

[Re $\nu > -\frac{1}{2}$] WA 409(1)

$$10. \quad \int_0^{\frac{\pi}{2}} J_{\mu}(z \sin^2 \theta) J_{\nu}(z \cos^2 \theta) \sin^{2\mu+1} \theta \cos^{2\nu+1} \theta d\theta = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2}) J_{\mu+\nu+\frac{1}{2}}(z)}{2\sqrt{\pi} \Gamma(\mu + \nu + 1) \sqrt{2z}} \\ [\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 417(1)}$$

6.684

$$1.^8 \quad \int_0^{\pi} (\sin x)^{2\nu} \frac{J_{\nu}(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})}{(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})^{\nu}} dx = 2^{\nu} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) \frac{J_{\nu}(\alpha)}{\alpha^{\nu}} \frac{J_{\nu}(\beta)}{\beta^{\nu}} \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 362(27)}$$

$$2. \quad \int_0^{\pi} (\sin x)^{2\nu} \frac{Y_{\nu}(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})}{(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})^{\nu}} dx = 2^{\nu} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) \frac{J_{\nu}(\alpha)}{\alpha^{\nu}} \frac{Y_{\nu}(\beta)}{\beta^{\nu}} \\ [|\alpha| < |\beta|, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 362(28)}$$

$$\mathbf{6.685} \quad \int_0^{\frac{\pi}{2}} \sec x \cos(2\lambda x) K_{2\mu}(a \sec x) dx = \frac{\pi}{2a} W_{\lambda, \mu}(a) W_{-\lambda, \mu}(a) \quad [\operatorname{Re} a > 0] \quad \text{ET II 378(41)}$$

6.686

$$1. \quad \int_0^{\infty} \sin(ax^2) J_{\nu}(bx) dx = -\frac{\sqrt{\pi}}{2\sqrt{a}} \sin\left(\frac{b^2}{8a} - \frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \\ [a > 0, b > 0, \operatorname{Re} \nu > -3] \quad \text{ET II 34(13)}$$

$$2. \quad \int_0^{\infty} \cos(ax^2) J_{\nu}(bx) dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \cos\left(\frac{b^2}{8a} - \frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 38(38)}$$

$$3. \quad \int_0^{\infty} \sin(ax^2) Y_{\nu}(bx) dx \\ = -\frac{\sqrt{\pi}}{4\sqrt{a}} \sec\left(\frac{\nu\pi}{2}\right) \\ \times \left[\cos\left(\frac{b^2}{8a} - \frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) - \sin\left(\frac{b^2}{8a} + \frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \right] \\ [a > 0, \quad b > 0, \quad -3 < \operatorname{Re} \nu < 3] \quad \text{ET II 107(7)}$$

$$4. \quad \int_0^{\infty} \cos(ax^2) Y_{\nu}(bx) dx \\ = \frac{\sqrt{\pi}}{4\sqrt{a}} \sec\left(\frac{\nu\pi}{2}\right) \\ \times \left[\sin\left(\frac{b^2}{8a} - \frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) + \cos\left(\frac{b^2}{8a} + \frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \right] \\ [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < 1] \quad \text{ET II 107(8)}$$

$$5. \quad \int_0^{\infty} \sin(ax^2) J_1(bx) dx = \frac{1}{b} \sin \frac{b^2}{4a} \quad [a > 0, \quad b > 0] \quad \text{ET II 19(16)}$$

$$6. \quad \int_0^\infty \cos(ax^2) J_1(bx) dx = \frac{2}{b} \sin^2\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET II 20(20)}$$

$$7. \quad \int_0^\infty \sin^2(ax^2) J_1(bx) dx = \frac{1}{2b} \cos\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET II 19(17)}$$

$$\begin{aligned} 6.687 \quad \int_0^\infty \cos\left(\frac{x^2}{2a}\right) K_{2\nu}(xe^{i\frac{\pi}{4}}) K_{2\nu}(xe^{-i\frac{\pi}{4}}) dx \\ = \frac{\Gamma\left(\frac{1}{4} + \nu\right) \Gamma\left(\frac{1}{4} - \nu\right) \sqrt{\pi}}{8\sqrt{a}} W_{\frac{1}{4}, \nu}(ae^{i\frac{\pi}{2}}) W_{\frac{1}{4}, \nu}(ae^{-i\frac{\pi}{2}}) \\ [a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{4}] \quad \text{ET II 372(1)} \end{aligned}$$

6.688

$$1. \quad \int_0^{\frac{\pi}{2}} J_\nu(\mu z \sin t) \cos(\mu x \cos t) dt = \frac{\pi}{2} J_{\frac{\nu}{2}}\left(\mu \frac{\sqrt{x^2 + z^2} + x}{2}\right) J_{\frac{\nu}{2}}\left(\mu \frac{\sqrt{x^2 + z^2} - x}{2}\right) \\ [\operatorname{Re} \nu > -1, \quad \operatorname{Re} z > 0] \quad \text{MO 46}$$

$$2. \quad \int_0^{\frac{\pi}{2}} (\sin x)^{\nu+1} \cos(\beta \cos x) J_\nu(\alpha \sin x) dx = 2^{-\frac{1}{2}} \sqrt{\pi} \alpha^\nu (\alpha^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{1}{4}} J_{\nu+\frac{1}{2}}\left[(\alpha^2 + \beta^2)^{\frac{1}{2}}\right] \\ [\operatorname{Re} \nu > -1] \quad \text{ET II 361(19)}$$

$$3. \quad \int_0^{\frac{\pi}{2}} \cos[(z - \zeta) \cos \theta] J_{2\nu}[2\sqrt{z\zeta} \sin \theta] d\theta = \frac{\pi}{2} J_\nu(z) J_\nu(\zeta) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{EH II 47(8)}$$

6.69–6.74 Combinations of Bessel and trigonometric functions and powers

$$6.691 \quad \int_0^\infty x \sin(bx) K_0(ax) dx = \frac{\pi b}{2} (a^2 + b^2)^{-\frac{3}{2}} \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 105(47)}$$

6.692

$$1. \quad \int_0^\infty x K_\nu(ax) I_\nu(bx) \sin(cx) dx = -\frac{1}{2} (ab)^{-\frac{3}{2}} c (u^2 - 1)^{-\frac{1}{2}} Q_{\nu-\frac{1}{2}}^1(u), \quad u = (2ab)^{-1} (a^2 + b^2 + c^2) \\ [\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET I 106(54)}$$

$$2. \quad \int_0^\infty x K_\nu(ax) K_\nu(bx) \sin(cx) dx = \frac{\pi}{4} (ab)^{-\frac{3}{2}} c (u^2 - 1)^{-\frac{1}{2}} \Gamma\left(\frac{3}{2} + \nu\right) \Gamma\left(\frac{3}{2} - \nu\right) P_{\nu-\frac{1}{2}}^{-1}(u) \\ u = (2ab)^{-1} (a^2 + b^2 + c^2) \quad [\operatorname{Re}(a+b) > 0, \quad c > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2}] \quad \text{ET I 107(61)}$$

6.693

$$1. \quad \int_0^\infty J_\nu(\alpha x) \sin \beta x \frac{dx}{x} = \frac{1}{\nu} \sin\left(\nu \arcsin \frac{\beta}{\alpha}\right) \quad [\beta \leq \alpha] \\ = \frac{\alpha^\nu \sin \frac{\nu\pi}{2}}{\nu (\beta + \sqrt{\beta^2 - \alpha^2})^\nu} \quad [\beta \geq \alpha] \\ [\operatorname{Re} \nu > -1] \quad \text{WA 443(2)}$$

$$\begin{aligned}
2.^8 \quad \int_0^\infty J_\nu(\alpha x) \cos \beta x \frac{dx}{x} &= \frac{1}{\nu} \cos \left(\nu \arcsin \frac{\beta}{\alpha} \right) & [\beta \leq \alpha] \\
&= \frac{\alpha^\nu \cos \frac{\nu\pi}{2}}{\nu \left(\beta + \sqrt{\beta^2 - \alpha^2} \right)^\nu} & [\beta \geq \alpha] \quad [\operatorname{Re} \nu > 0]
\end{aligned}$$

WA 443(3)

$$\begin{aligned}
3. \quad \int_0^\infty Y_\nu(ax) \sin(bx) \frac{dx}{x} &= -\frac{1}{\nu} \tan \left(\frac{\nu\pi}{2} \right) \sin \left[\nu \arcsin \left(\frac{b}{a} \right) \right] \\
&= \frac{1}{2\nu} \sec \left(\frac{\nu\pi}{2} \right) \left\{ a^{-\nu} \cos(\nu\pi) \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu - a^\nu \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^{-\nu} \right\} \\
&\quad [0 < b < a, \quad |\operatorname{Re} \nu| < 1] \\
&\quad [0 < a < b, \quad |\operatorname{Re} \nu| < 1] \\
&\quad \text{ET I 103(35)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty J_\nu(ax) \sin(bx) \frac{dx}{x^2} &= \frac{\sqrt{a^2 - b^2} \sin \left[\nu \arcsin \left(\frac{b}{a} \right) \right]}{\nu^2 - 1} - \frac{b \cos \left[\nu \arcsin \left(\frac{b}{a} \right) \right]}{\nu (\nu^2 - 1)} & [0 < b < a, \quad \operatorname{Re} \nu > 0] \\
&= \frac{-a^\nu \cos \left(\frac{\nu\pi}{2} \right) \left[b + \nu \sqrt{b^2 - a^2} \right]}{\nu (\nu^2 - 1) \left[b + \sqrt{b^2 - a^2} \right]^\nu} & [0 < a < b, \quad \operatorname{Re} \nu > 0]
\end{aligned}$$

ET I 99(6)

$$\begin{aligned}
5. \quad \int_0^\infty J_\nu(ax) \cos(bx) \frac{dx}{x^2} &= \frac{a \cos \left[(\nu - 1) \arcsin \left(\frac{b}{a} \right) \right]}{2\nu(\nu - 1)} + \frac{a \cos \left[(\nu + 1) \arcsin \left(\frac{b}{a} \right) \right]}{2\nu(\nu + 1)} & [0 < b < a, \quad \operatorname{Re} \nu > 1] \\
&= \frac{a^\nu \sin \left(\frac{\nu\pi}{2} \right)}{2\nu(\nu - 1) \left[b + \sqrt{b^2 - a^2} \right]^{\nu-1}} - \frac{a^{\nu+2} \sin \left(\frac{\nu\pi}{2} \right)}{2\nu(\nu + 1) \left[b + \sqrt{b^2 - a^2} \right]^{\nu+1}} & [0 < a < b, \quad \operatorname{Re} \nu > 1]
\end{aligned}$$

ET I 44(6)

$$\begin{aligned}
6. \quad \int_0^\infty J_0(\alpha x) \sin x \frac{dx}{x} &= \frac{\pi}{2} & [0 < \alpha < 1] \\
&= \operatorname{arccosec} \alpha & [\alpha > 1]
\end{aligned}$$

WH

$$\begin{aligned}
7. \quad \int_0^\infty J_0(x) \sin \beta x \frac{dx}{x} &= \frac{\pi}{2} & [\beta > 1] \\
&= \arcsin \beta & [\beta^2 < 1] \\
&= -\frac{\pi}{2} & [\beta < -1]
\end{aligned}$$

$$8. \quad \int_0^\infty [J_0(x) - \cos \alpha x] \frac{dx}{x} = \ln 2\alpha \quad \text{NT 66(13)}$$

$$9. \quad \int_0^z J_\nu(x) \sin(z-x) \frac{dx}{x} = \frac{2}{\nu} \sum_{k=0}^\infty (-1)^k J_{\nu+2k+1}(z) \quad [\operatorname{Re} \nu > 0] \quad \text{WA 416(4)}$$

$$10. \quad \int_0^z J_\nu(x) \cos(z-x) \frac{dx}{x} = \frac{1}{\nu} J_\nu(z) + \frac{2}{\nu} \sum_{k=1}^{\infty} (-1)^k J_{\nu+2k}(z) \quad [\operatorname{Re} \nu > 0] \quad \text{WA 416(5)}$$

$$\begin{aligned} 6.694^{10} \quad & \int_0^\infty \left[\frac{J_1(ax)}{x} \right]^2 \sin(bx) dx \\ &= \frac{1}{2}b - \left(\frac{4a}{3\pi} \right) \left[\left(1 + \frac{b^2}{4a^2} \right) E\left(\frac{b}{2a} \right) + \left(1 - \frac{b^2}{4a^2} \right) K\left(\frac{b}{2a} \right) \right] \quad [0 \leq b \leq 2a] \quad \text{ET I 102(22)} \\ &= \frac{1}{2}b - \frac{2b}{3\pi} \left[\left(1 + \frac{b^2}{4a^2} \right) E\left(\frac{2a}{b} \right) - \left(1 - \left(\frac{4a^2}{b^2} \right)^{-1} \right) K\left(\frac{2a}{b} \right) \right] \quad [0 \leq 2a \leq b] \end{aligned}$$

6.695

$$1. \quad \int_0^\infty \frac{\sin \alpha x}{\beta^2 + x^2} J_0(ux) dx = \frac{\sinh \alpha \beta}{\beta} K_0(\beta u) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad u > \alpha] \quad \text{MO 46}$$

$$2. \quad \int_0^\infty \frac{\cos \alpha x}{\beta^2 + x^2} J_0(ux) dx = \frac{\pi e^{-\alpha \beta}}{2\beta} I_0(\beta u) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad -\alpha < u < \alpha] \quad \text{MO 46}$$

$$3. \quad \int_0^\infty \frac{x}{x^2 + \beta^2} \sin(\alpha x) J_0(\gamma x) dx = \frac{\pi}{2} e^{-\alpha \beta} I_0(\gamma \beta) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad 0 < \gamma < \alpha] \quad \text{ET II 10(36)}$$

$$4. \quad \int_0^\infty \frac{x}{x^2 + \beta^2} \cos(\alpha x) J_0(\gamma x) dx = \cosh(\alpha \beta) K_0(\beta \gamma) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \alpha < \gamma] \quad \text{ET II 11(45)}$$

$$\begin{aligned} 6.696 \quad & \int_0^\infty [1 - \cos(\alpha x)] J_0(\beta x) \frac{dx}{x} = \operatorname{arccosh} \left(\frac{\alpha}{\beta} \right) \quad [0 < \beta < \alpha] \\ &= 0 \quad [0 < \alpha < \beta] \quad \text{ET II 11(43)} \end{aligned}$$

6.697

$$\begin{aligned} 1. \quad & \int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_0(x) dx = 2 \int_0^\alpha \frac{\cos \beta u}{\sqrt{1-u^2}} du \quad [0 \leq \alpha \leq 1] \quad \text{WA 463(2)} \\ &= \pi J_0(\beta) \quad [1 \leq \alpha < \infty] \quad \text{WA 463(1), ET II 345(42)} \end{aligned}$$

$$2. \quad \int_0^\infty \frac{\sin(x+t)}{x+t} J_0(t) dt = \frac{\pi}{2} J_0(x) \quad [x > 0] \quad \text{WA 475(4)}$$

$$3. \quad \int_0^\infty \frac{\cos(x+t)}{x+t} J_0(t) dt = -\frac{\pi}{2} Y_0(x) \quad [x > 0] \quad \text{WA 475(5)}$$

$$4. \quad \int_{-\infty}^\infty \frac{|x|}{x+\beta} \sin[\alpha(x+\beta)] J_0(bx) dx = 0 \quad [0 \leq \alpha < b] \quad \text{WA 464(5), ET II 345(43)a}$$

$$5. \quad \int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta} \left[J_{n+\frac{1}{2}}(x) \right]^2 dx = \pi \left[J_{n+\frac{1}{2}}(\beta) \right]^2 \quad [2 \leq \alpha < \infty, \quad n = 0, 1, \dots] \quad \text{ET II 346(45)}$$

$$6. \quad \int_{-\infty}^{\infty} \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_{n+\frac{1}{2}}(x) J_{-n-\frac{1}{2}}(x) dx = \pi J_{n+\frac{1}{2}}(\beta) J_{-n-\frac{1}{2}}(\beta) \\ [2 \leq \alpha < \infty, \quad n = 0, 1, \dots] \quad \text{ET II 346(46)}$$

$$7. \quad \int_{-\infty}^{\infty} \frac{J_{\mu}[a(z+x)]}{(z+x)^{\mu}} \frac{J_{\nu}[a(\zeta+x)]}{(\zeta+x)^{\nu}} dx = \frac{\Gamma(\mu+\nu) \sqrt{\pi} \sqrt{\frac{2}{a}}}{\Gamma(\mu+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})} \cdot \frac{J_{\mu+\nu-\frac{1}{2}}[a(z-\zeta)]}{(z-\zeta)^{\mu+\nu-\frac{1}{2}}} \\ [\operatorname{Re}(\mu+\nu) > 0] \quad \text{WA 463(3)}$$

6.698

$$1. \quad \int_0^{\infty} \sqrt{x} J_{\nu+\frac{1}{4}}(ax) J_{-\nu+\frac{1}{4}}(ax) \sin(bx) dx = \sqrt{\frac{2}{\pi b}} \frac{\cos[2\nu \arccos(\frac{b}{2a})]}{\sqrt{4a^2 - b^2}} \quad [0 < b < 2a] \\ = 0 \quad [0 < 2a < b] \quad \text{ET I 102(26)}$$

$$2. \quad \int_0^{\infty} \sqrt{x} J_{\nu-\frac{1}{4}}(ax) J_{-\nu-\frac{1}{4}}(ax) \cos(bx) dx = \sqrt{\frac{2}{\pi b}} \frac{\cos[2\nu \arccos(\frac{b}{2a})]}{\sqrt{4a^2 - b^2}} \quad [0 < b < 2a] \\ = 0 \quad [0 < 2a < b] \quad \text{ET I 46(24)}$$

$$3. \quad \int_0^{\infty} \sqrt{x} I_{\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \sin(bx) dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b + \sqrt{a^2 + b^2})^{2\nu}}{\sqrt{a^2 + b^2}} \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{5}{4}] \quad \text{ET I 106(56)}$$

$$4. \quad \int_0^{\infty} \sqrt{x} I_{-\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{-\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \cos(bx) dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b + \sqrt{a^2 + b^2})^{2\nu}}{\sqrt{a^2 + b^2}} \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{3}{4}] \quad \text{ET I 50(49)}$$

6.699

$$1. \quad \int_0^{\infty} x^{\lambda} J_{\nu}(ax) \sin(bx) dx = 2^{1+\lambda} a^{-(2+\lambda)} b \frac{\Gamma(\frac{2+\lambda+\nu}{2})}{\Gamma(\frac{\nu-\lambda}{2})} F\left(\frac{2+\lambda+\nu}{2}, \frac{2+\lambda-\nu}{2}; \frac{3}{2}; \frac{b^2}{a^2}\right) \\ [0 < b < a, \quad -\operatorname{Re} \nu - 1 < 1 + \operatorname{Re} \lambda < \frac{3}{2}] \\ = \left(\frac{1}{2}a\right)^{\nu} b^{-(\nu+\lambda+1)} \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)} \sin\left[\pi\left(\frac{1+\lambda+\nu}{2}\right)\right] \\ \times F\left(\frac{2+\lambda+\nu}{2}, \frac{1+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right) \\ [0 < a < b, \quad -\operatorname{Re} \nu - 1 < 1 + \operatorname{Re} \lambda < \frac{3}{2}] \quad \text{ET I 100(11)}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^\lambda J_\nu(ax) \cos(bx) dx &= \frac{2^\lambda a^{-(1+\lambda)} \Gamma\left(\frac{1+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda+1}{2}\right)} F\left(\frac{1+\lambda+\nu}{2}, \frac{1+\lambda-\nu}{2}; \frac{1}{2}; \frac{b^2}{a^2}\right) \\
&\quad [0 < b < a, \quad -\operatorname{Re} \nu < 1 + \operatorname{Re} \lambda < \frac{3}{2}] \\
&= \frac{\left(\frac{a}{2}\right)^\nu b^{-(\nu+1+\lambda)} \Gamma(1+\lambda+\nu) \cos\left[\frac{\pi}{2}(1+\lambda+\nu)\right]}{\Gamma(\nu+1)} F\left(\frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right) \\
&\quad [0 < a < b, \quad -\operatorname{Re} \nu < 1 + \operatorname{Re} \lambda < \frac{3}{2}] \\
&\quad \text{ET I 45(13)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^\lambda K_\mu(ax) \sin(bx) dx &= \frac{2^\lambda b \Gamma\left(\frac{2+\mu+\lambda}{2}\right) \Gamma\left(\frac{2+\lambda-\mu}{2}\right)}{a^{2+\lambda}} F\left(\frac{2+\mu+\lambda}{2}, \frac{2+\lambda-\mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right) \\
&\quad [\operatorname{Re}(-\lambda \pm \mu) < 2, \quad \operatorname{Re} a > 0, \quad b > 0] \\
&\quad \text{ET I 106(50)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^\lambda K_\mu(ax) \cos(bx) dx &= 2^{\lambda-1} a^{-\lambda-1} \Gamma\left(\frac{\mu+\lambda+1}{2}\right) \Gamma\left(\frac{1+\lambda-\mu}{2}\right) \\
&\quad \times F\left(\frac{\mu+\lambda+1}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2}\right) \\
&\quad [\operatorname{Re}(-\lambda \pm \mu) < 1, \quad \operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 49(42)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^\nu \sin(ax) J_\nu(bx) dx &= \frac{\sqrt{\pi} 2^\nu b^\nu (a^2 - b^2)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \nu\right)} \quad [0 < b < a, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \\
&= 0 \quad [0 < a < b, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \\
&\quad \text{ET II 32(4)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x^\nu \cos(ax) J_\nu(bx) dx &= -2^\nu \frac{\sin(\nu\pi)}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \nu\right) b^\nu (a^2 - b^2)^{-\nu-\frac{1}{2}} \quad [0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\
&= 2^\nu \frac{b^\nu}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \nu\right) (b^2 - a^2)^{-\nu-\frac{1}{2}} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\
&\quad \text{ET II 36(29)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x^{\nu+1} \sin(ax) J_\nu(bx) dx &= -2^{1+\nu} a \frac{\sin(\nu\pi)}{\sqrt{\pi}} b^\nu \Gamma\left(\nu + \frac{3}{2}\right) (a^2 - b^2)^{-\nu-\frac{3}{2}} \quad [0 < b < a, \quad -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}] \\
&= -\frac{2^{1+\nu}}{\sqrt{\pi}} a b^\nu \Gamma\left(\nu + \frac{3}{2}\right) (b^2 - a^2)^{-\nu-\frac{3}{2}} \quad [0 < a < b, \quad -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}] \\
&\quad \text{ET II 32(3)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_0^\infty x^{\nu+1} \cos(ax) J_\nu(bx) dx &= 2^{1+\nu} \sqrt{\pi} a b^\nu \frac{(a^2 - b^2)^{-\nu-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2} - \nu\right)} \quad [0 < b < a, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}] \\
&= 0 \quad [0 < a < b, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}] \\
&\quad \text{ET II 36(28)}
\end{aligned}$$

$$9. \quad \int_0^1 x^\nu \sin(ax) J_\nu(ax) dx = \frac{1}{2\nu+1} [\sin a J_\nu(a) - \cos a J_{\nu+1}(a)]$$

$$[\operatorname{Re} \nu > -1] \quad \text{ET II 334(9)a}$$

$$10. \quad \int_0^1 x^\nu \cos(ax) J_\nu(ax) dx = \frac{1}{2\nu+1} [\cos a J_\nu(a) + \sin a J_{\nu+1}(a)]$$

$$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 335(20)}$$

$$11. \quad \int_0^\infty x^{1+\nu} K_\nu(ax) \sin(bx) dx = \sqrt{\pi} (2a)^\nu \Gamma\left(\frac{3}{2} + \nu\right) b (b^2 + a^2)^{-\frac{3}{2}-\nu}$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}]$$

$$\text{ET I 105(49)}$$

$$12. \quad \int_0^\infty x^\mu K_\mu(ax) \cos(bx) dx = \frac{1}{2} \sqrt{\pi} (2a)^\mu \Gamma\left(\mu + \frac{1}{2}\right) (b^2 + a^2)^{-\mu-\frac{1}{2}}$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}]$$

$$\text{ET I 49(41)}$$

$$13. \quad \int_0^\infty x^\nu Y_{\nu-1}(ax) \sin(bx) dx = 0 \quad [0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$$

$$= \frac{2^\nu \sqrt{\pi} a^{\nu-1} b}{\Gamma(\frac{1}{2} - \nu)} (b^2 - a^2)^{-\nu-\frac{1}{2}} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$$

$$\text{ET I 104(36)}$$

$$14. \quad \int_0^\infty x^\nu Y_\nu(ax) \cos(bx) dx = 0 \quad [0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$$

$$= -2^\nu \sqrt{\pi} a^\nu \frac{(b^2 - a^2)^{-\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2} - \nu)} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$$

$$\text{ET I 47(30)}$$

6.711

$$1. \quad \int_0^\infty x^{\nu-\mu} J_\mu(ax) J_\nu(bx) \sin(cx) dx = 0 \quad [0 < c < b - a, \quad -1 < \operatorname{Re} \nu < 1 + \operatorname{Re} \mu]$$

$$\text{ET I 103(28)}$$

$$2. \quad \int_0^\infty x^{\nu-\mu+1} J_\mu(ax) J_\nu(bx) \cos(cx) dx = 0$$

$$[0 < c < b - a, \quad a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} \mu] \quad \text{ET I 47(25)}$$

$$3. \quad \int_0^\infty x^{\nu-\mu-2} J_\mu(ax) J_\nu(bx) \sin(cx) dx = 2^{\nu-\mu-1} a^\mu b^{-\nu} \frac{c \Gamma(\nu)}{\Gamma(\mu+1)}$$

$$[0 < a, \quad 0 < b, \quad 0 < c < b - a, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu + 3] \quad \text{ET I 103(29)}$$

$$4. \quad \int_0^\infty x^{\varrho-\mu-1} J_\mu(ax) J_\varrho(bx) \cos(cx) dx = 2^{\varrho-\mu-1} b^{-\varrho} a^\mu \frac{\Gamma(\varrho)}{\Gamma(\mu+1)}$$

$$[b > 0, \quad a > 0, \quad 0 < c < b - a, \quad 0 < \operatorname{Re} \varrho < \operatorname{Re} \mu + 2] \quad \text{ET I 47(26)}$$

$$5. \quad \int_0^\infty x^{1-2\nu} \sin(2ax) J_\nu(x) Y_\nu(x) dx = -\frac{\Gamma\left(\frac{3}{2}-\nu\right) a}{2\Gamma\left(2\nu-\frac{1}{2}\right)\Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right) \\ [0 < \operatorname{Re} \nu < \frac{3}{2}, \quad 0 < a < 1] \quad \text{ET II 348(63)}$$

$$6.^{10} \quad \int_0^\infty \arg \sin(zx) x^{\nu-\mu-4} J_\mu(ax) J_\nu(\rho x) dx = z \frac{\Gamma(\nu) a^\mu \rho^{-\nu}}{2^{\mu-\nu+3} \Gamma(\mu+1)} \left[\frac{\rho^2}{\nu-1} - \frac{a^2}{\mu+1} - \frac{2z^2}{3} \right]$$

$$7.^{10} \quad \int_0^\infty \cos(zx) x^{\nu-\mu-3} J_\mu(ax) J_\nu(\rho x) dx = \frac{\Gamma(\nu) a^\mu \rho^{-\nu}}{2^{\mu-\nu+3} \Gamma(\mu+1)} \left[\frac{\rho^2}{\nu-1} - \frac{a^2}{\mu+1} - 2z^2 \right]$$

6.712

$$1. \quad \int_0^\infty x^\nu [J_\nu(ax) \cos(ax) + Y_\nu(ax) \sin(ax)] \sin(bx) dx = \frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)} (b^2 + 2ab)^{-\nu-\frac{1}{2}} \\ [b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET I 104(40)}$$

$$2. \quad \int_0^\infty x^\nu [Y_\nu(ax) \cos(ax) - J_\nu(ax) \sin(ax)] \cos(bx) dx = -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)} (b^2 + 2ab)^{-\nu-\frac{1}{2}} \\ \text{ET I 48(35)}$$

$$3. \quad \int_0^\infty x^\nu [J_\nu(ax) \cos(ax) - Y_\nu(ax) \sin(ax)] \sin(bx) dx \\ = 0 \quad [0 < b < 2a, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \\ = \frac{2^\nu \sqrt{\pi} b^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)} (b^2 - 2ab)^{-\nu-\frac{1}{2}} \quad [2a < b, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \\ \text{ET I 104(41)}$$

$$4. \quad \int_0^\infty x^\nu [J_\nu(ax) \sin(ax) + Y_\nu(ax) \cos(ax)] \cos(bx) dx \\ = 0 \quad [0 < b < 2a, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ = -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)} (b^2 - 2ab)^{-\nu-\frac{1}{2}} \quad [0 < 2a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ \text{ET I 48(33)}$$

6.713

$$1. \quad \int_0^\infty x^{1-2\nu} \sin(2ax) \{ [J_\nu(x)]^2 - [Y_\nu(x)]^2 \} dx \\ = \frac{\sin(2\nu\pi) \Gamma\left(\frac{3}{2}-\nu\right) \Gamma\left(\frac{3}{2}-2\nu\right) a}{\pi \Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right) \\ [0 < \operatorname{Re} \nu < \frac{3}{4}, \quad 0 < a < 1] \quad \text{ET II 348(64)}$$

$$2. \quad \int_0^\infty x^{2-2\nu} \sin(2ax) [J_\nu(x) J_{\nu-1}(x) - Y_\nu(x) Y_{\nu-1}(x)] dx \\ = -\frac{\sin(2\nu\pi) \Gamma\left(\frac{3}{2}-\nu\right) \Gamma\left(\frac{5}{2}-2\nu\right) a}{\pi \Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right) \\ \left[\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{4}, \quad 0 < a < 1\right] \quad \text{ET II 348(65)}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^{2-2\nu} \sin(2ax) [J_\nu(x) Y_{\nu-1}(x) + Y_\nu(x) J_{\nu-1}(x)] dx \\
= -\frac{\Gamma\left(\frac{3}{2}-\nu\right) a}{\Gamma\left(2\nu-\frac{3}{2}\right) \Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right) \\
\left[\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{2}, \quad 0 < a < 1\right] \quad \text{ET II 349(66)}
\end{aligned}$$

6.714

$$\begin{aligned}
1. \quad \int_0^\infty \sin(2ax) [x^\nu J_\nu(x)]^2 dx \\
= \frac{a^{-2\nu} \Gamma\left(\frac{1}{2}+\nu\right)}{2\sqrt{\pi} \Gamma(1-\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}; 1-\nu; a^2\right) \quad \left[0 < a < 1, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right] \\
= \frac{a^{-4\nu-1} \Gamma\left(\frac{1}{2}+\nu\right)}{2\Gamma(1+\nu) \Gamma\left(\frac{1}{2}-2\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \quad \left[a > 1, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right] \\
\text{ET II 343(31)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty \cos(2ax) [x^\nu J_\nu(x)]^2 dx \\
= \frac{a^{-2\nu} \Gamma(\nu)}{2\sqrt{\pi} \Gamma\left(\frac{1}{2}-\nu\right)} F\left(\nu+\frac{1}{2}, \frac{1}{2}; 1-\nu; a^2\right) \\
+ \frac{\Gamma(-\nu) \Gamma\left(\frac{1}{2}+2\nu\right)}{2\pi \Gamma\left(\frac{1}{2}-\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; a^2\right) \quad \left[0 < a < 1, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}\right] \\
= -\frac{\sin(\nu\pi) a^{-4\nu-1} \Gamma\left(\frac{1}{2}+2\nu\right)}{\Gamma(1+\nu) \Gamma\left(\frac{1}{2}-\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \quad \left[a > 1, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}\right] \\
\text{ET II 344(33)}
\end{aligned}$$

6.715

$$1. \quad \int_0^\infty \frac{x^\nu}{x+\beta} \sin(x+\beta) J_\nu(x) dx = \frac{\pi}{2} \sec(\nu\pi) \beta^\nu J_{-\nu}(\beta) \quad \left[|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right] \quad \text{ET II 340(8)}$$

$$2. \quad \int_0^\infty \frac{x^\nu}{x+\beta} \cos(x+\beta) J_\nu(x) dx = -\frac{\pi}{2} \sec(\nu\pi) \beta^\nu Y_{-\nu}(\beta) \quad \left[|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right] \quad \text{ET II 340(9)}$$

6.716

$$1. \quad \int_0^a x^\lambda \sin(a-x) J_\nu(x) dx = 2a^{\lambda+1} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\nu-\lambda+2n) \Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda) \Gamma(\nu+\lambda+3+2n)} (\nu+2n+1) J_{\nu+2n+1}(a) \\
\left[\operatorname{Re}(\lambda+\nu) > -1\right] \quad \text{ET II 335(16)}$$

$$2. \quad \int_0^a x^\lambda \cos(a-x) J_\nu(x) dx = \frac{a^{\lambda+1} J_\nu(a)}{\lambda+\nu+1} + 2a^{\lambda+1} \\
\times \sum_{n=1}^\infty \frac{(-1)^n \Gamma(\nu-\lambda+2n-1) \Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda) \Gamma(\nu+\lambda+2n+2)} (\nu+2n) J_{\nu+2n}(a) \\
\left[\operatorname{Re}(\lambda+\nu) > -1\right] \quad \text{ET II 335(26)}$$

$$6.717 \quad \int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x^{\nu}(x+\beta)} J_{\nu+2n}(x) dx = \pi \beta^{-\nu} J_{\nu+2n}(\beta) \quad [1 \leq a < \infty, n = 0, 1, 2, \dots; \operatorname{Re} \nu > -\tfrac{3}{2}] \quad \text{ET II 345(44)}$$

6.718

$$1. \quad \int_0^{\infty} \frac{x^{\nu}}{x^2 + \beta^2} \sin(\alpha x) J_{\nu}(\gamma x) dx = \beta^{\nu-1} \sinh(\alpha \beta) K_{\nu}(\beta \gamma) \quad [0 < \alpha \leq \gamma, \operatorname{Re} \beta > 0, -1 < \operatorname{Re} \nu < \tfrac{3}{2}] \quad \text{ET II 33(8)}$$

$$2. \quad \int_0^{\infty} \frac{x^{\nu+1}}{x^2 + \beta^2} \cos(\alpha x) J_{\nu}(\gamma x) dx = \beta^{\nu} \cosh(\alpha \beta) K_{\nu}(\beta \gamma) \quad [0 < \alpha \leq \gamma, \operatorname{Re} \beta > 0, -1 < \operatorname{Re} \nu < \tfrac{1}{2}] \quad \text{ET II 37(33)}$$

$$3. \quad \int_0^{\infty} \frac{x^{1-\nu}}{x^2 + \beta^2} \sin(\alpha x) J_{\nu}(\gamma x) dx = \frac{\pi}{2} \beta^{-\nu} e^{-\alpha \beta} I_{\nu}(\beta \gamma) \quad [0 < \gamma \leq \alpha, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 33(9)}$$

$$4. \quad \int_0^{\infty} \frac{x^{-\nu}}{x^2 + \beta^2} \cos(\alpha x) J_{\nu}(\gamma x) dx = \frac{\pi}{2} \beta^{-\nu-1} e^{-\alpha \beta} I_{\nu}(\beta \gamma) \quad [0 < \gamma \leq \alpha, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -\tfrac{3}{2}] \quad \text{ET II 37(34)}$$

6.719

$$1.^6 \quad \int_0^{\alpha} \frac{\sin(\beta x)}{\sqrt{\alpha^2 - x^2}} J_{\nu}(x) dx = \pi \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\alpha \beta) J_{\frac{1}{2}\nu+n+\frac{1}{2}}(\tfrac{1}{2}\alpha) J_{\frac{1}{2}\nu-n-\frac{1}{2}}(\tfrac{1}{2}\alpha) \quad [\operatorname{Re} \nu > -2] \quad \text{ET II 335(17)}$$

$$2. \quad \int_0^{\alpha} \frac{\cos(\beta x)}{\sqrt{\alpha^2 - x^2}} J_{\nu}(x) dx = \frac{\pi}{2} J_0(\alpha \beta) \left[J_{\frac{1}{2}\nu}(\tfrac{1}{2}\alpha) \right]^2 + \pi \sum_{n=1}^{\infty} (-1)^n J_{2n}(\alpha \beta) J_{\frac{1}{2}\nu+n}(\tfrac{1}{2}\alpha) J_{\frac{1}{2}\nu-n}(\tfrac{1}{2}\alpha) \quad [\operatorname{Re} \nu > -1] \quad \text{ET II 336(27)}$$

6.721

$$1. \quad \int_0^{\infty} \sqrt{x} J_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = 2^{-3/2} a^{-2} \sqrt{\pi b} J_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \quad [b > 0] \quad \text{ET I 108(1)}$$

$$2. \quad \int_0^{\infty} \sqrt{x} J_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = 2^{-3/2} a^{-2} \sqrt{\pi b} J_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \quad [b > 0] \quad \text{ET I 51(1)}$$

$$3. \quad \int_0^{\infty} \sqrt{x} Y_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = -2^{-3/2} \sqrt{\pi b} a^{-2} \mathbf{H}_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \quad \text{ET I 108(7)}$$

$$4. \quad \int_0^{\infty} \sqrt{x} Y_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = -2^{-3/2} \sqrt{\pi b} a^{-2} \mathbf{H}_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \quad \text{ET I 52(7)}$$

$$5. \quad \int_0^\infty \sqrt{x} K_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = 2^{-5/2} \sqrt{\pi^3 b} a^{-2} \left[I_{\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) - \mathbf{L}_{\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) \right] \\ \left[|\arg a| < \frac{\pi}{4}, \quad b > 0 \right] \quad \text{ET I 109(11)}$$

$$6. \quad \int_0^\infty \sqrt{x} K_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = 2^{-5/2} \sqrt{\pi^3 b} a^{-2} \left[I_{-\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) - \mathbf{L}_{-\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) \right] \\ [b > 0] \quad \text{ET I 52(10)}$$

6.722

$$1. \quad \int_0^\infty \sqrt{x} K_{\frac{1}{8}+\nu}(a^2 x^2) I_{\frac{1}{8}-\nu}(a^2 x^2) \sin(bx) dx = \sqrt{2\pi} b^{-3/2} \frac{\Gamma(\frac{5}{8}-\nu)}{\Gamma(\frac{5}{4})} W_{\nu, \frac{1}{8}} \left(\frac{b^2}{8a^2} \right) M_{-\nu, \frac{1}{8}} \left(\frac{b^2}{8a^2} \right) \\ \left[\operatorname{Re} \nu < \frac{5}{8}, \quad |\arg a| < \frac{\pi}{4}, \quad b > 0 \right] \quad \text{ET I 109(13)}$$

$$2.^{10} \quad \int_0^\infty \sqrt{x} J_{-\frac{1}{8}-\nu}(a^2 x^2) J_{-\frac{1}{8}+\nu}(a^2 x^2) \cos(bx) dx \\ = \frac{\sqrt{\pi}}{2^{3/4} a^{3/2}} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}) \Gamma(\frac{5}{8}-\nu) \Gamma(\frac{5}{8}+\nu)} {}_2F_3 \left(\frac{3}{8}-\nu, \frac{3}{8}+\nu; \frac{3}{8}, \frac{3}{4}, \frac{7}{8}; -\left(\frac{b}{4a}\right)^4 \right) \\ - \frac{1}{a^2} \sqrt{\frac{2b}{\pi}} \cos(\pi\nu) {}_2F_3 \left(\frac{1}{2}-\nu, \frac{1}{2}+\nu; \frac{1}{2}, \frac{7}{8}, \frac{9}{8}; -\left(\frac{b}{4a}\right)^4 \right) \\ - \frac{b^{5/2}\nu}{15a^4} \sqrt{\frac{2}{\pi}} \sin(\pi\nu) {}_2F_3 \left(1-\nu, 1+\nu; \frac{11}{8}, \frac{3}{2}, \frac{13}{8}; -\left(\frac{b}{4a}\right)^4 \right) \\ [a^2 > 0, \quad \operatorname{Im} b = 0] \quad \text{MC}$$

$$3. \quad \int_0^\infty \sqrt{x} J_{\frac{1}{8}-\nu}(a^2 x^2) J_{\frac{1}{8}+\nu}(a^2 x^2) \sin(bx) dx \\ = \sqrt{\frac{2}{\pi}} b^{-3/2} \left[e^{\pi i/8} W_{\nu, \frac{1}{8}} \left(\frac{b^2 e^{\pi i/2}}{8a^2} \right) W_{-\nu, \frac{1}{8}} \left(\frac{b^2 e^{\pi i/2}}{8a^2} \right) \right. \\ \left. + e^{-i\pi/8} W_{\nu, \frac{1}{8}} \left(\frac{b^2 e^{-\pi i/2}}{8a^2} \right) W_{-\nu, \frac{1}{8}} \left(\frac{b^2 e^{-\pi i/2}}{8a^2} \right) \right] \\ [b > 0] \quad \text{ET I 108(6)}$$

$$4. \quad \int_0^\infty \sqrt{x} K_{\frac{1}{8}-\nu}(a^2 x^2) I_{-\frac{1}{8}-\nu}(a^2 x^2) \cos(bx) dx \\ = \sqrt{2\pi} b^{-3/2} \frac{\Gamma(\frac{3}{8}-\nu)}{\Gamma(\frac{3}{4})} W_{\nu, -\frac{1}{8}} \left(\frac{b^2}{8a^2} \right) M_{-\nu, -\frac{1}{8}} \left(\frac{b^2}{8a^2} \right) \\ [\operatorname{Re} \nu < \frac{3}{8}, \quad b > 0] \quad \text{ET I 52(12)}$$

$$\mathbf{6.723} \quad \int_0^\infty x J_\nu(x^2) [\sin(\nu\pi) J_\nu(x^2) - \cos(\nu\pi) Y_\nu(x^2)] J_{4\nu}(4ax) dx = \frac{1}{4} J_\nu(a^2) J_{-\nu}(a^2) \\ [a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 375(20)}$$

6.724

$$\begin{aligned}
 1. \quad \int_0^\infty x^{2\lambda} J_{2\nu} \left(\frac{a}{x} \right) \sin(bx) dx \\
 = \frac{\sqrt{\pi} a^{2\nu} \Gamma(\lambda - \nu + 1) b^{2\nu - 2\lambda - 1}}{4^{2\nu - \lambda} \Gamma(2\nu + 1) \Gamma \left(\nu - \lambda + \frac{1}{2} \right)} {}_0F_3 \left(2\nu + 1, \nu - \lambda, \nu - \lambda + \frac{1}{2}; \frac{a^2 b^2}{16} \right) \\
 + \frac{a^{2\lambda + 2} \Gamma(\nu - \lambda - 1) b}{2^{2\lambda + 3} \Gamma(\nu + \lambda + 2)} {}_0F_3 \left(\frac{3}{2}, \lambda - \nu + 2, \lambda + \nu + 2; \frac{a^2 b^2}{16} \right) \\
 \left[-\frac{5}{4} < \operatorname{Re} \lambda < \operatorname{Re} \nu, \quad a > 0, \quad b > 0 \right] \quad \text{ET I 109(15)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty x^{2\lambda} J_{2\nu} \left(\frac{a}{x} \right) \cos(bx) dx \\
 = 4^{\lambda - 2\nu} \sqrt{\pi} a^{2\nu} b^{2\nu - 2\lambda - 1} \frac{\Gamma(\lambda - \nu + \frac{1}{2})}{\Gamma(2\nu + 1) \Gamma(\nu - \lambda)} {}_0F_3 \left(2\nu + 1, \nu - \lambda + \frac{1}{2}, \nu - \lambda; \frac{a^2 b^2}{16} \right) \\
 + 4^{-\lambda - 1} a^{2\lambda + 1} \frac{\Gamma(\nu - \lambda - \frac{1}{2})}{\Gamma(\nu + \lambda + \frac{3}{2})} {}_0F_3 \left(\frac{1}{2}, \lambda - \nu + \frac{3}{2}, \nu + \lambda + \frac{3}{2}; \frac{a^2 b^2}{16} \right) \\
 \left[-\frac{3}{4} < \operatorname{Re} \lambda < \operatorname{Re} \nu - \frac{1}{2}, \quad a > 0, \quad b > 0 \right] \quad \text{ET I 53(14)}
 \end{aligned}$$

6.725

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin(bx)}{\sqrt{x}} J_\nu(a\sqrt{x}) dx = -\sqrt{\frac{\pi}{b}} \sin \left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4} \right) J_{\frac{\nu}{2}} \left(\frac{a^2}{8b} \right) \\
 [\operatorname{Re} \nu > -3, \quad a > 0, \quad b > 0] \quad \text{ET I 110(27)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{\cos(bx)}{\sqrt{x}} J_\nu(a\sqrt{x}) dx = \sqrt{\frac{\pi}{b}} \cos \left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4} \right) J_{\frac{1}{2}\nu} \left(\frac{a^2}{8b} \right) \\
 [\operatorname{Re} \nu > -1, \quad a > 0, \quad b > 0] \quad \text{ET I 54(25)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty x^{\frac{1}{2}\nu} J_\nu(a\sqrt{x}) \sin(bx) dx = 2^{-\nu} a^\nu b^{-\nu-1} \cos \left(\frac{a^2}{4b} - \frac{\nu\pi}{2} \right) \\
 [-2 < \operatorname{Re} \nu < \frac{1}{2}, \quad a > 0, \quad b > 0] \quad \text{ET I 110(28)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_0^\infty x^{\frac{1}{2}\nu} J_\nu(a\sqrt{x}) \cos(bx) dx = 2^{-\nu} b^{-\nu-1} a^\nu \sin \left(\frac{a^2}{4b} - \frac{\nu\pi}{2} \right) \\
 [-1 < \operatorname{Re} \nu < \frac{1}{2}, \quad a > 0, \quad b > 0] \quad \text{ET I 54(26)}
 \end{aligned}$$

6.726

$$\begin{aligned}
 1. \quad \int_0^\infty x (x^2 + b^2)^{-\frac{1}{2}\nu} J_\nu(a\sqrt{x^2 + b^2}) \sin(cx) dx \\
 = \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu + \frac{3}{2}} c (a^2 - c^2)^{\frac{1}{2}\nu - \frac{3}{4}} J_{\nu - \frac{3}{2}}(b\sqrt{a^2 - c^2}) \quad \left[0 < c < a, \quad \operatorname{Re} \nu > \frac{1}{2} \right] \\
 = 0 \quad \left[0 < a < c, \quad \operatorname{Re} \nu > \frac{1}{2} \right] \quad \text{ET I 111(37)}
 \end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty (x^2 + b^2)^{-\frac{1}{2}\nu} J_\nu \left(a\sqrt{x^2 + b^2} \right) \cos(cx) dx \\
&= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{1}{2}} (a^2 - c^2)^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}} \left(b\sqrt{a^2 - c^2} \right) \quad [0 < c < a, \quad b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
&= 0 \quad [0 < a < c, \quad b > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
&\hspace{15em} \text{ET I 55(37)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x (x^2 + b^2)^{\frac{1}{2}\nu} K_{\pm\nu} \left(a\sqrt{x^2 + b^2} \right) \sin(cx) dx \\
&= \sqrt{\frac{\pi}{2}} a^\nu b^{\nu+\frac{3}{2}} c (a^2 + c^2)^{-\frac{1}{2}\nu-\frac{3}{4}} K_{-\nu-\frac{3}{2}} \left(b\sqrt{a^2 + c^2} \right) \\
&\hspace{15em} [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c > 0] \quad \text{ET I 113(45)}
\end{aligned}$$

$$\begin{aligned}
4.11 \quad & \int_0^\infty (x^2 + b^2)^{\mp\frac{1}{2}\nu} K_\nu \left(a\sqrt{x^2 + b^2} \right) \cos(cx) dx \\
&= \sqrt{\frac{\pi}{2}} a^{\mp\nu} b^{\frac{1}{2}\mp\nu} (a^2 + c^2)^{\pm\frac{1}{2}\nu-\frac{1}{4}} K_{\pm\nu-\frac{1}{2}} \left(b\sqrt{a^2 + c^2} \right) \\
&\hspace{15em} [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c \text{ is real}] \quad \text{ET I 56(45)}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty (x^2 + a^2)^{-\frac{1}{2}\nu} Y_\nu \left(b\sqrt{x^2 + a^2} \right) \cos(cx) dx \\
&= \sqrt{\frac{a\pi}{2}} (ab)^{-\nu} (b^2 - c^2)^{\frac{1}{2}\nu-\frac{1}{4}} Y_{\nu-\frac{1}{2}} \left(a\sqrt{b^2 - c^2} \right) \quad [0 < c < b, \quad a > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
&= -\sqrt{\frac{2a}{\pi}} (ab)^{-\nu} (c^2 - b^2)^{\frac{1}{2}\nu-\frac{1}{4}} K_{\nu-\frac{1}{2}} \left(a\sqrt{c^2 - b^2} \right) \quad [0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
&\hspace{15em} \text{ET I 56(41)}
\end{aligned}$$

6.727

$$\begin{aligned}
1.^9 \quad & \int_0^a \frac{\cos(cx)}{\sqrt{a^2 - x^2}} J_\nu \left(b\sqrt{a^2 - x^2} \right) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} \left(\sqrt{b^2 + c^2} - c \right) \right] J_{\frac{1}{2}\nu} \left[\frac{a}{2} \left(\sqrt{b^2 + c^2} + c \right) \right] \\
&\hspace{15em} [\operatorname{Re} \nu > -1, \quad c > 0, \quad a > 0] \\
&\hspace{15em} \text{ET I 113(48)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_a^\infty \frac{\sin(cx)}{\sqrt{x^2 - a^2}} J_\nu \left(b\sqrt{x^2 - a^2} \right) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} \left(c - \sqrt{c^2 + b^2} \right) \right] J_{-\frac{1}{2}\nu} \left[\frac{a}{2} \left(c + \sqrt{c^2 + b^2} \right) \right] \\
&\hspace{15em} [0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -1] \\
&\hspace{15em} \text{ET I 113(49)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_a^\infty \frac{\cos(cx)}{\sqrt{x^2 - a^2}} J_\nu \left(b\sqrt{x^2 - a^2} \right) dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} \left(c - \sqrt{c^2 - b^2} \right) \right] Y_{-\frac{1}{2}\nu} \left[\frac{a}{2} \left(c + \sqrt{c^2 - b^2} \right) \right] \\
&\hspace{15em} [0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -1] \\
&\hspace{15em} \text{ET I 58(54)}
\end{aligned}$$

$$\begin{aligned}
4.^8 \quad & \int_0^a (a^2 - x^2)^{\frac{1}{2}\nu} \cos x I_\nu \left(\sqrt{a^2 - x^2} \right) dx = \frac{\sqrt{\pi} a^{2\nu+1}}{2^{\nu+1} \Gamma \left(\nu + \frac{3}{2} \right)} \\
&\hspace{15em} [\operatorname{Re} \nu > -\tfrac{1}{2}] \hspace{15em} \text{WA 409(2)}
\end{aligned}$$

6.728

$$\begin{aligned}
 1. \quad \int_0^\infty x \sin(ax^2) J_\nu(bx) dx \\
 = \frac{\sqrt{\pi}b}{8a^{3/2}} \left[\cos\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu - \frac{1}{2}}\left(\frac{b^2}{8a}\right) - \sin\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu + \frac{1}{2}}\left(\frac{b^2}{8a}\right) \right] \\
 [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -4] \quad \text{ET II 34(14)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty x \cos(ax^2) J_\nu(bx) dx \\
 = \frac{\sqrt{\pi}b}{8a^{3/2}} \left[\cos\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu + \frac{1}{2}}\left(\frac{b^2}{8a}\right) + \sin\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu - \frac{1}{2}}\left(\frac{b^2}{8a}\right) \right] \\
 [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 38(39)}
 \end{aligned}$$

$$3. \quad \int_0^\infty J_0(\beta x) \sin(\alpha x^2) x dx = \frac{1}{2\alpha} \cos \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \quad \beta > 0] \quad \text{MO 47}$$

$$4. \quad \int_0^\infty J_0(\beta x) \cos(\alpha x^2) x dx = \frac{1}{2\alpha} \sin \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \quad \beta > 0] \quad \text{MO 47}$$

$$\begin{aligned}
 5. \quad \int_0^\infty x^{\nu+1} \sin(ax^2) J_\nu(bx) dx = \frac{b^\nu}{2^{\nu+1}a^{\nu+1}} \cos\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right) \\
 [a > 0, \quad b > 0, \quad -2 < \operatorname{Re} \nu < \tfrac{1}{2}] \\
 \text{ET II 34(15)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_0^\infty x^{\nu+1} \cos(ax^2) J_\nu(bx) dx = \frac{b^\nu}{2^{\nu+1}a^{\nu+1}} \sin\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right) \\
 [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \tfrac{1}{2}] \\
 \text{ET II 38(40)}
 \end{aligned}$$

6.729

$$\begin{aligned}
 1. \quad \int_0^\infty x \sin(ax^2) J_\nu(bx) J_\nu(cx) dx = \frac{1}{2a} \cos\left(\frac{b^2 + c^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{bc}{2a}\right) \\
 [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -2] \\
 \text{ET II 51(26)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty x \cos(ax^2) J_\nu(bx) J_\nu(cx) dx = \frac{1}{2a} \sin\left(\frac{b^2 + c^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{bc}{2a}\right) \\
 [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -1] \\
 \text{ET II 51(27)}
 \end{aligned}$$

6.731

$$\begin{aligned}
 1.^{11} \quad \int_0^\infty x \sin(ax^2) J_\nu(bx^2) J_{2\nu}(2cx) dx \\
 = \frac{1}{2\sqrt{b^2 - a^2}} \sin\left(\frac{ac^2}{b^2 - a^2}\right) J_\nu\left(\frac{bc^2}{b^2 - a^2}\right) \quad [0 < a < b, \quad \operatorname{Re} \nu > -1] \\
 = \frac{1}{2\sqrt{a^2 - b^2}} \cos\left(\frac{ac^2}{a^2 - b^2}\right) J_\nu\left(\frac{bc^2}{a^2 - b^2}\right) \quad [0 < b < a, \quad \operatorname{Re} \nu > -1] \\
 \text{ET II 356(41)a}
 \end{aligned}$$

$$\begin{aligned}
2.10 \quad & \int_0^\infty x \cos(ax^2) J_\nu(bx^2) J_{2\nu}(2cx) dx \\
&= \frac{1}{2\sqrt{b^2 - a^2}} \cos\left(\frac{ac^2}{b^2 - a^2}\right) J_\nu\left(\frac{bc^2}{b^2 - a^2}\right) \quad [0 < a < b, \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \\
&= \frac{1}{2\sqrt{a^2 - b^2}} \sin\left(\frac{ac^2}{a^2 - b^2}\right) J_\nu\left(\frac{bc^2}{a^2 - b^2}\right) \quad [0 < b < a, \quad \operatorname{Re} \nu > -\tfrac{1}{2}]
\end{aligned}$$

ET II 356(42)_a

$$6.732 \quad \int_0^\infty x^2 \cos\left(\frac{x^2}{2a}\right) Y_1(x) K_1(x) dx = -a^3 K_0(a) \quad [a > 0] \quad \text{ET II 371(52)}$$

6.733

$$1. \quad \int_0^\infty \sin\left(\frac{a}{2x}\right) [\sin x J_0(x) + \cos x Y_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}) \quad [a > 0] \quad \text{ET II 346(51)}$$

$$2. \quad \int_0^\infty \cos\left(\frac{a}{2x}\right) [\sin x Y_0(x) - \cos x J_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}) \quad [a > 0] \quad \text{ET II 347(52)}$$

$$3. \quad \int_0^\infty x \sin\left(\frac{a}{2x}\right) K_0(x) dx = \frac{\pi a}{2} J_1(\sqrt{a}) K_1(\sqrt{a}) \quad [a > 0] \quad \text{ET II 368(34)}$$

$$4. \quad \int_0^\infty x \cos\left(\frac{a}{2x}\right) K_0(x) dx = -\frac{\pi a}{2} Y_1(\sqrt{a}) K_1(\sqrt{a}) \quad [a > 0] \quad \text{ET II 369(35)}$$

$$\begin{aligned}
6.734 \quad & \int_0^\infty \cos(a\sqrt{x}) K_\nu(bx) \frac{dx}{\sqrt{x}} \\
&= \frac{\pi}{2\sqrt{b}} \sec(\nu\pi) \left[D_{\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) + D_{\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right) \right] \\
&\quad [\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 132(27)}
\end{aligned}$$

6.735

$$1. \quad \int_0^\infty x^{1/4} \sin(2a\sqrt{x}) J_{-\frac{1}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{\frac{3}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(10)}$$

$$2. \quad \int_0^\infty x^{1/4} \cos(2a\sqrt{x}) J_{\frac{1}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{-\frac{3}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(12)}$$

$$3. \quad \int_0^\infty x^{1/4} \sin(2a\sqrt{x}) J_{\frac{3}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{-\frac{1}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(11)}$$

$$4. \quad \int_0^\infty x^{1/4} \cos(2a\sqrt{x}) J_{-\frac{3}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{\frac{1}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(13)}$$

6.736

$$1.11 \quad \int_0^\infty x^{-1/2} \sin x \cos(4a\sqrt{x}) J_0(x) dx = -2^{-3/2} \sqrt{\pi} \left[\cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) - \sin\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right] \quad [a > 0] \quad \text{ET II 341(18)}$$

$$2. \quad \int_0^\infty x^{-1/2} \cos x \cos(4a\sqrt{x}) J_0(x) dx = -2^{-3/2} \sqrt{\pi} \left[\sin\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) + \cos\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right] \quad [a > 0] \quad \text{ET II 342(22)}$$

$$3. \quad \int_0^\infty x^{-1/2} \sin x \sin(4a\sqrt{x}) J_0(x) dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 + \frac{\pi}{4}\right) J_0(a^2) \\ [a > 0] \quad \text{ET II 341(16)}$$

$$4. \quad \int_0^\infty x^{-1/2} \cos x \sin(4a\sqrt{x}) J_0(x) dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) \\ [a > 0] \quad \text{ET II 342(20)}$$

$$5. \quad \int_0^\infty x^{-1/2} \sin x \cos(4a\sqrt{x}) Y_0(x) dx = 2^{-3/2} \sqrt{\pi} \left[3 \sin\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) - \cos\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right] \\ [a > 0] \quad \text{ET II 347(55)}$$

$$6. \quad \int_0^\infty x^{-1/2} \cos x \cos(4a\sqrt{x}) Y_0(x) dx \\ = -2^{-3/2} \sqrt{\pi} \left[3 \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) + \sin\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right] \\ [a > 0] \quad \text{ET II 347(56)}$$

6.737

$$1. \quad \int_0^\infty \frac{\sin(a\sqrt{x^2 + b^2})}{\sqrt{x^2 + b^2}} J_\nu(cx) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{b}{2} (a - \sqrt{a^2 - c^2}) \right] J_{-\frac{1}{2}\nu} \left[\frac{b}{2} (a + \sqrt{a^2 - c^2}) \right] \\ [a > 0, \quad \text{Re } b > 0, \quad c > 0, \quad a > c, \quad \text{Re } \nu > -1] \quad \text{ET II 35(19)}$$

$$2. \quad \int_0^\infty \frac{\cos(a\sqrt{x^2 + b^2})}{\sqrt{x^2 + b^2}} J_\nu(cx) dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{b}{2} (a - \sqrt{a^2 - c^2}) \right] Y_{-\frac{1}{2}\nu} \left[\frac{b}{2} (a + \sqrt{a^2 - c^2}) \right] \\ [a > 0, \quad \text{Re } b > 0, \quad c > 0, \quad a > c, \quad \text{Re } \nu > -1] \quad \text{ET II 39(44)}$$

$$3. \quad \int_0^a \frac{\cos(b\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} J_\nu(cx) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - b) \right] J_{\frac{1}{2}\nu} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + b) \right] \\ [c > 0, \quad \text{Re } \nu > -1] \quad \text{ET II 39(47)}$$

$$4. \quad \int_0^a x^{\nu+1} \frac{\cos(\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} I_\nu(x) dx = \frac{\sqrt{\pi} a^{2\nu+1}}{2^{\nu+1} \Gamma\left(\nu + \frac{3}{2}\right)} \quad [\text{Re } \nu > -1] \quad \text{ET II 365(9)}$$

$$5. \quad \int_0^\infty x^{\nu+1} \frac{\sin(a\sqrt{b^2 + x^2})}{\sqrt{b^2 + x^2}} J_\nu(cx) dx \\ = \sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2 - c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} J_{-\nu-\frac{1}{2}}(b\sqrt{a^2 - c^2}) \quad [0 < c < a, \quad \text{Re } b > 0, \quad -1 < \text{Re } \nu < \frac{1}{2}] \\ = 0 \quad [0 < a < c, \quad \text{Re } b > 0, \quad -1 < \text{Re } \nu < \frac{1}{2}] \\ \text{ET II 35(20)}$$

$$\begin{aligned}
6. \quad \int_0^\infty x^{\nu+1} \frac{\cos(a\sqrt{x^2+b^2})}{\sqrt{x^2+b^2}} J_\nu(cx) dx &= -\sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2 - c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} Y_{-\nu-\frac{1}{2}}(b\sqrt{a^2-c^2}) \\
&\quad \left[0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right] \\
&= \sqrt{\frac{2}{\pi}} b^{\frac{1}{2}+\nu} c^\nu (c^2 - a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} K_{\nu+\frac{1}{2}}(b\sqrt{c^2-a^2}) \\
&\quad \left[0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right] \\
&\hspace{15em} \text{ET II 39(45)}
\end{aligned}$$

6.738

$$\begin{aligned}
1. \quad \int_0^a x^{\nu+1} \sin(b\sqrt{a^2-x^2}) J_\nu(x) dx &= \sqrt{\frac{\pi}{2}} a^{\nu+\frac{3}{2}} b (1+b^2)^{-\frac{1}{2}\nu-\frac{3}{4}} J_{\nu+\frac{3}{2}}(a\sqrt{1+b^2}) \\
&\hspace{15em} [\operatorname{Re} \nu > -1] \hspace{5em} \text{ET II 335(19)} \\
2. \quad \int_0^\infty x^{\nu+1} \cos(a\sqrt{x^2+b^2}) J_\nu(cx) dx \\
&= \sqrt{\frac{\pi}{2}} a b^{\nu+\frac{3}{2}} c^\nu (a^2 - c^2)^{-\frac{1}{2}\nu-\frac{3}{4}} \left[\cos(\pi\nu) J_{\nu+\frac{3}{2}}(b\sqrt{a^2-c^2}) - \sin(\pi\nu) Y_{\nu+\frac{3}{2}}(b\sqrt{a^2-c^2}) \right] \\
&\quad \left[0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2} \right] \\
&= 0 \\
&\quad \left[0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2} \right] \\
&\hspace{15em} \text{ET II 39(43)}
\end{aligned}$$

$$\begin{aligned}
6.739 \quad \int_0^t x^{-1/2} \frac{\cos(b\sqrt{t-x})}{\sqrt{t-x}} J_{2\nu}(a\sqrt{x}) dx &= \pi J_\nu \left[\frac{\sqrt{t}}{2} (\sqrt{a^2+b^2} + b) \right] J_\nu \left[\frac{\sqrt{t}}{2} (\sqrt{a^2+b^2} - b) \right] \\
&\hspace{15em} [\operatorname{Re} \nu > -\frac{1}{2}] \hspace{5em} \text{EH II 47(7)}
\end{aligned}$$

6.741

$$\begin{aligned}
1. \quad \int_0^1 \frac{\cos(\mu \arccos x)}{\sqrt{1-x^2}} J_\nu(ax) dx &= \frac{\pi}{2} J_{\frac{1}{2}(\mu+\nu)}\left(\frac{a}{2}\right) J_{\frac{1}{2}(\nu-\mu)}\left(\frac{a}{2}\right) \\
&\hspace{15em} [\operatorname{Re}(\mu+\nu) > -1, \quad a > 0] \hspace{5em} \text{ET II 41(54)} \\
2. \quad \int_0^1 \frac{\cos[(\nu+1) \arccos x]}{\sqrt{1-x^2}} J_\nu(ax) dx &= \sqrt{\frac{\pi}{a}} \cos\left(\frac{a}{2}\right) J_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right) \\
&\hspace{15em} [\operatorname{Re} \nu > -1, \quad a > 0] \hspace{5em} \text{ET II 40(53)} \\
3. \quad \int_0^1 \frac{\cos[(\nu-1) \arccos x]}{\sqrt{1-x^2}} J_\nu(ax) dx &= \sqrt{\frac{\pi}{a}} \sin\left(\frac{a}{2}\right) J_{\nu-\frac{1}{2}}\left(\frac{a}{2}\right) \\
&\hspace{15em} [\operatorname{Re} \nu > 0, \quad a > 0] \hspace{5em} \text{ET II 40(52)a}
\end{aligned}$$

6.75 Combinations of Bessel, trigonometric, and exponential functions and powers

$$6.751 \quad \text{Notation: } \ell_1 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \ell_2 = \frac{1}{2} \left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$$

$$1. \quad \int_0^\infty e^{-\frac{1}{2}ax} \sin(bx) I_0\left(\frac{1}{2}ax\right) dx = \frac{1}{\sqrt{2b}} \frac{1}{\sqrt{b^2 + a^2}} \sqrt{b + \sqrt{b^2 + a^2}} \\ [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 105(44)}$$

$$2. \quad \int_0^\infty e^{-\frac{1}{2}ax} \cos(bx) I_0\left(\frac{1}{2}ax\right) dx = \frac{a}{\sqrt{2b}} \frac{1}{\sqrt{a^2 + b^2} \sqrt{b + \sqrt{a^2 + b^2}}} \\ [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 48(38)}$$

$$3.^{10} \quad \int_0^\infty e^{-bx} \cos(ax) J_0(cx) dx = \frac{\left[\sqrt{(b^2 + c^2 - a^2)^2 + 4a^2b^2} + b^2 + c^2 - a^2 \right]^{1/2}}{\sqrt{2} \sqrt{(b^2 + c^2 - a^2)^2 + 4a^2b^2}} \\ [c > 0] \quad \text{ET II 11(46)}$$

alternatively, with a and b interchanged,

$$\int_0^\infty e^{-ax} \cos(bx) J_0(cx) dx = \frac{\sqrt{\ell_2^2 - b^2}}{\ell_2^2 - \ell_1^2} \quad [c > 0]$$

6.752

$$1.^{10} \quad \int_0^\infty e^{-ax} J_0(bx) \sin(cx) \frac{dx}{x} = \arcsin \left(\frac{2c}{\sqrt{a^2 + (c+b)^2} + \sqrt{a^2 + (c-b)^2}} \right) = \arcsin \left(\frac{c}{\ell_2} \right) \\ [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0] \quad \text{ET I 101(17)}$$

$$2.^{10} \quad \int_0^\infty e^{-ax} J_1(cx) \sin(bx) \frac{dx}{x} = \frac{b}{c} (1-r) = \frac{b - \sqrt{b^2 - \ell_1^2}}{c}, \\ \left[b^2 = \frac{c^2}{1-r^2} - \frac{a^2}{r^2}, \quad c > 0 \right] \quad \text{ET II 19(15)}$$

Notation: For integrals 6.752 3–6.752 5 we define the auxiliary functions

$$\ell_1(a) \equiv \ell_1(a, \rho, z) = \frac{1}{2} \left[\sqrt{(a+\rho)^2 + z^2} - \sqrt{(a-\rho)^2 + z^2} \right] \\ \ell_2(a) \equiv \ell_1(a, \rho, z) = \frac{1}{2} \left[\sqrt{(a+\rho)^2 + z^2} + \sqrt{(a-\rho)^2 + z^2} \right]$$

when $a \geq 0$, $\rho \geq 0$, and $z \geq 0$.

$$3.^{10} \quad \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_{\nu+1}(\rho x) \sqrt{x} dx \\ = a^{-\nu-3/2} \rho^{-\nu-1} \frac{\ell_1^{2\nu+2}}{\sqrt{\rho^2 - \ell_1^2}} \frac{a(\rho^2 - \ell_1^2)}{\ell_1(\ell_2^2 - \ell_1^2)} \\ = a^{\nu+1/2} \frac{\rho^{\nu+1}}{\ell_2^{2\nu+2}} \frac{\sqrt{\ell_2^2 - a^2}}{\ell_2^2 - \ell_1^2} \quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]$$

$$\begin{aligned}
4.10 \quad & \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} \\
&= a^{\nu+1/2} \rho^\nu \int_0^{1/\ell_2} \frac{1}{\ell_2^{2\nu}} \frac{1}{\sqrt{1-a^2/\ell_2^2}} d\left(\frac{1}{\ell_2}\right) \\
&= a^{-\nu-1/2} \rho^\nu \int_0^{a/\ell_2} x^{2\nu} \frac{dx}{\sqrt{1-x^2}} \quad \left[\nu > -\frac{1}{2}, \quad \operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|\right]
\end{aligned}$$

$$5.10 \quad \int_0^\infty e^{-zx} \sin(ax) J_1(\rho x) \frac{dx}{x^2} = \frac{\sqrt{\ell_2^2 - a^2} \left(a - \sqrt{a^2 - \ell_1^2}\right)^2}{2a\rho} + \frac{\rho}{2} \arcsin\left(\frac{a}{\ell_2}\right)$$

$[\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]$

6.753

$$1.8 \quad \int_0^\infty \frac{\sin(xa \sin \psi)}{x} e^{-xa \cos \varphi \cos \psi} J_\nu(xa \sin \varphi) dx = \nu^{-1} \left(\tan \frac{\varphi}{2}\right)^\nu \sin(\nu \psi)$$

$\left[\operatorname{Re} \nu > -1, \quad a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}\right] \quad \text{ET II 33(10)}$

$$2. \quad \int_0^\infty \frac{\cos(xa \sin \psi)}{x} e^{-xa \cos \varphi \cos \psi} J_\nu(xa \sin \varphi) dx = \nu^{-1} \left(\tan \frac{\varphi}{2}\right)^\nu \cos(\nu \psi)$$

$\left[\operatorname{Re} \nu > 0, \quad a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}\right]$
ET II 38(35)

$$3.8 \quad \int_0^\infty x^{\nu+1} e^{-sx} \sin(bx) J_\nu(ax) dx = -\frac{2(2a)^\nu}{\sqrt{\pi}} \Gamma\left(\nu + \frac{3}{2}\right) R^{-2\nu-3} \left[b \cos\left(\nu + \frac{3}{2}\right)\varphi + s \sin\left(\nu + \frac{3}{2}\right)\varphi\right]$$

$\left[\operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|,\right.$

$$\left. R^4 = (s^2 + a^2 - b^2)^2 + 4b^2 s^2, \quad \varphi = \arg(s^2 + a^2 - b^2 - 2ibs)\right]$$

$$4.8 \quad \int_0^\infty x^{\nu+1} e^{-sx} \cos(bx) J_\nu(ax) dx = \frac{2(2a)^\nu}{\sqrt{\pi}} \Gamma\left(\nu + \frac{3}{2}\right) R^{-2\nu-3} \left[s \cos\left(\nu + \frac{3}{2}\right)\varphi - b \sin\left(\nu + \frac{3}{2}\right)\varphi\right],$$

$\left[\operatorname{Re} \nu > -1, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|,\right.$

$$\left. R^4 = (s^2 + a^2 - b^2)^2 + 4b^2 s^2, \quad \varphi = \arg(s^2 + a^2 - b^2 - 2ibs)\right]$$

$$5.10 \quad \int_0^\infty x^\nu e^{-ax \cos \varphi \cos \psi} \sin(ax \sin \psi) J_\nu(ax \sin \varphi) dx$$

$$= 2^\nu \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}} a^{-\nu-1} (\sin \varphi)^\nu (\cos^2 \psi + \sin^2 \psi \cos^2 \varphi)^{-\nu-\frac{1}{2}} \sin\left[\left(\nu + \frac{1}{2}\right)\beta\right]$$

$\tan \frac{\beta}{2} = \tan \psi \cos \varphi \quad \left[a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -1\right] \quad \text{ET II 34(12)}$

$$\begin{aligned}
6. \quad \int_0^\infty x^\nu e^{-ax \cos \varphi \cos \psi} \cos(ax \sin \psi) J_\nu(ax \sin \varphi) dx \\
= 2^\nu \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} a^{-\nu-1} (\sin \varphi)^\nu (\cos^2 \psi + \sin^2 \psi \cos^2 \varphi)^{-\nu-\frac{1}{2}} \cos[(\nu + \frac{1}{2})\beta] \\
\tan \frac{\beta}{2} = \tan \psi \cos \varphi \quad \left[a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 38(37)}
\end{aligned}$$

6.754

$$\begin{aligned}
1. \quad \int_0^\infty e^{-x^2} \sin(bx) I_0(x^2) dx &= \frac{\sqrt{\pi}}{2^{3/2}} e^{-\frac{b^2}{8}} I_0\left(\frac{b^2}{8}\right) \quad [b > 0] \quad \text{ET I 108(9)} \\
2. \quad \int_0^\infty e^{-ax} \cos(x^2) J_0(x^2) dx &= \frac{1}{4} \sqrt{\frac{\pi}{2}} \left[J_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - Y_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} + \frac{\pi}{4}\right) \right] \\
&\quad [a > 0] \quad \text{MI 42} \\
3. \quad \int_0^\infty e^{-ax} \sin(x^2) J_0(x^2) dx &= \frac{1}{4} \sqrt{\frac{\pi}{2}} \left[J_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - Y_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} + \frac{\pi}{4}\right) \right] \\
&\quad [a > 0] \quad \text{MI 42}
\end{aligned}$$

6.755

$$\begin{aligned}
1. \quad \int_0^\infty x^{-\nu} e^{-x} \sin(4a\sqrt{x}) I_\nu(x) dx &= (2^{3/2}a)^{\nu-1} e^{-a^2} W_{\frac{1}{2}-\frac{3}{2}\nu, \frac{1}{2}-\frac{1}{2}\nu}(2a^2) \\
&\quad [a > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 366(14)} \\
2. \quad \int_0^\infty x^{-\nu-\frac{1}{2}} e^{-x} \cos(4a\sqrt{x}) I_\nu(x) dx &= 2^{\frac{3}{2}\nu-1} a^{\nu-1} e^{-a^2} W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}(2a^2) \\
&\quad [a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 366(16)} \\
3. \quad \int_0^\infty x^{-\nu} e^x \sin(4a\sqrt{x}) K_\nu(x) dx &= (2^{3/2}a)^{\nu-1} \pi \frac{\Gamma(\frac{3}{2}-2\nu)}{\Gamma(\frac{1}{2}+\nu)} e^{a^2} W_{\frac{3}{2}\nu-\frac{1}{2}, \frac{1}{2}-\frac{1}{2}\nu}(2a^2) \\
&\quad [a > 0, \quad 0 < \operatorname{Re} \nu < \frac{3}{4}] \quad \text{ET II 369(38)} \\
4. \quad \int_0^\infty x^{-\nu-\frac{1}{2}} e^x \cos(4a\sqrt{x}) K_\nu(x) dx &= 2^{\frac{3}{2}\nu-1} \pi a^{\nu-1} \frac{\Gamma(\frac{1}{2}-2\nu)}{\Gamma(\frac{1}{2}+\nu)} e^{a^2} W_{\frac{3}{2}\nu, -\frac{1}{2}\nu}(2a^2) \\
&\quad [a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{4}] \quad \text{ET II 369(42)} \\
5. \quad \int_0^\infty x^{\varrho-\frac{3}{2}} e^{-x} \sin(4a\sqrt{x}) K_\nu(x) dx &= \frac{\sqrt{\pi} a \Gamma(\varrho+\nu) \Gamma(\varrho-\nu)}{2^{\varrho-2} \Gamma(\varrho+\frac{1}{2})} {}_2F_2\left(\varrho+\nu, \varrho-\nu; \frac{3}{2}, \varrho+\frac{1}{2}; -2a^2\right) \\
&\quad [\operatorname{Re} \varrho > |\operatorname{Re} \nu|] \quad \text{ET II 369(39)} \\
6. \quad \int_0^\infty x^{\varrho-1} e^{-x} \cos(4a\sqrt{x}) K_\nu(x) dx &= \frac{\sqrt{\pi} \Gamma(\varrho+\nu) \Gamma(\varrho-\nu)}{2^\varrho \Gamma(\varrho+\frac{1}{2})} {}_2F_2\left(\varrho+\nu, \varrho-\nu; \frac{1}{2}, \varrho+\frac{1}{2}; -2a^2\right) \\
&\quad [\operatorname{Re} \varrho > |\operatorname{Re} \nu|] \quad \text{ET II 370(43)} \\
7. \quad \int_0^\infty x^{-1/2} e^{-x} \cos(4a\sqrt{x}) I_0(x) dx &= \frac{1}{\sqrt{2\pi}} e^{-a^2} K_0(a^2) \\
&\quad [a > 0] \quad \text{ET II 366(15)}
\end{aligned}$$

$$8. \quad \int_0^\infty x^{-1/2} e^x \cos(4a\sqrt{x}) K_0(x) dx = \sqrt{\frac{\pi}{2}} e^{a^2} K_0(a^2) \quad [a > 0] \quad \text{ET II 369(40)}$$

$$9. \quad \int_0^\infty x^{-1/2} e^{-x} \cos(4a\sqrt{x}) K_0(x) dx = \frac{1}{\sqrt{2}} \pi^{3/2} e^{-a^2} I_0(a^2) \quad \text{ET II 369(41)}$$

6.756

$$1. \quad \int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \sin(a\sqrt{x}) J_\nu(bx) dx \\ = \frac{i}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) - D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 34(17)}$$

$$2. \quad \int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \cos(a\sqrt{x}) J_\nu(bx) dx \\ = \frac{1}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) + D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 39(42)}$$

$$3. \quad \int_0^\infty x^{-1/2} e^{-a\sqrt{x}} \sin(a\sqrt{x}) J_0(bx) dx = \frac{1}{2b} a I_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \\ \left[|\arg a| < \frac{\pi}{4}, \quad b > 0 \right] \quad \text{ET II 11(40)}$$

$$4. \quad \int_0^\infty x^{-1/2} e^{-a\sqrt{x}} \cos(a\sqrt{x}) J_0(bx) dx = \frac{a}{2b} I_{-\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \\ \left[|\arg a| < \frac{\pi}{4}, \quad b > 0 \right] \quad \text{ET II 12(49)}$$

6.757

$$1. \quad \int_0^\infty e^{-bx} \sin[a(1 - e^{-x})] J_\nu(ae^{-x}) dx \\ = 2 \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\nu - b + 2n + 1) \Gamma(\nu + b)}{\Gamma(\nu - b + 1) \Gamma(\nu + b + 2n + 2)} (\nu + 2n - 1) J_{\nu+2n+1}(a) \\ [\operatorname{Re} b > -\operatorname{Re} \nu] \quad \text{ET I 193(26)}$$

$$2. \quad \int_0^\infty e^{-bx} \cos[a(1 - e^{-x})] J_\nu(ae^{-x}) dx \\ = \frac{J_\nu(a)}{\nu + b} + \sum_{n=0}^\infty 2(-1)^n \frac{\Gamma(\nu - b + 2n) \Gamma(\nu + b)}{\Gamma(\nu - b + 1) \Gamma(\nu + b + 2n + 1)} (\nu + 2n) J_{\nu+2n}(a) \\ [\operatorname{Re} b > -\operatorname{Re} \nu] \quad \text{ET I 193(27)}$$

6.758

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\mu-\nu)\theta} (\cos \theta)^{\nu+\mu} (\lambda z)^{-\nu-\mu} J_{\nu+\mu}(\lambda z) d\theta \\ = \pi (2az)^{-\mu} (2bz)^{-\nu} J_\mu(az) J_\nu(bz); \lambda = \sqrt{2 \cos \theta (a^2 e^{i\theta} + b^2 e^{-i\theta})} \\ \lambda = \sqrt{2 \cos \theta (a^2 e^{i\theta} + b^2 e^{-i\theta})} \quad [\operatorname{Re}(\nu + \mu) > -1] \quad \text{EH II 48(12)}$$

6.76 Combinations of Bessel, trigonometric, and hyperbolic functions

$$\begin{aligned}
 \mathbf{6.761} \quad \int_0^\infty \cosh x \cos(2a \sinh x) J_\nu(b e^x) J_\nu(b e^{-x}) dx &= \frac{J_{2\nu}(2\sqrt{b^2 - a^2})}{2\sqrt{b^2 - a^2}} & [0 < a < b, \quad \operatorname{Re} \nu > -1] \\
 &= 0 & [0 < b < a, \quad \operatorname{Re} \nu > -1] \\
 && \text{ET II 359(10)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6.762} \quad \int_0^\infty \cosh x \sin(2a \sinh x) [J_\nu(b e^x) Y_\nu(b e^{-x}) - Y_\nu(b e^x) J_\nu(b e^{-x})] dx \\
 &= 0 & [0 < a < b, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \\
 &= -\frac{2}{\pi} \cos(\nu\pi) (a^2 - b^2)^{-1/2} K_{2\nu} [2(a^2 - b^2)^{1/2}] & [0 < b < a, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \\
 && \text{ET II 360(12)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6.763} \quad \int_0^\infty \cosh x \cos(2a \sinh x) Y_\nu(b e^x) Y_\nu(b e^{-x}) dx \\
 &= -\frac{1}{2} (b^2 - a^2)^{-1/2} J_{2\nu} [2(b^2 - a^2)^{1/2}] & [0 < a < b, \quad |\operatorname{Re} \nu| < 1] \\
 &= \frac{2}{\pi} \cos(\nu\pi) (a^2 - b^2)^{-1/2} K_{2\nu} [2(a^2 - b^2)^{1/2}] & [0 < b < a, \quad |\operatorname{Re} \nu| < 1] \\
 && \text{ET II 360(11)}
 \end{aligned}$$

6.77 Combinations of Bessel functions and the logarithm, or arctangent

$$\begin{aligned}
 \mathbf{6.771} \quad \int_0^\infty x^{\mu+\frac{1}{2}} \ln x J_\nu(ax) dx &= \frac{2^{\mu-\frac{1}{2}} \Gamma(\frac{\mu+\nu}{2} + \frac{3}{4})}{\Gamma(\frac{\nu-\mu}{2} + \frac{1}{4}) a^{\mu+\frac{3}{2}}} \left[\psi\left(\frac{\mu+\nu}{2} + \frac{3}{4}\right) + \psi\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right) - \ln \frac{a^2}{4} \right] \\
 && [a > 0, \quad -\operatorname{Re} \nu - \tfrac{3}{2} < \operatorname{Re} \mu < 0] \\
 && \text{ET II 32(25)}
 \end{aligned}$$

$$\mathbf{6.772} \quad 1. \quad \int_0^\infty \ln x J_0(ax) dx = -\frac{1}{a} [\ln(2a) + \mathbf{C}] \quad \text{WA 430(4)a, ET II 10(27)}$$

$$2. \quad \int_0^\infty \ln x J_1(ax) dx = -\frac{1}{a} \left[\ln\left(\frac{a}{2}\right) + \mathbf{C} \right] \quad \text{ET II 19(11)}$$

$$3. \quad \int_0^\infty \ln(a^2 + x^2) J_1(bx) dx = \frac{2}{b} [K_0(ab) + \ln a] \quad \text{ET II 19(12)}$$

$$4. \quad \int_0^\infty J_1(tx) \ln \sqrt{1+t^4} dt = \frac{2}{x} \ker x \quad \text{MO 46}$$

$$\begin{aligned}
 \mathbf{6.773} \quad \int_0^\infty \frac{\ln(x + \sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} J_0(bx) dx &= \left[\frac{1}{2} K_0^2\left(\frac{ab}{2}\right) + \ln a I_0\left(\frac{ab}{2}\right) K_0\left(\frac{ab}{2}\right) \right] \\
 && [a > 0, \quad b > 0] \\
 && \text{ET II 10(28)}
 \end{aligned}$$

$$\mathbf{6.774} \quad \int_0^\infty \ln \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} - x} J_0(bx) \frac{dx}{\sqrt{x^2 + a^2}} = K_0^2\left(\frac{ab}{2}\right) \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 10(29)}$$

$$\begin{aligned}
 \mathbf{6.775} \quad \int_0^\infty x \left[\ln(1 + \sqrt{a^2 + x^2}) - \ln x \right] J_0(bx) dx &= \frac{1}{b^2} (1 - e^{-ab}) \\
 && [\operatorname{Re} a > 0, \quad b > 0] \\
 && \text{ET II 12(55)}
 \end{aligned}$$

$$6.776 \quad \int_0^\infty x \ln \left(1 + \frac{a^2}{x^2} \right) J_0(bx) dx = \frac{2}{b} \left[\frac{1}{b} - a K_1(ab) \right] \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 10(30)}$$

$$6.777 \quad \int_0^\infty J_1(tx) \arctan t^2 dt = -\frac{2}{x} \operatorname{kei} x \quad \text{MO 46}$$

6.78 Combinations of Bessel and other special functions

$$6.781 \quad \int_0^\infty \operatorname{si}(ax) J_0(bx) dx = -\frac{1}{b} \arcsin \left(\frac{b}{a} \right) \quad [0 < b < a]$$

$$= 0 \quad [0 < a < b]$$

ET II 13(6)

6.782

$$1. \quad \int_0^\infty \operatorname{Ei}(-x) J_0(2\sqrt{zx}) dx = \frac{e^{-z} - 1}{z} \quad \text{NT 60(4)}$$

$$2. \quad \int_0^\infty \operatorname{si}(x) J_0(2\sqrt{zx}) dx = -\frac{\sin z}{z} \quad \text{NT 60(6)}$$

$$3. \quad \int_0^\infty \operatorname{ci}(x) J_0(2\sqrt{zx}) dx = \frac{\cos z - 1}{z} \quad \text{NT 60(5)}$$

$$4. \quad \int_0^\infty \operatorname{Ei}(-x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\operatorname{Ei}(-z) - C - \ln z}{\sqrt{z}} \quad \text{NT 60(7)}$$

$$5. \quad \int_0^\infty \operatorname{si}(x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = -\frac{\frac{\pi}{2} - \operatorname{si}(z)}{\sqrt{z}} \quad \text{NT 60(9)}$$

$$6. \quad \int_0^\infty \operatorname{ci}(z) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\operatorname{ci}(z) - C - \ln z}{\sqrt{z}} \quad \text{NT 60(8)}$$

$$7. \quad \int_0^\infty \operatorname{Ei}(-x) Y_0(2\sqrt{zx}) dx = \frac{C + \ln z - e^2 \operatorname{Ei}(-z)}{\pi z} \quad \text{NT 63(5)}$$

6.783

$$1. \quad \int_0^\infty x \operatorname{si}(a^2 x^2) J_0(bx) dx = -\frac{2}{b^2} \sin \left(\frac{b^2}{4a^2} \right) \quad [a > 0] \quad \text{ET II 13(7)a}$$

$$2. \quad \int_0^\infty x \operatorname{ci}(a^2 x^2) J_0(bx) dx = \frac{2}{b^2} \left[1 - \cos \left(\frac{b^2}{4a^2} \right) \right] \quad [a > 0] \quad \text{ET II 13(8)a}$$

$$3. \quad \int_0^\infty \operatorname{ci}(a^2 x^2) J_0(bx) dx = \frac{1}{b} \left[\operatorname{ci} \left(\frac{b^2}{4a^2} \right) + \ln \left(\frac{b^2}{4a^2} \right) + 2C \right]$$

[a > 0] ET II 13(8)a

$$4. \quad \int_0^\infty \operatorname{si}(a^2 x^2) J_1(bx) dx = \frac{1}{b} \left[-\operatorname{si} \left(\frac{b^2}{4a^2} \right) - \frac{\pi}{2} \right] \quad [a > 0] \quad \text{ET II 20(25)a}$$

6.784

$$1. \quad \int_0^\infty x^{\nu+1} [1 - \Phi(ax)] J_\nu(bx) dx = a^{-\nu} \frac{\Gamma(\nu + \frac{3}{2})}{b^2 \Gamma(\nu + 2)} \exp \left(-\frac{b^2}{8a^2} \right) M_{\frac{1}{2}\nu + \frac{1}{2}, \frac{1}{2}\nu + \frac{1}{2}} \left(\frac{b^2}{4a^2} \right)$$

[arg a] < $\frac{\pi}{4}$, $b > 0$, $\operatorname{Re} \nu > -1$ ET II 92(22)

2.
$$\int_0^\infty x^\nu [1 - \Phi(ax)] J_\nu(bx) dx = \sqrt{\frac{2}{\pi}} \frac{a^{\frac{1}{2}-\nu} \Gamma(\nu + \frac{1}{2})}{b^{3/2} \Gamma(\nu + \frac{3}{2})} \exp\left(-\frac{b^2}{8a^2}\right) M_{\frac{1}{2}\nu - \frac{1}{4}, \frac{1}{2}\nu + \frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \\ \left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad b > 0\right] \quad \text{ET II 92(23)}$$
- 6.785
$$\int_0^\infty \frac{\exp\left(\frac{a^2}{2x} - x\right)}{x} \left[1 - \Phi\left(\frac{a}{\sqrt{2x}}\right)\right] K_\nu(x) dx = \frac{\pi^{5/2}}{4} \sec(\nu\pi) \left\{[J_\nu(a)]^2 + [Y_\nu(a)]^2\right\} \\ [\operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 370(46)}$$
- 6.786
$$\int_0^\infty x^{\nu-2\mu+2n+2} e^{x^2} \Gamma(\mu, x^2) Y_\nu(bx) dx \\ = (-1)^n \frac{\Gamma(\frac{3}{2} - \mu + \nu + n) \Gamma(\frac{3}{2} - \mu + n)}{b \Gamma(1 - \mu)} \exp\left(\frac{b^2}{8}\right) W_{\mu - \frac{1}{2}\nu - n - 1, \frac{1}{2}\nu}\left(\frac{b^2}{4}\right) \\ [n \text{ is an integer, } b > 0, \quad \operatorname{Re}(\nu - \mu + n) > -\frac{3}{2}, \quad \operatorname{Re}(-\mu + n) > -\frac{3}{2}, \quad \operatorname{Re} \nu < \frac{1}{2} - 2n] \\ \text{ET II 108(2)}$$
- 6.787
$$\int_{-\infty}^\infty \frac{x^{\nu+2n-\frac{1}{2}}}{B(a+x, a-x)} J_\nu(bx) dx = 0 \\ [\pi \leq b < \infty, \quad -1 < \operatorname{Re} \nu < 2a - 2n - \frac{7}{2}] \quad \text{ET II 92(21)}$$

6.79 Integration of Bessel functions with respect to the order

6.791

1.
$$\int_{-\infty}^\infty K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi K_{iy-iz}(a+b) \quad [|\arg a| + |\arg b| < \pi] \quad \text{ET II 382(21)}$$
2.
$$\int_{-\infty}^\infty J_{\nu-x}(a) J_{\mu+x}(a) dx = J_{\mu+\nu}(2a) \quad [\operatorname{Re}(\mu + \nu) > 1] \quad \text{ET II 379(1)}$$
3.
$$\int_{-\infty}^\infty J_{\kappa+x}(a) J_{\lambda-x}(a) J_{\mu+x}(a) J_{\nu-x}(a) dx \\ = \frac{\Gamma(\kappa + \lambda + \mu + \nu + 1)}{\Gamma(\kappa + \lambda + 1) \Gamma(\lambda + \mu + 1) \Gamma(\mu + \nu + 1) \Gamma(\nu + \kappa + 1)} \\ \times {}_4F_5\left(\frac{\kappa + \lambda + \mu + \nu + 1}{2}, \frac{\kappa + \lambda + \mu + \nu + 1}{2}, \frac{\kappa + \lambda + \mu + \nu}{2} + 1, \frac{\kappa + \lambda + \mu + \nu}{2} + 1; \right. \\ \left. \kappa + \lambda + \mu + \nu + 1, \kappa + \lambda + 1, \lambda + \mu + 1, \mu + \nu + 1, \nu + \kappa + 1; -4a^2\right) \\ [\operatorname{Re}(\kappa + \lambda + \mu + \nu) > -1] \quad \text{ET II 379(3)}$$

6.792

1.
$$\int_{-\infty}^\infty e^{\pi x} K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\pi z} K_{i(y-z)}(a-b) \\ [a > b > 0] \quad \text{ET II 382(22)}$$

$$2. \quad \int_{-\infty}^{\infty} e^{i\varrho x} K_{\nu+ix}(\alpha) K_{\nu-ix}(\beta) dx = \pi \left(\frac{\alpha e^{\rho} + \beta}{\alpha + \beta e^{\rho}} \right)^{\nu} K_{2\nu} \left(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cosh \varrho} \right) \\ [|\arg \alpha| + |\arg \beta| + |\operatorname{Im} \varrho| < \pi] \quad \text{ET II 382(23)}$$

$$3. \quad \int_{-\infty}^{\infty} e^{(\pi-\gamma)x} K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\beta y - \alpha z} K_{iy-iz}(c) \\ [0 < \gamma < \pi, \quad a > 0, \quad b > 0, \quad c > 0, \quad \alpha, \beta, \gamma \text{—the angles of the triangle with sides } a, b, c] \\ \text{ET II 382(24), EH II 55(44)a}$$

$$4.11 \quad \int_{-\infty}^{\infty} e^{-cxi} H_{\nu-ix}^{(2)}(a) H_{\nu+ix}^{(2)}(b) dx = 2i \left(\frac{h}{k} \right)^{2\nu} H_{2\nu}^{(2)}(hk) \\ h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, \quad c \text{ is real}] \quad \text{ET II 380(11)}$$

$$5. \quad \int_{-\infty}^{\infty} a^{-\mu-x} b^{-\nu+x} e^{cxi} J_{\mu+x}(a) J_{\nu-x}(b) dx \\ = \left[\frac{2 \cos \left(\frac{c}{2} \right)}{a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci}} \right]^{\frac{1}{2}\mu + \frac{1}{2}\nu} \exp \left[\frac{c}{2}(\nu - \mu)i \right] J_{\mu+\nu} \left\{ \left[2 \cos \left(\frac{c}{2} \right) \left(a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci} \right) \right]^{1/2} \right\} \\ [a > 0, \quad b > 0, \quad |c| < \pi, \quad \operatorname{Re}(\mu + \nu) > 1] \\ = 0 \\ [a > 0, \quad b > 0, \quad |c| \geq \pi, \quad \operatorname{Re}(\mu + \nu) > 1] \\ \text{EH II 54(41), ET II 379(2)}$$

6.793

$$1. \quad \int_{-\infty}^{\infty} e^{-cxi} [J_{\nu-ix}(a) Y_{\nu+ix}(b) + Y_{\nu-ix}(a) J_{\nu+ix}(b)] dx = -2 \left(\frac{h}{k} \right)^{2\nu} J_{2\nu}(hk) \\ h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, \quad \operatorname{Im} c = 0] \quad \text{ET II 380(9)}$$

$$2. \quad \int_{-\infty}^{\infty} e^{-cxi} [J_{\nu-ix}(a) J_{\nu+ix}(b) - Y_{\nu-ix}(a) Y_{\nu+ix}(b)] dx = 2 \left(\frac{h}{k} \right)^{2\nu} Y_{2\nu}(hk) \\ h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, \quad \operatorname{Im} c = 0] \quad \text{ET II 380(10)}$$

$$3.10 \quad \int_{-\infty}^{\infty} e^{i\gamma x} \operatorname{sech}(\pi x) [J_{-ix}(\alpha) J_{ix}(\beta) - J_{ix}(\alpha) J_{-ix}(\beta)] dx = 2i H(\sigma) \operatorname{sign}(\beta - \alpha) J_0 \left(\sigma^{1/2} \right) \\ [\alpha, \beta, \gamma \in \mathbb{R}, \quad \alpha, \beta > 0, \quad \sigma = \alpha^2 + \beta^2 - 2\alpha\beta \cosh \gamma, \quad H(\sigma) \text{ the Heaviside step function}]$$

6.794

$$1. \quad \int_0^{\infty} K_{ix}(a) K_{ix}(b) \cosh[(\pi - \varphi)x] dx = \frac{\pi}{2} K_0 \left(\sqrt{a^2 + b^2 - 2ab \cos \varphi} \right) \quad \text{EH II 55(42)}$$

$$2. \quad \int_0^{\infty} \cosh \left(\frac{\pi}{2} x \right) K_{ix}(a) dx = \frac{\pi}{2} \quad [a > 0] \quad \text{ET II 382(19)}$$

3.
$$\int_0^\infty \cosh(\varrho x) K_{ix+\nu}(a) K_{-ix+\nu}(a) dx = \frac{\pi}{2} K_{2\nu} \left[2a \cos \left(\frac{\varrho}{2} \right) \right]$$

$[2|\arg a| + |\operatorname{Re} \varrho| < \pi]$ ET II 383(28)
4.
$$\int_{-\infty}^\infty \operatorname{sech} \left(\frac{\pi}{2} x \right) J_{ix}(a) dx = 2 \sin a$$

$[a > 0]$ ET II 380(6)
5.
$$\int_{-\infty}^\infty \operatorname{cosech} \left(\frac{\pi}{2} x \right) J_{ix}(a) dx = -2i \cos a$$

$[a > 0]$ ET II 380(7)
6.
$$\int_0^\infty \operatorname{sech}(\pi x) \left\{ [J_{ix}(a)]^2 + [Y_{ix}(a)]^2 \right\} dx = -Y_0(2a) - \mathbf{E}_0(2a)$$

$[a > 0]$ ET II 380(12)
7.
$$\int_0^\infty x \sinh \left(\frac{\pi}{2} x \right) K_{ix}(a) dx = \frac{\pi a}{2}$$

$[a > 0]$ ET II 382(20)
8.
$$\int_0^\infty x \tanh(\pi x) K_{ix}(\beta) K_{ix}(\alpha) dx = \frac{\pi}{2} \sqrt{\alpha\beta} \frac{\exp(-\beta - \alpha)}{\alpha + \beta}$$

$[|\arg \beta| < \pi, \quad |\arg \alpha| < \pi]$ ET II 175(4)
9.
$$\int_0^\infty x \sinh(\pi x) K_{2ix}(\alpha) K_{ix}(\beta) dx = \frac{\pi^{3/2} \alpha}{2^{5/2} \sqrt{\beta}} \exp \left(-\beta - \frac{\alpha^2}{8\beta} \right)$$

$\left[\beta > 0, \quad |\arg \alpha| < \frac{\pi}{4} \right]$ ET II 175(5)
10.
$$\begin{aligned} \int_0^\infty \frac{x \sinh(\pi x)}{x^2 + n^2} K_{ix}(\alpha) K_{ix}(\beta) dx &= \frac{\pi^2}{2} I_n(\beta) K_n(\alpha) \quad [0 < \beta < \alpha; \quad n = 0, 1, 2, \dots] \\ &= \frac{\pi^2}{2} I_n(\alpha) K_n(\beta) \quad [0 < \alpha < \beta; \quad n = 0, 1, 2, \dots] \end{aligned}$$

ET II 176(8)
11.
$$\int_0^\infty x \sinh(\pi x) K_{ix}(\alpha) K_{ix}(\beta) K_{ix}(\gamma) dx = \frac{\pi^2}{4} \exp \left[-\frac{\gamma}{2} \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\gamma^2} \right) \right]$$

$\left[|\arg \alpha| + |\arg \beta| < \frac{\pi}{2}, \quad \gamma > 0 \right]$ ET II 176(9)
12.
$$\int_0^\infty x \sinh \left(\frac{\pi}{2} x \right) K_{\frac{1}{2}ix}(\alpha) K_{\frac{1}{2}ix}(\beta) K_{ix}(\gamma) dx = \frac{\pi^2 \gamma}{2\sqrt{\gamma^2 + 4\alpha\beta}} \exp \left[-\frac{(\alpha + \beta)\sqrt{\gamma^2 + 4\alpha\beta}}{2\sqrt{\alpha\beta}} \right]$$

$[|\arg \alpha| + |\arg \beta| < \pi, \quad \gamma > 0]$ ET II 176(10)
13.
$$\int_0^\infty x \sinh(\pi x) K_{\frac{1}{2}ix+\lambda}(\alpha) K_{\frac{1}{2}ix-\lambda}(\alpha) K_{ix}(\gamma) dx = 0$$

$[0 < \gamma < 2\alpha]$

$$= \frac{\pi^2 \gamma}{2^{2\lambda+1} \alpha^{2\lambda} z} \left[(\gamma + z)^{2\lambda} + (\gamma - z)^{2\lambda} \right]$$

$z = \sqrt{\gamma^2 - 4\alpha^2} \quad [0 < 2\alpha < \gamma]$ ET II 176(11)

6.795

1. $\int_0^\infty \cos(bx) K_{ix}(a) dx = \frac{\pi}{2} e^{-a \cosh b} \quad \left[|\operatorname{Im} b| < \frac{\pi}{2}, \quad a > 0 \right]$
EH II 55(46), ET II 175(2)
2. $\int_0^\infty J_x(ax) J_{-x}(ax) \cos(\pi x) dx = \frac{1}{4} (1 - a^2)^{-1/2} \quad [|a| < 1]$ ET II 380(4)
3. $\int_0^\infty x \sin(ax) K_{ix}(b) dx = \frac{\pi b}{2} \sinh a \exp(-b \cosh a) \quad \left[|\operatorname{Im} a| < \frac{\pi}{2}, \quad b > 0 \right]$ ET II 175(1)
4. $\int_{-\infty}^\infty \frac{\sin[(\nu + ix)\pi]}{n + \nu + ix} K_{\nu+ix}(a) K_{\nu-ix}(b) dx = \pi^2 I_n(a) K_{n+2\nu}(b) \quad [0 < a < b; \quad n = 0, 1, \dots]$
 $= \pi^2 K_{n+2\nu}(a) I_n(b) \quad [0 < b < a; \quad n = 0, 1, \dots]$
ET II 382(25)
5. $\int_0^\infty x \sin\left(\frac{1}{2}\pi x\right) K_{\frac{1}{2}ix}(a) K_{ix}(b) dx = \frac{\pi^{3/2}b}{\sqrt{2a}} \exp\left(-a - \frac{b^2}{8a}\right)$
 $\left[|\arg a| < \frac{\pi}{2}, \quad b > 0 \right]$ ET II 175(6)

6.796

1. $\int_{-\infty}^\infty \frac{e^{\frac{1}{2}\pi x} \cos(bx)}{\sinh(\pi x)} J_{ix}(a) dx = -i \exp(ia \cosh b) \quad [a > 0, \quad b > 0]$ ET II 380(8)
2. $\int_0^\infty \cos(bx) \cosh\left(\frac{1}{2}\pi x\right) K_{ix}(a) dx = \frac{\pi}{2} \cos(a \sinh b)$ EH II 55(47)
3. $\int_0^\infty \sin(bx) \sinh\left(\frac{1}{2}\pi x\right) K_{ix}(a) dx = \frac{\pi}{2} \sin(a \sinh b)$ EH II 55(48)
4. $\int_0^\infty \cos(bx) \cosh(\pi x) [K_{ix}(a)]^2 dx = -\frac{\pi^2}{4} Y_0 \left[2a \sinh\left(\frac{b}{2}\right) \right]$
 $[a > 0, \quad b > 0]$ ET II 383(27)
5. $\int_0^\infty \sin(bx) \sinh(\pi x) [K_{ix}(a)]^2 dx = \frac{\pi^2}{4} J_0 \left[2a \sinh\left(\frac{b}{2}\right) \right]$
 $[a > 0, \quad b > 0]$ ET II 382(26)

6.797

1. $\int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx$
 $= i2^\nu \sqrt{\pi} \Gamma\left(\frac{1}{2} + \nu\right) (ab)^\nu (a+b)^{-\nu} K_\nu(a+b)$
 $[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > 0]$ ET II 381(14)
2. $\int_0^\infty x e^{\pi x} \sinh(\pi x) \cosh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx = \frac{i\pi^{3/2}2^\nu}{\Gamma\left(\frac{1}{2} - \nu\right)} (b-a)^{-\nu} H_\nu^{(2)}(b-a)$
 $[0 < a < b, \quad 0 < \operatorname{Re} \nu < \frac{1}{2}]$
ET II 381(15)

3.
$$\int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma\left(\frac{\nu + ix}{2}\right) \Gamma\left(\frac{\nu - ix}{2}\right) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx$$

$$= i\pi 2^{2-\nu} (ab)^\nu (a^2 + b^2)^{-\frac{1}{2}\nu} H_\nu^{(2)}\left(\sqrt{a^2 + b^2}\right)$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 381(16)}$$
- 4.¹¹
$$\int_0^\infty x \sinh(\pi x) \Gamma(\lambda + ix) \Gamma(\lambda - ix) K_{ix}(a) K_{ix}(b) dx = 2^{\lambda-1} \pi^{3/2} (ab)^\lambda (a+b)^{-\lambda} \Gamma\left(\lambda + \frac{1}{2}\right) K_\lambda(a+b)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0, \quad b > 0]$$

$$\text{ET II 176(12)}$$
5.
$$\int_0^\infty x \sinh(2\pi x) \Gamma(\lambda + ix) \Gamma(\lambda - ix) K_{ix}(a) K_{ix}(b) dx = \frac{2^\lambda \pi^{\frac{5}{2}}}{\Gamma\left(\frac{1}{2} - \lambda\right)} \left(\frac{ab}{|b-a|}\right)^\lambda K_\lambda(|b-a|)$$

$$[a > 0, \quad 0 < \operatorname{Re} \lambda < \frac{1}{2}, \quad b > 0]$$

$$\text{ET II 176(13)}$$
6.
$$\int_0^\infty x \sinh(\pi x) \Gamma\left(\lambda + \frac{1}{2}ix\right) \Gamma\left(\lambda - \frac{1}{2}ix\right) K_{ix}(a) K_{ix}(b) dx = 2\pi^2 \left(\frac{ab}{2\sqrt{a^2 + b^2}}\right) K_{2\lambda}\left(\sqrt{a^2 + b^2}\right)$$

$$[|\arg a| < \frac{\pi}{2}, \quad \operatorname{Re} \lambda > 0, \quad b > 0]$$

$$\text{ET II 177(14)}$$
7.
$$\int_0^\infty \frac{x \tanh(\pi x) K_{ix}(a) K_{ix}(b)}{\Gamma\left(\frac{3}{4} + \frac{1}{2}ix\right) \Gamma\left(\frac{3}{4} - \frac{1}{2}ix\right)} dx = \frac{1}{2} \sqrt{\frac{\pi ab}{a^2 + b^2}} \exp\left(-\sqrt{a^2 + b^2}\right)$$

$$[|\arg a| < \frac{\pi}{2}, \quad b > 0], \quad \text{ET II 177(15)}$$

6.8 Functions Generated by Bessel Functions

6.81 Struve functions

6.811

1.
$$\int_0^\infty \mathbf{H}_\nu(bx) dx = -\frac{\cot\left(\frac{\nu\pi}{2}\right)}{b}$$

$$[-2 < \operatorname{Re} \nu < 0, \quad b > 0] \quad \text{ET II 158(1)}$$
2.
$$\int_0^\infty \mathbf{H}_\nu\left(\frac{a^2}{x}\right) \mathbf{H}_\nu(bx) dx = -\frac{J_{2\nu}(2a\sqrt{b})}{b}$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}]$$

$$\text{ET II 170(37)}$$
3.
$$\int_0^\infty \mathbf{H}_{\nu-1}\left(\frac{a^2}{x}\right) \mathbf{H}_\nu(bx) \frac{dx}{x} = -\frac{1}{a\sqrt{b}} J_{2\nu-1}(2a\sqrt{b})$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$$

$$\text{ET II 170(38)}$$

6.812

1.
$$\int_0^\infty \frac{\mathbf{H}_1(bx) dx}{x^2 + a^2} = \frac{\pi}{2a} [I_1(ab) - \mathbf{L}_1(ab)]$$

$$[\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 158(6)}$$

$$2. \quad \int_0^\infty \frac{\mathbf{H}_\nu(bx)}{x^2 + a^2} dx = -\frac{\pi}{2a \sin\left(\frac{\nu\pi}{2}\right)} \mathbf{L}_\nu(ab) + \frac{b \cot\left(\frac{\nu\pi}{2}\right)}{1 - \nu^2} {}_1F_2\left(1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{a^2 b^2}{2}\right)$$

[Re $a > 0$, $b > 0$, $|\operatorname{Re} \nu| < 2$]
ET II 159(7)

6.813

$$1. \quad \int_0^\infty x^{s-1} \mathbf{H}_\nu(ax) dx = \frac{2^{s-1} \Gamma\left(\frac{s+\nu}{2}\right)}{a^s \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}s + 1\right)} \tan\left(\frac{s+\nu}{2}\pi\right)$$

[$a > 0$, $-1 - \operatorname{Re} \nu < \operatorname{Re} s < \min\left(\frac{3}{2}, 1 - \operatorname{Re} \nu\right)$] WA 429(2), ET I 335(52)

$$2. \quad \int_0^\infty x^{-\nu-1} \mathbf{H}_\nu(x) dx = \frac{2^{-\nu-1}\pi}{\Gamma(\nu+1)} \quad [\operatorname{Re} \nu > -\tfrac{3}{2}] \quad \text{ET II 383(2)}$$

$$3. \quad \int_0^\infty x^{-\mu-\nu} \mathbf{H}_\mu(x) \mathbf{H}_\nu(x) dx = \frac{2^{-\mu-\nu} \sqrt{\pi} \Gamma(\mu+\nu)}{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\mu + \nu + \frac{1}{2}\right)}$$

[Re($\mu + \nu$) > 0] WA 435(2), ET II 384(8)

$$4. \quad \int_0^1 x^{\nu+1} \mathbf{H}_\nu(ax) dx = \frac{1}{a} \mathbf{H}_{\nu+1}(a) \quad [a > 0, \operatorname{Re} \nu > -\tfrac{3}{2}] \quad \text{ET II 158(2)a}$$

$$5. \quad \int_0^1 x^{1-\nu} \mathbf{H}_\nu(ax) dx = \frac{a^{\nu-1}}{2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} - \frac{1}{a} \mathbf{H}_{\nu-1}(a)$$

[$a > 0$] ET II 158(3)a

6.814

$$1. \quad \int_0^\infty \frac{x^{\nu+1} \mathbf{H}_\nu(bx)}{(x^2 + a^2)^{1-\mu}} dx = \frac{2^{\mu-1} \pi a^{\mu+\nu} b^{-\mu}}{\Gamma(1-\mu) \cos[(\mu+\nu)\pi]} [I_{-\mu-\nu}(ab) - \mathbf{L}_{\mu+\nu}(ab)]$$

[Re $a > 0$, $b > 0$, $\operatorname{Re} \nu > -\frac{3}{2}$, $\operatorname{Re}(\mu + \nu) < \frac{1}{2}$, $\operatorname{Re}(2\mu + \nu) < \frac{3}{2}$] ET II 159(8)

6.815

$$1. \quad \int_0^1 x^{\frac{1}{2}\nu} (1-x)^{\mu-1} \mathbf{H}_\nu(a\sqrt{x}) dx = 2^\mu a^{-\mu} \Gamma(\mu) \mathbf{H}_{\mu+\nu}(a)$$

[Re $\nu > -\frac{3}{2}$, $\operatorname{Re} \mu > 0$] ET II 199(88)a

$$2. \quad \int_0^1 x^{\lambda - \frac{1}{2}\nu - \frac{3}{2}} (1-x)^{\mu-1} \mathbf{H}_\nu(a\sqrt{x}) dx = \frac{\mathbf{B}(\lambda, \mu) a^{\nu+1}}{2^\nu \sqrt{\pi} \Gamma\left(\nu + \frac{3}{2}\right)} {}_2F_3\left(1, \lambda; \frac{3}{2}, \nu + \frac{3}{2}, \lambda + \mu; -\frac{a^2}{4}\right)$$

[Re $\lambda > 0$, $\operatorname{Re} \mu > 0$] ET II 199(89)a

6.82 Combinations of Struve functions, exponentials, and powers**6.821**

$$1.^6 \quad \int_0^\infty e^{-\alpha x} \mathbf{H}_{-n-\frac{1}{2}}(\beta x) dx = (-1)^n \beta^{n+\frac{1}{2}} \left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)^{-n-\frac{1}{2}} \frac{1}{\sqrt{\alpha^2 + \beta^2}}$$

[Re $\alpha > |\operatorname{Im} \beta|$] ET I 206(6)

$$2.^6 \quad \int_0^\infty e^{-\alpha x} \mathbf{L}_{-n-\frac{1}{2}}(\beta x) dx = \beta^{n+\frac{1}{2}} \left(\alpha + \sqrt{\alpha^2 - \beta^2} \right)^{-n-\frac{1}{2}} \frac{1}{\sqrt{\alpha^2 - \beta^2}} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|] \quad \text{ET I 208(26)}$$

$$3. \quad \int_0^\infty e^{-\alpha x} \mathbf{H}_0(\beta x) dx = \frac{2}{\pi} \frac{\ln \left(\frac{\sqrt{\alpha^2 + \beta^2} + \beta}{\alpha} \right)}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{ET II 205(1)}$$

$$4. \quad \int_0^\infty e^{-\alpha x} \mathbf{L}_0(\beta x) dx = \frac{2}{\pi} \frac{\arcsin \left(\frac{\beta}{\alpha} \right)}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|] \quad \text{ET II 207(18)}$$

$$6.822 \quad \int_0^\infty e^{(\nu+1)x} \mathbf{H}_\nu(a \sinh x) dx = \sqrt{\frac{\pi}{a}} \operatorname{cosec}(\nu\pi) \left[\sinh \left(\frac{a}{2} \right) I_{\nu+\frac{1}{2}} \left(\frac{a}{2} \right) - \cosh \left(\frac{a}{2} \right) I_{-\nu-\frac{1}{2}} \left(\frac{a}{2} \right) \right] \quad [\operatorname{Re} a > 0, \quad -2 < \operatorname{Re} \nu < 0] \quad \text{ET II 385(11)}$$

6.823

$$1. \quad \int_0^\infty x^\lambda e^{-\alpha x} \mathbf{H}_\nu(bx) dx = \frac{b^{\nu+1} \Gamma(\lambda + \nu + 2)}{2^\nu a^{\lambda+\nu+2} \sqrt{\pi} \Gamma \left(\nu + \frac{3}{2} \right)} {}_3F_2 \left(1, \frac{\lambda + \nu}{2} + 1, \frac{\lambda + \nu + 3}{2}; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{b^2}{a^2} \right) \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\lambda + \nu) > -2] \quad \text{ET II 161(19)}$$

$$2. \quad \int_0^\infty x^\nu e^{-\alpha x} \mathbf{L}_\nu(\beta x) dx = \frac{(2\beta)^\nu \Gamma \left(\nu + \frac{1}{2} \right)}{\sqrt{\pi} \left(\sqrt{\alpha^2 - \beta^2} \right)^{2\nu+1}} - \frac{\Gamma(2\nu + 1) \left(\frac{\beta}{\alpha} \right)^\nu}{\sqrt{\frac{\pi}{2}} \alpha (\beta^2 - \alpha^2)^{\frac{1}{2}\nu + \frac{1}{4}}} P_{-\nu-\frac{1}{2}}^{-\nu-\frac{1}{2}} \left(\frac{\beta}{\alpha} \right) \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 209(35)a}$$

6.824

$$1. \quad \int_0^\infty t^\nu e^{-at} \mathbf{L}_{2\nu}(2\sqrt{t}) dt = \frac{1}{a^{2\nu+1}} e^{\frac{1}{a}} \Phi \left(\frac{1}{\sqrt{a}} \right) \quad \text{MI 51}$$

$$2. \quad \int_0^\infty t^\nu e^{-at} \mathbf{L}_{-2\nu}(\sqrt{t}) dt = \frac{1}{\Gamma \left(\frac{1}{2} - 2\nu \right) a^{2\nu+1}} e^{\frac{1}{a}} \gamma \left(\frac{1}{2} - 2\nu, \frac{1}{a} \right) \quad \text{MI 51}$$

$$6.825 \quad \int_0^\infty x^{s-1} e^{-\alpha^2 x^2} \mathbf{H}_\nu(\beta x) dx = \frac{\beta^{\nu+1} \Gamma \left(\frac{1}{2} + \frac{s}{2} + \frac{\nu}{2} \right)}{2^{\nu+1} \sqrt{\pi} \alpha^{\nu+s+1} \Gamma \left(\nu + \frac{3}{2} \right)} {}_2F_2 \left(1, \frac{\nu + s + 1}{2}; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{\beta^2}{4\alpha^2} \right) \quad \left[\operatorname{Re} s > -\operatorname{Re} \nu - 1, \quad |\arg \alpha| < \frac{\pi}{4} \right] \quad \text{ET I 335(51)a, ET II 162(20)}$$

6.83 Combinations of Struve and trigonometric functions

$$6.831 \quad \int_0^\infty x^{-\nu} \sin(ax) \mathbf{H}_\nu(bx) dx = 0 \quad [0 < b < a, \quad \operatorname{Re} \nu > -\frac{1}{2}]$$

$$= \sqrt{\pi} 2^{-\nu} b^{-\nu} \frac{(b^2 - a^2)^{\nu-\frac{1}{2}}}{\Gamma \left(\nu + \frac{1}{2} \right)} \quad [0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 162(21)}$$

$$6.832 \quad \int_0^\infty \sqrt{x} \sin(ax) \mathbf{H}_{\frac{1}{4}}(b^2 x^2) dx = -2^{-3/2} \sqrt{\pi} \frac{\sqrt{a}}{b^2} Y_{\frac{1}{4}}\left(\frac{a^2}{4b^2}\right) \quad [a > 0] \quad \text{ET I 109(14)}$$

6.84–6.85 Combinations of Struve and Bessel functions

$$6.841 \quad \int_0^\infty \mathbf{H}_{\nu-1}(ax) Y_\nu(bx) dx = -a^{\nu-1} b^{-\nu} \quad [0 < b < a, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}]$$

$$= 0 \quad [0 < a < b, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 114(36)}$$

$$6.842 \quad \int_0^\infty [\mathbf{H}_0(ax) - Y_0(ax)] J_0(bx) dx = \frac{4}{\pi(a+b)} \mathbf{K}\left(\frac{|a-b|}{a+b}\right) \quad [a > 0, \quad b > 0] \quad \text{ET II 15(22)}$$

$$6.843 \quad 1. \quad \int_0^\infty J_{2\nu}(a\sqrt{x}) \mathbf{H}_\nu(bx) dx = -\frac{1}{b} Y_\nu\left(\frac{a^2}{4b}\right) \quad [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \tfrac{5}{4}] \quad \text{ET II 164(10)}$$

$$2. \quad \int_0^\infty K_{2\nu}(2a\sqrt{x}) \mathbf{H}_\nu(bx) dx = \frac{2^\nu}{\pi b} \Gamma(\nu+1) S_{-\nu-1,\nu}\left(\frac{a^2}{b}\right) \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 168(27)}$$

$$6.844 \quad \int_0^\infty \left[\cos\left(\frac{\mu-\nu}{2}\pi\right) J_\mu(a\sqrt{x}) - \sin\left(\frac{\mu-\nu}{2}\pi\right) Y_\mu(a\sqrt{x}) \right] K_\mu(a\sqrt{x}) \mathbf{H}_\nu(bx) dx$$

$$= \frac{1}{a^2} W_{\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) \quad \left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \operatorname{Re} \nu > |\operatorname{Re} \mu| - 2\right] \quad \text{ET II 169(35)}$$

$$6.845 \quad 1. \quad \int_0^\infty \left[\mathbf{H}_{-\nu}\left(\frac{a}{x}\right) - Y_{-\nu}\left(\frac{a}{x}\right) \right] J_\nu(bx) dx = \frac{4}{\pi b} \cos(\nu\pi) K_{2\nu}(2\sqrt{ab}) \quad [|\arg a| < \pi, \quad b > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 73(7)}$$

$$2. \quad \int_0^\infty \left[J_{-\nu}\left(\frac{a^2}{x}\right) + \sin(\nu\pi) \mathbf{H}_\nu\left(\frac{a^2}{x}\right) \right] \mathbf{H}_\nu(bx) dx = \frac{1}{b} \left[\frac{2}{\pi} K_{2\nu}(2a\sqrt{b}) - Y_{2\nu}(2a\sqrt{b}) \right] \quad [a > 0, \quad b > 0, \quad -\tfrac{3}{2} < \operatorname{Re} \nu < 0] \quad \text{ET II 170(39)}$$

$$6.846 \quad \int_0^\infty \left[\frac{2}{\pi} K_{2\nu}(2a\sqrt{x}) + Y_{2\nu}(2a\sqrt{x}) \right] \mathbf{H}_\nu(bx) dx = \frac{1}{b} J_\nu\left(\frac{a^2}{b}\right) \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 169(30)}$$

$$6.847 \quad \int_0^\infty \left[\cos \frac{\nu\pi}{2} J_\nu(ax) + \sin \frac{\nu\pi}{2} \mathbf{H}_\nu(ax) \right] \frac{dx}{x^2 + k^2} = \frac{\pi}{2k} [I_\nu(ak) - \mathbf{L}_\nu(ak)] \quad [a > 0, \quad \operatorname{Re} k > 0, \quad -\tfrac{1}{2} < \operatorname{Re} \nu < 2] \quad \text{ET II 384(5)a, WA 467(8)}$$

6.848

$$1. \quad \int_0^\infty x [I_\nu(ax) - \mathbf{L}_{-\nu}(ax)] J_\nu(bx) dx = \frac{2}{\pi} \left(\frac{a}{b}\right)^{\nu-1} \cos(\nu\pi) \frac{1}{a^2 + b^2} \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2} \right] \\ \text{ET II 74(12)}$$

$$2. \quad \int_0^\infty x [\mathbf{H}_{-\nu}(ax) - Y_{-\nu}(ax)] J_\nu(bx) dx = 2 \frac{\cos(\nu\pi)}{a^\nu \pi} b^{\nu-1} \frac{1}{a+b} \\ \left[|\arg a| < \pi, \quad -\frac{1}{2} < \operatorname{Re} \nu, \quad b > 0 \right] \\ \text{ET II 73(5)}$$

6.849

$$1. \quad \int_0^\infty x K_\nu(ax) \mathbf{H}_\nu(bx) dx = a^{-\nu-1} b^{\nu+1} \frac{1}{a^2 + b^2} \quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right] \\ \text{ET II 164(12)}$$

$$2. \quad \int_0^\infty x [K_\mu(ax)]^2 \mathbf{H}_0(bx) dx = -2^{-\mu-1} \pi a^{-2\mu} \frac{[(z+b)^{2\mu} + (z-b)^{2\mu}]}{bz} \sec(\mu\pi), \\ z = \sqrt{4a^2 + b^2} \quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < \frac{3}{2} \right] \quad \text{ET II 166(18)}$$

6.851

$$1. \quad \int_0^\infty x \left\{ [J_{\frac{1}{2}\nu}(ax)]^2 - [Y_{\frac{1}{2}\nu}(ax)]^2 \right\} \mathbf{H}_\nu(bx) dx \\ = 0 \quad \left[0 < b < 2a, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0 \right] \\ = \frac{4}{\pi b} \frac{1}{\sqrt{b^2 - 4a^2}} \quad \left[0 < 2a < b, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0 \right] \\ \text{ET II 164(7)}$$

$$2. \quad \int_0^\infty x^{\nu+1} \left\{ [J_\nu(ax)]^2 - [Y_\nu(ax)]^2 \right\} \mathbf{H}_\nu(bx) dx \\ = 0 \quad \left[0 < b < 2a, \quad -\frac{3}{4} < \operatorname{Re} \nu < 0 \right] \\ = \frac{2^{3\nu+2} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \nu\right)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \quad \left[0 < 2a < b, \quad -\frac{3}{4} < \operatorname{Re} \nu < 0 \right] \\ \text{ET II 163(6)}$$

6.852

$$1. \quad \int_0^\infty x^{1-\mu-\nu} J_\nu(x) \mathbf{H}_\mu(x) dx = \frac{(2\nu-1)2^{-\mu-\nu}}{(\mu+\nu-1) \Gamma\left(\mu+\frac{1}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right)} \\ \left[\operatorname{Re} \nu > \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > 1 \right] \\ \text{ET II 383(4)}$$

$$2. \quad \int_0^\infty x^{\mu-\nu+1} Y_\mu(ax) \mathbf{H}_\nu(bx) dx \\ = 0 \quad \left[0 < b < a, \quad \operatorname{Re}(\nu-\mu) > 0, \quad -\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2} \right] \\ = \frac{2^{1+\mu-\nu} a^\mu b^{-\nu}}{\Gamma(\nu-\mu)} (b^2 - a^2)^{\nu-\mu-1} \quad \left[0 < a < b, \quad \operatorname{Re}(\nu-\mu) > 0, \quad -\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2} \right] \\ \text{ET II 163(3)}$$

$$3. \quad \int_0^\infty x^{\mu+\nu+1} K_\mu(ax) \mathbf{H}_\nu(bx) dx = \frac{2^{\mu+\nu+1} b^{\nu+1}}{\sqrt{\pi} a^{\mu+2\nu+3}} \Gamma\left(\mu + \nu + \frac{3}{2}\right) F\left(1, \mu + \nu + \frac{3}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2}] \quad \text{ET II 165(13)}$$

6.853

$$1. \quad \int_0^\infty x^{1-\mu} [\sin(\mu\pi) J_{\mu+\nu}(ax) + \cos(\mu\pi) Y_{\mu+\nu}(ax)] \mathbf{H}_\nu(bx) dx \\ = 0 \quad [0 < b < a, \quad 1 < \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu - \mu) < \frac{1}{2}] \\ = \frac{b^\nu (b^2 - a^2)^{\mu-1}}{2^{\mu-1} a^{\mu+\nu} \Gamma(\mu)} \quad [0 < a < b, \quad 1 < \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu - \mu) < \frac{1}{2}] \\ \text{ET II 163(4)}$$

$$2. \quad \int_0^\infty x^{\lambda+\frac{1}{2}} [I_\mu(ax) - \mathbf{L}_{-\mu}(ax)] J_\nu(bx) dx \\ = 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi} b^{-\lambda-\frac{3}{2}} G_{33}^{22} \left(\frac{b^2}{a^2} \left| \begin{matrix} \frac{1+\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, \frac{1+\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\mu + \nu + \lambda) > -\frac{3}{2}, \quad -\operatorname{Re} \nu - \frac{5}{2} < \operatorname{Re}(\lambda - \mu) < 1] \quad \text{ET II 76(21)}$$

$$3. \quad \int_0^\infty x^{\lambda+\frac{1}{2}} [\mathbf{H}_\mu(ax) - Y_\mu(ax)] J_\nu(bx) dx \\ = 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi^2} b^{-\lambda-\frac{3}{2}} G_{33}^{23} \left(\frac{b^2}{a^2} \left| \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right) \\ [b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\lambda + \mu) < 1, \quad \operatorname{Re}(\lambda + \nu) + \frac{3}{2} > |\operatorname{Re} \mu|] \quad \text{ET II 73(6)}$$

$$4. \quad \int_0^\infty \sqrt{x} [I_{\nu-\frac{1}{2}}(ax) - \mathbf{L}_{\nu-\frac{1}{2}}(ax)] J_\nu(bx) dx = \sqrt{\frac{2}{\pi}} a^{\nu-\frac{1}{2}} b^{-\nu} \frac{1}{\sqrt{a^2 + b^2}} \\ [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 74(11)}$$

$$5. \quad \int_0^\infty x^{\mu-\nu+1} [I_\mu(ax) - \mathbf{L}_\mu(ax)] J_\nu(bx) dx = \frac{2^{\mu-\nu+1} a^{\mu-1} b^{\nu-2\mu-1}}{\sqrt{\pi} \Gamma(\nu - \mu + \frac{1}{2})} F\left(1, \frac{1}{2}; \nu - \mu + \frac{1}{2}; -\frac{b^2}{a^2}\right) \\ [-1 < 2 \operatorname{Re} \mu + 1 < \operatorname{Re} \nu + \frac{1}{2}, \quad \operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 74(13)}$$

$$6. \quad \int_0^\infty x^{\mu-\nu+1} [I_\mu(ax) - \mathbf{L}_{-\mu}(ax)] J_\nu(bx) dx = \frac{2^{\mu-\nu+1} a^{-\mu-1} b^{\nu-1}}{\Gamma(\frac{1}{2} - \mu) \Gamma(\frac{1}{2} + \nu)} F\left(1, \frac{1}{2} + \mu; \frac{1}{2} + \nu; -\frac{b^2}{a^2}\right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > -1, \quad b > 0] \quad \text{ET II 75(18)}$$

6.854

$$1. \quad \int_0^\infty x \mathbf{H}_{\frac{1}{2}\nu}(ax^2) K_\nu(bx) dx = \frac{\Gamma(\frac{1}{2}\nu + 1)}{2^{1-\frac{1}{2}\nu} a \pi} S_{-\frac{1}{2}\nu-1, \frac{1}{2}\nu} \left(\frac{b^2}{4a} \right) \\ [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 150(75)}$$

$$2. \quad \int_0^\infty x \mathbf{H}_{\frac{1}{2}\nu}(ax^2) J_\nu(bx) dx = -\frac{1}{2a} Y_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \quad [a > 0, \quad b > 0, \quad -2 < \operatorname{Re} \nu < \tfrac{3}{2}]$$

ET II 73(3)

6.855

$$1. \quad \int_0^\infty x^{2\nu+\frac{1}{2}} \left[I_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) - \mathbf{L}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) \right] J_\nu(bx) dx = 2^{\frac{3}{2}} \frac{a^{\nu+\frac{1}{2}}}{\sqrt{\pi} b^{\nu+1}} J_{2\nu+1}(\sqrt{2ab}) K_{2\nu+1}(\sqrt{2ab})$$

[Re $a > 0$, $b > 0$, $-1 < \operatorname{Re} \nu < \frac{1}{2}$]
ET II 76(22)

$$2. \quad \int_0^\infty \left[\mathbf{H}_{-\nu-1}\left(\frac{a}{x}\right) - Y_{-\nu-1}\left(\frac{a}{x}\right) \right] J_\nu(bx) \frac{dx}{x} = -\frac{4}{\pi \sqrt{ab}} \cos(\nu\pi) K_{-2\nu-1}(2\sqrt{ab})$$

[$|\arg a| < \pi$, $b > 0$, $|\operatorname{Re} \nu| < \frac{1}{2}$]
ET II 74(8)

$$3. \quad \int_0^\infty x^{2\nu+\frac{1}{2}} \left[\mathbf{H}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) - Y_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) \right] J_\nu(bx) dx$$

$$= -2^{5/2} \pi^{-3/2} a^{\nu+\frac{1}{2}} b^{-\nu-1} \sin(\nu\pi) K_{2\nu+1}(\sqrt{2abe}^{\frac{1}{4}\pi i}) K_{2\nu+1}(\sqrt{2abe}^{-\frac{1}{4}\pi i})$$

[$|\arg a| < \pi$, $b > 0$, $-1 < \operatorname{Re} \nu < -\frac{1}{6}$] ET II 74(9)

$$6.856 \quad \int_0^\infty x Y_\nu(a\sqrt{x}) K_\nu(a\sqrt{x}) \mathbf{H}_\nu(bx) dx = \frac{1}{2b^2} \exp\left(-\frac{a^2}{2b}\right)$$

[$b > 0$, $|\arg a| < \frac{\pi}{4}$, $\operatorname{Re} \nu > -\frac{3}{2}$]
ET II 169(32)

6.857

$$1. \quad \int_0^\infty x \exp\left(\frac{a^2 x^2}{8}\right) K_{\frac{1}{2}\nu}\left(\frac{a^2 x^2}{8}\right) \mathbf{H}_\nu(bx) dx$$

$$= \frac{2}{\sqrt{\pi}} a^{-\frac{\nu}{2}-1} b^{\frac{\nu}{2}-1} \cos\left(\frac{\nu\pi}{2}\right) \Gamma\left(-\frac{1}{2}\nu\right) \exp\left(\frac{b^2}{2a^2}\right) W_{k,m}\left(\frac{b^2}{a^2}\right)$$

$k = \frac{1}{4}\nu$, $m = \frac{1}{2} + \frac{1}{4}\nu$ [$|\arg a| < \frac{3}{4}\pi$, $b > 0$, $-\frac{3}{2} < \operatorname{Re} \nu < 0$] ET II 167(24)

$$2. \quad \int_0^\infty x^{\sigma-2} \exp\left(-\frac{1}{2}a^2 x^2\right) K_\mu\left(\frac{1}{2}a^2 x^2\right) \mathbf{H}_\nu(bx) dx$$

$$= \frac{\sqrt{\pi}}{2^{\nu+2}} a^{-\nu-\sigma} b^{\nu+1} \frac{\Gamma\left(\frac{\nu+\sigma}{2} + \mu\right) \Gamma\left(\frac{\nu+\sigma}{2} - \mu\right)}{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\nu + \frac{3}{2}\right) \Gamma\left(\frac{\nu+\sigma}{2}\right)}$$

$$\times {}_3F_3\left(1, \frac{\nu+\sigma}{2} + \mu, \frac{\nu+\sigma}{2} - \mu; \frac{3}{2}, \nu + \frac{3}{2}, \frac{\nu+\sigma}{2}; -\frac{b^2}{4a^2}\right)$$

[$b > 0$, $|\arg a| < \frac{\pi}{4}$, $\operatorname{Re}(\sigma + \nu) > 2|\operatorname{Re} \mu|$] ET II 167(23)

6.86 Lommel functions

6.861

$$1. \quad \int_0^\infty x^{\lambda-1} S_{\mu,\nu}(x) dx = \frac{\Gamma\left[\frac{1}{2}(1+\lambda+\mu)\right] \Gamma\left[\frac{1}{2}(1-\lambda-\mu)\right] \Gamma\left[\frac{1}{2}(1+\mu+\nu)\right] \Gamma\left[\frac{1}{2}(1+\mu-\nu)\right]}{2^{2-\lambda-\mu} \Gamma\left[\frac{1}{2}(\nu-\lambda)+1\right] \Gamma\left[1-\frac{1}{2}(\lambda+\nu)\right]} \\ [-\operatorname{Re} \mu < \operatorname{Re} \lambda + 1 < \tfrac{5}{2}] \quad \text{ET II 385(17)}$$

6.862

$$1. \quad \int_0^u x^{\lambda-\frac{1}{2}\mu-\frac{1}{2}} (u-x)^{\sigma-1} s_{\mu,\nu}(a\sqrt{x}) dx \\ = \Gamma(\sigma) \frac{a^{\mu+1} u^{\lambda+\sigma} \Gamma(\lambda+1)}{(\mu-\nu+1)(\mu+\nu+1)\Gamma(\lambda+\sigma+1)} \\ \times {}_2F_3\left(1, 1+\lambda; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}, \lambda+\sigma+1; -\frac{a^2 u}{4}\right) \\ [\operatorname{Re} \lambda > -1, \quad \operatorname{Re} \sigma > 0] \quad \text{ET II 199(92)}$$

$$2. \quad \int_u^\infty x^{\frac{1}{2}\nu} (x-u)^{\mu-1} s_{\lambda,\nu}(a\sqrt{x}) dx = \frac{\operatorname{B}\left[\mu, \frac{1}{2}(1-\lambda-\nu)-\mu\right] u^{\frac{1}{2}\mu+\frac{1}{2}\nu}}{a^\mu} S_{\lambda+\mu,\mu+\nu}(a\sqrt{u}) \\ [|\arg(a\sqrt{u})| < \pi, \quad 0 < 2\operatorname{Re} \mu < 1 - \operatorname{Re}(\lambda+\nu)] \quad \text{ET II 211(71)}$$

$$6.863 \quad \int_0^\infty \sqrt{x} e^{-\alpha x} s_{\mu,\frac{1}{4}}\left(\frac{x^2}{2}\right) dx = 2^{-2\mu-1} \sqrt{\alpha} \Gamma\left(2\mu + \frac{3}{2}\right) S_{-\mu-1,\frac{1}{4}}\left(\frac{\alpha^2}{2}\right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -\tfrac{3}{4}] \quad \text{ET I 209(38)}$$

$$6.864 \quad \int_0^\infty \exp[(\mu+1)x] s_{\mu,\nu}(a \sinh x) dx = 2^{\mu-2} \pi \operatorname{cosec}(\mu\pi) \Gamma(\varrho) \Gamma(\sigma) \\ \times \left[I_\varrho\left(\frac{a}{2}\right) I_\sigma\left(\frac{a}{2}\right) - I_{-\varrho}\left(\frac{a}{2}\right) I_{-\sigma}\left(\frac{a}{2}\right) \right] \\ 2\varrho = \mu + \nu + 1, \quad 2\sigma = \mu - \nu + 1 \quad [a > 0, \quad -2 < \operatorname{Re} \mu < 0] \quad \text{ET II 386(22)}$$

$$6.865 \quad \int_0^\infty \sqrt{\sinh x} \cosh(\nu x) S_{\mu,\frac{1}{2}}(a \cosh x) dx = \frac{\operatorname{B}\left(\frac{1}{4} - \frac{\mu+\nu}{2}, \frac{1}{4} - \frac{\mu-\nu}{2}\right)}{\sqrt{a} 2^{\mu+\frac{3}{2}}} S_{\mu+\frac{1}{2},\nu}(a) \\ [|\arg a| < \pi, \quad \operatorname{Re} \mu + |\operatorname{Re} \nu| < \tfrac{1}{2}] \\ \text{ET II 388(31)}$$

6.866

$$1. \quad \int_0^\infty x^{-\mu-1} \cos(ax) s_{\mu,\nu}(x) dx \\ = 0 \quad [a > 1] \\ = 2^{\mu-\frac{1}{2}} \sqrt{\pi} \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right) (1-a^2)^{\frac{1}{2}\mu+\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a) \quad [0 < a < 1] \\ \text{ET II 386(18)}$$

$$2. \quad \int_0^\infty x^{-\mu} \sin(ax) S_{\mu,\nu}(x) dx = 2^{-\mu-\frac{1}{2}} \sqrt{\pi} \Gamma\left(1 - \frac{\mu+\nu}{2}\right) \Gamma\left(1 - \frac{\mu-\nu}{2}\right) (a^2-1)^{\frac{1}{2}\mu-\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a) \\ [a > 1, \quad \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|] \\ \text{ET II 387(23)}$$

6.867

$$\begin{aligned}
1. \quad \int_0^{\pi/2} \cos(2\mu x) S_{2\mu-1, 2\nu}(a \cos x) dx \\
= \frac{\pi 2^{2\mu-3} a^{2\mu} \operatorname{cosec}(2\nu\pi)}{\Gamma(1-\mu-\nu)\Gamma(1-\mu+\nu)} \left[J_{\mu+\nu}\left(\frac{a}{2}\right) Y_{\mu-\nu}\left(\frac{a}{2}\right) - J_{\mu-\nu}\left(\frac{a}{2}\right) Y_{\mu+\nu}\left(\frac{a}{2}\right) \right] \\
[\operatorname{Re} \mu > -2, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 388(29)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^{\pi/2} \cos[(\mu+1)x] s_{\mu, \nu}(a \cos x) dx = 2^{\mu-2} \pi \Gamma(\varrho) \Gamma(\sigma) J_{\varrho}\left(\frac{a}{2}\right) J_{\sigma}\left(\frac{a}{2}\right) \\
2\varrho = \mu + \nu + 1, \quad 2\sigma = \mu - \nu + 1 \quad [\operatorname{Re} \mu > -2] \quad \text{ET II 386(21)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{6.868} \quad \int_0^{\pi/2} \frac{\cos(2\mu x)}{\cos x} S_{2\mu, 2\nu}(a \sec x) dx = \frac{\pi 2^{2\mu-1}}{a} W_{\mu, \nu}(ae^{i\frac{\pi}{2}}) W_{\mu, \nu}(ae^{-i\frac{\pi}{2}}) \\
[|\arg a| < \pi, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 388(30)}
\end{aligned}$$

6.869

$$\begin{aligned}
1. \quad \int_0^{\infty} x^{1-\mu-\nu} J_{\nu}(ax) S_{\mu, -\mu-2\nu}(x) dx = \frac{\sqrt{\pi} a^{\nu-1} \Gamma(1-\mu-\nu)}{2^{\mu+2\nu} \Gamma(\nu + \frac{1}{2})} (a^2 - 1)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a) \\
[a > 1, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re}(\mu + \nu) < 1] \\
\text{ET II 388(28)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^{\infty} x^{-\mu} J_{\nu}(ax) s_{\nu+\mu, -\nu+\mu+1}(x) dx \\
= 2^{\nu-1} \Gamma(\nu) a^{-\nu} (1-a^2)^{\mu} \quad [0 < a < 1, \quad \operatorname{Re} \mu > -1, \quad -1e < \operatorname{Re} \nu < \frac{3}{2}] \\
= 0 \quad [1 < a, \quad \operatorname{Re} \mu > -1, \quad -1 < \operatorname{Re} \nu < \frac{3}{2}] \\
\text{ET II 388(28)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^{\infty} x K_{\nu}(bx) s_{\mu, \frac{1}{2}\nu}(ax^2) dx = \frac{1}{4a} \Gamma\left(\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(\mu - \frac{1}{2}\nu + 1\right) S_{-\mu-1, \frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \\
[\operatorname{Re} \mu > \frac{1}{2}|\operatorname{Re} \nu| - 2, \quad a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 151(78)}
\end{aligned}$$

6.87 Thomson functions**6.871**

$$1. \quad \int_0^{\infty} e^{-\beta x} \operatorname{ber} x dx = \frac{(\sqrt{\beta^4 + 1} + \beta^2)^{1/2}}{\sqrt{2}(\beta^4 + 1)} \quad \text{ME 40}$$

$$2. \quad \int_0^{\infty} e^{-\beta x} \operatorname{bei} x dx = \frac{(\sqrt{\beta^4 + 1} - \beta^2)^{1/2}}{\sqrt{2}(\beta^4 + 1)} \quad \text{ME 40}$$

6.872

$$1. \quad \int_0^\infty e^{-\beta x} \operatorname{ber}_\nu(2\sqrt{x}) \, dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \left[J_{\frac{1}{2}(\nu-1)} \left(\frac{1}{2\beta} \right) \cos \left(\frac{1}{2\beta} + \frac{3\nu\pi}{4} \right) \right. \\ \left. - J_{\frac{1}{2}(\nu+1)} \left(\frac{1}{2\beta} \right) \cos \left(\frac{1}{2\beta} + \frac{3\nu+6}{4}\pi \right) \right]$$

MI 49

$$2. \quad \int_0^\infty e^{-\beta x} \operatorname{bei}_\nu(2\sqrt{x}) \, dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \left[J_{\frac{1}{2}(\nu-1)} \left(\frac{1}{2\beta} \right) \sin \left(\frac{1}{2\beta} + \frac{3\nu\pi}{4} \right) \right. \\ \left. - J_{\frac{1}{2}(\nu+1)} \left(\frac{1}{2\beta} \right) \sin \left(\frac{1}{2\beta} + \frac{3\nu+6}{4}\pi \right) \right]$$

MI 49

$$3. \quad \int_0^\infty e^{-\beta x} \operatorname{ber}(2\sqrt{x}) \, dx = \frac{1}{\beta} \cos \frac{1}{\beta}$$

ME 40

$$4. \quad \int_0^\infty e^{-\beta x} \operatorname{bei}(2\sqrt{x}) \, dx = \frac{1}{\beta} \sin \frac{1}{\beta}$$

ME 40

$$5. \quad \int_0^\infty e^{-\beta x} \operatorname{ker}(2\sqrt{x}) \, dx = -\frac{1}{2\beta} \left[\cos \frac{1}{\beta} \operatorname{ci} \frac{1}{\beta} + \sin \frac{1}{\beta} \operatorname{si} \frac{1}{\beta} \right]$$

MI 50

$$6. \quad \int_0^\infty e^{-\beta x} \operatorname{kei}(2\sqrt{x}) \, dx = -\frac{1}{2\beta} \left[\sin \frac{1}{\beta} \operatorname{ci} \frac{1}{\beta} - \cos \frac{1}{\beta} \operatorname{si} \frac{1}{\beta} \right]$$

MI 50

$$7. \quad \int_0^\infty e^{-\beta x} \operatorname{ber}_\nu(2\sqrt{x}) \operatorname{bei}_\nu(2\sqrt{x}) \, dx = \frac{1}{2\beta} J_\nu \left(\frac{2}{\beta} \right) \sin \left(\frac{2}{\beta} + \frac{3\nu\pi}{2} \right) \\ [\operatorname{Re} \nu > -1]$$

MI 49

$$6.873 \quad \int_0^\infty [\operatorname{ber}_\nu^2(2\sqrt{x}) + \operatorname{bei}_\nu^2(2\sqrt{x})] e^{-\beta x} \, dx = \frac{1}{\beta} I_\nu \left(\frac{2}{\beta} \right) \\ [\operatorname{Re} \nu > -1]$$

ME 40

6.874

$$1. \quad \int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{ber}_{2\nu}(2\sqrt{2x}) \, dx = \sqrt{\frac{\pi}{\beta}} J_\nu \left(\frac{1}{\beta} \right) \cos \left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2} \right) \\ [\operatorname{Re} \nu > -\frac{1}{2}]$$

MI 49

$$2. \quad \int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{bei}_{2\nu}(2\sqrt{2x}) \, dx = \sqrt{\frac{\pi}{\beta}} J_\nu \left(\frac{1}{\beta} \right) \sin \left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2} \right) \\ [\operatorname{Re} \nu > -\frac{1}{2}]$$

MI 49

$$3. \quad \int_0^\infty x^{\frac{\nu}{2}} \operatorname{ber}_\nu(\sqrt{x}) e^{-\beta x} \, dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \cos \left(\frac{1}{4\beta} + \frac{3\nu\pi}{4} \right) \quad [\operatorname{Re} \nu > -1]$$

ME 40

$$4. \quad \int_0^\infty x^{\frac{\nu}{2}} \operatorname{bei}_\nu(\sqrt{x}) e^{-\beta x} \, dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \sin \left(\frac{1}{4\beta} + \frac{3\nu\pi}{4} \right) \quad [\operatorname{Re} \nu > -1]$$

ME 40

6.875

1. $\int_0^\infty e^{-\beta x} \left[\ker(2\sqrt{x}) - \frac{1}{2} \ln x \operatorname{ber}(2\sqrt{x}) \right] dx = \frac{1}{\beta} \left[\ln \beta \cos \frac{1}{\beta} + \frac{\pi}{4} \sin \frac{1}{\beta} \right]$ MI 50
2. $\int_0^\infty e^{-\beta x} \left[\operatorname{kei}(2\sqrt{x}) - \frac{1}{2} \ln x \operatorname{bei}(2\sqrt{x}) \right] dx = \frac{1}{\beta} \left[\ln \beta \sin \frac{1}{\beta} - \frac{\pi}{4} \cos \frac{1}{\beta} \right]$ MI 50

6.876

1. $\int_0^\infty x \operatorname{kei} x J_1(ax) dx = -\frac{1}{2a} \arctan a^2$ $[a > 0]$ ET II 21(32)
2. $\int_0^\infty x \operatorname{ker} x J_1(ax) dx = \frac{1}{2a} \ln \sqrt{1+a^4}$ $[a > 0]$ ET II 21(33)

6.9 Mathieu Functions

Notation: $k^2 = q$. For definition of the coefficients $A_p^{(m)}$ and $B_p^{(m)}$, see section 8.6.

6.91 Mathieu functions

6.911

1. $\int_0^{2\pi} \operatorname{ce}_m(z, q) \operatorname{ce}_p(z, q) dz = 0$ $[m \neq p]$ MA
2. $\int_0^{2\pi} [\operatorname{ce}_{2n}(z, q)]^2 dz = 2\pi \left[A_0^{(2n)} \right]^2 + \pi \sum_{r=1}^\infty \left[A_{2r}^{(2n)} \right]^2 = \pi$ MA
3. $\int_0^{2\pi} [\operatorname{ce}_{2n+1}(z, q)]^2 dz = \pi \sum_{r=0}^\infty \left[A_{2r+1}^{(2n+1)} \right]^2 = \pi$ MA
4. $\int_0^{2\pi} \operatorname{se}_m(z, q) \operatorname{se}_p(z, q) dz = 0$ $[m \neq p]$ MA
5. $\int_0^{2\pi} [\operatorname{se}_{2n+1}(z, q)]^2 dz = \pi \sum_{r=0}^\infty \left[B_{2r+1}^{(2n+1)} \right]^2 = \pi$ MA
6. $\int_0^{2\pi} [\operatorname{se}_{2n+2}(z, q)]^2 dz = \pi \sum_{r=0}^\infty \left[B_{2r+2}^{(2n+2)} \right]^2 = \pi$ MA
7. $\int_0^{2\pi} \operatorname{se}_m(z, q) \operatorname{ce}_p(z, q) dz = 0$ $[m = 1, 2, \dots; \quad p = 1, 2, \dots]$ MA

6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions

6.921

1. $\int_0^\pi \cosh(2k \cos u \sinh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(\frac{\pi}{2}, q)} (-1)^n \operatorname{Ce}_{2n}(z, -q)$ $[q > 0]$ MA

$$2. \quad \int_0^\pi \cosh(2k \sin u \cosh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q)} (-1)^n \operatorname{Ce}_{2n}(z, -q) \\ [q > 0] \quad \text{MA}$$

$$3. \quad \int_0^\pi \sinh(2k \sin u \cosh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}_{2n+1}'(0, q)} (-1)^n \operatorname{Ce}_{2n+1}(z, -q) \\ [q > 0] \quad \text{MA}$$

$$4. \quad \int_0^\pi \sinh(2k \cos u \sinh z) \operatorname{ce}_{2n+1}(u, q) du = \frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}_{2n+1}'(\frac{\pi}{2}, q)} (-1)^{n+1} \operatorname{Se}_{2n+1}(z, -q) \\ [q > 0] \quad \text{MA}$$

$$5. \quad \int_0^\pi \sinh(2k \sin u \sin z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}_{2n+1}'(0, q)} \operatorname{se}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

6.922

$$1. \quad \int_0^\pi \cos u \cosh z \cos(2k \sin u \sinh z) \operatorname{ce}_{2n+1}(u, q) du = \frac{\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}(0, q)} \operatorname{Ce}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

$$2. \quad \int_0^\pi \sin u \sinh z \cos(2k \cos u \cosh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}(\frac{\pi}{2}, q)} \operatorname{Se}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

$$3. \quad \int_0^\pi \sin u \sinh z \sin(2k \cos u \cosh z) \operatorname{se}_{2n+2}(u, q) du = -\frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}_{2n+2}'(\frac{\pi}{2}, q)} \operatorname{Se}_{2n+2}(z, q) \\ [q > 0] \quad \text{MA}$$

$$4. \quad \int_0^\pi \cos u \cosh z \sin(2k \sin u \sinh z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}_{2n+2}'(0, q)} \operatorname{Se}_{2n+2}(z, q) \\ [q > 0] \quad \text{MA}$$

$$5. \quad \int_0^\pi \sin u \cosh z \cosh(2k \cos u \sinh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}(\frac{\pi}{2}, q)} (-1)^n \operatorname{Ce}_{2n+1}(z, -q) \\ [q > 0] \quad \text{MA}$$

$$6. \quad \int_0^\pi \cos u \sinh z \cosh(2k \sin u \cosh z) \operatorname{ce}_{2n+1}(u, q) du = \frac{\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}(0, q)} (-1)^n \operatorname{Se}_{2n+1}(z, -q) \\ [q > 0] \quad \text{MA}$$

$$7. \quad \int_0^\pi \sin u \cosh z \sinh(2k \cos u \sinh z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}_{2n+2}'(\frac{\pi}{2}, q)} (-1)^{n+1} \operatorname{Se}_{2n+2}(z, -q) \\ [q > 0] \quad \text{MA}$$

$$8. \quad \int_0^\pi \cos u \sinh z \sinh (2k \sin u \cosh z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}_{2n+2}'(0, q)} (-1)^n \operatorname{Se}_{2n+2}(z, -q) \\ [q > 0] \quad \text{MA}$$

6.923

$$1. \quad \int_0^\infty \sin (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+1}(u, q) du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

$$2. \quad \int_0^\infty \cos (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+1}(u, q) du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Gey}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

$$3. \quad \int_0^\infty \sin (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+2}(u, q) du = -\frac{k\pi B_2^{(2n+2)}}{4 \operatorname{se}_{2n+2}'(\frac{1}{2}\pi, q)} \operatorname{Gey}_{2n+2}(z, q) \\ [q > 0] \quad \text{MA}$$

$$4. \quad \int_0^\infty \cos (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+2}(u, q) du = -\frac{k\pi B_2^{(2n+2)}}{4 \operatorname{se}_{2n+2}(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+2}(z, q) \\ [q > 0] \quad \text{MA}$$

$$5. \quad \int_0^\infty \sin (2k \cosh z \cosh u) \operatorname{Ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n}(z, q) \\ [q > 0] \quad \text{MA}$$

$$6. \quad \int_0^\infty \cos (2k \cosh z \cosh u) \operatorname{Ce}_{2n}(u, q) du = -\frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Fey}_{2n}(z, q) \\ [q > 0] \quad \text{MA}$$

$$7. \quad \int_0^\infty \sin (2k \cosh z \cosh u) \operatorname{Ce}_{2n+1}(u, q) du = \frac{k\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}'(\frac{1}{2}\pi, q)} \operatorname{Fey}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

$$8. \quad \int_0^\infty \cos (2k \cosh z \cosh u) \operatorname{Ce}_{2n+1}(u, q) du = \frac{k\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}'(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

6.924

$$1. \quad \int_0^\pi \cos (2k \cos u \cos z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{ce}_{2n}(z, q) \\ [q > 0] \quad \text{MA}$$

$$2. \quad \int_0^\pi \sin (2k \cos u \cos z) \operatorname{ce}_{2n+1}(u, q) du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}_{2n+1}'(\frac{1}{2}\pi, q)} \operatorname{ce}_{2n+1}(z, q) \\ [q > 0] \quad \text{MA}$$

$$3. \quad \int_0^\pi \cos(2k \cos u \cosh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n}(z, q) \quad [q > 0] \quad \text{MA}$$

$$4. \quad \int_0^\pi \cos(2k \sin u \sinh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q)} \operatorname{Ce}_{2n}(z, q) \quad [q > 0] \quad \text{MA}$$

$$5. \quad \int_0^\pi \sin(2k \cos u \cosh z) \operatorname{ce}_{2n+1}(u, q) du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}_{2n+1}'(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n+1}(z, q) \quad [q > 0] \quad \text{MA}$$

$$6. \quad \int_0^\pi \sin(2k \sin u \sinh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}_{2n+1}''(0, q)} \operatorname{Se}_{2n+1}(z, q) \quad [q > 0] \quad \text{MA}$$

6.925 Notation: $z_1 = 2k\sqrt{\cosh^2 \xi - \sin^2 \eta}$, and $\tan \alpha = \tanh \xi \tan \eta$

$$1. \quad \int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n}(\theta, q) d\theta = 0. \quad \text{MA}$$

$$2. \quad \int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n}(\theta, q) d\theta = \frac{2\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q) \operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n}(\xi, q) \operatorname{ce}_{2n}(\eta, q) \quad \text{MA}$$

$$3. \quad \int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n+1}(\theta, q) d\theta = -\frac{2\pi k A_1^{(2n+1)}}{\operatorname{ce}_{2n+1}(0, q) \operatorname{ce}_{2n+1}'(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n+1}(\xi, q) \operatorname{ce}_{2n+1}(\eta, q) \quad \text{MA}$$

$$4. \quad \int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n+1}(\theta, q) d\theta = 0 \quad \text{MA}$$

$$5. \quad \int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+1}(\theta, q) d\theta = \frac{2\pi k B_1^{(2n+1)}}{\operatorname{se}_{2n+1}(0, q) \operatorname{se}_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+1}(\xi, q) \operatorname{se}_{2n+1}(\eta, q) \quad \text{MA}$$

$$6. \quad \int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+1}(\theta, q) d\theta = 0 \quad \text{MA}$$

$$7. \quad \int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+2}(\theta, q) d\theta = 0 \quad \text{MA}$$

$$8. \quad \int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+2}(\theta, q) d\theta = \frac{2\pi k^2 B_2^{(2n+2)}}{\operatorname{se}_{2n+2}'(0, q) \operatorname{se}_{2n+2}'(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+2}(\xi, q) \operatorname{se}_{2n+2}(\eta, q) \quad \text{MA}$$

6.926 $\int_0^\pi \sin u \sin z \sin(2k \cos u \cos z) \operatorname{se}_{2n+2}(u, q) du = -\frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}_{2n+2}'(\frac{\pi}{2}, q)} \operatorname{se}_{2n+2}(z, q) \quad [q > 0] \quad \text{MA}$

6.93 Combinations of Mathieu and Bessel functions

6.931

1.
$$\int_0^\pi J_0 \left\{ k [2 (\cos 2u + \cos 2z)]^{1/2} \right\} \text{ce}_{2n}(u, q) du = \frac{\pi [A_0^{(2n)}]^2}{\text{ce}_{2n}(0, q) \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \text{ce}_{2n}(z, q) \quad \text{MA}$$
2.
$$\int_0^{2\pi} Y_0 \left\{ k [2 (\cos 2u + \cosh 2z)]^{1/2} \right\} \text{ce}_{2n}(u, q) du = \frac{2\pi [A_0^{(2n)}]^2}{\text{ce}_{2n}(0, q) \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \text{Fey}_{2n}(z, q) \quad \text{MA}$$

6.94 Relationships between eigenfunctions of the Helmholtz equation in different coordinate systems

Notation: Particular solutions of the Helmholtz equation in three-dimensional infinite space

$$\nabla^2 \Psi + k^2 \Psi = 0$$

in Cartesian (x, y, z) , spherical (r, θ, ϕ) , and cylindrical (ρ, z, ϕ) coordinates are

$$\Psi_{k_x k_y k_z}(x, y, z) \propto e^{i(k_x x + k_y y + k_z z)} \quad \text{with} \quad k^2 = k_x^2 + k_y^2 + k_z^2$$

$$\Psi_{lm}(r, \theta, \phi) \propto e^{im\phi} \sqrt{\frac{k}{r}} Z_{l+1/2}(kr) P_l^m(\cos \theta)$$

$$\Psi_{mk_z}(\rho, z, \phi) \propto e^{i(m\phi + k_z z)} Z_{l+1/2}\left(\rho \sqrt{k^2 - k_z^2}\right)$$

with $P_l^m(\cos \theta)$ the associated Legendre function, Z is any Bessel function, $m = 0, 1, \dots, l$; $l \in \mathbb{N}$, $r^2 = \rho^2 + z^2$, $\rho = r \sin \theta$, $z = r \cos \theta$, $\phi = \arccot(x/y)$, and $k_t^2 = k^2 - k_z^2$.

6.941

1.
$$\int_{-k}^k e^{i\rho z} J_m\left(\rho \sqrt{k^2 - \rho^2}\right) P_l^m\left(\frac{p}{k}\right) dp = i^{l-m} \sqrt{\frac{2\pi k}{r}} J_{l+1/2}(kr) P_l^m\left(\frac{z}{r}\right) \quad [\rho > 0, \quad l \geq m \geq 0]$$
2.
$$\int_{-\infty}^{\infty} e^{-i\rho z} J_{l+1/2}(kr) P_l^m\left(\frac{z}{r}\right) dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} J_m\left(\rho \sqrt{k^2 - \rho^2}\right) P_l^m\left(\frac{\rho}{k}\right) \quad [\rho > 0, \quad l \geq m \geq 0]$$
3.
$$\begin{aligned} \int_0^\infty J_m(\rho k_t) \cos\left[k_x x + m \arcsin\left(\frac{x}{\rho}\right)\right] dx \\ = \frac{(-1)^m}{\sqrt{k_t^2 - k_x^2}} \cos\left[y \sqrt{k_t^2 - k_x^2} + m \arccos\left(\frac{k_x}{k_t}\right)\right] \quad [k_x^2 < k_t^2] \\ = 0 \quad [k_x^2 > k_t^2] \end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty Y_m(\rho k_t) \cos \left[k_x x + m \arcsin \left(\frac{x}{\rho} \right) \right] dx \\
= \frac{(-1)^m}{\sqrt{k_t^2 - k_x^2}} \sin \left[y \sqrt{k_t^2 - k_x^2} + m \arccos \left(\frac{k_x}{k_t} \right) \right] \quad [k_x^2 < k_t^2] \\
= \frac{(-1)^m}{\sqrt{k_x^2 - k_t^2}} \exp \left[-y \sqrt{k_x^2 - k_t^2} - m \operatorname{sign}(k_x) \operatorname{arccosh} \left(\frac{|k_x|}{k_t} \right) \right] \quad [k_x^2 > k_t^2]
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_{-\infty}^\infty H_{l+1/2}^{(j)}(kr) P_l^m \left(\frac{z}{r} \right) e^{-ik_z z} dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} H_m^{(j)} \left(\rho \sqrt{k^2 - k_z^2} \right) P_l^m \left(\frac{k_z}{k} \right) \\
[\rho > 0]
\end{aligned}$$

The result is true for $j = 1$ if $\pi > \arg \sqrt{k^2 - k_z^2} \geq 0$, for $j = 2$ if $-\pi < \arg \sqrt{k^2 - k_z^2} \leq 0$.

$$6. \quad \int_{-\infty}^\infty H_m^{(j)} \left(\rho \sqrt{k^2 - k_z^2} \right) P_l^m \left(\frac{k_z}{k} \right) e^{ik_z z} dk_z = i^{l-m} \sqrt{\frac{2\pi k}{r}} H_{l+1/2}^{(j)}(kr) P_l^m \left(\frac{z}{r} \right)$$

The result is true for $j = 1$ if $\pi > \arg \sqrt{k^2 - k_z^2} \geq 0$, for $j = 2$ if $-\pi < \arg \sqrt{k^2 - k_z^2} \leq 0$.

$$\begin{aligned}
7. \quad \int_{-\infty}^\infty J_{l+1/2}(kr) P_l^m \left(\frac{z}{r} \right) e^{-ik_z z} dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} J_m \left(\rho \sqrt{k^2 - k_z^2} \right) P_l^m \left(\frac{k_z}{k} \right) \quad [k_z^2 < k^2] \\
= 0 \quad [k_z^2 > k^2]
\end{aligned}$$

$$8. \quad \int_{-k}^k J_m \left(\rho \sqrt{k^2 - k_z^2} \right) P_l^m \left(\frac{k_z}{k} \right) e^{ik_z z} dk_z = i^{l-m} \sqrt{\frac{2\pi k}{r}} J_{l+1/2}(kr) P_l^m \left(\frac{z}{r} \right)$$

$$\begin{aligned}
9. \quad \int_{-\infty}^\infty Y_{l+1/2}(kr) P_l^m \left(\frac{z}{r} \right) e^{-ik_z z} dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} Y_m \left(\rho \sqrt{k^2 - k_z^2} \right) P_l^m \left(\frac{k_z}{k} \right) \quad [k_z^2 < k^2] \\
= -2i^{m-l} \sqrt{\frac{2r}{k\pi}} K_m \left(\rho \sqrt{k_z^2 - k^2} \right) P_l^m \left(\frac{k_z}{k} \right) \quad [k_z^2 > k^2]
\end{aligned}$$

$$\begin{aligned}
10. \quad i^{l-m} \int_{-k}^k Y_m \left(\rho \sqrt{k^2 - k_z^2} \right) P_l^m \left(\frac{k_z}{k} \right) e^{ik_z z} dk_z \\
- \frac{4}{\pi} \int_k^\infty \cos [k_z z + \tfrac{1}{2}\pi(m-l)] P_l^m \left(\frac{k_z}{k} \right) K_m \left(\rho \sqrt{k_z^2 - k^2} \right) e^{ik_z z} dk_z \\
= \sqrt{\frac{2\pi k}{r}} Y_{l+1/2}(kr) P_l^m \left(\frac{z}{r} \right)
\end{aligned}$$

7.1–7.2 Associated Legendre Functions

7.11 Associated Legendre functions

$$\mathbf{7.111} \quad \int_{\cos \varphi}^1 P_{\nu}(x) dx = \sin \varphi P_{\nu}^{-1}(\cos \varphi) \quad \text{MO 90}$$

7.112

$$\begin{aligned} 1. \quad \int_{-1}^1 P_n^m(x) P_k^m(x) dx &= 0 & [n \neq k] \\ &= \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} & [n = k] \end{aligned}$$

SM III 185, WH

$$2. \quad \int_{-1}^1 Q_n^m(x) P_k^m(x) dx = (-1)^m \frac{1 - (-1)^{n+k} (n+m)!}{(k-n)(k+n+1)(n-m)!} \quad \text{EH I 171(18)}$$

$$\begin{aligned} 3. \quad \int_{-1}^1 P_{\nu}(x) P_{\sigma}(x) dx &= \frac{2\pi \sin \pi(\sigma - \nu) + 4 \sin(\pi\nu) \sin(\pi\sigma) [\psi(\nu+1) - \psi(\sigma+1)]}{\pi^2(\sigma - \nu)(\sigma + \nu + 1)} \quad [\sigma + \nu + 1 \neq 0] \quad \text{EH I 170(7)} \\ &= \frac{\pi^2 - 2(\sin \pi\nu)^2 \psi'(\nu+1)}{\pi^2 \left(\nu + \frac{1}{2} \right)} \quad [\sigma = \nu] \quad \text{EH I 170(9)a} \end{aligned}$$

$$\begin{aligned} 4. \quad \int_{-1}^1 Q_{\nu}(x) Q_{\sigma}(x) dx &= \frac{[\psi(\nu+1) - \psi(\sigma+1)] [1 + \cos(\pi\sigma) \cos(\nu\pi)] - \frac{\pi}{2} \sin \pi(\nu - \sigma)}{(\sigma - \nu)(\sigma + \nu + 1)} \quad [\sigma + \nu + 1 \neq 0; \quad \nu, \quad \sigma \neq -1, -2, -3, \dots] \quad \text{EH I 170(11)} \\ &= \frac{\frac{1}{2}\pi^2 - \psi'(\nu+1) [1 + (\cos \nu\pi)^2]}{2\nu + 1} \quad [\nu = \sigma, \quad \nu \neq -1, -2, -3, \dots] \quad \text{EH I 170(12)} \end{aligned}$$

$$\begin{aligned} 5. \quad \int_{-1}^1 P_{\nu}(x) Q_{\sigma}(x) dx &= \frac{1 - \cos \pi(\sigma - \nu) - 2\pi^{-1} \sin(\pi\nu) \cos(\pi\sigma) [\psi(\nu+1) - \psi(\sigma+1)]}{(\nu - \sigma)(\nu + \sigma + 1)} \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \sigma > 0, \quad \sigma \neq \nu] \quad \text{EH I 170(13)} \\ &= -\frac{\sin(2\nu\pi) \psi'(\nu+1)}{\pi(2\nu+1)} \quad [\operatorname{Re} \nu > 0, \quad \sigma = \nu] \quad \text{EH I 171(14)} \end{aligned}$$

$$\mathbf{7.113} \quad \text{Notation:} \quad A = \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\sigma}{2}\right) \Gamma\left(1 + \frac{\nu}{2}\right)}$$

$$1. \quad \int_0^1 P_{\nu}(x) P_{\sigma}(x) dx = \frac{A \sin \frac{\pi\sigma}{2} \cos \frac{\pi\nu}{2} - A^{-1} \sin \frac{\pi\nu}{2} \cos \frac{\pi\sigma}{2}}{\frac{1}{2}\pi(\sigma - \nu)(\sigma + \nu + 1)} \quad \text{EH I 171(15)}$$

$$2. \quad \int_0^1 Q_\nu(x) Q_\sigma(x) dx = \frac{\psi(\nu+1) - \psi(\sigma+1) - \frac{\pi}{2} \left[(A - A^{-1}) \sin \frac{\pi(\sigma+\nu)}{2} (A + A^{-1}) \sin \frac{\pi(\sigma-\nu)}{2} \right]}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \sigma > 0] \quad \text{EH I 171(16)}$$

$$3. \quad \int_0^1 P_\nu(x) Q_\sigma(x) dx = \frac{A^{-1} \cos \frac{\pi(\nu-\sigma)}{2} - 1}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \sigma > 0] \quad \text{EH I 171(17)}$$

7.114

$$1. \quad \int_1^\infty P_\nu(x) Q_\sigma(x) dx = \frac{1}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re}(\sigma-\nu) > 0, \quad \operatorname{Re}(\sigma+\nu) > -1] \quad \text{ET II 324(19)}$$

$$2. \quad \int_1^\infty Q_\nu(x) Q_\sigma(x) dx = \frac{\psi(\sigma+1) - \psi(\nu+1)}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re}(\nu+\sigma) > -1; \quad \sigma, \nu \neq -1, -2, -3, \dots] \quad \text{EH I 170(5)}$$

$$3. \quad \int_1^\infty [Q_\nu(x)]^2 dx = \frac{\psi'(\nu+1)}{2\nu+1} \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{EH I 170(6)}$$

$$7.115 \quad \int_1^\infty Q_\nu(x) dx = \frac{1}{\nu(\nu+1)} \quad [\operatorname{Re} \nu > 0] \quad \text{ET II 324(18)}$$

7.12–7.13 Combinations of associated Legendre functions and powers

$$7.121 \quad \int_{\cos \varphi}^1 x P_\nu(x) dx = \frac{-\sin \varphi}{(\nu-1)(\nu+2)} [\sin \varphi P_\nu(\cos \varphi) + \cos \varphi P_\nu^1(\cos \varphi)] \quad \text{MO 90}$$

7.122

$$1. \quad \int_0^1 \frac{[P_n^m(x)]^2}{1-x^2} dx = \frac{1}{2m} \frac{(n+m)!}{(n-m)!} \quad [0 < m \leq n] \quad \text{MO 74}$$

$$2. \quad \int_0^1 [P_\nu^\mu(x)]^2 \frac{dx}{1-x^2} = -\frac{\Gamma(1+\mu+\nu)}{2\mu \Gamma(1-\mu+\nu)} \quad [\operatorname{Re} \mu < 0, \quad \nu + \mu \text{ is a positive integer}] \quad \text{EH I 172(26)}$$

$$3. \quad \int_0^1 [P_\nu^{n-\nu}(x)]^2 \frac{dx}{1-x^2} = -\frac{n!}{2(n-\nu) \Gamma(1-n+2\nu)} \quad [n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > n] \quad \text{ET II 315(9)}$$

$$7.123 \quad \int_{-1}^1 P_n^m(x) P_n^k(x) \frac{dx}{1-x^2} = 0 \quad [0 \leq m \leq n, \quad 0 \leq k \leq n; \quad m \neq k] \quad \text{MO 74}$$

$$7.124 \quad \int_{-1}^1 x^k (z-x)^{-1} (1-x^2)^{\frac{1}{2}m} P_n^m(x) dx = (-2)^m (z^2-1)^{\frac{1}{2}m} Q_n^m(z) \cdot z^k$$

[$m \leq n; k = 0, 1, \dots, n-m;$
 z is in the complex plane with a cut along the interval $(-1, 1)$ on the real axis]
 ET II 279(26)

$$\begin{aligned}
7.125 \quad \int_{-1}^1 (1-x^2)^{\frac{1}{2}m} P_k^m(x) P_l^m(x) P_n^m(x) dx &= (-1)^m \pi^{-3/2} \frac{(k+m)!(l+m)!(n+m)!(s-m)!}{(k-m)!(l-m)!(n-m)!(s-k)!} \\
&\times \frac{\Gamma(m+\frac{1}{2}) \Gamma(t-k+\frac{1}{2}) \Gamma(t-l+\frac{1}{2}) \Gamma(t-n+\frac{1}{2})}{(s-l)!(s-n)!\Gamma(s+\frac{3}{2})} \\
&\quad [2s = k+l+n+m \text{ and } 2t = k+l-n-m \text{ are both even} \\
&\quad \quad l \geq m, \quad m \leq k-l-m \leq n \leq k+l+m] \\
&\quad \text{ET II 280(32)}
\end{aligned}$$

7.126

$$1. \quad \int_0^1 P_\nu(x) x^\sigma dx = \frac{\sqrt{\pi} 2^{-\sigma-1} \Gamma(1+\sigma)}{\Gamma(1+\frac{1}{2}\sigma-\frac{1}{2}\nu) \Gamma(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2})} \quad [\operatorname{Re} \sigma > -1] \quad \text{EH I 171(23)}$$

$$\begin{aligned}
2. \quad \int_0^1 x^\sigma P_\nu^m(x) dx &= \frac{(-1)^m \pi^{1/2} 2^{-2m-1} \Gamma(\frac{1+\sigma}{2}) \Gamma(1+m+\nu)}{\Gamma(\frac{1}{2}+\frac{1}{2}m) \Gamma(\frac{3}{2}+\frac{\sigma}{2}+\frac{m}{2}) \Gamma(1-m+\nu)} \\
&\times {}_3F_2\left(\frac{m+\nu+1}{2}, \frac{m-\nu}{2}, \frac{m}{2}+1; m+1, \frac{3+\sigma+m}{2}; 1\right) \\
&\quad [\operatorname{Re} \sigma > -1; \quad m = 0, 1, 2, \dots] \quad \text{ET II 313(2)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^1 x^\sigma P_\nu^\mu(x) dx &= \frac{\pi^{1/2} 2^{2\mu-1} \Gamma(\frac{1+\sigma}{2})}{\Gamma(\frac{1-\mu}{2}) \Gamma(\frac{3+\sigma-\mu}{2})} {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1-\frac{\mu}{2}; 1-\mu, \frac{3+\sigma-\mu}{2}; 1\right) \\
&\quad [\operatorname{Re} \sigma > -1, \quad \operatorname{Re} \mu < 2] \quad \text{ET II 313(3)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_1^\infty x^{\mu-1} Q_\nu(ax) dx &= e^{\mu\pi i} \Gamma(\mu) a^{-\mu} (a^2-1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(a) \\
&\quad [|\arg(a-1)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(\nu-\mu) > -1] \quad \text{ET II 325(26)}
\end{aligned}$$

$$7.127 \quad \int_{-1}^1 (1+x)^\sigma P_\nu(x) dx = \frac{2^{\sigma+1} [\Gamma(\sigma+1)]^2}{\Gamma(\sigma+\nu+2) \Gamma(1+\sigma-\nu)} \quad [\operatorname{Re} \sigma > -1] \quad \text{ET II 316(15)}$$

7.128

$$\begin{aligned}
1. \quad \int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{3}{2}} P_\nu^\mu(x) dx \\
&= -\frac{\Gamma(\mu-\frac{1}{2}) (z-1)^{\mu-\frac{1}{2}} (z+1)^{-1/2}}{\pi^{1/2} e^{2\mu\pi i} \Gamma(\mu+\nu) \Gamma(\mu-\nu-1)} \\
&\quad \times \left\{ Q_\nu^\mu \left[\left(\frac{1+z}{2} \right)^{1/2} \right] Q_{-\nu-1}^{\mu-1} \left[\left(\frac{1+z}{2} \right)^{1/2} \right] + Q_\nu^{\mu-1} \left[\left(\frac{1+z}{2} \right)^{1/2} \right] Q_{-\nu-1}^\mu \left[\left(\frac{1+z}{2} \right)^{1/2} \right] \right\} \\
&\quad \quad \quad [-\tfrac{1}{2} < \operatorname{Re} \mu < 1, \\
&\quad \quad \quad z \text{ is in the complex plane with a cut along the interval } (-1, 1) \text{ of the real axis}]
\end{aligned}$$

ET II 317(20)

$$\begin{aligned}
2. \quad \int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_\nu^\mu(x) dx \\
&= \frac{2e^{-2\mu\pi i} \Gamma(\frac{1}{2}+\mu)}{\pi^{1/2} \Gamma(\mu-\nu) \Gamma(\mu+\nu+1)} (z-1)^\mu Q_\nu^\mu \left[\left(\frac{1+z}{2} \right)^{1/2} \right] Q_{-\nu-1}^\mu \left[\left(\frac{1+z}{2} \right)^{1/2} \right] \\
&\quad \quad \quad [-\tfrac{1}{2} < \operatorname{Re} \mu < 1, \\
&\quad \quad \quad z \text{ is in the complex plane with a cut along the interval } (-1, 1) \text{ of the real axis}]
\end{aligned}$$

ET II 316(18)

$$7.129 \quad \int_{-1}^1 P_{\nu}(x) P_{\lambda}(x) (1+x)^{\lambda+\nu} dx = \frac{2^{\lambda+\nu+1} [\Gamma(\lambda+\nu+1)]^4}{[\Gamma(\lambda+1) \Gamma(\nu+1)]^2 \Gamma(2\lambda+2\nu+2)} \\ [\operatorname{Re}(\nu+\lambda+1) > 0] \quad \text{EH I 172(30)}$$

7.131

$$1. \quad \int_1^{\infty} (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_{\nu}^{\mu}(x) dx \\ = \pi^{1/2} \frac{\Gamma(-\mu-\nu) \Gamma(1-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)} (z-1)^{\mu} \left\{ P_{\nu}^{\mu} \left[\left(\frac{1+z}{2} \right)^{1/2} \right] \right\}^2 \\ [\operatorname{Re}(\mu+\nu) < 0, \quad \operatorname{Re}(\mu-\nu) < 1, \quad |\arg(z+1)| < \pi] \quad \text{ET II 321(6)}$$

$$2. \quad \int_1^{\infty} (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{3}{2}} P_{\nu}^{\mu}(x) dx \\ = \frac{\pi^{1/2} \Gamma(1-\mu-\nu) \Gamma(2-\mu+\nu) (z-1)^{\mu-\frac{1}{2}} (z+1)^{-1/2}}{\Gamma(\frac{3}{2}-\mu)} P_{\nu}^{\mu} \left[\left(\frac{1+z}{2} \right)^{1/2} \right] P_{\nu}^{\mu-1} \left[\left(\frac{1+z}{2} \right)^{1/2} \right] \\ [\operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+\nu) < 1, \quad \operatorname{Re}(\mu-\nu) < 2, \quad |\arg(1+z)| < \pi] \quad \text{ET II 321(7)}$$

7.132

$$1. \quad \int_{-1}^1 (1-x^2)^{\lambda-1} P_{\nu}^{\mu}(x) dx = \frac{\pi 2^{\mu} \Gamma(\lambda+\frac{1}{2}\mu) \Gamma(\lambda-\frac{1}{2}\mu)}{\Gamma(\lambda+\frac{1}{2}\nu+\frac{1}{2}) \Gamma(\lambda-\frac{1}{2}\nu) \Gamma(-\frac{1}{2}\mu+\frac{1}{2}\nu+1) \Gamma(-\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2})} \\ [2 \operatorname{Re} \lambda > |\operatorname{Re} \mu|] \quad \text{ET II 316(16)}$$

$$2. \quad \int_1^{\infty} (x^2-1)^{\lambda-1} P_n^{\mu}(x) dx = \frac{2^{\mu-1} \Gamma(\lambda-\frac{1}{2}\mu) \Gamma(1-\lambda+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\lambda-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu) \Gamma(1-\lambda-\frac{1}{2}\mu)} \\ [\operatorname{Re} \lambda > \operatorname{Re} \mu, \quad \operatorname{Re}(1-2\lambda-\nu) > 0, \quad \operatorname{Re}(2-2\lambda+\nu) > 0] \quad \text{ET II 320(2)}$$

$$3.^9 \quad \int_1^{\infty} (x^2-1)^{\lambda-1} Q_{\nu}^{\mu}(x) dx = e^{\mu\pi i} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu) \Gamma(1-\lambda+\frac{1}{2}\nu) \Gamma(\lambda+\frac{1}{2}\mu) \Gamma(\lambda-\frac{1}{2}\mu)}{2^{2-\mu} \Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(\frac{1}{2}+\lambda+\frac{1}{2}\nu)} \\ [|\operatorname{Re} \mu| < 2 \operatorname{Re} \lambda < \operatorname{Re} \nu + 2] \quad \text{ET II 324(23)}$$

$$4. \quad \int_0^1 x^{\sigma} (1-x^2)^{-\frac{1}{2}\mu} P_{\nu}^{\mu}(x) dx = \frac{2^{\mu-1} \Gamma(\frac{1}{2}+\frac{1}{2}\sigma) \Gamma(1+\frac{1}{2}\sigma)}{\Gamma(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{3}{2})} \\ [\operatorname{Re} \mu < 1, \quad \operatorname{Re} \sigma > -1] \quad \text{EH I 172(24)}$$

$$5. \quad \int_0^1 x^{\sigma} (1-x^2)^{\frac{1}{2}m} P_{\nu}^m(x) dx = \frac{(-1)^m 2^{-m-1} \Gamma(\frac{1}{2}+\frac{1}{2}\sigma) \Gamma(1+\frac{1}{2}\sigma) \Gamma(1+m+\nu)}{\Gamma(1-m+\nu) \Gamma(1+\frac{1}{2}\sigma+\frac{1}{2}m-\frac{1}{2}\nu) \Gamma(\frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}m+\frac{1}{2}\nu)} \\ [\operatorname{Re} \sigma > -1, \quad m \text{ is a positive integer}] \quad \text{EH I 172(25), ET II 313(4)}$$

$$6. \quad \int_0^1 (1-x^2)^{\eta} P_{\nu}^{\mu}(x) dx = \frac{2^{\mu-1} \Gamma(1+\eta-\frac{1}{2}\mu) \Gamma(\frac{1}{2}+\frac{1}{2}\sigma)}{\Gamma(1-\mu) \Gamma(\frac{3}{2}+\eta+\frac{1}{2}\sigma-\frac{1}{2}\mu)} \\ \times {}_3F_2 \left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1+\eta-\frac{\mu}{2}; 1-\mu, \frac{3+\sigma-\mu}{2}+\eta; 1 \right) \\ [\operatorname{Re}(\eta-\frac{1}{2}\mu) > -1, \operatorname{Re} \sigma > -1] \quad \text{ET II 314(6)}$$

$$7. \quad \int_1^\infty x^{-\rho} (x^2 - 1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\rho+\mu-2} \Gamma\left(\frac{\rho+\mu+\nu}{2}\right) \Gamma\left(\frac{\rho+\mu-\nu-1}{2}\right)}{\sqrt{\pi} \Gamma(\rho)}$$

[Re $\mu < 1$, Re($\rho + \mu + \nu$) > 0 , Re($\rho + \mu - \nu$) > 1] ET II 320(3)

7.133

$$1. \quad \int_u^\infty Q_\nu(x)(x-u)^{\mu-1} dx = \Gamma(\mu) e^{\mu\pi i} (u^2 - 1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(u)$$

[|arg($u - 1$)| $< \pi$, $0 < \text{Re } \mu < 1 + \text{Re } \nu$] MO 90a

$$2. \quad \int_u^\infty (x^2 - 1)^{\frac{1}{2}\lambda} Q_\nu^{-\lambda}(x)(x-u)^{\mu-1} dx = \Gamma(\mu) e^{\mu\pi i} (u^2 - 1)^{\frac{1}{2}\lambda + \frac{1}{2}\mu} Q_\nu^{-\lambda-\mu}(u)$$

[|arg($u - 1$)| $< \pi$, $0 < \text{Re } \mu < 1 + \text{Re}(\nu - \lambda)$] ET II 204(30)

7.134

$$1. \quad \int_1^\infty (x-1)^{\lambda-1} (x^2 - 1)^{\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\lambda+\mu} \Gamma(\lambda) \Gamma(-\lambda - \mu - \nu) \Gamma(1 - \lambda - \mu + \nu)}{\Gamma(1 - \mu + \nu) \Gamma(-\mu - \nu) \Gamma(1 - \lambda - \mu)}$$

[Re $\lambda > 0$, Re($\lambda + \mu + \nu$) < 0 , Re($\lambda + \mu - \nu$) < 1] ET II 321(4)

$$2. \quad \int_1^\infty (x-1)^{\lambda-1} (x^2 - 1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = -\frac{2^{\lambda-\mu} \sin \pi \nu \Gamma(\lambda - \mu) \Gamma(-\lambda + \mu - \nu) \Gamma(1 - \lambda + \mu + \nu)}{\pi \Gamma(1 - \lambda)}$$

[Re($\lambda - \mu$) > 0 , Re($\mu - \lambda - \nu$) > 0 , Re($\mu - \lambda + \nu$) > -1] ET II 321(5)

7.135

$$1. \quad \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}\mu} (z - x)^{-1} P_{\mu+n}^\mu(x) dx = 2e^{-i\mu\pi} (z^2 - 1)^{-\frac{1}{2}\mu} Q_{\mu+n}^\mu(z)$$

[$n = 0, 1, 2, \dots$, Re $\mu + n > -1$, z is in the complex plane with a cut along the interval $(-1, 1)$ of the real axis.] ET II 316(17)

$$2. \quad \int_1^\infty (x-1)^{\lambda-1} (x^2 - 1)^{\mu/2} (x+z)^{-\rho} P_\nu^\mu(x) dx$$

$$= \frac{2^{\lambda+\mu-\rho} \Gamma(\lambda - \rho) \Gamma(\rho - \lambda - \mu - \nu) \Gamma(\rho - \lambda - \mu + \nu + 1)}{\Gamma(1 - \mu + \nu) \Gamma(-\mu - \nu) \Gamma(1 + \rho - \lambda - \mu)}$$

$$\times {}_3F_2\left(\rho, \rho - \lambda - \mu - \nu, \rho - \lambda - \mu + \nu + 1; \rho - \lambda + 1, \rho - \lambda - \mu + 1; \frac{1+z}{2}\right)$$

$$+ \frac{\Gamma(\rho - \lambda) \Gamma(\lambda)}{\Gamma(\rho) \Gamma(1 - \mu)} 2^\mu (z+1)^{\lambda-\rho} {}_3F_2\left(\lambda, -\mu - \nu, 1 - \mu + \nu; 1 - \mu, 1 - \rho + \lambda; \frac{1+z}{2}\right)$$

[Re $\lambda > 0$, Re($\rho - \lambda - \mu - \nu$) > 0 , Re($\rho - \lambda - \mu + \nu + 1$) > 0 , |arg($z + 1$)| $< \pi$]
ET II 322(9)

$$\begin{aligned}
3. \quad & \int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\mu/2} (x+z)^{-\rho} P_\nu^\mu(x) dx \\
&= -\frac{\sin(\nu\pi) \Gamma(\lambda-\mu-\rho) \Gamma(\rho-\lambda+\mu-\nu) \Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu} \pi \Gamma(1+\rho-\lambda)} \\
&\quad \times {}_3F_2\left(\rho, \rho-\lambda+\mu-\nu, \rho-\lambda+\mu+\nu+1; 1+\rho-\lambda, 1+\rho-\lambda+\mu; \frac{1+z}{2}\right) \\
&\quad + \frac{\Gamma(\lambda-\mu) \Gamma(\rho-\lambda+\mu)}{\Gamma(\rho) \Gamma(1-\mu)} (z+1)^{\lambda-\rho-\mu} \\
&\quad \times {}_3F_2\left(\lambda-\mu, -\nu, \nu+1; 1+\lambda-\mu-\rho, 1-\mu; \frac{1+z}{2}\right) \\
&[\operatorname{Re}(\lambda-\mu) > 0, \quad \operatorname{Re}(\rho-\lambda+\mu-\nu) > 0, \quad \operatorname{Re}(\rho-\lambda+\mu+\nu+1) > 0, \quad |\arg(z+1)| < \pi] \\
&\hspace{15em} \text{ET II 322(10)}
\end{aligned}$$

7.136

$$\begin{aligned}
1. \quad & \int_{-1}^1 (1-x^2)^{\lambda-1} (1-a^2x^2)^{\mu/2} P_\nu(ax) dx \\
&= \frac{\pi 2^\mu \Gamma(\lambda)}{\Gamma(\frac{1}{2}+\lambda) \Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu) \Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)} {}_2F_1\left(-\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}; \frac{1}{2}+\lambda; a^2\right) \\
&\hspace{15em} [\operatorname{Re} \lambda > 0, \quad -1 < a < 1] \quad \text{ET II 318(31)} \\
2. \quad & \int_1^\infty (x^2-1)^{\lambda-1} (a^2x^2-1)^{\mu/2} P_\nu^\mu(ax) dx \\
&= \frac{\Gamma(\lambda) \Gamma(1-\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(1-\lambda-\mu)} \\
&\quad \times 2^{\mu-1} a^{\mu-\nu-1} {}_2F_1\left(\frac{1-\mu+\nu}{2}, 1-\lambda-\frac{\mu-\nu}{2}; 1-\lambda-\mu; 1-\frac{1}{a^2}\right) \\
&[\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\nu-\mu-2\lambda) > -2, \quad \operatorname{Re}(2\lambda+\mu+\nu) < 1] \quad \text{ET II 325(25)} \\
3. \quad & \int_1^\infty (x^2-1)^{\lambda-1} (a^2x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(ax) dx = \frac{\Gamma(\frac{\mu+\nu+1}{2}) \Gamma(\lambda) \Gamma(1-\lambda+\frac{\mu+\nu}{2}) 2^{\mu-2} e^{\mu\pi i} a^{-\mu-\nu-1}}{\Gamma(\nu+\frac{3}{2})} \\
&\quad \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, 1-\lambda+\frac{\mu+\nu}{2}; \nu+\frac{3}{2}; a^{-2}\right) \\
&\hspace{15em} [|\arg(a-1)| < \pi, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\lambda-\mu-\nu) < 2] \quad \text{ET II 325(27)}
\end{aligned}$$

7.137

$$\begin{aligned}
1. \quad & \int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}} (x-1)^{-\mu-\frac{1}{2}} (1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx \\
&= \pi^{-1/2} e^{-\mu\pi i} \Gamma(\frac{1}{2}-\mu) a^{\frac{1}{2}\mu} \left\{ Q_\nu^\mu \left[(1+a)^{1/2} \right] \right\}^2 \\
&\hspace{15em} [|\arg a| < \pi, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > -1] \quad \text{ET II 325(28)} \\
2. \quad & \int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}} (x-1)^{-\mu-\frac{3}{2}} (1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx \\
&= -\pi^{-1/2} e^{-\mu\pi i} \Gamma(-\mu-\frac{1}{2}) a^{\frac{1}{2}\mu+\frac{1}{2}} (1+a^2)^{-1/2} Q_\nu^{\mu+1} \left[(1+a)^{1/2} \right] Q_\nu^\mu \left[(1+a)^{1/2} \right] \\
&\hspace{15em} [|\arg a| < \pi, \quad \operatorname{Re} \mu < -\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu+2) > 0] \quad \text{ET II 326(29)}
\end{aligned}$$

$$3. \quad \int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx = \pi^{1/2} \Gamma\left(\frac{1}{2}-\mu\right) a^{\frac{1}{2}\mu} \left\{ P_\nu^\mu \left[(1+a)^{1/2} \right] \right\}^2$$

$$[\operatorname{Re} \mu < \tfrac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(32)}$$

$$4. \quad \int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx$$

$$= \pi^{1/2} \Gamma\left(-\frac{1}{2}-\mu\right) a^{\frac{1}{2}\mu+\frac{1}{2}} P_\nu^{\mu+1} \left[(1+a)^{1/2} \right] P_\nu^\mu \left[(1+a)^{1/2} \right]$$

$$[\operatorname{Re} \mu < -\tfrac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(33)}$$

$$5. \quad \int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}(1+ax)^{-\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx$$

$$= \pi^{1/2} \Gamma\left(\frac{1}{2}+\mu\right) a^{-\frac{1}{2}\mu} P_\nu^\mu \left[(1+a)^{1/2} \right] P_\nu^{-\mu} \left[(1+a)^{1/2} \right]$$

$$[\operatorname{Re} \mu > -\tfrac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(34)}$$

$$6. \quad \int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{3}{2}}(1+ax)^{-\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx$$

$$= \frac{1}{2} \pi^{1/2} \Gamma\left(\mu-\frac{1}{2}\right) a^{\frac{1}{2}-\frac{1}{2}\mu} (1+a)^{-1/2} \left\{ P_\nu^{1-\mu} \left[(1+a)^{1/2} \right] P_\nu^\mu \left[(1+a)^{1/2} \right] \right\}$$

$$+ (\mu+\nu)(1-\mu+\nu) P_\nu^{-\mu} \left[(1+a)^{1/2} \right] P_\nu^\mu \left[(1+a)^{1/2} \right]$$

$$[\operatorname{Re} \mu > \tfrac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(35)}$$

$$7. \quad \int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx$$

$$= \pi^{1/2} \Gamma\left(\frac{1}{2}-\mu\right) a^{\frac{1}{2}\mu} P_\nu^\mu \left[(1+a)^{1/2} \right] Q_\nu^\mu \left[(1+a)^{1/2} \right]$$

$$[\operatorname{Re} \mu < \tfrac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 320(38)}$$

$$8. \quad \int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx$$

$$= \frac{1}{2} \pi^{1/2} \Gamma\left(-\mu-\frac{1}{2}\right) (1+a)^{-1/2} a^{\frac{1}{2}\mu+\frac{1}{2}}$$

$$\times \left\{ P_\nu^{\mu+1} \left[(1+a)^{1/2} \right] Q_\nu^\mu \left[(1+a)^{1/2} \right] + P_\nu^\mu \left[(1+a)^{1/2} \right] Q_\nu^{\mu+1} \left[(1+a)^{1/2} \right] \right\}$$

$$[\operatorname{Re} \mu < -\tfrac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 320(39)}$$

$$9. \quad \int_0^y (y-x)^{\mu-1} \left[x \left(1 + \frac{1}{2}\gamma x \right) \right]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x) dx$$

$$= \Gamma(\mu) \left(\frac{2}{\gamma} \right)^{\frac{1}{2}\mu} \left[y \left(1 + \frac{1}{2}\gamma y \right) \right]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1+\gamma y)$$

$$[\operatorname{Re} \lambda < 1, \quad \operatorname{Re} \mu > 0, \quad |\arg \gamma y| < \pi] \quad \text{ET II 193(52)}$$

$$10. \quad \int_0^y (y-x)^{\mu-1} x^{\sigma+\frac{1}{2}\lambda-1} \left(1 + \frac{1}{2}\gamma x \right)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x) dx$$

$$= \frac{\left(\frac{\gamma}{2} \right)^{-\frac{1}{2}\lambda} \Gamma(\sigma) \Gamma(\mu) y^{\sigma+\mu-1}}{\Gamma(1-\lambda) \Gamma(\sigma+\mu)} {}_3F_2 \left(-\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; -\frac{1}{2}\gamma y \right)$$

$$[\operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \quad |\gamma y| < 1] \quad \text{ET II 193(53)}$$

$$11. \quad \int_0^y (y-x)^{\mu-1} [x(1-x)]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x) dx = \Gamma(\mu) [y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1-2y) \\ [\operatorname{Re} \lambda < 1, \quad \operatorname{Re} \mu > 0, \quad 0 < y < 1] \\ \text{ET II 193(54)}$$

$$12. \quad \int_0^y (y-x)^{\mu-1} x^{\sigma+\frac{1}{2}\lambda-1} (1-x)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x) dx \\ = \frac{\Gamma(\mu)\Gamma(\sigma)y^{\sigma+\mu-1}}{\Gamma(\sigma+\mu)\Gamma(1-\lambda)} {}_3F_2(-\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; y) \\ [\operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \quad 0 < y < 1] \quad \text{ET II 193(155)}$$

$$7.138 \quad \int_0^\infty (a+x)^{-\mu-\nu-2} P_\mu\left(\frac{a-x}{a+x}\right) P_\nu\left(\frac{a-x}{a+x}\right) dx = \frac{a^{-\mu-\nu-1} [\Gamma(\mu+\nu+1)]^4}{[\Gamma(\mu+1)\Gamma(\nu+1)]^2 \Gamma(2\mu+2\nu+2)} \\ [|\arg a| < \pi, \quad \operatorname{Re}(\mu+\nu) > -1] \\ \text{ET II 326(3)}$$

7.14 Combinations of associated Legendre functions, exponentials, and powers

7.141

$$1. \quad \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{a^{-\lambda-\mu} e^{-a}}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} G_{23}^{31} \left(2a \left| \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, -\nu, 1+\nu \end{matrix} \right. \right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 323(13)}$$

$$2. \quad \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} Q_\nu^\mu(x) dx \\ = \frac{\Gamma(\nu+\mu+1)e^{\mu\pi i}}{2\Gamma(\nu-\mu+1)} a^{-\lambda-\mu} e^{-a} G_{23}^{22} \left(2a \left| \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, \nu+1, -\nu \end{matrix} \right. \right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0] \quad \text{ET II 325(24)}$$

$$3. \quad \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = -\pi^{-1} \sin(\nu\pi) a^{\mu-\lambda} e^{-a} G_{23}^{31} \left(2a \left| \begin{matrix} 1, 1-\mu \\ \lambda-\mu, 1+\nu, -\nu \end{matrix} \right. \right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda-\mu) > 0] \\ \text{ET II 323(15)}$$

$$4. \quad \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(x) dx = \frac{1}{2} e^{\mu\pi i} a^{\mu-\lambda} e^{-a} G_{23}^{22} \left(2a \left| \begin{matrix} 1-\mu, 1 \\ \lambda-\mu, \nu+1, -\nu \end{matrix} \right. \right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda-\mu) > 0] \\ \text{ET II 323(14)}$$

$$5. \quad \int_1^\infty e^{-ax} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = 2^{1/2} \pi^{-1/2} a^{\mu-\frac{1}{2}} K_{\nu+\frac{1}{2}}(a) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu < 1] \\ \text{ET II 323(11), MO 90}$$

$$7.142 \quad \int_1^\infty e^{-\frac{1}{2}ax} \left(\frac{x+1}{x-1} \right)^{\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) dx = \frac{2}{a} W_{\mu,\nu}(a) \quad [\operatorname{Re} \mu < 1, \quad \nu - \frac{1}{2} \neq 0, \pm 1, \pm 2, \dots]$$

BU 79(34), MO 118

7.143

$$1. \quad \int_0^\infty [x(1+x)]^{-\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x) dx = \frac{\beta^{\mu-\frac{1}{2}}}{\sqrt{\pi}} e^{\frac{1}{2}\beta} K_{\nu+\frac{1}{2}}\left(\frac{\beta}{2}\right) \\ [\operatorname{Re} \mu < 1, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 179(1)}$$

$$2. \quad \int_0^\infty \left(1 + \frac{1}{x}\right)^{\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x) dx = \frac{e^{\frac{1}{2}\beta}}{\beta} W_{\mu, \nu+\frac{1}{2}}(\beta) \\ [\operatorname{Re} \mu < 1, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 179(2)}$$

7.144

$$1. \quad \int_0^\infty e^{-\beta x} x^{\lambda+\frac{1}{2}\mu-1} (x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x) dx \\ = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left\{ \frac{\sin(\nu\pi)}{2\beta^{\lambda+\mu} \sin(\mu\pi)} E(-\nu, \nu+1, \lambda+\mu; \mu+1: 2\beta) \right. \\ \left. - \frac{\sin[(\mu+\nu)\pi]}{2^{1-\mu}\beta^\lambda \sin(\mu\pi)} E(\nu-\mu+1, -\nu-\mu, \lambda: 1-\mu: 2\beta) \right\} \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0] \quad \text{ET I 181(16)}$$

$$2. \quad \int_0^\infty e^{-\beta x} x^{\lambda-\frac{1}{2}\mu-1} (x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x) dx = -\frac{\sin(\nu\pi)}{2\beta^{\lambda-\mu} \sin(\mu\pi)} E(-\nu, \nu+1, \lambda-\mu: 1-\mu: 2\beta) \\ - \frac{\sin[(\mu-\nu)\pi]}{2^{1+\mu}\beta^\lambda \sin(\mu\pi)} E(\mu+\nu+1, \mu-\nu, \lambda: 1+\mu: 2\beta) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda-\mu) > 0] \quad \text{ET I 181(17)}$$

7.145

$$1. \quad \int_0^\infty \frac{e^{-\beta x}}{1+x} P_\nu \left[\frac{1}{(1+x)^2} - 1 \right] dx = \frac{e^\beta}{\beta} W_{\nu+\frac{1}{2}, 0}(\beta) W_{-\nu-\frac{1}{2}, 0}(\beta) \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 180(6)}$$

$$2. \quad \int_0^\infty x^{-1} e^{-\beta x} Q_{-\frac{1}{2}}(1+2x^{-2}) dx = \frac{\pi^2}{8} \left\{ \left[J_0\left(\frac{1}{2}\beta\right) \right]^2 + \left[Y_0\left(\frac{1}{2}\beta\right) \right]^2 \right\} \\ [\operatorname{Re} \beta > 0] \quad \text{ET II 327(5)}$$

$$3. \quad \int_0^\infty x^{-1} e^{-ax} Q_\nu(1+2x^{-2}) dx = \frac{1}{2} [\Gamma(\nu+1)]^2 a^{-1} W_{-\nu-\frac{1}{2}, 0}(ai) W_{-\nu-\frac{1}{2}, 0}(-ai) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 327(6)}$$

7.146

$$1. \quad \int_0^\infty x^{-\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(\sqrt{1+x}) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{5}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu+\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(\beta) \\ [\operatorname{Re} \mu < 1, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 180(7)}$$

$$2. \quad \int_0^\infty x^{-\frac{1}{2}\mu} \frac{e^{-\beta x}}{\sqrt{1+x}} P_\nu^\mu(\sqrt{1+x}) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{3}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu+\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(\beta) \\ [\operatorname{Re} \mu < 1, \operatorname{Re} \beta > 0] \quad \text{ET I 180(8)a}$$

$$3. \quad \int_0^\infty \sqrt{x} e^{-\beta x} P_\nu^{1/4} \left(\sqrt{1+x^2} \right) P_\nu^{-1/4} \left(\sqrt{1+x^2} \right) dx = \frac{1}{2} \sqrt{\frac{\pi}{2\beta}} H_{\nu+\frac{1}{2}}^{(1)} \left(\frac{1}{2}\beta \right) H_{\nu+\frac{1}{2}}^{(2)} \left(\frac{1}{2}\beta \right) \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 180(9)}$$

$$7.147 \quad \int_0^\infty x^{\lambda-1} (x^2 + a^2)^{\frac{1}{2}\nu} e^{-\beta x} P_\nu^\mu \left[\frac{x}{(x^2 + a^2)^{1/2}} \right] dx \\ = \frac{2^{-\nu-2} a^{\lambda+\nu}}{\pi \Gamma(-\mu-\nu)} G_{24}^{32} \left(\frac{a^2 \beta^2}{4} \left| \begin{matrix} 1 - \frac{\lambda}{2}, \frac{1-\lambda}{2} \\ 0, \frac{1}{2}, -\frac{\lambda+\mu+\nu}{2}, -\frac{\lambda-\mu+\nu}{2} \end{matrix} \right. \right) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 327(7)}$$

$$7.148 \quad \int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu+\nu-1} \exp \left(-\frac{1-x}{1+x} y \right) P_\nu^\mu(x) dx = 2^\nu y^{\frac{1}{2}\mu+\nu-\frac{1}{2}} e^{\frac{1}{2}y} W_{\frac{1}{2}\mu-\nu-\frac{1}{2}, \frac{1}{2}\mu}(y) \\ [\operatorname{Re} y > 0] \quad \text{ET II 317(21)}$$

$$7.149 \quad \int_1^\infty (\alpha^2 + \beta^2 + 2\alpha\beta x)^{-1/2} \exp \left[-(\alpha^2 + \beta^2 + 2\alpha\beta x)^{1/2} \right] P_\nu(x) dx \\ = 2\pi^{-1} (\alpha\beta)^{-1/2} K_{\nu+\frac{1}{2}}(\alpha) K_{\nu+\frac{1}{2}}(\beta) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 323(16)}$$

7.15 Combinations of associated Legendre and hyperbolic functions

7.151

$$1. \quad \int_0^\infty (\sinh x)^{\alpha-1} P_\nu^{-\mu} (\cosh x) dx = \frac{2^{-1-\mu} \Gamma(\frac{1}{2}\alpha + \frac{1}{2}\mu) \Gamma(\frac{1}{2}\nu - \frac{1}{2}\alpha + 1) \Gamma(\frac{1}{2} - \frac{1}{2}\alpha - \frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + 1) \Gamma(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu) \Gamma(1 + \frac{1}{2}\mu - \frac{1}{2}\alpha)} \\ [\operatorname{Re}(\alpha + \mu) > 0, \quad \operatorname{Re}(\nu - \alpha + 2) > 0, \quad \operatorname{Re}(1 - \alpha - \nu) > 0] \quad \text{EH I 172(28)}$$

$$2. \quad \int_0^\infty (\sinh x)^{\alpha-1} Q_\nu^\mu (\cosh x) dx = \frac{e^{i\mu\pi} 2^{\mu-\alpha} \Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu) \Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\mu) \Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\alpha)} \\ \times \Gamma(\frac{1}{2}\alpha + \frac{1}{2}\mu) \Gamma(\frac{1}{2}\alpha - \frac{1}{2}\mu) \\ [\operatorname{Re}(\alpha \pm \mu) > 0, \quad \operatorname{Re}(\nu - \alpha + 2) > 0] \quad \text{EH I 172(29)}$$

$$7.152 \quad \int_0^\infty e^{-\alpha x} \sinh^{2\mu} \left(\frac{1}{2}x \right) P_{2n}^{-2\mu} [\cosh \left(\frac{1}{2}x \right)] dx = \frac{\Gamma(2\mu + \frac{1}{2}) \Gamma(\alpha - n - \mu) \Gamma(\alpha + n - \mu + \frac{1}{2})}{4^\mu \sqrt{\pi} \Gamma(\alpha + n + \mu + 1) \Gamma(\alpha - n + \mu + \frac{1}{2})} \\ [\operatorname{Re} \alpha > n + \operatorname{Re} \mu, \quad \operatorname{Re} \mu > -\frac{1}{4}] \\ \text{ET I 181(15)}$$

7.16 Combinations of associated Legendre functions, powers, and trigonometric functions

7.161

$$\begin{aligned}
 1. \quad \int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \sin(ax) P_\nu^\mu(x) dx \\
 = \frac{\pi^{1/2} 2^{\mu-\lambda-1} \Gamma(\lambda+1) a}{\Gamma\left(1 + \frac{\lambda-\mu-\nu}{2}\right) \Gamma\left(\frac{3+\lambda-\mu+\nu}{2}\right)} \\
 \times {}_2F_3\left(\frac{1+\lambda}{2}, 1 + \frac{\lambda}{2}; \frac{3}{2}, 1 + \frac{\lambda-\mu-\nu}{2}, \frac{3+\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right) \\
 [\operatorname{Re} \lambda > -1, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 314(7)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \cos(ax) P_\nu^\mu(x) dx \\
 = \frac{\pi^{1/2} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma\left(1 + \frac{\lambda-\mu+\nu}{2}\right) \Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)} \\
 \times {}_2F_3\left(\frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{1}{2}, \frac{1+\lambda-\mu-\nu}{2}, 1 + \frac{\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right) \\
 [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 314(8)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty (x^2-1)^{\frac{1}{2}\mu} \sin(ax) P_\nu^\mu(x) dx = \frac{2^\mu \pi^{1/2} a^{-\mu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu\right) \Gamma\left(1 - \frac{1}{2}\mu + \frac{1}{2}\nu\right)} S_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) \\
 [a > 0, \quad \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re}(\mu+\nu) < 1] \\
 \text{ET II 320(1)}
 \end{aligned}$$

7.162

$$\begin{aligned}
 1. \quad \int_a^\infty P_\nu(2x^2a^{-2}-1) \sin(bx) dx = -\frac{\pi a}{4 \cos(\nu\pi)} \left\{ \left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) \right]^2 - \left[J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) \right]^2 \right\} \\
 [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < 0] \\
 \text{ET II 326(1)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_a^\infty P_\nu(2x^2a^{-2}-1) \cos(bx) dx \\
 = -\frac{\pi}{4} a \left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) - Y_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) Y_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) \right] \\
 [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < 0] \quad \text{ET II 326(2)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty (x^2+2)^{-1/2} \sin(ax) P_\nu^{-1}(x^2+1) dx = 2^{-1/2} \pi^{-1} a \sin(\nu\pi) \left[K_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right) \right]^2 \\
 [a > 0, \quad -2 < \operatorname{Re} \nu < 1] \quad \text{ET I 98(22)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_0^\infty (x^2+2)^{-1/2} \sin(ax) Q_\nu^1(x^2+1) dx = -2^{-3/2} \pi a K_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right) I_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right) \\
 [a > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET 98(23)}
 \end{aligned}$$

$$5. \quad \int_0^\infty \cos(ax) P_\nu(1+x^2) dx = -\frac{\sqrt{2}}{\pi} \sin(\nu\pi) \left[K_{\nu+\frac{1}{2}} \left(\frac{a}{\sqrt{2}} \right) \right]^2$$

$$[a > 0, \quad -1 < \operatorname{Re} \nu < 0] \quad \text{ET I 42(23)}$$

$$6. \quad \int_0^\infty \cos(ax) Q_\nu(1+x^2) dx = \frac{\pi}{\sqrt{2}} K_{\nu+\frac{1}{2}} \left(\frac{a}{\sqrt{2}} \right) I_{\nu+\frac{1}{2}} \left(\frac{a}{\sqrt{2}} \right)$$

$$[a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 42(24)}$$

$$7. \quad \int_0^1 \cos(ax) P_\nu(2x^2-1) dx = \frac{\pi}{2} J_{\nu+\frac{1}{2}} \left(\frac{a}{2} \right) J_{-\nu-\frac{1}{2}} \left(\frac{a}{2} \right)$$

$$[a > 0] \quad \text{ET I 42(25)}$$

7.163

$$1. \quad \int_a^\infty (x^2 - a^2)^{\frac{1}{2}\nu - \frac{1}{4}} \sin(bx) P_0^{\frac{1}{2}-\nu}(ax^{-1}) dx = b^{-\nu-\frac{1}{2}} \cos\left(ab - \frac{\nu\pi}{2} + \frac{\pi}{4}\right)$$

$$[a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET I 98(24)}$$

$$2. \quad \int_0^1 x^{-1} \cos(ax) P_\nu(2x^{-2}-1) dx = -\frac{1}{2} \pi \operatorname{cosec}(\nu\pi) {}_1F_1((\nu+1; 1; ai)) {}_1F_1(\nu+1; 1; -ai)$$

$$[a > 0, \quad -1 < \operatorname{Re} \nu < 0] \quad \text{ET II 327(4)}$$

7.164

$$1. \quad \int_0^\infty x^{1/2} \sin(bx) \left[P_\nu^{-1/4}(\sqrt{1+a^2x^2}) \right]^2 dx = \frac{\sqrt{\frac{2}{\pi}} a^{-1} b^{-1/2}}{\Gamma(\frac{5}{4}+\nu) \Gamma(\frac{1}{4}-\nu)} \left[K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) \right]^2$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re} \nu < \frac{1}{4}] \quad \text{ET II 327(8)}$$

$$2. \quad \int_0^\infty x^{1/2} \sin(bx) P_\nu^{-1/4}(\sqrt{1+a^2x^2}) Q_{\nu-1}^{-1/4}(\sqrt{1+a^2x^2}) dx$$

$$= \frac{\sqrt{\frac{\pi}{2}} e^{-\frac{1}{4}\pi i} \Gamma(\nu + \frac{5}{4})}{ab^{\frac{1}{2}} \Gamma(\nu + \frac{3}{4})} I_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right)$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{5}{4}] \quad \text{ET II 328(9)}$$

$$3. \quad \int_0^\infty x^{1/2} \sin(bx) P_\nu^{-1/4}(\sqrt{1+a^2x^2}) P_{\nu-1}^{-1/4}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}}$$

$$= \frac{a^{-2} b^{1/2}}{\sqrt{2\pi} \Gamma(\frac{5}{4}+\nu) \Gamma(\frac{5}{4}-\nu)} K_{\nu-\frac{1}{2}} \left(\frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right)$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re} \nu < \frac{5}{4}] \quad \text{ET II 328(10)}$$

$$4. \quad \int_0^\infty x^{1/2} \sin(bx) P_\nu^{1/4}(\sqrt{1+a^2x^2}) P_\nu^{-3/4}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}}$$

$$= \frac{a^{-2} b^{1/2}}{\sqrt{2\pi} \Gamma(\frac{7}{4}+\nu) \Gamma(\frac{3}{4}-\nu)} \left[K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) \right]^2$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{7}{4} < \operatorname{Re} \nu < \frac{3}{4}] \quad \text{ET II 328(11)}$$

$$5. \quad \int_0^\infty x^{1/2} \cos(bx) \left[P_\nu^{1/4} \left(\sqrt{1+a^2x^2} \right) \right]^2 dx = \frac{a^{-1} \left(\frac{\pi b}{2} \right)^{-1/2}}{\Gamma\left(\frac{3}{4} + \nu\right) \Gamma\left(-\frac{1}{4} - \nu\right)} \left[K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) \right]^2$$

[Re $\nu > 0$, $b > 0$, $-\frac{3}{4} < \text{Re } \nu < -\frac{1}{4}$]
ET II 328(12)

$$6. \quad \int_0^\infty x^{1/2} \cos(bx) P_\nu^{1/4} \left(\sqrt{1+a^2x^2} \right) Q_\nu^{1/4} \left(\sqrt{1+a^2x^2} \right) dx$$

$$= \frac{\sqrt{\frac{\pi}{2}} e^{\frac{1}{4}\pi i} \Gamma\left(\nu + \frac{3}{4}\right)}{ab^{1/2} \Gamma\left(\nu + \frac{5}{4}\right)} I_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right)$$

[Re $\nu > 0$, $b > 0$, $\text{Re } \nu > -\frac{3}{4}$] ET II 328(13)

$$7. \quad \int_0^\infty x^{1/2} \cos(bx) P_\nu^{-1/4} \left(\sqrt{1+a^2x^2} \right) P_\nu^{3/4} \left(\sqrt{1+a^2x^2} \right) \frac{dx}{\sqrt{1+a^2x^2}}$$

$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi} \Gamma\left(\frac{5}{4} + \nu\right) \Gamma\left(\frac{1}{4} - \nu\right)} \left[K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) \right]^2$$

[Re $\nu > 0$, $b > 0$, $-\frac{5}{4} < \text{Re } \nu < \frac{1}{4}$] ET II 328(14)

$$8. \quad \int_0^\infty x^{1/2} \cos(bx) P_\nu^{1/4} \left(\sqrt{1+a^2x^2} \right) P_{\nu-1}^{1/4} \left(\sqrt{1+a^2x^2} \right) \frac{dx}{\sqrt{1+a^2x^2}}$$

$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi} \Gamma\left(\frac{3}{4} + \nu\right) \Gamma\left(\frac{3}{4} - \nu\right)} K_{\nu-\frac{1}{2}} \left(\frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right)$$

[Re $\nu > 0$, $b > 0$, $|\text{Re } \nu| < \frac{3}{4}$] ET II 329(15)

$$7.165 \quad \int_0^\infty \cos(ax) P_\nu(\cosh x) dx$$

$$= -\frac{\sin(\nu\pi)}{4\pi^2} \Gamma\left(\frac{1+\nu+i\alpha}{2}\right) \Gamma\left(\frac{1+\nu-i\alpha}{2}\right) \Gamma\left(-\frac{\nu+i\alpha}{2}\right) \Gamma\left(-\frac{\nu-i\alpha}{2}\right)$$

[$a > 0$, $-1 < \text{Re } \nu < 0$] ET II 329(18)

$$7.166 \quad \int_0^\pi P_\nu^{-\mu}(\cos \varphi) \sin^{\alpha-1} \varphi d\varphi = \frac{2^{-\mu} \pi \Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2}\alpha - \frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}\alpha - \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)}$$

[Re $(\alpha \pm \mu) > 0$] MO 90, EH I 172(27)

$$7.167 \quad \int_0^a P_\nu^{-\mu}(\cos x) P_\nu^{-\eta}[\cos(a-x)] \left[\frac{\sin(a-x)}{\sin x} \right]^\eta \frac{dx}{\sin x} = \frac{2^\eta \Gamma(\mu-\eta) \Gamma\left(\eta + \frac{1}{2}\right) (\sin a)^\eta}{\sqrt{\pi} \Gamma(\eta + \mu + 1)} P_\nu^{-\mu}(\cos a)$$

[Re $\mu > \text{Re } \eta > -\frac{1}{2}$] ET II 329(16)

7.17 A combination of an associated Legendre function and the probability integral

$$7.171 \quad \int_1^\infty (x^2 - 1)^{-\frac{1}{2}\mu} \exp(a^2x^2) [1 - \Phi(ax)] P_\nu^\mu(x) dx$$

$$= \pi^{-1} 2^{\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{\mu-\nu}{2}\right) a^{\mu-\frac{3}{2}} e^{\frac{a^2}{2}} W_{\frac{1}{4}-\frac{1}{2}\mu, \frac{1}{4}+\frac{1}{2}\nu}(a^2)$$

[Re $a > 0$, $\text{Re } \mu < 1$, $\text{Re }(\mu + \nu) > -1$, $\text{Re }(\mu - \nu) > 0$]
ET II 324(17)

7.18 Combinations of associated Legendre and Bessel functions

7.181

$$1. \quad \int_1^\infty P_{\nu-\frac{1}{2}}(x) x^{1/2} Y_\nu(ax) dx = 2^{-1/2} a^{-1} [\cos(\tfrac{1}{2}a) J_\nu(\tfrac{1}{2}a) - \sin(\tfrac{1}{2}a) Y_\nu(\tfrac{1}{2}a)]$$

$$[a > 0, \quad \operatorname{Re} \nu < \tfrac{1}{2}] \quad \text{ET II 108(3)a}$$

$$2. \quad \int_1^\infty P_{\nu-\frac{1}{2}}(x) x^{1/2} J_\nu(ax) dx = -\frac{1}{\sqrt{2}a} [\cos(\tfrac{1}{2}a) Y_\nu(\tfrac{1}{2}a) + \sin(\tfrac{1}{2}a) J_\nu(\tfrac{1}{2}a)]$$

$$[|\operatorname{Re} \nu| < \tfrac{1}{2}] \quad \text{ET II 344(36)a}$$

7.182

$$1. \quad \int_1^\infty x^\nu (x^2 - 1)^{\frac{1}{2}\lambda - \frac{1}{2}} P_\lambda^{\lambda-1}(x) J_\nu(ax) dx = \frac{2^{\lambda+\nu} a^{-\lambda} \Gamma(\frac{1}{2} + \nu)}{\pi^{1/2} \Gamma(1 - \lambda)} S_{\lambda-\nu, \lambda+\nu}(a)$$

$$[a > 0, \quad \operatorname{Re} \nu < \tfrac{5}{2}, \quad \operatorname{Re}(2\lambda + \nu) < \tfrac{3}{2}]$$

ET II 345(38)a

$$2. \quad \int_1^\infty x^{\frac{1}{2}-\mu} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) J_\nu(ax) dx$$

$$= -2^{-3/2} \pi^{1/2} a^{\mu-\frac{1}{2}} \left[J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) Y_\nu\left(\frac{a}{2}\right) + Y_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) J_\nu\left(\frac{a}{2}\right) \right]$$

$$\left[-\frac{1}{4} < \operatorname{Re} \mu < 1, \quad a > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2} + 2 \operatorname{Re} \mu\right] \quad \text{ET II 344(37)a}$$

$$3. \quad \int_1^\infty x^{\frac{1}{2}-\mu} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) Y_\nu(ax) dx$$

$$= 2^{-3/2} \pi^{1/2} a^{\mu-\frac{1}{2}} \left[J_\nu\left(\frac{a}{2}\right) J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - Y_\nu\left(\frac{a}{2}\right) Y_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) \right]$$

$$\left[-\frac{1}{4} < \operatorname{Re} \mu < 1, \quad a > 0, \quad \operatorname{Re}(2\mu - \nu) > -\tfrac{1}{2}\right] \quad \text{ET II 349(67)a}$$

$$4. \quad \int_0^1 x^{\frac{1}{2}-\mu} (1 - x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) J_{\nu+\frac{1}{2}}(ax) dx = \sqrt{\frac{\pi}{2}} a^{\mu-\frac{1}{2}} J_{\frac{1}{2}-\mu}\left(\frac{1}{2}a\right) J_{\nu+\frac{1}{2}}\left(\frac{1}{2}a\right)$$

$$[\operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu - \nu) < 2]$$

ET II 337(33)a

$$5. \quad \int_1^\infty x^{\frac{1}{2}-\mu} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) K_\nu(ax) dx = (2\pi)^{-1/2} a^{\mu-\frac{1}{2}} K_\nu\left(\frac{1}{2}a\right) K_{\mu-\frac{1}{2}}\left(\frac{1}{2}a\right)$$

$$[\operatorname{Re} \mu < 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 135(5)a}$$

$$6. \quad \int_1^\infty x^{\mu+\frac{1}{2}} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) K_\nu(ax) dx = \sqrt{\frac{\pi}{2}} a^{-3/2} e^{-\frac{1}{2}a} W_{\mu, \nu}(a)$$

$$[\operatorname{Re} \mu < 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 135(3)a}$$

$$7. \quad \int_1^\infty x^{\mu-\frac{3}{2}} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) K_\nu(ax) dx = \sqrt{\frac{\pi}{2}} a^{-1/2} e^{-\frac{1}{2}a} W_{\mu-1, \nu}(a)$$

$$[\operatorname{Re} \mu < 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 135(4)a}$$

$$8. \quad \int_1^\infty x^{\mu-\frac{1}{2}} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{3}{2}}^\mu(x) K_\nu(ax) dx = \sqrt{\frac{\pi}{2}} a^{-1} e^{-\frac{1}{2}a} W_{\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a)$$

$$[\operatorname{Re} \mu < 1] \quad \text{ET II 135(6)a}$$

$$9. \quad \int_1^\infty x^{1/2} (x^2 - 1)^{\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{\frac{1}{2}-\nu} (2x^2 - 1) K_\nu(ax) dx = \pi^{-1/2} a^{-\nu} 2^{\nu-1} \left[K_{\mu+\frac{1}{2}} \left(\frac{a}{2} \right) \right]^2$$

$$[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} a > 0] \quad \text{ET II 136(11)a}$$

$$10. \quad \int_1^\infty x^{1/2} (x^2 - 1)^{\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{\frac{1}{2}-\nu} (2x^2 - 1) Y_\nu(ax) dx$$

$$= \pi^{1/2} 2^{\nu-2} a^{-\nu} \left[J_{\mu+\frac{1}{2}} \left(\frac{a}{2} \right) J_{-\mu-\frac{1}{2}} \left(\frac{a}{2} \right) - Y_{\mu+\frac{1}{2}} \left(\frac{a}{2} \right) Y_{-\mu-\frac{1}{2}} \left(\frac{a}{2} \right) \right]$$

$$[\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re} \nu + |2 \operatorname{Re} \mu + 1| < \frac{3}{2}] \quad \text{ET II 108(5)a}$$

$$11. \quad \int_1^\infty x^{1/2} (x^2 - 1)^{\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{\frac{1}{2}-\nu} (2x^2 - 1) J_\nu(ax) dx$$

$$= -2^{\nu-2} a^{-\nu} \pi^{1/2} \sec(\mu\pi) \left\{ \left[J_{\mu+\frac{1}{2}} \left(\frac{a}{2} \right) \right]^2 - \left[J_{-\mu-\frac{1}{2}} \left(\frac{a}{2} \right) \right]^2 \right\}$$

$$[\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re} \nu - \frac{3}{2} < 2 \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re} \nu] \quad \text{ET II 345(39)a}$$

$$12. \quad \int_1^\infty x (x^2 - 1)^{-\frac{1}{2}\nu} P_\mu^\nu (2x^2 - 1) K_\nu(ax) dx = 2^{-\nu} a^{\nu-1} K_{\mu+1}(a)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 136(10)a}$$

$$13. \quad \int_0^\infty x (x^2 + a^2)^{\frac{1}{2}\nu} P_\mu^\nu (1 + 2x^2 a^{-2}) K_\nu(xy) dx = 2^{-\nu} a y^{-\nu-1} S_{2\nu, 2\mu+1}(ay)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1]$$

$$\text{ET II 135(7)}$$

$$14. \quad \int_0^\infty x (x^2 + a^2)^{\frac{1}{2}\nu} [(\mu - \nu) P_\mu^\nu (1 + 2x^2 a^{-2}) + (\mu + \nu) P_{-\mu}^\nu (1 + 2x^2 a^{-2})] K_\nu(xy) dx$$

$$= 2^{1-\nu} \mu y^{-\nu-2} S_{2\nu+1, 2\mu}(ay)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 136(8)}$$

$$15. \quad \int_0^\infty x (x^2 + a^2)^{\frac{1}{2}\nu-1} [P_\mu^\nu (1 + 2x^2 a^{-2}) + P_{-\mu}^\nu (1 + 2x^2 a^{-2})] K_\nu(xy) dx = 2^{1-\nu} y^{-\nu} S_{2\nu-1, 2\mu}(ay)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1]$$

$$\text{ET II 136(9)}$$

$$16. \quad \int_0^\infty x^{1/2} (x^2 + 2)^{-\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{-\nu - \frac{1}{2}} (x^2 + 1) J_\nu(xy) dx = \frac{y^{-1/2} 2^{\frac{1}{2}-\nu} \pi^{-1/2} \left[K_{\mu+\frac{1}{2}} (2^{-1/2} y) \right]^2}{\Gamma(\nu + \mu + \frac{3}{2}) \Gamma(\nu - \mu + \frac{1}{2})}$$

$$[-\frac{3}{2} - \operatorname{Re} \nu < \operatorname{Re} \mu < \operatorname{Re} \nu + \frac{1}{2}, \quad y > 0]$$

$$\text{ET II 44(1)}$$

$$17. \quad \int_0^\infty x^{1/2} (x^2 + 2)^{-\frac{1}{2}\nu - \frac{1}{4}} Q_\mu^{\nu+\frac{1}{2}} (x^2 + 1) J_\nu(xy) dx$$

$$= 2^{-\nu - \frac{1}{2}} \pi^{1/2} e^{(\nu+\frac{1}{2})\pi i} y^\nu K_{\mu+\frac{1}{2}} (2^{-1/2} y) I_{\mu+\frac{1}{2}} (2^{-1/2} y)$$

$$[\operatorname{Re} \nu > -1, \quad \operatorname{Re}(2\mu + \nu) > -\frac{5}{2}, \quad y > 0] \quad \text{ET II 46(12)}$$

$$\begin{aligned}
7.183 \quad \int_0^\infty x^{1-\mu} (1+a^2x^2)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\pm iax) J_\nu(xy) dx \\
= i(2\pi)^{1/2} e^{i\pi(\mu \mp \frac{1}{2}\nu \mp \frac{1}{4})} a^{-1} y^{\mu-1} I_\nu\left(\frac{1}{2}a^{-1}y\right) K_\mu\left(\frac{1}{2}a^{-1}y\right) \\
\left[-\frac{3}{4}-\frac{1}{2}\operatorname{Re}\nu < \operatorname{Re}\mu < 1+\operatorname{Re}\nu, \quad y > 0, \quad \operatorname{Re}a > 0\right] \quad \text{ET II 46(11)}
\end{aligned}$$

7.184

$$\begin{aligned}
1. \quad \int_1^\infty x^{1/2} (x^2-1)^{\frac{1}{2}\mu-\frac{1}{4}} P_{-\frac{1}{2}+\nu}^{-\frac{1}{2}-\mu}(x^{-1}) J_\nu(xa) dx = 2^{1/2} a^{-1-\mu} \pi^{-1/2} \cos\left[a + \frac{1}{2}(\nu-\mu)\pi\right] \\
[|\operatorname{Re}\mu| < \frac{1}{2}, \quad \operatorname{Re}\nu > -1, \quad a > 0] \quad \text{ET II 44(2)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_1^\infty x^{-\nu} (x^2-1)^{\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu-\frac{1}{2}}(2x^{-2}-1) K_\nu(ax) dx \\
= \pi^{1/2} 2^{-\nu} a^{-2+\nu} W_{\mu+\frac{1}{2}, \nu-\frac{1}{2}}(a) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a) \\
[\operatorname{Re}\nu < \frac{3}{2}, \quad a > 0] \quad \text{ET II 370(45)a}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^\nu (1+x^2)^{\frac{1}{4}+\frac{\nu}{2}} Q_\mu^{\nu+\frac{1}{2}}\left(1+\frac{2}{x^2}\right) J_\nu(ax) dx \\
= -ie^{i\pi\nu} \pi^{-\frac{1}{2}} 2^\nu a^{-\nu-2} \left[\Gamma\left(\frac{3}{2}+\mu+\nu\right)\right]^2 \Gamma\left(\frac{1}{2}+\nu-\mu\right) \\
\times W_{-\mu-\frac{1}{2}, \nu+\frac{1}{2}}(a) \left[\frac{\cos(\mu\pi)}{\Gamma(2+2\nu)} M_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) + \frac{\sin(\mu\pi)}{\Gamma\left(\nu+\mu+\frac{3}{2}\right)} W_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a)\right] \\
[a > 0, \quad \operatorname{Re}(\mu+\nu) > -\frac{3}{2}, \quad \operatorname{Re}(\mu-\nu) < \frac{1}{2}] \quad \text{ET II 46(14)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^1 x^\nu (1-x^2)^{\frac{1}{2}\nu+\frac{1}{4}} P_\mu^{-\nu-\frac{1}{2}}(2x^{-2}-1) J_\nu(xy) dx \\
= 2^{\nu+\frac{1}{2}} y^\nu \frac{\Gamma\left(\frac{3}{2}+\mu+\nu\right) \Gamma\left(\frac{1}{2}+\nu-\mu\right)}{(2\pi)^{1/2} \left[\Gamma\left(\frac{3}{2}+\nu\right)\right]^2} \\
\times {}_1F_1\left(\nu+\mu+\frac{3}{2}; 2\nu+2; iy\right) {}_1F_1\left(\nu+\mu+\frac{3}{2}; 2\nu+2; -iy\right) \\
[y > 0, \quad -\frac{3}{2}-\operatorname{Re}\nu < \operatorname{Re}\mu < \operatorname{Re}\nu+\frac{1}{2}] \quad \text{ET II 45(3)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{-\nu} (x^2+a^2)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}(1+2a^2x^{-2}) K_\nu(xy) dx \\
= ie^{-i\pi\nu} \pi^{1/2} 2^{-\nu-1} a^{-\nu-\frac{1}{2}} y^{\nu-2} \left[\Gamma\left(\frac{3}{2}+\mu-\nu\right)\right]^2 W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(ia y) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(-ia y) \\
[\operatorname{Re}a > 0, \quad \operatorname{Re}y > 0, \quad \operatorname{Re}\mu > -\frac{3}{2}, \quad \operatorname{Re}(\mu-\nu) > -\frac{3}{2}] \quad \text{ET II 137(13)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x^{-\nu} (x^2+1)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}(1+2x^{-2}) J_\nu(ax) dx \\
= 2^{-\nu} a^{-\nu-2} \frac{ie^{-i\pi\nu} \pi^{1/2} \Gamma\left(\frac{3}{2}+\mu-\nu\right)}{\Gamma(2\nu)} M_{\mu+\frac{1}{2}, \nu-\frac{1}{2}}(a) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a) \\
[a > 0, \quad 0 < \operatorname{Re}\nu < \operatorname{Re}\mu + \frac{3}{2}] \quad \text{ET II 47(15)a}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x^{-\nu} (x^2 + a^2)^{\frac{1}{4} - \frac{1}{2}\nu} Q_{-\frac{1}{2}}^{\frac{1}{2} - \nu} (1 + 2a^2 x^{-2}) K_\nu(xy) dx \\
= i e^{-i\pi\nu} \pi^{3/2} 2^{-\nu-3} a^{\frac{1}{2}-\nu} y^{\nu-1} [\Gamma(1-\nu)]^2 \times \left\{ \left[J_{\nu-\frac{1}{2}} \left(\frac{ay}{2} \right) \right]^2 + \left[Y_{\nu-\frac{1}{2}} \left(\frac{ay}{2} \right) \right]^2 \right\} \\
[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 136(12)}
\end{aligned}$$

$$\begin{aligned}
7.185 \quad \int_0^\infty x^{1/2} Q_{\nu-\frac{1}{2}} [(a^2 + x^2) x^{-1}] J_\nu(xy) dx = 2^{-1/2} \pi y^{-1} \exp \left[- (a^2 - \tfrac{1}{4})^{1/2} y \right] J_\nu(\tfrac{1}{2}y) \\
[\operatorname{Re} \nu > -\tfrac{1}{2}, \quad y > 0] \quad \text{ET II 46(10)}
\end{aligned}$$

$$\begin{aligned}
7.186 \quad \int_0^\infty x (1 + x^2)^{-\nu-1} P_\nu \left(\frac{1 - x^2}{1 + x^2} \right) J_0(xy) dx = y^{2\nu} [2^\nu \Gamma(\nu + 1)]^{-2} K_0(y) \\
[\operatorname{Re} \nu > 0] \quad \text{ET II 13(10)}
\end{aligned}$$

$$\begin{aligned}
7.187 \quad 1. \quad \int_0^\infty x P_\mu^\nu (\sqrt{1 + x^2}) K_\nu(xy) dx = y^{-3/2} S_{\nu+\frac{1}{2}, \mu+\frac{1}{2}}(y) \\
[\operatorname{Re} \nu < 1, \quad \operatorname{Re} y > 0] \quad \text{ET II 137(14)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x \left[P_{\lambda-\frac{1}{2}} (\sqrt{1 + a^2 x^2}) \right]^2 J_0(xy) dx = 2\pi^{-2} y^{-1} a^{-1} \cos(\lambda\pi) \left[K_\lambda \left(\frac{y}{2a} \right) \right]^2 \\
[\operatorname{Re} a > 0, \quad |\operatorname{Re} \lambda| < \tfrac{1}{4}, \quad y > 0] \\
\text{ET II 13(11)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x (1 + x^2)^{-1/2} P_\mu^\nu (\sqrt{1 + x^2}) K_\nu(xy) dx = y^{-1/2} S_{\nu-\frac{1}{2}, \mu+\frac{1}{2}}(y) \\
[\operatorname{Re} \nu < 1, \quad \operatorname{Re} y > 0] \quad \text{ET II 137(15)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x P_\mu^{-\frac{1}{2}\nu} (\sqrt{1 + a^2 x^2}) Q_\mu^{-\frac{1}{2}\nu} (\sqrt{1 + a^2 x^2}) J_\nu(xy) dx \\
= \frac{y^{-1} e^{-\frac{1}{2}\nu\pi i} \Gamma(1 + \mu + \frac{1}{2}\nu)}{a \Gamma(1 + \mu - \frac{1}{2}\nu)} I_{\mu+\frac{1}{2}} \left(\frac{y}{2a} \right) K_{\mu+\frac{1}{2}} \left(\frac{y}{2a} \right) \\
[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \mu > -\tfrac{3}{4}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 47(16)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x P_{\sigma-\frac{1}{2}}^\mu (\sqrt{1 + a^2 x^2}) Q_{\sigma-\frac{1}{2}}^\mu (\sqrt{1 + a^2 x^2}) J_0(xy) dx \\
= y^{-2} e^{\mu\pi i} \frac{\Gamma(\frac{1}{2} + \sigma - \mu)}{\Gamma(1 + 2\sigma)} W_{\mu, \sigma} \left(\frac{y}{a} \right) M_{-\mu, \sigma} \left(\frac{y}{a} \right) \\
[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \sigma > -\tfrac{1}{4}, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 14(15)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x P_{\sigma-\frac{1}{2}}^\mu (\sqrt{1 + a^2 x^2}) P_{\sigma-\frac{1}{2}}^{-\mu} (\sqrt{1 + a^2 x^2}) J_0(xy) dx \\
= 2\pi^{-1} y^{-2} \cos(\sigma\pi) W_{\mu, \sigma} \left(\frac{y}{a} \right) W_{-\mu, \sigma} \left(\frac{y}{a} \right) \\
[\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re} \sigma| < \tfrac{1}{4}] \quad \text{ET II 14(14)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x \left\{ P_{\sigma-\frac{1}{2}}^\mu (\sqrt{1 + a^2 x^2}) \right\}^2 J_0(xy) dx = -i\pi^{-1} y^{-2} W_{\mu, \sigma} \left(\frac{y}{a} \right) \left[W_{\mu, \sigma} \left(e^{\pi i} \frac{y}{a} \right) - W_{\mu, \sigma} \left(e^{-\pi i} \frac{y}{a} \right) \right] \\
[\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re} \sigma| < \tfrac{1}{4}, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 14(13)}
\end{aligned}$$

$$\begin{aligned}
 8. \quad \int_0^\infty x (1 + a^2 x^2)^{-1/2} P_\mu^{-\frac{1}{2} - \frac{1}{2}\nu} \left(\sqrt{1 + a^2 x^2} \right) P_\mu^{\frac{1}{2} - \frac{1}{2}\nu} \left(\sqrt{1 + a^2 x^2} \right) J_\nu(xy) dx \\
 = \frac{\left[K_{\mu+\frac{1}{2}} \left(\frac{y}{2a} \right) \right]^2}{\pi a^2 \Gamma \left(\frac{\nu}{2} + \mu + \frac{3}{2} \right) \Gamma \left(\frac{\nu}{2} - \mu + \frac{1}{2} \right)} \\
 [\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{5}{4} < \operatorname{Re} \mu < \frac{1}{4}] \quad \text{ET II 46(9)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_0^\infty x \left\{ P_\mu^{-\frac{1}{2}\nu} \left(\sqrt{1 + a^2 x^2} \right) \right\}^2 J_\nu(xy) dx = \frac{2 \left[K_{\mu+\frac{1}{2}} \left(\frac{y}{2a} \right) \right]^2 y^{-1}}{\pi a \Gamma \left(1 + \mu + \frac{1}{2}\nu \right) \Gamma \left(\frac{1}{2}\nu - \mu \right)} \\
 [\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{3}{4} < \operatorname{Re} \mu < -\frac{1}{4}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 45(7)}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \int_0^\infty x (1 + a^2 x^2)^{-1/2} P_\mu^{-\frac{1}{2}\nu} \left(\sqrt{1 + a^2 x^2} \right) P_{\mu+1}^{-\frac{1}{2}\nu} \left(\sqrt{1 + a^2 x^2} \right) J_\nu(xy) dx \\
 = \frac{K_{\mu+\frac{1}{2}} \left(\frac{y}{2a} \right) K_{\mu+\frac{3}{2}} \left(\frac{y}{2a} \right)}{\pi a^2 \Gamma \left(2 + \frac{1}{2}\nu + \mu \right) \Gamma \left(\frac{1}{2}\nu - \mu \right)} \\
 [\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{7}{4} < \operatorname{Re} \mu < -\frac{1}{4}] \quad \text{ET II 45(8)}
 \end{aligned}$$

7.188

$$\begin{aligned}
 1. \quad \int_0^\infty x (a^2 + x^2)^{-\frac{1}{2}\mu} P_{\mu-1}^{-\nu} \left[\frac{a}{\sqrt{a^2 + x^2}} \right] J_\nu(xy) dx = \frac{y^{\mu-2} e^{-ay}}{\Gamma(\mu + \nu)} \\
 [\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > \frac{1}{2}] \quad \text{ET II 45(4)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty x^{\nu+1} (x^2 + a^2)^{\frac{1}{2}\nu} P_\nu \left(\frac{x^2 + 2a^2}{2a\sqrt{x^2 + a^2}} \right) J_\nu(xy) dx = \frac{(2a)^{\nu+1} y^{-\nu-1}}{\pi \Gamma(-\nu)} \left[K_{\nu+\frac{1}{2}} \left(\frac{ya}{2} \right) \right]^2 \\
 [\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 0, \quad y > 0] \\
 \text{ET II 45(5)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty x^{1-\nu} (x^2 + a^2)^{-\frac{1}{2}\nu} P_{\nu-1} \left(\frac{x^2 + 2a^2}{2a\sqrt{x^2 + a^2}} \right) J_\nu(xy) dx = \frac{(2a)^{1-\nu} y^{\nu-1}}{\Gamma(\nu)} I_{\nu-\frac{1}{2}} \left(\frac{ay}{2} \right) K_{\nu-\frac{1}{2}} \left(\frac{ay}{2} \right) \\
 [\operatorname{Re} a > 0, \quad y > 0, \quad 0 < \operatorname{Re} \nu < 1] \\
 \text{ET II 45(6)}
 \end{aligned}$$

7.189

$$\begin{aligned}
 1. \quad \int_0^\infty (a+x)^\mu e^{-x} P_\nu^{-2\mu} \left(1 + \frac{2x}{a} \right) I_\mu(x) dx = 0 \\
 \left[-\frac{1}{2} < \operatorname{Re} \mu < 0, \quad -\frac{1}{2} + \operatorname{Re} \mu < \operatorname{Re} \nu < -\frac{1}{2} - \operatorname{Re} \mu \right] \quad \text{ET II 366(18)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu} \left(1 + \frac{2x}{a} \right) I_\mu(x) dx \\
 = \frac{2^{\mu-1} \Gamma \left(\mu + \nu + \frac{1}{2} \right) \Gamma \left(\mu - \nu - \frac{1}{2} \right) e^a}{\pi^{1/2} \Gamma(2\mu + \nu + 1) \Gamma(2\mu - \nu)} W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a) \\
 [|\arg a| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu + \frac{1}{2}|] \quad \text{ET II 367(19)}
 \end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^{-\mu} e^x P_\nu^{2\mu} \left(1 + \frac{2x}{a}\right) K_\mu(x+a) dx \\
= \pi^{-1/2} 2^{\mu-1} \cos(\mu\pi) \Gamma\left(\mu + \nu + \frac{1}{2}\right) \Gamma\left(\mu - \nu + \frac{1}{2}\right) W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a) \\
[|\arg a| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu + \frac{1}{2}|] \quad \text{ET II 373(11)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{-\frac{1}{2}\mu} (x+a)^{-1/2} e^{-x} P_{\nu-\frac{1}{2}}^\mu \left(\frac{a-x}{a+x}\right) K_\nu(a+x) dx = \sqrt{\frac{\pi}{2}} a^{-\frac{1}{2}\mu} \Gamma(\mu, 2a) \\
[a > 0, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 374(12)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{-2\mu-\frac{3}{2}} P_\nu^{-\mu} [\cosh(2x)] I_{\mu-\frac{1}{2}}(a \operatorname{sech} x) dx \\
= \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu-\nu) \Gamma(\mu+\nu+1)}{\pi^{1/2} a^{\mu+\frac{3}{2}} [\Gamma(\mu+1)]^2} M_{\nu+\frac{1}{2}, \mu}(a) M_{-\nu-\frac{1}{2}, \mu}(a) \\
[\operatorname{Re} \mu > \operatorname{Re} \nu, \quad \operatorname{Re} \mu > -\operatorname{Re} \nu - 1] \quad \text{ET II 378(44)}
\end{aligned}$$

7.19 Combinations of associated Legendre functions and functions generated by Bessel functions

7.191

$$\begin{aligned}
1. \quad \int_a^\infty x^{1/2} (x^2 - a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu+\frac{1}{2}} (2x^2 a^{-2} - 1) [\mathbf{H}_\nu(x) - Y_\nu(x)] dx \\
= 2^{-\nu-2} \pi^{1/2} a \operatorname{cosec}(\mu\pi) \cos(\nu\pi) \left\{ [Y_\nu(\tfrac{1}{2}a)]^2 - [J_\nu(\tfrac{1}{2}a)]^2 \right\} \\
[-1 < \operatorname{Re} \mu < 0, \quad \operatorname{Re} \nu < \tfrac{1}{2}] \quad \text{ET II 384(6)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{1/2} (x^2 - a^2)^{-1/4-\nu/2} P_\mu^{\nu+1/2} (2x^2 a^{-2} - 1) [I_{-\nu}(x) - \mathbf{L}_\nu(x)] dx \\
= 2^{-\nu-1} \pi^{1/2} a \operatorname{cosec}(2\mu\pi) \cos(\nu\pi) \left\{ [I_\nu(\tfrac{1}{2}a)]^2 - [I_{-\nu}(\tfrac{1}{2}a)]^2 \right\} \\
[-1 < \operatorname{Re} \mu < 0, \quad \operatorname{Re} \nu < \tfrac{1}{2}] \quad \text{ET II 385(15)}
\end{aligned}$$

7.192

$$\begin{aligned}
1. \quad \int_0^1 x^{(\nu-\mu-1)/2} (1-x^2)^{(\nu-\mu-2)/4} P_{\nu-1/2}^{(\mu-\nu+2)/2}(x) S_{\mu, \nu}(ax) dx \\
= 2^{\mu-3/2} \pi^{1/2} a^{-(\nu-\mu-1)/2} \Gamma\left(\frac{\mu+\nu+3}{4}\right) \Gamma\left(\frac{\mu-3\nu+3}{4}\right) \cos\left(\frac{\mu-\nu}{2}\pi\right) \\
\times [J_\nu(\tfrac{1}{2}a) Y_{-(\mu-\nu+1)/2}(\tfrac{1}{2}a) - Y_\nu(\tfrac{1}{2}a) J_{-(\mu-\nu+1)/2}(\tfrac{1}{2}a)] \\
[\operatorname{Re}(\mu-\nu) < 0, \quad a > 0, \quad |\operatorname{Re}(\mu+\nu)| < 1, \quad \operatorname{Re}(\mu-3\nu) < 1] \quad \text{ET II 387(24)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_1^\infty x^{1/2} (x^2 - 1)^{-\beta/2} P_\nu^\beta(x) S_{\mu,1/2}(ax) dx \\
&= \frac{2^{-3/2+\beta-\mu} a^{\beta-1} \Gamma\left(\frac{\beta-\mu+\nu}{2} + \frac{1}{4}\right) \Gamma\left(\frac{\beta-\mu-\nu}{2} - \frac{1}{4}\right)}{\pi^{1/2} \Gamma\left(\frac{1}{2} - \mu\right)} S_{\mu-\beta+1, \nu+1/2}(a) \\
& \quad [\operatorname{Re} \beta < 1, \quad a > 0, \quad \operatorname{Re}(\mu + \nu - \beta) < -\frac{1}{2}, \quad \operatorname{Re}(\mu - \nu - \beta) < \frac{1}{2}] \quad \text{ET II 387(25)a}
\end{aligned}$$

7.193

$$\begin{aligned}
1. \quad & \int_1^\infty x^{-\nu} (x^2 - 1)^{1/4-\nu/2} P_{\mu/2-\nu/2}^{\nu-1/2}(2x^{-2} - 1) S_{\mu,\nu}(ax) dx \\
&= \frac{2^{\mu-\nu} a^{\nu-2} \pi^{1/2} \Gamma\left(\frac{3\nu-\mu-1}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)} W_{\rho,\sigma}\left(ae^{i\pi/2}\right) W_{\rho,\sigma}\left(ae^{-i\pi/2}\right) \\
& \quad \rho = \frac{1}{2}(\mu + 1 - \nu), \quad \sigma = \nu - \frac{1}{2}, \quad [\operatorname{Re}(\mu - \nu) < 0, \quad a > 0, \quad \operatorname{Re} \nu < \frac{3}{2}, \quad \operatorname{Re}(3\nu - \mu) > 1] \\
& \quad \text{ET II 387(27)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_1^\infty x (x^2 - 1)^{-\nu/2} P_\lambda^\nu(2x^2 - 1) S_{\mu,\nu}(ax) dx \\
&= \frac{a^{\nu-1} \Gamma\left(\frac{\nu-\mu+1}{2} + \lambda\right) \Gamma\left(\frac{\nu-\mu-1}{2} - \lambda\right)}{2 \Gamma\left(\frac{1-\mu-\nu}{2}\right) \Gamma\left(\frac{1-\mu+\nu}{2}\right)} S_{\mu-\nu+1, 2\lambda+1}(a) \\
& \quad [\operatorname{Re} \nu < 1, \quad a > 0, \quad \operatorname{Re}(\mu - \nu + \lambda) < -1, \quad \operatorname{Re}(\mu - \nu + \lambda) < 0] \quad \text{ET II 387(26)a}
\end{aligned}$$

7.21 Integration of associated Legendre functions with respect to the order**7.211**

$$\begin{aligned}
1. \quad & \int_0^\infty P_{-x-\frac{1}{2}}(\cos \theta) dx = \frac{1}{2} \operatorname{cosec}\left(\frac{1}{2}\theta\right) \quad [0 < \theta < \pi] \quad \text{ET II 329(19)} \\
2. \quad & \int_{-\infty}^\infty P_x(\cos \theta) dx = \operatorname{cosec}\left(\frac{1}{2}\theta\right) \quad [0 < \theta < \pi] \quad \text{ET II 329(20)}
\end{aligned}$$

$$\begin{aligned}
7.212 \quad & \int_0^\infty x^{-1} \tanh(\pi x) P_{-\frac{1}{2}+ix}(\cosh a) dx = 2e^{-\frac{1}{2}a} K(e^{-a}) \\
& \quad [a > 0] \quad \text{ET II 330(22)}
\end{aligned}$$

$$\begin{aligned}
7.213 \quad & \int_0^\infty \frac{x \tanh(\pi x)}{a^2 + x^2} P_{-\frac{1}{2}+ix}(\cosh b) dx = Q_{a-\frac{1}{2}}(\cosh b) \quad [\operatorname{Re} a > 0] \quad \text{ET II 387(23)}
\end{aligned}$$

$$\begin{aligned}
7.214 \quad & \int_0^\infty \sinh(\pi x) \cos(ax) P_{-\frac{1}{2}+ix}(b) dx = \frac{1}{\sqrt{2(b + \cosh a)}} \\
& \quad [a > 0, \quad |b| < 1] \quad \text{ET I 42(27)}
\end{aligned}$$

$$\begin{aligned}
7.215 \quad & \int_0^\infty \cos(bx) P_{-\frac{1}{2}+ix}^\mu(\cosh a) dx = 0 \quad [0 < a < b] \\
&= \frac{\sqrt{\frac{\pi}{2}} (\sinh a)^\mu}{\Gamma\left(\frac{1}{2} - \mu\right) (\cosh a - \cosh b)^{\mu+\frac{1}{2}}} \quad [0 < b < a] \\
& \quad \text{ET II 330(21)}
\end{aligned}$$

$$7.216 \quad \int_0^\infty \cos(bx) \Gamma(\mu + ix) \Gamma(\mu - ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\mu} (\cosh a) dx = \frac{\sqrt{\frac{\pi}{2}} \Gamma(\mu) (\sinh a)^{\mu-\frac{1}{2}}}{(\cosh a + \cosh b)^\mu} \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 330(24)}$$

$$7.217 \quad 1. \quad \int_{-\infty}^\infty \left(\nu - \frac{1}{2} + ix\right) \Gamma\left(\frac{1}{2} - ix\right) \Gamma\left(2\nu - \frac{1}{2} + ix\right) P_{\nu+ix-1}^{\frac{1}{2}-\nu} (\cos \theta) I_{\nu-\frac{1}{2}+ix}(a) K_{\nu-\frac{1}{2}+ix}(b) dx \\ = \sqrt{2\pi} (\sin \theta)^{\nu-\frac{1}{2}} \left(\frac{ab}{\omega}\right)^\nu K_\nu(\omega) \\ \left[\omega = (a^2 + b^2 + 2ab \cos \theta)^{1/2}\right] \quad \text{ET II 383(29)}$$

$$2. \quad \int_0^\infty x e^{\pi x} \tanh(\pi x) P_{-\frac{1}{2}+ix}(-\cos \theta) H_{ix}^{(2)}(ka) H_{ix}^{(2)}(kb) dx = -\frac{2(ab)^{1/2}}{\pi R} e^{-ikR}; \\ R = (a^2 + b^2 - 2ab \cos \theta)^{1/2} \quad [a > 0, \quad b > 0, \quad 0 < \theta < \pi, \quad \operatorname{Im} k \leq 0] \quad \text{ET II 381(17)}$$

$$3. \quad \int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\nu} (-\cos \theta) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx \\ = i(2\pi)^{1/2} (\sin \theta)^{\nu-\frac{1}{2}} \left(\frac{ab}{R}\right)^\nu H_\nu^{(2)}(R) \\ R = (a^2 + b^2 - 2ab \cos \theta)^{1/2} \quad [a > 0, \quad b > 0, \quad 0 < \theta < \pi, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 381 (18)}$$

$$4. \quad \int_0^\infty x \sinh(\pi x) \Gamma(\lambda + ix) \Gamma(\lambda - ix) K_{ix}(a) K_{ix}(b) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\lambda}(\beta) dx = \frac{\pi^{1/2}}{\sqrt{2}} \left(\frac{ab}{z}\right)^\lambda (\beta^2 - 1)^{\frac{1}{2}\lambda - \frac{1}{4}} K_\lambda(z) \\ z = \sqrt{a^2 + b^2 + 2ab\beta} \quad \left[|\arg a| < \frac{\pi}{2}, \quad |\arg(\beta - 1)| < \pi, \quad \operatorname{Re} \lambda > 0\right] \quad \text{ET II 177(16)}$$

7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions

7.221

$$1. \quad \int_{-1}^1 P_n(x) P_m(x) dx = 0 \quad [m \neq n] \\ = \frac{2}{2n+1} \quad [m = n] \quad \text{WH, EH I 170(8, 10)}$$

$$2.^6 \quad \int_0^1 P_n(x) P_m(x) dx = \frac{1}{2n+1} \quad [m = n] \\ = 0 \quad [n - m \text{ is even, } m \neq n] \\ = \frac{(-1)^{\frac{1}{2}(m+n-1)} m! n!}{2^{m+n-1} (m-n) (n+m+1) \left[\left(\frac{n}{2}\right)! \left(\frac{m-1}{2}\right)!\right]^2} \quad [n \text{ is even, } m \text{ is odd}]$$

WH

$$3. \quad \int_0^{2\pi} P_{2n}(\cos \varphi) d\varphi = 2\pi \left[\binom{2n}{n} 2^{-2n}\right]^2. \quad \text{MO 70, EH II 183(50)}$$

7.222

1. $\int_{-1}^1 x^m P_n(x) dx = 0 \quad [m < n]$
2. $\int_{-1}^1 (1+x)^{m+n} P_m(x) P_n(x) dx = \frac{2^{m+n+1} [(m+n)!]^4}{(m!n!)^2 (2m+2n+1)!} \quad \text{ET II 277(15)}$
3. $\int_{-1}^1 (1+x)^{m-n-1} P_m(x) P_n(x) dx = 0 \quad [m > n] \quad \text{ET II 278(16)}$
4. $\int_{-1}^1 (1-x^2)^n P_{2m}(x) dx = \frac{2n^2}{(n-m)(2m+2n+1)} \int_{-1}^1 (1-x^2)^{n-1} P_{2m}(x) dx$
 $[m < n] \quad \text{WH}$
5. $\int_0^1 x^2 P_{n+1}(x) P_{n-1}(x) dx = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)} \quad \text{WH}$

7.223 $\int_{-1}^1 \frac{1}{z-x} \{P_n(x) P_{n-1}(x) - P_{n-1}(x) P_n(z)\} dx = -\frac{2}{n} \quad \text{WH}$

7.224 $[z \text{ belongs to the complex plane with a discontinuity along the interval from } -1 \text{ to } +1.]$

1. $\int_{-1}^1 (z-x)^{-1} P_n(x) dx = 2 Q_n(z) \quad \text{ET II 277(7)}$
2. $\int_{-1}^1 x(z-x)^{-1} P_0(x) dx = 2 Q_1(z) \quad \text{ET II 277(8)}$
3. $\int_{-1}^1 x^{n+1}(z-x)^{-1} P_n(x) dx = 2z^{n+1} Q_n(z) - \frac{2^{n+1} (n!)^2}{(2n+1)!} \quad \text{ET II 277(9)}$
4. $\int_{-1}^1 x^m(z-x)^{-1} P_n(x) dx = 2z^m Q_n(z) \quad [m \leq n] \quad \text{ET II 277(10)a}$
5. $\int_{-1}^1 (z-x)^{-1} P_m(x) P_n(x) dx = 2 P_m(z) Q_n(z) \quad [m \leq n] \quad \text{ET II 278(18)a}$
6. $\int_{-1}^1 (z-x)^{-1} P_n(x) P_{n+1}(x) dx = 2 P_{n+1}(z) Q_n(z) - \frac{2}{n+1} \quad \text{ET II 278(19)}$
7. $\int_{-1}^1 x(z-x)^{-1} P_m(x) P_n(x) dx = 2z P_m(z) Q_n(z) \quad [m < n] \quad \text{ET II 278(21)}$
8. $\int_{-1}^1 x(z-x)^{-1} [P_n(x)]^2 dx = 2z P_n(z) Q_n(z) - \frac{2}{2n+1} \quad \text{ET II 278(20)}$

7.225

1. $\int_{-1}^x (x-t)^{-1/2} P_n(t) dt = \left(n + \frac{1}{2}\right)^{-1} (1+x)^{-1/2} [T_n(x) + T_{n+1}(x)] \quad \text{EH II 187(43)}$
2. $\int_x^1 (t-x)^{-1/2} P^{-1/2} P_n(t) dt = \left(n + \frac{1}{2}\right)^{-1} (1-x)^{-1/2} [T_n(x) - T_{n+1}(x)] \quad \text{EH II 187(44)}$

$$3. \quad \int_{-1}^1 (1-x)^{-1/2} P_n(x) dx = \frac{2^{3/2}}{2n+1} \quad \text{EH II 183(49)}$$

$$4. \quad \int_{-1}^1 (\cosh 2p - x)^{-1/2} P_n(x) dx = \frac{2\sqrt{2}}{2n+1} \exp[-(2n+1)p] \\ [p > 0] \quad \text{WH}$$

$$5.^{10} \quad \frac{1}{2} \int_{-1}^1 \frac{P_\ell(z) dz}{\sqrt{(xy-z)^2 - (x^2-1)(y^2-1)}} = P_\ell(x) Q_\ell(y) \quad (1 < x \leq y) \\ = P_\ell(y) Q_\ell(x) \quad (1 < y \leq x)$$

7.226

$$1. \quad \int_{-1}^1 (1-x^2)^{-1/2} P_{2m}(x) dx = \left[\frac{\Gamma(\frac{1}{2} + m)}{m!} \right]^2 \quad \text{ET II 276(4)}$$

$$2. \quad \int_{-1}^1 x (1-x^2)^{-1/2} P_{2m+1}(x) dx = \frac{\Gamma(\frac{1}{2} + m) \Gamma(\frac{3}{2} + m)}{m!(m+1)!} \quad \text{ET II 276(5)}$$

$$3. \quad \int_{-1}^1 (1+px^2)^{-m-3/2} P_{2m}(x) dx = \frac{2}{2m+1} (-p)^m (1+p)^{-m-1/2} \\ [|p| < 1] \quad \text{MO 71}$$

$$7.227 \quad \int_0^1 x (a^2 + x^2)^{-1/2} P_n(1-2x^2) dx = \frac{[a + (a^2 + 1)^{1/2}]^{-2n-1}}{2n+1} \\ [\operatorname{Re} a > 0] \quad \text{ET II 278(23)}$$

$$7.228^6 \quad \frac{1}{2} \Gamma(1+\mu) \int_{-1}^1 P_l(x) (z-x)^{-\mu-1} dx = (z^2-1)^{-\mu/2} e^{-i\pi\mu} Q_l^\mu(z) \\ [l = 0, 1, 2, \dots, \quad |\arg(z-1)| < \pi]$$

7.23 Combinations of Legendre polynomials and powers**7.231**

$$1. \quad \int_0^1 x^\lambda P_{2m}(x) dx = \frac{(-1)^m \Gamma(m - \frac{1}{2}\lambda) \Gamma(\frac{1}{2} + \frac{1}{2}\lambda)}{2 \Gamma(-\frac{1}{2}\lambda) \Gamma(m + \frac{3}{2} + \frac{1}{2}\lambda)} \quad [\operatorname{Re} \lambda > -1] \quad \text{EH II 183(51)}$$

$$2.^6 \quad \int_0^1 x^\lambda P_{2m+1}(x) dx = \frac{(-1)^m \Gamma(m + \frac{1}{2} - \frac{1}{2}\lambda) \Gamma(1 + \frac{1}{2}\lambda)}{2 \Gamma(\frac{1}{2} - \frac{1}{2}\lambda) \Gamma(m + 2 + \frac{1}{2}\lambda)} \\ [\operatorname{Re} \lambda > -2] \quad \text{EH II 183(52)}$$

7.232

$$1. \quad \int_{-1}^1 (1-x)^{a-1} P_m(x) P_n(x) dx \\ = \frac{2^a \Gamma(a) \Gamma(n-a+1)}{\Gamma(1-a) \Gamma(n+a+1)} {}_4F_3(-m, m+1, a, a; 1, a+n+1, a-n; 1) \\ [\operatorname{Re} a > 0] \quad \text{ET II 278(17)}$$

$$2. \quad \int_{-1}^1 (1-x)^{a-1} (1+x)^{b-1} P_n(x) dx = \frac{2^{a+b-1} \Gamma(a) \Gamma(b)}{\Gamma(a+b)} {}_3F_2(-n, 1+n, a; 1, a+b; 1)$$

[Re $a > 0$, Re $b > 0$] ET II 276(6)

$$3. \quad \int_0^1 (1-x)^{\mu-1} P_n(1-\gamma x) dx = \frac{\Gamma(\mu)n!}{\Gamma(\mu+n+1)} P_n^{(\mu, -\mu)}(1-\gamma)$$

[Re $\mu > 0$] ET II 190(37)a

$$4. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} P_n(1-\gamma x) dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_3F_2\left(-n, n+1, \nu; 1, \mu+\nu; \frac{1}{2}\gamma\right)$$

[Re $\mu > 0$, Re $\nu > 0$] ET II 190(38)

$$7.233 \quad \int_0^1 x^{2\mu-1} P_n(1-2x^2) dx = \frac{(-1)^n [\Gamma(\mu)]^2}{2\Gamma(\mu+n+1)\Gamma(\mu-n)}$$

[Re $\mu > 0$] ET II 278(22)

7.24 Combinations of Legendre polynomials and other elementary functions

$$7.241 \quad \int_0^\infty P_n(1-x)e^{-ax} dx = e^{-a} a^n \left(\frac{1}{a} \frac{d}{da}\right)^n \left(\frac{e^a}{a}\right)$$

$$= a^n \left(1 + \frac{1}{2} \frac{d}{da}\right)^n \left(\frac{1}{a^{n+1}}\right)$$

[Re $a > 0$] ET I 171(2)

$$7.242 \quad \int_0^\infty P_n(e^{-x}) e^{-ax} dx = \frac{(a-1)(a-2)\cdots(a-n+1)}{(a+n)(a+n-2)\cdots(a-n+2)}$$

[$n \geq 2$, Re $a > 0$] ET I 171(3)

$$7.243 \quad 1. \quad \int_0^\infty P_{2n}(\cosh x) e^{-ax} dx = \frac{(a^2-1^2)(a^2-3^2)\cdots[a^2-(2n-1)^2]}{a(a^2-2^2)(a^2-4^2)\cdots[a^2-(2n)^2]}$$

[Re $a > 2n$] ET I 171(6)

$$2. \quad \int_0^\infty P_{2n+1}(\cosh x) e^{-ax} dx = \frac{a(a^2-2^2)(a^2-4^2)\cdots[a^2-(2n)^2]}{(a^2-1)(a^2-3^2)\cdots[a^2-(2n+1)^2]}$$

[Re $a > 2n+1$] ET I 171(7)

$$3. \quad \int_0^\infty P_{2n}(\cos x) e^{-ax} dx = \frac{(a^2+1^2)(a^2+3^2)\cdots[a^2+(2n-1)^2]}{a(a^2+2^2)(a^2+4^2)\cdots[a^2+(2n)^2]}$$

[Re $a > 0$] ET I 171(4)

$$4. \quad \int_0^\infty P_{2n+1}(\cos x) e^{-ax} dx = \frac{a(a^2+2^2)(a^2+4^2)\cdots[a^2+(2n)^2]}{(a^2+1^2)(a^2+3^2)\cdots[a^2+(2n+1)^2]}$$

[Re $a > 0$] ET I 171(5)

$$5.^{11} \quad \int_{-1}^1 e^{ix\alpha} P_n(x) dx = i^n \sqrt{\frac{2\pi}{\alpha}} J_{n+\frac{1}{2}}(\alpha)$$

[$n = 0, 1, 2, \dots$, $a > 0$]

7.244

$$1. \quad \int_0^1 P_n(1-2x^2) \sin ax \, dx = \frac{\pi}{2} \left[J_{n+\frac{1}{2}} \left(\frac{a}{2} \right) \right]^2 \quad [a > 0] \quad \text{ET I 94(2)}$$

$$2. \quad \int_0^1 P_n(1-2x^2) \cos ax \, dx = \frac{\pi}{2} (-1)^n J_{n+\frac{1}{2}} \left(\frac{a}{2} \right) J_{-n-\frac{1}{2}} \left(\frac{a}{2} \right) \quad [a > 0] \quad \text{ET I 38(1)}$$

7.245

$$1. \quad \int_0^{2\pi} P_{2m+1}(\cos \theta) \cos \theta \, d\theta = \frac{\pi}{2^{4m+1}} \binom{2m}{m} \binom{2m+2}{m+1} \quad \text{MO 70, EH II 183(5)}$$

$$2. \quad \int_0^\pi P_m(\cos \theta) \sin n\theta \, d\theta = \frac{2(n-m+1)(n-m+3) \cdots (n+m-1)}{(n-m)(n-m+2) \cdots (n+m)} \quad [n > m \text{ and } n+m \text{ is odd}]$$

$$= 0 \quad [n \leq m \text{ or } n+m \text{ is even}]$$

MO 71

$$3.^{10} \quad \int_0^{2\pi} P_{2n+1}(\sin \alpha \sin \phi) \sin \phi \, d\phi = (-1)^{n+1} \frac{2\sqrt{\pi} \Gamma(n+\frac{3}{2})}{(2n+1) \Gamma(n+2)} P_{2n+1}^1(\cos \alpha)$$

$$[\alpha \neq \frac{1}{2}(2n+1)\pi, \quad n \text{ an integer}]$$

$$4. \quad \int_{-1}^1 \cos(\alpha x) P_n(x) \, dx = 0 \quad [n \text{ is odd}]$$

$$= (-1)^v \sqrt{\frac{2\pi}{\alpha}} J_{2v+\frac{1}{2}}(\alpha) \quad [n = 2v \text{ is even}]$$

GH2 24 (171.10a)

$$\mathbf{7.246} \quad \int_0^\pi P_n(1-2\sin^2 x \sin^2 \theta) \sin x \, dx = \frac{2 \sin(2n+1)\theta}{(2n+1) \sin \theta} \quad \text{MO 71}$$

$$\mathbf{7.247} \quad \int_0^1 P_{2n+1}(x) \sin ax \frac{dx}{\sqrt{x}} = (-1)^{n+1} \sqrt{\frac{\pi}{2a}} J_{2n+\frac{3}{2}}(a) \quad [a > 0] \quad \text{ET I 94(1)}$$

7.248

$$1. \quad \int_{-1}^1 (a^2 + b^2 - 2abx)^{-1/2} \sin \left[\lambda (a^2 + b^2 - 2abx)^{1/2} \right] P_n(x) \, dx = \pi(ab)^{-1/2} J_{n+\frac{1}{2}}(a\lambda) J_{n+\frac{1}{2}}(b\lambda)$$

$$[a > 0, \quad b > 0] \quad \text{ET II 277(11)}$$

$$2. \quad \int_{-1}^1 (a^2 + b^2 - 2abx)^{-1/2} \cos \left[\lambda (a^2 + b^2 - 2abx)^{1/2} \right] P_n(x) \, dx = -\pi(ab)^{-1/2} J_{n+\frac{1}{2}}(a\lambda) Y_{n+\frac{1}{2}}(b\lambda)$$

$$[0 \leq a \leq b] \quad \text{ET II 277(12)}$$

7.249

$$1. \quad \int_{-1}^1 P_n(x) \arcsin x \, dx = 0 \quad [n \text{ is even}]$$

$$= \pi \left\{ \frac{(n-2)!!}{2^{\frac{1}{2}(n+1)} \left(\frac{n+1}{2} \right)!} \right\}^2 \quad [n \text{ is odd}]$$

$$2. \quad P_n(x) = \frac{1}{t} \sum_{t=0}^{t-1} \left(x + \sqrt{x^2 - 1} \cos \frac{2\pi r}{t} \right)^n \quad [t > n]$$

7.25 Combinations of Legendre polynomials and Bessel functions

7.251

$$1. \quad \int_0^1 x P_n(1-2x^2) Y_\nu(xy) dx = \pi^{-1} y^{-1} [S_{2n+1}(y) + \pi Y_{2n+1}(y)]$$

[$n = 0, 1, \dots; \quad y > 0, \quad \nu > 0$] ET II 108(1)

$$2. \quad \int_0^1 x P_n(1-2x^2) K_0(xy) dx = y^{-1} \left[(-1)^{n+1} K_{2n+1}(y) + \frac{i}{2} S_{2n+1}(iy) \right]$$

[$y > 0$] ET II 134(1)

$$3. \quad \int_0^1 x P_n(1-2x^2) J_0(xy) dx = y^{-1} J_{2n+1}(y) \quad [y > 0] \quad \text{ET II 13(1)}$$

$$4. \quad \int_0^1 x P_n(1-2x^2) [J_0(ax)]^2 dx = \frac{1}{2(2n+1)} \left\{ [J_n(a)]^2 + [J_{n+1}(a)]^2 \right\} \quad \text{ET II 338(39)a}$$

$$5. \quad \int_0^1 x P_n(1-2x^2) J_0(ax) Y_0(ax) dx = \frac{1}{2(2n+1)} [J_n(a) Y_n(a) + J_{n+1}(a) Y_{n+1}(a)]$$

ET II 339(48)a

$$6. \quad \int_0^1 x^2 P_n(1-2x^2) J_1(xy) dx = y^{-1} (2n+1)^{-1} [(n+1) J_{2n+2}(y) - n J_{2n}(y)]$$

[$y > 0$] ET II 20(23)

$$7. \quad \int_0^1 x^{\mu-1} P_n(2x^2-1) J_\nu(ax) dx = \frac{2^{-\nu-1} a^\nu [\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu)]^2}{\Gamma(\nu+1) \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + n+1) \Gamma(\frac{1}{2} + \frac{1}{2}\nu - n)}$$

$$\times {}_2F_3 \left(\frac{\mu+\nu}{2}, \frac{\mu+\nu}{2}; \nu+1, \frac{\mu+\nu}{2} + n+1, \frac{\mu+\nu}{2} - n; -\frac{a^2}{4} \right)$$

[$a > 0, \quad \text{Re}(\mu+\nu) > 0$] ET II 337(32)a

$$\mathbf{7.252} \quad \int_0^1 e^{-ax} P_n(1-2x) I_0(ax) dx = \frac{e^{-a}}{2n+1} [I_n(a) + I_{n+1}(a)]$$

[$a > 0$] ET II 366(11)a

$$\mathbf{7.253} \quad \int_0^{\pi/2} \sin(2x) P_n(\cos 2x) J_0(a \sin x) dx = a^{-1} J_{2n+1}(a) \quad \text{ET II 361(20)}$$

$$\mathbf{7.254} \quad \int_0^1 x P_n(1-2x^2) [I_0(ax) - \mathbf{I}_0(ax)] dx = (-1)^n [I_{2n+1}(a) - \mathbf{I}_{2n+1}(a)]$$

[$a > 0$] ET II 385(14)a

7.3–7.4 Orthogonal Polynomials

7.31 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and powers

7.311

$$1. \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = 0 \quad \left[n > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 280(1)}$$

$$2. \quad \int_0^1 x^{n+2\rho} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\Gamma(2\nu+n) \Gamma(2\rho+n+1) \Gamma(\nu+\frac{1}{2}) \Gamma(\rho+\frac{1}{2})}{2^{n+1} \Gamma(2\nu) \Gamma(2\rho+1) n! \Gamma(n+\nu+\rho+1)} \\ \left[\operatorname{Re} \rho > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 280(2)}$$

$$3. \quad \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^\beta C_n^\nu(x) dx = \frac{2^{\beta+\nu+\frac{1}{2}} \Gamma(\beta+1) \Gamma(\nu+\frac{1}{2}) \Gamma(2\nu+n) \Gamma(\beta-\nu+\frac{3}{2})}{n! \Gamma(2\nu) \Gamma(\beta-\nu-n+\frac{3}{2}) \Gamma(\beta+\nu+n+\frac{3}{2})} \\ \left[\operatorname{Re} \beta > -1, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 280(3)}$$

$$4. \quad \int_{-1}^1 (1-x)^\alpha (1+x)^\beta C_n^\nu(x) dx = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1) \Gamma(n+2\nu)}{n! \Gamma(2\nu) \Gamma(\alpha+\beta+2)} \\ \times {}_3F_2 \left(-n, n+2\nu, \alpha+1; \nu+\frac{1}{2}, \alpha+\beta+2; 1 \right) \\ \left[\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1 \right] \quad \text{ET II 281(4)}$$

7.312 In the following integrals, z belongs to the complex plane with a cut along the interval of the real axis from -1 to 1 .

$$1. \quad \int_{-1}^1 x^m (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\pi^{1/2} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^m (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \\ \left[m \leq n, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 281(5)}$$

$$2. \quad \int_{-1}^1 x^{n+1} (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\pi^{1/2} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^{n+1} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \\ - \frac{\pi 2^{1-2\nu-n} n!}{\Gamma(\nu) \Gamma(\nu+n+1)} \\ \left[\operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 281(6)}$$

$$3.^6 \quad \int_{-1}^1 (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = \frac{\pi^{1/2} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} C_m^\nu(z) Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \\ \left[m \leq n, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \quad \text{ET II 283(17)}$$

7.313

$$1. \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = 0 \quad \left[m \neq n, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \\ \text{ET II 282(12), MO 98a, EH I 177(16)}$$

$$2. \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi 2^{1-2\nu} \Gamma(2\nu+n)}{n! (n+\nu) [\Gamma(\nu)]^2} \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right] \\ \text{ET II 281(8), MO 98a, EH I 177(17)}$$

7.314

$$1. \quad \int_{-1}^1 (1-x)^{\nu-\frac{3}{2}} (1+x)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi^{1/2} \Gamma(\nu - \frac{1}{2}) \Gamma(2\nu + n)}{n! \Gamma(\nu) \Gamma(2\nu)} \quad [\operatorname{Re} \nu > \frac{1}{2}] \quad \text{ET II 281(9)}$$

$$2. \quad \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{2\nu-1} [C_n^\nu(x)]^2 dx = \frac{2^{3\nu-\frac{1}{2}} [\Gamma(2\nu + n)]^2 \Gamma(2n + \nu + \frac{1}{2})}{(n!)^2 \Gamma(2\nu) \Gamma(3\nu + 2n + \frac{1}{2})} \quad [\operatorname{Re} \nu > 0] \quad \text{ET II 282(10)}$$

$$3. \quad \int_{-1}^1 (1-x)^{3\nu+2n-\frac{3}{2}} (1+x)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi^{1/2} [\Gamma(\nu + \frac{1}{2})]^2 \Gamma(\nu + 2n + \frac{1}{2}) \Gamma(2\nu + 2n) \Gamma(3\nu + 2n - \frac{1}{2})}{2^{2\nu+2n} [n! \Gamma(\nu + n + \frac{1}{2}) \Gamma(2\nu)]^2 \Gamma(2\nu + 2n + \frac{1}{2})} \quad [\operatorname{Re} \nu > \frac{1}{6}] \quad \text{ET II 282(11)}$$

$$4. \quad \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{\nu+m-n-\frac{3}{2}} C_m^\nu(x) C_n^\nu(x) dx = (-1)^m \frac{2^{2-2\nu-m+n} \pi^{3/2} \Gamma(2\nu + n)}{m!(n-m)! [\Gamma(\nu)]^2 \Gamma(\frac{1}{2} + \nu + m)} \frac{\Gamma(\nu - \frac{1}{2} + m - n) \Gamma(\frac{1}{2} - \nu + m - n)}{\Gamma(\frac{1}{2} - \nu - n) \Gamma(\frac{1}{2} + m - n)} \quad [\operatorname{Re} \nu > -\frac{1}{2}; \quad n \geq m] \quad \text{ET II 282(13)a}$$

$$5. \quad \int_{-1}^1 (1-x)^{2\nu-1} (1+x)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = \frac{2^{3\nu-\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) \Gamma(2\nu + m) \Gamma(2\nu + n)}{m!n! \Gamma(2\nu) \Gamma(\frac{1}{2} - \nu)} \frac{\Gamma(\nu + \frac{1}{2} + m + n) \Gamma(\frac{1}{2} - \nu + n - m)}{\Gamma(\nu + \frac{1}{2} + n - m) \Gamma(3\nu + \frac{1}{2} + m + n)} \quad [\operatorname{Re} \nu > 0] \quad \text{ET II 282(14)}$$

$$6. \quad \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{3\nu+m+n-\frac{3}{2}} C_m^\nu(x) C_n^\nu(x) dx = \frac{2^{4\nu+m+n-1} [\Gamma(\nu + \frac{1}{2}) \Gamma(2\nu + m + n)]^2 \Gamma(\nu + m + n + \frac{1}{2}) \Gamma(3\nu + m + n - \frac{1}{2})}{\Gamma(\nu + m + \frac{1}{2}) \Gamma(\nu + n + \frac{1}{2}) \Gamma(2\nu + m) \Gamma(2\nu + n) \Gamma(4\nu + 2m + 2n)} \quad [\operatorname{Re} \nu > \frac{1}{6}] \quad \text{ET II 282(15)}$$

$$7. \quad \int_{-1}^1 (1-x)^\alpha (1+x)^{\nu-\frac{1}{2}} C_m^\mu(x) C_n^\nu(x) dx = \frac{2^{\alpha+\nu+\frac{1}{2}} \Gamma(\alpha + 1) \Gamma(\nu + \frac{1}{2}) \Gamma(\nu - \alpha + n - \frac{1}{2}) \Gamma(2\mu + m) \Gamma(2\nu + n)}{m!n! \Gamma(\nu - \alpha - \frac{1}{2}) \Gamma(\nu - \alpha + n + \frac{3}{2}) \Gamma(2\mu) \Gamma(2\nu)} \times {}_4F_3 \left(-m, m + 2\mu, \alpha + 1, \alpha - \nu + \frac{3}{2}; \mu + \frac{1}{2}, \nu + \alpha + n + \frac{3}{2}, \alpha - \nu - n + \frac{3}{2}; 1 \right) \quad [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 283(16)}$$

$$7.315 \quad \int_{-1}^1 (1-x^2)^{\frac{1}{2}\nu-1} C_{2n}^\nu(ax) dx = \frac{\pi^{1/2} \Gamma(\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2})} C_n^{\frac{1}{2}\nu}(2a^2 - 1) \quad [\operatorname{Re} \nu > 0] \quad \text{ET II 283(19)}$$

$$7.316 \quad \int_{-1}^1 (1-x^2)^{\nu-1} C_n^\nu(\cos \alpha \cos \beta + x \sin \alpha \sin \beta) dx = \frac{2^{2\nu-1} n! [\Gamma(\nu)]^2}{\Gamma(2\nu+n)} C_n^\nu(\cos \alpha) C_n^\nu(\cos \beta) \\ [\operatorname{Re} \nu > 0] \quad \text{ET II 283(20)}$$

7.317

$$1. \quad \int_0^1 (1-x)^{\mu-1} x^{\lambda-\frac{1}{2}} C_n^\lambda(1-\gamma x) dx = \frac{\Gamma(2\lambda+n) \Gamma(\lambda+\frac{1}{2}) \Gamma(\mu)}{\Gamma(2\lambda) \Gamma(\lambda+\mu+n+\frac{1}{2})} P_n^{(\alpha,\beta)}(1-\gamma) \\ \alpha = \lambda + \mu - \frac{1}{2}, \quad \beta = \lambda - \mu - \frac{1}{2} \quad [\operatorname{Re} \lambda > -1, \quad \lambda \neq 0, \quad -\frac{1}{2}, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 190(39)a}$$

$$2. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_n^\lambda(1-\gamma x) dx = \frac{\Gamma(2\lambda+n) \Gamma(\mu) \Gamma(\nu)}{n! \Gamma(2\lambda) \Gamma(\mu+\nu)} \\ \times {}_3F_2\left(-n, n+2\lambda, \nu; \lambda+\frac{1}{2}, \mu+\nu; \frac{\gamma}{2}\right) \\ [2\lambda \neq 0, -1, -2, \dots, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 191(40)a}$$

$$7.318 \quad \int_0^1 x^{2\nu} (1-x^2)^{\sigma-1} C_n^\nu(1-x^2 y) dx = \frac{\Gamma(2\nu+n) \Gamma(\nu+\frac{1}{2}) \Gamma(\sigma)}{2 \Gamma(2\nu) \Gamma(n+\nu+\sigma+\frac{1}{2})} P_n^{(\alpha,\beta)}(1-y), \\ \alpha = \nu + \sigma - \frac{1}{2}, \quad \beta = \nu - \sigma - \frac{1}{2} \quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \sigma > 0] \quad \text{ET II 283(21)}$$

7.319

$$1. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda(\gamma x^{1/2}) dx = (-1)^n \frac{\Gamma(\lambda+n) \Gamma(\mu) \Gamma(\nu)}{n! \Gamma(\lambda) \Gamma(\mu+\nu)} {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 191(41)a}$$

$$2. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda(\gamma x^{1/2}) dx = \frac{(-1)^n 2\gamma \Gamma(\mu) \Gamma(\lambda+n+1) \Gamma(\nu+\frac{1}{2})}{n! \Gamma(\lambda) \Gamma(\mu+\nu+\frac{1}{2})} \\ \times {}_3F_2\left(-n, n+\lambda+1, \nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 191(42)}$$

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

$$7.321 \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 281(7), MO 99a}$$

$$7.322 \quad \int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu\left(\frac{x}{a}-1\right) e^{-bx} dx = (-1)^n \frac{\pi \Gamma(2\nu+n)}{n! \Gamma(\nu)} \left(\frac{a}{2b}\right)^\nu e^{-ab} I_{\nu+n}(ab) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 171(9)}$$

7.323

$$1. \quad \int_0^\pi C_n^\nu(\cos \varphi) (\sin \varphi)^{2\nu} d\varphi = 0 \quad [n = 1, 2, 3, \dots] \\ = 2^{-2\nu} \pi \Gamma(2\nu+1) [\Gamma(1+\nu)]^{-2} \quad [n = 0]$$

$$\begin{aligned}
2.11 \quad \int_0^\pi C_n^\nu (\cos \psi \cos \psi' + \sin \psi \sin \psi' \cos \varphi) (\sin \varphi)^{2\nu-1} d\varphi \\
= 2^{2\nu-1} n! [\Gamma(\nu)]^2 C_n^\nu(\cos \psi) C_n^\nu(\cos \psi') [\Gamma(2\nu + n)]^{-1} \\
[\operatorname{Re} \nu > 0] \quad \text{EH I 177(20)}
\end{aligned}$$

7.324

$$\begin{aligned}
1. \quad \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n+1}^\nu(x) \sin ax \, dx &= (-1)^n \pi \frac{\Gamma(2n+2\nu+1) J_{2n+\nu+1}(a)}{(2n+1)! \Gamma(\nu) (2a)^\nu} \\
&[\operatorname{Re} \nu > -\tfrac{1}{2}, \quad a > 0] \quad \text{ET I 94(4)} \\
2. \quad \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n}^\nu(x) \cos ax \, dx &= \frac{(-1)^n \pi \Gamma(2n+2\nu) J_{\nu+2n}(a)}{(2n)! \Gamma(\nu) (2a)^\nu} \\
&[\operatorname{Re} \nu > -\tfrac{1}{2}, \quad a > 0] \quad \text{ET I 38(3)a}
\end{aligned}$$

7.325* Complete System of Orthogonal Step Functions

Let $s_j(x) = (-1)^{\lfloor 2jx \rfloor}$ for $j \in \mathbb{N}$ and $c_j(x) = (-1)^{\lfloor 2jx+1/2 \rfloor}$ for $j \in 0 + \mathbb{N}$ where $\lfloor z \rfloor$ denotes the integer part of z . Thus, $c_j(z)$ and $s_j(z)$ have minimal period j^{-1} and manifest even and odd symmetry about $x = 1/2$, respectively, and so are the discrete analogues of $\cos 2\pi jx$ and $\sin 2\pi jx$. Furthermore, for $j \in \mathbb{N}$ let $\underline{j}$ denote its odd part: the quotient of j by its highest power-of-two factor. Then for all j and $k \in \mathbb{N}$, if $(\underline{j}, \underline{k})$ denotes their highest common factor and $[j, k]$ denotes their lowest common multiple:

$$\begin{aligned}
1. \quad \int_0^1 s_j(x) s_k(x) \, dx &= \begin{cases} \frac{(\underline{j}, \underline{k})}{[j, k]} & \text{if } j/\underline{j} = k/\underline{k} \\ 0 & \text{otherwise} \end{cases} \\
2. \quad \int_0^1 c_j(x) c_k(x) \, dx &= \begin{cases} (-1)^{(j+k)/2+1} \frac{(\underline{j}, \underline{k})}{[j, k]} & \text{if } j/\underline{j} = k/\underline{k} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

7.33 Combinations of the polynomials $C_n^\nu(x)$ and Bessel functions; Integration of Gegenbauer functions with respect to the index**7.331**

$$\begin{aligned}
1. \quad \int_1^\infty x^{2n+1-\nu} (x^2-1)^{\nu-2n-\frac{1}{2}} C_{2n}^{\nu-2n} \left(\frac{1}{x} \right) J_\nu(xy) \, dx \\
= (-1)^n 2^{2n-\nu+1} y^{-\nu+2n-1} [(2n)!]^{-1} \Gamma(2\nu-2n) [\Gamma(\nu-2n)]^{-1} \cos y \\
[y > 0, \quad 2n - \tfrac{1}{2} < \operatorname{Re} \nu < 2n + \tfrac{1}{2}] \quad \text{ET II 44(10)a}
\end{aligned}$$

7.332

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\nu+1} (x^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{3}{4}} C_{2n+1}^{\nu+\frac{1}{2}} \left[(x^2 + \beta^2)^{-1/2} \beta \right] J_{\nu+\frac{3}{2}+2n} \left[(x^2 + \beta^2)^{1/2} a \right] J_\nu(xy) dx \\
&= (-1)^n 2^{1/2} \pi^{-1/2} a^{\frac{1}{2}-\nu} y^\nu (a^2 - y^2)^{-1/2} \sin \left[\beta (a^2 - y^2)^{1/2} \right] C_{2n+1}^{\nu+\frac{1}{2}} \left[\left(1 - \frac{y^2}{a^2} \right)^{1/2} \right] \\
&\quad [0 < y < a] \\
&= 0 \\
&\quad [a < y < \infty] \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1] \\
&\quad \text{ET II 59(23)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\nu+1} (x^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{3}{4}} C_{2n}^{\nu+\frac{1}{2}} \left[\beta (x^2 + \beta^2)^{-1/2} \right] J_{\nu+\frac{1}{2}+2n} \left[(x^2 + \beta^2)^{1/2} a \right] J_\nu(xy) dx \\
&= (-1)^n 2^{1/2} \pi^{-1/2} a^{\frac{1}{2}-\nu} y^\nu (a^2 - y^2)^{-1/2} \cos \left[\beta (a^2 - y^2)^{1/2} \right] C_{2n}^{\nu+\frac{1}{2}} \left[\left(1 - \frac{y^2}{a^2} \right)^{1/2} \right] \\
&\quad [0 < y < a] \\
&= 0 \\
&\quad [a < y < \infty] \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1] \\
&\quad \text{ET II 59(24)}
\end{aligned}$$

7.333

$$\begin{aligned}
1. \quad & \int_0^\pi (\sin x)^{\nu+1} \cos(a \cos \theta \cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx \\
&= (-1)^{\frac{n}{2}} \left(\frac{2\pi}{a} \right)^{1/2} (\sin \theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+\frac{1}{2}+n}(a) \quad [n = 0, 2, 4, \dots] \\
&= 0 \quad [n = 1, 3, 5, \dots] \\
&\quad [\operatorname{Re} \nu > -1] \quad \text{WA 414(2)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\pi (\sin x)^{\nu+1} \sin(a \cos \theta \cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx \\
&= 0 \quad [n = 0, 2, 4, \dots] \\
&= (-1)^{\frac{n-1}{2}} \left(\frac{2\pi}{a} \right)^{1/2} (\sin \theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+\frac{1}{2}+n}(a) \quad [n = 1, 3, 5, \dots] \\
&\quad [\operatorname{Re} \nu > -1] \quad \text{WA 414(3)a}
\end{aligned}$$

7.334

$$\begin{aligned}
1. \quad & \int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{J_\nu(\omega)}{\omega^\nu} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(\alpha)}{\alpha^\nu} \frac{J_{\nu+n}(\beta)}{\beta^\nu}, \\
&\quad \omega = (\alpha^2 + \beta^2 - 2\alpha\beta \cos x)^{1/2} \quad [n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > -\tfrac{1}{2}] \quad \text{ET II 362(29)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{Y_\nu(\omega)}{\omega^\nu} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(\alpha)}{\alpha^\nu} \frac{Y_{\nu+n}(\beta)}{\beta^\nu}, \\
&\quad \omega = (\alpha^2 + \beta^2 - 2\alpha\beta \cos x)^{1/2} \quad [|\alpha| < |\beta|, \quad \operatorname{Re} \nu - \tfrac{1}{2}] \quad \text{ET II 362(30)}
\end{aligned}$$

Integration of Gegenbauer functions with respect to the index

$$7.335 \quad \int_{c-i\infty}^{c+i\infty} [\sin(\alpha\pi)]^{-1} t^\alpha C_\alpha^\nu(z) d\alpha = -2i (1 + 2tz + t^2)^{-\nu} \\ [-2 < \operatorname{Re} \nu < c < 0, \quad |\arg(z \pm 1)| < \pi] \\ \text{EH I 178(25)}$$

$$7.336 \quad \int_{-\infty}^{\infty} \operatorname{sech}(\pi x) \left(\nu - \frac{1}{2} + ix \right) K_{\nu-\frac{1}{2}+ix}(a) I_{\nu-\frac{1}{2}+ix}(b) C_{-\frac{1}{2}+ix}^\nu(-\cos \varphi) dx \\ = \frac{2^{-\nu+1}(ab)^\nu}{\Gamma(\nu)} \omega^{-\nu} K_\nu(\omega) \\ \omega = \sqrt{a^2 + b^2 - 2ab \cos \varphi} \quad \text{EH II 55(45)}$$

7.34 Combinations of Chebyshev polynomials and powers

$$7.341 \quad \int_{-1}^1 [T_n(x)]^2 dx = 1 - (4n^2 - 1)^{-1} \quad \text{ET II 271(6)}$$

$$7.342 \quad \int_{-1}^1 U_n \left[x (1 - y^2)^{1/2} (1 - z^2)^{1/2} + yz \right] dx = \frac{2}{n+1} U_n(y) U_n(z) \\ [|y| < 1, \quad |z| < 1] \quad \text{ET II 275(34)}$$

7.343

$$1. \quad \int_{-1}^1 T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = 0 \quad [m \neq n] \\ = \frac{\pi}{2} \quad [m = n \neq 0] \\ = \pi \quad [m = n = 0]$$

MO 104

$$2. \quad \int_{-1}^1 \sqrt{1-x^2} U_n(x) U_m(x) dx = 0 \quad [m \neq n] \quad \text{ET II 274(28)} \\ = \frac{\pi}{2} \quad [m = n] \quad \text{ET II 274(27), MO 105a}$$

7.344

$$1. \quad \int_{-1}^1 (y-x)^{-1} (1-y^2)^{-1/2} T_n(y) dy = \pi U_{n-1}(x) \quad [n = 1, 2, \dots] \quad \text{EH II 187(47)}$$

$$2. \quad \int_{-1}^1 (y-x)^{-1} (1-y^2)^{1/2} U_{n-1}(y) dy = -\pi T_n(x) \quad [n = 1, 2, \dots] \quad \text{EH II 187(48)}$$

7.345

$$1. \quad \int_{-1}^1 (1-x)^{-1/2} (1+x)^{m-n-\frac{3}{2}} T_m(x) T_n(x) dx = 0 \quad [m > n] \quad \text{ET II 272(10)}$$

$$2. \quad \int_{-1}^1 (1-x)^{-1/2} (1+x)^{m+n-\frac{3}{2}} T_m(x) T_n(x) dx = \frac{\pi(2m+2n-2)!}{2^{m+n}(2m-1)!(2n-1)!} \\ [m+n \neq 0] \quad \text{ET II 272(11)}$$

3. $\int_{-1}^1 (1-x)^{1/2}(1+x)^{m+n+\frac{3}{2}} U_m(x) U_n(x) dx = \frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!}$ ET II 274(31)
4. $\int_{-1}^1 (1-x)^{1/2}(1+x)^{m-n-\frac{1}{2}} U_m(x) U_n(x) dx = 0 \quad [m > n]$ ET II 274(30)
5. $\int_{-1}^1 (1-x)(1+x)^{1/2} U_m(x) U_n(x) dx = \frac{2^{5/2}(m+1)(n+1)}{(m+n+\frac{3}{2})(m+n+\frac{5}{2})[1-4(m-n)^2]}$ ET II 274(29)
6. $\int_{-1}^1 (1+x)^{-1/2}(1-x)^{\alpha-1} T_m(x) T_n(x) dx$
 $= \frac{\pi^{1/2} 2^{\alpha-\frac{1}{2}} \Gamma(\alpha) \Gamma(n-\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2}-\alpha) \Gamma(\alpha+n+\frac{1}{2})} {}_4F_3 \left(-m, m, \alpha, \alpha + \frac{1}{2}; \frac{1}{2}, \alpha+n+\frac{1}{2}, \alpha-n+\frac{1}{2}; 1 \right)$
 $[\operatorname{Re} \alpha > 0]$ ET II 272(12)
7. $\int_{-1}^1 (1+x)^{1/2}(1-x)^{\alpha-1} U_m(x) U_n(x) dx$
 $= \frac{\pi^{1/2} 2^{\alpha-\frac{1}{2}} (m+1)(n+1) \Gamma(\alpha) \Gamma(n-\alpha+\frac{3}{2})}{\Gamma(\frac{3}{2}-\alpha) \Gamma(\frac{3}{2}+\alpha+n)}$
 $\times {}_4F_3 \left(-m, m+2, \alpha, \alpha - \frac{1}{2}; \frac{3}{2}, \alpha+n+\frac{3}{2}, \alpha-n-\frac{1}{2}; 1 \right)$
 $[\operatorname{Re} \alpha > 0]$ ET II 275(32)
- 7.346** $\int_0^1 x^{s-1} T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{s 2^s B(\frac{1}{2} + \frac{1}{2}s + \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}s - \frac{1}{2}n)}$
 $[\operatorname{Re} s > 0]$ ET II 324(2)
- 7.347**
1. $\int_{-1}^1 (1-x)^\alpha (1+x)^\beta T_n(x) dx = \frac{2^{\alpha+\beta+2n+1} (n!)^2 \Gamma(\alpha+1) \Gamma(\beta+1)}{(2n)! \Gamma(\alpha+\beta+2)}$
 $\times {}_3F_2 \left(-n, n, \alpha+1; \frac{1}{2}, \alpha+\beta+2; 1 \right)$
 $[\operatorname{Re} \alpha > -1, \operatorname{Re} \beta > -1]$ ET II 271(2)
2. $\int_{-1}^1 (1-x)^\alpha (1+x)^\beta U_n(x) dx = \frac{2^{\alpha+\beta+2n+2} [(n+1)!]^2 \Gamma(\alpha+1) \Gamma(\beta+1)}{(2n+2)! \Gamma(\alpha+\beta+2)}$
 $\times {}_3F_2 \left(-n, n+1, \alpha+1; \frac{3}{2}, \alpha+\beta+2; 1 \right)$
 ET II 273(22)
- 7.348** $\int_{-1}^1 (1-x^2)^{-1/2} U_{2n}(xz) dx = \pi P_n(2z^2-1) \quad [|z| < 1]$ ET II 275(33)
- 7.349** $\int_{-1}^1 (1-x^2)^{-1/2} T_n(1-x^2y) dx = \frac{1}{2} \pi [P_n(1-y) + P_{n-1}(1-y)]$ ET II 222(14)

7.35 Combinations of Chebyshev polynomials and elementary functions

$$7.351 \quad \int_0^1 x^{-1/2} (1-x^2)^{-\frac{1}{2}} e^{-\frac{2a}{x}} T_n(x) dx = \pi^{1/2} D_{n-\frac{1}{2}} \left(2a^{1/2} \right) D_{-n-\frac{1}{2}} \left(2a^{1/2} \right) \quad [\operatorname{Re} a > 0] \quad \text{ET II 272(13)}$$

7.352

$$1. \quad \int_0^\infty \frac{x U_n \left[a (a^2 + x^2)^{-1/2} \right]}{(a^2 + x^2)^{\frac{1}{2}n+1} (e^{\pi x} + 1)} dx = \frac{a^{-n}}{2n} - 2^{-n-1} \zeta \left(n+1, \frac{a+1}{2} \right) \quad [\operatorname{Re} a > 0] \quad \text{ET II 275(39)}$$

$$2. \quad \int_0^\infty \frac{x U_n \left[a (a^2 + x^2)^{-1/2} \right]}{(a^2 + x^2)^{\frac{1}{2}n+1} (e^{2\pi x} - 1)} dx = \frac{1}{2} \zeta(n+1, a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n} \quad [\operatorname{Re} a > 0] \quad \text{ET II 276(40)}$$

7.353

$$1. \quad \int_0^\infty (a^2 + x^2)^{-\frac{1}{2}n} \operatorname{sech} \left(\frac{1}{2} \pi x \right) T_n \left[a (a^2 + x^2)^{-1/2} \right] dx = 2^{1-2n} \left[\zeta \left(n, \frac{a+1}{4} \right) - \zeta \left(n, \frac{a+3}{4} \right) \right] \\ = 2^{1-n} \Phi \left(-1, n, \frac{a+1}{2} \right) \quad [\operatorname{Re} a > 0] \quad \text{ET II 273(19)}$$

$$2. \quad \int_0^\infty (a^2 + x^2)^{-\frac{1}{2}n} \left[\cosh \left(\frac{1}{2} \pi x \right) \right]^{-2} T_n \left[a (a^2 + x^2)^{-1/2} \right] dx = \pi^{-1} n 2^{1-n} \zeta \left(n+1, \frac{a+1}{2} \right) \quad [\operatorname{Re} a > 0] \quad \text{ET II 273(20)}$$

7.354

$$1. \quad \int_{-1}^1 \sin(xyz) \cos \left[(1-x^2)^{1/2} (1-y^2)^{1/2} z \right] T_{2n+1}(x) dx = (-1)^n \pi T_{2n+1}(y) J_{2n+1}(z) \quad \text{ET II 271(4)}$$

$$2. \quad \int_{-1}^1 \sin(xyz) \sin \left[(1-x^2)^{1/2} (1-y^2)^{1/2} z \right] U_{2n+1}(x) dx = (-1)^n \pi (1-y^2)^{1/2} U_{2n+1}(y) J_{2n+2}(z) \quad \text{ET II 274(25)}$$

$$3. \quad \int_{-1}^1 \cos(xyz) \cos \left[(1-x^2)^{1/2} (1-y^2)^{1/2} z \right] T_{2n}(x) dx = (-1)^n \pi T_{2n}(y) J_{2n}(z) \quad \text{ET II 271(5)}$$

$$4. \quad \int_{-1}^1 \cos(xyz) \sin \left[(1-x^2)^{1/2} (1-y^2)^{1/2} z \right] U_{2n}(x) dx = (-1)^n \pi (1-y^2)^{1/2} U_{2n}(y) J_{2n+1}(z) \quad \text{ET II 274(24)}$$

7.355

$$1. \quad \int_0^1 T_{2n+1}(x) \sin ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n+1}(a) \quad [a > 0] \quad \text{ET I 94(3)a}$$

$$2. \quad \int_0^1 T_{2n}(x) \cos ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n}(a) \quad [a > 0] \quad \text{ET I 38(2)a}$$

7.36 Combinations of Chebyshev polynomials and Bessel functions

$$7.361 \quad \int_0^1 (1-x^2)^{-1/2} T_n(x) J_\nu(xy) dx = \frac{1}{2} \pi J_{\frac{1}{2}(\nu+n)} \left(\frac{1}{2} y \right) J_{\frac{1}{2}(\nu-n)} \left(\frac{1}{2} y \right) \\ [y > 0, \quad \operatorname{Re} \nu > -n-1] \quad \text{ET II 42(1)}$$

$$7.362 \quad \int_1^\infty (x^2-1)^{-\frac{1}{2}} T_n \left(\frac{1}{x} \right) K_{2\mu}(ax) dx = \frac{\pi}{2a} W_{\frac{1}{2}n, \mu}(a) W_{-\frac{1}{2}n, \mu}(a) \\ [\operatorname{Re} a > 0] \quad \text{ET II 366(17)a}$$

7.37–7.38 Hermite polynomials

$$7.371 \quad \int_0^x H_n(y) dy = [2(n+1)]^{-1} [H_{n+1}(x) - H_{n+1}(0)] \quad \text{EH II 194(27)}$$

$$7.372 \quad \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} H_{2n}(\sqrt{x}t) dx = \frac{(-1)^n \pi^{1/2} (2n)! \Gamma(\alpha + \frac{1}{2}) L_n^\alpha(x)}{\Gamma(n+\alpha+1)} \\ [\operatorname{Re} \alpha > -\frac{1}{2}] \quad \text{EH II 195(34)}$$

7.373

$$1. \quad \int_0^x e^{-y^2} H_n(y) dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x) \quad [\text{see 8.956}] \quad \text{EH II 194(26)}$$

$$2. \quad \int_{-\infty}^\infty e^{-x^2} H_{2m}(xy) dx = \sqrt{\pi} \frac{(2m)!}{m!} (y^2-1)^m \quad \text{EH II 195(28)}$$

7.374

$$1. \quad \int_{-\infty}^\infty e^{-x^2} H_n(x) H_m(x) dx = 0 \quad [m \neq n] \quad \text{SM II 567} \\ = 2^n \cdot n! \sqrt{\pi} \quad [m = n]$$

SM II 568

$$2.^{11} \quad \int_{-\infty}^\infty e^{-2x^2} H_m(x) H_n(x) dx = (-1)^{[\frac{m}{2}] + [\frac{n}{2}]} 2^{\frac{m+n-1}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \quad [m+n \text{ is even}] \\ = 0 \quad [m+n \text{ is odd}]$$

ET II 289(10)a

$$3. \quad \int_{-\infty}^\infty e^{-x^2} H_m(ax) H_n(x) dx = 0 \quad [m < n] \quad \text{ET II 290(20)a}$$

$$4. \quad \int_{-\infty}^\infty e^{-x^2} H_{2m+n}(ax) H_n(x) dx = \sqrt{\pi} 2^n \frac{(2m+n)!}{m!} (a^2-1)^m a^n \quad \text{ET II 291(21)a}$$

$$5. \quad \int_{-\infty}^\infty e^{-2\alpha^2 x^2} H_m(x) H_n(x) dx = 2^{\frac{m+n-1}{2}} \alpha^{-m-n-1} (1-2\alpha^2)^{\frac{m+n}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \\ \times {}_2F_1\left(-m, n; \frac{1-m-n}{2}; \frac{\alpha^2}{2\alpha^2-1}\right) \\ [\operatorname{Re} \alpha^2 > 0, \quad \alpha^2 \neq \frac{1}{2}, \quad m+n \text{ is even}] \quad \text{ET II 289(12)a}$$

$$6. \quad \int_{-\infty}^\infty e^{-(x-y)^2} H_n(x) dx = \pi^{1/2} y^n 2^n \quad \text{ET II 288(2)a, EH II 195(31)}$$

$$7. \quad \int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x) dx = 2^n \pi^{1/2} m! y^{n-m} L_m^{n-m}(-2y^2) \quad [m \leq n] \quad \text{BU 148(15), ET II 289(13)a}$$

$$8. \quad \int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(\alpha x) dx = \pi^{1/2} (1 - \alpha^2)^{\frac{n}{2}} H_n \left[\frac{\alpha y}{(1 - \alpha^2)^{1/2}} \right] \quad \text{ET II 290(17)a}$$

$$9. \quad \int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(\alpha x) H_n(\alpha x) dx = \pi^{1/2} \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k} \binom{n}{k} (1 - \alpha^2)^{\frac{m+n}{2}-k} H_{m+n-2k} \left[\frac{\alpha y}{(1 - \alpha^2)^{1/2}} \right] \quad \text{ET II 291(26)a}$$

$$10. \quad \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2u}} H_n(x) dx = (2\pi u)^{1/2} (1 - 2u)^{\frac{n}{2}} H_n \left[y(1 - 2u)^{-1/2} \right] \quad [0 \leq u < \frac{1}{2}] \quad \text{EH II 195(30)}$$

7.375

$$1. \quad \int_{-\infty}^{\infty} e^{-2x^2} H_k(x) H_m(x) H_n(x) dx = \pi^{-1} 2^{\frac{1}{2}(m+n+k-1)} \Gamma(s-k) \Gamma(s-m) \Gamma(s-n) \quad 2s = k + m + n + 1 \quad [k + m + n \text{ is even}] \quad \text{ET II 290(14)a}$$

$$2. \quad \int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_m(x) H_n(x) dx = \frac{2^{\frac{m+n+k}{2}} \pi^{1/2} k! m! n!}{(s-k)!(s-m)!(s-n)!}, \quad 2s = m + n + k \quad [k + m + n \text{ is even}] \quad \text{ET II 290(15)a}$$

7.376

$$1. \quad \int_{-\infty}^{\infty} e^{ixy} e^{-\frac{x^2}{2}} H_n(x) dx = (2\pi)^{1/2} e^{-\frac{y^2}{2}} H_n(y) i^n \quad \text{MO 165a}$$

$$2. \quad \int_0^{\infty} e^{-2\alpha x^2} x^\nu H_{2n}(x) dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma(\frac{\nu+1}{2}) \Gamma(n+\frac{1}{2})}{\sqrt{\pi} \alpha^{\frac{1}{2}(\nu+1)}} F\left(-n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha}\right) \quad [\text{Re } \alpha > 0, \quad \text{Re } \nu > -1] \quad \text{BU 150(18a)}$$

$$3. \quad \int_0^{\infty} e^{-2\alpha x^2} x^\nu H_{2n+1}(x) dx = (-1)^n 2^{2n-\frac{1}{2}\nu} \frac{\Gamma(\frac{\nu}{2}+1) \Gamma(n+\frac{3}{2})}{\sqrt{\pi} \alpha^{\frac{1}{2}\nu+1}} F\left(-n, \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2\alpha}\right) \quad [\text{Re } \alpha > 0, \quad \text{Re } \nu > -2] \quad \text{BU 150(18b)}$$

$$7.377^8 \quad \int_{-\infty}^{\infty} e^{-x^2} H_m(x+y) H_n(x+z) dx = 2^n \pi^{1/2} m! z^{n-m} L_m^{n-m}(-2yz) \quad [m \leq n] \quad \text{ET II 292(30)a}$$

$$7.378 \quad \int_0^{\infty} x^{\alpha-1} e^{-\beta x} H_n(x) dx = 2^n \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n! \Gamma(\alpha + n - 2m)}{m! (n - 2m)!} (-1)^m 2^{-2m} \beta^{2m-\alpha-n} \quad [\text{Re } \alpha > 0, \text{ if } n \text{ is even; } \text{Re } \alpha > -1, \text{ if } n \text{ is odd; } \text{Re } \beta > 0] \quad \text{ET I 172(11)a}$$

7.379

$$1. \quad \int_{-\infty}^{\infty} x e^{-x^2} H_{2m+1}(xy) dx = \pi^{1/2} \frac{(2m+1)!}{m!} y (y^2 - 1)^m \quad \text{EH II 195(28)}$$

$$2. \quad \int_{-\infty}^{\infty} x^n e^{-x^2} H_n(xy) dx = \pi^{1/2} n! P_n(y) \quad \text{EH II 195(29)}$$

$$7.381 \quad \int_{-\infty}^{\infty} (x \pm ic)^\nu e^{-x^2} H_n(x) dx = 2^{n-1-\nu} \pi^{1/2} \frac{\Gamma(\frac{n-\nu}{2})}{\Gamma(-\nu)} \exp[\pm \frac{1}{2} \pi(\nu + n)i] \\ [c > 0] \quad \text{ET II 288(3)a}$$

$$7.382 \quad \int_0^{\infty} x^{-1} (x^2 + a^2)^{-1} e^{-x^2} H_{2n+1}(x) dx = (-2)^n \pi^{1/2} a^{-2} \left[2^\nu n! - (2n+1)! e^{\frac{1}{2}a^2} D_{-2n-2}(a\sqrt{2}) \right] \\ \text{ET II 288(4)a}$$

7.383

$$1. \quad \int_0^{\infty} e^{-xp} H_{2n+1}(\sqrt{x}) dx = (-1)^n 2^n (2n+1)!! \pi^{1/2} (p-1)^n p^{-n-\frac{3}{2}} \\ [\text{Re } p > 0] \quad \text{EF 151(261)a, ET I 172(12)a}$$

$$2. \quad \int_0^{\infty} e^{-(b-\beta x)} H_{2n+1}(\sqrt{(\alpha-\beta)x}) dx = (-1)^n \sqrt{\pi} \sqrt{\alpha-\beta} \frac{(2n+1)!}{n!} \frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{3}{2}}} \\ [\text{Re}(b-\beta) > 0] \quad \text{ET I 172(15)a}$$

$$3. \quad \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-(b-\beta)x} H_{2n}(\sqrt{(\alpha-\beta)x}) dx = (-1)^n \sqrt{\pi} \frac{(2n)!}{n!} \frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{1}{2}}} \\ [\text{Re}(b-\beta) > 0] \quad \text{ET I 172(16)a}$$

$$4. \quad \int_0^{\infty} x^{a-\frac{1}{2}n-1} e^{-bx} H_n(\sqrt{x}) dx = 2^n \Gamma(a) b^{-a} {}_2F_1\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1-a; b\right) \\ \left[\text{Re } a > \frac{1}{2}n, \text{ if } n \text{ is even, } \text{Re } a > \frac{1}{2}n - \frac{1}{2}, \text{ if } n \text{ is odd, } \text{Re } b > 0, \right. \\ \left. \text{If } a \text{ is even, only the first } 1 + \left\lfloor \frac{n}{2} \right\rfloor \text{ terms are kept in the series for } {}_2F_1 \right] \\ \text{ET I 172(14)a}$$

$$5. \quad \int_0^{\infty} x^{-1/2} e^{-px} H_{2n}(\sqrt{x}) dx = (-1)^n 2^n (2n-1)!! \pi^{1/2} (p-1)^n p^{-n-\frac{1}{2}} \quad \text{MO 177a}$$

$$7.384 \quad \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-bx} \left[H_n\left(\frac{\alpha + \sqrt{x}}{\lambda}\right) + H_n\left(\frac{a - \sqrt{x}}{\lambda}\right) \right] dx = \sqrt{\frac{2\pi}{b}} (1 - \lambda^{-2} b^{-1})^{\frac{n}{2}} H_n\left(\frac{\alpha}{\sqrt{\lambda^2 - \frac{1}{b}}}\right) \\ [\text{Re } b > 0] \quad \text{ET I 173(17)a}$$

7.385

$$1. \quad \int_0^{\infty} \frac{e^{-bx}}{\sqrt{e^x - 1}} H_{2n}[\sqrt{s(1 - e^{-x})}] dx = (-1)^n 2^{2n} \sqrt{\pi} \frac{(2n)! \Gamma(b + \frac{1}{2})}{\Gamma(n + b + 1)} L_n^n(s) \\ [\text{Re } b > -\frac{1}{2}] \quad \text{ET I 174(23)a}$$

$$2. \quad \int_0^\infty e^{-bx} H_{2n+1} \left[\sqrt{s} \sqrt{1 - e^{-x}} \right] dx = (-1)^n 2^{2n} \sqrt{\pi s} \frac{(2n+1)! \Gamma(b)}{\Gamma\left(n+b+\frac{3}{2}\right)} L_n^b(s) \quad [\operatorname{Re} b > 0] \quad \text{ET I 174(24)a}$$

$$\mathbf{7.386} \quad \int_0^\infty x^{-\frac{n+1}{2}} e^{-\frac{q^2}{4x}} H_n \left(\frac{q}{2\sqrt{x}} \right) e^{-px} dx = 2^n \pi^{1/2} p^{\frac{n-1}{2}} e^{-q\sqrt{p}} \quad \text{EF 129(117)}$$

$$\mathbf{7.387} \quad 1. \quad \int_0^\infty e^{-x^2} \sinh(\sqrt{2}\beta x) H_{2n+1}(x) dx = 2^{n-\frac{1}{2}} \pi^{1/2} \beta^{2n+1} e^{\frac{1}{2}\beta^2} \quad \text{ET II 289(7)a}$$

$$2. \quad \int_0^\infty e^{-x^2} \cosh(\sqrt{2}\beta x) H_{2n}(x) dx = 2^{n-1} \pi^{1/2} \beta^{2n} e^{\frac{1}{2}\beta^2} \quad \text{ET II 289(8)a}$$

$$\mathbf{7.388} \quad 1. \quad \int_0^\infty e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(x) dx = (-1)^n 2^{n-\frac{1}{2}} \pi^{1/2} \beta^{2n+1} e^{-\frac{1}{2}\beta^2} \quad \text{ET II 288(5)a}$$

$$2. \quad \int_0^\infty e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(ax) dx = (-1)^n 2^{-1} \pi^{1/2} (a^2 - 1)^{n+\frac{1}{2}} e^{-\frac{1}{2}\beta^2} H_{2n+1} \left(\frac{a\beta}{\sqrt{2}(a^2 - 1)^{1/2}} \right) \quad \text{ET II 290(18)a}$$

$$3. \quad \int_0^\infty e^{-x^2} \cos(\sqrt{2}\beta x) H_{2n}(x) dx = (-1)^n 2^{n-1} \pi^{1/2} \beta^{2n} e^{-\frac{1}{2}\beta^2} \quad \text{ET II 289(6)a}$$

$$4. \quad \int_0^\infty e^{-x^2} \cos(\sqrt{2}\beta x) H_{2n}(ax) dx = 2^{-1} \pi^{1/2} (1 - a^2)^n e^{-\frac{1}{2}\beta^2} H_{2n} \left[\frac{a\beta}{\sqrt{2}(a^2 - 1)^{1/2}} \right] \quad \text{ET II 290(19)a}$$

$$5. \quad \int_0^\infty e^{-y^2} [H_n(y)]^2 \cos(\sqrt{2}\beta y) dy = \pi^{1/2} 2^{n-1} n! e^{-\frac{\beta^2}{2}} L_n(\beta^2) \quad \text{EH II 195(33)}$$

$$6.^{11} \quad \int_0^\infty e^{-x^2} \sin(bx) H_n(x) H_{n+2m+1}(x) dx = 2^{n-1} (-1)^m \sqrt{\pi} n! b^{2m+1} e^{-\frac{b^2}{4}} L_n^{2m+1} \left(\frac{b^2}{2} \right) \quad [b > 0] \quad \text{ET I 39(11)a}$$

$$7. \quad \int_0^\infty e^{-x^2} \cos(bx) H_n(x) H_{n+2m}(x) dx = 2^{n-\frac{1}{2}} \sqrt{\frac{\pi}{2}} n! (-1)^m b^{2m} e^{-\frac{b^2}{4}} L_n^{2m} \left(\frac{b^2}{2} \right) \quad [b > 0] \quad \text{ET I 39(11)a}$$

$$\mathbf{7.389} \quad \int_0^\pi (\cos x)^n H_{2n} [a(1 - \sec x)^{1/2}] dx = 2^{-n} (-1)^n \pi \frac{(2n)!}{(n!)^2} [H_n(a)]^2 \quad \text{ET II 292(31)}$$

7.39 Jacobi polynomials

$$\mathbf{7.391} \quad 1. \quad \int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx$$

$$= 0 \quad [m \neq n, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1]$$

$$= \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! (\alpha+\beta+1+2n) \Gamma(\alpha+\beta+n+1)} \quad [m = n, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1]$$

ET II 285(5, 9)

2.
$$\int_{-1}^1 (1-x)^\rho (1+x)^\sigma P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1) \Gamma(n+1+\alpha)}{n! \Gamma(\rho+\sigma+2) \Gamma(1+\alpha)} \\ \times {}_3F_2(-n, \alpha+\beta+n+1, \rho+1; \alpha+1, \rho+\sigma+2; 1) \\ [\operatorname{Re} \rho > -1, \quad \operatorname{Re} \sigma > -1] \quad \text{ET II 284(3)}$$
- 3.⁶
$$\int_{-1}^1 (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\alpha+\sigma+1} \Gamma(\sigma+1) \Gamma(\alpha+1) \Gamma(\sigma-\beta+1)}{n! \Gamma(\sigma-\beta-n+1) \Gamma(\alpha+\sigma+n+2)} \\ [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \sigma > -1] \quad \text{ET II 284(1)}$$
4.
$$\int_{-1}^1 (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\beta+\rho+1} \Gamma(\rho+1) \Gamma(\beta+n+1) \Gamma(\alpha-\rho+n)}{n! \Gamma(\alpha-\rho) \Gamma(\beta+\rho+n+2)} \\ [\operatorname{Re} \rho > -1, \quad \operatorname{Re} \beta > -1] \quad \text{ET II 284(2)}$$
5.
$$\int_{-1}^1 (1-x)^{\alpha-1} (1+x)^\beta \left[P_n^{(\alpha,\beta)}(x) \right]^2 dx = \frac{2^{\alpha+\beta} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! \alpha \Gamma(\alpha+\beta+n+1)} \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > -1] \quad \text{ET II 285(6)}$$
6.
$$\int_{-1}^1 (1-x)^{2\alpha} (1+x)^\beta \left[P_n^{(\alpha,\beta)}(x) \right]^2 dx = \frac{2^{4\alpha+\beta+1} \Gamma(\alpha+\frac{1}{2}) [\Gamma(\alpha+n+1)]^2 \Gamma(\beta+2n+1)}{\sqrt{\pi} (n!)^2 \Gamma(\alpha+1) \Gamma(2\alpha+\beta+2n+2)} \\ [\operatorname{Re} \alpha > -\frac{1}{2}, \quad \operatorname{Re} \beta > -1] \quad \text{ET II 285(7)}$$
7.
$$\int_{-1}^1 (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_n^{(\rho,\beta)}(x) dx \\ = \frac{2^{\rho+\beta+1} \Gamma(\rho+n+1) \Gamma(\beta+n+1) \Gamma(\alpha+\beta+2n+1)}{n! \Gamma(\beta+\rho+2n+2) \Gamma(\alpha+\beta+n+1)} \\ [\operatorname{Re} \rho > -1, \quad \operatorname{Re} \beta > -1] \quad \text{ET II 285(10)}$$
8.
$$\int_{-1}^1 (1-x)^{\rho-1} (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_n^{(\rho,\beta)}(x) dx = \frac{2^{\rho+\beta} \Gamma(\alpha+n+1) \Gamma(\beta+n+1) \Gamma(\rho)}{n! \Gamma(\alpha+1) \Gamma(\rho+\beta+n+1)} \\ [\operatorname{Re} \beta > -1, \quad \operatorname{Re} \rho > 0] \quad \text{ET II 286(11)}$$
- 9.⁷
$$\int_{-1}^1 (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\sigma)}(x) dx \\ = \frac{2^{\alpha+\sigma+1} \Gamma(\alpha+n+1) \Gamma(\alpha+\beta+m+n+1) \Gamma(\sigma+m+1) \Gamma(\sigma-\beta+1)}{m!(n-m)! \Gamma(\alpha+\beta+n+1) \Gamma(\alpha+\sigma+m+n+2) \Gamma(\alpha-\beta+m-n+1)} \\ [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \sigma > -1] \quad \text{ET II 286(12)}$$
- 10.⁶
$$\int_{-1}^1 (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\beta)}(x) dx \\ = \frac{2^{\beta+\rho+1} \Gamma(\alpha+\beta+m+n+1) \Gamma(\beta+n+1) \Gamma(\rho+m+1) \Gamma(\alpha-\rho-m+n)}{m!(n-m)! \Gamma(\alpha+\beta+n+1) \Gamma(\beta+\rho+m+n+2) \Gamma(\alpha-\rho)} \\ [\operatorname{Re} \beta > -1, \quad \operatorname{Re} \rho > -1] \quad \text{ET II 287(16)}$$
11.
$$\int_0^x (1-y)^\alpha (1+y)^\beta P_n^{(\alpha,\beta)}(y) dy = \frac{1}{2n} \left[P_{n-1}^{(\alpha+1,\beta+1)}(0) - (1-x)^{\alpha+1} (1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(x) \right] \\ \text{EH II 173(38)}$$

7.392

1.
$$\int_0^1 x^{\lambda-1} (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(1-\gamma x) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\alpha+1)\Gamma(\lambda+\mu)} {}_3F_2\left(-n, n+\alpha+\beta+1, \lambda; \alpha+1, \lambda+\mu; \frac{1}{2}\gamma\right)$$

[Re $\lambda > 0$, Re $\mu > 0$] ET II 192(46)a
2.
$$\int_0^1 x^{\lambda-1} (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(\gamma x - 1) dx = (-1)^n \frac{\Gamma(\beta+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\beta+1)\Gamma(\lambda+\mu)} {}_3F_2\left(-n, n+\alpha+\beta+1, \lambda; \beta+1, \lambda+\mu; \frac{1}{2}\gamma\right) a$$

[Re $\lambda > 0$, Re $\mu > 0$] ET II 192(47)a
3.
$$\int_0^1 x^\alpha (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(1-\gamma x) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} P_n^{(\alpha+\mu, \beta-\mu)}(1-\gamma)$$

[Re $\alpha > -1$, Re $\mu > 0$] ET II 191(43)a
4.
$$\int_0^1 x^\beta (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(\gamma x - 1) dx = \frac{\Gamma(\beta+n+1)\Gamma(\mu)}{\Gamma(\beta+\mu+n+1)} P_n^{(\alpha-\mu, \beta+\mu)}(\gamma - 1)$$

[Re $\beta > -1$, Re $\mu > 0$] ET II 191(44)a

7.393

1.
$$\int_0^1 (1-x^2)^\nu \sin bx P_{2n+1}^{(\nu, \nu)}(x) dx = \frac{(-1)^n \sqrt{\pi} \Gamma(2n+\nu+2) J_{2n+\nu+\frac{3}{2}}(b)}{2^{\frac{1}{2}-\nu} (2n+1)! b^{\nu+\frac{1}{2}}}$$

[$b > 0$, Re $\nu > -1$] ET I 94(5)
2.
$$\int_0^1 (1-x^2)^\nu \cos bx P_{2n}^{(\nu, \nu)}(x) dx = \frac{(-1)^n 2^{\nu-\frac{1}{2}} \sqrt{\pi} \Gamma(2n+\nu+1) J_{2n+\nu+\frac{1}{2}}(b)}{(2n)! b^{\nu+\frac{1}{2}}}$$

[$b > 0$, Re $\nu > -1$] ET I 38(4)

7.41–7.42 Laguerre polynomials**7.411**

1.
$$\int_0^t L_n(x) dx = L_n(t) - L_{n+1}(t)/(n+1)$$
 MO 110
2.
$$\int_0^t L_n^\alpha(x) dx = L_n^\alpha(t) - L_{n+1}^\alpha(t) - \binom{n+\alpha}{n} + \binom{n+1+\alpha}{n+1}$$
 EH II 189(16)a
3.
$$\int_0^t L_{n-1}^{\alpha+1}(x) dx = -L_n^\alpha(t) + \binom{n+\alpha}{n}$$
 EH II 189(15)a
4.
$$\int_0^t L_m(x) L_n(t-x) dx = L_{m+n}(t) - L_{m+n+1}(t)$$
 EH II 191(31)
5.
$$\sum_{k=0}^{\infty} \left[\int_0^t \frac{L_k(x)}{k!} dx \right]^2 = e^t - 1$$
 [$t \geq 0$] MO 110

7.412

$$1. \quad \int_0^1 (1-x)^{\mu-1} x^\alpha L_n^\alpha(ax) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} L_n^{\alpha+\mu}(a) \quad [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \mu > 0] \\ \text{EH II 191(30)a, BU 129(14c)}$$

$$2. \quad \int_0^1 (1-x)^{\mu-1} x^{\lambda-1} L_n^\alpha(\beta x) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\alpha+1)\Gamma(\lambda+\mu)} {}_2F_2(-n, \lambda; \alpha+1, \lambda+\mu; \beta) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 192(50)a}$$

$$7.413 \quad \int_0^1 x^\alpha (1-x)^\beta L_m^\alpha(xy) L_n^\beta[(1-x)y] dx = \frac{(m+n)!\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{m!n!\Gamma(\alpha+\beta+m+n+2)} L_{m+n}^{\alpha+\beta+1}(y) \\ [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1] \quad \text{ET II 293(7)}$$

7.414

$$1.^{11} \quad \int_y^\infty e^{-x} L_n^\alpha(x) dx = e^{-y} [L_n^\alpha(y) - L_{n-1}^\alpha(y)] \quad \text{EH II 191(29)}$$

$$2. \quad \int_0^\infty e^{-bx} L_n(\lambda x) L_n(\mu x) dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}} P_n \left[\frac{b^2 - (\lambda+\mu)b + 2\lambda\mu}{b(b-\lambda-\mu)} \right] \\ [\operatorname{Re} b > 0] \quad \text{ET I 175(34)}$$

$$3.^8 \quad \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = 0 \quad [m \neq n, \quad \operatorname{Re} \alpha > -1] \quad \text{BU 115(8), ET II 293(3)} \\ = \frac{\Gamma(\alpha+n+1)}{n!} \quad [m = n, \quad \operatorname{Re} \alpha > 0] \quad \text{BU 115(8), ET II 292(2)}$$

$$4. \quad \int_0^\infty e^{-bx} x^\alpha L_n^\alpha(\lambda x) L_m^\alpha(\mu x) dx = \frac{\Gamma(m+n+\alpha+1)}{m!n!} \frac{(b-\lambda)^n (b-\mu)^m}{b^{m+n+\alpha+1}} \\ \times F \left[-m, -n; -m-n-\alpha, \frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)} \right] \\ [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} b > 0] \quad \text{ET I 175(35)}$$

$$4(1)^9. \quad \int_0^\infty e^{-x} x^{\alpha+1/2} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(\alpha+n+1)^2 \Gamma(\alpha+m+1) \Gamma(\alpha+\frac{3}{2}) \Gamma(m-\frac{1}{2})}{n!m! \Gamma(\alpha+1) \Gamma(-\frac{1}{2})} \\ \times {}_3F_2(-n, \alpha+\frac{3}{2}, \frac{3}{2}; \alpha+1, \frac{3}{2}-m; 1)$$

$$5. \quad \int_0^\infty e^{-bx} L_n^a(x) dx = \sum_{m=0}^n \binom{a+m-1}{m} \frac{(b-1)^{n-m}}{b^{n-m+1}} \quad [\operatorname{Re} b > 0] \quad \text{ET I 174(27)}$$

$$6. \quad \int_0^\infty e^{-bx} L_n(x) dx = (b-1)^n b^{-n-1} \quad [\operatorname{Re} b > 0] \quad \text{ET I 174(25)}$$

$$7. \quad \int_0^\infty e^{-st} t^\beta L_n^\alpha(t) dt = \frac{\Gamma(\beta+1)\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} s^{-\beta-1} F \left(-n, \beta+1; \alpha+1; \frac{1}{s} \right) \\ [\operatorname{Re} \beta > -1, \quad \operatorname{Re} s > 0] \\ \text{BU 119(4b), EH II 191(133)}$$

$$8. \quad \int_0^\infty e^{-st} t^\alpha L_n^\alpha(t) dt = \frac{\Gamma(\alpha+n+1)(s-1)^n}{n!s^{\alpha+n+1}} \quad [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} s > 0] \\ \text{EH II 191(32), MO 176a}$$

$$9. \quad \int_0^\infty e^{-x} x^{\alpha+\beta} L_m^\alpha(x) L_n^\beta(x) dx = (-1)^{m+n} (\alpha+\beta)! \binom{\alpha+m}{n} \binom{\beta+n}{m} \\ [\operatorname{Re}(\alpha+\beta) > -1] \quad \text{ET II 293(4)}$$

$$10.^6 \quad \int_0^\infty e^{-bx} x^{2a} [L_n^a(x)]^2 dx = \frac{2^{2a} \Gamma(a + \frac{1}{2}) \Gamma(n + \frac{1}{2})}{\pi (n!)^2 b^{2a+1}} \\ \times F\left(-n, a + \frac{1}{2}; \frac{1}{2} - n; \left(1 - \frac{2}{b}\right)^2\right) \Gamma(a + n + 1) \\ \left[\operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re} b > 0\right] \quad \text{ET I 174(30)}$$

$$11. \quad \int_0^\infty e^{-x} x^{\gamma-1} L_n^\mu(x) dx = \frac{\Gamma(\gamma) \Gamma(1 + \mu + n - \gamma)}{n! \Gamma(1 + \mu - \gamma)} \quad [\operatorname{Re} \gamma > 0] \quad \text{BU 120(4b)}$$

$$12. \quad \int_0^\infty e^{-x(s + \frac{a_1+a_2}{2})} x^{\mu+\beta} L_k^\mu(a_1x) L_k^\mu(a_2x) dx \\ = \frac{\Gamma(1 + \mu + \beta) \Gamma(1 + \mu + k)}{k! k! \Gamma(1 + \mu)} \left\{ \frac{d^k}{dh^k} \left[\frac{F\left(\frac{1+\mu+\beta}{2}, 1 + \frac{\mu+\beta}{2}; 1 + \mu; \frac{A^2}{B^2}\right)}{(1-h)^{1+\mu} B^{1+\mu+\beta}} \right] \right\}_{h=0} \\ A^2 = \frac{4a_1a_2h}{(1-h)^2}; \quad B = s + \frac{a_1 + a_2}{2} \frac{1+h}{1-h} \\ \left[\operatorname{Re}\left(s + \frac{a_1+a_2}{2}\right) > 0, \quad a_1 > 0, \quad a_2 > 0, \quad \operatorname{Re}(\mu + \beta) > -1\right] \quad \text{BU 142(19)}$$

$$13. \quad \int_0^\infty \exp\left[-x\left(s + \frac{a_1+a_2}{2}\right)\right] x^\mu L_k^\mu(a_1x) L_k^\mu(a_2x) dx = \frac{\Gamma(1 + \mu + k)}{b_0^{1+\mu+k}} \cdot \frac{b_0^k}{k!} \cdot P_k^{(\mu,0)}\left(\frac{b_1^2}{b_0b_2}\right) \\ b_0 = s + \frac{a_1+a_2}{2}, \quad b_1^2 = b_0b_2 + 2a_1a_2, \quad b_2 = s - \frac{a_1+a_2}{2} \\ \left[\operatorname{Re} \mu > -1, \quad \operatorname{Re}\left(s + \frac{a_1+a_2}{2}\right) > 0\right] \\ \text{BU 144(22)}$$

$$7.415 \quad \int_0^1 (1-x)^{\mu-1} x^{\lambda-1} e^{-\beta x} L_n^\alpha(\beta x) dx = \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + 1)} B(\lambda, \mu) {}_2F_2(\alpha + n + 1, \lambda; \alpha + 1, \lambda + \mu; -\beta) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 193(51)a}$$

$$7.416 \quad \int_{-\infty}^\infty x^{m-n} \exp\left[-\frac{1}{2}(x-y)^2\right] L_n^{m-n}(x^2) dx = \frac{(2\pi)^{1/2}}{n!} i^{n-m} 2^{-\frac{n+m}{2}} H_n\left(\frac{iy}{\sqrt{2}}\right) H_m\left(\frac{iy}{\sqrt{2}}\right) \\ \text{BU 149(15b), ET II 293(8)a}$$

7.417

$$1. \quad \int_0^\infty x^{\nu-2n-1} e^{-ax} \sin(bx) L_{2n}^{\nu-2n-1}(ax) dx = (-1)^n i \Gamma(\nu) \frac{b^{2n} [(a-ib)^{-\nu} - (a+ib)^{-\nu}]}{2(2n)!} \\ [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 2n] \\ \text{ET I 95(12)}$$

$$2. \quad \int_0^\infty x^{\nu-2n-2} e^{-ax} \sin(bx) L_{2n+1}^{\nu-2n-2}(ax) dx = (-1)^{n+1} \Gamma(\nu) \frac{b^{2n+1} [(a+ib)^{-\nu} + (a-ib)^{-\nu}]}{2(2n+1)!} \\ [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 2n+1] \\ \text{ET I 95(13)}$$

$$3. \quad \int_0^\infty x^{\nu-2n} e^{-ax} \cos(bx) L_{\nu-2n}^{2n-1}(ax) dx = i(-1)^{n+1} \Gamma(\nu) \frac{b^{2n-1} [(a-ib)^{-\nu} - (a+ib)^{-\nu}]}{2(2n-1)!}$$

[$b > 0$, $\operatorname{Re} a > 0$, $\operatorname{Re} \nu > 2n-1$] ET I 39(12)

$$4. \quad \int_0^\infty x^{\nu-2n-1} e^{-ax} \cos(bx) L_{2n}^{\nu-2n-1}(ax) dx = (-1)^n \Gamma(\nu) \frac{b^{2n} [(a+ib)^{-\nu} + (a-ib)^{-\nu}]}{2(2n)!}$$

[$b > 0$, $\operatorname{Re} \nu > 2n$, $\operatorname{Re} a > 0$] ET I 39(13)

7.418

$$1. \quad \int_0^\infty e^{-\frac{1}{2}x^2} \sin(bx) L_n(x^2) dx = (-1)^n \frac{1}{2} n! \frac{1}{\sqrt{2\pi}} \left\{ [D_{-n-1}(ib)]^2 - [D_{-n-1}(-ib)]^2 \right\}$$

[$b > 0$] ET I 95(14)

$$2. \quad \int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx) L_n(x^2) dx = \sqrt{\frac{\pi}{2}} (n!)^{-1} e^{-\frac{1}{2}b^2} 2^{-n} \left[H_n \left(\frac{b}{\sqrt{2}} \right) \right]^2$$

[$b > 0$] ET I 39(14)

$$3. \quad \int_0^\infty x^{2n+1} e^{-\frac{1}{2}x^2} \sin(bx) L_n^{n+\frac{1}{2}} \left(\frac{1}{2}x^2 \right) dx = \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}} \left(\frac{b^2}{2} \right)$$

[$b > 0$] ET I 95(15)

$$4. \quad \int_0^\infty x^{2n} e^{-\frac{1}{2}x^2} \cos(bx) L_n^{n-\frac{1}{2}} \left(\frac{1}{2}x^2 \right) dx = \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}} \left(\frac{1}{2}b^2 \right)$$

[$b > 0$] ET I 39(16)

$$5. \quad \int_0^\infty x e^{-\frac{1}{2}x^2} L_n^\alpha \left(\frac{1}{2}x^2 \right) L_n^{\frac{1}{2}-\alpha} \left(\frac{1}{2}x^2 \right) \sin(xy) dx = \left(\frac{\pi}{2} \right)^{1/2} y e^{-\frac{1}{2}y^2} L_n^\alpha \left(\frac{1}{2}y^2 \right) L_n^{\frac{1}{2}-\alpha} \left(\frac{1}{2}y^2 \right)$$

ET II 294(11)

$$6. \quad \int_0^\infty e^{-\frac{1}{2}x^2} L_n^\alpha \left(\frac{1}{2}x^2 \right) L_n^{-\frac{1}{2}-\alpha} \left(\frac{1}{2}x^2 \right) \cos(xy) dx = \left(\frac{\pi}{2} \right)^{1/2} e^{-\frac{1}{2}y^2} L_n^\alpha \left(\frac{1}{2}y^2 \right) L_n^{-\alpha-\frac{1}{2}} \left(\frac{1}{2}y^2 \right)$$

ET II 294(12)

$$7.419 \quad \int_0^\infty x^{n+2\nu-\frac{1}{2}} \exp[-(1+a)x] L_n^{2\nu}(ax) K_\nu(x) dx$$

$$= \frac{\pi^{1/2} \Gamma(n+\nu+\frac{1}{2}) \Gamma(n+3\nu+\frac{1}{2})}{2^{n+2\nu+\frac{1}{2}} n! (2\nu+1)} F \left(n+\nu+\frac{1}{2}, n+3\nu+\frac{1}{2}; 2\nu+1; -\frac{1}{2}a \right)$$

[$\operatorname{Re} a > -2$, $\operatorname{Re}(n+\nu) > -\frac{1}{2}$, $\operatorname{Re}(n+3\nu) > -\frac{1}{2}$] ET II 370(44)

7.421

$$1. \quad \int_0^\infty x e^{-\frac{1}{2}\alpha x^2} L_n \left(\frac{1}{2}\beta x^2 \right) J_0(xy) dx = \frac{(\alpha-\beta)^n}{\alpha^{n+1}} e^{-\frac{1}{2\alpha}y^2} L_n \left[\frac{\beta y^2}{2\alpha(\beta-\alpha)} \right]$$

[$y > 0$, $\operatorname{Re} \alpha > 0$] ET II 13(4)a

$$2. \quad \int_0^\infty x e^{-x^2} L_n(x^2) J_0(xy) dx = \frac{2^{-2n-1}}{n!} y^{2n} e^{-\frac{1}{4}y^2}$$

ET II 13(5)

$$3. \quad \int_0^\infty x^{2n+\nu+1} e^{-\frac{1}{2}x^2} L_n^{\nu+n} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx = y^{2n+\nu} e^{-\frac{1}{2}y^2} L_n^{\nu+n} \left(\frac{1}{2}y^2 \right) \\ [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{MO 183}$$

$$4. \quad \int_0^\infty x^{\nu+1} e^{-\beta x^2} L_n^\nu (\alpha x^2) J_\nu(xy) dx = 2^{-\nu-1} \beta^{-\nu-n-1} (\beta - \alpha)^n y^\nu e^{-\frac{y^2}{4\beta}} L_n^\nu \left[\frac{\alpha y^2}{4\beta(\alpha - \beta)} \right] \\ \text{ET II 43(5)}$$

$$5. \quad \int_0^\infty e^{-\frac{1}{2q}x^2} x^{\nu+1} L_n^\nu \left[\frac{x^2}{2q(1-q)} \right] J_\nu(xy) dx = \frac{q^{n+\nu+1}}{(q-1)^n} e^{-\frac{qy^2}{2}} y^\nu L_n^\nu \left(\frac{y^2}{2} \right) \\ [\nu > 0] \quad \text{MO 183}$$

$$6.* \quad \int_0^\infty x^{\nu+1} e^{-x^2} L_n^\nu (x^2) J_\nu(xy) dx = \frac{1}{2n!} \left(\frac{y}{2} \right)^{2n+\nu} e^{-\frac{1}{4}y^2}$$

7.422

$$1. \quad \int_0^\infty x^{\nu+1} e^{-\beta x^2} \left[L_n^{\frac{1}{2}\nu} (\alpha x^2) \right]^2 J_\nu(xy) dx \\ = \frac{y^\nu}{\pi n!} \Gamma \left(n + 1 + \frac{1}{2}\nu \right) (2\beta)^{-\nu-1} e^{-\frac{y^2}{4\beta}} \\ \times \sum_{l=0}^n \frac{(-1)^l \Gamma \left(n - l + \frac{1}{2} \right) \Gamma \left(l + \frac{1}{2} \right)}{\Gamma \left(l + 1 + \frac{1}{2}\nu \right) (n-l)!} \left(\frac{2\alpha - \beta}{\beta} \right)^{2l} L_{2l}^\nu \left[\frac{\alpha y^2}{2\beta(2\alpha - \beta)} \right] \\ [y > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 43(7)}$$

$$2.^9 \quad \int_0^\infty x^{\nu+1} e^{-\alpha x^2} L_m^{\nu-\sigma} (\alpha x^2) L_n^\sigma (\alpha x^2) J_\nu(xy) dx \\ = (-1)^{m+n} (2\alpha)^{-\nu-1} y^\nu e^{-\frac{y^2}{4\alpha}} L_n^{m-n-\sigma} \left(\frac{y^2}{4\alpha} \right) L_m^{n-m+\sigma-\nu} \left(\frac{y^2}{4\alpha} \right) \\ [y > 0, \quad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1, \quad n \neq 0, \quad \sigma \neq 0, \quad \alpha \neq 1] \quad \text{ET II 43(8)}$$

7.423

$$1. \quad \int_0^\infty e^{-\frac{1}{2}x^2} L_n \left(\frac{1}{2}x^2 \right) H_{2n+1} \left(\frac{x}{2\sqrt{2}} \right) \sin(xy) dx = \left(\frac{\pi}{2} \right)^{1/2} e^{-\frac{1}{2}y^2} L_n \left(\frac{1}{2}y^2 \right) H_{2n+1} \left(\frac{y}{2\sqrt{2}} \right) \\ \text{ET II 294(13)a}$$

$$2. \quad \int_0^\infty e^{-\frac{1}{2}x^2} L_n \left(\frac{1}{2}x^2 \right) H_{2n} \left(\frac{x}{2\sqrt{2}} \right) \cos(xy) dx = \left(\frac{\pi}{2} \right)^{1/2} e^{-\frac{1}{2}y^2} L_n \left(\frac{1}{2}y^2 \right) H_{2n} \left(\frac{y}{2\sqrt{2}} \right) \\ \text{ET II 294(14)a}$$

7.5 Hypergeometric Functions**7.51 Combinations of hypergeometric functions and powers**

$$7.511 \quad \int_0^\infty F(a, b; c; -z) z^{-s-1} dz = \frac{\Gamma(a+s) \Gamma(b+s) \Gamma(c) \Gamma(-s)}{\Gamma(a) \Gamma(b) \Gamma(c+s)} \\ [c \neq 0, -1, -2, \dots, \quad \operatorname{Re} s < 0, \quad \operatorname{Re}(a+s) > 0, \quad \operatorname{Re}(b+s) > 0] \quad \text{EH I 79(4)}$$

7.512

$$1. \quad \int_0^1 x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma\left(1+\frac{\alpha}{2}\right) \Gamma(\gamma) \Gamma(\alpha-\gamma+1) \Gamma\left(\gamma-\frac{\alpha}{2}-\beta\right)}{\Gamma(1+\alpha) \Gamma\left(1+\frac{\alpha}{2}-\beta\right) \Gamma\left(\gamma-\frac{\alpha}{2}\right)} \\ \left[\operatorname{Re} \alpha + 1 > \operatorname{Re} \gamma > \operatorname{Re} \beta, \quad \operatorname{Re}\left(\gamma-\frac{\alpha}{2}-\beta\right) > 0 \right] \quad \text{ET II 398(1)}$$

$$2. \quad \int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-n} F(-n, \beta; \gamma; x) dx = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\beta-\gamma+1) \Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n) \Gamma(\gamma-\rho) \Gamma(\beta-\gamma+\rho+1)} \\ [n = 0, 1, 2, \dots; \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\beta-\gamma) > n-1] \quad \text{ET II 398(2)}$$

$$3. \quad \int_0^1 x^{\rho-1}(1-x)^{\beta-\rho-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\beta-\rho) \Gamma(\gamma-\alpha-\rho)}{\Gamma(\beta) \Gamma(\gamma-\alpha) \Gamma(\gamma-\rho)} \\ [\operatorname{Re} \rho > 0, \quad \operatorname{Re}(\beta-\rho) > 0, \quad \operatorname{Re}(\gamma-\alpha-\rho) > 0] \quad \text{ET II 399(3)}$$

$$4. \quad \int_0^1 x^{\gamma-1}(1-x)^{\rho-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha) \Gamma(\gamma+\rho-\beta)} \\ [\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\gamma+\rho-\alpha-\beta) > 0] \quad \text{ET II 399(4)}$$

$$5. \quad \int_0^1 x^{\rho-1}(1-x)^{\sigma-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\rho) \Gamma(\sigma)}{\Gamma(\rho+\sigma)} {}_3F_2(\alpha, \beta, \rho; \gamma, \rho+\sigma; 1) \\ [\operatorname{Re} \rho > 0, \quad \operatorname{Re} \sigma > 0, \quad \operatorname{Re}(\gamma+\sigma-\alpha-\beta) > 0] \quad \text{ET II 399(5)}$$

$$6.^{10} \quad \int_0^1 x^{\lambda-1}(1-x)^{\beta-\lambda-1} F\left(\alpha, \beta; \lambda; \frac{zx}{b}\right) dx = B(\lambda, \beta-\lambda)(1-z/b)^{-\alpha} \quad \text{BU 9}$$

$$7.^{11} \quad \int_0^1 x^{\gamma-1}(1-x)^{\delta-\gamma-1} F(\alpha, \beta; \gamma; xz) F(\delta-\alpha, \delta-\beta; \delta-\gamma; (1-x)\zeta) dx \\ = \frac{\Gamma(\gamma) \Gamma(\delta-\gamma)}{\Gamma(\delta)} (1-\zeta)^{\alpha+\beta-\delta} F(\alpha, \beta; \delta; z+\zeta-z\zeta) \\ [0 < \operatorname{Re} \gamma < \operatorname{Re} \delta, \quad |\arg(1-z)| < \pi, \quad |\arg(1-\zeta)| < \pi] \quad \text{ET II 400(11)}$$

$$8. \quad \int_0^1 x^{\gamma-1}(1-x)^{\epsilon-1}(1-xz)^{-\delta} F(\alpha, \beta; \gamma; xz) F\left[\delta, \beta-\gamma; \epsilon; \frac{(1-x)z}{(1-xz)}\right] dx \\ = \frac{\Gamma(\gamma) \Gamma(\epsilon)}{\Gamma(\gamma+\epsilon)} F(\alpha+\delta, \beta; \gamma+\epsilon; z) \\ [\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \epsilon > 0, \quad |\arg(z-1)| < \pi] \quad \text{ET II 400(12), Eh I 78(3)}$$

$$9. \quad \int_0^1 x^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) dx \\ = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha) \Gamma(\gamma+\rho-\beta)} (1-z)^{-\sigma} \\ \times {}_3F_2\left(\rho, \sigma, \gamma+\rho-\alpha-\beta; \gamma+\rho-\alpha, \gamma+\rho-\beta; \frac{z}{z-1}\right) \\ [\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\gamma+\rho-\alpha-\beta) > 0, \quad |\arg(1-z)| < \pi] \quad \text{ET II 399(6)}$$

$$10. \quad \int_0^\infty x^{\gamma-1} (x+z)^{-\sigma} F(\alpha, \beta; \gamma; -x) dx = \frac{\Gamma(\gamma) \Gamma(\alpha - \gamma + \sigma) \Gamma(\beta - \gamma + \sigma)}{\Gamma(\sigma) \Gamma(\alpha + \beta - \gamma + \sigma)} \\ \times F(\alpha - \gamma + \sigma, \beta - \gamma + \sigma; \alpha + \beta - \gamma + \sigma; 1 - z) \\ [\operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\alpha - \gamma + \sigma) > 0, \quad \operatorname{Re}(\beta - \gamma + \sigma) > 0, \quad |\arg z| < \pi] \quad \text{ET II 400(10)}$$

$$11. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(a_1, \dots, a_p; \nu, b_2, \dots, b_q; ax) dx \\ = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)} {}_pF_q(a_1, \dots, a_p; \mu + \nu, b_2, \dots, b_q; a) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad p \leq q + 1; \text{ if } p = q + 1, \text{ then } |a| < 1] \quad \text{ET II 200(94)}$$

$$12. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; ax) dx \\ = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)} {}_{p+1}F_{q+1}(\nu, a_1, \dots, a_p; \mu + \nu, b_1, \dots, b_q; a) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad p \leq q + 1, \text{ if } p = q + 1, \text{ then } |a| < 1] \quad \text{ET II 200(95)}$$

$$7.513 \quad \int_0^1 x^{s-1} (1-x^2)^\nu F(-n, a; b; x^2) dx = \frac{1}{2} B\left(\nu + 1, \frac{s}{2}\right) {}_3F_2\left(-n, a, \frac{s}{2}; b, \nu + 1 + \frac{s}{2}; 1\right) \\ [\operatorname{Re} s > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 336(4)}$$

7.52 Combinations of hypergeometric functions and exponentials

$$7.521 \quad \int_0^\infty e^{-st} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t) dt = \frac{1}{s} {}_{p+1}F_q(1, a_1, \dots, a_p; b_1, \dots, b_q; s^{-1}) \\ [p \leq q] \quad \text{EH I 192}$$

$$7.522 \quad 1.11 \quad \int_0^\infty e^{-\lambda x} x^{\gamma-1} {}_2F_1(\alpha, \beta; \delta; -x) dx = \frac{\Gamma(\delta) \lambda^{-\gamma}}{\Gamma(\alpha) \Gamma(\beta)} E(\alpha, \beta, \gamma; \delta; \lambda) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{EH I 205(10)}$$

$$2.6 \quad \int_0^\infty e^{-bx} x^{a-1} F\left(\frac{1}{2} + \nu, \frac{1}{2} - \nu; a; -\frac{x}{2}\right) dx = 2^a e^b \frac{1}{\sqrt{\pi}} \Gamma(a) (2b)^{\frac{1}{2}-a} K_\nu(b) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET I 212(1)}$$

$$3. \quad \int_0^\infty e^{-bx} x^{\gamma-1} F(2\alpha, 2\beta; \gamma; -\lambda x) dx = \Gamma(\gamma) b^{-\gamma} \left(\frac{b}{\lambda}\right)^{\alpha+\beta-\frac{1}{2}} e^{\frac{b}{2\lambda}} W_{\frac{1}{2}-\alpha-\beta, \alpha-\beta}\left(\frac{b}{\lambda}\right) \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \gamma > 0, \quad |\arg \lambda| < \pi] \\ \text{BU 78(30), ET I 212(4)}$$

$$4.6 \quad \int_0^\infty e^{-xt} t^{b-1} F(a, a-c+1; b; -t) dt = x^{a-b} \Gamma(b) \Psi(a, c; x) \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} x > 0] \quad \text{EH I 273(11)}$$

$$5. \quad \int_0^\infty e^{-x} x^{s-1} {}_pF_q(a_1, \dots, a_p, b_1, \dots, b_q; ax) dx = \Gamma(s) {}_{p+1}F_q(s, a_1, \dots, a_p; b_1, \dots, b_q; a) \\ [p < q, \quad \operatorname{Re} s > 0] \quad \text{ET I 337(11)}$$

$$6. \quad \int_0^\infty x^{\beta-1} e^{-\mu x} {}_2F_2(-n, n+1; 1, \beta; x) dx = \Gamma(\beta) \mu^{-\beta} P_n \left(1 - \frac{2}{\mu} \right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 218(6)}$$

$$7. \quad \int_0^\infty x^{\beta-1} e^{-\mu x} {}_2F_2 \left(-n, n; \beta, \frac{1}{2}; x \right) dx = \Gamma(\beta) \mu^{-\beta} \cos \left[2n \arcsin \left(\frac{1}{\sqrt{\mu}} \right) \right] \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 218(7)}$$

$$8. \quad \int_0^\infty x^{\rho_n-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \rho_1, \dots, \rho_n; \lambda x) dx \\ = \Gamma(\rho_n) \mu^{-\rho_n} {}_mF_{n-1} \left(a_1, \dots, a_m; \rho_1, \dots, \rho_{n-1}; \frac{\lambda}{\mu} \right) \\ [m \leq n; \quad \operatorname{Re} \rho_n > 0, \quad \operatorname{Re} \mu > 0, \text{ if } m < n; \operatorname{Re} \mu > \operatorname{Re} \lambda, \text{ if } m = n] \quad \text{ET I 219(16)a}$$

$$9. \quad \int_0^\infty x^{\sigma-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \rho_1, \dots, \rho_n; \lambda x) dx \\ = \Gamma(\sigma) \mu^{-\sigma} {}_{m+1}F_n \left(a_1, \dots, a_m, \sigma; \rho_1, \dots, \rho_n; \frac{\lambda}{\mu} \right) \\ [m \leq n, \quad \operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \text{ if } m < n; \operatorname{Re} \mu > \operatorname{Re} \lambda, \text{ if } m = n] \quad \text{ET I 219(17)}$$

$$\mathbf{7.523} \quad \int_1^\infty (x-1)^{\mu-1} x^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}ax} W_{2\mu+\frac{1}{2}, \lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{\mu+\frac{1}{2}, \lambda}(a) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0]$$

7.524

$$1. \quad \int_0^\infty e^{-\lambda x} F \left(\alpha, \beta; \frac{1}{2}; -x^2 \right) dx = \lambda^{\alpha+\beta-1} S_{1-\alpha-\beta, \alpha-\beta}(\lambda) \\ [\operatorname{Re} \lambda > 0] \quad \text{ET II 401(13)}$$

$$2. \quad \int_0^\infty e^{-st} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t^2) dx = s^{-1} {}_{p+2}F_q \left(a_1, \dots, a_p, 1, \frac{1}{2}; b_1, \dots, b_q; \frac{4}{s^2} \right) \\ [p < q] \quad \text{MO 176}$$

$$3. \quad \int_0^\infty e^{-st} {}_0F_q \left(\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}, 1; \frac{t^q}{q^q} \right) dt = s^{-1} \exp(s^{-q}) \quad \text{MO 176}$$

7.525

$$1. \quad \int_0^\infty x^{\sigma-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \rho_1, \dots, \rho_n; (\lambda x)^k) dx \\ = \Gamma(\sigma) \mu^{-\sigma} {}_{m+k}F_n \left(a_1, \dots, a_m, \frac{\sigma}{k}, \frac{\sigma+1}{k}, \dots, \frac{\sigma+k-1}{k}; \rho_1, \dots, \rho_n; \left(\frac{k\lambda}{\mu} \right)^k \right) \\ \left[m+k \leq n+1, \quad \operatorname{Re} \sigma > 0; \quad \operatorname{Re} \mu > 0, \text{ if } m+k \leq n; \right. \\ \left. \operatorname{Re} \left(\mu + k\lambda e^{\frac{2\pi i}{k}} \right) > 0; \quad r = 0, 1, \dots, k-1 \text{ for } m+k = n+1 \right] \\ \text{ET I 220(19)}$$

$$2. \quad \int_0^\infty x e^{-\lambda x} F\left(\alpha, \beta; \frac{3}{2}; -x^2\right) dx = \lambda^{\alpha+\beta-2} S_{1-\alpha-\beta, \alpha-\beta}(\lambda)$$

[Re $\lambda > 0$] ET II 401(14)

7.526

$$1. \quad \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} s^{-b} F\left(a, b; a+b-c+1; 1-\frac{1}{s}\right) dx = 2\pi i \frac{\Gamma(a+b-c+1)}{\Gamma(b)\Gamma(b-c+1)} t^{b-1} \Psi(a; c; t)$$

[Re $b > 0$, Re $(b-c) > -1$, $\gamma > \frac{1}{2}$] EH I 273(12)

$$2. \quad \int_0^\infty e^{-t} t^{\gamma-1} (x+t)^{-\alpha} (y+t)^{-a'} F\left[a, a'; \gamma; \frac{t(x+y+t)}{(x+t)(y+t)}\right] dt = \Gamma(\gamma) \Psi(a, c; x) \Psi(a', c; y),$$

$\gamma = a + a' - c + 1$ [Re $\gamma > 0$, $xy \neq 0$] EH I 287(21)

$$3. \quad \int_0^\infty x^{\gamma-1} (x+y)^{-\alpha} (x+z)^{-\beta} e^{-x} F\left[\alpha, \beta; \gamma; \frac{x(x+y+z)}{(x+y)(x+z)}\right] dx$$

$= \Gamma(\gamma) (zy)^{-\frac{1}{2}-\mu} e^{\frac{y+z}{2}} W_{\nu, \mu}(y) W_{\lambda, \mu}(z)$
 $2\nu = 1 - \alpha + \beta - \gamma; \quad 2\lambda = 1 + \alpha - \beta - \gamma; \quad 2\mu = \alpha + \beta - \gamma$
[Re $\gamma > 0$, $|\arg y| < \pi$, $|\arg z| < \pi$]
ET II 401(15)

7.527

$$1. \quad \int_0^\infty (1 - e^{-x})^{\lambda-1} e^{-\mu x} F(\alpha, \beta; \gamma; \delta e^{-x}) dx = B(\mu, \lambda) {}_3F_2(\alpha, \beta, \mu; \gamma, \mu + \lambda; \delta)$$

[Re $\lambda > 0$, Re $\mu > 0$, $|\arg(1 - \delta)| < \pi$] ET I 213(9)

$$2. \quad \int_0^\infty (1 - e^{-x})^\mu e^{-\alpha x} F(-n, \mu + \beta + n; \beta; e^{-x}) dx = \frac{B(\alpha, \mu + n + 1) B(\alpha, \beta + n - \alpha)}{B(\alpha, \beta - \alpha)}$$

[Re $\alpha > 0$, Re $\mu > -1$] ET I 213(10)

$$3. \quad \int_0^\infty (1 - e^{-x})^{\gamma-1} e^{-\mu x} F(\alpha, \beta; \gamma; 1 - e^{-x}) dx = \frac{\Gamma(\mu) \Gamma(\gamma - \alpha - \beta + \mu) \Gamma(\gamma)}{\Gamma(\gamma - \alpha + \mu) \Gamma(\gamma - \beta + \mu)}$$

[Re $\mu > 0$, Re $\mu > \text{Re}(\alpha + \beta - \gamma)$, Re $\gamma > 0$] ET I 213(11)

$$4. \quad \int_0^\infty (1 - e^{-x})^{\gamma-1} e^{-\mu x} F[\alpha, \beta; \gamma; \delta(1 - e^{-x})] dx = B(\mu, \gamma) F(\alpha, \beta; \mu + \gamma; \delta)$$

[Re $\mu > 0$, Re $\gamma > 0$, $|\arg(1 - \delta)| < \pi$] ET I 213(12)

7.53 Hypergeometric and trigonometric functions

7.531

$$1. \quad \int_0^\infty x \sin \mu x F\left(\alpha, \beta; \frac{3}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-2} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \\ [\mu > 0, \quad \operatorname{Re} \alpha > \frac{1}{2}, \quad \operatorname{Re} \beta > \frac{1}{2}] \\ \text{ET I 115(6)}$$

$$2. \quad \int_0^\infty \cos \mu x F\left(\alpha, \beta; \frac{1}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-1} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \\ [\mu > 0, \quad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad c > 0] \\ \text{ET I 61(9)}$$

7.54 Combinations of hypergeometric and Bessel functions

$$7.541 \quad \int_0^\infty x^{\alpha+\beta-2\nu-1} (x+1)^{-\nu} e^{xz} K_\nu[(x+1)z] F(\alpha, \beta; \alpha+\beta-2\nu; -x) dx \\ = \pi^{-\frac{1}{2}} \cos(\nu\pi) \Gamma\left(\frac{1}{2} - \alpha + \nu\right) \Gamma\left(\frac{1}{2} - \beta + \nu\right) \Gamma(\gamma) (2z)^{-\frac{1}{2}-\frac{1}{2}\gamma} W_{\frac{1}{2}\gamma, \frac{1}{2}(\beta-\alpha)}(2z) \\ \gamma = \alpha + \beta - 2\nu \quad [\operatorname{Re}(\alpha + \beta - 2\nu) > 0, \quad \operatorname{Re}\left(\frac{1}{2} - \alpha + \nu\right) > 0, \quad \operatorname{Re}\left(\frac{1}{2} - \beta + \nu\right) > 0, \quad |\arg z| < \frac{3}{2}\pi] \\ \text{ET II 401(16)}$$

7.542

$$1. \quad \int_0^\infty x^{\sigma-1} {}_pF_{p-1}(a_1, \dots, a_p; b_1, \dots, b_{p-1}; -\lambda x^2) Y_\nu(xy) dx \\ = \frac{\Gamma(b_1) \dots \Gamma(b_{p-1})}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+2, p+3}^{p+2, 1} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} b_0^*, \dots, b_{p+1}^*, \\ h, k, a_1^*, \dots, a_p^*, l \end{matrix} \right. \right) \\ a_j^* = a_j - \frac{\sigma}{2}, \quad j = 1, \dots, p; \quad b_0^* = 1 - \frac{\sigma}{2}; \quad b_j^* = b_j - \frac{\sigma}{2}, \\ j = 1, \dots, p-1; \quad h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2} \\ [|\arg \lambda| < \pi, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|, \quad \operatorname{Re} a_j > \frac{1}{2} \operatorname{Re} \sigma - \frac{3}{4}, \quad y > 0] \\ \text{ET II 118(53)}$$

$$2. \quad \int_0^\infty x^{\sigma-1} {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2) Y_\nu(xy) dx \\ = \frac{\Gamma(b_1) \dots \Gamma(b_p)}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+2, p+3}^{p+2, 1} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} b_0^*, \dots, b_p^*, l \\ h, k, a_1^*, \dots, a_p^*, l \end{matrix} \right. \right) \\ b_0^* = 1 - \frac{\sigma}{2}; \quad a_j^* = a_j - \frac{\sigma}{2}, \quad b_j^* = b_j - \frac{\sigma}{2}; \quad j = 1, \dots, p; \quad h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2} \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|, \quad \operatorname{Re} a_j > \frac{1}{2} \operatorname{Re} \sigma - \frac{3}{4}, \quad y > 0] \\ \text{ET II 119(54)}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^{\sigma-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2) Y_\nu(xy) dx \\
= -\pi^{-1} 2^{\sigma-1} y^{-\sigma} \cos\left[\frac{\pi}{2}(\sigma - \nu)\right] \Gamma\left(\frac{\sigma + \nu}{2}\right) \Gamma\left(\frac{\sigma - \nu}{2}\right) \\
\times {}_{p+2}F_q\left(a_1, \dots, a_p, \frac{\sigma + \nu}{2}, \frac{\sigma - \nu}{2}; b_1, \dots, b_q; -\frac{4\lambda}{y^2}\right) \\
[y > 0, \quad p \leq q - 1, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|] \quad \text{ET II 119(55)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{\sigma-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2) K_\nu(xy) dx \\
= 2^{\sigma-2} y^{-\sigma} \Gamma\left(\frac{\sigma + \nu}{2}\right) \Gamma\left(\frac{\sigma - \nu}{2}\right) {}_{p+2}F_q\left(a_1, \dots, a_p, \frac{\sigma + \nu}{2}, \frac{\sigma - \nu}{2}; b_1, \dots, b_q; -\frac{4\lambda}{y^2}\right) \\
[\operatorname{Re} y > 0, \quad p \leq q - 1, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|] \quad \text{ET II 153(88)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{2\rho} {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2) J_\nu(xy) dx \\
= \frac{2^{2\rho} \Gamma(b_1) \dots \Gamma(b_p)}{y^{2\rho+1} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+1, p+2}^{p+1, 1}\left(\frac{y^2}{4\lambda} \left| \begin{matrix} 1, & b_1, \dots, b_p \\ h, & a_1, \dots, a_p, & k \end{matrix} \right. \right) \\
h = \frac{1}{2} + \rho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2}\nu \\
[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 - \operatorname{Re} \nu < 2 \operatorname{Re} \rho < \frac{1}{2} + 2 \operatorname{Re} a_r, \quad r = 1, \dots, p] \quad \text{ET II 91(18)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x^{2\rho} {}_{m+1}F_m(a_1, \dots, a_{m+1}; b_1, \dots, b_m; -\lambda^2 x^2) J_\nu(xy) dx \\
= \frac{2^{2\rho} \Gamma(b_1) \dots \Gamma(b_m) y^{-2\rho-1}}{\Gamma(a_1) \dots \Gamma(a_{m+1})} G_{m+1, m+3}^{m+2, 1}\left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} 1, & b_1, \dots, b_m \\ h, & a_1, \dots, a_{m+1}, & k \end{matrix} \right. \right) \\
h = \frac{1}{2} + \rho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2}\nu, \\
[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\rho + \nu) > -1, \quad \operatorname{Re}(\rho - a_r) < \frac{1}{4}; \quad r = 1, \dots, m+1] \quad \text{ET II 91(19)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x^\delta F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx \\
= \frac{2^\delta \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} y^{-\delta-1} G_{24}^{22}\left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} 1 - \alpha, & 1 - \beta \\ \frac{1 + \delta + \nu}{2}, & 0, & 1 - \gamma, & \frac{1 + \delta - \nu}{2} \end{matrix} \right. \right) \\
[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 - \operatorname{Re} \nu - 2 \min(\operatorname{Re} \alpha, \operatorname{Re} \beta) < \operatorname{Re} \delta < -\frac{1}{2}] \quad \text{ET II 82(9)}
\end{aligned}$$

$$\begin{aligned}
8. \quad \int_0^\infty x^\delta F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^\delta y^{-\delta-1} \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} G_{24}^{31}\left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} 1, & \gamma \\ \frac{1 + \delta + \nu}{2}, & \alpha, \beta, & \frac{1 + \delta - \nu}{2} \end{matrix} \right. \right) \\
[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\operatorname{Re} \nu - 1 < \operatorname{Re} \delta < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{1}{2}] \quad \text{ET II 81(6)}
\end{aligned}$$

$$\begin{aligned}
9. \quad \int_0^\infty x^{\nu+1} F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^{\nu+1} \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} y^{-\nu-2} G_{13}^{30}\left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} \gamma \\ \nu + 1, & \alpha, & \beta \end{matrix} \right. \right) \\
[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 < \operatorname{Re} \nu < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2}] \quad \text{ET II 81(5)}
\end{aligned}$$

$$10. \quad \int_0^\infty x^{\nu+1} F(\alpha, \beta; \nu+1; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^{\nu-\alpha-\beta+2} \Gamma(\nu+1)}{\lambda^{\alpha+\beta} \Gamma(\alpha) \Gamma(\beta)} y^{\alpha+\beta-\nu-2} K_{\alpha-\beta} \left(\frac{y}{\lambda} \right) \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 < \operatorname{Re} \nu < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2}] \quad \text{ET II 81(3)}$$

$$11. \quad \int_0^\infty x^{\nu+1} F(\alpha, \beta; \nu+1; -\lambda^2 x^2) K_\nu(xy) dx = 2^{\nu+1} \lambda^{-\alpha-\beta} y^{\alpha+\beta-\nu-2} \Gamma(\nu+1) S_{1-\alpha-\beta, \alpha-\beta} \left(\frac{y}{\lambda} \right) \\ [\operatorname{Re} y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 152(86)}$$

$$12. \quad \int_0^\infty x^{\nu+1} F\left(\alpha, \beta; \frac{\beta+\nu}{2} + 1; -\lambda^2 x^2\right) J_\nu(xy) dx = \frac{\Gamma\left(\frac{\beta+\nu+2}{2}\right) y^{\beta-1} \lambda^{-\nu-\beta-1}}{\pi^{\frac{1}{2}} \Gamma(\alpha) \Gamma(\beta) 2^{\beta-1}} K_{\frac{1}{2}(\nu+\beta+1)} \left(\frac{y}{2\lambda} \right)^2 \\ [y > 0, \quad -1 < \operatorname{Re} \nu < (2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2})] \quad \text{ET II 81(4)}$$

$$13. \quad \int_0^\infty x^{\sigma+\frac{1}{2}} F(\alpha, \beta; \gamma; -\lambda^2 x^2) Y_\nu(xy) dx = \frac{\lambda^{-\sigma-1} y^{-\frac{1}{2}} \Gamma(\gamma)}{\sqrt{2} \Gamma(\alpha) \Gamma(\beta)} G_{35}^{41} \left(\frac{y^2}{4\lambda^2} \middle| \begin{matrix} 1-p, \gamma-p, l \\ h, k, \alpha-p, \beta-p, l \end{matrix} \right) \\ h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu, \quad p = \frac{1}{2} + \frac{1}{2}\sigma \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu| - \frac{3}{2}, \quad \operatorname{Re} \sigma < 2 \operatorname{Re} \alpha, \quad \operatorname{Re} \sigma < 2 \operatorname{Re} \beta] \quad \text{ET II 118(52)}$$

$$14. \quad \int_0^\infty x^{\nu+2} F\left(\frac{1}{2}, \frac{1}{2} - \nu; \frac{3}{2}; -\lambda^2 x^2\right) Y_\nu(xy) dx = \frac{2^\nu y^{-\nu-1}}{\pi^{\frac{1}{2}} \lambda^2 \Gamma\left(\frac{1}{2} - \nu\right)} K_\nu\left(\frac{y}{2\lambda}\right) K_{\nu+1}\left(\frac{y}{2\lambda}\right) \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}] \\ \text{ET II 117(49)}$$

$$15. \quad \int_0^\infty x^{\nu+2} F\left(1, 2\nu + \frac{3}{2}; \nu+2; -\lambda^2 x^2\right) Y_\nu(xy) dx = \pi^{-\frac{1}{2}} 2^{-\nu} \lambda^{-2\nu-3} \frac{\Gamma(\nu+2)}{\Gamma\left(2\nu + \frac{3}{2}\right)} \left[K_\nu\left(\frac{y}{2\lambda}\right) \right]^2 \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}] \\ \text{ET II 117(50)}$$

$$16. \quad \int_0^\infty x^{\nu+2} F\left(1, \mu + \nu + \frac{3}{2}; \frac{3}{2}; -\lambda^2 x^2\right) Y_\nu(xy) dx = \frac{\pi^{\frac{1}{2}} 2^{-\mu-\nu-1} \lambda^{-\mu-2\nu-3} y^{\mu+\nu}}{\Gamma\left(\mu + \nu + \frac{3}{2}\right)} K_\mu\left(\frac{y}{\lambda}\right) \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -\frac{3}{2}] \quad \text{ET II 118(51)}$$

$$17. \quad \int_0^\infty x^{2\alpha+\nu} F\left(\alpha - \nu - \frac{1}{2}, \alpha; 2\alpha; -\lambda^2 x^2\right) J_\nu(xy) dx \\ = \frac{i \Gamma\left(\frac{1}{2} + \alpha\right) \Gamma\left(\frac{1}{2} + \alpha + \nu\right)}{\pi 2^{1-\nu-2\alpha} \lambda^{2\alpha-1} y^{\nu+2}} W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(\frac{y}{\lambda}\right) \left[W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(e^{-i\pi} \frac{y}{\lambda}\right) - W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(e^{i\pi} \frac{y}{\lambda}\right) \right] \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu < -\frac{1}{2}, \quad \operatorname{Re}(\alpha + \nu) > -\frac{1}{2}] \quad \text{ET II 80(1)}$$

$$18. \quad \int_0^\infty x^{2\alpha-\nu} F\left(\nu + \alpha - \frac{1}{2}, \alpha; 2\alpha; -\lambda^2 x^2\right) J_\nu(xy) dx \\ = \frac{2^{2\alpha-\nu} \Gamma\left(\frac{1}{2} + \alpha\right) y^{\nu-2}}{\lambda^{2\alpha-1} \Gamma(2\nu)} M_{\alpha-\frac{1}{2}, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right) W_{\frac{1}{2}-\alpha, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right) \\ \text{ET II 80(2)}$$

7.543

$$1. \quad \int_0^\infty x^{-2\alpha-1} F\left(\frac{1}{2} + \alpha, 1 + \alpha; 1 + 2\alpha; -\frac{4\lambda^2}{x^2}\right) J_\nu(xy) dx = \lambda^{-2\alpha} I_{\frac{1}{2}\nu+\alpha}(\lambda y) K_{\frac{1}{2}\nu-\alpha}(\lambda y) \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > -\frac{1}{2}] \quad \text{ET II 81(7)}$$

$$2. \quad \int_0^\infty x^{\nu+1-4\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \nu + 1; -\frac{\lambda^2}{x^2}\right) J_\nu(xy) dx \\ = \frac{\Gamma(\nu)}{\Gamma(2\alpha)} 2^\nu \lambda^{1-2\alpha} y^{2\alpha-\nu-1} I_\nu\left(\frac{1}{2}\lambda y\right) K_{2\alpha-\nu-1}\left(\frac{1}{2}\lambda y\right) \\ [y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \alpha - 1 < \operatorname{Re} \nu < 4\operatorname{Re} \alpha - \frac{3}{2}] \quad \text{ET II 81(8)}$$

$$\mathbf{7.544} \quad \int_0^\infty x^{\nu+1}(1+x)^{-2\alpha} F\left[\alpha, \nu + \frac{1}{2}; 2\nu + 1; \frac{4x}{(1+x)^2}\right] J_\nu(xy) dx \\ = \frac{\Gamma(\nu+1)\Gamma(\nu-\alpha+1)}{\Gamma(\alpha)} 2^{2\nu-2\alpha+1} y^{2(\alpha-\nu-1)} J_\nu(y) \\ [y > 0, \quad -1 < \operatorname{Re} \nu < 2\operatorname{Re} \alpha - \frac{3}{2}] \quad \text{ET II 82(10)}$$

7.6 Confluent Hypergeometric Functions**7.61 Combinations of confluent hypergeometric functions and powers****7.611**

$$1. \quad \int_0^\infty x^{-1} W_{k,\mu}(x) dx = \frac{\pi^{\frac{3}{2}} 2^k \sec(\mu\pi)}{\Gamma\left(\frac{3}{4} - \frac{1}{2}k + \frac{1}{2}\mu\right) \Gamma\left(\frac{3}{4} - \frac{1}{2}k - \frac{1}{2}\mu\right)} \\ [|\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{ET II 406(22)}$$

$$2. \quad \int_0^\infty x^{-1} M_{k,\mu}(x) W_{\lambda,\mu}(x) dx = \frac{\Gamma(2\mu+1)}{(k-\lambda)\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} \\ [\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\lambda) > 0] \\ \text{BU 116(11), ET II 409(39)}$$

$$3. \quad \int_0^\infty x^{-1} W_{k,\mu}(x) W_{\lambda,\mu}(x) dx \\ = \frac{1}{(k-\lambda)\sin(2\mu\pi)} \left[\frac{1}{\Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - \lambda - \mu\right)} - \frac{1}{\Gamma\left(\frac{1}{2} - k - \mu\right) \Gamma\left(\frac{1}{2} - \lambda + \mu\right)} \right] \\ [|\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{BU 116(12), ET II 409(40)}$$

$$4. \quad \int_0^\infty \{W_{\kappa,\mu}(z)\}^2 \frac{dz}{z} = \frac{\pi}{\sin 2\pi\mu} \frac{\psi\left(\frac{1}{2} + \mu - \kappa\right) - \psi\left(\frac{1}{2} - \mu - \kappa\right)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right) \Gamma\left(\frac{1}{2} - \mu - \kappa\right)} \\ [|\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{BU 117(12a)}$$

$$5. \quad \int_0^\infty \frac{1}{z} [W_{\kappa,0}(z)]^2 dx = \frac{\psi'\left(\frac{1}{2} - \kappa\right)}{[\Gamma\left(\frac{1}{2} - \kappa\right)]^2} \quad \text{BU 117(12b)}$$

$$6. \quad \int_0^\infty x^{\rho-1} W_{k,\mu}(x) W_{-k,\mu}(x) dx = \frac{\Gamma(\rho+1) \Gamma\left(\frac{1}{2}\rho + \frac{1}{2} + \mu\right) \Gamma\left(\frac{1}{2}\rho + \frac{1}{2} - \mu\right)}{2\Gamma\left(1 + \frac{1}{2}\rho + k\right) \Gamma\left(1 + \frac{1}{2}\rho - k\right)} \\ [\operatorname{Re} \rho > 2|\operatorname{Re} \mu| - 1] \quad \text{ET II 409(41)}$$

$$7.11 \quad \int_0^\infty x^{\rho-1} W_{k,\mu}(x) W_{\lambda,\nu}(x) dx \\ = \frac{\Gamma(1-\mu+\nu+\rho) \Gamma(1+\mu+\nu+\rho) \Gamma(-2\nu)}{\Gamma\left(\frac{1}{2} - \lambda - \nu\right) \Gamma\left(\frac{3}{2} - k + \nu + \rho\right)} \\ \times {}_3F_2\left(1-\mu+\nu+\rho, 1+\mu+\nu+\rho, \frac{1}{2} - \lambda + \nu; 1+2\nu, \frac{3}{2} - k + \nu + \rho; 1\right) \\ + \frac{\Gamma(1+\mu-\nu+\rho) \Gamma(1-\mu-\nu+\rho) \Gamma(2\nu)}{\Gamma\left(\frac{1}{2} - \lambda + \nu\right) \Gamma\left(\frac{3}{2} - k - \nu + \rho\right)} \\ \times {}_3F_2\left(1+\mu-\nu+\rho, 1-\mu-\nu+\rho, \frac{1}{2} - \lambda - \nu; 1-2\nu, \frac{3}{2} - k - \nu + \rho; 1\right) \\ [|\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \rho + 1] \quad \text{ET II 410(42)}$$

7.612

$$1. \quad \int_0^\infty t^{b-1} {}_1F_1(a; c; -t) dt = \frac{\Gamma(b) \Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} \quad [0 < \operatorname{Re} b < \operatorname{Re} a] \quad \text{EH I 285(10)}$$

$$2. \quad \int_0^\infty t^{b-1} \Psi(a, c; t) dt = \frac{\Gamma(b) \Gamma(a-b) \Gamma(b-c+1)}{\Gamma(a) \Gamma(a-c+1)} \quad [0 < \operatorname{Re} b < \operatorname{Re} a \quad \operatorname{Re} c < \operatorname{Re} b + 1] \\ \text{EH I 285(11)}$$

7.613

$$1. \quad \int_0^t x^{\gamma-1} (t-x)^{c-\gamma-1} {}_1F_1(a; \gamma; x) dx = t^{c-1} \frac{\Gamma(\gamma) \Gamma(c-\gamma)}{\Gamma(c)} {}_1F_1(a; c; t) \\ [\operatorname{Re} c > \operatorname{Re} \gamma > 0] \quad \text{BU 9(16)a, EH I 271(16)}$$

$$2. \quad \int_0^t x^{\beta-1} (t-x)^{\gamma-1} {}_1F_1(t; \beta; x) dx = \frac{\Gamma(\beta) \Gamma(\gamma)}{\Gamma(\beta+\gamma)} t^{\beta+\gamma-1} {}_1F_1(t; \beta+\gamma; t) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{ET II 401(1)}$$

$$3. \quad \int_0^1 x^{\lambda-1} (1-x)^{2\mu-\lambda} {}_1F_1\left(\frac{1}{2} + \mu - \nu; \lambda; xz\right) dx = B(\lambda, 1+2\mu-\lambda) e^{\frac{1}{2}z} z^{-\frac{1}{2}-\mu} M_{\nu,\mu}(z) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\mu-\lambda) > -1] \quad \text{BU 14(14)}$$

$$4. \quad \int_0^t x^{\beta-1} (t-x)^{\delta-1} {}_1F_1(t; \beta; x) {}_1F_1(\gamma; \delta; t-x) dx = \frac{\Gamma(\beta) \Gamma(\delta)}{\Gamma(\beta+\delta)} t^{\beta+\delta-1} {}_1F_1(t+\gamma; \beta+\delta; t) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \delta > 0] \quad \text{ET II 402(2), EH I 271(15)}$$

$$5. \quad \int_0^t x^{\mu-\frac{1}{2}} (t-x)^{\nu-\frac{1}{2}} M_{k,\mu}(x) M_{\lambda,\nu}(t-x) dx = \frac{\Gamma(2\mu+1) \Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)} t^{\mu+\nu} M_{k+\lambda, \mu+\nu+\frac{1}{2}}(t) \\ \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}\right] \quad \text{BU 128(14), ET II 402(7)}$$

$$\begin{aligned}
 6. \quad \int_0^1 x^{\beta-1} (1-x)^{\sigma-\beta-1} {}_1F_1(\alpha; \beta; \lambda x) {}_1F_1[\sigma-\alpha; \sigma-\beta; \mu(1-x)] dx \\
 = \frac{\Gamma(\beta) \Gamma(\sigma-\beta)}{\Gamma(\sigma)} e^\lambda {}_1F_1(\alpha; \sigma; \mu-\lambda) \\
 [0 < \operatorname{Re} \beta < \operatorname{Re} \sigma] \quad \text{ET II 402(3)}
 \end{aligned}$$

7.62–7.63 Combinations of confluent hypergeometric functions and exponentials

7.621

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-st} t^\alpha M_{\mu, \nu}(t) dt = \frac{\Gamma(\alpha + \nu + \frac{3}{2})}{(\frac{1}{2} + s)^{\alpha + \nu + \frac{3}{2}}} F\left(\alpha + \nu + \frac{3}{2}, -\mu + \nu + \frac{1}{2}; 2\nu + 1; \frac{2}{2s + 1}\right) \\
 [\operatorname{Re}(\alpha + \mu + \frac{3}{2}) > 0, \quad \operatorname{Re} s > \frac{1}{2}] \\
 \text{BU 118(1), MO 176a, EH I 270(12)a}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty e^{-st} t^{\mu-\frac{1}{2}} M_{\lambda, \mu}(qt) dt = q^{\mu+\frac{1}{2}} \Gamma(2\mu + 1) (s - \frac{1}{2}q)^{\lambda-\mu-\frac{1}{2}} (s + \frac{1}{2}q)^{-\lambda-\mu-\frac{1}{2}} \\
 \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} s > \frac{|\operatorname{Re} q|}{2} \right] \\
 \text{BU 119(4c), MO 176a, EH I 271(13)a}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty e^{-st} t^\alpha W_{\lambda, \mu}(qt) dt = \frac{\Gamma(\alpha + \mu + \frac{3}{2}) \Gamma(\alpha - \mu + \frac{3}{2}) q^{\mu+\frac{1}{2}}}{\Gamma(\alpha - \lambda + 2)} \left(s + \frac{1}{2}q\right)^{-\alpha-\mu-\frac{3}{2}} \\
 \times F\left(\alpha + \mu + \frac{3}{2}, \mu - \lambda + \frac{1}{2}; \alpha - \lambda + 2; \frac{2s - q}{2s + q}\right) \\
 \left[\operatorname{Re}\left(\alpha \pm \mu + \frac{3}{2}\right) > 0, \quad \operatorname{Re} s > -\frac{q}{2}, \quad q > 0 \right] \quad \text{EH I 271(14)a, BU 121(6), MO 176}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_0^\infty e^{-st} t^{b-1} {}_1F_1(a; c; kt) dt = \Gamma(b) s^{-b} F(a, b; c; ks^{-1}) \quad [|s| > |k|] \\
 = \Gamma(b) (s - k)^{-b} F\left(c - a, b; c; \frac{k}{k - s}\right) \quad [|s - k| > |k|] \\
 [\operatorname{Re} b > 0, \quad \operatorname{Re} s > \max(0, \operatorname{Re} k)] \quad \text{EH I 269(5)}
 \end{aligned}$$

$$5. \quad \int_0^\infty t^{c-1} {}_1F_1(a; c; t) e^{-st} dt = \Gamma(c) s^{-c} (1 - s^{-1})^{-a} \quad [\operatorname{Re} c > 0, \quad \operatorname{Re} s > 1] \quad \text{EH I 270(6)}$$

$$\begin{aligned}
 6. \quad \int_0^\infty t^{b-1} \Psi(a, c; t) e^{-st} dt = \frac{\Gamma(b) \Gamma(b - c + 1)}{\Gamma(a + b - c + 1)} F(b, b - c + 1; a + b - c + 1; 1 - s) \\
 [\operatorname{Re} b > 0, \quad \operatorname{Re} c < \operatorname{Re} b + 1, \quad |1 - s| < 1] \\
 = \frac{\Gamma(b) \Gamma(b - c + 1)}{\Gamma(a + b - c + 1)} s^{-b} F(a, b; a + b - c + 1; 1 - s^{-1}) \\
 [\operatorname{Re} s > \frac{1}{2}] \\
 \text{EH I 270(7)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \int_0^\infty e^{-\frac{b}{2}x} x^{\nu-1} M_{\kappa, \mu}(bx) dx = \frac{\Gamma(1 + 2\mu) \Gamma(\kappa - \nu) \Gamma(\frac{1}{2} + \mu + \nu)}{\Gamma(\frac{1}{2} + \mu + \kappa) \Gamma(\frac{1}{2} + \mu - \nu)} b^\nu \\
 [\operatorname{Re}(\nu + \frac{1}{2} + \mu) > 0, \quad \operatorname{Re}(\kappa - \nu) > 0] \\
 \text{BU 119(3)a, ET I 215(11)a}
 \end{aligned}$$

$$8. \quad \int_0^\infty e^{-sx} M_{\kappa,\mu}(x) \frac{dx}{x} = \frac{2\Gamma(1+2\mu)e^{-i\pi\kappa}}{\Gamma(\frac{1}{2}+\mu+\kappa)} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} Q_{\mu-\frac{1}{2}}^\kappa(2s) \\ [\operatorname{Re}(\frac{1}{2}+\mu) > 0, \quad \operatorname{Re} s > \frac{1}{2}] \quad \text{BU 119(4a)}$$

$$9. \quad \int_0^\infty e^{-sx} W_{\kappa,\mu}(x) \frac{dx}{x} = \frac{\pi}{\cos(\frac{\pi\mu}{2})} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} P_{\mu-\frac{1}{2}}^\kappa(2s) \\ [\operatorname{Re}(\frac{1}{2}\pm\mu) > 0, \quad \operatorname{Re} s > -\frac{1}{2}] \quad \text{BU 121(7)}$$

$$10. \quad \int_0^\infty x^{k+2\mu-1} e^{-\frac{3}{2}x} W_{k,\mu}(x) dx = \frac{\Gamma(k+\mu+\frac{1}{2})\Gamma[\frac{1}{4}(2k+6\mu+5)]}{(k+3\mu+\frac{1}{2})\Gamma[\frac{1}{4}(2\mu-2k+3)]} \\ [\operatorname{Re}(k+\mu) > -\frac{1}{2}, \quad \operatorname{Re}(k+3\mu) > -\frac{1}{2}] \quad \text{BU 122(8a), ET II 406(23)}$$

$$11. \quad \int_0^\infty e^{-\frac{1}{2}x} x^{\nu-1} W_{\kappa,\mu}(x) dx = \frac{\Gamma(\nu+\frac{1}{2}-\mu)\Gamma(\nu+\frac{1}{2}+\mu)}{\Gamma(\nu-\kappa+1)} \\ [\operatorname{Re}(\nu+\frac{1}{2}\pm\mu) > 0] \quad \text{BU 122(8b)}$$

$$12. \quad \int_0^\infty e^{\frac{1}{2}x} x^{\nu-1} W_{\kappa,\mu}(x) dx = \Gamma(-\kappa-\mu) \frac{\Gamma(\frac{1}{2}+\mu+\nu)\Gamma(\frac{1}{2}-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu-\kappa)\Gamma(\frac{1}{2}+\mu-\kappa)} \\ [\operatorname{Re}(\nu+\frac{1}{2}\pm\mu) > 0, \quad \operatorname{Re}(\kappa+\nu) < 0] \quad \text{BU 122(8c)a}$$

7.622

$$1. \quad \int_0^\infty e^{-st} t^{c-1} {}_1F_1(a; c; t) {}_1F_1(\alpha; c; \lambda t) dt \\ = \Gamma(c)(s-\lambda)^{-a}(s-\lambda)^{-\alpha} s^{a+\alpha-c} F[a, \alpha; c; \lambda(s-1)^{-1}(s-\lambda)^{-1}] \\ [\operatorname{Re} c > 0, \quad \operatorname{Re} s > \operatorname{Re} \lambda + 1] \quad \text{EH I 287(22)}$$

$$2. \quad \int_0^\infty e^{-t} t^\rho {}_1F_1(a; c; t) \Psi(a'; c'; \lambda t) dt \\ = C \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(\gamma)} \lambda^\sigma F(c-a, \beta; \gamma; 1-\lambda^{-1}), \\ \rho = c-1, \quad \sigma = -c, \quad \beta = c-c'+1, \quad \gamma = c-a+a'-c'+1, \quad C = \frac{\Gamma(a'-a)}{\Gamma(a')}, \text{ or}$$

$$\rho = c+c'-2, \quad \sigma = 1-c-c', \quad \beta = c+c'-1, \quad \gamma = a'-a+c, \quad C = \frac{\Gamma(a'-a-c'+1)}{\Gamma(a'-c'+1)} \\ \text{EH I 287(24)}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\nu-1} e^{-bx} M_{\lambda_1, \mu_1 - \frac{1}{2}}(a_1 x) \dots M_{\lambda_n, \mu_n - \frac{1}{2}}(a_n x) dx \\
& = a_1^{\mu_1} \dots a_n^{\mu_n} (b+A)^{-\nu-M} \Gamma(\nu+M) \\
& \quad \times F_A \left(\nu+M; \mu_1 - \lambda_1, \dots, \mu_n - \lambda_n; 2\mu_1, \dots, 2\mu_n; \frac{a_1}{b+A}, \dots, \frac{a_n}{b+A} \right), \\
& \quad M = \mu_1 + \dots + \mu_n, \quad A = \frac{1}{2} (a_1 + \dots + a_n) \\
& \quad [\operatorname{Re}(\nu+M) > 0, \quad \operatorname{Re}(b \pm \tfrac{1}{2}a_1 \pm \dots \pm \tfrac{1}{2}a_n) > 0] \quad \text{ET I 216(14)}
\end{aligned}$$

7.623

$$\begin{aligned}
1. \quad & \int_0^\infty e^{-x} x^{c+n-1} (x+y)^{-1} {}_1F_1(a; c; x) dx = (-1)^n \Gamma(c) \Gamma(1-a) y^{c+n-1} \Psi(c-a, c; y) \\
& \quad [-\operatorname{Re} c < n < 1 - \operatorname{Re} a, \quad n = 0, 1, 2, \dots, \quad |\arg y| < \pi] \quad \text{EH I 285(16)} \\
2. \quad & \int_0^t x^{-1} (t-x)^{k-1} e^{\frac{1}{2}(t-x)} M_{k, \mu}(x) dx = \frac{\Gamma(k) \Gamma(2\mu+1)}{\Gamma(k+\mu+\frac{1}{2})} \pi^{\frac{1}{2}} t^{k-\frac{1}{2}} l_\mu \left(\frac{1}{2}t \right) \\
& \quad [\operatorname{Re} k > 0, \quad \operatorname{Re} \mu > -\tfrac{1}{2}] \quad \text{ET II 402(5)} \\
3. \quad & \int_0^t x^{k-1} (t-x)^{\lambda-1} e^{\frac{1}{2}(t-x)} M_{k+\lambda, \mu}(x) dx = \frac{\Gamma(\lambda) \Gamma(k+\mu+\frac{1}{2}) t^{k+\lambda-1}}{\Gamma(k+\lambda+\mu+\frac{1}{2})} M_{k, \mu}(t) \\
& \quad [\operatorname{Re}(k+\mu) > -\tfrac{1}{2}, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 402(6)} \\
4. \quad & \int_0^t x^{-k-\lambda-1} (t-x)^{\lambda-1} e^{\frac{1}{2}x} W_{k, \mu}(x) dx = \frac{\Gamma(\lambda) \Gamma(\frac{1}{2}-k-\lambda+\mu) \Gamma(\frac{1}{2}-k-\lambda-\mu)}{t^{k+1} \Gamma(\frac{1}{2}-k+\mu) \Gamma(\frac{1}{2}-k-\mu)} W_{k+\lambda, \mu}(t) \\
& \quad [\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k+\lambda) < \tfrac{1}{2} - |\operatorname{Re} \mu|] \quad \text{ET II 405(21)} \\
5. \quad & \int_1^\infty (x-1)^{\mu-1} x^{\lambda-\frac{1}{2}} e^{\frac{1}{2}ax} W_{k, \lambda}(ax) dx = \frac{\Gamma(\mu) \Gamma(\frac{1}{2}-k-\lambda-\mu)}{\Gamma(\frac{1}{2}-k-\lambda)} a^{-\frac{1}{2}\mu} e^{\frac{1}{2}a} W_{k+\frac{1}{2}\mu, \lambda+\frac{1}{2}\mu}(a) \\
& \quad [|\arg(a)| < \tfrac{3}{2}\pi, \quad 0 < \operatorname{Re} \mu < \tfrac{1}{2} - \operatorname{Re}(k+\lambda)] \quad \text{ET II 211(72)a} \\
6.^{11} \quad & \int_1^\infty (x-1)^{\mu-1} x^{\mu-\frac{1}{2}} e^{-\frac{1}{2}ax} W_{2\mu+\frac{1}{2}, \lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{\mu+\frac{1}{2}, \lambda}(a) \\
& \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 211(74)a} \\
7. \quad & \int_1^\infty (x-1)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k, \lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{k-\mu, \lambda}(a) \\
& \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 211(73)a}
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int_0^1 (1-x)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\lambda}(ax) dx \\
&= \Gamma(\mu) e^{-\frac{1}{2}a} \sec[(k-\mu-\lambda)\pi] \\
&\quad \times \left\{ \sin(\mu\pi) \frac{\Gamma(k-\mu+\lambda+\frac{1}{2})}{\Gamma(2\lambda+1)} M_{k-\mu,\lambda}(a) + \cos[(k-\lambda)\pi] W_{k-\mu,\lambda}(a) \right\} \\
&\quad [0 < \operatorname{Re} \mu < \operatorname{Re} k - |\operatorname{Re} \lambda| + \tfrac{1}{2}] \quad \text{ET II 200(93)a}
\end{aligned}$$

7.624

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} M_{k,\mu}(x) dx \\
&= \frac{-\sigma \Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} + k + \mu)} G_{34}^{23} \left(a \left| \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2} + \mu + \rho, -\sigma, \sigma, \frac{1}{2} - \mu + \rho \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re}(\mu + \rho) > -\tfrac{1}{2}, \quad \operatorname{Re}(k - \rho - \sigma) > 0] \quad \text{ET II 403(8)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx = -\pi^{-\frac{1}{2}} \sigma a^\sigma G_{34}^{32} \left(a \left| \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2} + \mu + \rho, \frac{1}{2} - \mu + \rho, -\sigma, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \tfrac{1}{2}] \quad \text{ET II 406(24)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx \\
&= -\frac{\sigma \pi^{-\frac{1}{2}} a^\sigma}{\Gamma(\frac{1}{2} - k + \mu) \Gamma(\frac{1}{2} - k - \mu)} G_{34}^{33} \left(a \left| \begin{matrix} \frac{1}{2}, 1, 1+k+\rho \\ \frac{1}{2} + \mu + \rho, \frac{1}{2} - \mu + \rho, -\sigma, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \tfrac{1}{2}, \quad \operatorname{Re}(k + \rho + \sigma) < 0] \quad \text{ET II 406(25)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} M_{k,\mu}(x) dx \\
&= \frac{\Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} + k + \mu)} G_{34}^{23} \left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2} - k - \rho \\ -\sigma, \rho + \mu, \rho - \mu, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re}(\rho + \mu) > -\tfrac{1}{2}, \quad \operatorname{Re}(k - \rho - \sigma) > -\tfrac{1}{2}] \quad \text{ET II 403(9)}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx \\
&= \frac{\pi^{-\frac{1}{2}} a^\sigma}{\Gamma(\frac{1}{2} - k + \mu) \Gamma(\frac{1}{2} - k - \mu)} G_{34}^{33} \left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2} + k + \rho \\ -\sigma, \rho + \mu, \rho - \mu, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \tfrac{1}{2}, \quad \operatorname{Re}(k + \rho + \sigma) < \tfrac{1}{2}] \quad \text{ET II 406(26)}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx = \pi^{-\frac{1}{2}} a^\sigma G_{34}^{32} \left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2} - k + \rho \\ -\sigma, \rho + \mu, \rho - \mu, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \tfrac{1}{2}] \quad \text{ET II 406(27)}
\end{aligned}$$

7.625

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\rho-1} \exp \left[-\frac{1}{2}(\alpha + \beta)x \right] M_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx \\
&= \frac{\Gamma(1 + \mu + \nu + \rho) \Gamma(1 + \mu - \nu + \rho)}{\Gamma\left(\frac{3}{2} - \lambda + \mu + \rho\right)} \alpha^{\mu+\frac{1}{2}} \beta^{-\mu-\rho-\frac{1}{2}} \\
&\quad \times {}_3F_2 \left(\frac{1}{2} + k + \mu, 1 + \mu + \nu + \rho, 1 + \mu - \nu + \rho; 2\mu + 1, \frac{3}{2} - \lambda + \mu + \rho; -\frac{\alpha}{\beta} \right) \\
&\quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\rho + \mu) > |\operatorname{Re} \nu| - 1] \quad \text{ET II 410(43)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\rho-1} \exp \left[-\frac{1}{2}(\alpha + \beta)x \right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx \\
&= \beta^{-\rho} \left[\Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right) \Gamma\left(\frac{1}{2} - \lambda + \nu\right) \Gamma\left(\frac{1}{2} - \lambda - \nu\right) \right]^{-1} \\
&\quad \times G_{33}^{33} \left(\frac{\beta}{\alpha} \left| \begin{matrix} \frac{1}{2} + \mu, \frac{1}{2} - \mu, 1 + \lambda + \rho \\ \frac{1}{2} + \nu + \rho, \frac{1}{2} - \nu + \rho, -k \end{matrix} \right. \right) \\
&\quad [|\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \rho + 1, \quad \operatorname{Re}(k + \lambda + \rho) < 0] \quad \text{ET II 410(44)a}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\rho-1} \exp \left[-\frac{1}{2}(\alpha + \beta)x \right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx = \beta^{-\rho} G_{33}^{22} \left(\frac{\beta}{\alpha} \left| \begin{matrix} \frac{1}{2} + \mu, \frac{1}{2} - \nu, 1 - \lambda + \rho \\ \frac{1}{2} + \nu + \rho, \frac{1}{2} - \nu + \rho, k \end{matrix} \right. \right) \\
&\quad \text{ET II 411(46)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{\rho-1} \exp \left[-\frac{1}{2}(\alpha - \beta)x \right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx \\
&= \beta^{-\rho} \left[\Gamma\left(\frac{1}{2} - \lambda + \nu\right) \Gamma\left(\frac{1}{2} - \lambda - \nu\right) \right]^{-1} G_{33}^{23} \left(\frac{\beta}{\alpha} \left| \begin{matrix} \frac{1}{2} + \mu, \frac{1}{2} - \mu, 1 + \lambda + \rho \\ \frac{1}{2} + \nu + \rho, \frac{1}{2} - \nu + \rho, k \end{matrix} \right. \right) \\
&\quad [\operatorname{Re} \alpha > 0, \quad |\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \rho + 1] \quad \text{ET II 411(45)}
\end{aligned}$$

7.626

$$\begin{aligned}
1. \quad & \int_0^1 \left[\frac{k}{x} - \frac{1}{4}(\xi + \eta) \exp \left[-\frac{1}{2}(\xi + \eta)x \right] x^c \right] {}_1F_1(a; c; \xi x) {}_1F_1(a; c; \eta x) dx \\
&= 0 \quad [\xi \neq \eta, \quad \operatorname{Re} c > 0] \\
&= \frac{a}{\xi} e^{-\xi} [{}_1F_1(a + 1; c; \xi)]^2 \quad [\xi = \eta, \quad \operatorname{Re} c > 0] \\
&\quad [\text{where } \xi \text{ and } \eta \text{ are any two zeros of the function } {}_1F_1(a; c; x)] \quad \text{EH I 285}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_1^\infty \left[\frac{k}{x} - \frac{1}{4}(\xi + \eta) \right] e^{-\frac{1}{2}(\xi + \eta)x} x^c \Psi(a, c; \xi x) \Psi(a, c; \eta x) dx = 0 \quad [\xi \neq \eta]; \\
&= -\xi^{-1} e^{-\xi} [\Psi(a - 1, c; \xi)]^2 \quad [\xi = \eta] \\
&\quad [\text{where } \xi \text{ and } \eta \text{ are any two zeros of the function } \Psi(a, c; x)] \quad \text{EH I 286}
\end{aligned}$$

7.627

1.
$$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{\frac{1}{2}x} W_{k,\mu}(a+x) dx = \frac{\Gamma(2\lambda)\Gamma(\frac{1}{2}-k+\mu-2\lambda)}{\Gamma(\frac{1}{2}-k+\mu)} a^{\lambda-\mu-\frac{1}{2}} W_{k+\lambda,\mu-\lambda}(a)$$

$$\left[|\arg a| < \pi, \quad 0 < 2\operatorname{Re} \lambda < \frac{1}{2} - \operatorname{Re}(k+\mu) \right] \quad \text{ET II 411(50)}$$
2.
$$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x} M_{k,\mu}^{-\frac{1}{2}x}(a+x) dx$$

$$= \frac{\Gamma(2\lambda)\Gamma(2\mu+1)\Gamma(k+\mu-2\lambda+\frac{1}{2})}{\Gamma(k+\mu+\frac{1}{2})\Gamma(1-2\lambda+2\mu)} a^{\lambda-\mu-\frac{1}{2}} M_{k-\lambda,\mu-\lambda}(a)$$

$$[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k+\mu-2\lambda) > -\frac{1}{2}] \quad \text{ET II 405(20)}$$
3.
$$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(2\lambda)a^{\lambda-\mu-\frac{1}{2}} W_{k-\lambda,\mu-\lambda}(a)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 411(47)}$$
4.
$$\int_0^\infty x^{\lambda-1}(a+x)^{k-\lambda-1}e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(\lambda)a^{k-1} W_{k-\lambda,\mu}(a)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 411(48)}$$
5.
$$\int_0^\infty x^{\rho-1}(a+x)^{-\sigma}e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(\rho)a^\rho e^{\frac{1}{2}a} G_{23}^{30} \left(a \left| \begin{matrix} 0, 1-k-\sigma \\ -\rho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma \end{matrix} \right. \right)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \rho > 0] \quad \text{ET II 411(49)}$$
6.
$$\int_0^\infty x^{\rho-1}(a+x)^{-\sigma}e^{\frac{1}{2}x} W_{k,\mu}(a+x) dx$$

$$= \frac{\Gamma(\rho)a^\rho e^{-\frac{1}{2}a}}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)} G_{23}^{31} \left(a \left| \begin{matrix} k-\sigma+1, 0 \\ -\rho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma \end{matrix} \right. \right)$$

$$[|\arg a| < \pi, \quad 0 < \operatorname{Re} \rho < \operatorname{Re}(\sigma-k)] \quad \text{ET II 412(51)}$$
7.
$$\int_0^\infty e^{-\frac{1}{2}(a+x)} \frac{(a+x)^{2\kappa-1}}{(ax)^\kappa} W_{\kappa,\mu}(x) \frac{dx}{x} = \frac{\Gamma(\frac{1}{2}-\mu-\kappa)\Gamma(\frac{1}{2}+\mu-\kappa)}{a\Gamma(1-2\kappa)} W_{\kappa,\mu}(a)$$

$$[\operatorname{Re}(\frac{1}{2} \pm \mu - \kappa) > 0] \quad \text{BU 126(7a)}$$
8.
$$\int_0^\infty e^{-\frac{1}{2}x} x^{\gamma+\alpha-1} M_{\kappa,\mu}(x) \frac{dx}{(x+a)^\alpha}$$

$$= \frac{\Gamma(1+2\mu)\Gamma(\frac{1}{2}+\mu+\gamma)\Gamma(\kappa-\gamma)}{\Gamma(\frac{1}{2}+\mu-\gamma)\Gamma(\frac{1}{2}+\mu+\kappa)} {}_2F_2 \left(\alpha, \kappa-\gamma; \frac{1}{2}+\mu-\gamma, \frac{1}{2}-\mu-\gamma; a \right)$$

$$+ \frac{\Gamma(\alpha+\gamma+\frac{1}{2}+\mu)\Gamma(-\gamma-\frac{1}{2}-\mu)}{\Gamma(\alpha)} a^{\gamma+\frac{1}{2}+\mu}$$

$$\times {}_2F_2 \left(\alpha+\gamma+\mu+\frac{1}{2}, \kappa+\mu+\frac{1}{2}; 1+2\mu, \frac{3}{2}+\mu+\gamma; a \right)$$

$$[\operatorname{Re}(\gamma+\alpha+\frac{1}{2}+\mu) > 0, \quad \operatorname{Re}(\gamma-\kappa) < 0] \quad \text{BU 126(8)a}$$

$$9. \quad \int_0^\infty e^{-\frac{1}{2}x} x^{n+\mu+\frac{1}{2}} M_{\kappa,\mu}(x) \frac{dx}{x+a} = (-1)^{n+1} a^{n+\mu+\frac{1}{2}} e^{\frac{1}{2}a} \Gamma(1+2\mu) \Gamma\left(\frac{1}{2}-\mu+\kappa\right) W_{-\kappa,\mu}(a) \\ \left[n = 0, 1, 2, \dots, \quad \operatorname{Re}\left(\mu+1+\frac{n}{2}\right) > 0, \quad \operatorname{Re}\left(\kappa-\mu-\frac{1}{2}\right) < n, \quad |\arg a| < \pi \right] \quad \text{BU 127(10a)a}$$

7.628

$$1. \quad \int_0^\infty e^{-st} e^{-t^2} t^{2c-2} {}_1F_1(a; c; t^2) dt = 2^{1-2c} \Gamma(2c-1) \Psi\left(c-\frac{1}{2}, a+\frac{1}{2}; \frac{1}{4}s^2\right) \\ [\operatorname{Re} c > \frac{1}{2}, \quad \operatorname{Re} s > 0] \quad \text{EH I 270(11)}$$

$$2. \quad \int_0^\infty t^{2\nu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{-3\nu,\nu}\left(\frac{t^2}{a}\right) dt = \frac{1}{2\sqrt{\pi}} \Gamma(4\nu+1) a^{-\nu} s^{-4\nu} e^{as^2/8} K_{2\nu}\left(\frac{as^2}{8}\right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{4}, \quad \operatorname{Re} s > 0] \quad \text{ET I 215(12)}$$

$$3. \quad \int_0^\infty t^{2\mu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{\lambda,\mu}\left(\frac{t^2}{a}\right) dt \\ = 2^{-3\mu-\lambda} \Gamma(4\mu+1) a^{\frac{1}{2}(\lambda+\mu-1)} s^{\lambda-\mu-1} e^{\frac{as^2}{8}} W_{-\frac{1}{2}(\lambda+3\mu), \frac{1}{2}(\lambda-\mu)}\left(\frac{as^2}{4}\right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{4}, \quad \operatorname{Re} s > 0] \quad \text{ET I 215(13)}$$

7.629

$$1.^8 \quad \int_0^\infty t^k \exp\left(\frac{a}{2t}\right) e^{-st} W_{k,\mu}\left(\frac{a}{t}\right) dt = 2^{1-2k} \sqrt{as}^{-k-\frac{1}{2}} S_{2k,2\mu}(2\sqrt{as}) \\ [|\arg a| < \pi, \quad \operatorname{Re}(k \pm \mu) > -\frac{1}{2}, \quad \operatorname{Re} s > 0] \quad \text{ET I 217(21)}$$

$$2. \quad \int_0^\infty t^{-k} \exp\left(-\frac{a}{2t}\right) e^{-st} W_{k,\mu}\left(\frac{a}{t}\right) dt = 2\sqrt{as}^{k-\frac{1}{2}} K_{2\mu}(2\sqrt{as}) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} s > 0] \quad \text{ET I 217(22)}$$

7.631

$$1. \quad \int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1})\right] W_{k,\mu}(\alpha^{-1}x) W_{\lambda,\nu}(\beta x^{-1}) dx \\ = \beta^\rho \left[\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1} \\ \times G_{24}^{41}\left(\frac{\beta}{\alpha} \left| \begin{matrix} 1+k, & 1-\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\rho, & \frac{1}{2}-\nu-\rho \end{matrix} \right. \right) \\ [|\arg \alpha| < \frac{3}{2}\pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(k+\rho) < -|\operatorname{Re} \nu| - \frac{1}{2}] \quad \text{ET II 412(55)}$$

$$2. \quad \int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1})\right] W_{k,\mu}(\alpha^{-1}x) W_{\lambda,\nu}(\beta x^{-1}) dx \\ = \beta^\rho \left[\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right) \Gamma\left(\frac{1}{2}-\lambda+\nu\right) \Gamma\left(\frac{1}{2}-\lambda-\nu\right)\right]^{-1} \\ \times G_{24}^{42}\left(\frac{\beta}{\alpha} \left| \begin{matrix} 1+k, & 1+\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\rho, & \frac{1}{2}-\nu-\rho \end{matrix} \right. \right) \\ [|\arg \alpha| < \frac{3}{2}\pi, \quad |\arg \beta| < \frac{3}{2}\pi, \quad \operatorname{Re}(\lambda-\rho) < \frac{1}{2}-|\operatorname{Re} \mu|, \quad \operatorname{Re}(k+\rho) < \frac{1}{2}-|\operatorname{Re} \nu|] \\ \text{ET II 412(57)}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\rho-1} \exp \left[\frac{1}{2} (\alpha^{-1} x + \beta x^{-1}) \right] W_{k,\mu}(\alpha^{-1} x) W_{\lambda,\nu}(\beta x^{-1}) dx \\
& = \beta^\rho G_{24}^{40} \left(\frac{\beta}{\alpha} \left| \begin{matrix} 1-k, & 1-\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\rho, & \frac{1}{2}-\nu-\rho \end{matrix} \right. \right) \\
& \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 412(54)} \\
7.632 \quad & \int_0^\infty e^{-st} (e^t - 1)^{\mu-\frac{1}{2}} \exp \left(-\frac{1}{2} \lambda e^t \right) M_{k,\mu}(\lambda e^t - \lambda) dt \\
& = \frac{\Gamma(2\mu+1) \Gamma(\frac{1}{2} + k - \mu + s)}{\Gamma(s+1)} W_{-k-\frac{1}{2}s, \mu-\frac{1}{2}s}(\lambda) \\
& \quad [\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} s > \operatorname{Re}(\mu - k) - \frac{1}{2}] \quad \text{ET I 216(15)}
\end{aligned}$$

7.64 Combinations of confluent hypergeometric and trigonometric functions

$$\begin{aligned}
7.641 \quad & \int_0^\infty \cos(ax) {}_1F_1(\nu+1; 1; ix) {}_1F_1(\nu+1; 1; -ix) dx \\
& = -a^{-1} \sin(\nu\pi) P_\nu(2a^{-2} - 1) \quad [0 < a < 1]; \\
& = 0 \quad [1 < a < \infty] \\
& \quad [-1 < \operatorname{Re} \nu < 0] \quad \text{ET II 402(4)} \\
7.642^{11} \quad & \int_0^\infty \cos(2xy) {}_1F_1(a; c; -x^2) dx = \frac{1}{2} \pi^{\frac{1}{2}} \frac{\Gamma(c)}{\Gamma(a)} |y|^{2\alpha-1} e^{-y^2} \Psi\left(c - \frac{1}{2}, a + \frac{1}{2}; y^2\right) \quad \text{EH I 285(12)} \\
7.643 \quad & \\
1. \quad & \int_0^\infty x^{4\nu} e^{-\frac{1}{2}x^2} \sin(bx) {}_1F_1\left(\frac{1}{2} - 2\nu; 2\nu + 1; \frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{4\nu} c^{-\frac{1}{2}b^2} {}_1F_1\left(\frac{1}{2} - 2\nu; 1 + 2\nu; \frac{1}{2}b^2\right) \\
& \quad [b > 0, \quad \operatorname{Re} \nu > -\frac{1}{4}] \quad \text{ET I 115(5)} \\
2. \quad & \int_0^\infty x^{2\nu-1} e^{-\frac{1}{4}x^2} \sin(bx) M_{3\nu,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{2\nu-1} e^{-\frac{1}{4}b^2} M_{3\nu,\nu}\left(\frac{1}{2}b^2\right) \\
& \quad [b > 0, \quad \operatorname{Re} \nu > -\frac{1}{4}] \quad \text{ET I 116(10)} \\
3. \quad & \int_0^\infty x^{-2\nu-1} e^{\frac{1}{4}x^2} \cos(bx) W_{3\nu,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{-2\nu-1} e^{\frac{1}{4}b^2} W_{3\nu,\nu}\left(\frac{1}{2}b^2\right) \\
& \quad [\operatorname{Re} \nu < \frac{1}{4}, \quad b > 0] \quad \text{ET I 61(7)} \\
4. \quad & \int_0^\infty x^{-2\nu} e^{\frac{1}{4}x^2} \sin(bx) W_{3\nu-1,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{-2\nu} e^{\frac{1}{4}b^2} W_{3\nu-1,\nu}\left(\frac{1}{2}b^2\right) \\
& \quad [\operatorname{Re} \nu < \frac{1}{2}, \quad b > 0] \quad \text{ET I 116(9)}
\end{aligned}$$

7.644

$$\begin{aligned}
1.^{11} \quad & \int_0^\infty x^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} \sin\left(2ax^{\frac{1}{2}}\right) M_{k,\mu}(x) dx = \pi^{\frac{1}{2}} a^{k+\mu-1} \frac{\Gamma(3-2\mu)}{\Gamma(\frac{1}{2} + k + \mu)} \exp\left(-\frac{a^2}{2}\right) W_{\rho,\sigma}(a^2), \\
& \quad 2\rho = k - 3\mu + 1, \quad 2\sigma = k + \mu - 1 \quad [a > 0, \quad \operatorname{Re}(k + \mu) > 0] \quad \text{ET II 403(10)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{\rho-1} \sin\left(cx^{\frac{1}{2}}\right) e^{-\frac{1}{2}x} W_{k,\mu}(x) dx &= \frac{c\Gamma(1+\mu+\rho)\Gamma(1-\mu+\rho)}{\Gamma\left(\frac{3}{2}-k+\rho\right)} \\
&\times {}_2F_2\left(1+\mu+\rho, 1-\mu+\rho; \frac{3}{2}, \frac{3}{2}-k+\rho; -\frac{c^2}{4}\right) \\
&[\operatorname{Re} \rho > |\operatorname{Re} \mu| - 1] \quad \text{ET II 407(28)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^{\rho-1} \sin\left(cx^{\frac{1}{2}}\right) e^{\frac{1}{2}x} W_{k,\mu}(x) dx \\
&= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G_{23}^{22}\left(\frac{c^2}{4} \left| \begin{matrix} \frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho \\ \frac{1}{2}, -k-\rho, 0 \end{matrix} \right. \right) \\
&[c > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - 1, \quad \operatorname{Re}(k+\rho) < \frac{1}{2}] \quad \text{ET II 407(29)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{\rho-1} \cos\left(cx^{\frac{1}{2}}\right) e^{-\frac{1}{2}x} W_{k,\mu}(x) dx &= \frac{\Gamma\left(\frac{1}{2}+\mu+\rho\right)\Gamma\left(\frac{1}{2}-\mu+\rho\right)}{\Gamma(1-k+\rho)} \\
&\times {}_2F_2\left(\frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho; \frac{1}{2}, 1-k+\rho; -\frac{c^2}{4}\right) \\
&[\operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}] \quad \text{ET II 407(30)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{\rho-1} \cos\left(cx^{\frac{1}{2}}\right) e^{\frac{1}{2}x} W_{k,\mu}(x) dx \\
&= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G_{23}^{22}\left(\frac{c^2}{4} \left| \begin{matrix} \frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho \\ 0, -k-\rho, \frac{1}{2} \end{matrix} \right. \right) \\
&[c > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\rho) < \frac{1}{2}] \quad \text{ET II 407(31)}
\end{aligned}$$

7.65 Combinations of confluent hypergeometric functions and Bessel functions

7.651

$$\begin{aligned}
1. \quad \int_0^\infty J_\nu(xy) M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ax) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ax) dx \\
&= ay^{-\mu-1} \frac{\Gamma(\nu+1)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu\right)} \left[a + (a^2 + y^2)^{\frac{1}{2}}\right]^\mu (a^2 + y^2)^{-\frac{1}{2}} \\
&[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} a > 0] \quad \text{ET II 85(19)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty M_{k, \frac{1}{2}\nu}(-iax) M_{-k, \frac{1}{2}\nu}(-iax) J_\nu(xy) dx \\
&= \frac{ae^{-\frac{1}{2}(\nu+1)\pi i} [\Gamma(1+\nu)]^2}{\Gamma\left(\frac{1}{2}+k+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-k+\frac{1}{2}\nu\right)} y^{-1-2k} \\
&\times (a^2 - y^2)^{-\frac{1}{2}} \left\{ \left[a + (a^2 - y^2)^{\frac{1}{2}}\right]^{2k} + \left[a - (a^2 - y^2)^{\frac{1}{2}}\right]^{2k} \right\} \quad [0 < y < a]; \\
&= 0 \quad [a < y < \infty] \\
&[a > 0, \quad \operatorname{Re} \nu > -1, \quad |\operatorname{Re} k| < \frac{1}{4}] \quad \text{ET II 85(18)}
\end{aligned}$$

$$\begin{aligned}
7.652 \quad \int_0^\infty M_{-\mu, \frac{1}{2}\nu} \left\{ a \left[(b^2 + x^2)^{\frac{1}{2}} - b \right] \right\} W_{\mu, \frac{1}{2}\nu} \left\{ a \left[(b^2 + x^2)^{\frac{1}{2}} + b \right] \right\} J_\nu(xy) dx \\
= \frac{ay^{-2\mu-1} \Gamma(1+\nu) \left[(a^2 + y^2)^{\frac{1}{2}} + a \right]^{2\mu}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \mu\right) (A^2 + Y^2)^{\frac{1}{2}}} \exp \left[-b(a^2 + y^2)^{\frac{1}{2}} \right] \\
[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{1}{4}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 87(29)}
\end{aligned}$$

7.66 Combinations of confluent hypergeometric functions, Bessel functions, and powers

7.661

$$\begin{aligned}
1. \quad \int_0^\infty x^{-1} W_{k,\mu}(ax) M_{-k,\mu}(ax) J_0(xy) dx \\
= e^{-ik\pi} \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2} + \mu + k\right)} P_{\mu-\frac{1}{2}}^k \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] Q_{\mu-\frac{1}{2}}^k \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] \\
[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} k < \frac{3}{4}] \quad \text{ET II 18(44)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{-1} W_{k,\mu}(ax) W_{-k,\mu}(ax) J_0(xy) dx = \frac{1}{2} \pi \cos(\mu\pi) P_{\mu-\frac{1}{2}}^k \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] P_{\mu-\frac{1}{2}}^{-k} \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] \\
[y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \mu| < \frac{1}{2}] \\
\text{ET II 18(45)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^{2\mu-\nu} W_{k,\mu}(ax) M_{-k,\mu}(ax) J_\nu(xy) dx \\
= 2^{2\mu-\nu+2k} a^{2k} y^{\nu-2\mu-2k-1} \frac{\Gamma(2\mu+1)}{\Gamma\left(\nu - k - \mu + \frac{1}{2}\right)} \\
\times {}_3F_2 \left(\frac{1}{2} - k, 1 - k, \frac{1}{2} - k + \mu; 1 - 2k, \frac{1}{2} - k - \mu + \nu; -\frac{y^2}{a^2} \right) \\
[y > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(2\mu + 2k - \nu) < \frac{1}{2}] \quad \text{ET II 85(20)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{2\rho-\nu} W_{k,\mu}(iax) W_{k,\mu}(-iax) J_\nu(xy) dx \\
= 2^{2\rho-\nu} y^{\nu-2\rho-1} \pi^{-\frac{1}{2}} \left[\Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right) \right]^{-1} G_{44}^{24} \left(\frac{y^2}{a^2} \left| \begin{matrix} \frac{1}{2}, 0, \frac{1}{2} - \mu, \frac{1}{2} + \mu \\ \rho + \frac{1}{2}, -k, k, \rho - \nu + \frac{1}{2} \end{matrix} \right. \right) \\
[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - 1, \quad \operatorname{Re}(2\rho + 2k - \nu) < \frac{1}{2}] \quad \text{ET II 86(23)a}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{2\rho-\nu} W_{k,\mu}(ax) M_{-k,\mu}(ax) J_\nu(xy) dx \\
= \frac{2^{2\rho-\nu} \Gamma(2\mu+1)}{\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2} - k + \mu\right)} y^{\nu-2\rho-1} G_{44}^{23} \left(\frac{y^2}{a^2} \left| \begin{matrix} \frac{1}{2}, 0, \frac{1}{2} - \mu, \frac{1}{2} + \mu \\ \rho + \frac{1}{2}, -k, k, \rho - \nu + \frac{1}{2} \end{matrix} \right. \right) \\
[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \rho > -1, \quad \operatorname{Re}(\rho + \mu) > -1, \quad \operatorname{Re}(2\rho + 2k + \nu) < \frac{1}{2}] \quad \text{ET II 86(21)a}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int_0^\infty x^{2\rho-\nu} W_{k,\mu}(ax) W_{-k,\mu}(ax) J_\nu(xy) dx \\
&= \frac{\Gamma(\rho+1+\mu)\Gamma(\rho+1-\mu)\Gamma(2\rho+2)}{\Gamma(\frac{3}{2}+k+\rho)\Gamma(\frac{3}{2}-k+\rho)\Gamma(1+\nu)} y^\nu 2^{-\nu-1} a^{-2\rho-1} \\
&\quad \times {}_4F_3\left(\rho+1, \rho+\frac{3}{2}, \rho+1+\mu, \rho+1-\mu; \frac{3}{2}+k+\rho, \frac{3}{2}-k+\rho, 1+\nu; -\frac{y^2}{a^2}\right) \\
&\quad [y > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 86(22)a}
\end{aligned}$$

7.662

$$\begin{aligned}
1. \quad & \int_0^\infty x^{-1} M_{-\mu, \frac{1}{4}\nu} \left(\frac{1}{2}x^2\right) W_{\mu, \frac{1}{4}\nu} \left(\frac{1}{2}x^2\right) J_\nu(xy) dx = \frac{\Gamma(1+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}+\frac{1}{4}\nu-\mu)} I_{\frac{1}{4}\nu-\mu} \left(\frac{1}{4}y^2\right) K_{\frac{1}{4}\nu+\mu} \left(\frac{1}{4}y^2\right) \\
&\quad [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 86(24)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{-1} M_{\alpha-\beta, \frac{1}{4}\nu-\gamma} \left(\frac{1}{2}x^2\right) W_{\alpha+\beta, \frac{1}{4}\nu+\gamma} \left(\frac{1}{2}x^2\right) J_\nu(xy) dx \\
&= \frac{\Gamma(1+\frac{1}{2}\nu-2\gamma)}{\Gamma(1+\frac{1}{2}\nu-2\beta)} y^{-2} M_{\alpha-\gamma, \frac{1}{4}\nu-\beta} \left(\frac{1}{2}y^2\right) W_{\alpha+\gamma, \frac{1}{4}\nu+\beta} \left(\frac{1}{2}y^2\right) \\
&\quad [y > 0, \quad \operatorname{Re} \beta < \frac{1}{8}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu-4\gamma) > -2] \quad \text{ET II 86(25)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{-1} M_{k,0}(iax^2) M_{k,0}(-iax^2) K_0(xy) dx = \frac{\pi}{16} \left\{ \left[J_k \left(\frac{y^2}{8a} \right) \right]^2 + \left[Y_k \left(\frac{y^2}{8a} \right) \right]^2 \right\} \\
&\quad [a > 0] \quad \text{ET II 152(83)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{-1} M_{k,\mu}(iax^2) M_{k,\mu}(-iax^2) K_0(xy) dx = ay^{-2} [\Gamma(2\mu+1)]^2 W_{-\mu,k} \left(\frac{iy^2}{4a} \right) W_{-\mu,k} \left(-\frac{iy^2}{4a} \right) \\
&\quad [a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{ET II 152(84)}
\end{aligned}$$

7.663

$$\begin{aligned}
1. \quad & \int_0^\infty x^{2\rho} {}_1F_1(a; b; -\lambda x^2) J_\nu(xy) dx = \frac{2^{2\rho} \Gamma(b)}{\Gamma(a) y^{2\rho+1}} G_{23}^{21} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} 1, b \\ \frac{1}{2} + \rho + \frac{1}{2}\nu, a, \frac{1}{2} + \rho - \frac{1}{2}\nu \end{matrix} \right. \right) \\
&\quad [y > 0, \quad -1 - \operatorname{Re} \nu < 2 \operatorname{Re} \rho < \frac{1}{2} + 2 \operatorname{Re} a, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 88(6)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\nu+1} {}_1F_1 \left(2a - \nu; a + 1; -\frac{1}{2}x^2 \right) J_\nu(xy) dx = \frac{2^{\nu-a+\frac{1}{2}} \Gamma(a+1)}{\pi^{\frac{1}{2}} \Gamma(2a-\nu)} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} K_{a-\nu-\frac{1}{2}} \left(\frac{1}{4}y^2 \right) \\
&\quad [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(4a-3\nu) > \frac{1}{2}] \quad \text{ET II 87(1)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^a {}_1F_1 \left(a; \frac{1+a+\nu}{2}; -\frac{1}{2}x^2 \right) J_\nu(xy) dx = y^{a-1} {}_1F_1 \left(a; \frac{1+a+\nu}{2}; -\frac{y^2}{2} \right) \\
&\quad [y > 0, \quad \operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re}(a+\nu) > -1] \quad \text{ET II 87(2)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{\nu+1-2a} {}_1F_1\left(a; 1+\nu-a; -\frac{1}{2}x^2\right) J_\nu(xy) dx \\
= \frac{\pi^{\frac{1}{2}} \Gamma(1+\nu-a)}{\Gamma(a)} 2^{-2a+\nu+\frac{1}{2}} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} I_{a-\frac{1}{2}}\left(\frac{1}{4}y^2\right) \\
[y > 0, \quad \operatorname{Re} a - 1 < \operatorname{Re} \nu < 4 \operatorname{Re} a - \frac{1}{2}] \quad \text{ET II 87(3)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x {}_1F_1(\lambda; 1; -x^2) J_0(xy) dx = [2^{2\lambda-1} \Gamma(\lambda)]^{-1} y^{2\lambda-2} e^{-\frac{1}{4}y^2} \\
[y > 0, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 18(46)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x^{\nu+1} {}_1F_1(a; b; -\lambda x^2) J_\nu(xy) dx \\
= \frac{2^{1-a} \Gamma(b)}{\Gamma(a) \lambda^{\frac{1}{2}a+\frac{1}{2}\nu}} y^{a-2} e^{-\frac{y^2}{8\lambda}} W_{k,\mu}\left(\frac{y^2}{4\lambda}\right), \quad 2k = a - 2b + \nu + 2, \quad 2\mu = a - \nu - 1 \\
[y > 0, \quad -1 < \operatorname{Re} \nu < 2 \operatorname{Re} a - \frac{1}{2}, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 88(4)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x^{2b-\nu-1} {}_1F_1(a; b; -\lambda x^2) J_\nu(xy) dx = \frac{2^{2b-2a-\nu-1} \Gamma(b)}{\Gamma(a-b+\nu+1)} \lambda^{-a} y^{2a-2b+\nu} \\
\times {}_1F_1\left(a; 1+a-b+\nu; -\frac{y^2}{4\lambda}\right) \\
[y > 0, \quad 0 < \operatorname{Re} b < \frac{3}{4} + \operatorname{Re}(a + \frac{1}{2}\nu), \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 88(5)}
\end{aligned}$$

7.664

$$\begin{aligned}
1. \quad \int_0^\infty x W_{\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right) K_\nu(xy) dx = 2ay^{-1} K_{2\mu}\left[(2ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\right] K_{2\mu}\left[(2ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}\right] \\
[\operatorname{Re} y > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 152(85)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) J_\nu(xy) dx \\
= -4y^{-1} \left\{ \sin\left[(\mu - \frac{1}{2}\nu)\pi\right] J_{2\mu}\left(2y^{\frac{1}{2}}\right) + \cos\left[(\mu - \frac{1}{2}\nu)\pi\right] Y_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right) \\
[y > 0, \quad \operatorname{Re}(\nu \pm 2\mu) > -1] \quad \text{ET II 87(27)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) Y_\nu(xy) dx \\
= 4y^{-1} \left\{ \left\{ \cos\left[(\mu - \frac{1}{2}\nu)\pi\right] J_{2\mu}\left(2y^{\frac{1}{2}}\right) - \sin\left[(\mu - \frac{1}{2}\nu)\pi\right] Y_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} \\
[y > 0, \quad |\operatorname{Re} \mu| < \frac{1}{4}] \quad \text{ET II 117(48)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) M_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) J_\nu(xy) dx = \frac{4\Gamma(1+2\mu)y^{-1}}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \mu)} J_{2\mu}\left(2y^{\frac{1}{2}}\right) K_{2\mu}\left(2y^{\frac{1}{2}}\right) \\
[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{4}] \\
\text{ET II 86(26)}
\end{aligned}$$

$$\begin{aligned}
 5. \quad \int_0^\infty x W_{-\frac{1}{2}\nu, \mu} \left(\frac{ia}{x} \right) W_{-\frac{1}{2}\nu, \mu} \left(-\frac{ia}{x} \right) J_\nu(xy) dx \\
 = 4ay^{-1} \left[\Gamma \left(\frac{1}{2} + \mu + \frac{1}{2}\nu \right) \Gamma \left(\frac{1}{2} - \mu + \frac{1}{2}\nu \right) \right]^{-1} K_\mu \left[(2ia y)^{\frac{1}{2}} \right] K_\mu \left[(-2ia y)^{\frac{1}{2}} \right] \\
 [y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \mu| < \frac{1}{2}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 87(28)}
 \end{aligned}$$

7.665

$$\begin{aligned}
 1. \quad \int_0^\infty x^{-\frac{1}{2}} J_\nu \left(ax^{\frac{1}{2}} \right) K_{\frac{1}{2}\nu - \mu} \left(\frac{1}{2}x \right) M_{k, \mu}(x) dx \\
 = \frac{\Gamma(2\mu + 1)}{a \Gamma \left(k + \frac{1}{2}\nu + 1 \right)} W_{\frac{1}{2}(k - \mu), \frac{1}{2}k - \frac{1}{4}\nu} \left(\frac{a^2}{2} \right) M_{\frac{1}{2}(k + \mu), \frac{1}{2}k + \frac{1}{4}\nu} \left(\frac{a^2}{2} \right) \\
 [a > 0, \quad \operatorname{Re} k > -\frac{1}{4}, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 405(18)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty x^{\frac{1}{2}c + \frac{1}{2}c' - 1} \Psi(a, c; x) {}_1F_1(a'; c'; -x) J_{c+c'-2} \left[2(xy)^{\frac{1}{2}} \right] dx \\
 = \frac{\Gamma(c')}{\Gamma(a+a')} y^{\frac{1}{2}c + \frac{1}{2}c' - 1} \Psi(c' - a', c + c' - a - a'; y) {}_1F_1(a'; a + a'; -y) \\
 [\operatorname{Re} c' > 0, \quad 1 < \operatorname{Re}(c + c') < 2 \operatorname{Re}(a + a') + \frac{1}{2}] \quad \text{EH I 287(23)}
 \end{aligned}$$

$$\begin{aligned}
 7.666 \quad \int_0^\infty x^{\frac{1}{2}c - \frac{1}{2}} {}_1F_1 \left(a; c; -2x^{\frac{1}{2}} \right) \Psi \left(a, c; 2x^{\frac{1}{2}} \right) J_{c-1} \left[2(xy)^{\frac{1}{2}} \right] dx \\
 = 2^{-c} \frac{\Gamma(c)}{\Gamma(a)} y^{a - \frac{1}{2}c - \frac{1}{2}} \left[1 + (1 + y)^{\frac{1}{2}} \right]^{c-2a} (1 + y)^{-\frac{1}{2}} \\
 [\operatorname{Re} c > 2, \quad \operatorname{Re}(c - 2a) < \frac{1}{2}] \quad \text{EH I 285(13)}
 \end{aligned}$$

7.67 Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers

7.671

$$\begin{aligned}
 1. \quad \int_0^\infty x^{k - \frac{3}{2}} \exp \left[-\frac{1}{2}(a+1)x \right] K_\nu \left(\frac{1}{2}ax \right) M_{k, \nu}(x) dx \\
 = \frac{\pi^{\frac{1}{2}} \Gamma(k) \Gamma(k + 2\nu)}{a^{k+\nu} \Gamma \left(k + \nu + \frac{1}{2} \right)} {}_2F_1 \left(k, k + 2\nu; 2\nu + 1; -a^{-1} \right) \\
 [\operatorname{Re} a > 0, \quad \operatorname{Re} k > 0, \quad \operatorname{Re}(k + 2\nu) > 0] \quad \text{ET II 405(17)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty x^{-k - \frac{3}{2}} \exp \left[-\frac{1}{2}(a-1)x \right] K_\mu \left(\frac{1}{2}ax \right) W_{k, \mu}(x) dx \\
 = \frac{\pi \Gamma(-k) \Gamma(2\mu - k) \Gamma(-2\mu - k)}{\Gamma \left(\frac{1}{2} - k \right) \Gamma \left(\frac{1}{2} + \mu - k \right) \Gamma \left(\frac{1}{2} - \mu - k \right)} 2^{2k+1} a^{k-\nu} {}_2F_1 \left(-k, 2\mu - k; -2k; 1 - a^{-1} \right) \\
 [\operatorname{Re} a > 0, \quad \operatorname{Re} k < 2 \operatorname{Re} \mu < -\operatorname{Re} k] \quad \text{ET II 408(36)}
 \end{aligned}$$

7.672

$$\begin{aligned}
1. \quad \int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} M_{k,\mu}(ax^2) J_\nu(xy) dx \\
= \frac{\Gamma(2\mu+1)}{\Gamma(\mu+k+\frac{1}{2})} 2^{2\rho} y^{-2\rho-1} G_{23}^{21} \left(\frac{y^2}{4a} \left| \begin{matrix} \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, k, \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix} \right. \right) \\
[y > 0, \quad -1 - \operatorname{Re}(\frac{1}{2}\nu + \mu) < \operatorname{Re} \rho < \operatorname{Re} k - \frac{1}{4}, \quad \operatorname{Re} a > 0] \quad \text{ET II 83(10)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} W_{k,\mu}(ax^2) J_\nu(xy) dx \\
= \frac{\Gamma(1+\mu+\frac{1}{2}\nu+\rho) \Gamma(1-\mu+\frac{1}{2}\nu+\rho) 2^{-\nu-1}}{\Gamma(\nu+1) \Gamma(\frac{3}{2}-k+\frac{1}{2}\nu+\rho)} a^{-\frac{1}{2}\nu-\rho-\frac{1}{2}} y^\nu \\
\times {}_2F_2 \left(\lambda+\mu, \lambda-\mu; \nu+1, \frac{1}{2}-k+\lambda; -\frac{y^2}{4a} \right), \\
\lambda = 1 + \frac{1}{2}\nu + \rho \quad [y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(\rho \pm \mu + \frac{1}{2}\nu) > -1] \quad \text{ET II 85(16)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^{2\rho} e^{\frac{1}{2}ax^2} W_{k,\mu}(ax^2) J_\nu(xy) dx = \frac{2^{2\rho} y^{-2\rho-1}}{\Gamma(\frac{1}{2}+\mu-k) \Gamma(\frac{1}{2}-\mu-k)} \\
\times G_{23}^{22} \left(\frac{y^2}{4a} \left| \begin{matrix} \frac{1}{2}-\mu, \quad \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, \quad -k, \quad \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix} \right. \right) \\
[y > 0, \quad |\arg a| < \pi, \quad -1 - \operatorname{Re}(\frac{1}{2}\nu \pm \mu) < \operatorname{Re} \rho < -\frac{1}{4} - \operatorname{Re} k] \quad \text{ET II 85(17)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy) dx = \frac{2^\lambda y^{-1/2} \Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+k+\mu)} G_{34}^{31} \left(\frac{y^2}{2} \left| \begin{matrix} -\mu-\lambda, \quad \mu-\lambda, \\ h, \quad \kappa, \quad -\lambda-\frac{1}{2}, \quad l \end{matrix} \right. \right) \\
h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu \\
[y > 0, \quad \operatorname{Re}(k-\lambda) > 0, \quad \operatorname{Re}(2\lambda+2\mu \pm \nu) > -\frac{5}{2}] \quad \text{ET II 116(45)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy) dx \\
= 2^\lambda \left[\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right) \right]^{-1} G_{34}^{32} \left(\frac{y^2}{2} \left| \begin{matrix} -\mu-\lambda, \quad \mu-\lambda, \quad l \\ h, \quad \kappa, \quad -\frac{1}{2}-k-\lambda, \quad l \end{matrix} \right. \right) y^{-1/2}, \\
h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu \\
[y > 0, \quad \operatorname{Re}(k+\lambda) < 0, \quad \operatorname{Re}(2\lambda \pm 2\mu \pm \nu) > -\frac{5}{2}] \quad \text{ET II 117(47)}
\end{aligned}$$

$$\begin{aligned}
6. \quad \int_0^\infty x^{-1/2} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu-\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(x^2) J_\nu(xy) dx = (2\nu+1) 2^{-\nu} y^{\nu-1} \left[1 - \Phi\left(\frac{1}{2}y\right) \right] \\
[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 82(1)}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int_0^\infty x^{-1} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}}(x^2) J_\nu(xy) dx = \frac{\Gamma(\nu+2) y^\nu}{\Gamma(\nu+\frac{3}{2}) 2^\nu} \left[1 - \Phi\left(\frac{1}{2}y\right) \right] \\
[y > 0, \operatorname{Re} \nu > -1] \quad \text{ET II 82(2)}
\end{aligned}$$

$$8. \quad \int_0^\infty e^{-\frac{1}{4}x^2} M_{k, \frac{1}{2}\nu} \left(\frac{1}{2} \right) x^2 J_\nu(xy) dx = \frac{2^{-k} \Gamma(\nu+1)}{\Gamma(k + \frac{1}{2}\nu + \frac{1}{2})} y^{2k-1} e^{-\frac{1}{2}y^2} \\ [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} k < \frac{1}{2}] \quad \text{ET II 83(7)}$$

$$9. \quad \int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} M_{k, \mu} \left(\frac{1}{2} \right) x^2 J_\nu(xy) dx \\ = 2^{\frac{1}{2}(\frac{1}{2}-k-3\mu+\nu)} \frac{\Gamma(2\mu+1)}{\Gamma(\mu+k+\frac{1}{2})} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} W_{\alpha, \beta} \left(\frac{1}{2}y^2 \right), \\ 2\alpha = k - 3\mu + \nu + \frac{1}{2}, \quad 2\beta = k + \mu - \nu - \frac{1}{2} \\ [y > 0, \quad -1 < \operatorname{Re} \nu < 2\operatorname{Re}(k + \mu) - \frac{1}{2}] \quad \text{ET II 83(9)}$$

$$10. \quad \int_0^\infty x^{\nu-2\mu} e^{\frac{1}{4}x^2} W_{k, \pm\mu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx = \frac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)} 2^{\beta-\mu} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha, \beta} \left(\frac{1}{2}y^2 \right) \\ 2\alpha = \frac{1}{2} + k + \nu - 3\mu, \quad 2\beta = \frac{1}{2} - k + \nu - \mu \\ [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - 2\mu) > -1] \\ \text{ET II 84(14)}$$

$$11. \quad \int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} W_{k, \pm\mu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx \\ = \frac{\Gamma(1+\nu-2\mu)}{\Gamma(\frac{1}{2}+\mu-k)} 2^{\frac{1}{2}(\frac{1}{2}+k-3\mu+\nu)} y^{\mu-k-\frac{3}{2}} e^{\frac{1}{4}y^2} W_{\alpha, \beta} \left(\frac{1}{2}y^2 \right), \\ 2\alpha = k + 3\mu - \nu - \frac{1}{2}, \quad 2\beta = k - \mu + \nu + \frac{1}{2} \\ [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - 2\mu) > -1, \quad \operatorname{Re}(k - \mu + \frac{1}{2}\nu) < -\frac{1}{4}] \quad \text{ET II 84(15)}$$

$$12. \quad \int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k, \mu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx \\ = \frac{\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+k-\mu+\nu)} 2^{\frac{1}{2}(\frac{1}{2}-k+3\mu-\nu)} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha, \beta} \left(\frac{1}{2}y^2 \right) \\ 2\alpha = \frac{1}{2} + k + 3\mu - \nu, \quad 2\beta = -\frac{1}{2} + k - \mu + \nu \\ [y > 0, \quad -\frac{1}{2} < \operatorname{Re} \mu < \operatorname{Re}(k + \frac{1}{2}\nu) - \frac{1}{4}] \quad \text{ET II 83(8)}$$

$$13. \quad \int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k, \mu} \left(\frac{1}{2}x^2 \right) Y_\nu(xy) dx \\ = \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} \Gamma(2\mu+1) \\ \times \Gamma\left(\frac{1}{2}-k-\mu\right) \left\{ \cos[(\nu-2\mu)\pi] \frac{\Gamma(2\mu-\nu-1)}{\Gamma(2\beta+1)} M_{\alpha, \beta} \left(\frac{1}{2}y^2 \right) \right. \\ \left. - \sin[(\nu+k-\mu)\pi] W_{\alpha, \beta} \left(\frac{1}{2}y^2 \right) \right\} \\ 2\alpha = 3\mu - \nu + k + \frac{1}{2}, \quad 2\beta = \mu - \nu - k + \frac{1}{2} \\ [y > 0, \quad -1 < 2\operatorname{Re} \mu < \operatorname{Re}(2k + \nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu - \nu) > -1] \quad \text{ET II 116(44)}$$

$$\begin{aligned}
14. \quad & \int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{4}x^2} M_{k,\mu} \left(\frac{1}{2}x^2 \right) Y_\nu(xy) dx \\
& = \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} \Gamma(2\mu+1) \\
& \quad \times \Gamma\left(\frac{1}{2}-\mu-k\right) e^{-\frac{1}{4}y^2} \left\{ \cos(2\mu\pi) \frac{\Gamma(2\mu+\nu+1)}{\Gamma\left(\mu+\nu-k+\frac{3}{2}\right)} M_{\alpha,\beta}\left(\frac{1}{2}y^2\right) \right. \\
& \quad \left. + \sin[(\mu-k)\pi] W_{\alpha,\beta}\left(\frac{1}{2}y^2\right) \right\} \\
& \quad 2\alpha = 3\mu + \nu + k + \frac{1}{2}, \quad 2\beta = \mu + \nu - k + \frac{1}{2} \\
& [y > 0, \quad -1 < 2\operatorname{Re} \mu < \operatorname{Re}(2k - \nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -1] \quad \text{ET II 116(43)}
\end{aligned}$$

$$\begin{aligned}
15. \quad & \int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{2}ax^2} M_{k,\mu}(ax^2) K_\nu(xy) dx = 2^{\mu-k-\frac{1}{2}} a^{\frac{1}{4}-\frac{1}{2}(\mu+\nu+k)} y^{k-\mu-\frac{3}{2}} \\
& \quad \times \Gamma(2\mu+1) \Gamma(2\mu+\nu+1) \exp\left(\frac{y^2}{8a}\right) W_{\kappa,m}\left(\frac{y^2}{4a}\right), \\
& \quad 2\kappa = -3\mu - \nu - k - \frac{1}{2}, \quad 2m = \mu + \nu - k + \frac{1}{2} \\
& [\operatorname{Re} y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -1] \quad \text{ET II 152(82)}
\end{aligned}$$

7.673

$$\begin{aligned}
1.^{10} \quad & \int_0^\infty e^{-\frac{1}{2}ax} x^{\frac{1}{2}(\mu-\nu-1)} M_{\kappa,\frac{1}{2}\mu}(ax) J_\nu(2\sqrt{bx}) dx \\
& = \left(\frac{b}{a}\right)^{\frac{\kappa-1}{2}-\frac{1+\mu}{4}} a^{-\frac{1}{2}(\mu+1-\nu)} \Gamma(1+\mu) e^{-\frac{b}{2a}} \frac{1}{\Gamma\left(1+\frac{\kappa+\nu}{2}-\frac{1+\mu}{4}\right)} \\
& \quad \times M_{\frac{1}{2}(\kappa-\nu-1)+\frac{3}{4}(1+\mu), \frac{\kappa+\nu}{2}-\frac{1+\mu}{4}}\left(\frac{b}{a}\right) \\
& \quad \left[\operatorname{Re}(1+\mu) > 0, \quad \operatorname{Re}\left(\kappa + \frac{\nu-\mu}{2}\right) > -\frac{3}{4}, \quad \operatorname{Im} b = 0 \right] \quad \text{BU 128(12)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty e^{\frac{1}{2}ax} x^{\frac{1}{2}(\nu-1\mp\mu)} W_{\kappa,\frac{1}{2}\mu}(ax) J_\nu(2\sqrt{bx}) dx = a^{-\frac{1}{2}(\nu+1\mp\mu)} \frac{\Gamma(\nu+1\mp\mu) e^{\frac{b}{2a}}}{\Gamma\left(\frac{1\pm\mu}{2}-\kappa\right)} \left(\frac{a}{b}\right)^{\frac{1}{2}(\kappa+1)+\frac{1}{4}(1\mp\nu)} \\
& \quad \times W_{\frac{1}{2}(\kappa+1-\nu)-\frac{3}{4}(1\mp\mu), \frac{1}{2}(\kappa+\nu)+\frac{1}{4}(1\mp\mu)}\left(\frac{b}{a}\right) \\
& \quad \left[\operatorname{Re}\left(\frac{\nu\mp\mu}{2} + \kappa\right) < \frac{3}{4}, \quad \operatorname{Re} \nu > -1 \right] \quad \text{BU 128(13)}
\end{aligned}$$

7.674

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{\rho-1} e^{-\frac{1}{2}\kappa} J_{\lambda+\nu} \left(ax^{1/2} \right) J_{\lambda-\nu} \left(ax^{1/2} \right) W_{k,\mu}(x) dx \\
 &= \frac{\left(\frac{1}{2}a\right)^{2\lambda} \Gamma\left(\frac{1}{2} + \lambda + \mu + \rho\right) \Gamma\left(\frac{1}{2} + \lambda - \mu + \rho\right)}{\Gamma(1 + \lambda + \nu) \Gamma(1 + \lambda - \nu) \Gamma(1 + \lambda - k + \rho)} \\
 &\quad \times {}_4F_4 \left(1 + \lambda, \frac{1}{2} + \lambda, \frac{1}{2} + \lambda + \mu + \rho, \frac{1}{2} + \lambda - \mu + \rho; 1 + \lambda + \nu, \right. \\
 &\quad \left. 1 + \lambda - \nu, 1 + 2\lambda, 1 + \lambda - k + \rho; -a^2 \right) \\
 &\quad \left[|\operatorname{Re} \mu| < \operatorname{Re}(\lambda + \rho) + \frac{1}{2} \right] \quad \text{ET II 409(37)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{\rho-1} e^{-\frac{1}{2}\kappa} I_{\lambda+\nu} \left(ax^{1/2} \right) K_{\lambda-\nu} \left(ax^{1/2} \right) W_{k,\mu}(x) dx \\
 &= \frac{\pi^{-1/2}}{2} G_{45}^{24} \left(a^2 \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2} + \mu - \rho, \frac{1}{2} - \mu - \rho \\ \lambda, \nu, -\lambda, -\nu, k - \rho \end{matrix} \right. \right) \\
 &\quad \left[|\operatorname{Re} \mu| < \operatorname{Re}(\lambda + \rho) + \frac{1}{2}, \quad |\operatorname{Re} \mu| < \operatorname{Re}(\nu + \rho) + \frac{1}{2} \right] \quad \text{ET II 409(38)}
 \end{aligned}$$

Combinations of Struve functions and confluent hypergeometric functions

7.675

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu} \left(\frac{1}{2}x^2 \right) \mathbf{H}_\nu(xy) dx = \frac{2^{-\lambda} \Gamma(2\mu + 1)}{y^{1/2} \Gamma\left(\frac{1}{2} + k + \mu\right)} G_{34}^{22} \left(\frac{y^2}{2} \left| \begin{matrix} l, -\mu - \lambda, \mu - \lambda \\ l, k - \lambda - \frac{1}{2}, h, \kappa \end{matrix} \right. \right) \\
 &\quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = \frac{3}{4} + \frac{1}{2}\nu \\
 &\quad \left[\operatorname{Re}(2\lambda + 2\mu + \nu) > -\frac{7}{2}, \quad \operatorname{Re}(k - \lambda) > 0, \quad y > 0, \quad \operatorname{Re}(2\lambda - 2k + \nu) < -\frac{1}{2} \right] \\
 &\quad \text{ET II 171(42)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,\mu} \left(\frac{1}{2}x^2 \right) \mathbf{H}_\nu(xy) dx \\
 &= 2^{\frac{1}{4}-\lambda-\frac{1}{2}\nu} \pi^{-1/2} y^{\nu+1} \frac{\Gamma\left(\frac{7}{4} + \frac{1}{2}\nu + \lambda + \mu\right) \Gamma\left(\frac{7}{4} + \frac{1}{2}\nu + \lambda - \mu\right)}{\Gamma\left(\nu + \frac{3}{2}\right) \Gamma\left(\frac{9}{4} + \lambda - k - \frac{1}{2}\nu\right)} \\
 &\quad \times {}_3F_3 \left(1, \frac{7}{4} + \frac{\nu}{2} + \lambda + \mu, \frac{7}{4} + \frac{\nu}{2} + \lambda - \mu; \frac{3}{2}, \nu + \frac{3}{2}, \frac{9}{4} + \lambda - k + \frac{\nu}{2}; -\frac{y^2}{2} \right) \\
 &\quad \left[\operatorname{Re}(2\lambda + \nu) > 2|\operatorname{Re} \mu| - \frac{7}{4}, \quad y > 0 \right] \quad \text{ET II 171(43)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu} \left(\frac{1}{2}x^2 \right) \mathbf{H}_\nu(xy) dx \\
 &= \left[2^\lambda \Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right) \right]^{-1} y^{-1/2} G_{34}^{23} \left(\frac{y^2}{2} \left| \begin{matrix} l, -\mu - \lambda, \mu - \lambda \\ l, -k - \lambda - \frac{1}{2}, h, \kappa \end{matrix} \right. \right) \\
 &\quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = \frac{3}{4} + \frac{1}{2}\nu \\
 &\quad \left[y > 0, \quad \operatorname{Re}(2\lambda + \nu) > 2|\operatorname{Re} \mu| - \frac{7}{2}, \quad \operatorname{Re}(2k + 2\lambda + \nu) < -\frac{1}{2}, \quad \operatorname{Re}(k + \lambda) < 0 \right] \quad \text{ET II 172(46)a}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_0^\infty e^{\frac{1}{2}x^2} W_{-\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu} (x^2) \mathbf{H}_\nu(xy) dx = 2^{-\nu-1} y^\nu \pi e^{\frac{1}{4}y^2} \left[1 - \Phi\left(\frac{y}{2}\right) \right] \\
 &\quad [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 171(44)}
 \end{aligned}$$

7.68 Combinations of confluent hypergeometric functions and other special functions

Combinations of confluent hypergeometric functions and associated Legendre functions

7.681

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{-1/2} (a+x)^\mu e^{-\frac{1}{2}x} P_\nu^{-2\mu} \left(1 + 2\frac{x}{a}\right) M_{k,\mu}(x) dx \\
 &= -\frac{\sin(\nu\pi)}{\pi \Gamma(k)} \Gamma(2\mu+1) \Gamma\left(k-\mu+\nu+\frac{1}{2}\right) \Gamma\left(k-\mu-\nu-\frac{1}{2}\right) e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \\
 & \quad \rho = \frac{1}{2} - k + \mu, \quad \sigma = \frac{1}{2} + \nu \\
 & \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\mu) > |\operatorname{Re} \nu + \frac{1}{2}|] \quad \text{ET II 403(11)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{-1/2} (a+x)^{-\mu} e^{-\frac{1}{2}x} P_\nu^{-2\mu} \left(1 + 2\frac{x}{a}\right) M_{k,\mu}(x) dx \\
 &= \frac{\Gamma(2\mu+1) \Gamma\left(k+\mu+\nu+\frac{1}{2}\right) \Gamma\left(k+\mu-\nu-\frac{1}{2}\right) e^{\frac{1}{2}a}}{\Gamma\left(k+\mu+\frac{1}{2}\right) \Gamma(2\mu+\nu+1) \Gamma(2\mu-\nu)} W_{\frac{1}{2}-k-\mu, \frac{1}{2}+\nu}(a) \\
 & \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k+\mu) > |\operatorname{Re} \nu + \frac{1}{2}|] \quad \text{ET II 403(12)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{k+\nu-\frac{3}{2}}^\mu \left(1 + 2\frac{x}{a}\right) W_{k,\nu}(x) dx \\
 &= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a) \\
 & \quad 2\rho = \frac{1}{2} + 2\mu + \nu - k, \quad 2\sigma = k + 3\nu - \frac{3}{2} \\
 & \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1] \\
 & \quad \text{ET II 407(32)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{k+\mu+\nu-\frac{3}{2}}^\mu \left(1 + 2\frac{x}{a}\right) W_{k,\nu}(x) dx \\
 &= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a) \\
 & \quad 2\rho = \frac{1}{2} - k + \nu, \quad 2\sigma = k + 2\mu + 3\nu - \frac{3}{2} \\
 & \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1] \\
 & \quad \text{ET II 408(33)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \int_0^\infty x^{\mu-\frac{1}{4}k-\frac{1}{2}\nu-\frac{1}{2}} (a+x)^{\frac{1}{2}\nu} e^{-\frac{1}{2}x} Q_{\mu-k+\frac{3}{2}}^\nu \left(1 + 2\frac{x}{a}\right) M_{k,\nu}(x) dx \\
 &= \frac{e^{\nu\pi i} \Gamma(1+2\mu-\nu) \Gamma(1+2\mu) \Gamma\left(\frac{5}{2}-k+\mu+\nu\right)}{2 \Gamma\left(\frac{1}{2}+k+\mu\right)} a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)} e^{\frac{1}{2}a} W_{\rho,\sigma}(a) \\
 & \quad 2\rho = \frac{1}{2} - k - \mu + 2\nu, \quad 2\sigma = k - 3\mu - \frac{3}{2} \\
 & \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu-\nu) > -1] \\
 & \quad \text{ET II 404(14)}
 \end{aligned}$$

7.682

$$\begin{aligned}
1. \quad & \int_0^\infty x^{-1/2} e^{-\frac{1}{2}x} P_\nu^{-2\mu} \left[\left(1 + \frac{x}{a}\right)^{1/2} \right] M_{k,\mu}(x) dx \\
&= \frac{\Gamma(2\mu+1) \Gamma(k + \frac{1}{2}\nu) \Gamma(k - \frac{1}{2}\nu - \frac{1}{2}) e^{\frac{1}{2}a}}{2^{2\mu} a^{1/4} \Gamma(k + \mu + \frac{1}{2}) \Gamma(\mu + \frac{1}{2}\nu + \frac{1}{2}) \Gamma(\mu - \frac{1}{2}\nu)} W_{\frac{3}{4}-k, \frac{1}{4}+\frac{1}{2}\nu}(a) \\
& \quad [|\arg a| < \pi, \quad \operatorname{Re} k > \frac{1}{2} \operatorname{Re} \nu - \frac{1}{2}, \quad \operatorname{Re} k > -\frac{1}{2} \operatorname{Re} \nu] \quad \text{ET II 404(13)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\frac{1}{2}(k+\mu+\nu)-1} (a+x)^{-1/2} e^{-\frac{1}{2}x} Q_{k-\mu-\nu-1}^{1-k+\mu-\nu} \left[\left(1 + \frac{x}{a}\right)^{1/2} \right] M_{k,\mu}(x) dx \\
&= e^{(1-k+\mu-\nu)\pi i} 2^{\mu-k-\nu} a^{\frac{1}{2}(k+\mu-1)} \frac{\Gamma(\frac{1}{2}-\nu) \Gamma(1+2\mu) \Gamma(k+\mu+\nu)}{\Gamma(k+\mu+\frac{1}{2})} e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \\
& \quad \rho = \frac{1}{2} - k - \frac{1}{2}\nu, \quad \sigma = \mu + \frac{1}{2}\nu \\
& \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k+\mu+\nu) > 0] \quad \text{ET II 404(15)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\nu-\frac{1}{2}} e^{-\frac{1}{2}x} Q_{2k-2\nu-3}^{2\mu-2\nu} \left[\left(1 + \frac{x}{a}\right)^{1/2} \right] M_{k,\mu}(x) dx \\
&= e^{2(\mu-\nu)\pi i} 2^{2\mu-2\nu-1} a^{\frac{1}{2}(k+\mu-1)} e^{\frac{1}{2}a} \frac{\Gamma(2\mu+1) \Gamma(\nu+1) \Gamma(k+\mu-2\nu-\frac{1}{2})}{\Gamma(k+\mu+\frac{1}{2})} W_{\rho,\sigma}(a), \\
& \quad 2\rho = 1 - k + \mu - 2\nu, \quad 2\sigma = k - \mu - 2\nu - 2 \\
& \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(k+\mu-2\nu) > \frac{1}{2}] \quad \text{ET II 404(16)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-3}^\mu \left[\left(1 + \frac{x}{a}\right)^{\frac{1}{2}} \right] W_{k,\nu}(x) dx \\
&= \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma(\frac{3}{2}-k-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \\
& \quad 2\rho = 1 - k + \mu + \nu, \quad 2\sigma = k + \mu + 3\nu - 2 \\
& \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1] \\
& \quad \text{ET II 408(34)}
\end{aligned}$$

$$\begin{aligned}
5.^8 \quad & \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-1/2} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-2}^\mu \left[\left(1 + \frac{x}{a}\right)^{1/2} \right] W_{k,\nu}(x) dx \\
&= \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma(\frac{3}{2}-k-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \quad 2\rho = \mu + \nu - k, \quad 2\sigma = k + \mu + 3\nu - 1 \\
& \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 408(35)}
\end{aligned}$$

A combination of confluent hypergeometric functions and orthogonal polynomials

$$\begin{aligned}
7.683^8 \quad & \int_0^1 e^{-\frac{1}{2}ax} x^\alpha (1-x)^{\frac{\mu-\alpha}{2}-1} L_n^\alpha(ax) M_{\alpha-\frac{1+\alpha}{2}, \frac{\mu-\alpha-1}{1}} [a(1-x)] dx \\
&= \frac{\Gamma(\mu-\alpha) \Gamma(1+n+\alpha)}{\Gamma(1+\mu)} \frac{1}{n!} a^{-\frac{1+\alpha}{2}} M_{\alpha+n, \frac{\mu}{2}}(a) \\
& \quad [\operatorname{Re} a > -1, \quad \operatorname{Re}(\mu-\alpha) > 0, \quad n = 0, 1, 2, \dots] \quad \text{BU 129(14b)}
\end{aligned}$$

A combination of hypergeometric and confluent hypergeometric functions

$$\begin{aligned}
 7.684 \quad \int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} M_{\gamma+\rho, \beta+\rho+\frac{1}{2}}(x) {}_2F_1\left(\alpha, \beta; \gamma; -\frac{\lambda}{x}\right) dx \\
 = \frac{\Gamma(\alpha + \beta + 2\rho) \Gamma(2\beta + 2\rho) \Gamma(\gamma)}{\Gamma(\beta) \Gamma(\beta + \gamma + 2\rho)} \lambda^{\frac{1}{2}\beta + \rho - \frac{1}{2}} e^{\frac{1}{2}\lambda} W_{k, \mu}(\lambda); \\
 k = \frac{1}{2} - \alpha - \frac{1}{2}\beta - \rho, \quad \mu = \frac{1}{2}\beta + \rho \\
 [|\arg \lambda| < \pi, \quad \operatorname{Re}(\beta + \rho) > 0, \quad \operatorname{Re}(\alpha + \beta + 2\rho) > 0, \quad \operatorname{Re} \gamma > 0] \\
 \text{ET II 405(19)}
 \end{aligned}$$

7.69 Integration of confluent hypergeometric functions with respect to the index

$$7.691 \quad \int_{-\infty}^\infty \operatorname{sech}(\pi x) W_{ix, 0}(\alpha) W_{-ix, 0}(\beta) dx = 2 \frac{(a\beta)^{1/2}}{\alpha + \beta} \exp\left[-\frac{1}{2}(\alpha + \beta)\right] \quad \text{ET II 414(61)}$$

$$7.692 \quad \int_{-\infty}^\infty \Gamma(-a) \Gamma(c-a) \Psi(a, c; x) \Psi(c-a, c; y) da = 2\pi i \Gamma(c) \Psi(c, 2c; x+y) \quad \text{EH I 285(15)}$$

7.693

$$\begin{aligned}
 1. \quad \int_{-\infty}^\infty \Gamma(ix) \Gamma(2k+ix) W_{k+ix, k-\frac{1}{2}}(\alpha) W_{-k-ix, k-\frac{1}{2}}(\beta) dx \\
 = 2\pi^{1/2} \Gamma(2k) (a\beta)^k (\alpha + \beta)^{\frac{1}{2}-2k} K_{2k-\frac{1}{2}}\left(\frac{\alpha + \beta}{2}\right) \\
 \text{ET II 414(62)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_{-\infty}^\infty \Gamma\left(\frac{1}{2} + \nu + \mu + x\right) \Gamma\left(\frac{1}{2} + \nu + \mu - x\right) \Gamma\left(\frac{1}{2} + \nu - \mu + x\right) \Gamma\left(\frac{1}{2} + \nu - \mu - x\right) \\
 \times M_{\mu+ix, \nu}(\alpha) M_{\mu-ix, \nu}(\beta) dx \\
 = \frac{2\pi (a\beta)^{\nu+\frac{1}{2}} [\Gamma(2\nu+1)]^2 \Gamma(2\nu+2\mu+1) \Gamma(2\nu-2\mu+1)}{(\alpha + \beta)^{2\nu+1} \Gamma(4\nu+2)} M_{2\mu, 2\nu+\frac{1}{2}}(\alpha + \beta) \\
 [\operatorname{Re} \nu > |\operatorname{Re} \mu| - \frac{1}{2}] \quad \text{ET II 413(59)}
 \end{aligned}$$

$$\begin{aligned}
 7.694^{11} \quad \int_{-\infty}^\infty e^{-2\rho x i} \Gamma\left(\frac{1}{2} + \nu + ix\right) \Gamma\left(\frac{1}{2} + \nu - ix\right) M_{ix, \nu}(\alpha) M_{ix, \nu}(\beta) dx \\
 = \pi \sqrt{\alpha\beta} [\Gamma(2\nu+1)]^2 \operatorname{sech} \rho \exp\left[-\frac{1}{2}(\alpha + \beta) \tanh \rho\right] J_{2\nu}\left(\sqrt{\alpha\beta} \operatorname{sech} \rho\right) \\
 [|\operatorname{Im} \rho| < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}]
 \end{aligned}$$

7.7 Parabolic Cylinder Functions

7.71 Parabolic cylinder functions

7.711

$$\begin{aligned}
 1. \quad \int_{-\infty}^\infty D_n(x) D_m(x) dx = 0 \quad [m \neq n] \\
 = n!(2\pi)^{1/2} \quad [m = n]
 \end{aligned}$$

$$2. \quad \int_0^\infty D_\mu(\pm t) D_\nu(t) dt = \frac{\pi 2^{\frac{1}{2}(\mu+\nu+1)}}{\mu-\nu} \left[\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu)} \mp \frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)\Gamma(-\frac{1}{2}\mu)} \right]$$

[when the lower sign is taken, $\operatorname{Re} \mu > \operatorname{Re} \nu$] BU 11 117(13a), EH II 122(21)

$$3. \quad \int_0^\infty [D_\nu(t)]^2 dt = \pi^{1/2} 2^{-3/2} \frac{\psi(\frac{1}{2}-\frac{1}{2}\nu) - \psi(-\frac{1}{2}\nu)}{\Gamma(-\nu)} \quad \text{BU 117(13b)a, EH II 122(22)a}$$

7.72 Combinations of parabolic cylinder functions, powers, and exponentials

7.721

$$1. \quad \int_{-\infty}^\infty e^{-\frac{1}{4}x^2} (x-z)^{-1} D_n(x) dx = \pm i e^{\mp n\pi i} (2\pi)^{1/2} n! e^{-\frac{1}{4}z^2} D_{-n-1}(\mp iz)$$

[The upper or lower sign is taken accordingly as the imaginary part of z is positive or negative.]
WH

$$2. \quad \int_1^\infty x^\nu (x-1)^{\frac{1}{2}\mu-\frac{1}{2}\nu-1} \exp\left[-\frac{(x-1)^2 a^2}{4}\right] D_\mu(ax) dx = 2^{\mu-\nu-2} a^{\frac{\mu}{2}-\frac{\nu}{2}-1} \Gamma\left(\frac{\mu-\nu}{2}\right) D_\nu(a)$$

[$\operatorname{Re}(\mu-\nu) > 0$] ET II 395(4)a

7.722

$$1. \quad \int_0^\infty e^{-\frac{3}{4}x^2} x^\nu D_{\nu+1}(x) dx = 2^{-\frac{1}{2}-\frac{1}{2}\nu} \Gamma(\nu+1) \sin \frac{1}{4}(1-\nu)\pi$$

[$\operatorname{Re} \nu > -1$] WH

$$2. \quad \int_0^\infty e^{-\frac{1}{4}x^2} x^{\mu-1} D_{-\nu}(x) dx = \frac{\pi^{1/2} 2^{-\frac{1}{2}\mu-\frac{1}{2}\nu} \Gamma(\mu)}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})}$$

[$\operatorname{Re} \mu > 0$] EH II 122(20)

$$3.^{11} \quad \int_0^\infty e^{-\frac{3}{4}x^2} x^\nu D_{\nu-1}(x) dx = 2^{-\frac{1}{2}\nu} \Gamma(\nu) \sin\left(\frac{1}{4}\pi\nu\right)$$

[$\operatorname{Re} \nu > -1$] ET II 395(2)

7.723

$$1. \quad \int_0^\infty e^{-\frac{1}{4}x^2} x^\nu (x^2+y^2)^{-1} D_\nu(x) dx = \left(\frac{\pi}{2}\right)^{1/2} \Gamma(\nu+1) y^{\nu-1} e^{\frac{1}{4}y^2} D_{-\nu-1}(y)$$

[$\operatorname{Re} y > 0, \operatorname{Re} \nu > -1$]
EH II 121(18)a, ET II 396(6)a

$$2. \quad \int_0^\infty e^{-\frac{1}{4}x^2} x^{\nu-1} (x^2+y^2)^{-1/2} D_\nu(x) dx = y^{\nu-1} \Gamma(\nu) e^{\frac{1}{4}y^2} D_{-\nu}(y)$$

[$\operatorname{Re} y > 0, \operatorname{Re} \nu > 0$] ET II 396(7)

$$3. \quad \int_0^1 x^{2\nu-1} (1-x^2)^{\lambda-1} e^{\frac{a^2 x^2}{4}} D_{-2\lambda-2\nu}(ax) dx = \frac{\Gamma(\lambda)\Gamma(2\nu)}{\Gamma(2\lambda+2\nu)} 2^{\lambda-1} e^{\frac{a^2}{4}} D_{-2\nu}(a)$$

[$\operatorname{Re} \lambda > 0, \operatorname{Re} \nu > 0$] ET II 395(3)a

$$7.724 \quad \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{2\mu}} e^{\frac{1}{4}x^2} D_\nu(x) dx = (2\pi\mu)^{1/2} (1-\mu)^{\frac{1}{2}\nu} e^{\frac{y^2}{4-4\mu}} D_\nu\left[y(1-\mu)^{-1/2}\right]$$

[$0 < \operatorname{Re} \mu < 1$]
EH II 121(15)

7.725

$$1. \quad \int_0^\infty e^{-pt} (2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu-2}(\sqrt{2t}) dt = \left(\frac{\pi}{2}\right)^{1/2} \frac{(\sqrt{p+1}-1)^{\nu+1}}{(\nu+1)p^{\nu+1}} \quad [\operatorname{Re} \nu > -1] \quad \text{MO 175}$$

$$2. \quad \int_0^\infty e^{-pt} (2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu}(\sqrt{2t}) dt = \left(\frac{\pi}{2}\right)^{1/2} \frac{(\sqrt{p+1}-1)^\nu}{p^\nu \sqrt{p+1}} \quad [\operatorname{Re} \nu > -1] \quad \text{MO 175}$$

$$3. \quad \int_0^\infty e^{-bx} D_{2n+1}(\sqrt{2x}) dx = (-2)^n \Gamma\left(n + \frac{3}{2}\right) \left(b - \frac{1}{2}\right)^n \left(b + \frac{1}{2}\right)^{-n-\frac{3}{2}} \quad [\operatorname{Re} b > -\frac{1}{2}] \quad \text{ET I 210(3)}$$

$$4. \quad \int_0^\infty (\sqrt{x})^{-1} e^{-bx} D_{2n}(\sqrt{2x}) dx = (-2)^n \Gamma\left(n + \frac{1}{2}\right) \left(b - \frac{1}{2}\right)^n \left(b + \frac{1}{2}\right)^{-n-\frac{1}{2}} \quad [\operatorname{Re} b > -\frac{1}{2}] \quad \text{ET I 210(5)}$$

$$5. \quad \int_0^\infty x^{-\frac{1}{2}(\nu+1)} e^{-sx} D_\nu(\sqrt{x}) dx = \sqrt{\pi} \left(1 + \sqrt{\frac{1}{2} + 2s}\right)^\nu \frac{1}{\sqrt{\frac{1}{4} + s}} \quad [\operatorname{Re} s > -\frac{1}{4}, \quad \operatorname{Re} \nu < 1] \quad \text{ET I 210(7)}$$

$$6. \quad \int_0^\infty e^{-zt} t^{-1+\frac{\beta}{2}} D_{-\nu} \left[2(kt)^{1/2}\right] dt = \frac{2^{1-\beta-\frac{\nu}{2}} \pi^{1/2} \Gamma(\beta)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}\beta + \frac{1}{2})} (z+k)^{-\frac{\beta}{2}} F\left(\frac{\nu}{2}, \frac{\beta}{2}; \frac{\nu+\beta+1}{2}; \frac{z-k}{z+k}\right) \quad [\operatorname{Re}(z+k) > 0, \quad \operatorname{Re} \frac{z}{k} > 0] \quad \text{EH II 121(11)}$$

$$\mathbf{7.726} \quad \int_{-\infty}^\infty e^{ixy - \frac{(1+\lambda)x^2}{4}} D_\nu \left[x(1-\lambda)^{1/2}\right] dx = (2\pi)^{1/2} \lambda^{\frac{1}{2}\nu} e^{-\frac{(1+\lambda)y^2}{4\lambda}} D_\nu \left[i(\lambda^{-1}-1)^{1/2} y\right] \quad [\operatorname{Re} \lambda > 0] \quad \text{EH II 121(16)}$$

$$\mathbf{7.727} \quad \int_0^\infty \frac{e^{\frac{1}{2}x} e^{-bx}}{(e^x-1)^{\mu+\frac{1}{2}}} \exp\left(-\frac{a}{1-e^{-x}}\right) D_{2\mu}\left(\frac{2\sqrt{a}}{\sqrt{1-e^{-x}}}\right) dx = e^{-a} 2^{b+\mu} \Gamma(b+\mu) D_{-2b}(2\sqrt{a}) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > -\operatorname{Re} \mu] \quad \text{ET I 211(13)}$$

$$\mathbf{7.728} \quad \int_0^\infty (2t)^{-\frac{\nu}{2}} e^{-pt} e^{-\frac{q^2}{8t}} D_{\nu-1}\left(\frac{q}{\sqrt{2t}}\right) dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} p^{\frac{1}{2}\nu-1} e^{-q\sqrt{p}} \quad \text{MO 175}$$

7.73 Combinations of parabolic cylinder and hyperbolic functions**7.731**

$$1. \quad \int_0^\infty \cosh(2\mu x) \exp\left[-(a \sinh x)^2\right] D_{2k}(2a \cosh x) dx = 2^{k-\frac{3}{2}} \pi^{1/2} a^{-1} W_{k,\mu}(2a^2) \quad [\operatorname{Re}^2 a > 0] \quad \text{ET II 398(20)}$$

$$2. \quad \int_0^\infty \cosh(2\mu x) \exp \left[(a \sinh x)^2 \right] D_{2k}(2a \cosh x) dx = \frac{\Gamma(\mu - k) \Gamma(-\mu - k)}{2^{k+\frac{5}{2}} a \Gamma(-2k)} W_{k+\frac{1}{2}, \mu}(2a^2) \\ \left[|\arg a| < \frac{3\pi}{4}, \quad \operatorname{Re} k + |\operatorname{Re} \mu| < 0 \right] \\ \text{ET II 398(21)}$$

7.74 Combinations of parabolic cylinder and trigonometric functions

7.741

$$1. \quad \int_0^\infty \sin(bx) \left\{ [D_{-n-1}(ix)]^2 - [D_{-n-1}(-ix)]^2 \right\} dx = (-1)^{n+1} \frac{i}{n!} \pi \sqrt{2\pi} e^{-\frac{1}{2}b^2} L_n(b^2) \\ [b > 0] \quad \text{ET I 115(3)}$$

$$2. \quad \int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx) D_{2n+1}(x) dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2} \\ [b > 0] \quad \text{ET I 115(1)}$$

$$3. \quad \int_0^\infty e^{-\frac{1}{4}x^2} \cos(bx) D_{2n}(x) dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} \quad [b > 0] \quad \text{ET I 60(2)}$$

$$4. \quad \int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx) \left[D_{2\nu-\frac{1}{2}}(x) - D_{2\nu-\frac{1}{2}}(-x) \right] dx = \sqrt{2\pi} \sin \left[\left(\nu - \frac{1}{4} \right) \pi \right] b^{2\nu-\frac{1}{2}} e^{-\frac{1}{2}b^2} \\ [\operatorname{Re} \nu > \frac{1}{4}, \quad b > 0] \quad \text{ET I 115(2)}$$

$$5. \quad \int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx) \left[D_{2\nu-\frac{1}{2}}(x) + D_{2\nu-\frac{1}{2}}(-x) \right] dx = \frac{2^{\frac{1}{4}-2\nu} \sqrt{\pi} b^{2\nu-\frac{1}{2}} e^{-\frac{1}{4}b^2}}{\operatorname{cosec} \left[\left(\nu + \frac{1}{4} \right) \pi \right]} \\ [\operatorname{Re} \nu > \frac{1}{4}, \quad b > 0] \quad \text{ET I 61(4)}$$

7.742

$$1. \quad \int_0^\infty x^{2\rho-1} \sin(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = 2^{\nu-\rho-\frac{1}{2}} \pi^{1/2} a \frac{\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)} \\ \times {}_2F_2 \left(\rho + \frac{1}{2}, \rho + 1; \frac{3}{2}, \rho - \nu + 1; -\frac{a^2}{2} \right) \\ [\operatorname{Re} \rho > -\frac{1}{2}] \quad \text{ET II 396(8)}$$

$$2. \quad \int_0^\infty x^{2\rho-1} \sin(ax) e^{\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22} \left(\frac{a^2}{2} \left| \begin{matrix} \frac{1}{2} - \rho, 1 - \rho \\ -\rho - \nu, \frac{1}{2}, 0 \end{matrix} \right. \right) \\ [a > 0, \quad \operatorname{Re} \rho > -\frac{1}{2}, \quad \operatorname{Re}(\rho + \nu) < \frac{1}{2}] \\ \text{ET II 396(9)}$$

$$3. \quad \int_0^\infty x^{2\rho-1} \cos(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\nu-\rho} \Gamma(2\rho) \pi^{1/2}}{\Gamma(\rho - \nu + \frac{1}{2})} {}_2F_2 \left(\rho, \rho + \frac{1}{2}; \frac{1}{2}, \rho - \nu + \frac{1}{2}; -\frac{a^2}{2} \right) \\ [\operatorname{Re} \rho > 0] \quad \text{ET II 396(10)a}$$

$$4. \quad \int_0^\infty x^{2\rho-1} \cos(ax) e^{\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22} \left(\frac{a^2}{2} \left| \begin{matrix} \frac{1}{2} - \rho, 1 - \rho \\ -\rho - \nu, 0, \frac{1}{2} \end{matrix} \right. \right) \\ [a > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\rho + \nu) < \frac{1}{2}] \quad \text{ET II 396(11)}$$

$$7.743 \quad \int_0^{\pi/2} (\cos x)^{-\mu-2} (\sin x)^{-\nu} D_\nu(a \sin x) D_\mu(a \cos x) dx = -\left(\frac{1}{2}\pi\right)^{1/2} (1+\mu)^{-1} D_{\mu+\nu+1}(a) \\ [\operatorname{Re} \nu < 1, \quad \operatorname{Re} \mu < -1] \quad \text{ET II 397(19)}$$

$$7.744 \quad 1. \quad \int_0^\infty \sin(bx) \left[D_{-\nu-\frac{1}{2}}(\sqrt{2x}) - D_{-\nu-\frac{1}{2}}(-\sqrt{2x}) \right] D_{\nu-\frac{1}{2}}(\sqrt{2x}) dx \\ = -\sqrt{2\pi} \sin\left[\left(\frac{1}{4} + \frac{1}{2}\nu\right)\pi\right] b^{-\nu-\frac{1}{2}} \frac{(1+\sqrt{1+b^2})^\nu}{\sqrt{1+b^2}} \\ [b > 0] \quad \text{ET I 115(4)}$$

$$2. \quad \int_0^\infty \cos(bx) \left[D_{-2\nu-\frac{1}{2}}(\sqrt{2x}) + D_{-2\nu-\frac{1}{2}}(-\sqrt{2x}) \right] D_{2\nu-\frac{1}{2}}(\sqrt{2x}) dx \\ = -\frac{\sqrt{\pi} \sin\left[\left(\nu - \frac{1}{4}\right)\pi\right] (1+\sqrt{1+b^2})^{2\nu}}{\sqrt{1+b^2} b^{2\nu+\frac{1}{2}}} \\ [b > 0] \quad \text{ET I 60(3)}$$

7.75 Combinations of parabolic cylinder and Bessel functions

7.751

$$1. \quad \int_0^\infty [D_n(ax)]^2 J_1(xy) dx = (-1)^{n-1} y^{-1} \left[D_n\left(\frac{y}{a}\right) \right]^2 \quad [y > 0] \quad \text{ET II 20(24)}$$

$$2. \quad \int_0^\infty J_0(xy) D_n(ax) D_{n+1}(ax) dx = (-1)^n y^{-1} D_n\left(\frac{y}{a}\right) D_{n+1}\left(\frac{y}{a}\right) \\ [y > 0, \quad |\arg a| < \frac{1}{4}\pi] \quad \text{ET II 17(42)}$$

$$3. \quad \int_0^\infty J_0(xy) D_\nu(x) D_{\nu+1}(x) dx = 2^{-1} y^{-1} [D_\nu(-y) D_{\nu+1}(y) - D_{\nu+1}(-y) D_\nu(y)] \quad \text{ET II 397(17)a}$$

7.752

$$1. \quad \int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{2\nu-1}(x) J_\nu(xy) dx = -\frac{1}{2} \sec(\nu\pi) y^{\nu-1} e^{-\frac{1}{4}y^2} [D_{2\nu-1}(y) - D_{2\nu-1}(-y)] \\ [y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 76(1), MO 183}$$

$$2. \quad \int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{2\nu-1}(x) J_\nu(xy) dx = 2^{\frac{1}{2}-\nu} \pi \sin(\nu\pi) y^{-\nu} \Gamma(2\nu) e^{\frac{1}{4}y^2} K_\nu\left(\frac{1}{4}y^2\right) \\ [y > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET II 77(4)}$$

$$3. \quad \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu}(x) J_\nu(xy) dx = \frac{1}{2} \sec(\nu\pi) y^{\nu-1} e^{-\frac{1}{4}y^2} [D_{2\nu+1}(y) - D_{2\nu+1}(-y)] \\ [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 78(13)}$$

$$4. \quad \int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{2\nu+1}(x) J_\nu(xy) dx = \frac{1}{2} \sec(\nu\pi) e^{-\frac{1}{4}y^2} y^\nu [D_{2\nu}(y) + D_{2\nu}(-y)]$$

$$[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(5)}$$

$$5. \quad \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu+2}(x) J_\nu(xy) dx = -\frac{1}{2} \sec(\nu\pi) y^\nu e^{-\frac{1}{4}y^2} [D_{2\nu+2}(y) + D_{2\nu+2}(-y)]$$

$$[\operatorname{Re} \nu > -1, \quad y > 0] \quad \text{ET II 78(16)}$$

$$6. \quad \int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{2\nu+2}(x) J_\nu(xy) dx = \pi^{-1} \sin(\nu\pi) \Gamma(2\nu+3) y^{-\nu-2} e^{\frac{1}{4}y^2} K_{\nu+1}\left(\frac{1}{4}y^2\right)$$

$$[y > 0, \quad -1 < \operatorname{Re} \nu < -\frac{5}{6}] \quad \text{ET II 78(19)}$$

$$7. \quad \int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{-2\nu}(x) J_\nu(xy) dx = 2^{-1/2} \pi^{1/2} y^{-\nu} e^{-\frac{1}{4}y^2} I_\nu\left(\frac{1}{4}y^2\right)$$

$$[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(8)}$$

$$8. \quad \int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu}(x) J_\nu(xy) dx = y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu}(y) \quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad y > 0]$$

$$\text{ET II 77(9), EH II 121(17)}$$

$$9. \quad \int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu-2}(x) J_\nu(xy) dx = (2\nu+1)^{-1} y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-1}(y)$$

$$[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(10)}$$

$$10. \quad \int_0^\infty x^\nu e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{2^{\mu-\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) y^\nu}{\Gamma(\nu-\mu+1) a^{1+2\nu}} {}_1F_1\left(\nu+\frac{1}{2}; \nu-\mu+1; -\frac{y^2}{2a^2}\right)$$

$$[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(11)}$$

$$11. \quad \int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{\Gamma(\frac{1}{2}+\nu) a^{2k} 2^{m+\mu}}{\Gamma(\frac{1}{2}-\mu) y^{\mu+\frac{3}{2}}} e^{\frac{y^2}{4a^2}} W_{k,m}\left(\frac{y^2}{4a^2}\right)$$

$$2k = \frac{1}{2} + \mu - \nu, \quad 2m = \frac{1}{2} + \mu + \nu$$

$$[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re} \nu < \operatorname{Re}(\frac{1}{2} - 2\mu)] \quad \text{ET II 78(12)}$$

$$12. \quad \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{2^\mu \Gamma(\nu+\frac{3}{2}) y^\nu}{\Gamma(\nu-\mu+\frac{3}{2}) a^{2\nu+2}} {}_1F_1\left(\nu+\frac{3}{2}; \nu-\mu+\frac{3}{2}; -\frac{y^2}{2a^2}\right)$$

$$[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 79(23)}$$

$$13. \quad \int_0^\infty x^{\nu+1} e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{\Gamma(\frac{3}{2}+\nu) 2^{\frac{1}{2}+m+\mu} a^{2k+1}}{\Gamma(-\mu) y^{\mu+2}} e^{\frac{y^2}{4a^2}} W_{k,m}\left(\frac{y^2}{2a^2}\right)$$

$$2k = \mu - \nu - 1, \quad 2m = \mu + \nu + 1$$

$$[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2} - 2\operatorname{Re} \mu] \quad \text{ET II 79(24)}$$

$$14. \quad \int_0^\infty x^{\lambda+\frac{1}{2}} e^{\frac{1}{4}a^2x^2} D_\mu(ax) J_\nu(xy) dx = \frac{2^{\lambda-\frac{1}{2}}\mu\pi^{-\frac{1}{2}}}{\Gamma(-\mu)y^{\lambda+\frac{3}{2}}} G_{23}^{22} \left(\frac{y^2}{2a^2} \left| \begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right) \\ [y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad \operatorname{Re} \mu < -\operatorname{Re} \lambda < \operatorname{Re} \nu + \frac{3}{2}] \quad \text{ET II 80(26)}$$

$$15. \quad \int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-1}(x) J_\nu(xy) dx = (2\nu+1)y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu-2}(y) \\ [y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 79(20)}$$

$$16. \quad \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy) dx = 2^{-1/2}\pi^{1/2}y^{-\nu-2} e^{-\frac{1}{4}y^2} I_{\nu+1}(\frac{1}{4}y^2) \\ [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 79(21)}$$

$$17. \quad \int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy) dx = y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-3}(y) \\ [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 79(22)}$$

$$18. \quad \int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{\frac{1}{2}\nu-\frac{1}{2}}(ax) Y_\nu(xy) dx = -\pi^{-1}2^{\frac{3}{4}\nu+\frac{3}{4}}a^{-\nu}y^{-1}\Gamma(\nu+1)e^{\frac{y^2}{4a^2}} W_{-\frac{1}{2}\nu-\frac{1}{2},\frac{1}{2}\nu} \left(\frac{y^2}{2a^2} \right) \\ [y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{2}{3}] \quad \text{ET II 115(39)}$$

7.753

$$1. \quad \int_0^\infty x^{\nu-\frac{1}{2}} e^{-(x+a)^2} I_{\nu-\frac{1}{2}}(2ax) D_\nu(2x) dx = \frac{1}{2}\pi^{-1/2}\Gamma(\nu)a^{\nu-\frac{1}{2}} D_{-\nu}(2a) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 397(12)}$$

$$2. \quad \int_0^\infty x^{\nu-\frac{3}{2}} e^{-(x+a)^2} I_{\nu-\frac{3}{2}}(2ax) D_\nu(2x) dx = \frac{1}{2}\pi^{-1/2}\Gamma(\nu)a^{\nu-\frac{3}{2}} D_{-\nu}(2a) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 1] \quad \text{ET II 397(13)}$$

7.754

$$1. \quad \int_0^\infty x^\nu e^{-\frac{1}{4}x^2} \{[1 \mp 2\cos(\nu\pi)] D_{2\nu-1}(x) - D_{2\nu-1}(-x)\} J_\nu(xy) dx \\ = \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{[1 \mp 2\cos(\nu\pi)] D_{2\nu-1}(y) - D_{2\nu-1}(-y)\} \\ [y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 76(2, 3)}$$

$$2. \quad \int_0^\infty x^\nu e^{-\frac{1}{4}x^2} \{[1 \mp 2\cos(\nu\pi)] D_{2\nu+1}(x) - D_{2\nu+1}(-x)\} J_\nu(xy) dx \\ = \mp y^\nu e^{-\frac{1}{4}y^2} \{[1 \mp 2\cos(\nu\pi)] D_{2\nu}(y) + D_{2\nu}(-y)\} \\ [y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(6, 7)}$$

$$3. \quad \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} \{[1 \pm 2\cos(\nu\pi)] D_{2\nu}(x) + D_{2\nu}(-x)\} J_\nu(xy) dx \\ = \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{[1 \pm 2\cos(\nu\pi)] D_{2\nu+1}(y) - D_{2\nu+1}(-y)\} \\ [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 78(14, 15)}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} \{[1 \mp 2 \cos(\nu\pi)] D_{2\nu+2}(x) + D_{2\nu+2}(-x)\} J_\nu(xy) dx \\
= \pm y^\nu e^{-\frac{1}{4}y^2} \{[1 \mp 2 \cos(\nu\pi)] D_{2\nu+2}(y) + D_{2\nu+2}(-y)\} \\
[y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 78(17, 18)}
\end{aligned}$$

7.755

$$\begin{aligned}
1. \quad \int_0^\infty x^{-1/2} D_\nu(\sqrt{ax}) D_{-\nu-1}(\sqrt{ax}) J_0(xy) dx \\
= 2^{-3/2} \pi a^{-1/2} P_{-\frac{1}{4}}^{\frac{1}{2}\nu+\frac{1}{4}} \left[\left(1 + \frac{4y^2}{a^2}\right)^{1/2} \right] P_{\frac{1}{4}}^{\frac{1}{2}\nu-\frac{1}{4}} \left[\left(1 + \frac{4y^2}{a^2}\right)^{1/2} \right] \\
[y > 0, \operatorname{Re} a > 0] \quad \text{ET II 17(43)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{1/2} D_{-\frac{1}{2}-\nu}(ae^{\frac{1}{4}\pi i} x^{1/2}) D_{-\frac{1}{2}-\nu}(ae^{-\frac{1}{4}\pi i} x^{1/2}) J_\nu(xy) dx \\
= 2^{-\nu} \pi^{1/2} y^{-\nu-1} (a^2 + 2y)^{-1/2} [\Gamma(\nu + \frac{1}{2})]^{-1} [(a^2 + 2y)^{1/2} - a]^{2\nu} \\
[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 80(27)}
\end{aligned}$$

$$\begin{aligned}
3. \quad a \int_0^\infty D_{-\frac{1}{2}-\nu}(ae^{\frac{1}{4}\pi i} x^{-1/2}) D_{-\frac{1}{2}-\nu}(ae^{-\frac{1}{4}\pi i} x^{-1/2}) J_\nu(xy) dx \\
= 2^{1/2} \pi^{1/2} y^{-1} [\Gamma(\nu + \frac{1}{2})]^{-1} \exp[-a(2y)^{1/2}] \\
[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 80(28)a}
\end{aligned}$$

$$\begin{aligned}
4. \quad \int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}}(ax^{-1/2}) D_{-\nu-\frac{1}{2}}(ax^{-1/2}) Y_\nu(xy) dx \\
= y^{-3/2} \exp(-ay^{1/2}) \sin\left[ay^{1/2} - \frac{1}{2}(\nu - \frac{1}{2})\pi\right] \\
[y > 0, \quad |\arg a| < \frac{1}{4}\pi] \quad \text{ET II 115(40)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}}(ax^{-1/2}) D_{-\nu-\frac{1}{2}}(ax^{-1/2}) K_\nu(xy) dx = 2^{-1} y^{-3/2} \pi \exp[-a(2y)^{1/2}] \\
[\operatorname{Re} y > 0, \quad |\arg a| < \frac{1}{4}\pi] \quad \text{ET II 151(81)}
\end{aligned}$$

Combinations of parabolic cylinder and Struve functions

$$\begin{aligned}
7.756 \quad \int_0^\infty x^{-\nu} e^{-\frac{1}{4}x^2} [D_\mu(x) - D_\mu(-x)] \mathbf{H}_\nu(xy) dx \\
= \frac{2^{3/2} \Gamma(\frac{1}{2}\mu + \frac{1}{2})}{\Gamma(\frac{1}{2}\mu + \nu + 1)} y^{\mu+\nu} \sin\left(\frac{1}{2}\mu\pi\right) {}_1F_1\left(\frac{1}{2}\mu + \frac{1}{2}; \frac{1}{2}\mu + \nu + 1; -\frac{1}{2}y^2\right) \\
[y > 0, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2}, \quad \operatorname{Re} \mu > -1] \quad \text{ET II 171(41)}
\end{aligned}$$

7.76 Combinations of parabolic cylinder functions and confluent hypergeometric functions

7.761

$$\begin{aligned}
 1. \quad \int_0^\infty e^{\frac{1}{4}t^2} t^{2c-1} D_{-\nu}(t) {}_1F_1\left(a; c; -\frac{1}{2}pt^2\right) dt \\
 = \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu}} \frac{\Gamma(2c) \Gamma(\frac{1}{2}\nu - c + a)}{\Gamma(\frac{1}{2}\nu) \Gamma(a + \frac{1}{2} + \frac{1}{2}\nu)} F\left(a, c + \frac{1}{2}; a + \frac{1}{2} + \frac{1}{2}\nu; 1-p\right) \\
 [|1-p| < 1, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > 2\operatorname{Re}(c-a)] \quad \text{EH II 121(12)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty e^{\frac{1}{4}t^2} t^{2c-2} D_{-\nu}(t) {}_1F_1\left(a; c; -\frac{1}{2}pt^2\right) dt \\
 = \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu-\frac{1}{2}}} \frac{\Gamma(2c-1) \Gamma(\frac{1}{2}\nu + \frac{1}{2} - c + a)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu) \Gamma(a + \frac{1}{2}\nu)} F\left(a, c - \frac{1}{2}; a + \frac{1}{2}\nu; 1-p\right) \\
 [|1-p| < 1, \quad \operatorname{Re} c > \frac{1}{2}, \quad \operatorname{Re} \nu > 2\operatorname{Re}(c-a) - 1] \quad \text{EH II 121(13)}
 \end{aligned}$$

7.77 Integration of a parabolic cylinder function with respect to the index

$$\begin{aligned}
 7.771 \quad \int_0^\infty \cos(ax) D_{x-\frac{1}{2}}(\beta) D_{-x-\frac{1}{2}}(\beta) dx &= \frac{1}{2} \left(\frac{\pi}{\cos a} \right)^{1/2} \exp\left(-\frac{\beta^2 \cos a}{2}\right) \quad [|a| < \frac{1}{2}\pi] \\
 &= 0 \quad [|a| > \frac{1}{2}\pi]
 \end{aligned}$$

ET II 298(22)

7.772

$$\begin{aligned}
 1. \quad \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left[\frac{(\tan \frac{1}{2}\varphi)^\nu}{\cos \frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\xi\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right) \right. \\
 \left. + \frac{(\cot \frac{1}{2}\varphi)^\nu}{\sin \frac{1}{2}\varphi} D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\xi\right) D_\nu\left(-e^{\frac{1}{4}i\pi}\eta\right) \right] \frac{d\nu}{\sin \nu\pi} \\
 = -2i(2\pi)^{1/2} \exp\left[-\frac{1}{4}i(\xi^2 - \eta^2) \cos \varphi - \frac{1}{2}i\xi\eta \sin \varphi\right] \\
 \text{EH II 125(7)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{(\tan \frac{1}{2}\varphi)^\nu}{\cos \frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\zeta\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right) \frac{d\nu}{\sin \nu\pi} \\
 = -2i D_0\left[e^{\frac{1}{4}i\pi}(\zeta \cos \frac{1}{2}\varphi + \eta \sin \frac{1}{2}\varphi)\right] D_{-1}\left[e^{\frac{1}{4}i\pi}(\eta \cos \frac{1}{2}\varphi - \zeta \sin \frac{1}{2}\varphi)\right] \\
 \text{EH II 125(8)}
 \end{aligned}$$

7.773

$$1. \quad \int_{c-i\infty}^{c+i\infty} D_\nu(z) t^\nu \Gamma(-\nu) d\nu = 2\pi i e^{-\frac{1}{4}z^2 - zt - \frac{1}{2}t^2} \quad \left[c < 0, \quad |\arg t| < \frac{\pi}{4} \right] \quad \text{EH II 126(10)}$$

$$\begin{aligned}
2. \quad & \int_{c-i\infty}^{c+i\infty} [D_\nu(x) D_{-\nu-1}(iy) + D_\nu(-x) D_{-\nu-1}(iy)] \frac{t^{-\nu-1} d\nu}{\sin(-\nu\pi)} \\
&= \frac{2\pi i}{\left(\frac{\pi}{2}\right)^{1/2}} (1+t^2)^{-\frac{1}{2}} \exp \left[\frac{1}{4} \frac{1-t^2}{1+t^2} (x^2+y^2) + i \frac{txy}{1+t^2} \right] \\
&\quad \left[-1 < c < 0, \quad |\arg t| < \frac{1}{2}\pi \right] \quad \text{EH II 126(11)}
\end{aligned}$$

$$\begin{aligned}
7.774 \quad & \int_{c-i\infty}^{c+i\infty} D_\nu \left[k^{\frac{1}{2}}(1+i)\xi \right] D_{-\nu-1} \left[k^{\frac{1}{2}}(1+i)\eta \right] \Gamma\left(-\frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu\right) d\nu = 2^{1/2} \pi^2 H_0^{(2)} \left[\frac{1}{2}k(\xi^2 + \eta^2) \right] \\
&\quad [-1 < c < 0, \quad \operatorname{Re} ik \geq 0] \quad \text{EH II 125(9)}
\end{aligned}$$

7.8 Meijer's and MacRobert's Functions (G and E)

7.81 Combinations of the functions G and E and the elementary functions

7.811

$$\begin{aligned}
1. \quad & \int_0^\infty G_{p,q}^{m,n} \left(\eta x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) G_{\sigma,\tau}^{\mu,\nu} \left(\omega x \left| \begin{matrix} c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau \end{matrix} \right. \right) dx \\
&= \frac{1}{\eta} G_{q+\sigma, p+\tau}^{n+\mu, m+\nu} \left(\frac{\omega}{\eta} \left| \begin{matrix} -b_1, \dots, -b_m, c_1, \dots, c_\sigma, -b_{m+1}, \dots, -b_q \\ -a_1, \dots, -a_n, d_1, \dots, d_\tau, -a_{n+1}, \dots, -a_p \end{matrix} \right. \right)
\end{aligned}$$

subject to the following constraints

- $m, n, p, q, \mu, \nu, \sigma, \tau$ are integers;
- $1 \leq n \leq p < q < p + \tau - \sigma$
- $\frac{1}{2}p + \frac{1}{2}q - n < m \leq q, \quad 0 \leq \nu \leq \sigma, \quad \frac{1}{2}\sigma + \frac{1}{2}\tau - \nu < \mu \leq \tau$
- $\operatorname{Re}(b_j + d_k) > -1 \quad (j = 1, \dots, m; k = 1, \dots, \mu)$
- $\operatorname{Re}(a_j + c_k) < 1 \quad (j = 1, \dots, n; k = 1, \dots, \tau)$
- $\omega \neq 0, \quad \eta \neq 0, \quad |\arg \eta| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad |\arg \omega| < (\mu + \nu - \frac{1}{2}\sigma - \frac{1}{2}\tau)\pi$
- The following must not be integers:

$$\begin{aligned}
& b_j - b_k \quad (j = 1, \dots, m; k = 1, \dots, m; j \neq k), \\
& a_j - a_k \quad (j = 1, \dots, n; k = 1, \dots, n; j \neq k), \\
& d_j - d_k \quad (j = 1, \dots, \mu; k = 1, \dots, \mu; j \neq k), \\
& a_j + d_k \quad (j = 1, \dots, n; k = 1, \dots, n);
\end{aligned}$$

- The following must not be positive integers:

$$\begin{aligned}
& a_j - b_k \quad (j = 1, \dots, n; k = 1, \dots, m) \\
& c_j - d_k \quad (j = 1, \dots, \nu; k = 1, \dots, \mu)
\end{aligned}$$

Formula **7.811** 1 also holds for four sets of restrictions. See C. S. Meijer, Neue Integraldarstellungen für Whittakersche Funktionen, Nederl. Akad. Wetensch. Proc. **44** (1941), 82–92.

ET II 422(14)

Hereafter, $G_{p,q}^{m,n}$ will be written as G_{pq}^{mn} , and commas will only be inserted in entries like $G_{p+1,q+1}^{m,n+1}$, where their omission could cause ambiguity.

$$2. \quad \int_0^1 x^{\rho-1} (1-x)^{\sigma-1} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \Gamma(\sigma) G_{p+1, q+1}^{m, n+1} \left(\alpha \left| \begin{matrix} 1-\rho, a_1, \dots, a_p \\ b_1, \dots, b_q, 1-\rho-\sigma \end{matrix} \right. \right)$$

where

- $(p+q) < 2(m+n)$
- $|\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q) \pi$
- $\operatorname{Re}(\rho + b_j) > 0, j = 1, \dots, m$
- $\operatorname{Re} \sigma > 0$
- either

$$\begin{aligned} p+q &\leq 2(m+n), \quad |\arg \alpha| \leq (m+n - \frac{1}{2}p - \frac{1}{2}q) \pi, \\ \operatorname{Re}(\rho + b_j) &> 0; \quad j = 1, \dots, m; \quad \operatorname{Re} \sigma > 0, \\ \operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p-q) \left(\rho - \frac{1}{2} \right) \right] &> -\frac{1}{2}, \end{aligned}$$

or

$$p < q \quad (\text{or } p \leq q \text{ for } |\alpha| < 1), \quad \operatorname{Re}(\rho + b_j) > 0; \quad j = 1, \dots, m; \quad \operatorname{Re} \sigma > 0$$

ET II 417(1)

$$3. \quad \int_1^\infty x^{-\rho} (x-1)^{\sigma-1} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \Gamma(\sigma) G_{p+1, q+1}^{m+1, n} \left(\alpha \left| \begin{matrix} a_1, \dots, a_p, \rho \\ \rho - \sigma, b_1, \dots, b_q \end{matrix} \right. \right)$$

where

- $p+q < 2(m+n)$
- $|\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q) \pi$
- $\operatorname{Re}(\rho - \sigma - a_j) > -1; \quad j = 1, \dots, n$
- $\operatorname{Re} \sigma > 0$
- either

$$\begin{aligned} p+q &\leq 2(m+n), \quad |\arg \alpha| \leq (m+n - \frac{1}{2}p - \frac{1}{2}q) \pi, \\ \operatorname{Re}(\rho - \sigma - a_j) &> -1; \quad j = 1, \dots, n; \quad \operatorname{Re} \sigma > 0, \\ \operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left(\rho - \sigma + \frac{1}{2} \right) \right] &> -\frac{1}{2}, \end{aligned}$$

or

$$q < p \quad (\text{or } q \leq p \text{ for } |\alpha| > 1), \quad \operatorname{Re}(\rho - \sigma - a_j) > -1; \quad j = 1, \dots, n; \quad \operatorname{Re} \sigma > 0$$

ET II 417(2)

$$4. \quad \int_0^\infty x^{\rho-1} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \frac{\prod_{j=1}^m \Gamma(b_j + \rho) \prod_{j=1}^n \Gamma(1 - a_j - \rho)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \rho) \prod_{j=n+1}^p \Gamma(a_j + \rho)} \alpha^{-\rho}$$

$$p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q) \pi, \quad -\min_{1 \leq j \leq m} \operatorname{Re} b_j < \operatorname{Re} \rho < 1 - \max_{1 \leq j \leq n} \operatorname{Re} a_j$$

ET II 418(3)a, ET I 337(14)

$$5. \quad \int_0^\infty x^{\rho-1} (x+\beta)^{-\sigma} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)} G_{p+1, q+1}^{m+1, n+1} \left(\alpha \beta \left| \begin{matrix} 1-\rho, a_1, \dots, a_p \\ \sigma-\rho, b_1, \dots, b_q \end{matrix} \right. \right)$$

where

- $p+q < 2(m+n)$
- $|\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q) \pi$
- $|\arg \beta| < \pi$
- $\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m$
- $\operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n$
- either

$$p \leq q, \quad p+q \leq 2(m+n), \quad |\arg \alpha| \leq (m+n - \frac{1}{2}p - \frac{1}{2}q) \pi, \quad |\arg \beta| < \pi$$

$$\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n,$$

$$\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j - (q-p) \left(\rho - \sigma - \frac{1}{2} \right) \right] > 1,$$

or

$$p \geq q, \quad p+q \leq 2(m+n), \quad |\arg \alpha| \leq \left(m+n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \quad |\arg \beta| < \pi,$$

$$\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n,$$

$$\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p-q) \left(\rho - \frac{1}{2} \right) \right] > 1$$

ET II 418(4)

7.812

$$1. \quad \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} E \left(a_1, \dots, a_p : \rho_1, \dots, \rho_q; \frac{z}{x^m} \right) dx$$

$$= \Gamma(\gamma - \beta) m^{\beta-\gamma} E(a_1, \dots, a_{p+m} : \rho_1, \dots, \rho_{q+m} : z)$$

$$a_{p+k} = \frac{\beta + k - 1}{m}, \quad \rho_{q+k} = \frac{\gamma + k - 1}{m}, \quad k = 1, \dots, m$$

$$[\operatorname{Re} \gamma > \operatorname{Re} \beta > 0, \quad m = 1, 2, \dots] \quad \text{ET II 414(2)}$$

$$2. \quad \int_0^\infty x^{\rho-1} (1+x)^{-\sigma} E[a_1, \dots, a_p : \rho_1, \dots, \rho_q : (1+x)z] dx$$

$$= \Gamma(\rho) E(a_1, \dots, a_p, \sigma - \rho; \rho_1, \dots, \rho_q, \sigma; z)$$

$$[\operatorname{Re} \sigma > \operatorname{Re} \rho > 0] \quad \text{ET II 415(3)}$$

$$\begin{aligned}
3. \quad \int_0^\infty (1+x)^{-\beta} x^{s-1} G_{pq}^{mn} \left(\frac{ax}{1+x} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \Gamma(\beta-s) G_{p+1, q+1}^{m, n+1} \left(a \left| \begin{matrix} 1-s, a_1, \dots, a_p \\ b_1, \dots, b_q, 1-\beta \end{matrix} \right. \right) \\
&\left[-\min \operatorname{Re} b_k < \operatorname{Re} s < \operatorname{Re} \beta, \quad 1 \leq k \leq m; \quad (p+q) < 2(m+n), \right. \\
&\quad \left. |\arg a| < (m+n - \tfrac{1}{2}p - \tfrac{1}{2}q) \pi \right] \\
&\text{ET I 338(19)}
\end{aligned}$$

7.813

$$\begin{aligned}
1. \quad \int_0^\infty x^{-\rho} e^{-\beta x} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \beta^{\rho-1} G_{p+1, q}^{m, n+1} \left(\frac{\alpha}{\beta} \left| \begin{matrix} \rho, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\left[p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \tfrac{1}{2}p - \tfrac{1}{2}q) \pi, \right. \\
&\quad \left. |\arg \beta| < \tfrac{1}{2}\pi, \quad \operatorname{Re}(b_j - \rho) > -1, \quad j = 1, \dots, m \right] \\
&\text{ET II 419(5)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty e^{-\beta x} G_{pq}^{mn} \left(\alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \pi^{-1/2} \beta^{-1} G_{p+2, q}^{m, n+2} \left(\frac{4\alpha}{\beta^2} \left| \begin{matrix} 0, \frac{1}{2}, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\left[p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \tfrac{1}{2}p - \tfrac{1}{2}q) \pi, \right. \\
&\quad \left. |\arg \beta| < \tfrac{1}{2}\pi, \quad \operatorname{Re} b_j > -\tfrac{1}{2}; \quad j = 1, \dots, m \right] \\
&\text{ET II 419(6)}
\end{aligned}$$

7.814

$$\begin{aligned}
1. \quad \int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; xz) dx \\
&= \pi \operatorname{cosec}(\beta\pi) \left[E(a_1, \dots, a_p; 1-\beta, \rho_1, \dots, \rho_q; e^{\pm i\pi} z) \right. \\
&\quad \left. - z^{-\beta} E(a_1 + \beta, \dots, a_p + \beta; 1+\beta, \rho_1 + \beta, \dots, \rho_l + \beta; e^{\pm i\pi} z) \right] \\
&[p \geq q+1, \operatorname{Re}(a_r + \beta) > 0, r = 1, \dots, p, |\arg z| < \pi. \text{ The formula holds also for } p < q+1, \\
&\text{provided the integral converges.}] \\
&\text{ET II 415(4)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; x^{-m} z) dx \\
&= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta-\frac{1}{2}} E(a_1, \dots, a_{p+m}; \rho_1, \dots, \rho_q; m^{-m} z) \\
&\left[\operatorname{Re} \beta > 0, \quad a_{p+k} = \frac{\beta+k-1}{m}, \quad k = 1, \dots, m; \quad m = 1, 2, \dots \right] \quad \text{ET II 415(5)}
\end{aligned}$$

7.815

$$\begin{aligned}
1. \quad \int_0^\infty \sin(cx) G_{pq}^{mn} \left(\alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \sqrt{\pi} c^{-1} G_{p+2, q}^{m, n+1} \left(\frac{4\alpha}{c^2} \left| \begin{matrix} 0, a_1, \dots, a_p, \frac{1}{2} \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\left[p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \tfrac{1}{2}p - \tfrac{1}{2}q) \pi, \right. \\
&\quad c > 0, \quad \operatorname{Re} b_j > -1, \quad j = 1, 2, \dots, m, \quad \operatorname{Re} a_j < \tfrac{1}{2}, \quad j = 1, \dots, n \left. \right] \\
&\text{ET II 420(7)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty \cos(cx) G_{pq}^{mn} \left(\alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \pi^{1/2} c^{-1} G_{p+2,q}^{m,n+1} \left(\frac{4\alpha}{c^2} \left| \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 0 \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\quad [p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \\
&\quad c > 0, \quad \operatorname{Re} b_j > -\frac{1}{2}, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j < \frac{1}{2}, \quad j = 1, \dots, n] \\
&\quad \text{ET II 420(8)}
\end{aligned}$$

7.82 Combinations of the functions G and E and Bessel functions

7.821

$$\begin{aligned}
1. \quad \int_0^\infty x^{-\rho} J_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= G_{p+2,q}^{m,n+1} \left(\alpha \left| \begin{matrix} \rho - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\quad \left[p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi \right. \\
&\quad \left. - \frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \rho < 1 + \frac{1}{2} \operatorname{Re} \nu + \min_{1 \leq j \leq m} \operatorname{Re} b_j \right] \\
&\quad \text{ET II 420(9)}
\end{aligned}$$

$$\begin{aligned}
2. \quad \int_0^\infty x^{-\rho} Y_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
&= G_{p+3,q+1}^{m,n+2} \left(\alpha \left| \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2} + \frac{1}{2}\nu \\ b_1, \dots, b_q, \rho + \frac{1}{2} + \frac{1}{2}\nu \end{matrix} \right. \right) \\
&\quad \left[p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right. \\
&\quad \left. - \frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \rho < \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2}|\operatorname{Re} \nu| + 1 \right] \\
&\quad \text{ET II 420(10)}
\end{aligned}$$

$$\begin{aligned}
3. \quad \int_0^\infty x^{-\rho} K_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \frac{1}{2} G_{p+2,q}^{m,n+2} \left(\alpha \left| \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\quad \left[p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right. \\
&\quad \left. \operatorname{Re} \rho < 1 - \frac{1}{2}|\operatorname{Re} \nu| + \min_{1 \leq j \leq m} \operatorname{Re} b_j \right] \\
&\quad \text{ET II 421(11)}
\end{aligned}$$

7.822

$$\begin{aligned}
1. \quad \int_0^\infty x^{2\rho} J_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \frac{2^{2\rho}}{y^{2\rho+1}} G_{p+2,q}^{m,n+1} \left(\frac{4\lambda}{y^2} \left| \begin{matrix} h, a_1, \dots, a_p, k \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\quad h = \frac{1}{2} - \rho - \frac{1}{2}\nu, \quad k = \frac{1}{2} - \rho + \frac{1}{2}\nu \\
&\quad \left[p+q < 2(m+n), \quad |\arg \lambda| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad \operatorname{Re}(b_j + \rho + \frac{1}{2}\nu) > -\frac{1}{2}, \right. \\
&\quad \left. j = 1, 2, \dots, m, \quad \operatorname{Re}(a_j + \rho) < \frac{3}{4}, \quad j = 1, \dots, n, \quad y > 0 \right] \\
&\quad \text{ET II 91(20)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{1/2} Y_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
&= (2\lambda)^{-1/2} y^{-1/2} G_{q+1, p+3}^{n+2, m} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q, l \\ h, k, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p, l \end{matrix} \right. \right) \\
&\quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu \\
&\quad \left[p + q < 2(m + n), \quad |\arg \lambda| < (m + n - \frac{1}{2}p - \frac{1}{2}q) \pi, \quad y > 0, \right. \\
&\quad \left. \operatorname{Re} a_j < 1, \quad j = 1, \dots, n, \quad \operatorname{Re} (b_j \pm \frac{1}{2}\nu) > -\frac{3}{4}, \quad j = 1, \dots, m \right]
\end{aligned}$$

ET II 119(56)

$$\begin{aligned}
3. \quad & \int_0^\infty x^{1/2} K_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
&= 2^{-3/2} \lambda^{-1/2} y^{-1/2} G_{q, p+2}^{n+2, m} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q \\ h, k, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p \end{matrix} \right. \right) \\
&\quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu \\
&\quad \left[\operatorname{Re} y > 0, \quad p + q < 2(m + n), \quad |\arg \lambda| < (m + n - \frac{1}{2}p - \frac{1}{2}q) \pi, \right. \\
&\quad \left. \operatorname{Re} b_j > \frac{1}{2} |\operatorname{Re} \nu| - \frac{3}{4}, \quad j = 1, \dots, m \right]
\end{aligned}$$

ET II 153(90)

7.823

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\beta-1} J_\nu(x) E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : x^{-2m} z) dx \\
&= (2\pi)^{-m} (2m)^{\beta-1} \left\{ \exp \left[\frac{1}{2} \pi (\beta - \nu - 1) i \right] E[a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_q : (2m)^{-2m} z e^{-m\pi i}] \right. \\
&\quad \left. + \exp \left[-\frac{1}{2} \pi (\beta - \nu - 1) i \right] E[a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_q : (2m)^{-2m} z e^{m\pi i}] \right\}, \\
&\quad a_{p+k} = \frac{\beta + \nu + 2k - 2}{2m}, \quad a_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}, \quad m = 1, 2, \dots; \quad k = 1, \dots, m \\
&\quad [\operatorname{Re}(\beta + \nu) > 0, \quad \operatorname{Re}(2a_r m - \beta) > -\frac{3}{2}, \quad r = 1, \dots, p] \quad \text{ET II 415(7)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\beta-1} K_\nu(x) E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : x^{-2m} z) dx \\
&= (2\pi)^{1-m} 2^{\beta-2} m^{\beta-1} \\
&\quad \times E[a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_q : (2m)^{-2m} z], \\
&\quad a_{p+k} = \frac{\beta + \nu + 2k - 2}{2m}, \quad a_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}, \quad k = 1, 2, \dots, m \\
&\quad [\operatorname{Re} \beta > |\operatorname{Re} \nu|, \quad m = 1, 2, \dots]
\end{aligned}$$

ET II 416(8)

7.824

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{1/2} \mathbf{H}_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
 &= (2\lambda y)^{-1/2} G_{q+1, p+3}^{n+1, m+1} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} l, \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q \\ l, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p, h, k \end{matrix} \right. \right) \\
 & \quad h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2} \\
 & \quad \left[p + q < 2(m + n), \quad |\arg \lambda| < (m + n - \frac{1}{2}p - \frac{1}{2}q) \pi, \quad y > 0, \right. \\
 & \quad \left. \operatorname{Re} a_j < \min \left(1, \frac{3}{4} - \frac{1}{2}\nu \right), \quad j = 1, \dots, n, \quad \operatorname{Re} (2b_j + \nu) > -\frac{5}{2}, \quad j = 1, \dots, m \right]
 \end{aligned}$$

ET II 172(47)

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{-\rho} \mathbf{H}_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
 &= G_{p+3, q+1}^{m+1, n+1} \left(\alpha \left| \begin{matrix} \rho - \frac{1}{2} - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu, \rho - \frac{1}{2}\nu \\ \rho - \frac{1}{2} - \frac{1}{2}\nu, b_1, \dots, b_q \end{matrix} \right. \right) \\
 & \quad \left[p + q < 2(m + n), \quad |\arg \alpha| < (m + n - \frac{1}{2}p - \frac{1}{2}q) \pi, \right. \\
 & \quad \left. \max \left(-\frac{3}{4}, \operatorname{Re} \frac{\nu - 1}{2} \right) + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \rho < \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2} \operatorname{Re} \nu + \frac{3}{2} \right]
 \end{aligned}$$

ET II 421(12)

7.83 Combinations of the functions G and E and other special functions

$$\begin{aligned}
 7.831 \quad & \int_1^\infty x^{-\rho} (x-1)^{\sigma-1} F(k + \sigma - \rho, \lambda + \sigma - \rho; \sigma; 1-x) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
 &= \Gamma(\sigma) G_{p+2, q+2}^{m+2, n} \left(\alpha \left| \begin{matrix} a_1, \dots, a_p, k + \lambda + \sigma - \rho, \rho \\ k, \lambda, b_1, \dots, b_q \end{matrix} \right. \right)
 \end{aligned}$$

where

ET II 421(13)

- $\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left(k + \frac{1}{2} \right) \right] > -\frac{1}{2}$
- $\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left(\lambda + \frac{1}{2} \right) \right] > -\frac{1}{2}$
- either

$$\begin{aligned}
 & p + q < 2(m + n), \quad |\arg \alpha| < (m + n - \frac{1}{2}p - \frac{1}{2}q) \pi, \\
 & \operatorname{Re} \sigma > 0, \quad \operatorname{Re} k \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n,
 \end{aligned}$$

or

$$p + q \leq 2(m + n), \quad |\arg \alpha| \leq \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi,$$

$$\operatorname{Re} \sigma > 0, \quad \operatorname{Re} k \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n,$$

7.832
$$\int_0^\infty x^{\beta-1} e^{-\frac{1}{2}x} W_{\kappa, \mu}(x) E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : x^{-m} z) dx$$

$$= (2\pi)^{\frac{1}{2} - \frac{1}{2}m} m^{\beta + \kappa - \frac{1}{2}} E(a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_{q+m} : m^{-m} z),$$

$$a_{p+k} = \frac{\beta + k + \mu - \frac{1}{2}}{m}, \quad a_{p+m+k} = \frac{\beta - \mu + k - \frac{1}{2}}{m}, \quad \rho_{q+k} = \frac{\beta - \kappa + k}{m}, \quad k = 1, \dots, m$$

$$[\operatorname{Re} \beta > |\operatorname{Re} \mu| - \frac{1}{2}, \quad m = 1, 2, \dots] \quad \text{ET II 416(10)}$$

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8–9 Special Functions

8.1 Elliptic Integrals and Functions

8.11 Elliptic integrals

8.110

1. Every integral of the form $\int R(x, \sqrt{P(x)}) dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals:

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx, \quad \int \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

which are called respectively *elliptic integrals of the first, second, and third kind in the Legendre normal form*. The results of this reduction for the more frequently encountered integrals are given in formulas **3.13–3.17**. The number k is called the *modulus*^{*} of these integrals; the number $k' = \sqrt{1-k^2}$ is called the complementary modulus, and the number n is called the parameter of the integral of the third kind. BY (110.04)

2. By means of the substitution $x = \sin \varphi$, elliptic integrals can be reduced to the normal trigonometric forms

$$\int \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}, \quad \int \sqrt{1-k^2 \sin^2 \varphi} d\varphi, \quad \int \frac{d\varphi}{(1-n \sin^2 \varphi) \sqrt{1-k^2 \sin^2 \varphi}}. \quad \text{BY (110.04)}$$

The results of reducing integrals of trigonometric functions to normal form are given in **2.58–2.62**.

- 3.¹¹ Elliptic integrals from 0 to 1 in the **8.110 1** formulation (or from 0 to $\frac{\pi}{2}$ in the **8.110 2** formulation) are called *complete elliptic integrals*.
- 4.^{*} Take note that in mathematical software, and elsewhere, the notation for elliptic integrals is often modified by replacing the parameter k^2 that is used here with k .

8.111

Notations:

1. $\Delta\varphi = \sqrt{1-k^2 \sin^2 \varphi}; \quad k' = \sqrt{1-k^2}; \quad k^2 < 1$

^{*}The quantity k is sometimes called the *module* of the functions.

2. The elliptic integral of the first kind:

$$F(\varphi, k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

3. The elliptic integral of the second kind:

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \alpha} d\alpha = \int_0^{\sin \varphi} \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}} dx \quad \text{FI II 135}$$

- 4.¹¹ The elliptic integral of the third kind:

$$\Pi(\varphi, n, k) = \int_0^\varphi \frac{d\alpha}{(1 - n \sin^2 \alpha) \sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{dx}{(1 - nx^2) \sqrt{(1 - x^2)(1 - k^2 x^2)}} \quad \text{BY (110.04)}$$

$$5. \quad D(\varphi, k) = \frac{F(\varphi, k) - E(\varphi, k)}{k^2} = \int_0^\varphi \frac{\sin^2 \alpha d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{x^2 dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

$$6.* \quad \int_0^{\pi/2} \frac{dx}{\sqrt{a^2 + \sin^2 x}} \arctan \left(\frac{b}{\sqrt{a^2 + \sin^2 x}} \right) = \frac{\pi}{2|a|} F \left(\arcsin \left(\frac{b}{\sqrt{a^2 + b^2 + 1}} \right), \frac{i}{a} \right)$$

[a and b are real]

8.112 Complete elliptic integrals

1. $\mathbf{K}(k) = F\left(\frac{\pi}{2}, k\right) = \mathbf{K}'(k')$
2. $\mathbf{E}(k) = E\left(\frac{\pi}{2}, k\right) = \mathbf{E}'(k')$
3. $\mathbf{K}'(k) = F\left(\frac{\pi}{2}, k'\right) = \mathbf{K}(k')$
4. $\mathbf{E}'(k) = E\left(\frac{\pi}{2}, k'\right) = \mathbf{E}(k')$
5. $\mathbf{D} = D\left(\frac{\pi}{2}, k\right) = \frac{\mathbf{K} - \mathbf{E}}{k^2}$

In writing complete elliptic integrals, the modulus k , which acts as an independent variable, is often omitted, and we write

$$\mathbf{K} (\equiv \mathbf{K}(k)), \quad \mathbf{K}' (\equiv \mathbf{K}'(k)), \quad \mathbf{E} (\equiv \mathbf{E}(k)), \quad \mathbf{E}' (\equiv \mathbf{E}'(k)).$$

Series representations

8.113

$$1. \quad \mathbf{K} = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 k^{2n} + \cdots \right\} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

FI II 487, WH 499

$$2. \quad K = \frac{\pi}{1+k'} \left\{ 1 + \left(\frac{1}{2}\right)^2 \left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\frac{1-k'}{1+k'}\right)^4 + \dots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \dots \right\}$$

DW

$$3. \quad K = \ln \frac{4}{k'} + \left(\frac{1}{2}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2}\right) k'^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4}\right) k'^4 \\ + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{2}{5 \cdot 6}\right) k'^6 + \dots$$

DW

See also **8.197** 1 and **8.197** 2.

8.114

$$1.^6 \quad E = \frac{\pi}{2} \left\{ 1 - \frac{1}{2^2} k^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} k^4 - \dots - \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \frac{k^{2n}}{2n-1} - \dots \right\} = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

WH 518, FI II 487

$$2. \quad E = \frac{(1+k')\pi}{4} \left\{ 1 + \frac{1}{2^2} \left(\frac{1-k'}{1+k'}\right)^2 + \frac{1^2}{2^2 \cdot 4^2} \left(\frac{1-k'}{1+k'}\right)^4 + \dots + \left(\frac{(2n-3)!!}{2^n n!}\right)^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \dots \right\}$$

DW

$$3. \quad E = 1 + \frac{1}{2} \left(\ln \frac{4}{k'} - \frac{1}{1 \cdot 2}\right) k'^2 + \frac{1^2 \cdot 3}{2^2 \cdot 4} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{1}{3 \cdot 4}\right) k'^4 \\ + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{1}{5 \cdot 6}\right) k'^6 + \dots$$

DW

$$8.115 \quad D = \pi \left\{ \frac{1}{1} \left(\frac{1}{2}\right)^2 + \frac{2}{3} \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^2 + \dots + \frac{n}{2n-1} \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2(n-1)} + \dots \right\}$$

ZH 43(158)

$$8.116 \quad \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-k^2 \sin^2 \varphi}}{1-n^2 \sin^2 \varphi} d\varphi = \sqrt{n'^2 - k'^2} \left(\frac{\arccos \frac{1}{n'}}{n' \sqrt{n'^2 - 1}} + \mathbf{R} \right), \quad \text{where}$$

ZH 44(163)

$$\mathbf{R} = \frac{k'^2}{2} \left(p + \frac{1}{2}\right) \frac{1}{n'^3} + \frac{k'^4}{16} \left[-1 + \left(p + \frac{1}{4}\right) \frac{1}{n'^3} \left(1 + \frac{6}{n'^2}\right)\right] \\ + \frac{k'^6}{16} \left[-\frac{7}{16} - \frac{1}{n'^2} + \left(p + \frac{1}{6}\right) \frac{1}{n'^3} \left(\frac{3}{8} + \frac{1}{n'^2} + \frac{5}{n'^4}\right)\right] \\ + \frac{15k'^8}{256} \left[-\frac{37}{144} - \frac{21}{40n'^2} - \frac{1}{n'^4} + \left(p + \frac{1}{8}\right) \frac{1}{n'^3} \left(\frac{5}{24} + \frac{9}{20n'^2} + \frac{1}{n'^4} + \frac{14}{3n'^6}\right)\right] + \dots, \\ p = \ln \frac{4}{k'}, \quad k' = 4e^{-p}, \quad k'^2 = 1 - k^2, \quad n'^2 = 1 - n^2 \quad \text{ZH 44(163)}$$

Trigonometric series

8.117 For *small* values of k and φ , we may use the series

$$1. \quad F(\varphi, k) = \frac{2}{\pi} \mathbf{K} \varphi - \sin \varphi \cos \varphi \left(a_0 + \frac{2}{3} a_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} a_2 \sin^4 \varphi + \dots \right), \quad \text{where}$$

$$a_0 = \frac{2}{\pi} \mathbf{K} - 1; \quad a_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 k^{2n} \quad \text{ZH 10(19)}$$

$$2. \quad E(\varphi, k) = \frac{2}{\pi} \mathbf{E} \varphi + \sin \varphi \cos \varphi \left(b_0 + \frac{2}{3} b_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} b_2 \sin^4 \varphi + \dots \right), \quad \text{where}$$

$$b_0 = 1 - \frac{2}{\pi} \mathbf{E}, \quad b_n = b_{n-1} - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 \frac{k^{2n}}{2n-1} \quad \text{ZH 27(86)}$$

8.118 For k close to 1, we may use the series

$$1. \quad F(\varphi, k) = \frac{2}{\pi} \mathbf{K}' \ln \tan \left(\frac{\varphi}{2} + \frac{\pi}{4} \right) - \frac{\tan \varphi}{\cos \varphi} \left(a'_0 - \frac{2}{3} a'_1 \tan^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} a'_2 \tan^4 \varphi - \dots \right), \quad \text{where}$$

$$a'_0 = \frac{2}{\pi} \mathbf{K}' - 1; \quad a'_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 k'^{2n} \quad \text{ZH 10(23)}$$

$$2. \quad E(\varphi, k) = \frac{2}{\pi} (\mathbf{K}' - \mathbf{E}') \ln \tan \left(\frac{\varphi}{2} + \frac{\pi}{2} \right) + \frac{\tan \varphi}{\cos \varphi} \left(b'_1 - \frac{2}{3} b'_2 \tan^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} b'_3 \tan^4 \varphi - \dots \right) + \frac{1}{\sin \varphi} \left[1 - \cos \varphi \sqrt{1 - k^2 \sin^2 \varphi} \right],$$

where

$$b'_0 = \frac{2}{\pi} (\mathbf{K}' - \mathbf{E}'), \quad b'_n = b'_{n-1} - \left[\frac{(2n-3)!!}{2^{n-1}(n-1)!} \right]^2 \left(\frac{2n-1}{2n} \right) k'^{2n} \quad \text{ZH 27(90)}$$

For the expansion of complete elliptic integrals in Legendre polynomials, see **8.928**.

8.119 Representation in the form of an infinite product:

$$1. \quad \mathbf{K}(k) = \frac{\pi}{2} \prod_{n=1}^{\infty} (1 + k_n), \quad \text{where}$$

$$k_n = \frac{1 - \sqrt{1 - k_{n-1}^2}}{1 + \sqrt{1 - k_{n-1}^2}}; \quad k_0 = k \quad \text{FI II 166}$$

See also **8.197**.

8.12 Functional relations between elliptic integrals

8.121

1. $F(-\varphi, k) = -F(\varphi, k)$ JA
2. $E(-\varphi, k) = -E(\varphi, k)$ JA
3. $F(n\pi \pm \varphi, k) = 2n \mathbf{K}(k) \pm F(\varphi, k)$ JA
4. $E(n\pi \pm \varphi, k) = 2n \mathbf{E}(k) \pm E(\varphi, k)$ JA

8.122 $\mathbf{E}(k) \mathbf{K}'(k) + \mathbf{E}'(k) \mathbf{K}(k) - \mathbf{K}(k) \mathbf{K}'(k) = \frac{\pi}{2}$ FI II 691, 791

8.123

1. $\frac{\partial F}{\partial k} = \frac{1}{k'^2} \left(\frac{E - k'^2 F}{k} - \frac{k \sin \varphi \cos \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \right)$ MO 138, BY (710.07)
2. $\frac{d \mathbf{K}(k)}{dk} = \frac{\mathbf{E}(k)}{k k'^2} - \frac{\mathbf{K}(k)}{k}$ FI II 691
3. $\frac{\partial E}{\partial k} = \frac{E - F}{k}$ MO 138
4. $\frac{d \mathbf{E}(k)}{dk} = \frac{\mathbf{E}(k) - \mathbf{K}(k)}{k}$ FI II 690

8.124

1. The functions $\mathbf{K}$ and $\mathbf{K}'$ satisfy the equation

$$\frac{d}{dk} \left\{ k k'^2 \frac{du}{dk} \right\} - k u = 0. \quad \text{WH 499, WH 502}$$

2. The functions $\mathbf{E}$ and $\mathbf{E}' - \mathbf{K}'$ satisfy the equation

$$k'^2 \frac{d}{dk} \left(k \frac{du}{dk} \right) + k u = 0. \quad \text{WH}$$

8.125

1. $F\left(\psi, \frac{1 - k'}{1 + k'}\right) = (1 + k') F(\varphi, k)$ $[\tan(\psi - \varphi) = k' \tan \varphi]$ MO 130
2. $E\left(\psi, \frac{1 - k'}{1 + k'}\right) = \frac{2}{1 + k'} [E(\varphi, k) + k' F(\varphi, k)] - \frac{1 - k'}{1 + k'} \sin \psi$
 $[\tan(\psi - \varphi) = k' \tan \varphi]$ MO 131
3. $F\left(\psi, \frac{2\sqrt{k}}{1 + k}\right) = (1 + k) F(\varphi, k)$ $\left[\sin \psi = \frac{(1 + k) \sin \varphi}{1 + k \sin^2 \varphi}\right]$
4. $E\left(\psi, \frac{2\sqrt{k}}{1 + k}\right) = \frac{1}{1 + k} \left[2E(\varphi, k) - k'^2 F(\varphi, k) + 2k \frac{\sin \varphi \cos \varphi}{1 + k \sin^2 \varphi} \sqrt{1 - k^2 \sin^2 \varphi} \right]$
 $\left[\sin \psi = \frac{(1 + k) \sin \varphi}{1 + k \sin^2 \varphi}\right]$ MO 131

8.126 In particular,

1. $\mathbf{K} \left(\frac{1-k'}{1+k'} \right) = \frac{1+k'}{2} \mathbf{K}(k)$ MO 130
2. $\mathbf{E} \left(\frac{1-k'}{1+k'} \right) = \frac{1}{1+k'} [\mathbf{E}(k) + k' \mathbf{K}(k)]$ MO 130
3. $\mathbf{K} \left(\frac{2\sqrt{k}}{1+k} \right) = (1+k) \mathbf{K}(k)$ MO 130
4. $\mathbf{E} \left(\frac{2\sqrt{k}}{1+k} \right) = \frac{1}{1+k} [2\mathbf{E}(k) - k'^2 \mathbf{K}(k)]$ MO 130

8.127¹¹

k_1	$\sin \varphi_1$	$\cos \varphi_1$	$F(\varphi_1, k_1)$	$E(\varphi_1, k_1)$
$i \frac{k}{k'}$	$k' \frac{\sin \varphi}{\Delta \varphi}$	$\frac{\cos \varphi}{\Delta \varphi}$	$k' F(\varphi, k)$	$\frac{1}{k'} \left[E(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta \varphi} \right]$
k'	$-i \tan \varphi$	$\sec \varphi$	$-i F(\varphi, k)$	$ia [E(\varphi, k) - F(\varphi, k) - \Delta \varphi \tan \varphi]$
$\frac{1}{k}$	$k \sin \varphi$	$\Delta \varphi$	$k F(\varphi, k)$	$\frac{1}{k} [E(\varphi, k) - k'^2 F(\varphi, k)]$
$\frac{1}{k'}$	$-ik' \tan \varphi$	$\frac{\Delta \varphi}{\cos \varphi}$	$-ik' F(\varphi, k)$	$\frac{i}{k'} [E(\varphi, k) - k'^2 F(\varphi, k) - \Delta \varphi \tan \varphi]$
$\frac{k'}{ik}$	$\frac{-ik \sin \varphi}{\Delta \varphi}$	$\frac{1}{\Delta \varphi}$	$-ik F(\varphi, k)$	$\frac{i}{k} \left[E(\varphi, k) - F(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta \varphi} \right]$

(see **8.111 1**) MO 131

8.128 In particular,

1. $\mathbf{K} \left(i \frac{k}{k'} \right) = k' \mathbf{K}(k)$ [Im(k) < 0] MO 130
2. $\mathbf{K} \left(i \frac{k}{k'} \right) = k' [\mathbf{K}'(k') - i \mathbf{K}(k)]$ [Im(k) < 0] MO 130
3. $\mathbf{K} \left(\frac{1}{k} \right) = k [\mathbf{K}(k) + i \mathbf{K}'(k)]$ [Im(k) < 0] MO 130

For integrals of elliptic integrals, see **6.11–6.15**. For indefinite integrals of complete elliptic integrals, see **5.11**.

8.129 Special values:

1. $\mathbf{K} \left(\sin \frac{\pi}{4} \right) = \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) = \mathbf{K}' \left(\frac{\sqrt{2}}{2} \right) = \sqrt{2} \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{4\sqrt{\pi}} \left[\Gamma \left(\frac{1}{4} \right) \right]^2$ MO 130
2. $\mathbf{K}' \left(\sqrt{2} - 1 \right) = \sqrt{2} \mathbf{K} \left(\sqrt{2} - 1 \right)$ MO 130

$$3. \quad K' \left(\sin \frac{\pi}{12} \right) = \sqrt{3} K \left(\sin \frac{\pi}{12} \right) \quad \text{MO 130}$$

$$4. \quad K' \left(\tan^2 \frac{\pi}{8} \right) = K' \left(\frac{2 - \sqrt{2}}{2 + \sqrt{2}} \right) = 2 K \left(\tan^2 \frac{\pi}{8} \right) \quad \text{MO 130}$$

$$5.* \quad K \left(\sin \frac{\pi}{12} \right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$6.* \quad E = \frac{\pi\sqrt{3}}{12K} + \sqrt{\frac{2}{3}} k' K$$

$$7.* \quad E' = \frac{\pi\sqrt{3}}{4K'} + \sqrt{\frac{2}{3}} k K'$$

8.13 Elliptic functions

8.130 Definition and general properties.

1. A single-valued function $f(z)$ of a complex variable, which is not a constant, is said to be elliptic if it has two periods $2\omega_1$ and $2\omega_2$, that is

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z) \quad [m, n \text{ integers}].$$

The ratio of the periods of an analytic function cannot be a real number. For an elliptic function $f(z)$, the z -plane can be partitioned into parallelograms—the period parallelograms—the vertices of which are the points $z_0 + 2m\omega_1 + 2n\omega_2$. At corresponding points of these parallelograms, the function $f(z)$ has the same value. ZH 117, SI 299

2. Suppose that α is the angle between the sides a and b of one of the period parallelograms. Then,

$$\tau = \frac{\omega_1}{\omega_2} = \frac{a}{b} e^{i\alpha}, \quad q = e^{i\pi\tau} = e^{-\frac{a}{b}\pi \sin \alpha} \left[\cos \left(\frac{a}{b}\pi \cos \alpha \right) + i \sin \left(\frac{a}{b}\pi \cos \alpha \right) \right].$$

3. The *derivative* of an elliptic function is also an elliptic function with the same periods.

SM III 598

4. A non-constant elliptic function has a finite number of poles in a period parallelogram: it can have no more than two simple and one second-order pole in such a parallelogram. Suppose that these poles lie at the points $a_1, a_2, \dots, a_n$ and that their orders are $\alpha_1, \alpha_2, \dots, \alpha_n$. Suppose that the zeros of an analytic function that occur in a single parallelogram are $b_1, b_2, \dots, b_m$ and that the orders of the zeros are $\beta_1, \beta_2, \dots, \beta_m$, respectively. Then,

$$\gamma = \alpha_1 + \alpha_2 + \dots + \alpha_n = \beta_1 + \beta_2 + \dots + \beta_m. \quad \text{ZH 118}$$

The number γ representing this sum is called the *order* of the elliptic function.

5. The sum of the residues of an elliptic function with respect to all the poles belonging to a period parallelogram is equal to zero.
6. The difference between the sum of all the zeros and the sum of all the poles of an elliptic function that are located in a period parallelogram is equal to one of its periods.
7. Every two elliptic functions with the same periods are related by an algebraic relationship.

GO II 151

- 8.⁷ A non-constant single-valued function which is not constant cannot have more than two periods.
GO II 147
9. An elliptic function of order γ assumes *an arbitrary value* γ times in a period parallelogram.
SM 601, SI 301

8.14 Jacobian elliptic functions

8.141 Consider the upper limit φ of the integral

$$u = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

as a function of u . Using the notation

$$\varphi = \operatorname{am} u$$

we call this upper limit the *amplitude*. The quantity u is called the *argument*, and its dependence on φ is written

$$u = \arg \varphi.$$

8.142 The amplitude is an *infinitely many-valued* function of u and has a period of $4\mathbf{K}i$. The *branch points* of the amplitude correspond to the values of the argument

$$u = 2m\mathbf{K} + (2n + 1)\mathbf{K}'i, \quad \text{ZH 67--69}$$

where m and n are arbitrary integers (see also **8.151**).

8.143 The first two of the following functions

$$\operatorname{sn} u = \sin \varphi = \sin \operatorname{am} u, \quad \operatorname{cn} u = \cos \varphi = \cos \operatorname{am} u,$$

$$\operatorname{dn} u = \Delta \varphi = \sqrt{1 - k^2 \sin^2 \varphi} = \frac{d\varphi}{du}$$

are called, respectively, the *sine-amplitude* and the *cosine-amplitude* while the third may be called the *delta amplitude*. All these elliptic functions were exhibited by Jacobi and they bear his name. SI 16

The Jacobian elliptic functions are *doubly periodic* functions and have *two simple poles* in a period parallelogram. ZH 69

8.144

$$1. \quad u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad \text{SI 21(23)}$$

$$2. \quad u = \int_1^{\operatorname{cn} u} \frac{dt}{\sqrt{(1-t^2)(k'^2 + k^2t^2)}} \quad \text{SI 21(23)}$$

$$3. \quad u = \int_1^{\operatorname{dn} u} \frac{dt}{\sqrt{(1-t^2)(t^2 - k'^2)}} \quad \text{SI 21(23)}$$

8.145 Power series representations:

$$\begin{aligned} 1.^{11} \quad \operatorname{sn} u = u &- \frac{1+k^2}{3!}u^3 + \frac{1+14k^2+k^4}{5!}u^5 - \frac{1+135k^2+135k^4+k^6}{7!}u^7 \\ &+ \frac{1+1228k^2+5478k^4+1228k^6+k^8}{9!}u^9 - \dots \\ &[|u| < |\mathbf{K}'|] \quad \text{ZH 81(97)} \end{aligned}$$

$$2. \quad \operatorname{cn} u = 1 - \frac{1}{2!}u^2 + \frac{1+4k^2}{4!}u^4 - \frac{1+44k^2+16k^4}{6!}u^6 + \frac{1+408k^2+912k^4+64k^6}{8!}u^8 - \dots$$

[|u| < |K'|] ZH 81(98)

$$3. \quad \operatorname{dn} u = 1 - \frac{k^2}{2!}u^2 + \frac{k^2(4+k^2)}{4!}u^4 - \frac{k^2(16+44k^2+k^4)}{6!}u^6 + \frac{k^2(64+912k^2+408k^4+k^6)}{8!}u^8 - \dots$$

[|u| < |K'|] ZH 81(99)

$$4. \quad \operatorname{am} u = u - \frac{k^2}{3!}u^3 + \frac{k^2(4+k^2)}{5!}u^5 - \frac{k^2(16+44k^2+k^4)}{7!}u^7 + \frac{k^2(64+912k^2+408k^4+k^6)}{9!}u^9 - \dots$$

[|u| < |K'|] LA 380(4)

8.146 Representation as a trigonometric series or a product $\left(q = e^{-\frac{\pi K'}{K}} = e^{\pi i \tau}\right)^*$

$$1.^{11} \quad \operatorname{sn} u = \frac{2\pi}{kK} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K}$$

WH 511a, ZH 84(108)

$$2.^{11} \quad \operatorname{cn} u = \frac{2\pi}{kK} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K}$$

WH 511a, ZH 84(109)

$$3. \quad \operatorname{dn} u = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{K}$$

WH 511a, ZH 84(110)

$$4.^{11} \quad \operatorname{am} u = \frac{\pi u}{2K} + 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^{2n}} \sin \frac{n\pi u}{K}$$

WH 511a

$$5. \quad \frac{1}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\frac{1}{\sin \frac{\pi u}{2K}} + 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K} \right]$$

LA 369(3)

$$6. \quad \frac{1}{\operatorname{cn} u} = \frac{\pi}{2k'K} \left[\frac{1}{\cos \frac{\pi u}{2K}} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1+q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K} \right]$$

LA 369(3)

$$7. \quad \frac{1}{\operatorname{dn} u} = \frac{\pi}{2k'K} \left[1 + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{K} \right]$$

LA 369(3)

$$8. \quad \frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\pi}{2k'K} \left[\tan \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1+q^{2n}} \sin \frac{n\pi u}{K} \right]$$

LA 369(4)

$$9.^{11} \quad \frac{\operatorname{sn} u}{\operatorname{dn} u} = -\frac{2\pi}{kk'K} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K}$$

LA 369(4)

$$10. \quad \frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\cot \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin \frac{\pi n u}{K} \right]$$

LA 369(5)

*The expansions 1–22 are valid in every strip of the form $\left| \operatorname{Im} \frac{\pi u}{2K} \right| < \frac{1}{2} \pi \operatorname{Im} \tau$. The expansions 23–25 are valid in an arbitrary bounded portion of u .

$$11. \quad \frac{\operatorname{cn} u}{\operatorname{dn} u} = -\frac{2\pi}{kK} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K} \quad \text{LA 369(5)}$$

$$12. \quad \frac{\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\frac{1}{\sin \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K} \right] \quad \text{LA 369(6)}$$

$$13. \quad \frac{\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left[\frac{1}{\cos \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K} \right] \quad \text{LA 369(6)}$$

$$14. \quad \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\cot \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^n} \sin \frac{n\pi u}{K} \right] \quad \text{LA 369(7)}$$

$$15. \quad \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left\{ \tan \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+(-1)^n q^n} \sin \frac{n\pi u}{K} \right\} \quad \text{LA 369(7)}$$

$$16. \quad \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{4\pi^2}{k^2 K} \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K} \quad \text{LA 369(7)}$$

$$17. \quad \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} = \frac{\pi}{2(1-k^2)K} \left[\tan \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^n} \sin \frac{n\pi u}{K} \right] \quad \text{LA 369(8)}$$

$$18. \quad \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} = \frac{\pi}{2K} \left[\cot \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+(-1)^n q^n} \sin \frac{n\pi u}{K} \right] \quad \text{LA 369(8)}$$

$$19. \quad \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} = \frac{\pi}{K} \left[\frac{1}{\sin \frac{\pi u}{K}} + 4 \sum_{n=1}^{\infty} \frac{q^{2(2n-1)}}{1-q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K} \right] \quad \text{LA 369(8)}$$

$$20.^{11} \quad \ln \operatorname{sn} u = \ln \frac{2K}{\pi} + \ln \sin \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^n} \sin^2 \frac{n\pi u}{2K} \quad \text{LA 369(2)}$$

$$21. \quad \ln \operatorname{cn} u = \ln \cos \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+(-1)^n q^n} \sin^2 \frac{n\pi u}{2K} \quad \text{LA 369(2)}$$

$$22. \quad \ln \operatorname{dn} u = -8 \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{q^{2n-1}}{1-q^{2(2n-1)}} \sin^2(2n-1) \frac{\pi u}{2K} \quad \text{LA 369(2)}$$

$$23.^{11} \quad \operatorname{sn} u = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1-2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}} \quad \text{WH 508a, ZH 86(145)}$$

$$24. \quad \operatorname{cn} u = \frac{2\sqrt{k'}\sqrt[4]{q}}{\sqrt{k}} \cos \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1+2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}} \quad \text{WH 508a, ZH 86(146)}$$

$$25. \quad \operatorname{dn} u = \sqrt{k'} \prod_{n=1}^{\infty} \frac{1+2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}} \quad \text{WH 508a, ZH 86(147)}$$

$$26. \quad \operatorname{sn}^3 u = \sum_{n=0}^{\infty} \left[\frac{1+k^2}{2k^3} - \frac{(2n+1)^2}{2k^3} \frac{\pi^2}{4K^2} \right] \frac{2\pi q^{n+\frac{1}{2}} \sin(2n+1) \frac{\pi u}{2K}}{K(1-q^{2n+1})}$$

$$\left[\left| \operatorname{Im} \frac{u}{2K} \right| < \operatorname{Im} \tau \right]$$

$$27. \quad \frac{1}{\operatorname{sn}^2 u} = \frac{\pi^2}{4K^2} \operatorname{cosec}^2 \frac{\pi u}{2K} + \frac{K-E}{K} - \frac{2\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{nq^{2n} \cos \frac{n\pi u}{K}}{1-q^{2n}}$$

$$\left[\left| \operatorname{Im} \frac{u}{2K} \right| < \frac{1}{2} \operatorname{Im} \tau \right]$$

MO 148

8.147

$$1. \quad \operatorname{sn} u = \frac{\pi}{2kK} \sum_{n=-\infty}^{\infty} \frac{1}{\sin \frac{\pi}{2K} [u - (2n-1)iK']}$$

MO 149

$$2. \quad \operatorname{cn} u = \frac{\pi i}{2kK} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sin \frac{\pi}{2K} [u - (2n-1)iK']}$$

MO 150

$$3. \quad \operatorname{dn} u = \frac{\pi i}{2K} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\tan \frac{\pi}{2K} [u - (2n-1)iK']}$$

MO150

8.148 The Weierstrass expansions of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$:

$$\operatorname{sn} u = \frac{B}{A}, \quad \operatorname{cn} u = \frac{C}{A}, \quad \operatorname{dn} u = \frac{D}{A}, \quad \text{ZH 82-83(105,106,107)}$$

where

$$\begin{aligned} A &= 1 - \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1} \frac{u^{2n+2}}{(2n+2)!} & B &= \sum_{n=0}^{\infty} (-1)^n b_n \frac{u^{2n+1}}{(2n+1)!} \\ C &= \sum_{n=0}^{\infty} (-1)^n c_n \frac{u^{2n}}{(2n)!} & D &= \sum_{n=0}^{\infty} (-1)^n d_n \frac{u^{2n}}{(2n)!} \end{aligned}$$

and

$$\begin{aligned} a_2 &= 2k^2, \quad a_3 = 8(k^2 + k^4), \quad a_4 = 32(k^2 + k^6) + 68k^4, \quad a_5 = 128(k^2 + k^8) + 480(k^4 + k^6), \\ a_6 &= 512(k^2 + k^{10}) + 3008(k^4 + k^8) + 5400k^6, \quad \dots \\ b_0 &= 1, \quad b_1 = 1 + k^2, \quad b_2 = 1 + k^4 + 4k^2, \quad b_3 = 1 + k^6 + 9(k^2 + k^4), \\ b_4 &= 1 + k^8 + 16(k^2 + k^6) - 6k^4, \quad b_5 = 1 + k^{10} + 25(k^2 + k^8) - 494(k^4 + k^6), \\ b_6 &= 1 + k^{12} + 36(k^2 + k^{10}) - 5781(k^4 + k^8) - 12184k^6, \quad \dots \\ c_0 &= 1, \quad c_1 = 1, \quad c_2 = 1 + 2k^2, \quad c_3 = 1 + 6k^2 + 8k^4, \quad c_4 = 1 + 12k^2 + 60k^4 + 32k^6, \\ c_5 &= 1 + 20k^2 + 348k^4 + 448k^6 + 128k^8, \quad c_6 = 1 + 30k^2 + 2372k^4 + 4600k^6 + 2880k^8 + 512k^{10}, \quad \dots \\ d_0 &= 1, \quad d_1 = k^2, \quad d_2 = 2k^2 + k^4, \quad d_3 = 8k^2 + 6k^4 + k^6, \quad d_4 = 32k^2 + 60k^4 + 12k^6 + k^8, \\ d_5 &= 128k^2 + 448k^4 + 348k^6 + 20k^8 + k^{10}, \\ d_6 &= 512k^2 + 2880k^4 + 4600k^6 + 2372k^8 + 30k^{10} + k^{12}, \quad \dots \end{aligned}$$

8.15 Properties of Jacobian elliptic functions and functional relationships between them

8.151 The periods, zeros, poles, and residues of Jacobian elliptic functions:

1.

	Periods	Zeros	Poles	Residues
$\operatorname{sn} u$	$4m\mathbf{K} + 2n\mathbf{K}'i$	$2m\mathbf{K} + 2n\mathbf{K}'i$	$2m\mathbf{K} + (2n+1)\mathbf{K}'i$	$(-1)^m \frac{1}{k}$
$\operatorname{cn} u$	$4m\mathbf{K} + 2n(\mathbf{K} + \mathbf{K}'i)$	$(2m+1)\mathbf{K} + 2n\mathbf{K}'i$	$2m\mathbf{K} + (2n+1)\mathbf{K}'i$	$(-1)^{m-1} \frac{i}{k}$
$\operatorname{dn} u$	$2m\mathbf{K} + 4n\mathbf{K}'i$	$(2m+1)\mathbf{K} + (2n+1)\mathbf{K}'i$	$2m\mathbf{K} + (2n+1)\mathbf{K}'i$	$(-1)^{n-1}i$

SM 630, ZH 69–72

2.

$u^* = u + \mathbf{K}$	$u + i\mathbf{K}$	$u + \mathbf{K} + i\mathbf{K}'$	$u + 2\mathbf{K}$	$u + 2i\mathbf{K}'$	$u + 2\mathbf{K} + 2i\mathbf{K}'$
$\operatorname{sn} u^* = \frac{\operatorname{cn} u}{\operatorname{dn} u}$	$\frac{1}{k \operatorname{sn} u}$	$\frac{1}{k} \frac{\operatorname{dn} u}{\operatorname{cn} u}$	$-\operatorname{sn} u$	$\operatorname{sn} u$	$-\operatorname{sn} u$
$\operatorname{cn} u^* = -k' \frac{\operatorname{sn} u}{\operatorname{dn} u}$	$-\frac{i}{k} \frac{\operatorname{dn} u}{\operatorname{sn} u}$	$-\frac{ik'}{k \operatorname{cn} u}$	$-\operatorname{cn} u$	$-\operatorname{cn} u$	$\operatorname{cn} u$
$\operatorname{dn} u^* = k' \frac{1}{\operatorname{dn} u}$	$-i \frac{\operatorname{cn} u}{\operatorname{sn} u}$	$ik' \frac{\operatorname{sn} u}{\operatorname{cn} u}$	$\operatorname{dn} u$	$-\operatorname{dn} u$	$-\operatorname{dn} u$

SM 630

3.

$u^* = 0$	$-u$	$\frac{1}{2}\mathbf{K}$	$\frac{1}{2}(\mathbf{K} + i\mathbf{K}')$	$\frac{1}{2}i\mathbf{K}'$	$u + 2m\mathbf{K} + 2n\mathbf{K}'i$
$\operatorname{sn} u^* = 0$	$-\operatorname{sn} u$	$\frac{1}{\sqrt{1+k'}}$	$\frac{\sqrt{1+k} + i\sqrt{1-k}}{\sqrt{2k}}$	$\frac{i}{\sqrt{k}}$	$(-1)^m \operatorname{sn} u$
$\operatorname{cn} u^* = 1$	$\operatorname{cn} u$	$\frac{\sqrt{k'}}{\sqrt{1+k'}}$	$\frac{(1-i)\sqrt{k'}}{\sqrt{2k}}$	$\frac{\sqrt{1+k}}{\sqrt{k}}$	$(-1)^{m+n} \operatorname{cn} u$
$\operatorname{dn} u^* = 1$	$\operatorname{dn} u$	$\sqrt{k'}$	$\frac{\sqrt{k'}(\sqrt{1+k'} - i\sqrt{1-k'})}{\sqrt{2}}$	$\sqrt{1+k}$	$(-1)^n \operatorname{dn} u$

SI 19, SI 18(13), WH,

WH

WH

WH

8.152 Transformation formulas

u_1	l_1	$sn(u_1, k_1)$	$cn(u_1, k_1)$	$dn(u_1, k_1)$
ku	$\frac{1}{k}$	$k \operatorname{sn}(u, k)$	$\operatorname{dn}(u, k)$	$\operatorname{cn}(u, k)$
iu	k'	$i \frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$
$k'u$	$i \frac{k}{k'}$	$k' \frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$
iku	$i \frac{k'}{k}$	$ik \frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$
$ik'u$	$\frac{1}{k'}$	$ik' \frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$\frac{(1+k) \operatorname{sn}(u, k)}{1+k \operatorname{sn}^2(u, k)}$	$\frac{\operatorname{cn}(u, k) \operatorname{dn}(u, k)}{1+k \operatorname{sn}^2(u, k)}$	$\frac{1-k \operatorname{sn}^2(u, k)}{1+k \operatorname{sn}^2(u, k)}$
$(1+k')u$	$\frac{1-k'}{1+k'}$	$(1+k') \frac{\operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1-(1+k') \operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1-(1-k') \operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}$
$\frac{(1+\sqrt{k'})^2}{2}u$	$\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^2$	$\frac{k^2 \operatorname{sn}(u, k) d \operatorname{cn}(u, k)}{\sqrt{k_1} [1+\operatorname{dn}(u, k)] [k'+\operatorname{dn}(u, k)]}$	$\frac{\operatorname{dn}(u, k) - \sqrt{k'}}{1 - \sqrt{k'}} \times \sqrt{\frac{2(1+k')}{[1+\operatorname{dn}(u, k)] [k'+\operatorname{dn}(u, k)]}}$	$\frac{\sqrt{1+k_1} (\operatorname{dn}(u, k) + \sqrt{k'})}{\sqrt{[1+\operatorname{dn}(u, k)] [k'+\operatorname{dn}(u, k)]}}$

8.153

$$1. \quad \operatorname{sn}(iu, k) = i \frac{\operatorname{sn}(u, k')}{\operatorname{cn}(u, k')} \quad \text{SI 50(64)}$$

$$2. \quad \operatorname{cn}(iu, k) = \frac{1}{\operatorname{cn}(u, k')} \quad \text{SI 50(65)}$$

$$3. \quad \operatorname{dn}(iu, k) = \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')} \quad \text{SI 50(65)}$$

$$4. \quad \operatorname{sn}(u, k) = k^{-1} \operatorname{sn}(ku, k^{-1})$$

$$5. \quad \operatorname{cn}(u, k) = \operatorname{dn}(ku, k^{-1})$$

$$6. \quad \operatorname{dn}(u, k) = \operatorname{cn}(ku, k^{-1})$$

$$7.^{11} \quad \operatorname{sn}(u, ik) = \frac{1}{\sqrt{1+k^2}} \frac{\operatorname{sn}\left(u\sqrt{1+k^2}, k(1+k^2)^{-1/2}\right)}{\operatorname{dn}\left(u\sqrt{1+k^2}, k(1+k^2)^{-1/2}\right)}$$

$$8.^{11} \quad \operatorname{cn}(u, ik) = \frac{\operatorname{sn}\left(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2}\right)}{\operatorname{dn}\left(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2}\right)}$$

$$9.^{11} \quad \operatorname{dn}(u, ik) = \frac{1}{\operatorname{dn}\left(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2}\right)}$$

Functional relations

8.154

$$1. \quad \operatorname{sn}^2 u = \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{dn} 2u} \quad \text{MO 146}$$

$$2. \quad \operatorname{cn}^2 u = \frac{\operatorname{cn} 2u + \operatorname{dn} 2u}{1 + \operatorname{dn} 2u} \quad \text{MO 146}$$

$$3. \quad \operatorname{dn}^2 u = \frac{\operatorname{dn} 2u + k^2 \operatorname{cn} 2u + k'^2}{1 + \operatorname{dn} 2u} \quad \text{MO 146}$$

$$4. \quad \operatorname{sn}^2 u + \operatorname{cn}^2 u = 1 \quad \text{SI 16(9)}$$

$$5. \quad \operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1 \quad \text{SI 16(9)}$$

8.155

$$1. \quad \frac{1 - \operatorname{dn} 2u}{1 + \operatorname{dn} 2u} = k^2 \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{\operatorname{dn}^2 u} \quad \text{MO 146}$$

$$2. \quad \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{cn} 2u} = \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u}{\operatorname{cn}^2 u} \quad \text{MO 146}$$

8.156

$$1. \quad \operatorname{sn}(u \pm v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \quad \text{SI 46(56)}$$

$$2. \quad \operatorname{cn}(u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \quad \text{SI 46(57)}$$

$$3. \quad \operatorname{dn}(u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \quad \text{SI 46(58)}$$

8.157

$$1. \quad \operatorname{sn} \frac{u}{2} = \pm \frac{1}{k} \sqrt{\frac{1 - \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm \sqrt{\frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}} \quad \text{SI 47(61), SU 67(15)}$$

$$2. \quad \operatorname{cn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{dn} u}} = \pm \frac{k'}{k} \sqrt{\frac{1 - \operatorname{dn} u}{\operatorname{dn} u - \operatorname{cn} u}} \quad \text{SI 48(62), SI 67(16)}$$

$$3. \quad \operatorname{dn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm k' \sqrt{\frac{1 - \operatorname{cn} u}{\operatorname{dn} u + \operatorname{cn} u}} \quad \text{SI 48(63), SI 67(17)}$$

8.158

$$1. \quad \frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u \quad \text{SI 21(21)}$$

$$2. \quad \frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u \quad \text{SI 21(21)}$$

$$3.^8 \quad \frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{dn} u \operatorname{cn} u \quad \text{SI 21(21)}$$

8.159 Jacobian elliptic functions are solutions of the following differential equations:

$$1. \quad \frac{d}{du} \operatorname{sn} u = \sqrt{(1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u)} \quad \text{SI 21(22)}$$

$$2. \quad \frac{d}{du} \operatorname{cn} u = -\sqrt{(1 - \operatorname{cn}^2 u)(k'^2 + k^2 \operatorname{cn}^2 u)}, \quad \text{SI 21(22)}$$

$$3. \quad \frac{d}{du} \operatorname{dn} u = -\sqrt{(1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)} \quad \text{SI 21(22)}$$

For the indefinite integrals of Jacobi's elliptic functions, see **5.13**.

8.16 The Weierstrass function $\wp(u)$

8.160 The Weierstrass elliptic function $\wp(u)$ is defined by

$$1. \quad \wp(u) = \frac{1}{u^2} + \sum'_{m,n} \left\{ \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\}, \quad \text{SI 307(6)}$$

where the symbol $\sum'$ means that the summation is made over all combinations of integers m and n except for the combination $m = n = 0$; $2\omega_1$ and $2\omega_2$ are the periods of the function $\wp(u)$. Obviously,

$$2. \quad \wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u) \text{ and } \operatorname{Im} \left(\frac{\omega_1}{\omega_2} \right) \neq 0,$$

$$3. \quad \frac{d}{du} \wp(u) = -2 \sum_{m,n} \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^3},$$

where the summation is made over all integral values of m and n .

The series **8.160** 1 and **8.160** 3 converge everywhere except at the poles, that is, at the points $2m\omega_1 + 2n\omega_2$ (where m and n are integers).

4. The function $\wp(u)$ is a *doubly periodic function* and has *one second-order pole* in a period parallelogram. SI 306

8.161 The function $\wp(u)$ satisfies the differential equation

$$1. \quad \left[\frac{d\wp(u)}{du} \right]^2 = 4\wp^3(u) - g_2\wp(u) - g_3, \quad \text{SI 142, 310, WH}$$

where

$$2. \quad g_2 = 60 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-4}; \quad g_3 = 140 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-6} \quad \text{WH, SI 310}$$

The functions g_2 and g_3 are called the *invariants* of the function $\wp(u)$.

$$\mathbf{8.162} \quad u = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4(z - e_1)(z - e_2)(z - e_3)}},$$

where e_1 , e_2 , and e_3 are the roots of the equation $4z^3 - g_2z - g_3 = 0$; that is,

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4} \quad \text{SI 142, 143, 144}$$

8.163 $\wp(\omega_1) = e_1$, $\wp(\omega_1) + \omega_2 = e_2$, $\wp(\omega_2) = e_3$. Here, it is assumed that if e_1 , e_2 , and e_3 lie on a straight line in the complex plane, e_2 lies between e_1 and e_3 .

8.164 The number $\Delta = g_2^3 - 27g_3^2$ is called the *discriminant* of the function $\wp(u)$. If $\Delta > 0$, all roots e_1 , e_2 , and e_3 of the equation $4z^3 - g_2z - g_3 = 0$ (where g_2 and g_3 are real numbers) are *real*. In this case, the roots e_1 , e_2 , and e_3 are numbered in such a way that $e_1 > e_2 > e_3$.

1. If $\Delta > 0$, then

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}, \quad \omega_2 = i \int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2z - 4z^3}},$$

where ω_1 is real and ω_2 is a purely imaginary number. Here, the values of the radical in the integrand are chosen in such a way that ω_1 and $\frac{\omega_2}{i}$ will be positive.

2. If $\Delta < 0$, the root e_2 of the equation $4z^3 - g_2z - g_3 = 0$ is *real*, and the remaining two roots (e_1 and e_3) are *complex conjugates*. Suppose that $e_1 = \alpha + i\beta$, and $e_3 = \alpha - i\beta$. In this case, it is convenient to take

$$\omega' = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} \quad \text{and} \quad \omega'' = \int_{e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}$$

as basic semiperiods.

In the first integral, the integration is taken over a path lying entirely in the upper half-plane and in the second over a path lying entirely in the lower half-plane. SI 151(21, 22)

8.165 Series representation:

$$1. \quad \wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{4 \cdot 5} + \frac{g_3 u^4}{4 \cdot 7} + \frac{g_2^2 u^6}{2^4 \cdot 3 \cdot 5^2} + \frac{3g_2 g_3 u^8}{2^4 \cdot 5 \cdot 7 \cdot 11} + \dots \quad \text{WH}$$

8.166 Functional relations

$$1. \quad \wp(u) = \wp(-u), \quad \wp'(u) = -\wp'(-u)$$

$$2. \quad \wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4} \left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2 \quad \text{SI 163(32)}$$

$$\mathbf{8.167} \quad \wp(u; g_2, g_3) = \mu^2 \wp\left(\mu u; \frac{g_2}{\mu^4}, \frac{g_3}{\mu^6}\right) \quad (\text{the formula for homogeneity})$$

SI 149(13)

The special case: $\mu = i$.

$$1. \quad \wp(u; g_2, g_3) = -\wp(iu; g_2, -g_3)$$

8.168 An arbitrary elliptic function can be expressed in terms of the elliptic function $\wp(u)$ having the same periods as the original function and its derivative $\wp'(u)$. This expression is rational with respect to $\wp(u)$ and linear with respect to $\wp'(u)$.

8.169 A connection with the Jacobian elliptic functions. For $\Delta > 0$ (see **8.164** 1).

$$1. \quad \wp\left(\frac{u}{\sqrt{e_1 - e_2}}\right) = e_1 + (e_1 - e_3) \frac{\text{cn}^2(u; k)}{\text{sn}^2(u; k)}$$

$$= e_2 + (e_1 - e_3) \frac{\text{dn}^2(u; k)}{\text{sn}^2(u; k)}$$

$$= e_3 + (e_1 - e_3) \frac{1}{\text{sn}^2(u; k)}$$

SI 145(5), ZH 120(197–199)a

$$2. \quad \omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_2 = \frac{iK'}{\sqrt{e_1 - e_3}}, \quad \text{SI 154(29)}$$

where

$$3. \quad k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad k' = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}} \quad \text{SI 145(7)}$$

For $\Delta < 0$ (see **8.164** 2)

$$4. \quad \wp\left(\frac{u}{\sqrt[4]{9\alpha^2 + \beta^2}}\right) = e_2 + \sqrt{9\alpha^2 + \beta^2} \frac{1 + \text{cn}(2u; k)}{1 - \text{cn}(2u; k)}; \quad \text{SI 147(12)}$$

$$5. \quad \omega' = \frac{K - iK'}{2\sqrt{9\alpha^2 + \beta^2}}, \quad \omega'' = \frac{K + iK'}{\sqrt[4]{9\alpha^2 + \beta^2}}, \quad \text{SI 153(28)}$$

where

$$6.^{11} \quad k = \sqrt{\frac{1}{2} - \frac{3e_2}{4\sqrt{9\alpha^2 + \beta^2}}}; \quad k' = \sqrt{\frac{1}{2} + \frac{3e_2}{4\sqrt{9\alpha^2 + \beta^2}}} \quad \text{SI 147}$$

For $\Delta = 0$, all the roots e_1, e_2 , and e_3 are real, and if $g_2 g_3 \neq 0$, two of them are equal to each other. If $e_1 = e_2 \neq e_3$, then

$$7. \quad \wp(u) = \frac{3g_3}{g_2} - \frac{9g_3}{2g_2} \coth^2 \left(u \sqrt{-\frac{9g_3}{2g_2}} \right) \quad \text{SI 148}$$

If $e_1 \neq e_2 = e_3$, then

$$8. \quad \wp(u) = -\frac{3g_3}{2g_2} + \frac{9g_3}{2g_2} \frac{1}{\sin^2 \left(u \sqrt{\frac{9g_3}{2g_2}} \right)} \quad \text{SI 149}$$

If $g_2 = g_3 = 0$, then $e_1 = e_2 = e_3 = 0$, and

$$9. \quad \wp(u) = \frac{1}{u^2} \quad \text{SI 149}$$

8.17 The functions $\zeta(u)$ and $\sigma(u)$

8.171 Definitions:

$$1. \quad \zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(z) - \frac{1}{z^2} \right) dz \quad \text{SI 181(45)}$$

$$2. \quad \sigma(u) = u \exp \left\{ \int_0^u \left(\wp(z) - \frac{1}{z^2} \right) dz \right\} \quad \text{SI 181(46)}$$

8.172 Series and infinite-product representation

$$1. \quad \zeta(u) = \frac{1}{u} + \sum'_{m,n} \left(\frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{(2m\omega_1 - 2n\omega_2)^2} \right) \quad \text{SI 307(8)}$$

$$2. \quad \sigma(u) = u \prod'_{m,n} \left(1 - \frac{u}{2m\omega_1 + 2n\omega_2} \right) \exp \left\{ \frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2(2m\omega_1 + 2n\omega_2)^2} \right\} \quad \text{SI 308(9)}$$

8.173

$$1. \quad \zeta(u) = u - \frac{g_2 u^3}{2^2 \cdot 3 \cdot 5} - \frac{g_3 u^5}{2^2 \cdot 5 \cdot 7} - \frac{g_2^2 u^7}{2^4 \cdot 3 \cdot 5^2 \cdot 7} - \frac{3g_2 g_3 u^9}{2^4 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \dots \quad \text{SI 181(49)}$$

$$2. \quad \sigma(u) = u - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^3 \cdot 3 \cdot 5 \cdot 7} - \frac{g_2^2 u^9}{2^9 \cdot 3^2 \cdot 5 \cdot 7} - \frac{3g_2 g_3 u^{11}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11} - \dots \quad \text{SI 181(49)}$$

$$8.174 \quad \zeta(u) = \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \cot \frac{\pi u}{2\omega_1} + \frac{\pi}{2\omega_1} \sum_{n=1}^{\infty} \left\{ \cot \left(\frac{\pi u}{2\omega_1} + n\pi \frac{\omega_2}{\omega_1} \right) + \cot \left(\frac{\pi u}{2\omega_1} - n\pi \frac{\omega_2}{\omega_1} \right) \right\} \quad \text{MO 154}$$

$$= \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \cot \frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin \frac{\pi n u}{\omega_1} \quad \text{MO 155}$$

Functional relations and properties

$$8.175 \quad \zeta(u) = -\zeta(-u), \quad \sigma(u) = -\sigma(-u) \quad \text{SI 181}$$

8.176

$$1. \quad \zeta(u + 2\omega_1) = \zeta(u) + 2\zeta(\omega_1) \quad \text{SI 184(57)}$$

2. $\zeta(u + 2\omega_2) = \zeta(u) + 2\zeta(\omega_2)$ SI 184(57)
3. $\sigma(u + 2\omega_1) = -\sigma(u) \exp\{2(u + \omega_1)\zeta(\omega_1)\}$. SI 185(60)
4. $\sigma(u + 2\omega_2) = -\sigma(u) \exp\{2(u + \omega_2)\zeta(\omega_2)\}$. SI 185(60)
5. $\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \frac{\pi}{2}i$ SI 186(62)

8.177

1. $\zeta(u + v) - \zeta(u) - \zeta(v) = \frac{1}{2} \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}$ SI 182(53)
2. $\wp(u) - \wp(v) = -\frac{\sigma(u - v)\sigma(u + v)}{\sigma^2(u)\sigma^2(v)}$ SI 183(54)
3. $\zeta(u - v) + \zeta(u + v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)}$ SI 182(51)

8.178

1. $\zeta(u; \omega_1, \omega_2) = t\zeta(tu; t\omega_1, t\omega_2)$ MO 154
- 2.⁸ $\sigma(u; \omega_1, \omega_2) = t^{-1}\sigma(tu; t\omega_1, t\omega_2)$ MO 156

For the indefinite integrals of Weierstrass elliptic functions, see **5.14**.

8.18–8.19 Theta functions

8.180 *Theta functions* are defined as the sums (for $|q| < 1$) of the following series:

1. $\vartheta_4(u) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nui} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nu$ WH
2. $\vartheta_1(u) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2 \sum_{n=1}^{\infty} (-1)^{n+1} q^{(n-\frac{1}{2})^2} \sin(2n-1)u$ WH
- 3.¹¹ $\vartheta_2(u) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2 \sum_{n=1}^{\infty} q^{(n-\frac{1}{2})^2} \cos(2n-1)u$ WH
4. $\vartheta_3(u) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nui} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nu$ WH

The notations $\vartheta(u, q)$ and $\vartheta(u \mid \tau)$, where τ and q are related by $q = e^{i\pi\tau}$, are also used. Here, q is called the *nome* of the theta function and τ its *parameter*.

8.181 Representation of theta functions in terms of infinite products

1. $\vartheta_4(u) = \prod_{n=1}^{\infty} \left(1 - 2q^{2n-1} \cos 2u + q^{2(2n-1)}\right) (1 - q^{2n})$ SI 200(9), ZH 90(9)
2. $\vartheta_3(u) = \prod_{n=1}^{\infty} \left(1 + 2q^{2n-1} \cos 2u + q^{2(2n-1)}\right) (1 - q^{2n})$ SI 200(9), ZH 90(9)

$$\begin{aligned}
3. \quad \vartheta_1(u) &= 2\sqrt[4]{q} \sin u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2u + q^{4n}) (1 - q^{2n}) & \text{SI 200(9), ZH 90(9)} \\
4.^8 \quad \vartheta_2(u) &= 2\sqrt[4]{q} \cos u \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2u + q^{4n}) (1 - q^{2n}) & \text{SI 200(0), ZH 90(9)}
\end{aligned}$$

Functional relations and properties

8.182 Quasiperiodicity. Suppose that $q = e^{\pi\tau i}$ ($\text{Im } \tau > 0$). Then, theta functions that are periodic functions of u are called *quasiperiodic functions* of τ and u . This property follows from the equations

$$\begin{aligned}
1. \quad \vartheta_4(u + \pi) &= \vartheta_4(u) & \text{SI 200(10)} \\
2. \quad \vartheta_4(u + \tau\pi) &= -\frac{1}{q} e^{-2iu} \vartheta_4(u) & \text{SI 200(10)} \\
3. \quad \vartheta_1(u + \pi) &= -\vartheta_1(u) & \text{SI 200(10)} \\
4. \quad \vartheta_1(u + \tau\pi) &= -\frac{1}{q} e^{-2iu} \vartheta_1(u) & \text{SI 200(10)} \\
5. \quad \vartheta_2(u + \pi) &= -\vartheta_2(u) & \text{SI 200(10)} \\
6. \quad \vartheta_2(u + \tau\pi) &= \frac{1}{q} e^{-2iu} \vartheta_2(u) & \text{SI 200(10)} \\
7. \quad \vartheta_3(u + \pi) &= \vartheta_3(u) & \text{SI 200(10)} \\
8. \quad \vartheta_3(u + \tau\pi) &= \frac{1}{q} e^{-2iu} \vartheta_3(u) & \text{SI 200(10)}
\end{aligned}$$

8.183

$$\begin{aligned}
1. \quad \vartheta_4\left(u + \frac{1}{2}\pi\right) &= \vartheta_3(u) & \text{WH} \\
2. \quad \vartheta_1\left(u + \frac{1}{2}\pi\right) &= \vartheta_2(u) & \text{WH} \\
3. \quad \vartheta_2\left(u + \frac{1}{2}\pi\right) &= -\vartheta_1(u) & \text{WH} \\
4. \quad \vartheta_3\left(u + \frac{1}{2}\pi\right) &= \vartheta_4(u) & \text{WH} \\
5. \quad \vartheta_4\left(u + \frac{1}{2}\pi\tau\right) &= iq^{-1/4} e^{-iu} \vartheta_1(u) & \text{WH} \\
6. \quad \vartheta_1\left(u + \frac{1}{2}\pi\tau\right) &= iq^{-1/4} e^{-iu} \vartheta_4(u) & \text{WH} \\
7. \quad \vartheta_2\left(u + \frac{1}{2}\pi\tau\right) &= q^{-1/4} e^{-iu} \vartheta_3(u) & \text{WH} \\
8. \quad \vartheta_3\left(u + \frac{1}{2}\pi\tau\right) &= q^{-1/4} e^{-iu} \vartheta_2(u) & \text{WH}
\end{aligned}$$

8.184 Even and odd theta functions

$$\begin{aligned}
1. \quad \vartheta_1(-u) &= -\vartheta_1(u) & \text{WH} \\
2. \quad \vartheta_2(-u) &= \vartheta_2(u) & \text{WH} \\
3. \quad \vartheta_3(-u) &= \vartheta_3(u) & \text{WH} \\
4. \quad \vartheta_4(-u) &= \vartheta_4(u) & \text{WH}
\end{aligned}$$

$$\mathbf{8.185} \quad \vartheta_4^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u) \quad \text{WH}$$

8.186⁷ Considering the theta functions as functions of two independent variables u and τ , we have

$$\pi i \frac{\partial^2 \vartheta_k(u | \tau)}{\partial u^2} + 4 \frac{\partial \vartheta_k(u | \tau)}{\partial \tau} = 0 \quad [k = 1, 2, 3, 4] \quad \text{WH}$$

8.187 We denote the partial derivatives of the theta functions with respect to u by a prime and consider them as functions of the single argument u . Then,

$$1. \quad \vartheta_1'(0) = \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) \quad \text{WH}$$

$$2. \quad \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)} \quad \text{WH}$$

$$\mathbf{8.188} \quad \vartheta_1(u) \vartheta_2(u) \vartheta_3(u) \vartheta_4(0) = \frac{1}{2} \vartheta_1(2u) \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) \quad \text{WH}$$

8.189 The zeros of the theta functions:

$$1.^8 \quad \vartheta_4(u) = 0 \text{ for } u = 2m \frac{\pi}{2} + (2n-1) \frac{\pi\tau}{2} \quad \text{SI 201}$$

$$2.^{10} \quad \vartheta_1(u) = 0 \text{ for } u = 2m \frac{\pi}{2} + 2n \frac{\pi\tau}{2} \quad \text{SI 201}$$

$$3. \quad \vartheta_2(u) = 0 \text{ for } u = (2m-1) \frac{\pi}{2} + 2n \frac{\pi\tau}{2} \quad \text{SI 201}$$

$$4. \quad \vartheta_3(u) = 0 \text{ for } u = (2m-1) \frac{\pi}{2} + (2n-1) \frac{\pi\tau}{2} \quad [m \text{ and } n \text{ are integers or zero}] \quad \text{SI 201}$$

For integrals of theta functions, see **6.16**.

8.191 Connections with the Jacobian elliptic functions:

For $\tau = i \frac{K'}{K}$, i.e. for $q = \exp\left(-\pi \frac{K'}{K}\right)$,

$$1. \quad \operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{\vartheta_1\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)} \quad \text{SI 206(22), SI 209(35)}$$

$$2. \quad \operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{\vartheta_2\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)} \quad \text{SI 207(23), SI 209(35)}$$

$$3. \quad \operatorname{dn} u = \sqrt{k'} \frac{\vartheta_3\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)} \quad \text{SI 207(24), SI 209(35)}$$

8.192 Series representation of the functions H , H_1 , Θ , Θ_1 .

In these formulas, $q = \exp\left(-\pi \frac{K'}{K}\right)$.

$$1. \quad \Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos \frac{n\pi u}{K} \quad \text{SI 207(25), SI 212(42)}$$

$$2. \quad H(u) = \vartheta_1\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt[4]{q^{(2n+1)^2}} \sin(2n-1) \frac{\pi u}{2K} \quad \text{SI 207(25), SI 212(43)}$$

$$3. \quad \Theta_1(u) = \vartheta_3\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos \frac{n\pi u}{K} \quad \text{SI 207(25), SI 212(45)}$$

$$4. \quad H_1(u) = \vartheta_2\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} \sqrt[4]{q^{(2n-1)^2}} \cos(2n-1) \frac{\pi u}{2K} \quad \text{SI 207(25), SI 212(44)}$$

8.193 Connections with the Weierstrass elliptic functions

$$1. \quad \wp(u) = e_1 + \left[\frac{H_1(u\sqrt{\lambda}) H'(0)}{H_1(0) H(u\sqrt{\lambda})} \right]^2 \lambda = e_2 + \left[\frac{\Theta_1(u\sqrt{\lambda}) H'(0)}{\Theta_1(0) H'(u\sqrt{\lambda})} \right]^2 \lambda = e_3 + \left[\frac{\Theta(u\sqrt{\lambda}) H'(0)}{\Theta(0) H'(u\sqrt{\lambda})} \right]^2 \lambda$$

SI 235(77,78)

$$2. \quad \zeta(u) = \frac{\eta_1 u}{\omega_1} + \sqrt{\lambda} \frac{H'(u\sqrt{\lambda})}{H(u\sqrt{\lambda})} \quad \text{SI 234(73)}$$

$$3. \quad \sigma(u) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{H(u\sqrt{\lambda})}{H'(0)} \quad \text{SI 234(72)}$$

where

$$\lambda = e_1 - e_3; \quad \eta_1 = \zeta(\omega_1) = -\frac{\omega_1 \lambda}{3} \frac{H'''(0)}{H'(0)} \quad \text{SI 236}$$

8.194 The connection with elliptic integrals:

$$1. \quad E(u, k) = u - u \frac{\Theta''(0)}{\Theta(0)} + \frac{\Theta'(u)}{\Theta(u)} \quad \text{SI 228(65)}$$

$$2.^{11} \quad \Pi(u, -k^2 \sin^2 a, k) = \int_0^u \frac{d\varphi}{1 - k^2 \sin^2 a \sin^2 \varphi} = u + \frac{\sin a}{\operatorname{cn} a \operatorname{dn} a} \left[\frac{\Theta'(a)}{\Theta(a)} u + \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)} \right]$$

SI 228(65)

q -series and products, $q = \exp\left(-\pi \frac{K'}{K}\right)$

$$8.195 \quad \frac{\pi}{2} \left[1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right]^2 = K = \frac{\pi}{2} \Theta^2(K) \quad (\text{cf. 8.197 1}) \quad \text{SI 219}$$

$$8.196 \quad E = K - K \frac{\Theta''(0)}{\Theta(0)} = K - \frac{2\pi^2}{K} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n^2 q^{n^2}}{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}} \quad \text{SI 230(67)}$$

8.197

$$1. \quad 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \sqrt{\frac{2K}{\pi}} = \vartheta_3(0) \quad (\text{cf. 8.195}) \quad \text{WH}$$

$$2. \quad \sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2} = \sqrt{\frac{kK}{2\pi}} = \frac{1}{2} \vartheta_2(0) \quad \text{WH}$$

3. $4\sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^4 = k$ SI 206(17, 18)
4. $\prod_{n=1}^{\infty} \left(\frac{1-q^{2n-1}}{1+q^{2n-1}} \right)^4 = k'$ SI 206(19, 20)
5. $2\sqrt[4]{q} \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1-q^{2n-1}} \right)^2 = 2\sqrt{k} \frac{K}{\pi}$ WH
6. $\prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1+q^{2n}} \right)^2 = 2\sqrt{k'} \frac{K}{\pi}$ WH

8.198

1. $\lambda = \frac{1}{2} \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \frac{\sum_{n=0}^{\infty} q^{(2n+1)^2}}{1 + 2 \sum_{n=1}^{\infty} q^{4n^2}}$ [for $0 < k < 1$, we have $0 < \lambda < \frac{1}{2}$] WH

The series

2. $q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \dots$ WH
- is used to determine q from the given modulus k .

8.199¹⁰ Identities involving products of theta functions

1. $\vartheta_1(x, q) \vartheta_1(y, q) = \vartheta_3(x+y, q^2) \vartheta_2(x-y, q^2) - \vartheta_2(x+y, q^2) \vartheta_3(x-y, q^2)$ LW 7(1.4.7)
2. $\vartheta_1(x, q) \vartheta_2(y, q) = \vartheta_1(x+y, q^2) \vartheta_4(x-y, q^2) + \vartheta_4(x+y, q^2) \vartheta_1(x-y, q^2)$ LW 8(1.4.8)
3. $\vartheta_2(x, q) \vartheta_2(y, q) = \vartheta_2(x+y, q^2) \vartheta_3(x-y, q^2) + \vartheta_3(x+y, q^2) \vartheta_2(x-y, q^2)$ LW 8(1.4.9)
4. $\vartheta_3(x, q) \vartheta_3(y, q) = \vartheta_3(x+y, q^2) \vartheta_3(x-y, q^2) + \vartheta_2(x+y, q^2) \vartheta_2(x-y, q^2)$ LW 8(1.4.10)
5. $\vartheta_3(x, q) \vartheta_4(y, q) = \vartheta_4(x+y, q^2) \vartheta_4(x-y, q^2) - \vartheta_1(x+y, q^2) \vartheta_1(x-y, q^2)$ LW 8(1.4.11)
6. $\vartheta_4(x, q) \vartheta_4(y, q) = \vartheta_3(x+y, q^2) \vartheta_3(x-y, q^2) - \vartheta_2(x+y, q^2) \vartheta_2(x-y, q^2)$ LW 8(1.4.12)
7. $\vartheta_1(x+y) \vartheta_1(x-y) \vartheta_4^2(0) = \vartheta_3^2(x) \vartheta_2^2(y) - \vartheta_2^2(x) \vartheta_3^2(y) = \vartheta_1^2(x) \vartheta_4^2(y) - \vartheta_4^2(x) \vartheta_1^2(y)$ LW 8(1.4.16)
8. $\vartheta_2(x+y) \vartheta_2(x-y) \vartheta_4^2(0) = \vartheta_4^2(x) \vartheta_2^2(y) - \vartheta_1^2(x) \vartheta_3^2(y) = \vartheta_2^2(x) \vartheta_4^2(y) - \vartheta_3^2(x) \vartheta_1^2(y)$ LW 8(1.4.17)
9. $\vartheta_3(x+y) \vartheta_3(x-y) \vartheta_4^2(0) = \vartheta_4^2(x) \vartheta_3^2(y) - \vartheta_1^2(x) \vartheta_2^2(y) = \vartheta_3^2(x) \vartheta_4^2(y) - \vartheta_2^2(x) \vartheta_1^2(y)$ LW 8(1.4.18)
10. $\vartheta_4(x+y) \vartheta_4(x-y) \vartheta_4^2(0) = \vartheta_4^2(x) \vartheta_4^2(y) - \vartheta_1^2(x) \vartheta_1^2(y)$ LW 8(1.4.15)
11. $\vartheta_4(x+y) \vartheta_4(x-y) \vartheta_4^2(0) = \vartheta_3^2(x) \vartheta_3^2(y) - \vartheta_2^2(x) \vartheta_2^2(y) = \vartheta_4^2(x) \vartheta_4^2(y) - \vartheta_1^2(x) \vartheta_1^2(y)$ LW 9(1.4.19)
12. $\vartheta_1(x+y) \vartheta_1(x-y) \vartheta_3^2(0) = \vartheta_1^2(x) \vartheta_3^2(y) - \vartheta_3^2(x) \vartheta_1^2(y) = \vartheta_4^2(x) \vartheta_2^2(y) - \vartheta_2^2(x) \vartheta_4^2(y)$ LW 9(1.4.23)
13. $\vartheta_2(x+y) \vartheta_2(x-y) \vartheta_3^2(0) = \vartheta_2^2(x) \vartheta_3^2(y) - \vartheta_4^2(x) \vartheta_1^2(y) = \vartheta_3^2(x) \vartheta_2^2(y) - \vartheta_1^2(x) \vartheta_4^2(y)$ LW 9(1.4.24)
14. $\vartheta_3(x+y) \vartheta_3(x-y) \vartheta_3^2(0) = \vartheta_1^2(x) \vartheta_1^2(y) + \vartheta_3^2(x) \vartheta_3^2(y) = \vartheta_2^2(x) \vartheta_2^2(y) + \vartheta_4^2(x) \vartheta_4^2(y)$ LW 9(1.4.25)
15. $\vartheta_4(x+y) \vartheta_4(x-y) \vartheta_3^2(0) = \vartheta_1^2(x) \vartheta_2^2(y) + \vartheta_3^2(x) \vartheta_4^2(y) = \vartheta_2^2(x) \vartheta_1^2(y) + \vartheta_4^2(x) \vartheta_3^2(y)$ LW 9(1.4.26)

16. $\vartheta_1(x+y) \vartheta_1(x-y) \vartheta_2^2(0) = \vartheta_1^2(x) \vartheta_2^2(y) - \vartheta_2^2(x) \vartheta_1^2(y) = \vartheta_4^2(x) \vartheta_3^2(y) - \vartheta_3^2(x) \vartheta_4^2(y)$ LW 9(1.4.30)
17. $\vartheta_2(x+y) \vartheta_2(x-y) \vartheta_2^2(0) = \vartheta_2^2(x) \vartheta_2^2(y) - \vartheta_1^2(x) \vartheta_1^2(y) = \vartheta_3^2(x) \vartheta_3^2(y) - \vartheta_4^2(x) \vartheta_4^2(y)$ LW 10(1.4.31)
18. $\vartheta_3(x+y) \vartheta_3(x-y) \vartheta_2^2(0) = \vartheta_3^2(x) \vartheta_2^2(y) + \vartheta_4^2(x) \vartheta_1^2(y) = \vartheta_2^2(x) \vartheta_3^2(y) + \vartheta_1^2(x) \vartheta_4^2(y)$ LW 10(1.4.32)
19. $\vartheta_4(x+y) \vartheta_4(x-y) \vartheta_2^2(0) = \vartheta_4^2(x) \vartheta_2^2(y) + \vartheta_3^2(x) \vartheta_1^2(y) = \vartheta_1^2(x) \vartheta_3^2(y) + \vartheta_2^2(x) \vartheta_4^2(y)$ LW 10(1.4.33)
20. $\vartheta_3^2(x) \vartheta_3^2(0) = \vartheta_4^2(x) \vartheta_4^2(0) + \vartheta_2^2(x) \vartheta_2^2(0)$ LW 11(1.4.49)
21. $\vartheta_4^2(x) \vartheta_3^2(0) = \vartheta_1^2(x) \vartheta_2^2(0) + \vartheta_3^2(x) \vartheta_4^2(0)$ LW 11(1.4.50)
22. $\vartheta_4^2(x) \vartheta_2^2(0) = \vartheta_1^2(x) \vartheta_3^2(0) + \vartheta_2^2(x) \vartheta_4^2(0)$ LW 11(1.4.51)
23. $\vartheta_3^2(x) \vartheta_2^2(0) = \vartheta_1^2(x) \vartheta_4^2(0) + \vartheta_2^2(x) \vartheta_3^2(0)$ LW 11(1.4.52)
- 24.⁸ $\vartheta_3^4(x) = \vartheta_2^4(0) + \vartheta_4^4(0)$ LW 11(1.4.53)

8.199(2)¹⁰ Derivatives of ratios of theta functions

1. $\frac{d}{dx} (\vartheta_1 / \vartheta_4) = \vartheta_4^2(0) \vartheta_2(x) \vartheta_3(x) / \vartheta_4^2(x)$ LW 19(1.9.3)
2. $\frac{d}{dx} (\vartheta_2 / \vartheta_4) = -\vartheta_3^2(0) \vartheta_1(x) \vartheta_3(x) / \vartheta_4^2(x)$ LW 19(1.9.6)
3. $\frac{d}{dx} (\vartheta_3 / \vartheta_4) = -\vartheta_2^2(0) \vartheta_1(x) \vartheta_2(x) / \vartheta_4^2(x)$ LW 19(1.9.7)
4. $\frac{d}{dx} (\vartheta_1 / \vartheta_3) = \vartheta_3^2(0) \vartheta_2(x) \vartheta_4(x) / \vartheta_3^2(x)$ LW 19(1.9.8)
5. $\frac{d}{dx} (\vartheta_2 / \vartheta_3) = -\vartheta_4^2(0) \vartheta_1(x) \vartheta_4(x) / \vartheta_3^2(x)$ LW 19(1.9.9)
6. $\frac{d}{dx} (\vartheta_1 / \vartheta_2) = \vartheta_2^2(0) \vartheta_3(x) \vartheta_4(x) / \vartheta_2^2(x)$ LW 19(1.9.10)
7. $\frac{d}{dx} (\vartheta_4 / \vartheta_1) = -\vartheta_4^2(0) \vartheta_2(x) \vartheta_3(x) / \vartheta_1^2(x)$ LW 19(1.9.11)
8. $\frac{d}{dx} (\vartheta_4 / \vartheta_2) = \vartheta_3^2(0) \vartheta_1(x) \vartheta_3(x) / \vartheta_2^2(x)$ LW 20(1.9.12)
9. $\frac{d}{dx} (\vartheta_4 / \vartheta_3) = \vartheta_2^2(0) \vartheta_1(x) \vartheta_2(x) / \vartheta_3^2(x)$ LW 20(1.9.13)
10. $\frac{d}{dx} (\vartheta_3 / \vartheta_1) = -\vartheta_3^2(0) \vartheta_2(x) \vartheta_4(x) / \vartheta_1^2(x)$ LW 20(1.9.14)
11. $\frac{d}{dx} (\vartheta_3 / \vartheta_2) = \vartheta_4^2(0) \vartheta_1(x) \vartheta_4(x) / \vartheta_2^2(x)$ LW 20(1.9.15)
12. $\frac{d}{dx} (\vartheta_2 / \vartheta_1) = -\vartheta_2^2(0) \vartheta_3(x) \vartheta_4(x) / \vartheta_1^2(x)$ LW 20(1.9.16)

8.199(3)¹⁰ Derivatives of theta functions

1. $\frac{d}{du} \ln \vartheta_1(u) = \cot u + 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - 2q^{2n} \cos 2u + q^{4n}}$

2. $\frac{d}{du} \ln \vartheta_2(u) = -\tan u - 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 + 2q^{2n} \cos 2u + q^{4n}}$
3. $\frac{d}{du} \ln \vartheta_3(u) = -4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 + 2q^{2n} \cos 2u + q^{4n-2}}$
4. $\frac{d}{du} \ln \vartheta_4(u) = 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 - 2q^{2n} \cos 2u + q^{4n-2}}$
5. $\frac{d^2}{du^2} \ln \vartheta_2(u) = - \sum_{n=-\infty}^{\infty} \text{sech}^2 \{i(u + n\pi\tau)\}$

8.2 The Exponential Integral Function and Functions Generated by It

8.21 The exponential integral function $\text{Ei}(x)$

8.211

1. $\text{Ei}(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt = \text{li}(e^x) \quad [x < 0]$
- 2.¹¹ $\text{Ei}(x) = - \lim_{\varepsilon \rightarrow 0+} \left[\int_{-x}^{-\varepsilon} \frac{e^{-t}}{t} dt + \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} dt \right] = \text{PV} \int_{-\infty}^x \frac{e^t}{t} dt$
 $[x > 0]$
- 3.⁷ $\text{Ei}(x) = \frac{1}{2} \{ \text{Ei}(x + i0) + \text{Ei}(x - i0) \} \quad [x > 0]$ ET I 386

8.212

- 1.⁸ $\text{Ei}(-x) = C + \ln x + \int_0^x \frac{e^{-t} - 1}{t} dt \quad [x > 0]$ NT 11(1)
 $= C + e^{-x} \ln x + \int_0^x e^{-t} \ln t dt \quad [x > 0]$ NT 11(10)
- 2.⁷ $\text{Ei}(x) = e^x \left[\frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x-t)^2} \right] \quad [x > 0] \quad (\text{cf. } \mathbf{8.211} \text{ } 1)$
3. $\text{Ei}(-x) = e^{-x} \left[-\frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x+t)^2} \right] \quad [x > 0] \quad (\text{cf. } \mathbf{8.211} \text{ } 1)$ LA 281(28)
4. $\text{Ei}(\pm x) = \pm e^{\pm x} \int_0^1 \frac{dt}{x \pm \ln t} \quad [x > 0] \quad (\text{cf. } \mathbf{8.211} \text{ } 1)$
5. $\text{Ei}(\pm xy) = \pm e^{\pm xy} \int_0^{\infty} \frac{e^{-xt}}{y \mp t} dt \quad [\text{Re } y > 0, \quad x > 0]$ NT 19(11)
6. $\text{Ei}(\pm x) = -e^{\pm x} \int_0^{\infty} \frac{e^{-it}}{t \pm ix} dt \quad [x > 0]$ NT 23(2, 3)
- 7.⁸ $\text{Ei}(xy) = e^{xy} \int_0^1 \frac{t^{y-1}}{x + \ln t} dt$ LA 282(44)a

8. $\text{Ei}(-xy) = -e^{-xy} \int_0^1 \frac{t^{y-1}}{x - \ln t} dt$ LA 282(45)a
 $= x^{-1} e^{-xy} \left[\int_0^1 \frac{t^{x-1}}{(y - \ln t)^2} dt - y^{-1} \right] \quad [x > 0, \quad y > 0]$ LA 283(47)a
9. $\text{Ei}(x) = e^x \int_1^\infty \frac{1}{x - \ln t} \frac{dt}{t^2}$ [x > 0] LA 283(48)
10. $\text{Ei}(-x) = -e^{-x} \int_1^\infty \frac{1}{x + \ln t} \frac{dt}{t^2}$ [x > 0] LA 283(48)
11. $\text{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t + x \sin t}{t^2 + x^2} dt$ [x > 0] NT 23(6)
12. $\text{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t - x \sin t}{t^2 + x^2} dt$ [x < 0] NT 23(6)
13. $\text{Ei}(-x) = \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} \arctan \frac{t}{x} dt$ [Re x > 0] NT 25(13)
14. $\text{Ei}(-x) = \frac{2e^{-x}}{\pi} \int_0^\infty \frac{x \cos t - t \sin t}{t^2 + x^2} \ln t dt$ [x > 0] NT 26(7)
15. $\text{Ei}(x) = 2 \ln x - \frac{2e^x}{\pi} \int_0^\infty \frac{x \cos t + t \sin t}{t^2 + x^2} \ln t dt$ [x > 0] NT 27(8)
16. $\text{Ei}(-x) = -x \int_1^\infty e^{-tx} \ln t dt$ [x > 0] NT 32(12)

See also **3.327**, **3.881** 8, **3.916** 2 and 3, **4.326** 1, **4.326** 2, **4.331** 2, **4.351** 3, **4.425** 3, **4.581**. For integrals of the exponential integral function, see **6.22–6.23**, **6.78**.

Series and asymptotic representations

8.213

1. $\text{li}(x) = C + \ln(-\ln x) + \sum_{k=1}^\infty \frac{(\ln x)^k}{k \cdot k!}$ [0 < x < 1] NT 3(9)
2. $\text{li}(x) = C + \ln \ln x + \sum_{k=1}^\infty \frac{(\ln x)^k}{k \cdot k!}$ [x > 1] NT 3(10)

8.214

1. $\text{Ei}(x) = C + \ln(-x) + \sum_{k=1}^\infty \frac{x^k}{k \cdot k!}$ [x < 0]
2. $\text{Ei}(x) = C + \ln x + \sum_{k=1}^\infty \frac{x^k}{k \cdot k!}$ [x > 0]
3. $\text{Ei}(x) - \text{Ei}(-x) = 2x \sum_{k=0}^\infty \frac{x^{2k}}{(2k+1)(2k+1)!}$ [x > 0] NT 39(13)

$$8.215^7 \quad \text{Ei}(z) = \frac{e^z}{z} \left[\sum_{k=0}^n \frac{k!}{z^k} + R_n(z) \right] \quad |R_n(z)| = O(|z|^{-n-1})$$

$$[z \rightarrow \infty, \quad |\arg(-z)| \leq \pi - \delta; \quad \delta > 0 \text{ small}], \quad |R_n(z)| \leq (n+1)!|z|^{-n-1} \quad [\text{Re } z \leq 0]$$

$$8.216^7 \quad \text{Ei}(nx) - \text{Ei}(-nx) = e^{nx'} \left(\frac{1}{nx} + \frac{1}{n^2 x^2} + \frac{k_n}{n^3 x^3} \right),$$

where $x' = x \operatorname{sign} \operatorname{Re}(x)$, $k_n = O(1)$, and $n \rightarrow \infty$ NT 39(15)

8.217 Functional relations:

$$1. \quad e^{x'} \text{Ei}(-x') - e^{-x'} \text{Ei}(x') = -2 \int_0^\infty \frac{x' \sin t}{t^2 + x^2} dt \quad \text{NT 24(11)}$$

$$= \frac{4}{\pi} \int_0^\infty \frac{x' \cos t}{t^2 + x^2} \ln t \, dt - 2e^{-x'} \ln x' \quad [x' = x \operatorname{sign} \operatorname{Re} x] \quad \text{NT 27(9)}$$

$$2. \quad e^{x'} \text{Ei}(-x') + e^{-x'} \text{Ei}(x') = -2 \int_0^\infty \frac{t \cos t}{t^2 + x^2} dt = 2e^{-x'} \ln x' - \frac{4}{\pi} \int_0^\infty \frac{t \sin t}{t^2 + x^2} \ln t \, dt$$

$[x' = x \operatorname{sign} \operatorname{Re} x] \quad \text{NT 24(10), NT 27(10)}$

$$3. \quad \text{Ei}(-x) - \text{Ei}\left(-\frac{1}{x}\right) = \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} \arctan \frac{t(x - \frac{1}{x})}{1 + t^2} dt$$

$[\operatorname{Re} x > 0] \quad \text{NT 25(14)}$

$$4. \quad \text{Ei}(-\alpha x) \text{Ei}(-\beta x) - \ln(\alpha\beta) \text{Ei}[-(\alpha + \beta)x] = e^{-(\alpha+\beta)x} \int_0^\infty \frac{e^{-tx} \ln[(\alpha+t)(\beta+t)]}{t + \alpha + \beta} dt \quad \text{NT 32(9)}$$

See also **3.723** 1 and 5, **3.742** 2 and 4, **3.824** 4, **4.573** 2.

- For a connection with a confluent hypergeometric function, see **9.237**.
- For integrals of the exponential integral function, see **5.21**, **5.22**, **5.23**, **6.22**, and **6.23**.

8.218 Two numerical values:

$$1. \quad \text{Ei}(-1) = -0.219 \, 383 \, 934 \, 395 \, 520 \, 273 \, 665 \dots \quad \text{NT 89}$$

$$2. \quad \text{Ei}(1) = 1.895 \, 117 \, 816 \, 355 \, 936 \, 755 \, 478 \dots \quad \text{NT 89}$$

8.219* Definite integrals of exponential functions

$$1.* \quad \int_0^\infty \text{Ei}^2(x) e^{-2x} dx = \frac{\pi^2}{4}$$

$$2.* \quad \int_0^\infty \text{Ei}^2(-x) e^{2x} dx = \frac{\pi^2}{4}$$

$$3.* \quad \int_0^\infty \text{Ei}(x) \text{Ei}(-x) dx = 0$$

8.22 The hyperbolic sine integral $\operatorname{shi} x$ and the hyperbolic cosine integral $\operatorname{chi} x$

8.221

$$1. \quad \operatorname{shi} x = \int_0^x \frac{\sinh t}{t} dt = -i \left[\frac{\pi}{2} + \operatorname{si}(ix) \right] \quad (\text{see } \mathbf{8.230} \ 1) \quad \text{EH II } 146(17)$$

$$2.^{11} \quad \operatorname{chi} x = C + \ln x + \int_0^x \frac{\cosh t - 1}{t} dt \quad \text{EH II } 146(18)$$

8.23 The sine integral and the cosine integral: $\operatorname{si} x$ and $\operatorname{ci} x$

8.230

$$1.^{10} \quad \operatorname{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt = -\frac{\pi}{2} + \operatorname{Si}(x), \text{ where } \operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad \text{NT } 11(3)$$

$$2.^{10} \quad \operatorname{ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt = C + \ln x + \int_0^x \frac{\cos t - 1}{t} dt \quad [\operatorname{ci}(x) \text{ is also written } \operatorname{Ci}(x)] \quad \text{NT } 11(2)$$

8.231

$$1. \quad \operatorname{si}(xy) = -\int_x^\infty \frac{\sin ty}{t} dt \quad \text{NT } 18(7)$$

$$2. \quad \operatorname{ci}(xy) = -\int_x^\infty \frac{\cos ty}{t} dt \quad \text{NT } 18(6)$$

$$3. \quad \operatorname{si}(x) = -\int_0^{\pi/2} e^{-x \cos t} \cos(x \sin t) dt \quad \text{NT } 13(26)$$

8.232

$$1. \quad \operatorname{si}(x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)(2k-1)!} \quad \text{NT } 7(4)$$

$$2.^7 \quad \operatorname{ci}(x) = C + \ln(x) + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!} \quad \text{NT } 7(3)$$

8.233

$$1. \quad \operatorname{ci}(x) \pm i \operatorname{si}(x) = \operatorname{Ei}(\pm ix) \quad \text{NT } 6a$$

$$2. \quad \operatorname{ci}(x) - \operatorname{ci}(xe^{\pm \pi i}) = \mp \pi i \quad \text{NT } 7(5)$$

$$3. \quad \operatorname{si}(x) + \operatorname{si}(-x) = -\pi \quad \text{NT } 7(7)$$

8.234

$$1.^7 \quad \operatorname{Ei}(-x) - \operatorname{ci}(x) = \int_0^{\pi/2} e^{-x \cos \varphi} \sin(s \sin \varphi) d\varphi \quad \text{NT } 13(27)$$

$$2. \quad [\operatorname{ci}(x)]^2 + [\operatorname{si}(x)]^2 = -2 \int_0^{\pi/2} \frac{\exp(-x \tan \varphi) \ln \cos \varphi}{\sin \varphi \cos \varphi} d\varphi \quad [\operatorname{Re} x > 0] \quad (\text{see also } \mathbf{4.366})$$

NT 32(11)

See also **3.341**, **3.351** 1 and 2, **3.354** 1 and 2, **3.721** 2 and 3, **3.722** 1, 3, 5 and 7, **3.723** 8 and 11, **4.338** 1, **4.366** 1.

8.235

1. $\lim_{x \rightarrow +\infty} (x^\varrho \operatorname{si}(x)) = 0, \quad \lim_{x \rightarrow +\infty} (x^\varrho \operatorname{ci}(x)) = 0 \quad [\varrho < 1] \quad \text{NT 38(5)}$
2. $\lim_{x \rightarrow -\infty} \operatorname{si}(x) = -\pi, \quad \lim_{x \rightarrow -\infty} \operatorname{ci}(x) = \pm \pi i \quad \text{NT 38(6)}$

- For integrals of the sine integral and cosine integral, see **6.24–6.26**, **6.781**, **6.782**, and **6.783**.
- For indefinite integrals of the sine integral and cosine integral, see **5.3**.

8.24 The logarithm integral $\operatorname{li}(x)$ **8.240**

1. $\operatorname{li}(x) = \int_0^x \frac{dt}{\ln t} = \operatorname{Ei}(\ln x) \quad [x < 1] \quad \text{JA}$
2. $\operatorname{li}(x) = \lim_{\varepsilon \rightarrow 0} \left[\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t} \right] = \operatorname{Ei}(\ln x) \quad [x > 1] \quad \text{JA}$
3. $\operatorname{li}\{\exp(-xe^{\pm i\pi})\} = \operatorname{Ei}(-xe^{\pm i\pi}) = \operatorname{Ei}(x \mp i0) = \operatorname{Ei}(x) \pm i\pi = \operatorname{li}(e^x) \pm i\pi$
 $[x > 0] \quad \text{JA, NT 2(6)}$

Integral representations**8.241**

1. $\operatorname{li}(x) = \int_{-\infty}^{\ln x} \frac{e^t}{t} dt = x \ln \ln \frac{1}{x} - \int_{-\ln x}^{\infty} e^{-t} \ln t dt \quad [x < 1] \quad \text{LA 281(33)}$
2. $\operatorname{li}(x) = x \int_0^1 \frac{dt}{\ln x + \ln t}$ LA 280(22)
 $= \frac{x}{\ln x} + x \int_0^1 \frac{dt}{(\ln x + \ln t)^2}$ LA 280(29)
 $= x \int_1^{\infty} \frac{1}{\ln x - \ln t} \frac{dt}{t^2} \quad [x < 1] \quad \text{LA 280(30)}$
3. $\operatorname{li}(a^x) = \frac{1}{\ln a} \int_{-\infty}^x \frac{a^t}{t} dt \quad [x > 0]$

For integrals of the logarithm integral, see **6.21**

8.25 The probability integral $\Phi(x)$, the Fresnel integrals $S(x)$ and $C(x)$, the error function $\operatorname{erf}(x)$, and the complementary error function $\operatorname{erfc}(x)$ **8.250** Definition:

- 1.¹¹ $\Phi(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (\text{called the error function})$
2. $S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 dt$

$$3. \quad C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 dt$$

$$4.^{11} \quad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \quad (\text{called the complementary error function})$$

$$5.^* \quad \int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \sin(a\sqrt{x}) dx \\ = -\sinh(a\sqrt{p}) + \frac{1}{2}e^{-a\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} - \sqrt{py}\right) + \frac{1}{2}e^{a\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right)$$

$$6.^* \quad \int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos(a\sqrt{x}) dx = \frac{1}{\sqrt{\pi y}} \exp\left(-\frac{a^2}{4y} - py\right) - \frac{\sqrt{p}}{2}e^{-a\sqrt{p}} \Phi\left(\frac{a}{a\sqrt{y}} - \sqrt{py}\right) \\ + \frac{\sqrt{p}}{2}e^{\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right) - \sqrt{p} \cosh(a\sqrt{p}) \\ [\operatorname{Re} p > 0, \quad a, b \text{ are real}]$$

$$7.^* \quad \int_0^p \exp(-x^2) \Phi(p-x) dx = \int_0^p \exp(-x^2) \operatorname{erf}(p-x) dx = \frac{\sqrt{\pi}}{2} \left[\Phi\left(\frac{p}{\sqrt{2}}\right) \right]^2$$

$$8.^* \quad \int_0^p x^2 \exp(-x^2) \Phi(p-x) dx = \int_0^p x^2 \exp(-x^2) \operatorname{erf}(p-x) dx \\ = \frac{\sqrt{\pi}}{4} \left[\Phi\left(\frac{p}{\sqrt{2}}\right) \right]^2 - \frac{p}{2\sqrt{2}} \Phi\left(-\frac{x^2}{2}\right) \operatorname{erf}\left(\frac{p}{\sqrt{2}}\right)$$

$$9.^* \quad \int_{(b-a)/\sqrt{2}}^{(b+a)/\sqrt{2}} \exp(-x^2) \Phi(b\sqrt{2}-x) dx + \int_{(a-b)/\sqrt{2}}^{(a+b)/\sqrt{2}} \exp(-x^2) \Phi(a\sqrt{2}-x) dx = \sqrt{\pi} \Phi(a) \Phi(b)$$

Integral representations

8.251

$$1. \quad \Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} dt \quad (\text{see also } \mathbf{3.361} \ 1)$$

$$2. \quad S(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\sin t}{\sqrt{t}} dt$$

$$3. \quad C(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos t}{\sqrt{t}} dt$$

8.252

$$1. \quad \Phi(xy) = \frac{2y}{\sqrt{\pi}} \int_0^x e^{-t^2 y^2} dt \quad [\operatorname{Re} y^2 > 0]$$

$$2. \quad S(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \sin(t^2 y^2) dt$$

$$3. \quad C(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \cos(t^2 y^2) dt$$

$$\begin{aligned}
4. \quad \Phi(xy) &= 1 - \frac{2}{\sqrt{\pi}} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} t y dt}{\sqrt{t^2 + x^2}} \quad [\operatorname{Re} y^2 > 0] & \text{NT 19(11)a} \\
&= 1 - \frac{2x}{\pi} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} dt}{t^2 + x^2} \quad [\operatorname{Re} y^2 > 0] & \text{NT 19(13)a}
\end{aligned}$$

$$5.^7 \quad \Phi\left(\frac{-y}{2xi}\right) - \Phi\left(\frac{y}{2xi}\right) = \frac{4xie^{\frac{y^2}{4x^2}}}{\sqrt{\pi}} \int_0^\infty e^{-t^2 y^2} \sin(ty) dt \quad [\operatorname{Re} x^2 > 0] \quad \text{NT 28(3)a}$$

$$6.^8 \quad \Phi\left(\frac{y}{2x}\right) = 1 - \frac{2}{\sqrt{\pi}} x e^{-\frac{y^2}{4}} \int_0^\infty e^{-t^2 x^2 - ty} dt \quad [\operatorname{Re} x^2 > 0] \quad \text{NT 27(1)a}$$

See also **3.322**, **3.362** 2, **3.363**, **3.468**, **3.897**, **6.511** 4 and 5.

8.253⁸ Series representations:

$$\begin{aligned}
1.^{11} \quad \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} e^{-x^2} x F_1\left(1; \frac{3}{2}; x^2\right) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^\infty (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!} & \text{NT 7(9)a} \\
&= \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^\infty \frac{2^k x^{2k+1}}{(2k+1)!!} & \text{NT 10(11)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad S(x) &= \frac{2}{\sqrt{2\pi}} \left(x \sin x^2 F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right) - \frac{2}{3}x^3 \cos x^2 F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right) \right) \\
&= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)} & \text{NT 8(14)a} \\
&= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} - \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} \right\} & \text{NT 10(13)a}
\end{aligned}$$

$$\begin{aligned}
3. \quad C(x) &= \frac{2}{\sqrt{2\pi}} \left(\frac{2}{3}x^3 \sin x^2 F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right) - x \cos x^2 F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right) \right) \\
&= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)} & \text{NT 8(13)a} \\
&= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} + \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} \right\} & \text{NT 10(12)a}
\end{aligned}$$

For the expansions in Bessel functions, see **8.515** 2, **8.515** 3.

Asymptotic representations

$$\begin{aligned}
\mathbf{8.254}^8 \quad \Phi(z) &= 1 - \frac{e^{-z^2}}{\sqrt{\pi}z} \left[\sum_{k=0}^n (-1)^k \frac{(2k-1)!!}{(2z^2)^k} + O\left(|z|^{-2n-z}\right) \right], \\
& \quad [z \rightarrow \infty, \quad |\arg(-z)| \leq \pi - \delta; \quad \delta > 0 \text{ small}]
\end{aligned}$$

where

$$|R_n| < \frac{\Gamma(n + \frac{1}{2})}{|x|^{n+\frac{1}{2}}} \cos \frac{\varphi}{2}, \quad x = |x|e^{i\varphi} \text{ and } \varphi^2 < \pi^2 \quad \text{NT 37(10)a}$$

8.255

$$1. \quad S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}x} \cos x^2 + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty] \quad \text{MO 127a}$$

$$2. \quad C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi x}} \sin x^2 + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty] \quad \text{MO 127a}$$

8.256 Functional relations:

$$1. \quad C(z) + i S(z) = \sqrt{\frac{i}{2}} \Phi\left(\frac{z}{\sqrt{i}}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{it^2} dt$$

$$2. \quad C(z) - i S(z) = \frac{1}{\sqrt{2i}} \Phi(z\sqrt{i}) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-it^2} dt$$

$$3. \quad [\cos^2 u C(u) + \sin u^2 S(u)] = \frac{1}{2} [\cos^2 u + \sin u^2] + \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \sin t^2 dt$$

$$[\operatorname{Re} u \geq 0] \quad \text{NT 28(6)a}$$

$$4. \quad [\cos^2 u S(u) - \sin u^2 C(u)] = \frac{1}{2} [\cos^2 u - \sin u^2] - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \cos t^2 dt$$

$$[\operatorname{Re} u \geq 0] \quad \text{NT 28(5)a}$$

$$5.^{11} \quad \left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2 = \frac{2}{\pi} \int_0^{\pi/2} \frac{\exp(-x^2 \tan \varphi) \sin \frac{\varphi}{2} \sqrt{\cos \varphi}}{\sin 2\varphi} d\varphi$$

$$(see also \textbf{6.322}) \quad \text{NT 33(18)a}$$

- For a connection with a confluent hypergeometric function, see **9.236**.
- For a connection with a parabolic cylinder function, see **9.254**.

8.257

$$1. \quad \lim_{x \rightarrow +\infty} (x^\varrho [S(x) - \tfrac{1}{2}]) = 0 \quad [\varrho < 1] \quad \text{NT 38(11)}$$

$$2. \quad \lim_{x \rightarrow +\infty} (x^\varrho [C(x) - \tfrac{1}{2}]) = 0 \quad [\varrho < 1] \quad \text{NT 38(11)}$$

$$3. \quad \lim_{x \rightarrow +\infty} S(x) = \frac{1}{2} \quad \text{NT 38(12)a}$$

$$4. \quad \lim_{x \rightarrow +\infty} C(x) = \frac{1}{2} \quad \text{NT 38(12)a}$$

- For integrals of the probability integral, see **6.28–6.31**.
- For integrals of Fresnel's sine integral and cosine integral, see **6.32**.

8.258¹⁰ Integrals involving the complementary error function

$$1. \quad \int_0^\infty \operatorname{erfc}^2(x) e^{-\beta x^2} dx = \frac{1}{\sqrt{\beta\pi}} \left(-\arccos\left(\frac{1}{1+\beta}\right) + 2 \arctan\left(\sqrt{\beta}\right) \right)$$

$$[\beta > 0]$$

$$2. \quad \int_0^\infty x \operatorname{erfc}^2(x) e^{-\beta x^2} dx = \frac{1}{2\beta} \left(1 - \frac{4}{\pi} \frac{\arctan(\sqrt{1+\beta})}{\sqrt{1+\beta}} \right)$$

$$[\beta > 0]$$

3.
$$\int_0^\infty x^3 \operatorname{erfc}^2(x) e^{-\beta x^2} dx = \frac{1}{2\beta^2} \left(1 - \frac{4}{\pi} \frac{\arctan(\sqrt{1+\beta})}{\sqrt{1+\beta}} \right) + \frac{1}{\beta\pi} \left(\frac{1}{(1+\beta)(\beta^2+2\beta+2)} - \frac{\arctan(\sqrt{1+\beta})}{(1+\beta)^{\frac{3}{2}}} \right)$$
 $[\beta > 0]$
4.
$$\int_0^\infty x \operatorname{erfc}(\sqrt{x}) e^{-\beta x} dx = \frac{1}{\beta^2} \left[1 - \frac{1 + \frac{3}{2}\beta}{(1+\beta)^{\frac{3}{2}}} \right] \quad [\beta > 0]$$
- 5.¹¹
$$\int_0^\infty \sqrt{x} \operatorname{erfc}(\sqrt{x}) e^{-\beta x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} \frac{\arctan(\sqrt{\beta})}{\beta^{\frac{3}{2}}} - \frac{1}{2\beta(1+\beta)} \right)$$
 $[\beta > 0]$

8.259* Integrals involving the error function and an exponential function

1.
$$\int_{-\infty}^\infty e^{-px^2} \Phi(a+bx) dx = \sqrt{\frac{\pi}{p}} \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right) \quad [\operatorname{Re} p > 0], \quad a, b \text{ real}$$
2.
$$\int_{-\infty}^\infty x^2 e^{-px^2} \Phi(a+bx) dx = \frac{1}{2p} \sqrt{\frac{\pi}{p}} \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right) - \frac{ab^2}{p(b^2+p)^{3/2}} \exp\left(-\frac{a^2 p}{b^2+p}\right)$$
 $[\operatorname{Re} p > 0, \quad a, b \text{ are real}]$
3.
$$\int_{-\infty}^\infty x^{2n} e^{-px^2} \Phi(a+bx) dx = (-1)^n \frac{\partial^n}{\partial p^n} \left[\sqrt{\frac{\pi}{p}} \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right) \right]$$
 $[n = 0, 1, \dots, \quad \operatorname{Re} p > 0, \quad a, b \text{ are real}]$

8.26 Lobachevskiy's function $L(x)$

8.260 Definition:

$$L(x) = -\int_0^x \ln \cos t \, dt \quad \text{LO III 184(10)}$$

For integral representations of the function $L(x)$, see also **3.531** 8, **3.532** 2, **3.533**, and **4.224**.

8.261 Representation in the form of a series:

$$L(x) = x \ln 2 - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin 2kx}{k^2} \quad \text{LO III 185(11)}$$

8.262 Functional relationships:

1.
$$L(-x) = -L(x) \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right] \quad \text{LO III 185(13)}$$
2.
$$L(\pi - x) = \pi \ln 2 - L(x) \quad \text{LO III 286}$$
3.
$$L(\pi + x) = \pi \ln 2 + L(x) \quad \text{LO III 286}$$
4.
$$L(x) - L\left(\frac{\pi}{2} - x\right) = \left(x - \frac{\pi}{4}\right) \ln 2 - \frac{1}{2} L\left(\frac{\pi}{2} - 2x\right) \quad \left[0 \leq x < \frac{\pi}{4}\right] \quad \text{LO III 186(14)}$$

8.3 Euler's Integrals of the First and Second Kinds and Functions Generated by Them

8.31 The gamma function (Euler's integral of the second kind): $\Gamma(z)$

8.310 Definition:

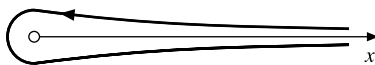
$$1. \quad \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad [\operatorname{Re} z > 0] \quad (\text{Euler}) \quad \text{FI II 777(6)}$$

Generalization:

$$2. \quad \Gamma(z) = -\frac{1}{2i \sin \pi z} \int_C (-t)^{z-1} e^{-t} dt$$

for z not an integer. The contour C is shown in the drawing:

WH



$\Gamma(z)$ is an analytic function z with simple poles at the points $z = -l$ (for $l = 0, 1, 2, \dots$) to which correspond to residues $\frac{(-1)^l}{l!}$. $\Gamma(z)$ satisfies the relation $\Gamma(1) = 1$.

WH, MO 1

Integral representations

$$8.311 \quad \Gamma(z) = \frac{1}{e^{2\pi iz} - 1} \int_{\infty}^{(0+)} e^{-t} t^{z-1} dt \quad \text{MO 2}$$

8.312

$$1. \quad \Gamma(z) = \int_0^1 \left(\ln \frac{1}{t} \right)^{z-1} dt \quad [\operatorname{Re} z > 0] \quad \text{FI II 778}$$

$$2. \quad \Gamma(z) = x^z \int_0^{\infty} e^{-xt} t^{z-1} dt \quad [\operatorname{Re} z > 0, \quad \operatorname{Re} x > 0] \quad \text{FI II 779(8)}$$

$$3. \quad \Gamma(z) = \frac{2a^z e^a}{\sin \pi z} \int_0^{\infty} e^{-at^2} (1+t^2)^{z-\frac{1}{2}} \cos [2at + (2z-1) \arctan t] dt$$

[$a > 0$] WH

$$4. \quad \Gamma(z) = \frac{1}{2 \sin \pi z} \int_0^{\infty} e^{-t^2} t^{z-1} (1+t^2)^{\frac{z}{2}} \{3 \sin [t + z \operatorname{arccot}(-t)] + \sin [t + (z-2) \operatorname{arccot}(-t)]\} dt$$

[arccot denotes an obtuse angle] WH

$$5. \quad \Gamma(y) = x^y e^{-i\beta y} \int_0^{\infty} t^{y-1} \exp(-xte^{-i\beta}) dt$$

[x, y, β real, $x > 0$, $y > 0$, $|\beta| < \frac{\pi}{2}$] MO 8

$$6. \quad \Gamma(z) = \frac{b^z}{2 \sin \pi z} \int_{-\infty}^{\infty} e^{bti} (it)^{z-1} dt \quad [b > 0, \quad 0 < \operatorname{Re} z < 1] \quad \text{NH 154(3)}$$

7.
$$\Gamma(z) = \frac{(\sqrt{a^2 + b^2})^z}{\cos\left(z \arctan \frac{b}{a}\right)} \int_0^\infty e^{-at} \cos(bt) t^{z-1} dt$$
 NH 152(1)a

$$= \frac{(\sqrt{a^2 + b^2})^z}{\sin\left(z \arctan \frac{b}{a}\right)} \int_0^\infty e^{-at} \sin(bt) t^{z-1} dt$$
 NH 152(2)

$$[a > 0, \quad b \geq 0, \quad \operatorname{Re} z > 0]$$
8.
$$\Gamma(z) = \frac{b^z}{\cos \frac{\pi z}{2}} \int_0^\infty \cos(bt) t^{z-1} dt$$

$$= \frac{b^z}{\sin \frac{\pi z}{2}} \int_0^\infty \sin(bt) t^{z-1} dt$$

$$[b > 0, \quad 0 < \operatorname{Re} z < 1]$$
 NH 152(5)
9.
$$\Gamma(z) = \int_0^\infty e^{-t} (t - z) t^{z-1} \ln t \, dt$$

$$[\operatorname{Re} z > 0]$$
 NH 173(7)
10.
$$\Gamma(z) = \int_{-\infty}^\infty \exp(zt - e^t) \, dt$$

$$[\operatorname{Re} z > 0]$$
 NH 145(14)
- 11.¹¹
$$\Gamma(x) \cos \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos \alpha} \cos(\lambda t \sin \alpha) \, dt$$

$$\left[\lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \right]$$
 WH
12.
$$\Gamma(x) \sin \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos \alpha} \sin(\lambda t \sin \alpha) \, dt$$

$$\left[\lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \right]$$
 WH
13.
$$\Gamma(-z) = \int_0^\infty \left[\frac{e^{-t} - \sum_{k=0}^n (-1)^k \frac{t^k}{k!}}{t^{z+1}} \right] dt$$

$$[n = [\operatorname{Re} z]]$$
 MO 2
- 8.313**
$$\Gamma\left(\frac{z+1}{v}\right) = vu^{\frac{z+1}{v}} \int_0^\infty \exp(-ut^v) t^z \, dt$$

$$[\operatorname{Re} u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} z > -1]$$

 JA, MO 7a
- 8.314***
$$\Gamma(z) = \int_1^\infty e^{-t} t^{z-1} \, dt + \sum_{n=0}^\infty \frac{(-1)^n}{k!(z+n)}$$

$$[z \rightarrow 0, \text{ in } |\arg z| < \pi]$$
- 8.315**
- 1.¹¹
$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t} \, dt$$

$$[\text{for the contour } C, \text{ see } \mathbf{8.310} \, 2]$$
- 2.⁸
$$\int_{-\infty}^\infty \frac{e^{bti}}{(a+it)^2} \, dt = \frac{2\pi e^{-ab} b^{z-1}}{\Gamma(z)}$$

$$\int_{-\infty}^\infty \frac{e^{-bti}}{(a+it)^z} \, dt = 0 \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} z > 0, \quad |\arg(a+it)| < \tfrac{1}{2}\pi]$$

$$3. \quad \frac{1}{\Gamma(z)} = a^{1-z} \frac{e^a}{\pi} \int_0^{\pi/2} \cos(a \tan \theta - z\theta) \cos^{z-2} \theta d\theta \quad [\operatorname{Re} z > 1] \quad \text{NH 157(14)}$$

See also **3.324** 2, **3.326**, **3.328**, **3.381** 4, **3.382** 2, **3.389** 2, **3.433**, **3.434**, **3.478** 1, **3.551** 1, 2, **3.827** 1, **4.267** 7, **4.272**, **4.353** 1, **4.369** 1, **6.214**, **6.223**, **6.246**, **6.281**.

8.32 Representation of the gamma function as series and products

8.321 Representation in the form of a series:

$$1.^6 \quad \Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k$$

$$\left[c_0 = 1, \quad c_{n+1} = \frac{\sum_{k=0}^n (-1)^{k+1} s_{k+1} c_{n-k}}{n+1}; \quad s_1 = C, \quad s_n = \zeta(n) \text{ for } n \geq 2, \quad |z| < 1 \right]$$

NH 40(1, 3)

$$2.^{11} \quad \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$

$$\left[d_0 = 1, \quad d_{n+1} = \frac{\sum_{k=0}^n (-1)^k s_{k+1} d_{n-k}}{n+1}; \quad s_1 = C, \quad s_n = \zeta(n) \text{ for } n \geq 2 \right] \quad \text{NH 41(4, 6)}$$

Infinite-product representation

$$8.322^{11} \quad \Gamma(z) = e^{-Cz} \frac{1}{z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1 + \frac{z}{k}} \quad [\operatorname{Re} z > 0] \quad \text{SM 269}$$

$$= \frac{1}{z} \prod_{k=1}^{\infty} \frac{(1 + \frac{1}{k})^z}{1 + \frac{z}{k}} \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

$$= \lim_{n \rightarrow \infty} \frac{n^z}{z} \prod_{k=1}^n \frac{k}{z+k} \quad [\operatorname{Re} z > 0] \quad \text{SM 267(130)}$$

$$8.323^7 \quad \Gamma(z) = 2z^z e^{-z} \prod_{k=1}^{\infty} 2^k \sqrt{B(2^{k-1}z, \frac{1}{2})} \quad \text{NH 98(12)}$$

$$8.324^7 \quad \Gamma(1+z) = 4^z \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{z}{2^k}\right)}{\sqrt{\pi}} \quad \text{MO 3}$$

8.325

$$1. \quad \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty} \left[\left(1 + \frac{\gamma}{\alpha+k}\right) \left(1 - \frac{\gamma}{\beta+k}\right) \right] \quad \text{NH 62(2)}$$

$$2.^{11} \quad \frac{e^{Cx} \Gamma(z+1)}{\Gamma(z-x+1)} = \prod_{k=1}^{\infty} \left[\left(1 - \frac{x}{z+k}\right) e^{x/k} \right] \quad [z \neq 0, -1, -2, \dots; \quad \operatorname{Re} z > 0, \quad \operatorname{Re}(z-x) > 0]$$

$$3.^7 \quad \frac{\sqrt{\pi}}{\Gamma(1+\frac{z}{2})\Gamma(\frac{1}{2}-\frac{z}{2})} = \prod_{k=1}^{\infty} \left(1 - \frac{z}{2k-1}\right) \left(1 + \frac{z}{2k}\right) \quad \text{MO 2}$$

8.326

$$1. \quad \frac{\frac{[\Gamma(x)]^2}{\Gamma(2x)}}{B(x+iy, x-iy)} = \left| \frac{\Gamma(x)}{\Gamma(x-iy)} \right|^2 = \prod_{k=0}^{\infty} \left(1 + \frac{y^2}{(x+k)^2} \right)$$

[x, y are real, $x \neq 0, -1, -2, \dots$]
LO V, NH 63(4)

$$2.^{11} \quad \frac{\Gamma(x+iy)}{\Gamma(x)} = \frac{xe^{-iCy}}{x+iy} \prod_{n=1}^{\infty} \frac{\exp\left(\frac{iy}{n}\right)}{1 + \frac{iy}{x+n}}$$

[x, y are real, $x \neq 0, -1, -2, \dots$]

MO 2

8.327 Asymptotic representation for large arguments:

$$1.^* \quad \Gamma(z) \sim z^{z-\frac{1}{2}} e^{-z} \sqrt{2\pi} \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + O(z^{-5}) \right\}$$

[$|\arg z| < \pi$] WH

For z real and positive, the remainder of the series is less than the last term that is retained.

$$2.^* \quad n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \text{ or equivalently } \Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

[Stirling's asymptotic formula for $n \gg 0$] AS 6.1.38

$$3.^* \quad \ln \Gamma(z) \sim \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \dots$$

[$z \rightarrow \infty$, $|\arg z| < \pi$] AS 6.1.38

8.328

$$1. \quad \lim_{|y| \rightarrow \infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi}$$

[x and y are real] MO 6

$$2. \quad \lim_{|z| \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)} e^{-a \ln z} = 1$$

MO 6

8.33 Functional relations involving the gamma function**8.331**

$$1. \quad \Gamma(x+1) = x \Gamma(x)$$

$$2.^* \quad \Gamma(x+a) = (x+a-1) \Gamma(x+a-1)$$

$$= \frac{\Gamma(x+a+1)}{(x+a)}$$

$$3.^* \quad \Gamma(x-a) = (x-a-1) \Gamma(x-a-1)$$

$$= \frac{\Gamma(x-a+1)}{(x-a)}$$

8.332

1. $|\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}$ [y is real] MO 3
2. $|\Gamma(\frac{1}{2} + iy)|^2 = \frac{\pi}{\cosh \pi y}$ [y is real]
3. $\Gamma(1 + ix) \Gamma(1 - ix) = \frac{\pi x}{\sinh x\pi}$ [x is real] LO V
4. $\Gamma(1 + x + iy) \Gamma(1 - x + iy) \Gamma(1 + x - iy) \Gamma(1 - x - iy) = \frac{2\pi^2 (x^2 + y^2)}{\cosh 2y\pi - \cos 2x\pi}$
[x and y are real] LO V

8.333 $[\Gamma(n+1)]^n = G(n+1) \prod_{k=1}^n k^k,$
 where n is a natural number and

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp \left[-\frac{z(z+1)}{2} - \frac{C}{2} z^2 \right] \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n \exp \left(-z + \frac{z^2}{2n} \right) \right\}$$
WH

8.334

1. $\prod_{k=1}^n \frac{1}{\Gamma(-z \exp \frac{2\pi k i}{n})} = -z^n \prod_{k=1}^{\infty} \left[1 - \left(\frac{z}{k} \right)^n \right]$ [$n = 2, 3, 3 \dots$] MO 2
2. $\Gamma(\frac{1}{2} + x) \Gamma(\frac{1}{2} - x) = \frac{\pi}{\cos \pi x}$
3. $\Gamma(1 - x) \Gamma(x) = \frac{\pi}{\sin \pi x}$ FI II 430

Special cases

8.335⁷ $\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right)$ [product theorem] FI II 782a, WH

1. $\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right)$ [doubling formula]
2. $\Gamma(3x) = \frac{3^{3x-\frac{1}{2}}}{2\pi} \Gamma(x) \Gamma\left(x + \frac{1}{3}\right) \Gamma\left(x + \frac{2}{3}\right)$
3. $\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) \Gamma\left(1 - \frac{k}{n}\right) = \frac{(2\pi)^{n-1}}{n}$ WH
- 4.¹⁰ $\sum_{n=0}^{\infty} \frac{\Gamma^2(n - \frac{1}{2})}{4(n!)^2 \Gamma^2(-\frac{1}{2})} = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{1024} + \frac{25}{65536} + \dots = \frac{1}{\pi}$

8.336 $\Gamma\left(-\frac{yz + xi}{2y}\right) \Gamma(1 - z) = (2i)^{z+1} y \Gamma\left(1 + \frac{yz - xi}{2y}\right) \int_0^{\infty} e^{-tx} \sin^z(ty) dt$
[$\operatorname{Re}(yi) > 0, \operatorname{Re}(x - yzi) > 0$] NH 133(10)

- For a connection with the psi function, see **8.361** 1.
- For a connection with the beta function, see **8.384** 1.
- For integrals of the gamma function, see **8.412** 4, **8.414**, **9.223**, **9.242** 3, **9.242** 4.

8.337

1. $[\Gamma'(x)]^2 < \Gamma(x) \Gamma''(x)$ $[x > 0]$ MO 1
2. For $x > 0$, $\min \Gamma(1+x) = 0.88560 \dots$ is attained when $x = 0.46163 \dots$ JA

Particular values**8.338**

1. $\Gamma(1) = \Gamma(2) = 1$
2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
3. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
4. $\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2 [(4k+1)^2 - 1]}{[(4k-1)^2 - 1] (4k+1)^2}$ MO 1a
5. $\prod_{k=1}^8 \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^6} \left(\frac{\pi}{\sqrt{3}}\right)^3$ WH

8.339 For n a natural number

1. $\Gamma(n) = (n-1)!$
2. $\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!$
3. $\Gamma\left(\frac{1}{2} - n\right) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}$
4. $\frac{\Gamma\left(p + n + \frac{1}{2}\right)}{\Gamma\left(p - n + \frac{1}{2}\right)} = \frac{(4p^2 - 1^2)(4p^2 - 3^2) \dots [4p^2 - (2n-1)^2]}{2^{2n}}$ WA 221
- 5.* $\Gamma(n+k) = (n+k-1)!$

$$= \frac{\Gamma(n+k+1)}{(n+k)} \quad [n+k \geq 0, 1, \dots]$$
- 6.* $\Gamma(n-k) = (n-k-1)!$

$$= \frac{\Gamma(n-k+1)}{(n-k)} \quad [n-k \geq 0, 1, \dots]$$

8.34 The logarithm of the gamma function

8.341 Integral representation:

$$1. \quad \ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tz}}{t} dt$$

[Re $z > 0$] WH

$$2.^{11} \quad \ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + 2 \int_0^\infty \frac{\arctan \frac{t}{z}}{e^{2\pi t} - 1} dt$$

[Re $z > 0$ and $\arctan w = \int_0^w \frac{du}{1+u^2}$ is taken over a rectangular path in the w -plane] WH

$$3. \quad \ln \Gamma(z) = \int_0^\infty \left\{ \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)e^{-t} \right\} \frac{dt}{t}$$

[Re $z > 0$] WH

$$4. \quad \ln \Gamma(z) = \int_0^\infty \left\{ (z-1)e^{-t} + \frac{(1+t)^{-z} - (1+t)^{-1}}{\ln(1+t)} \right\} \frac{dt}{t}$$

[Re $z > 0$] WH

$$5. \quad \ln \Gamma(x) = \frac{\ln \pi - \ln \sin \pi x}{2} + \frac{1}{2} \int_0^\infty \left\{ \frac{\sinh\left(\frac{1}{2} - x\right)t}{\sinh \frac{t}{2}} - (1-2x)e^{-t} \right\} \frac{dt}{t}$$

[0 < $x < 1$] WH

$$6. \quad \ln \Gamma(z) = \int_0^1 \left\{ \frac{t^z - t}{t-1} - t(z-1) \right\} \frac{dt}{t \ln t}$$

[Re $z > 0$] WH

$$7. \quad \ln \Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}} \right] \frac{dt}{t}$$

[Re $z > 0$] NH 187(7)

See also **3.427** 9, **3.554** 5.

8.342 Series representations:

$$1.^{11} \quad \ln \Gamma(z+1)$$

$$= \frac{1}{2} \left[\ln \left(\frac{\pi z}{\sin \pi z} \right) - \ln \frac{1+z}{1-z} \right] + (1 - \mathcal{C})z + \sum_{k=1}^\infty \frac{1 - \zeta(2k+1)}{2k+1} z^{2k+1}$$

$$= -\mathcal{C}z + \sum_{k=2}^\infty (-1)^k \frac{z^k}{k} \zeta(k)$$

[| z | < 1] NH 38(16, 12)

$$2. \quad \ln \Gamma(1+x) = \frac{1}{2} \ln \frac{\pi x}{\sin \pi x} - \mathcal{C}x - \sum_{n=1}^\infty \frac{x^{2n+1}}{2n+1} \zeta(2n+1)$$

[| x | < 1] NH 38(14)

8.343

$$1. \quad \ln \Gamma(x) = \ln \sqrt{2\pi} + \sum_{n=1}^\infty \left\{ \frac{1}{2n} \cos 2n\pi x + \frac{1}{n\pi} (\mathcal{C} + \ln 2n\pi) \sin 2n\pi x \right\}$$

[0 < $x < 1$] FI III 558

$$2. \quad \ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{(m+1)(m+2)} \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}} \\ [|\arg z| < \pi]$$

MO 9

8.344⁷ Asymptotic expansion for large values of $|z|$:

$$\ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R_n(z),$$

where

$$|R_n(z)| < \frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1} \cos^{2n-1}(\frac{1}{2} \arg z)}$$

MO5

For integrals of $\ln \Gamma(x)$, see **6.44**.

8.35 The incomplete gamma function

8.350 Definition:

$$1. \quad \gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt \quad [\operatorname{Re} \alpha > 0] \quad \text{EH II 133(1), NH 1(1)}$$

$$2.^{11} \quad \Gamma(\alpha, x) = \int_x^{\infty} e^{-t} t^{\alpha-1} dt \quad \text{EH II 133(2), NH 2(2), LE 339}$$

$$3.* \quad \Gamma(z, 0) = \Gamma(z)$$

$$4.* \quad \Gamma(a, \infty) = 0$$

$$5.* \quad \gamma(a, 0) = 0$$

8.351

$$1. \quad \gamma^*(\alpha, x) = \frac{x^{-\alpha}}{\Gamma(\alpha)} \gamma(\alpha, x) \text{ is an analytic function with respect to } \alpha \text{ and } x \quad \text{EH II 133(5)}$$

2. Another definition of $\Gamma(\alpha, x)$ that is also suitable for the case $\operatorname{Re} \alpha \leq 0$:

$$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha} e^{-x} \Phi(1, 1 + \alpha; x) = \frac{x^\alpha}{\alpha} \Phi(a, 1 + a; -x) \quad \text{EH II 133(3)}$$

3. For fixed x , $\Gamma(\alpha, x)$ is an entire function of α . For non-integral α , $\Gamma(\alpha, x)$ is a multiple-valued function of x with a branch point at $x = 0$.

4. A second definition of $\Gamma(\alpha, x)$:

$$\Gamma(\alpha, x) = x^\alpha e^{-x} \Psi(1, 1 + \alpha; x) = e^{-x} \Psi(1 - \alpha, 1 - \alpha; x) \quad \text{EH II 133(4)}$$

8.352 Special cases:

$$1. \quad \gamma(1 + n, x) = n! \left[1 - e^{-x} \left(\sum_{m=0}^n \frac{x^m}{m!} \right) \right] \quad [n = 0, 1, \dots]$$

EH II 136(17, 16), NH 6(11)

$$2. \quad \Gamma(1 + n, x) = n! e^{-x} \sum_{m=0}^n \frac{x^m}{m!} \quad [n = 0, 1, \dots] \quad \text{EH II 136(16, 18)}$$

$$3.^{11} \quad \Gamma(-n, x) = \frac{(-1)^n}{n!} \left[\text{Ei}(-z) - \frac{1}{2} \ln(-z) + \frac{1}{2} \ln\left(-\frac{1}{z}\right) - \ln z \right] - e^{-z} \sum_{k=1}^n \frac{z^{k-n-1}}{(-n)_k}$$

$$[n = 1, 2, \dots]$$

$$4.^* \quad \Gamma(n, x) = (n-1)! e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!}$$

$$5.^* \quad \Gamma(-n+1, x) = \frac{(-1)^{n+1}}{(n-1)!} \left[\Gamma(0, x) - e^{-x} \sum_{m=0}^{n-2} (-1)^m \frac{m!}{x^{m+1}} \right]$$

$$[n = 2, 3, \dots]$$

$$6.^* \quad \gamma(n, x) = (n-1)! \left[1 - e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!} \right]$$

$$[n = 1, 2, \dots]$$

$$7.^* \quad \Gamma(n, x) = (n-1)! e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!}$$

$$[n = 1, 2, \dots]$$

$$8.^* \quad \Gamma(-n+k, x) = \frac{(-1)^{n-k}}{(n-k)!} \left[\Gamma(0, x) - e^{-x} \sum_{m=0}^{n-k-1} (-1)^m \frac{m!}{x^{m+1}} \right]$$

$$[n-k \geq 1, \quad k = 0, 1, \dots]$$

8.353 Integral representations:

$$1. \quad \gamma(\alpha, x) = x^\alpha \operatorname{cosec} \pi \alpha \int_0^\pi e^x \cos \theta \cos(\alpha \theta + x \sin \theta) d\theta \quad [x \neq 0, \quad \operatorname{Re} \alpha > 0, \quad \alpha \neq 1, 2, \dots]$$

EH II 137(2)

$$2. \quad \gamma(\alpha, x) = x^{\frac{1}{2}\alpha} \int_0^\infty e^{-t} t^{\frac{1}{2}\alpha-1} J_\alpha(2\sqrt{xt}) dt \quad [\operatorname{Re} \alpha > 0]$$

EH II 138(4)

$$3. \quad \Gamma(\alpha, x) = \frac{\rho^{-x} x^\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{e^{-t} t^{-\alpha}}{x+t} dt \quad [\operatorname{Re} \alpha < 1, \quad x > 0]$$

EH II 137(3), NH 19(12)

$$4. \quad \Gamma(\alpha, x) = \frac{2x^{\frac{1}{2}\alpha} e^{-x}}{\Gamma(1-\alpha)} \int_0^\infty e^{-t} t^{-\frac{1}{2}\alpha} K_\alpha[2\sqrt{xt}] dt \quad [\operatorname{Re} \alpha < 1]$$

EH II 138(5)

$$5. \quad \Gamma(\alpha, xy) = y^\alpha e^{-xy} \int_0^\infty e^{-ty} (t+x)^{\alpha-1} dt$$

$$[\operatorname{Re} y > 0, \quad x > 0, \quad \operatorname{Re} \alpha > 1] \quad (\text{See also } \mathbf{3.936} \text{ } 5, \mathbf{3.944} \text{ } 1-4) \quad \text{NH } 19(10)$$

For integrals of the gamma function, see **6.45**.

8.354 Series representations:

$$1. \quad \gamma(\alpha, x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)}$$

EH II 135(4)

$$2. \quad \Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \quad [\alpha \neq 0, -1, -2, \dots]$$

EH II 135(5), LE 340(2)

$$3. \quad \begin{aligned} \Gamma(\alpha, x) - \Gamma(\alpha, x+y) &= \gamma(\alpha, x+y) - \gamma(\alpha, x) \\ &= e^{-x} x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k [1 - e^{-y} e_k(y)] \Gamma(1-\alpha+k)}{x^k \Gamma(1-\alpha)} \\ e_k(x) &= \sum_{m=0}^k \frac{x^m}{m!} \quad [|y| < |x|] \end{aligned} \quad \text{EH II 139(2)}$$

$$4. \quad \gamma(\alpha, x) = \Gamma(\alpha) e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} x^{\frac{1}{2}n} I_{n+\alpha}(2\sqrt{x}) \sum_{m=0}^n \frac{(-1)^m}{m!} \quad [x \neq 0, \quad \alpha \neq 0, \quad -1, -2, \dots]$$

EH II 139(3)

$$5. \quad \Gamma(\alpha, x) = e^{-x} x^{\alpha} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{n+1} \quad [x > 0] \quad \text{EH II 140(5)}$$

$$\mathbf{8.355} \quad \Gamma(\alpha, x) \gamma(\alpha, y) = e^{-x-y} (xy)^{\alpha} \sum_{n=0}^{\infty} \frac{n! \Gamma(\alpha)}{(n+1) \Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\alpha}(y)$$

[y > 0, x ≥ y, α ≠ 0, -1, ...] EH II 139(4)

8.356 Functional relations:

$$1.^{11} \quad \gamma(\alpha+1, x) = \alpha \gamma(\alpha, x) - x^{\alpha} e^{-x} \quad \text{EH II 134(2)}$$

$$2. \quad \Gamma(\alpha+1, x) = \alpha \Gamma(\alpha, x) + x^{\alpha} e^{-x} \quad \text{EH II 134(3)}$$

$$3. \quad \Gamma(\alpha, x) + \gamma(\alpha, x) = \Gamma(\alpha) \quad \text{EH II 134(1)}$$

$$4. \quad \frac{d\gamma(\alpha, x)}{dx} = -\frac{d\Gamma(\alpha, x)}{dx} = x^{\alpha-1} e^{-x} \quad \text{EH II 135(8)}$$

$$5. \quad \frac{\Gamma(\alpha+n, x)}{\Gamma(\alpha+n)} = \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)} + e^{-x} \sum_{s=0}^{n-1} \frac{x^{\alpha+s}}{\Gamma(\alpha+s+1)} \quad \text{NH 4(3)}$$

$$6.^{11} \quad \Gamma(\alpha) \Gamma(\alpha+n, x) - \Gamma(\alpha+n) \Gamma(\alpha, x) = \Gamma(\alpha+n) \gamma(\alpha, x) - \Gamma(\alpha) \gamma(\alpha+n, x) \quad \text{NH 5}$$

$$7.* \quad \begin{aligned} \Gamma(a+k, x) &= (a+k-1) \Gamma(a+k-1, x) + x^{a+k-1} e^{-x} \\ &= \frac{1}{a+k} [\Gamma(a+k+1, x) - x^{a+k} e^{-x}] \end{aligned}$$

$$8.* \quad \begin{aligned} \Gamma(a-k, x) &= (a-k-1) \Gamma(a-k-1, x) + x^{a-k-1} e^{-x} \\ &= \frac{1}{a-k} [\Gamma(a-k+1, x) - x^{a-k} e^{-x}] \end{aligned}$$

$$9.* \quad \begin{aligned} \gamma(a+k, x) &= (a+k-1) \gamma(a+k-1, x) - x^{a+k-1} e^{-x} \\ &= \frac{1}{a+k} [\Gamma(a+k+1, x) + x^{a+k} e^{-x}] \end{aligned}$$

$$\begin{aligned}
 10.^* \quad \gamma(a-k, x) &= (a-k-1)\gamma(a-k-1, x) - x^{a-k-1}e^{-x} \\
 &= \frac{1}{a-k} [\gamma(a-k+1, x) + x^{a-k}e^{-x}]
 \end{aligned}$$

8.357 Asymptotic representation for large values of $|x|$:

$$\begin{aligned}
 1. \quad \Gamma(\alpha, x) &= x^{\alpha-1}e^{-x} \left[\sum_{m=0}^{M-1} \frac{(-1)^m \Gamma(1-\alpha+m)}{x^m \Gamma(1-\alpha)} + O(|x|^{-M}) \right] \\
 &\quad \left[|x| \rightarrow \infty, -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}, \quad M = 1, 2, \dots \right] \quad \text{EH II 135(6), NH 37(7), LE 340(3)}
 \end{aligned}$$

8.358 Representation as a continued fraction:

$$\Gamma(\alpha, x) = \frac{e^{-x}x^\alpha}{x + \frac{1-\alpha}{1 + \frac{1}{x + \frac{2-\alpha}{1 + \frac{2}{x + \frac{3-\alpha}{1 + \dots}}}}}} \quad \text{EH II 136(13), NH 42(9)}$$

8.359 Relationships with other functions:

$$\begin{aligned}
 1. \quad \Gamma(0, x) &= -\text{Ei}(-x) && \text{EH II 143(1)} \\
 2. \quad \Gamma\left(0, \ln \frac{1}{x}\right) &= -\text{li}(x) && \text{EH II 143(2)} \\
 3. \quad \Gamma\left(\frac{1}{2}, x^2\right) &= \sqrt{\pi} - \sqrt{\pi} \Phi(x) && \text{EH II 147(2)} \\
 4.^{11} \quad \gamma\left(\frac{1}{2}, x^2\right) &= \sqrt{\pi} \Phi(x) && \text{EH II 147(1)}
 \end{aligned}$$

8.36 The psi function $\psi(x)$

8.360 Definition:

$$1. \quad \psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

8.361 Integral representations:

$$\begin{aligned}
 1.^8 \quad \psi(z) &= \frac{d \ln \Gamma(z)}{dz} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt && [\text{Re } z > 0] \quad \text{NH 183(1), WH} \\
 2. \quad \psi(z) &= \int_0^\infty \left\{ e^{-t} - \frac{1}{(1+t)^z} \right\} \frac{dt}{t} && [\text{Re } z > 0] \quad \text{NH 184(7), WH} \\
 3. \quad \psi(z) &= \ln z - \frac{1}{2z} - 2 \int_0^\infty \frac{t dt}{(t^2 + z^2)(e^{2\pi t} - 1)} && [\text{Re } z > 0] \quad \text{WH} \\
 4. \quad \psi(z) &= \int_0^1 \left(\frac{1}{-\ln t} - \frac{t^{z-1}}{1-t} \right) dt && [\text{Re } z > 0] \quad \text{WH}
 \end{aligned}$$

5. $\psi(z) = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt - C,$ WH
6. $\psi(z) = \int_0^\infty \left\{ (1+t)^{-1} - (1+t)^{-z} \right\} \frac{dt}{t} - C,$ [Re $z > 0$] WH
7. $\psi(z) = \int_0^1 \frac{t^{z-1} - 1}{t - 1} dt - C$ FI II 796, WH
8. $\psi(z) = \ln z + \int_0^\infty e^{-tz} \left[\frac{1}{t} - \frac{1}{1 - e^{-t}} \right] dt$ [Re $z > 0$] MO 4

See also **3.244** 3, **3.311** 6, **3.317** 1, **3.457**, **3.458** 2, **3.471** 14, **4.253** 1 and 6, **4.275** 2, **4.281** 4, **4.482** 5.

For integrals of the psi function, see **6.46**, **6.47**.

Series representation

8.362

1. $\psi(x) = -C - \sum_{k=0}^\infty \left(\frac{1}{x+k} - \frac{1}{k+1} \right)$ FI II 799(26), KU 26(1)
 $= -C - \frac{1}{x} + x \sum_{k=1}^\infty \frac{1}{k(x+k)}$ FI II 495
2. $\psi(x) = \ln x - \sum_{k=0}^\infty \left[\frac{1}{x+k} - \ln \left(1 + \frac{1}{x+k} \right) \right]$ MO 4
3. $\psi(x) = -C + \frac{\pi^2}{6}(x-1) - (x-1) \sum_{k=1}^\infty \left(\frac{1}{k+1} - \frac{1}{x+k} \right) \sum_{n=0}^{k-1} \frac{1}{x+n}$ NH 54(12)

8.363

1. $\psi(x+1) = -C + \sum_{k=2}^\infty (-1)^k \zeta(k) x^{k-1}$ NH 37(5)
2. $\psi(x+1) = \frac{1}{2x} - \frac{\pi}{2} \cot \pi x - \frac{x^2}{1-x^2} - C + \sum_{k=1}^\infty [1 - \zeta(2k+1)] x^{2k}$ NH 38(10)
3. $\psi(x) - \psi(y) = \sum_{k=0}^\infty \left(\frac{1}{y+k} - \frac{1}{x+k} \right)$
(see also **3.219**, **3.231** 5, **3.311** 7, **3.688** 20, **4.253** 1, **4.295** 37) NH 99(3)
4. $\psi(x+iy) - \psi(x-iy) = \sum_{k=0}^\infty \frac{2yi}{y^2 + (x+k)^2}$
5. $\psi\left(\frac{p}{q}\right) = -C + \sum_{k=0}^\infty \left(\frac{1}{k+1} - \frac{q}{p+kq} \right)$ (see also **3.244** 3) NH 29(1)

$$6.^8 \quad \psi\left(\frac{p}{q}\right) = -C - \ln(2q) - \frac{\pi}{2} \cot \frac{p\pi}{q} + 2 \sum_{k=1}^{\left[\frac{q+1}{2}\right]-1} \left[\cos \frac{2kp\pi}{q} \ln \sin \frac{k\pi}{q} \right] \\ [q = 2, 3, \dots, p = 1, 2, \dots, q-1] \quad \text{MO 4, EH I 19(29)}$$

$$7. \quad \psi\left(\frac{p}{q}\right) - \psi\left(\frac{p-1}{q}\right) = q \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(p+kq)^n - 1} \quad \text{NH 59(3)}$$

$$8. \quad \psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, x) \quad \text{NH 37(1)}$$

Infinite-product representation

8.364

$$1. \quad e^{\psi(x)} = x \prod_{k=0}^{\infty} \left(1 + \frac{1}{x+k}\right) e^{-\frac{1}{x+k}} \quad \text{NH 65(12)}$$

$$2. \quad e^{y\psi(x)} = \frac{\Gamma(x+y)}{\Gamma(x)} \prod_{k=0}^{\infty} \left(1 + \frac{y}{x+k}\right) e^{-\frac{y}{x+k}} \quad \text{NH 65(11)}$$

See also **8.37**.

- For a connection with Riemann's zeta function, see **9.533** 2.
- For a connection with the gamma function, see **4.325** 12 and **4.352** 1.
- For a connection with the beta function, see **4.253** 1.
- For series of psi functions, see **8.403** 2, **8.446**, and **8.447** 3 (Bessel functions), **8.761** (derivatives of associated Legendre functions with respect to the degree), **9.153**, **9.154** (hypergeometric function), **9.237** (confluent hypergeometric function).
- For integrals containing psi functions, see **6.46–6.47**.

8.365 Functional relations:

$$1. \quad \psi(x+1) = \psi(x) + \frac{1}{x} \quad \text{JA}$$

$$2. \quad \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2\beta(x) \quad (\text{cf. } \mathbf{8.37} \text{ 0})$$

$$3. \quad \psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k} \quad \text{GA 154(64)a}$$

$$4. \quad \psi(n+1) = -C + \sum_{k=1}^n \frac{1}{k} \quad \text{MO 4}$$

$$5. \quad \lim_{n \rightarrow \infty} [\psi(z+n) - \ln n] = 0 \quad \text{MO 3}$$

$$6. \quad \psi(nz) = \frac{1}{n} \sum_{k=0}^{n-1} \psi\left(z + \frac{k}{n}\right) + \ln n \quad [n = 2, 3, 4, \dots] \quad \text{MO 3}$$

7. $\psi(x - n) = \psi(x) - \sum_{k=1}^n \frac{1}{x - k}$
8. $\psi(1 - z) = \psi(z) + \pi \cot \pi z$ GA 155(68)a
9. $\psi\left(\frac{1}{2} + z\right) = \psi\left(\frac{1}{2} - z\right) + \pi \tan \pi z$ JA
10. $\psi\left(\frac{3}{4} - n\right) = \psi\left(\frac{1}{4} + n\right) + \pi$ $[n = 0, \pm 1, \pm 2, \dots]$

8.366 Particular values

1. $\psi(1) = -C$ (cf. **8.367** 1)
2. $\psi\left(\frac{1}{2}\right) = -C - 2 \ln 2 = -1.963\,510\,026\dots$ GA 155a
3. $\psi\left(\frac{1}{2} \pm n\right) = -C + 2 \left[\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right]$ JA
4. $\psi\left(\frac{1}{4}\right) = -C - \frac{\pi}{2} - 3 \ln 2$ GA 157a
5. $\psi\left(\frac{3}{4}\right) = -C + \frac{\pi}{2} - 3 \ln 2$ GA 157a
6. $\psi\left(\frac{1}{3}\right) = -C - \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3$ GA 157a
7. $\psi\left(\frac{2}{3}\right) = -C + \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3$ GA 157a
8. $\psi'(1) = \frac{\pi^2}{6} = 1.644\,934\,066\,848\dots$ JA
9. $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} = 4.934\,802\,200\,5\dots$ JA
10. $\psi'(-n) = \infty$ $[n \text{ is a natural number}]$ JA
11. $\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}$ $[n \text{ is a natural number}]$ JA
12. $\psi'\left(\frac{1}{2} + n\right) = \frac{\pi^2}{2} - 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}$ $[n \text{ is a natural number}]$ JA
13. $\psi'\left(\frac{1}{2} - n\right) = \frac{\pi^2}{2} + 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}$ $[n \text{ is a natural number}]$ JA

8.367 Euler's constant (also denoted by γ):

1. $C = -\psi(1) = 0.577\,215\,664\,90\dots$ FI II 319, 795
2. $C = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^{n-1} \frac{1}{k} - \ln n \right]$ FI II 801a
3. $C = \lim_{x \rightarrow 1+0} \left[\zeta(x) - \frac{1}{x-1} \right]$ FI II 804

Integral representations:

$$4. \quad C = -\int_0^\infty e^{-t} \ln t \, dt \quad \text{FI II 807}$$

$$5. \quad C = -\int_0^1 \ln \left(\ln \frac{1}{t} \right) dt \quad \text{FI II 807}$$

$$6. \quad C = \int_0^1 \left[\frac{1}{\ln t} + \frac{1}{1-t} \right] dt \quad \text{DW}$$

$$7. \quad C = -\int_0^\infty \left[\cos t - \frac{1}{1+t} \right] \frac{dt}{t} \quad \text{MO 10}$$

$$8. \quad C = 1 - \int_0^\infty \left[\frac{\sin t}{t} - \frac{1}{1+t} \right] \frac{dt}{t} \quad \text{MO 10}$$

$$9. \quad C = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t} \right] \frac{dt}{t} \quad \text{FI II 795, 802}$$

$$10. \quad C = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t^2} \right] \frac{dt}{t} \quad \text{DW, MO 10}$$

$$11. \quad C = \int_0^\infty \left[\frac{1}{e^t - 1} - \frac{1}{te^t} \right] dt \quad \text{DW}$$

$$12. \quad C = \int_0^1 (1 - e^{-t}) \frac{dt}{t} - \int_1^\infty \frac{e^{-t}}{t} dt \quad \text{FI II 802}$$

See also **8.361** 5–**8.361** 7, **3.311** 6, **3.435** 3 and 4, **3.476** 2, **3.481** 1 and 2, **3.951** 10, **4.283** 9, **4.331** 1, **4.421** 1, **4.424** 1, **4.553**, **4.572**, **6.234**, **6.264** 1, **6.468**.

13. Asymptotic expansions

$$C = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \dots$$

$$\dots + \frac{B_{2r}}{2r} \frac{1}{n^{2r}} + \frac{B_{2r+2}}{2(r+1)} \frac{\theta}{n^{2r+2}} \quad [0 < \theta < 1] \quad \text{FI II 827}$$

8.37 The function $\beta(x)$

8.370 Definition:

$$\beta(x) = \frac{1}{2} \left[\psi \left(\frac{x+1}{2} \right) - \psi \left(\frac{x}{2} \right) \right] \quad \text{NH 16(13)}$$

8.371 Integral representations:

$$1.^3 \quad \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt \quad [\operatorname{Re} x > 0] \quad \text{WH}$$

$$2. \quad \beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt \quad [\operatorname{Re} x > 0] \quad \text{MO 4}$$

$$3. \quad \beta \left(\frac{x+1}{2} \right) = \int_0^\infty \frac{e^{-xt}}{\cosh t} dt \quad [\operatorname{Re} x > -1]$$

See also **3.241** 1, **3.251** 7, **3.522** 2 and 4, **3.623** 2 and 3, **4.282** 2, **4.389** 3, **4.532** 1 and 3.

Series representation

8.372

$$1.^7 \quad \beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k} \quad [-x \notin \mathbb{N}] \quad \text{NH 37, 101(1)}$$

$$2.^7 \quad \beta(x) = \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)} \quad [-x \notin \mathbb{N}] \quad \text{NH 101(2)}$$

$$3.^8 \quad \beta(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{x(x+1)\dots(x+k)} \frac{1}{2^k} \quad [-x \notin \mathbb{N}]$$

[β has simple poles at $x = -n$ with residue $(-1)^n$] NH 246(7)

8.373

$$1.^6 \quad \beta(x+1) = \ln 2 + \sum_{k=1}^{\infty} (-1)^k (1-2^{-k}) \zeta(k+1) x^k \quad [|x| < 1] \quad \text{NH 37(5)}$$

$$2.^6 \quad \beta(x+1) = \ln 2 - 1 + \frac{1}{2x} - \frac{\pi}{2 \sin \pi x} + \frac{1}{1-x^2} - \sum_{k=1}^{\infty} [1 - (1-2^{-2k}) \zeta(2k+1)] x^{2k}$$

[$0 < |x| < 2$; $x \neq \pm 1$] NH 38(11)

$$8.374 \quad \frac{d^n}{dx^n} \beta(x) = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{n+1}} \quad [-x \in \mathbb{N}] \quad \text{NH 37(2)}$$

8.375 Representation in the form of a finite sum:

$$1.^6 \quad \beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin \frac{p\pi}{q}} - \sum_{k=0}^{\lfloor \frac{q-1}{2} \rfloor} \cos \frac{p(2k+1)\pi}{q} \ln \sin \frac{(2k+1)\pi}{2q}$$

[$q = 2, 3, \dots, p = 1, 2, 3, \dots, q-1$] (see also 8.362 5–7) NH 23(9)

$$2. \quad \beta(n) = (-1)^{n+1} \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k}$$

Functional relations

$$8.376 \quad \sum_{k=0}^{2n} (-1)^k \beta\left(\frac{x+k}{2n+1}\right) = (2n+1) \beta(x) \quad \text{NH 19}$$

$$8.377 \quad \sum_{k=1}^n \beta(2^k x) = \psi(2^n x) - \psi(x) - n \ln 2 \quad \text{NH 20(10)}$$

8.38 The beta function (Euler's integral of the first kind): $B(x, y)$

Integral representation

8.380

$$\begin{aligned} 1. \quad B(x, y) &= \int_0^1 t^{x-1} (1-t)^{y-1} dt^* \\ &= 2 \int_0^1 t^{2x-1} (1-t^2)^{y-1} dt \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0] \end{aligned} \quad \text{FI II 774(1)}$$

$$2. \quad B(x, y) = 2 \int_0^{\pi/2} \sin^{2x-1} \varphi \cos^{2y-1} \varphi d\varphi \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0] \quad \text{KU 10}$$

$$3. \quad B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt = 2 \int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}} dt \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0] \quad \text{FI II 775}$$

$$4. \quad B(x, y) = 2^{2-y-x} \int_{-1}^1 \frac{(1+t)^{2x-1} (1-t)^{2y-1}}{(1+t^2)^{x+y}} dt \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0] \quad \text{MO 7}$$

$$5. \quad B(x, y) = \int_0^1 \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt = \int_1^\infty \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0] \quad \text{BI (1)(15)}$$

$$\begin{aligned} 6. \quad B(x, y) &= \frac{1}{2^{x+y-1}} \int_0^1 \left[(1+t)^{x-1} (1-t)^{y-1} + (1+t)^{y-1} (1-t)^{x-1} \right] dt \\ & \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0] \end{aligned} \quad \text{BI (1)(15)}$$

$$\begin{aligned} 7. \quad B(x, y) &= z^y (1+z)^x \int_0^1 \frac{t^{x-1} (1-t)^{y-1}}{(t+z)^{x+y}} dt \\ & \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad 0 > z > -1, \quad \operatorname{Re}(x+y) < 1] \end{aligned} \quad \text{NH 163(8)}$$

$$\begin{aligned} 8. \quad B(x, y) &= z^y (1+z)^x \int_0^{\pi/2} \frac{\cos^{2x-1} \varphi \sin^{2y-1} \varphi}{(z + \cos^2 \varphi)^{x+y}} d\varphi \\ & \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad 0 > z > -1, \quad \operatorname{Re}(x+y) < 1] \end{aligned} \quad \text{NH 163(8)}$$

See also **3.196** 3, **3.198**, **3.199**, **3.215**, **3.238** 3, **3.251** 1–3, 11, **3.253**, **3.312** 1, **3.512** 1 and 2, **3.541** 1, **3.542** 1, **3.621** 5, **3.623** 1, **3.631** 1, 8, 9, **3.632** 2, **3.633** 1, 4, **3.634** 1, 2, **3.637**, **3.642** 1, **3.667** 8, **3.681** 2.

$$9. \quad B(x, x) = \frac{1}{2^{2x-2}} \int_0^1 (1-t^2)^{x-1} dt = \frac{1}{2^{2x-1}} \int_0^1 \frac{(1-t)^{x-1}}{\sqrt{t}} dt$$

See **8.384** 4, **8.382** 3, and also **3.621** 1, **3.642** 2, **3.665** 1, **3.821** 6, **3.839** 6.

$$10. \quad B(x+y, x-y) = 4^{1-x} \int_0^\infty \frac{\cosh 2yt}{\cosh^{2x} t} dt \quad [\operatorname{Re} x > |\operatorname{Re} y|, \quad \operatorname{Re} x > 0] \quad \text{MO 9}$$

$$\begin{aligned} 11. \quad B\left(x, \frac{y}{z}\right) &= z \int_0^1 (1-t^z)^{x-1} t^{y-1} dt \\ & \quad \left[\operatorname{Re} z > 0, \quad \operatorname{Re} \frac{y}{z} > 0, \quad \operatorname{Re} x > 0 \right] \end{aligned} \quad \text{FI II 787a}$$

*This equation is used as the definition of the function $B(x, y)$.

8.381

$$1. \quad \int_{-\infty}^{\infty} \frac{dt}{(a+it)^x(b-it)^y} = \frac{2\pi(a+b)^{1-x-y}}{(x+y-1)B(x, y)} \quad [a > 0, \quad b > 0; \quad x \text{ and } y \text{ are real, } \quad x+y > 1] \quad \text{MO 7}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{dt}{(a-it)^x(b-it)^y} = 0 \quad [a > 0, \quad b > 0; \quad x \text{ and } y \text{ are real, } \quad x+y > 1] \quad \text{MO 7}$$

$$3. \quad B(x+iy, x-iy) = 2^{1-2x} \alpha e^{-2i\gamma y} \int_{-\infty}^{\infty} \frac{e^{2i\alpha y t} dt}{\cosh^{2x}(\alpha t - \gamma)} \quad [y, \alpha, \gamma \text{ are real, } \quad \alpha > 0; \quad \operatorname{Re} x > 0] \quad \text{MI 8a}$$

For an integral representation of $\ln B(x, y)$, see **3.428 7**.

$$\begin{aligned} 4. \quad \frac{1}{B(x, y)} &= \frac{2^{x+y-1}(x+y-1)}{\pi} \int_0^{\pi/2} \cos[(x-y)t] \cos^{x+y-2} t \, dt & \text{NH 158(5)a} \\ &= \frac{2^{x+y-2}(x+y-1)}{\pi \cos[(x-y)\frac{\pi}{2}]} \int_0^{\pi} \cos[(x-y)t] \sin^{x+y-2} t \, dt & \text{NH 159(8)a} \\ &= \frac{2^{x+y-2}(x+y-1)}{\pi \sin[(x-y)\frac{\pi}{2}]} \int_0^{\pi} \sin[(x-y)t] \sin^{x+y-2} t \, dt & \text{NH 159(9)a} \end{aligned}$$

Series representation**8.382**

$$1. \quad B(x, y) = \frac{1}{y} \sum_{n=0}^{\infty} (-1)^n y \frac{(y-1) \dots (y-n)}{n!(x+n)} \quad [y > 0] \quad \text{WH}$$

$$2. \quad \ln B\left(\frac{1+x}{2}, \frac{1}{2}\right) \ln \sqrt{2\pi} + \frac{1}{2} \left[\ln\left(\frac{\tan \frac{\pi x}{2}}{x}\right) - \ln\left(\frac{1+x}{1-x}\right) \right] + \sum_{k=0}^{\infty} \frac{1 - (1-2^{-2k}) \zeta(2k+1)}{2k+1} x^{2k+1} \quad [|x| < 2] \quad \text{NH 39(17)}$$

$$3. \quad B\left(z, \frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{1}{z+k} + \frac{1}{z} \quad (\text{see also } \mathbf{8.384} \text{ and } \mathbf{8.380 9}) \quad \text{WH}$$

8.383 Infinite-product representation:

$$(x+y+1)B(x+1, y+1) = \prod_{k=1}^{\infty} \frac{k(x+y+k)}{(x+k)(y+k)} \quad [x, \quad y \neq -1, \quad -2, \dots] \quad \text{MO 2}$$

8.384 Functional relations involving the beta function:

$$1. \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y, x) \quad \text{FI II 779}$$

$$2. \quad B(x, y)B(x+y, z) = B(y, z)B(y+z, x) \quad \text{MO 6}$$

$$3. \quad \sum_{k=0}^{\infty} B(x, y+k) = B(x-1, y) \quad \text{WH}$$

$$4. \quad B(x, x) = 2^{1-2x} B\left(\frac{1}{2}, x\right) \quad (\text{see also } \mathbf{8.380} \text{ 9 and } \mathbf{8.382} \text{ 3})$$

FI II 784

$$5. \quad B(x, x) B\left(x + \frac{1}{2}, x + \frac{1}{2}\right) = \frac{\pi}{2^{4x-1} x} \quad \text{WH}$$

$$6. \quad \frac{1}{B(n, m)} = m \binom{n+m-1}{n-1} = n \binom{n+m-1}{m-1} \quad [m \text{ and } n \text{ are natural numbers}]$$

For a connection with the psi function, see **4.253** 1.

8.39 The incomplete beta function $B_x(p, q)$

$$\mathbf{8.391}^7 \quad B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt = \frac{x^p}{p} {}_2F_1(p, 1-q; p+1; x) \quad \text{ET I 373}$$

$$\mathbf{8.392} \quad I_x(p, q) = \frac{B_x(p, q)}{B(p, q)} \quad \text{ET II 429}$$

8.4–8.5 Bessel Functions and Functions Associated with Them

8.40 Definitions

8.401 Bessel functions $Z_\nu(z)$ are solutions of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z} \frac{d Z_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0 \quad \text{KU 37(1)}$$

Special types of Bessel functions are what are called Bessel functions of the first kind $J_\nu(z)$, Bessel functions of the second kind $Y_\nu(z)$ (also called Neumann functions and often written $N_\nu(z)$), and Bessel functions of the third kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ (also called Hankel's functions).

$$\mathbf{8.402} \quad J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)} \quad [|\arg z| < \pi] \quad \text{KU 55(1)}$$

8.403

$$1. \quad Y_\nu(z) = \frac{1}{\sin \nu \pi} [\cos \nu \pi J_\nu(z) - J_{-\nu}(z)] \quad [\text{for non-integer } \nu, \quad |\arg z| < \pi]$$

KU 41(3)

$$\begin{aligned}
2. \quad \pi Y_n(z) &= 2 J_n(z) \ln \frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} \\
&\quad - \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! (k+n)!} \left(\frac{z}{2}\right)^{n+2k} [\psi(k+1) + \psi(k+n+1)] \\
&\quad = 2 J_n(z) \left(\ln \frac{z}{2} + C\right) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} \\
&\quad - \left(\frac{z}{2}\right)^n \frac{1}{n!} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{n+2k}}{k! (k+n)!} \left[\sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^k \frac{1}{m} \right] \\
&\quad \quad \quad [n+1 \text{ a natural number, } |\arg z| < \pi] \\
&\quad \quad \quad \text{KU 44, WA 75(3)a}
\end{aligned}$$

8.404

$$\begin{aligned}
1. \quad Y_{-n}(z) &= (-1)^n Y_n(z) & [n \text{ is a natural number}] & \text{KU 41(2)} \\
2. \quad J_{-n}(z) &= (-1)^n J_n(z) & [n \text{ is a natural number}] & \text{KU 41(2)}
\end{aligned}$$

8.405⁷

$$\begin{aligned}
1. \quad H_{\nu}^{(1)}(z) &= J_{\nu}(z) + i Y_{\nu}(z) & \text{KU 44(1)} \\
2. \quad H_{\nu}^{(2)}(z) &= J_{\nu}(z) - i Y_{\nu}(z) & \text{KU 44(1)}
\end{aligned}$$

In all relationships that hold for an arbitrary Bessel function $Z_{\nu}(z)$, that is, for the functions $J_{\nu}(z)$, $Y_{\nu}(z)$, and linear combinations of them, for example, $H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$, we shall write simply the letter Z instead of the letters J , Y , $H^{(1)}$, and $H^{(2)}$.

Modified Bessel functions of imaginary argument $I_{\nu}(z)$ and $K_{\nu}(z)$ **8.406**

$$\begin{aligned}
1. \quad I_{\nu}(z) &= e^{-\frac{\pi}{2}\nu i} J_{\nu}(e^{\frac{\pi}{2}i} z) & \left[-\pi < \arg z \leq \frac{\pi}{2}\right] & \text{WA 92} \\
2. \quad I_{\nu}(z) &= e^{\frac{3}{2}\pi\nu i} J_{\nu}(e^{-\frac{3}{2}\pi i} z) & \left[\frac{\pi}{2} < \arg z \leq \pi\right] & \text{WA 92}
\end{aligned}$$

For integer ν ,

$$3. \quad I_n(z) = i^{-n} J_n(iz) \quad \text{KU 46(1)}$$

8.407

$$\begin{aligned}
1.^8 \quad K_{\nu}(z) &= \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_{\nu}^{(1)}(ze^{\frac{1}{2}\pi i}) & [-\pi < \arg z \leq \frac{1}{2}\pi] \\
2.^8 \quad K_{\nu}(z) &= \frac{-\pi i}{2} e^{-\frac{\pi}{2}\nu i} H_{-\nu}^{(2)}(ze^{-\frac{1}{2}\pi i}) & [-\frac{1}{2}\pi < \arg z \leq \pi] & \text{WA 92(8)}
\end{aligned}$$

For the differential equation defining these functions, see **8.494**.

8.41 Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$

8.411

$$\begin{aligned} 1.^{11} \quad J_n(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ni\theta + iz \sin \theta} d\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin \theta) d\theta \quad [n = 0, 1, 2, \dots] \end{aligned} \quad \text{WH}$$

$$2. \quad J_{2n}(z) = \frac{1}{\pi} \int_0^{\pi} \cos 2n\theta \cos(z \sin \theta) d\theta = \frac{2}{\pi} \int_0^{\pi/2} \cos 2n\theta \cos(z \sin \theta) d\theta$$

[n an integer] WA 30(7)

$$\begin{aligned} 3.^{11} \quad J_{2n+1}(z) &= \frac{1}{\pi} \int_0^{\pi} \sin(2n+1)\theta \sin(z \sin \theta) d\theta \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin(2n+1)\theta \sin(z \sin \theta) d\theta \quad [n \text{ an integer}] \end{aligned} \quad \text{WA 30(6)}$$

$$4. \quad J_\nu(z) = 2 \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\pi/2} \sin^{2\nu} \theta \cos(z \cos \theta) d\theta$$

[$\text{Re } \nu > -\frac{1}{2}$] WH

$$5. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\pi} \sin^{2\nu} \theta \cos(z \cos \theta) d\theta \quad [\text{Re } \nu > -\frac{1}{2}]$$

$$6. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-\pi/2}^{\pi/2} \cos(z \sin \theta) \cos^{2\nu} \theta d\theta$$

[$\text{Re } \nu > -\frac{1}{2}$] KU 65(5), WA 35(4)a

$$7. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\pi} e^{\pm iz \cos \varphi} \sin^{2\nu} \varphi d\varphi \quad [\text{Re } (\nu + \frac{1}{2}) > 0] \quad \text{WH}$$

$$8. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} \cos zt dt \quad [\text{Re } \nu > -\frac{1}{2}] \quad \text{KU 65(6), WH}$$

$$9. \quad J_\nu(x) = 2 \frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2}\right)} \int_1^{\infty} \frac{\sin xt}{(t^2 - 1)^{\nu+\frac{1}{2}}} dt \quad \left[-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}, \quad x > 0\right] \quad \text{MO 37}$$

$$10. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{izt} (1-t^2)^{\nu-\frac{1}{2}} dt \quad [\text{Re } \nu > -\frac{1}{2}] \quad \text{WA 34(3)}$$

$$11. \quad J_\nu(x) = \frac{2}{\pi} \int_0^{\infty} \sin\left(x \cosh t - \frac{\nu\pi}{2}\right) \cosh \nu t dt \quad \text{WA 199(12)}$$

$$12. \quad J_\nu(z) = \frac{2^{\nu+1} z^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\pi/2} \frac{\left(\cos^{\nu-\frac{1}{2}} \theta\right) \sin\left(z - \nu\theta + \frac{1}{2}\theta\right)}{\sin^{2\nu+1} \theta} e^{-2z \cot \theta} d\theta$$

[$|\arg z| < \frac{\pi}{2}, \quad \text{Re } (\nu + \frac{1}{2}) > 0$] WH

$$13.^{10} \quad J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta - \frac{\sin \nu\pi}{\pi} \int_0^\infty e^{-\nu\theta - z \sinh \theta} d\theta$$

[Re $z > 0$] WA 195(4)

$$14. \quad J_\nu(z) = \frac{e^{\pm \nu\pi i}}{\pi} \left[\int_0^\pi \cos(\nu\theta + z \sin \theta) d\theta - \sin \nu\pi \int_0^\infty e^{-\nu\theta + z \sinh \theta} d\theta \right]$$

[for $\frac{\pi}{2} < |\arg z| < \pi$, with the upper sign taken for $|\arg z| > \frac{\pi}{2}$
and the lower sign taken for $|\arg z| < -\frac{\pi}{2}$]
WH

8.412

$$1. \quad J_\nu(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp \left[\frac{z}{2} \left(t - \frac{1}{t} \right) \right] dt \quad \left[|\arg z| < \frac{\pi}{2} \right] \quad \text{WH, WA 195(2)}$$

$$2. \quad J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp \left(t - \frac{z^2}{4t} \right) dt \quad \text{WA 195(1)}$$

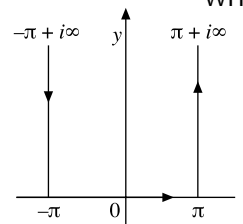
$$3.^8 \quad J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k!} \int_{-\infty}^{(0+)} e^t t^{-\nu-k-1} dt \quad \text{WA 195(1)}$$

$$4. \quad J_\nu(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-t)}{\Gamma(\nu+t+1)} \left(\frac{x}{2} \right)^{\nu+2t} dt \quad [\text{Re } \nu > 0, \quad x > 0] \quad \text{WA 214(7)}$$

$$5.^7 \quad J_\nu(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \left(\frac{z}{2}\right)^\nu}{2\pi i \Gamma\left(\frac{1}{2}\right)} \int_A^{(1+, -1-)} (t^2 - 1)^{\nu-\frac{1}{2}} \cos(zt) dt$$

[$\nu \neq \frac{1}{2}, \frac{3}{2}, \dots$; The point A falls to the right of the point $t = 1$,
and $\arg(t-1) = \arg(t+1) = 0$ at the point A]
WH

$$6.^8 \quad J_\nu(z) = \frac{1}{2\pi} \int_{-\pi+\infty i}^{\pi+\infty i} e^{-iz \sin \theta + i\nu\theta} d\theta \quad [\text{Re } z > 0]$$



The path of integration being taken around the semi-infinite strip $y \geq 0, -\pi \leq x \leq \pi$.

$$8.413^8 \quad \frac{J_\nu \left(\sqrt{z^2 + \zeta^2} \right)}{(z^2 - \zeta^2)^{\frac{\nu}{2}}} = \frac{1}{\pi(z + \zeta)^\nu} \left\{ \int_0^\infty e^{\zeta \cos t} \cos(z \sin t - \nu t) dt \right. \\ \left. - \sin \nu\pi \int_0^\infty \exp(-z \sinh t - \zeta \cosh t - \nu t) dt \right\}$$

[Re $(z + \zeta) > 0$] MO 40

$$8.414 \quad \int_{2x}^\infty \frac{J_0(t)}{t} dt = \frac{1}{4\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-t)}{t \Gamma(1+t)} x^{2t} dt \quad [x > 0] \quad \text{MO 41}$$

See **3.715** 2, 9, 10, 13, 14, 19–21, **3.865** 1, 2, 4, **3.996** 4.

- For an integral representation of $J_0(z)$, see **3.714** 2, **3.753** 2, 3, and **4.124**.
- For an integral representation of $J_1(z)$, see **3.697**, **3.711**, **3.752** 2, and **3.753** 5.

8.415

1.
$$Y_0(x) = \frac{4}{\pi^2} \int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}} \sin(xt) dt - \frac{4}{\pi^2} \int_1^\infty \frac{\ln(t + \sqrt{t^2-1})}{\sqrt{t^2-1}} \sin(xt) dt$$

 $[x > 0]$ MO 37
2.
$$Y_\nu(x) = -2 \frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right) \Gamma\left(\frac{1}{2}\right)} \int_1^\infty \frac{\cos xt}{(t^2-1)^{\nu+\frac{1}{2}}} dt$$

 $\left[-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0\right]$
 KU 89(28)a, MO 38
3.
$$Y_\nu(x) = -\frac{2}{\pi} \int_0^\infty \cos\left(x \cosh t - \frac{\nu\pi}{2}\right) \cosh \nu t dt$$

 $[-1 < \operatorname{Re} \nu < 1, \quad x > 0]$ WA 199(13)
- 4.⁸
$$Y_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \nu\theta) d\theta - \frac{1}{\pi} \int_0^\infty (e^{\nu t} + e^{-\nu t} \cos \nu\pi) e^{-z \sinh t} dt$$

 $[\operatorname{Re} z > 0]$ WA 197(1)
5.
$$Y_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \left[\int_0^{\pi/2} \sin(z \sin \theta) \cos^{2\nu} \theta d\theta - \int_0^\infty e^{-z \sinh \theta} \cosh^{2\nu} \theta d\theta \right]$$

 $[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0]$ WA 181(5)a
6.
$$Y_\nu(z) = -\frac{2^{\nu+1} z^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}} \theta \cos\left(z - \nu\theta + \frac{1}{2}\theta\right)}{\sin^{2\nu+1} \theta} e^{-2z \cot \theta} d\theta$$

 $\left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re}\left(\nu + \frac{1}{2}\right) > 0\right]$
 WA 186(8)

For an integral representation of $Y_0(z)$, see **3.714** 3, **3.753** 4, **3.864**. See also **3.865** 3.

8.42 Integral representations of the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ **8.421**

1.
$$H_\nu^{(1)}(x) = \frac{e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^\infty e^{ix \cosh t - \nu t} dt$$

$$= \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_0^\infty e^{ix \cosh t} \cosh \nu t dt$$

 $[-1 < \operatorname{Re} \nu < 1, \quad x > 0]$ WA 199(10)
2.
$$H_\nu^{(2)}(x) = -\frac{e^{\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^\infty e^{-ix \cosh t - \nu t} dt$$

$$= -\frac{2e^{\frac{\nu\pi i}{2}}}{\pi i} \int_0^\infty e^{-ix \cosh t} \cosh \nu t dt$$

 $[-1 < \operatorname{Re} \nu < 1, \quad x > 0]$ WA 199(11)

$$3. \quad H_\nu^{(1)}(z) = -\frac{2^{\nu+1} i z^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t e^{i(z-\nu t+\frac{i}{2})}}{\sin^{2\nu+1} t} \exp(-2z \cot t) dt$$

[$\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0$] WA 186(5)

$$4. \quad H_\nu^{(2)}(z) = \frac{2^{\nu+1} i z^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t e^{-i(z-\nu t+\frac{i}{2})}}{\sin^{2\nu+1} t} \exp(-2z \cot t) dt$$

[$\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0$] WA 186(6)

$$5. \quad H_\nu^{(1)}(x) = -\frac{2i \left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \nu)} \int_1^\infty \frac{e^{ixt}}{(t^2 - 1)^{\nu+\frac{1}{2}}} dt$$

[$-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0$] WA 87(1)

$$6. \quad H_\nu^{(2)}(x) = \frac{2i \left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \nu)} \int_1^\infty \frac{e^{-ixt}}{(t^2 - 1)^{\nu+\frac{1}{2}}} dt$$

[$-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0$] WA 187(2)

$$7. \quad H_\nu^{(1)}(z) = -\frac{i}{\pi} e^{-\frac{1}{2} i \nu \pi} \int_0^\infty \exp \left[\frac{1}{2} i z \left(t + \frac{1}{t} \right) \right] t^{-\nu-1} dt$$

[$0 < \arg z < \pi$; or $\arg z = 0$ and $-1 < \operatorname{Re} \nu < 1$] MO 38

$$8. \quad H_\nu^{(1)}(xz) = -\frac{i}{\pi} e^{-\frac{1}{2} i \nu \pi} z^\nu \int_0^\infty \exp \left[\frac{1}{2} i x \left(t + \frac{z^2}{t} \right) \right] t^{-\nu-1} dt$$

[$0 < \arg z < \frac{\pi}{2}, \quad x > 0, \quad \operatorname{Re} \nu > -1$; or $\arg z = \frac{\pi}{2}, \quad x > 0$ and $-1 < \operatorname{Re} \nu < 1$] MO 38

$$9. \quad H_\nu^{(1)}(xz) = \sqrt{\frac{2}{\pi z}} \frac{x^\nu \exp \left[i \left(xz - \frac{\pi}{2} \nu - \frac{\pi}{4} \right) \right]}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty \left(1 + \frac{it}{2z} \right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}} e^{-xt} dt$$

[$\operatorname{Re} \nu > -\frac{1}{2}, \quad -\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi, \quad x > 0$] MO 39

$$10. \quad H_\nu^{(1)}(z) = \frac{-2ie^{-i\nu\pi} \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{iz \cosh t} \sinh^{2\nu} t dt$$

[$0 < \arg z < \pi, \quad \operatorname{Re} \nu > -\frac{1}{2}$ or $\arg z = 0$ and $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$] MO 38

$$11. \quad H_0^{(1)}(x) = -\frac{i}{\pi} \int_{-\infty}^\infty \frac{\exp(i\sqrt{x^2+t^2})}{\sqrt{x^2+t^2}} dt$$

[$x > 0$] MO 38

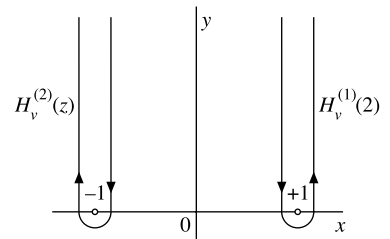
8.422

$$1. \quad H_\nu^{(1)}(z) = \frac{\Gamma(\frac{1}{2} - \nu) \left(\frac{z}{2}\right)^\nu}{\pi i \Gamma(\frac{1}{2})} \int_{1+\infty i}^{(1+)} e^{izt} (t^2 - 1)^{\nu-\frac{1}{2}} dt$$

[$-\pi < \arg z < 2\pi$] WA 183(4)

$$2. \quad H_\nu^{(2)}(z) = \frac{\Gamma(\frac{1}{2} - \nu) \left(\frac{z}{2}\right)^\nu}{\pi i \Gamma(\frac{1}{2})} \int_{-1+\infty i}^{(-1-)} e^{izt} (t^2 - 1)^{\nu-\frac{1}{2}} dt$$

[$-2\pi < \arg z < \pi$]



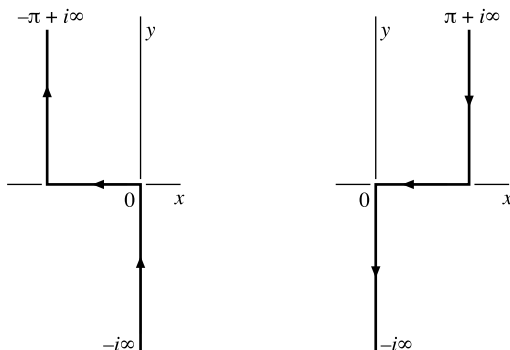
The paths of integration are shown in the drawing.

8.423

$$1. \quad H_{\nu}^{(1)}(z) = -\frac{1}{\pi} \int_{-\infty i}^{-\pi+\infty i} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0] \quad \text{WA 197(2)a}$$

$$2. \quad H_{\nu}^{(2)}(z) = -\frac{1}{\pi} \int_{\pi+\infty i}^{-\infty i} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0] \quad \text{WA 197(3)a}$$

The path of integration for **8.423 1** is shown in the left-hand drawing and for **8.423 2** in the right-hand drawing.

**8.424**

$$1. \quad H_{\nu}^{(1)}(z) J_{\nu}(\zeta) = \frac{1}{\pi i} \int_0^{\gamma+i\infty} \exp \left[\frac{1}{2} \left(t - \frac{z^2 + \zeta^2}{t} \right) \right] I_{\nu} \left(\frac{z\zeta}{t} \right) \frac{dt}{t} \quad [\gamma > 0, \quad \operatorname{Re} \nu > -1, \quad |\zeta| < |z|] \quad \text{MO 45}$$

$$2. \quad H_{\nu}^{(2)}(z) J_{\nu}(\zeta) = \frac{i}{\pi} \int_0^{\gamma-i\infty} \exp \left[\frac{1}{2} \left(t - \frac{z^2 + \zeta^2}{t} \right) \right] I_{\nu} \left(\frac{z\zeta}{t} \right) \frac{dt}{t} \quad [\gamma > 0, \quad \operatorname{Re} \nu > -1, \quad |\zeta| < |z|] \quad \text{MO 45}$$

8.43 Integral representations of the functions $I_{\nu}(z)$ and $K_{\nu}(z)$

The function $I_{\nu}(z)$

8.431

$$1. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{\pm zt} dt \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$2. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} \cosh zt dt \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$3. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi} e^{\pm z \cos \theta} \sin^{2\nu} \theta d\theta \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$4. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi} \cosh(z \cos \theta) \sin^{2\nu} \theta d\theta \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$5. \quad I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos \nu \theta \, d\theta - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-z \cosh t - \nu t} \, dt$$

$$\left[|\arg z| \leq \frac{\pi}{2}, \quad \operatorname{Re} \nu > 0 \right] \quad \text{WA 201(4)}$$

See also **3.383** 2, **3.387** 1, **3.471** 6, **3.714** 5.

For an integral representation of $I_0(z)$ and $I_1(z)$, see **3.366** 1, **3.534** **3.856** 6.

The function $K_\nu(z)$

8.432

1. $K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh \nu t \, dt$ $\left[|\arg z| < \frac{\pi}{2} \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0 \right]$ MO 39
2. $K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-z \cosh t} \sinh^{2\nu} t \, dt$
 $\left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0; \text{ or } \operatorname{Re} z = 0 \text{ and } -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2} \right]$ WA 190(5), WH
3. $K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} \, dt$
 $\left[\operatorname{Re}\left(\nu + \frac{1}{2}\right) > 0, \quad |\arg z| < \frac{\pi}{2}; \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0 \right]$ WA 190(4)
4. $K_\nu(x) = \frac{1}{\cos \frac{\nu\pi}{2}} \int_0^\infty \cos(x \sinh t) \cosh \nu t \, dt$ $[x > 0, \quad -1 < \operatorname{Re} \nu < 1]$ WA 202(13)
5. $K_\nu(xz) = \frac{\Gamma\left(\nu + \frac{1}{2}\right) (2z)^\nu}{x^\nu \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos xt \, dt}{(t^2 + z^2)^{\nu + \frac{1}{2}}}$ $\left[\operatorname{Re}\left(\nu + \frac{1}{2}\right) \geq 0, \quad x > 0, \quad |\arg z| < \frac{\pi}{2} \right]$
WA 191(1)
- 6.¹¹ $K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t - z^2/4t} \, dt}{t^{\nu+1}}$ $\left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re} z^2 > 0 \right]$ WA 203(15)
- 7.⁷ $K_\nu(xz) = \frac{z^\nu}{2} \int_0^\infty \exp\left[-\frac{x}{2} \left(t + \frac{z^2}{t}\right)\right] t^{-\nu-1} \, dt$
 $\left[|\arg z| < \frac{\pi}{4} \text{ or } |\arg z| = \frac{\pi}{4} \text{ and } \operatorname{Re} \nu < 1 \right]$ MO 39
8. $K_\nu(xz) = \sqrt{\frac{\pi}{2z}} \frac{x^\nu e^{-xz}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-xt} t^{\nu - \frac{1}{2}} \left(1 + \frac{t}{2z}\right)^{\nu - \frac{1}{2}} \, dt$
 $\left[|\arg z| < \pi, \quad \operatorname{Re} \nu > -\frac{1}{2}, x > 0 \right]$ MO 39
9. $K_\nu(xz) = \frac{\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{x}{2z}\right)^\nu \int_0^\infty \frac{\exp(-x\sqrt{t^2 + z^2})}{\sqrt{t^2 + z^2}} t^{2\nu} \, dt$
 $\left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0, \quad \operatorname{Re} \sqrt{t^2 + z^2} > 0, \quad x > 0 \right]$ MO 39

See also **3.383** 3, **3.387** 3, 6, **3.388** 2, **3.389** 4, **3.391**, **3.395** 1, **3.471** 9, **3.483**, **3.547** 2, **3.856**, **3.871** 3, 4, **7.141** 5.

$$\mathbf{8.433} \quad K_{\frac{1}{3}} \left(\frac{2x\sqrt{x}}{3\sqrt{3}} \right) = \frac{3}{\sqrt{x}} \int_0^{\infty} \cos(t^3 + xt) dt \quad \text{KU 98(31), WA 211(2)}$$

For an integral representation of $K_0(z)$, see **3.754** 2, **3.864**, **4.343**, **4.356**, **4.367**.

8.44 Series representation

The function $J_{\nu}(z)$

$$\mathbf{8.440} \quad J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad [|\arg z| < \pi] \quad \text{WH 358 a}$$

8.441 Special cases:

$$\begin{aligned} 1. \quad J_0(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} (k!)^2} \\ 2. \quad J_1(z) &= -J'_0(z) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k! (k+1)!} \\ 3. \quad J_{\frac{1}{3}}(z) &= \frac{1}{\Gamma(\frac{4}{3})} \sqrt[3]{\frac{z}{2}} \sum_{k=0}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 1 \cdot 4 \cdot 7 \cdots (3k+1)} \\ 4. \quad J_{-\frac{1}{3}}(z) &= \frac{1}{\Gamma(\frac{2}{3})} \sqrt[3]{\frac{2}{z}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 2 \cdot 5 \cdot 8 \cdots (3k-1)} \right\} \end{aligned}$$

For the expansion of $J_{\nu}(z)$ in Laguerre polynomials, see **8.975** 3.

8.442

$$\begin{aligned} 1.^7 \quad J_{\nu}(z) J_{\mu}(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{\mu+\nu+2m} (\mu+\nu+m+1)_m}{m! \Gamma(\mu+m+1) \Gamma(\nu+m+1)} \\ 2.^8 \quad J_{\nu}(az) J_{\mu}(bz) &= \frac{\left(\frac{az}{2}\right)^{\nu} \left(\frac{bz}{2}\right)^{\mu}}{\Gamma(\mu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu-1; \frac{b^2}{a^2}\right)}{k! \Gamma(\nu+k+1)} \end{aligned} \quad \text{MO 28}$$

The function $Y_{\nu}(z)$

$$\mathbf{8.443}^{11} \quad Y_{\nu}(z) = \frac{1}{\sin \nu \pi} \left\{ \cos \nu \pi \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu+k+1)} - \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(k-\nu+1)} \right\} \quad [\nu \neq \text{an integer}] \quad (\text{cf. } \mathbf{8.403} \text{ 1})$$

For $\nu+1$ a natural number, see **8.403** 2.; for ν a negative integer, see **8.404** 1

8.444 Special cases,

$$1. \quad \pi Y_0(z) = 2 J_0(z) \left(\ln \frac{z}{2} + C \right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2} \right)^{2k} \sum_{m=1}^k \frac{1}{m} \quad \text{KU 44}$$

$$2.^{11} \quad \pi Y_1(z) = 2 J_1(z) \left(\ln \frac{z}{2} + C \right) - \frac{2}{z} - \frac{z}{2} - \sum_{k=2}^{\infty} \frac{(-1)^{k+1} \left(\frac{z}{2} \right)^{2k-1}}{k!(k-1)!} \left(\frac{1}{k} + 2 \sum_{m=1}^{k-1} \frac{1}{m} \right)$$

The functions $I_\nu(z)$ and $K_n(z)$

$$8.445 \quad I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2} \right)^{\nu+2k} \quad \text{WH 372a}$$

$$8.446^8 \quad K_n(z) = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k! \left(\frac{z}{2} \right)^{n-2k}} \\ + (-1)^{n+1} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{n+2k}}{k!(n+k)!} \left[\ln \frac{z}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(n+k+1) \right] \quad \text{WA 95(15)}$$

$$= (-1)^{n+1} I_n(z) \left(\ln \frac{1}{2} z + C \right) + \frac{1}{2} (-1)^n \sum_{l=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{n+2l}}{l!(n+l)!} \left(\sum_{k=1}^l \frac{1}{k} + \sum_{k=1}^{n+l} \frac{1}{k} \right)$$

$$+ \frac{1}{2} \sum_{l=0}^{n-1} \frac{(-1)^l (n-l-1)!}{l!} \left(\frac{z}{2} \right)^{2l-n}$$

$[n+1 \text{ is a natural number}]$

MO 29

8.447 Special cases:

$$1. \quad I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{2k}}{(k!)^2}$$

$$2. \quad I_1(z) = I'_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{2k+1}}{k!(k+1)!}$$

$$3. \quad K_0(z) = -\ln \frac{z}{2} I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k} (k!)^2} \psi(k+1) \quad \text{WA 95(14)}$$

8.45 Asymptotic expansions of Bessel functions

8.451 For large values of $|z|$ *

$$1. \quad J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma(\nu + 2k + \frac{1}{2})}{(2k)! \Gamma(\nu - 2k + \frac{1}{2})} + R_1 \right] \right. \\ \left. - \sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma(\nu + 2k + \frac{3}{2})}{(2k+1)! \Gamma(\nu - 2k - \frac{1}{2})} + R_2 \right] \right\} \\ [|\arg z| < \pi] \quad (\text{see } \mathbf{8.339} \text{ 4}) \quad \text{WA 222(1, 3)}$$

$$2.^{11} \quad Y_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma(\nu + 2k + \frac{1}{2})}{(2k)! \Gamma(\nu - 2k + \frac{1}{2})} + R_1 \right] \right. \\ \left. + \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma(\nu + 2k + \frac{3}{2})}{(2k+1)! \Gamma(\nu - 2k - \frac{1}{2})} + R_2 \right] \right\} \\ [|\arg z| < \pi] \quad (\text{see } \mathbf{8.339} \text{ 4}) \quad \text{WA 222(2, 4, 5)}$$

$$3.^{11} \quad H_{\nu}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\pi}{2}\nu - \frac{\pi}{4})} \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2iz)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k! \Gamma(\nu - k + \frac{1}{2})} + \theta_1 \frac{(-1)^n}{(2iz)^n} \frac{\Gamma(\nu + n + \frac{1}{2})}{n! \Gamma(\nu - n + \frac{1}{2})} \right] \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad |\arg z| < \pi] \quad (\text{see } \mathbf{8.339} \text{ 4}) \quad \text{WA 221(5)}$$

$$4.^{11} \quad H_{\nu}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\pi}{2}\nu - \frac{\pi}{4})} \left[\sum_{k=0}^{n-1} \frac{1}{(2iz)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k! \Gamma(\nu - k + \frac{1}{2})} + \theta_2 \frac{1}{(2iz)^n} \frac{\Gamma(\nu + n + \frac{1}{2})}{n! \Gamma(\nu - n + \frac{1}{2})} \right] \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad |\arg z| < \pi] \quad (\text{see } \mathbf{8.339} \text{ 4}) \quad \text{WA 221(6)}$$

For indices of the form $\nu = \frac{2n-1}{2}$ (where n is a natural number), the series **8.451** terminate. In this case, the closed formulas **8.46** are valid for all values.

$$5. \quad I_{\nu}(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k! \Gamma(\nu - k + \frac{1}{2})} \\ + \frac{\exp[-z \pm (\nu + \frac{1}{2})\pi i]}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{1}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k! \Gamma(\nu - k + \frac{1}{2})} \\ [\text{The } + \text{ sign is taken for } -\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi, \text{ the } - \text{ sign for } -\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi]^* \quad (\text{see } \mathbf{8.339} \text{ 4})] \\ \text{WA 226(2,3)}$$

$$6.^{11} \quad K_{\nu}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[\sum_{k=0}^{n-1} \frac{1}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k! \Gamma(\nu - k + \frac{1}{2})} + \theta_3 \frac{\Gamma(\nu + n + \frac{1}{2})}{(2z)^n n! \Gamma(\nu - n + \frac{1}{2})} \right] \\ (\text{see } \mathbf{8.339} \text{ 4}) \quad \text{WA 231, 245(9)}$$

An estimate of the remainders of the asymptotic series in formulas **8.451**:

*An estimate of the remainders in formulas **8.451** is given in **8.451 7** and **8.451 8**.

*The contradiction that this condition contains at first glance is explained by the so-called Stokes phenomenon (see Watson, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge Univ. Press, 1944, page 201).

$$7. \quad |R_1| < \left| \frac{\Gamma(\nu + 2n + \frac{1}{2})}{(2z)^{2n}(2n)! \Gamma(\nu - 2n + \frac{1}{2})} \right| \quad \left[n > \frac{\nu}{2} - \frac{1}{4} \right] \quad \text{WA 231}$$

$$8. \quad |R_2| < \left| \frac{\Gamma(\nu + 2n + \frac{3}{2})}{(2z)^{2n+1}(2n+1)! \Gamma(\nu - 2n - \frac{1}{2})} \right| \quad \left[n \geq \frac{\nu}{2} - \frac{3}{4} \right] \quad \text{WA 231}$$

$$\text{For } -\frac{\pi}{2} < \arg z < \frac{3}{2}\pi, \nu \text{ real, and } n + \frac{1}{2} > |\nu| \quad \text{WA 245}$$

$$|\theta_1| < \begin{cases} 1, & \text{if } \operatorname{Im} z \geq 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \leq 0 \end{cases}$$

$$\text{For } -\frac{3}{2}\pi < \arg z < \frac{\pi}{2}, \nu \text{ real, and } n + \frac{1}{2} > |\nu| \quad \text{WA 246}$$

$$|\theta_2| < \begin{cases} 1, & \text{if } \operatorname{Im} z \leq 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \geq 0 \end{cases}$$

$$\text{For } \nu \text{ real,} \quad \text{WA 245}$$

$$|\theta_3| < \begin{cases} 1 & \text{if } \operatorname{Re} z \geq 0 \\ |\operatorname{cosec}(\arg z)|, & \text{if } \operatorname{Re} z < 0 \end{cases}$$

$$\operatorname{Re} \theta_3 \geq 0, \quad \text{if } \operatorname{Re} z \geq 0$$

$$\text{For } \nu \text{ and } z \text{ real and } n \geq \nu - \frac{1}{2}, \quad \text{WA 231}$$

$$0 \leq |\theta_3| \leq 1$$

In particular, it follows from **8.451 7** and **8.451 8** that for real positive values of z and ν , the errors $|R_1|$ and $|R_2|$ are less than the absolute value of the first discarded term. For values of $|\arg z|$ close to π , the series **8.451 1** and **8.451 2** may not be suitable for calculations. In particular, the error for $|\arg z| > \pi$ can be greater in absolute value than the first discarded term.

“Approximation by tangents”

8.452¹¹ For large values of the index (where the argument is less than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \cosh \alpha$. Then, for large values of ν , the following expansions are valid:

$$1. \quad J_\nu \left(\frac{\nu}{\cosh \alpha} \right) \sim \frac{\exp(\nu \tanh \alpha - \nu \alpha)}{\sqrt{2\nu\pi \tanh \alpha}} \left\{ 1 + \frac{1}{\nu} \left(\frac{1}{8} \coth \alpha - \frac{5}{24} \coth^3 \alpha \right) \right. \\ \left. + \frac{1}{\nu^2} \left(\frac{9}{128} \coth^2 \alpha - \frac{231}{576} \coth^4 \alpha + \frac{1155}{3456} \coth^6 \alpha \right) + \dots \right\}$$

WA 269(3)

$$2. \quad Y_\nu \left(\frac{\nu}{\cosh \alpha} \right) \sim - \frac{\exp(\nu \alpha - \nu \tanh \alpha)}{\sqrt{\frac{\pi}{2} \nu \tanh \alpha}} \left\{ 1 - \frac{1}{\nu} \left(\frac{1}{8} \coth \alpha - \frac{5}{24} \coth^3 \alpha \right) + \frac{1}{\nu^2} \left(\frac{9}{128} \coth^2 \alpha - \frac{231}{576} \coth^4 \alpha + \frac{1155}{3456} \coth^6 \alpha \right) + \dots \right\}$$

WA 270(5)

8.453 For large values of the index (where the argument is greater than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \cos \beta$. Then, for large values of ν , the following expansions are valid:

$$1. \quad J_\nu(\nu \sec \beta) \sim \sqrt{\frac{2}{\nu \pi \tan \beta}} \left\{ \left[1 - \frac{1}{\nu^2} \left(\frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right] \cos \left(\nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) + \left[\frac{1}{\nu} \left(\frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \dots \right] \sin \left(\nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) \right\}$$

WA 271(4)

$$2. \quad Y_\nu(\nu \sec \beta) \sim \sqrt{\frac{2}{\nu \pi \tan \beta}} \left\{ \left[1 - \frac{1}{\nu^2} \left(\frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right] \sin \left(\nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) - \left[\frac{1}{\nu} \left(\frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \dots \right] \cos \left(\nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) \right\}$$

WA 271(5)

$$3. \quad H_\nu^{(1)}(\nu \sec \beta) \sim \frac{\exp \left[\nu i (\tan \beta - \beta) - \frac{\pi}{4} i \right]}{\sqrt{\frac{\pi}{2} \nu \tan \beta}} \left\{ 1 - \frac{i}{\nu} \left(\frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \frac{1}{\nu^2} \left(\frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right\}$$

WA 271(1)

$$4. \quad H_\nu^{(2)}(\nu \sec \beta) \sim \frac{\exp \left[-\nu i (\tan \beta - \beta) + \frac{\pi}{4} i \right]}{\sqrt{\frac{\pi}{2} \nu \tan \beta}} \left\{ 1 + \frac{i}{\nu} \left(\frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \frac{1}{\nu^2} \left(\frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right\}$$

WA 271(2)

Formulas **8.453** are not valid when $|x - \nu|$ is of a size comparable to $x^{\frac{1}{3}}$. For arbitrary small (and also large) values of $|x - \nu|$, we may use the following formulas:

8.454 Suppose that $x > 0$ and $\nu > 0$, we set

$$w = \sqrt{\frac{x^2}{\nu^2} - 1};$$

Then,

$$\begin{aligned} 1. \quad H_{\nu}^{(1)}(x) &= \frac{w}{\sqrt{3}} \exp \left\{ \left[\frac{\pi}{6} + \nu \left(w - \frac{w^3}{3} - \arctan w \right) \right] i \right\} H_{\frac{1}{3}}^{(1)} \left(\frac{\nu}{3} w^3 \right) + O \left(\frac{1}{|\nu|} \right) \\ 2. \quad H_{\nu}^{(2)}(x) &= \frac{w}{\sqrt{3}} \exp \left\{ \left[-\frac{\pi}{6} - \nu \left(w - \frac{w^3}{3} - \arctan w \right) \right] i \right\} H_{\frac{1}{3}}^{(2)} \left(\frac{\nu}{3} w^3 \right) + O \left(\frac{1}{|\nu|} \right) \end{aligned} \quad \text{MO 34}$$

The absolute value of the error $O \left(\frac{1}{|\nu|} \right)$ is then less than $24\sqrt{2} \left| \frac{1}{\nu} \right|$.

8.455 For x real and ν a natural number ($\nu = n$), if $n \gg 1$, the following approximations are valid:

$$\begin{aligned} 1.^7 \quad J_n(x) &\approx \frac{1}{\pi} \sqrt{\frac{2(n-x)}{3x}} K_{\frac{1}{3}} \left\{ \frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}} \right\} \\ &\quad [n > x] \quad (\text{see also } \mathbf{8.433}) \\ &\quad \text{WA 276(1)} \end{aligned}$$

$$\approx \frac{1}{2} e^{\frac{2}{3}\pi i} \sqrt{\frac{2(n-x)}{3x}} H_{\frac{1}{3}}^{(1)} \left\{ \frac{i [2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}} \right\} \quad [n > x]$$

$$\approx \frac{1}{\sqrt{3}} \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] + J_{-\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\} \quad \text{MO 34}$$

(see also **8.441** 3, **8.441** 4)
WA 276(2)

$$\begin{aligned} 2. \quad Y_n(x) &\approx \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{-\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] - J_{\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\} \\ &\quad [x > n] \quad \text{WA 276(3)} \end{aligned}$$

An estimate of the error in formulas **8.455** has not yet been achieved.

$$\mathbf{8.456}^{11} \quad J_{\nu}^2(z) + Y_{\nu}^2(z) \approx \frac{2}{\pi z} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k z^{2k}} \frac{\Gamma(\nu+k+\frac{1}{2})}{k! \Gamma(\nu-k+\frac{1}{2})} [|\arg z| < \pi] \quad (\text{see also } \mathbf{8.479} \text{ 1})$$

WA 250(5)

$$\mathbf{8.457} \quad J_{\nu}^2(x) + J_{\nu+1}^2(x) \approx \frac{2}{\pi x} \quad [x \gg |\nu|] \quad \text{WA 223}$$

8.46 Bessel functions of order equal to an integer plus one-half

The function $J_\nu(z)$

8.461

$$1.^{11} \quad J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!} (2z)^{-2k} \right. \\ \left. + \cos\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!} (2z)^{-(2k+1)} \right\} \\ [n+1 \text{ is a natural number}] \quad (\text{cf. } \mathbf{8.451} \ 1) \quad \text{KU } 59(6), \text{ WA } 66(2)$$

$$2. \quad J_{-n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)! (2z)^{2k}} \right. \\ \left. - \sin\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)! (2z)^{2k+1}} \right\} \\ [n+1 \text{ is a natural number}] \quad (\text{cf. } \mathbf{8.451} \ 1) \quad \text{KU } 58(7), \text{ WA } 67(5)$$

8.462

$$1. \quad J_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{-n+k-1} (n+k)!}{k!(n-k)! (2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{-n+k-1} (n+k)!}{k!(n-k)! (2z)^k} \right\} \\ [n+1 \text{ is a natural number}] \\ \text{KU } 59(6), \text{ WA } 66(1)$$

$$2. \quad J_{-n-\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{n+k} (n+k)!}{k!(n-k)! (2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{n+k} (n+k)!}{k!(n-k)! (2z)^k} \right\} \\ [n+1 \text{ is a natural number}] \\ \text{KU } 59(7), \text{ WA } 67(4)$$

8.463

$$1. \quad J_{n+\frac{1}{2}}(z) = (-1)^n z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left(\frac{\sin z}{z} \right) \quad \text{KU } 58(4)$$

$$2. \quad J_{-n-\frac{1}{2}}(z) = z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left(\frac{\cos z}{z} \right) \quad \text{KU } 58(5)$$

8.464 Special cases:

$$1. \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{DW}$$

$$2. \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z \quad \text{DW}$$

3. $J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right)$ DW
4. $J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(-\sin z - \frac{\cos z}{z} \right)$ DW
- 5.⁸ $J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \left(\frac{3}{z^2} - 1 \right) \sin z - \frac{3}{z} \cos z \right\}$ DW
6. $J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \frac{3}{z} \sin z + \left(\frac{3}{z^2} - 1 \right) \cos z \right\}$ DW

The function $Y_{n+\frac{1}{2}}(z)$

8.465

1. $Y_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z)$ JA
2. $Y_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z)$ JA

The functions $H_{n+\frac{1}{2}}^{(1,2)}(z)$, $I_{n+\frac{1}{2}}(z)$, $K_{n+\frac{1}{2}}(z)$

8.466

1. $H_{n-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} i^{-n} e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$
(cf. 8.451 3)

2. $H_{n-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} i^n e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$ (cf. 8.451 4)

- 8.467 $I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \left[e^z \sum_{k=0}^n \frac{(-1)^k (n+k)!}{k!(n-k)!(2z)^k} \pm (-1)^{n+1} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2z)^k} \right]$
(cf. 8.451 5) KU 60a

- 8.468 $K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2z)^k}$ (cf. 8.451 6) KU 60

8.469 Special cases:

1. $Y_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \cos z$
2. $Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$
3. $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ WA 95(13)
4. $H_{\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{iz}}{i}$ MO 27

5. $H_{\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{-iz}}{-i}$ MO 27
6. $H_{-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{iz}$ MO 27
7. $H_{-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-iz}$ MO 27

8.47–8.48 Functional relations

8.471⁸ Recursion formulas:

1. $z Z_{\nu-1}(z) + z Z_{\nu+1}(z) = 2\nu Z_{\nu}(z)$ KU 56(13), WA 56(1), WA 79(1), WA 88(3)
2. $Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2 \frac{d}{dz} Z_{\nu}(z)$ KU 56(12), WA 56(2), WA 79(2), We 88(4)

Sonin and Nielsen, in their construction of the theory of Bessel functions, defined Bessel functions as analytic functions of z that satisfy the recursion relations **8.471**. Z denotes J , N , $H^{(1)}$, $H^{(2)}$ or any linear combination of these functions, the coefficients of which are independent of z and ν .

8.472 Consequences of the recursion formulas for Z defined as above:

1. $z \frac{d}{dz} Z_{\nu}(z) + \nu Z_{\nu}(z) = z Z_{\nu-1}(z)$ KU 56(11), WA 56(3), WA 79(3), WA 88(5)
2. $z \frac{d}{dz} Z_{\nu}(z) - \nu Z_{\nu}(z) = -z Z_{\nu+1}(z)$ KU 56(10), WA 56(4), WA 79(4), WA 88(6)
3. $\left(\frac{d}{z dz}\right)^m (z^{\nu} Z_{\nu}(z)) = z^{\nu-m} Z_{\nu-m}(z)$ KU 56(8), WA 57(5), WA 89(9)
4. $\left(\frac{d}{z dz}\right)^m (z^{-\nu} Z_{\nu}(z)) = (-1)^m z^{-\nu-m} Z_{\nu+m}(z)$ WA 89(10), Ku 55(5), WA 57(6)
5. $Z_{-n}(z) = (-1)^n Z_n(z)$ [n is a natural number] (cf. **8.404**)

8.473 Special cases:

1. $J_2(z) = \frac{2}{z} J_1(z) - J_0(z)$
2. $Y_2(z) = \frac{2}{z} Y_1(z) - Y_0(z)$
3. $H_2^{(1,2)}(z) = \frac{2}{z} H_1^{(1,2)}(z) - H_0^{(1,2)}(z)$
4. $\frac{d}{dz} J_0(z) = -J_1(z)$
5. $\frac{d}{dz} Y_0(z) = -Y_1(z)$
6. $\frac{d}{dz} H_0^{(1,2)}(z) = -H_1^{(1,2)}(z)$

8.474⁸ Each of the pairs of functions $J_{\nu}(z)$ and $J_{-\nu}(z)$ (for $\nu \neq 0, \pm 1, \pm 2, \dots$), $J_{\nu}(z)$ and $Y_{\nu}(z)$, and $H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$, which are solutions of equation **8.401**, and also the pair $I_{\nu}(z)$ and $K_{\nu}(z)$ is a pair of linearly independent functions. The Wronskians of these pairs are, respectively,

$$-\frac{2}{\pi z} \sin \nu \pi, \quad \frac{2}{\pi z}, \quad -\frac{4i}{\pi z}, \quad -\frac{1}{z} \quad \text{KU 52(10, 11, 12), WA 90(1, 4)}$$

8.475⁶ The functions $J_\nu(z)$, and $Y_\nu(z)$, $H_\nu^{(1,2)}(z)$, $I_\nu(z)$, $K_\nu(z)$, with the exception of $J_n(z)$ and $I_n(z)$, for n an integer are *non-single-valued*: $z = 0$ is a branch point for these functions. The branches of these functions that lie on opposite sides of the cut $(-\infty, 0)$ are connected by the relations

8.476

$$1. \quad J_\nu(e^{m\pi i} z) = e^{m\nu\pi i} J_\nu(z) \quad \text{WA 90(1)}$$

$$2. \quad Y_\nu(e^{m\pi i} z) = e^{-m\nu\pi i} Y_\nu(z) + 2i \sin m\nu\pi \cot \nu\pi J_\nu(z) \quad \text{WA 90(3)}$$

$$3. \quad Y_{-\nu}(e^{m\pi i} z) = e^{-m\nu\pi i} Y_{-\nu}(z) + 2i \sin m\nu\pi \operatorname{cosec} \nu\pi J_\nu(z) \quad \text{WA 90(4)}$$

$$4. \quad I_\nu(e^{m\pi i} z) = e^{m\nu\pi i} I_\nu(z) \quad \text{WA 95(17)}$$

$$5. \quad K_\nu(e^{m\pi i} z) = e^{-m\nu\pi i} K_\nu(z) - i\pi \frac{\sin m\nu\pi}{\sin \nu\pi} I_\nu(z) \quad [\nu \text{ not an integer}] \quad \text{WA 95(18)}$$

$$6. \quad H_\nu^{(1)}(e^{m\pi i} z) = e^{-m\nu\pi i} H_\nu^{(1)}(z) - 2e^{-\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} J_\nu(z) \\ = \frac{\sin(1-m)\nu\pi}{\sin \nu\pi} H_\nu^{(1)}(z) - e^{-\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_\nu^{(2)}(z) \quad \text{WA 95(5)}$$

$$7. \quad H_\nu^{(2)}(e^{m\pi i} z) = e^{-m\nu\pi i} H_\nu^{(2)}(z) + 2e^{\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} J_\nu(z) \\ = \frac{\sin(1+m)\nu\pi}{\sin \nu\pi} H_\nu^{(2)}(z) + e^{\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_\nu^{(1)}(z) \quad [m \text{ an integer}] \quad \text{WA 90(6)}$$

$$8. \quad H_\nu^{(1)}(e^{i\pi} z) = -H_{-\nu}^{(2)}(z) = -e^{-i\pi\nu} H_\nu^{(2)}(z) \quad \text{MO 26}$$

$$9. \quad H_\nu^{(2)}(e^{-i\pi} z) = -H_{-\nu}^{(1)}(z) = -e^{i\pi\nu} H_\nu^{(1)}(z) \quad \text{MO 26}$$

$$10.^8 \quad \overline{H}_\nu^{(2)}(z) = H_{\bar{\nu}}^{(1)}(\bar{z}) \quad \text{MO 26}$$

8.477

$$1. \quad J_\nu(z) Y_{\nu+1}(z) - J_{\nu+1}(z) Y_\nu(z) = -\frac{2}{\pi z} \quad \text{WA 91(12)}$$

$$2. \quad I_\nu(z) K_{\nu+1}(z) + I_{\nu+1}(z) K_\nu(z) = \frac{1}{z} \quad \text{WA 95(20)}$$

See also **3.863**.

- For a connection with Legendre functions, see **8.722**.
- For a connection with the polynomials $C_n^\lambda(t)$, see **8.936 4**.
- For a connection with a confluent hypergeometric function, see **9.235**.

8.478 For $\nu > 0$ and $x > 0$, the product

$$x [J_\nu^2(x) + Y_\nu^2(x)],$$

considered as a function of x , decreases monotonically, if $\nu > \frac{1}{2}$ and increases monotonically if $0 < \nu < \frac{1}{2}$.

8.479

$$1.^{11} \quad \frac{1}{\sqrt{x^2 - \nu^2}} > \frac{\pi}{2} [J_\nu^2(x) + Y_\nu^2(x)] \geq \frac{1}{x} \quad [x \geq \nu \geq \tfrac{1}{2}] \quad \text{MO 35}$$

$$2. \quad |J_n(nz)| \leq 1 \quad \left[\left| \frac{z \exp \sqrt{1 - z^2}}{1 + \sqrt{1 - z^2}} \right| < 1, n \text{ a natural number} \right] \quad \text{MO 35}$$

Relations between Bessel functions of the first, second, and third kinds

$$\begin{aligned} 8.481 \quad J_\nu(z) &= \frac{Y_{-\nu}(z) - Y_\nu(z) \cos \nu\pi}{\sin \nu\pi} = H_\nu^{(1)}(z) - i Y_\nu(z) \\ &= H_\nu^{(2)}(z) + i Y_\nu(z) = \frac{1}{2} (H_\nu^{(1)}(z) + H_\nu^{(2)}(z)) \end{aligned} \quad (\text{cf. } \mathbf{8.403} \text{ 1, } \mathbf{8.405}) \quad \text{WA 89(1), JA}$$

$$\begin{aligned} 8.482 \quad Y_\nu(z) &= \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi} = i J_\nu(z) - i H_\nu^{(1)}(z) \\ &= i H_\nu^{(2)}(z) - i J_\nu(z) = \frac{i}{2} (H_\nu^{(2)}(z) - H_\nu^{(1)}(z)) \end{aligned} \quad (\text{cf. } \mathbf{8.403} \text{ 1, } \mathbf{8.405}) \quad \text{WA 89(3), JA}$$

8.483

$$\begin{aligned} 1. \quad H_\nu^{(1)}(z) &= \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi} = \frac{Y_{-\nu}(z) - e^{-\nu\pi i} Y_\nu(z)}{\sin \nu\pi} = J_\nu(z) + i Y_\nu(z) \quad \text{WA 89(5)} \\ 2. \quad H_\nu^{(2)}(z) &= \frac{e^{\nu\pi i} J_\nu(z) - J_{-\nu}(z)}{i \sin \nu\pi} = \frac{Y_{-\nu}(z) - e^{\nu\pi i} Y_\nu(z)}{\sin \nu\pi} = J_\nu(z) - i Y_\nu(z) \\ & \quad (\text{cf. } \mathbf{8.405}) \quad \text{WA 89(6)} \end{aligned}$$

8.484

$$\begin{aligned} 1. \quad H_{-\nu}^{(1)}(z) &= e^{\nu\pi i} H_\nu^{(1)}(z) \quad \text{WA 89(7)} \\ 2. \quad H_{-\nu}^{(2)}(z) &= e^{-\nu\pi i} H_\nu^{(2)}(z) \quad \text{WA 89(7)} \end{aligned}$$

$$8.485^7 \quad K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \quad [\nu \text{ not an integer}] \quad (\text{see also } \mathbf{8.407}) \quad \text{WA 92(6)}$$

8.486 Recursion formulas for the functions $I_\nu(z)$ and $K_\nu(z)$ and their consequences:

$$\begin{aligned} 1. \quad z I_{\nu-1}(z) - z I_{\nu+1}(z) &= 2\nu I_\nu(z) \quad \text{WA 93(1)} \\ 2. \quad I_{\nu-1}(z) + I_{\nu+1}(z) &= 2 \frac{d}{dz} I_\nu(z) \quad \text{WA 93(2)} \\ 3. \quad z \frac{d}{dz} I_\nu(z) + \nu I_\nu(z) &= z I_{\nu-1}(z) \quad \text{WA 93(3)} \\ 4. \quad z \frac{d}{dz} I_\nu(z) - \nu I_\nu(z) &= z I_{\nu+1}(z) \quad \text{WA 93(4)} \\ 5. \quad \left(\frac{d}{z dz} \right)^m \{ z^\nu I_\nu(z) \} &= z^{\nu-m} I_{\nu-m}(z) \quad \text{WA 93(5)} \end{aligned}$$

6. $\left(\frac{d}{z dz}\right)^m \{z^{-\nu} I_\nu(z)\} = z^{-\nu-m} I_{\nu+m}(z)$ WA 93(6)
7. $I_{-n}(z) = I_n(z)$ [n a natural number] WA 93(8)
8. $I_2(z) = -\frac{2}{z} I_1(z) + I_0(z)$
9. $\frac{d}{dz} I_0(z) = I_1(z)$ WA 93(7)
10. $z K_{\nu-1}(z) - z K_{\nu+1}(z) = -2\nu K_\nu(z)$ WA 93(1)
11. $K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{d}{dz} K_\nu(z)$ WA 93(2)
12. $z\frac{d}{dz} K_\nu(z) + \nu K_\nu(z) = -z K_{\nu-1}(z)$ WA 93(3)
13. $z\frac{d}{dz} K_\nu(z) - \nu K_\nu(z) = -z K_{\nu+1}(z)$ WA 93(4)
14. $\left(\frac{d}{z dz}\right)^m \{z^\nu K_\nu(z)\} = (-1)^m z^{\nu-m} K_{\nu-m}(z)$ WA 93(5)
15. $\left(\frac{d}{z dz}\right)^m \{z^{-\nu} K_\nu(z)\} = (-1)^m z^{-\nu-m} K_{\nu+m}(z)$ WA 93(6)
16. $K_{-\nu}(z) = K_\nu(z)$ WA 93(8)
17. $K_2(z) = \frac{2}{z} K_1(z) + K_0(z)$
18. $\frac{d}{dz} K_0(z) = -K_1(z)$ WA 93(7)
19. $\frac{\partial J_\nu(z)}{\partial \nu} = \left[\ln \frac{z}{2} - \psi(\nu+1)\right] J_\nu(z) + \frac{(z/2)^{\nu+1}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(z/2)^n J_{n+1}(z)}{n!(\nu+n+1)^2}$ LUKE 360

8.486(1)⁷ Differentiation with respect to order

1. $\frac{\partial J_\nu(z)}{\partial \nu} = J_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k! \Gamma(\nu+k+1)}$
[$\nu \neq n$ or $n + \frac{1}{2}$, n integer] MS 3.1.3
2. $\frac{\partial J_{-\nu}(z)}{\partial \nu} = -J_{-\nu}(z) \ln\left(\frac{1}{2}z\right) + \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{-\nu+2k} \frac{\psi(-\nu+k+1)}{k! \Gamma(-\nu+k+1)}$
[$\nu \neq n$ or $n + \frac{1}{2}$, n integer] MS 3.1.3
3. $\frac{\partial Y_\nu(z)}{\partial \nu} = \cot \pi \nu \frac{\partial J_\nu(z)}{\partial \nu} - \operatorname{cosec} \pi \nu \frac{\partial J_{-\nu}(z)}{\partial \nu} - \pi \operatorname{cosec} \pi \nu Y_\nu(z)$
[$\nu \neq n$ or $n + \frac{1}{2}$, n integer] MS 3.1.3
4. $\frac{\partial I_\nu(z)}{\partial \nu} = I_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k! \Gamma(\nu+k+1)}$ [$\nu \neq n$ or $n + \frac{1}{2}$, n integer]

$$5. \quad \frac{\partial K_\nu(z)}{\partial \nu} = -\pi \cot \pi \nu K_\nu(z) + \frac{1}{2} \pi \operatorname{cosec} \pi \nu \left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu} \right] \quad \left[\nu \neq n \text{ or } n + \frac{1}{2}, \quad n \text{ integer} \right] \quad \text{MS 3.1.3}$$

$$6. \quad \left[\frac{\partial J_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = \frac{1}{2} \pi (\pm 1)^n Y_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} J_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{MS 3.2.3}$$

$$7. \quad \left[\frac{\partial Y_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = -\frac{1}{2} \pi (\pm 1)^n J_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} Y_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{MS 3.2.3}$$

$$8. \quad \left[\frac{\partial I_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = (-1)^{n+1} K_n(z) \pm (-1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{MS 3.2.3}$$

$$9. \quad \left[\frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = \pm \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} K_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{MS 3.2.3}$$

$$10. \quad (-1)^n \left[\frac{\partial}{\partial \nu} I_\nu(z) \right]_{\nu=n} = -K_n(z) + \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{AS 9.6.44}$$

$$11.^{11} \quad \left[\frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=n} = \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} K_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{AS 9.6.45}$$

Special cases

$$12. \quad \left[\frac{\partial J_\nu(z)}{\partial \nu} \right]_{\nu=0} = \frac{1}{2} \pi Y_0(z) \quad \text{MS 3.2.3}$$

$$13. \quad \left[\frac{\partial Y_\nu(z)}{\partial \nu} \right]_{\nu=0} = -\frac{1}{2} \pi J_0(z) \quad \text{MS 3.2.3}$$

$$14. \quad \left[\frac{\partial I_\nu(z)}{\partial \nu} \right]_{\nu=0} = -K_0(z) \quad \text{MS 3.2.3}$$

$$15. \quad \left[\frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=0} = 0 \quad \text{MS 3.2.3}$$

$$16. \quad \left[\frac{\partial J_\nu(x)}{\partial \nu} \right]_{\nu=\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-1/2} [\sin x \operatorname{Ci}(3x) - \cos x \operatorname{Si}(2x)] \quad \text{MS 3.3.3}$$

$$17. \quad \left[\frac{\partial J_\nu(x)}{\partial \nu} \right]_{\nu=-\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-1/2} [\cos x \operatorname{Ci}(2x) + \sin x \operatorname{Si}(2x)] \quad \text{MS 3.3.3}$$

$$18. \quad \left[\frac{\partial Y_\nu(x)}{\partial \nu} \right]_{\nu=\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-1/2} \{\cos x \operatorname{Ci}(2x) + \sin x [\operatorname{Si}(2x) - \pi]\} \quad \text{MS 3.3.3}$$

$$19. \quad \left[\frac{\partial Y_\nu(x)}{\partial \nu} \right]_{\nu=-\frac{1}{2}} = -\left(\frac{1}{2}\pi x\right)^{-1/2} \{\sin x \operatorname{Ci}(2x) - \cos x [\operatorname{Si}(2x) - \pi]\} \quad \text{MS 3.3.3}$$

$$20. \quad \left[\frac{\partial I_\nu(x)}{\partial \nu} \right]_{\nu=\pm\frac{1}{2}} = (2\pi x)^{-1/2} [e^x \operatorname{Ei}(-2x) \mp e^{-x} \overline{\operatorname{Ei}}(2x)] \quad \text{MS 3.3.3}$$

$$21. \quad \left[\frac{\partial K_\nu(x)}{\partial \nu} \right]_{\nu=\pm\frac{1}{2}} = \mp \left(\frac{\pi}{2x} \right)^{\frac{1}{2}} e^x \operatorname{Ei}(-2x) \quad \text{MS 3.3.3}$$

8.487 Continuity with respect to the order*:

$$1. \quad \lim_{\nu \rightarrow n} Y_\nu(z) = Y_n(z) \quad [n \text{ an integer}] \quad \text{WA 76}$$

$$2. \quad \lim_{\nu \rightarrow n} H_\nu^{(1,2)}(z) = H_n^{(1,2)}(z) \quad [n \text{ an integer}] \quad \text{WA 183}$$

$$3. \quad \lim_{\nu \rightarrow n} K_\nu(z) = K_n(z) \quad [n \text{ an integer}] \quad \text{WA 92}$$

8.49 Differential equations leading to Bessel functions

See also **8.401**

8.491

$$1. \quad \frac{1}{z} \frac{d}{dz} (zu') + \left(\beta^2 - \frac{\nu^2}{z^2} \right) u = 0 \quad u = Z_\nu(\beta z) \quad \text{JA}$$

$$2. \quad \frac{1}{z} \frac{d}{dz} (zu') + \left[(\beta \gamma z^{\gamma-1})^2 - \left(\frac{\nu \gamma}{z} \right)^2 \right] u = 0 \quad u = Z_\nu(\beta z^\gamma) \quad \text{JA}$$

$$3. \quad u'' + \frac{1-2\alpha}{z} u' + \left[(\beta \gamma z^{\gamma-1})^2 + \frac{\alpha^2 - \nu^2 \gamma^2}{z^2} \right] u = 0 \quad u = z^\alpha Z_\nu(\beta z^\gamma) \quad \text{JA}$$

$$4. \quad u'' + \left[(\beta \gamma z^{\gamma-1})^2 - \frac{4\nu^2 \gamma^2 - 1}{4z^2} \right] u = 0 \quad u = \sqrt{z} Z_\nu(\beta z^\gamma) \quad \text{JA}$$

$$5. \quad u'' + \left(\beta^2 - \frac{4\nu^2 - 1}{4z^2} \right) u = 0 \quad u = \sqrt{z} Z_\nu(\beta z) \quad \text{JA}$$

$$6. \quad u'' + \frac{1-2\alpha}{z} u' + \left(\beta^2 + \frac{\alpha^2 - \nu^2}{z^2} \right) u = 0 \quad u = z^\alpha Z_\nu(\beta z) \quad \text{JA}$$

$$7. \quad u'' + bz^m u = 0 \quad u = \sqrt{z} Z_{\frac{1}{m+2}} \left(\frac{2\sqrt{b}}{m+2} z^{\frac{m+2}{2}} \right) \quad \text{JA 111(5)}$$

$$8. \quad u'' + \frac{1}{z} u' + 4 \left(z^2 - \frac{\nu^2}{z^2} \right) u = 0 \quad u = Z_\nu(z^2) \quad \text{WA 111(6)}$$

$$9. \quad u'' + \frac{1}{z} u' + \frac{1}{4z} \left(1 - \frac{\nu^2}{z} \right) u = 0 \quad u = Z_\nu(\sqrt{z}) \quad \text{WA 111(7)}$$

$$10. \quad u'' + \frac{1-\nu}{z} u' + \frac{1}{4} \frac{u}{z} = 0 \quad u = z^{\frac{\nu}{2}} Z_\nu(\sqrt{z}) \quad \text{WA 111(9)a}$$

$$11. \quad u'' + \beta^2 \gamma^2 z^{2\beta-2} u = 0 \quad u = z^{1/2} Z_{\frac{1}{2\beta}}(\gamma z^\beta) \quad \text{WA 110(3)}$$

*The continuity of the functions $J_\nu(z)$ and $I_\nu(z)$ follows directly from the series representations of these functions.

$$12. \quad z^2 u'' + (2\alpha - 2\beta\nu + 1)zu' + [\beta^2 \gamma^2 z^{2\beta} + \alpha(\alpha - 2\beta\nu)]u = 0$$

$$u = z^{\beta\nu - \alpha} Z_\nu(\gamma z^\beta) \quad \text{WA 112(21)}$$

8.492

$$1. \quad u'' + (e^{2z} - \nu^2)u = 0 \quad u = Z_\nu(e^z) \quad \text{WA 112(22)}$$

$$2. \quad u'' + \frac{e^{2/z} - \nu^2}{z^4}u = 0 \quad u = z Z_\nu(e^{1/z}) \quad \text{WA 112(22)}$$

8.493

$$1. \quad u'' + \left(\frac{1}{z} - 2 \tan z\right)u' - \left(\frac{\nu^2}{z^2} + \frac{\tan z}{z}\right)u = 0 \quad u = \sec z Z_\nu(z) \quad \text{JA}$$

$$2. \quad u'' + \left(\frac{1}{z} + 2 \cot z\right)u' - \left(\frac{\nu^2}{z^2} - \frac{\cot z}{z}\right)u = 0 \quad u = \operatorname{cosec} z Z_\nu(z) \quad \text{JA}$$

8.494

$$1. \quad u'' + \frac{1}{z}u' - \left(1 + \frac{\nu^2}{z^2}\right)u = 0 \quad u = Z_\nu(iz) = C_1 I_\nu(z) + C_2 K_\nu(z) \quad \text{JA}$$

$$2. \quad u'' + \frac{1}{z}u' - \left[\frac{1}{z} + \left(\frac{\nu}{2z}\right)^2\right]u = 0 \quad u = Z_\nu(2i\sqrt{z}) \quad \text{JA}$$

$$3. \quad u'' + u' + \frac{1}{z^2}\left(\frac{1}{4} - \nu^2\right)u = 0 \quad u = \sqrt{z}e^{-\frac{z}{2}} Z_\nu\left(\frac{iz}{2}\right) \quad \text{JA}$$

$$4.^{10} \quad u'' + \left(\frac{2\nu+1}{z} - k\right)u' - \frac{2\nu+1}{2z}ku = 0 \quad u = z^{-\nu}e^{\frac{1}{2}kx} Z_\nu\left(\frac{ikz}{2}\right) \quad \text{JA}$$

$$5. \quad u'' + \frac{1-\nu}{z}u' - \frac{1}{4}\frac{u}{z} = 0 \quad u = z^{\frac{\nu}{2}} Z_\nu(i\sqrt{z}) \quad \text{WA 111(8)}$$

$$6. \quad u'' \pm \frac{u}{\sqrt{z}} = 0$$

$$u = \sqrt{z} Z_{\frac{2}{3}}\left(\frac{4}{3}z^{\frac{3}{4}}\right), \quad u = \sqrt{z} Z_{\frac{2}{3}}\left(\frac{4}{3}iz^{\frac{3}{4}}\right) \quad \text{WA 111(10)}$$

$$7. \quad u'' \pm zu = 0$$

$$u = \sqrt{z} Z_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right), \quad u = \sqrt{z} Z_{\frac{1}{3}}\left(\frac{2}{3}iz^{\frac{3}{2}}\right) \quad \text{WA 111(10)}$$

$$8. \quad u'' - \left(c^2 + \frac{\nu(\nu+1)}{z^2}\right)u = 0 \quad u = \sqrt{z} Z_{\nu+\frac{1}{2}}(icz) \quad \text{WA 108(1)}$$

$$9. \quad u'' - \frac{2\nu}{z}u' - c^2u = 0 \quad u = z^{\nu+\frac{1}{2}} Z_{\nu+\frac{1}{2}}(icz) \quad \text{WA 109(3, 4)}$$

$$10. \quad u'' - c^2 z^{2\nu-2}u = 0 \quad u = \sqrt{z} Z_{\frac{1}{2\nu}}\left(i\frac{c}{\nu}z^\nu\right) \quad \text{WA 109(5, 6)}$$

8.495

$$1. \quad u'' + \frac{1}{z}u' + \left(i - \frac{\nu^2}{z^2}\right)u = 0 \quad u = Z_\nu(z\sqrt{i}) \quad \text{JA}$$

$$2. \quad u'' + \left(\frac{1}{z} \mp 2i\right) u' - \left(\frac{\nu^2}{z^2} \pm \frac{i}{z}\right) u = 0 \quad u = e^{\pm iz} Z_\nu(z) \quad \text{JA}$$

$$3. \quad u'' + \frac{1}{z} u' + s e^{i\alpha} u = 0 \quad u = Z_0 \left(\sqrt{s} z e^{\frac{i}{2}\alpha} \right) \quad \text{JA}$$

$$4. \quad u'' + \left(s e^{i\alpha} + \frac{1}{4z^2} \right) u = 0 \quad u = \sqrt{z} Z_0 \left(\sqrt{s} z e^{\frac{i}{2}\alpha} \right) \quad \text{JA}$$

8.496

$$1. \quad \frac{d^2}{dz^2} \left(z^4 \frac{d^2 u}{dz^2} \right) - z^2 u = 0 \quad u = \frac{1}{z} \left\{ Z_2(2\sqrt{z}) + \overline{Z_2(2i\sqrt{z})} \right\} \quad \text{WA 122(7)}$$

$$2. \quad \frac{d^2}{dz^2} \left(z^{\frac{16}{5}} \frac{d^2 u}{dz^2} \right) - z^{\frac{8}{5}} u = 0 \quad u = z^{-7/10} \left\{ Z_{\frac{5}{6}} \left(\frac{5}{3} z^{\frac{3}{5}} \right) + \overline{Z_{\frac{5}{6}} \left(\frac{5}{3} i z^{\frac{3}{5}} \right)} \right\} \quad \text{WA 122(8)}$$

$$3. \quad \frac{d^2}{dz^2} \left(z^{12} \frac{d^2 u}{dz^2} \right) - z^6 u = 0 \quad u = z^{-4} \left\{ Z_{10} \left(2z^{-1/2} \right) + \overline{Z_{10} \left(2i z^{-1/2} \right)} \right\} \quad \text{WA 122(9)}$$

$$4. \quad \frac{d^4 u}{dz^4} + \frac{2}{z} \frac{d^3 u}{dz^3} - \frac{2\nu^2 + 1}{z^2} \frac{d^2 u}{dz^2} + \frac{2\nu^2 + 1}{z^3} \frac{du}{dz} + \left(\frac{\nu^4 - 4\nu^2}{z^4} - 1 \right) u = 0, \\ u = A_1 J_\nu(z) + A_2 Y_\nu(z) + A_3 I_\nu(z) + A_4 K_\nu(z), \text{ where } A_1, A_2, A_3, A_4 \text{ are constants} \quad \text{MO 29}$$

8.51–8.52 Series of Bessel functions

8.511 Generating functions for Bessel functions:

$$1. \quad \exp \frac{1}{2} \left(t - \frac{1}{t} \right) z = J_0(z) + \sum_{k=1}^{\infty} [t^k + (-t)^{-k}] J_k(z) = \sum_{k=-\infty}^{\infty} J_k(z) t^k \quad [|z| < |t|] \quad \text{KU 119(12)}$$

$$2. \quad \exp \left(t - \frac{1}{t} \right) z = \left\{ \sum_{k=-\infty}^{\infty} t^k J_k(z) \right\} \left\{ \sum_{m=-\infty}^{\infty} t^m J_m(z) \right\} \quad \text{WA 40}$$

$$3. \quad \exp (\pm i z \sin \varphi) = J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\varphi \pm 2i \sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\varphi \quad \text{KU 120(13)}$$

$$4. \quad \exp (i z \cos \varphi) = \sqrt{\frac{\pi}{2z}} \sum_{k=0}^{\infty} (2k+1) i^k J_{k+\frac{1}{2}}(z) P_k(\cos \varphi) \quad \text{WA 401(1)} \\ = \sum_{k=-\infty}^{\infty} i^k J_k(z) e^{ik\varphi} \quad \text{MO 27} \\ = J_0(z) + 2 \sum_{k=1}^{\infty} i^k J_k(z) \cos k\varphi \quad \text{MO 27}$$

$$5. \quad \sqrt{\frac{i}{\pi}} e^{iz \cos 2\varphi} \int_{-\infty}^{\sqrt{2z} \cos \varphi} e^{-it^2} dt = \frac{1}{2} J_0(z) + \sum_{k=1}^{\infty} e^{\frac{1}{4}k\pi i} J_{\frac{k}{2}}(z) \cos k\varphi \quad \text{MO 28}$$

The series $\sum J_k(z)$

8.512

$$1. \quad J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) = 1 \quad \text{WA 44}$$

$$2. \quad \sum_{k=0}^{\infty} \frac{(n+2k)(n+k-1)!}{k!} J_{n+2k}(z) = \left(\frac{z}{2}\right)^n \quad [n = 1, 2, \dots] \quad \text{WA 45}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{(4k+1)(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}}$$

8.513

Notation: In formulas **8.513** $Q_k^{(p)} = \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^m \binom{k}{m} (k-2m)^p}{2^k k!}$

$$1. \quad \sum_{k=1}^{\infty} (2k)^{2p} J_{2k}(z) = \sum_{k=0}^p Q_{2k}^{(2p)} z^{2k} \quad [p = 1, 2, 3, \dots] \quad \text{WA 46(1)}$$

$$2. \quad \sum_{k=0}^{\infty} (2k+1)^{2p+1} J_{2k+1}(z) = \sum_{k=0}^p Q_{2k+1}^{(2p+1)} z^{2k+1} \quad [p = 0, 1, 2, 3, \dots] \quad \text{WA 46(2)}$$

In particular:

$$3. \quad \sum_{k=0}^{\infty} (2k+1)^3 J_{2k+1}(z) = \frac{1}{2} (z + z^3) \quad \text{WA 47(4)}$$

$$4. \quad \sum_{k=1}^{\infty} (2k)^2 J_{2k}(z) = \frac{1}{2} z^2 \quad \text{WA 47(4)}$$

$$5. \quad \sum_{k=1}^{\infty} 2k(2k+1)(2k+2) J_{2k+1}(z) = \frac{1}{2} z^3 \quad \text{WA 47(4)}$$

8.514

$$1. \quad \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) = \frac{\sin z}{2} \quad \text{WH}$$

$$2. \quad J_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(z) = \cos z \quad \text{WH}$$

$$3. \quad \sum_{k=1}^{\infty} (-1)^{k+1} (2k)^2 J_{2k}(z) = \frac{z \sin z}{2} \quad \text{WA 32(9)}$$

4. $\sum_{k=0}^{\infty} (-1)^k (2k+1)^2 J_{2k+1}(z) = \frac{z \cos z}{2}$ WA 32(10)
5. $J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\theta = \cos(z \sin \theta)$ KU 120(14), WA 32
6. $\sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\theta = \frac{\sin(z \sin \theta)}{2}$ KU 120(15), WA 32
7. $\sum_{k=0}^{\infty} J_{2k+1}(x) = \frac{1}{2} \int_0^x J_0(t) dt$ [x is real] WA 638

8.515

1. $\sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \left(\frac{2z+t}{2z} \right)^k J_{\nu+k}(z) = \left(\frac{z}{z+t} \right)^{\nu} J_{\nu}(z+t)$ AD (9140)
 2. $\sum_{k=1}^{\infty} J_{2k-\frac{1}{2}}(x^2) = S(x)$ MO 127a
 3. $\sum_{k=0}^{\infty} J_{2k+\frac{1}{2}}(x^2) = C(x)$ MO 127a
- 8.516 $\sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{2n+2k}(2z \sin \theta) = (z \sin \theta)^{2n}$ WA 47

The series $\sum A_k J_k(kx)$ and $\sum A_k J'_k(kx)$

8.517

1. $\sum_{k=1}^{\infty} J_k(kz) = \frac{z}{2(1-z)}$ $\left[\left| \frac{z \exp \sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| < 1 \right]$ WA 615(1)
2. $\sum_{k=1}^{\infty} (-1)^k J_k(kz) = -\frac{z}{2(1+z)}$ $\left[\left| \frac{z \exp \sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| < 1 \right]$ WA 622(1)
3. $\sum_{k=1}^{\infty} J_{2k}(2kz) = \frac{z^2}{2(1-z^2)}$ $\left[\left| \frac{z \exp \sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| < 1 \right]$ MO 58

8.518

- 1.¹¹ $\sum_{k=1}^{\infty} \frac{J'_k(kx)}{k} = \frac{1}{2} + \frac{x}{4}$ $[0 \leq x < 1]$ MO 58
- 2.¹¹ $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{J'_k(kx)}{k} = \frac{1}{2} - \frac{x}{4}$ $[0 \leq x < 1]$ MO 58
3. $\sum_{k=1}^{\infty} k J'_k(kx) = \frac{1}{2(1-x)^2}$ $[0 \leq x < 1]$ MO 58

$$4. \quad \sum_{k=1}^{\infty} (-1)^{k-1} J'_k(kx) k = \frac{1}{2(1+x)^2} \quad [0 \leq x < 1] \quad \text{MO 58}$$

The series $\sum A_k J_0(kx)$

8.519 If, on the interval $[0 \leq x \leq \pi]$, a function $f(x)$ possesses a continuous derivative with respect to x that is of bounded variation, then

$$1. \quad f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k J_0(kx) \quad [0 < x < \pi]$$

where

$$2. \quad a_0 = 2f(0) + \frac{2}{\pi} \int_0^{\pi} du \int_0^{\pi/2} u f'(u \sin \varphi) d\varphi$$

$$3. \quad a_n = \frac{2}{\pi} \int_0^{\pi} du \int_0^{\pi/2} u f'(u \sin \varphi) \cos nu d\varphi \quad \text{WH}$$

8.521 Examples:

$$1. \quad \sum_{k=1}^{\infty} J_0(kx) = -\frac{1}{2} + \frac{1}{x} + 2 \sum_{m=1}^n \frac{1}{\sqrt{x^2 - 4m^2\pi^2}} \quad [2n\pi < x < 2(n+1)\pi] \quad \text{MO 59}$$

$$2. \quad \sum_{k=1}^{\infty} (-1)^{k+1} J_0(kx) = \frac{1}{2} \quad [0 < x < \pi] \quad \text{KU 124(12)}$$

$$3. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} J_0\{(2k-1)x\} \begin{cases} \frac{\pi^2}{8} - \frac{|x|}{2} & [-\pi < x < \pi] \\ \frac{\pi^2}{8} + \sqrt{x^2 - \pi^2} - \frac{x}{2} - \pi \arccos \frac{\pi}{x} & [\pi < x < 2\pi] \end{cases} \quad \begin{matrix} \text{KU 124} \\ \text{MO 59} \end{matrix}$$

$$4. \quad \sum_{k=1}^{\infty} e^{-kz} J_0\left(k\sqrt{x^2 + y^2}\right) = \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{(2ki\pi + z)^2 + x^2 + y^2}} - \frac{1}{\sqrt{(2ki\pi - z)^2 + x^2 + y^2}} \right\} \\ = \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} B_{2k} r^{2k-1} P_{2k-1}\left(\frac{z}{r}\right) \quad [0 < r < 2\pi] \quad \text{MO 59}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and where the radical indicates the square root with a positive real part. In formula **8.521** 4, the first equation holds when x and y are real and $\operatorname{Re} z > 0$; the second equation holds when x , y , and z are all real.

The series $\sum A_k Z_0(kx) \sin kx$ and $\sum A_k Z_0(kx) \cos kx$

8.522

$$1. \quad \sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - (2\pi l + tx)^2}} + \frac{1}{x\sqrt{1-t^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - (2\pi l - tx)^2}}$$

MO 59

$$2. \quad \sum_{k=1}^{\infty} J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}$$

MO 59

$$3. \quad \sum_{k=1}^{\infty} Y_0(kx) \cos kxt = -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} \\ - \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}$$

MO 60

In formulas **8.522**, $x > 0$, $0 \leq t < 1$, $2\pi m < x(1-t) < 2(m+1)\pi$, $2n\pi < x(1+t) < 2(n+1)\pi$, $m+1$ and $n+1$ are natural numbers.

8.523

$$1. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - [(2l-1)\pi + tx]^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}$$

MO 60

$$2. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}$$

MO 60

$$\begin{aligned}
3. \quad \sum_{k=1}^{\infty} (-1)^k Y_0(kx) \cos kxt &= -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} \\
&- \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} \\
&- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}
\end{aligned}$$

MO 60

In formulas **8.523**, $x > 0$, $0 \leq t < 1$, $(2m-1)\pi < x(1-t) < (2m+1)\pi$, $(2n-1)\pi < x(1+t) < (2n+1)\pi$, m and n are natural numbers.

8.524

$$\begin{aligned}
1. \quad \sum_{k=1}^{\infty} J_0(kx) \cos kxt &= -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2 - (2l\pi - tx)^2}} \\
2. \quad \sum_{k=1}^{\infty} J_0(kx) \sin kxt &\sum_{l=0}^m \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} + \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} \\
&- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l}
\end{aligned}$$

MO 60

MO 60

$$\begin{aligned}
3.^6 \quad \sum_{k=1}^{\infty} Y_0(kx) \cos kxt &= -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) - \sum_{l=0}^m \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} \\
&- \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} \\
&- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\}
\end{aligned}$$

MO 61

In formulas **8.524**, $x > 0$, $t > 1$, $2m\pi < x(t-1) < 2(m+1)\pi$, $2n\pi < x(t+1) < 2(n+1)\pi$, $m+1$ and $n+1$ are natural numbers.

8.525

$$1. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}$$

MO 61

$$\begin{aligned}
 2. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt &= \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} \\
 &+ \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} \\
 &- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}
 \end{aligned}$$

MO 61

$$\begin{aligned}
 3. \quad \sum_{k=1}^{\infty} (-1)^k Y_0(kx) \cos kxt &= -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} \\
 &- \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} \\
 &- \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} \\
 &- \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}
 \end{aligned}$$

MO 61

In formulas **8.525**, $x > 0, t > 1$, $(2m-1)\pi < x(t-1) < (2m+1)\pi$, $(2n-1)\pi < x(t+1) < (2n+1)\pi$, m and n are natural numbers.

8.526

$$\begin{aligned}
 1. \quad \sum_{k=1}^{\infty} K_0(kx) \cos kxt &= \frac{1}{2} \left(C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2x\sqrt{1+t^2}} + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi - tx)^2}} - \frac{1}{2l\pi} \right\} \\
 &+ \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi + tx)^2}} - \frac{1}{2l\pi} \right\}
 \end{aligned}$$

MO 61

$$\begin{aligned}
 2. \quad \sum_{k=1}^{\infty} (-1)^k K_0(kx) \cos kxt &= \frac{1}{2} \left(C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi - xt]^2}} - \frac{1}{2l\pi} \right\} \\
 &+ \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi + xt]^2}} - \frac{1}{2l\pi} \right\} \\
 &\quad [x > 0, \quad t \text{ real}] \quad (\text{see also } \mathbf{8.66}) \quad \text{MO 62}
 \end{aligned}$$

8.53 Expansion in products of Bessel functions

“Summation theorems”

8.530 Suppose that $r > 0, \varrho > 0, \varphi > 0$, and $R = \sqrt{r^2 + \varrho^2 - 2r\varrho \cos \varphi}$; that is, suppose that r, ϱ , and R are the sides of a triangle such that the angle between the sides r and ϱ is equal to φ . Suppose also that $\varrho < r$ and that ψ is the angle opposite the side ϱ , so that

$$1. \quad 0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \varrho e^{-i\varphi}}{r - \varrho e^{i\varphi}}$$

When these conditions are satisfied, we have the “summation theorem” for Bessel functions:

$$1. \quad e^{i\nu\psi} Z_\nu(mR) = \sum_{k=-\infty}^{\infty} J_k(m\varrho) Z_{\nu+k}(mr) e^{ik\varphi} \quad [m \text{ is an arbitrary complex number}]$$

WA 394(6)

For $Z_\nu = J_\nu$ and ν an integer, the restriction $\varrho < r$ is superfluous.

MO 31

8.531 Special cases:

$$1. \quad J_0(mR) = J_0(m\varrho) J_0(mr) + 2 \sum_{k=1}^{\infty} J_k(m\varrho) J_k(mr) \cos k\varphi \quad \text{WA 391(1)}$$

$$2. \quad H_0^{(1,2)}(mR) = J_0(m\varrho) H_0^{(1,2)}(mr) + 2 \sum_{k=1}^{\infty} J_k(m\varrho) H_k^{(1,2)}(mr) \cos k\varphi \quad \text{MO 31}$$

$$\begin{aligned} 3. \quad J_0(z \sin \alpha) &= J_0^2\left(\frac{z}{2}\right) + 2 \sum_{k=1}^{\infty} J_k^2\left(\frac{z}{2}\right) \cos 2k\alpha \\ &= \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \left(2k + \frac{1}{2}\right) \frac{(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) P_{2k}(\cos \alpha) \end{aligned} \quad \text{MO 31}$$

8.532 The term “summation theorem” is also applied to the formula

$$1. \quad \frac{Z_\nu(mR)}{R^\nu} = 2^\nu m^{-\nu} \Gamma(\nu) \sum_{k=0}^{\infty} (\nu + k) \frac{J_{\nu+k}(m\varrho)}{\varrho^\nu} \frac{Z_{\nu+k}(mr)}{r^\nu} C_k^\nu(\cos \varphi)$$

$[\nu \neq -1, -2, -3, \dots; \text{the conditions on } r, \varrho, R, \varphi, \text{ and } m \text{ are the same as in formula 8.530; for } Z_\nu = J_\nu \text{ and } \nu \text{ an integer, formula 8.532 1 is valid for arbitrary } r, \varrho, \text{ and } \varphi].$

WA 398(4)

8.533 Special cases:

$$1. \quad \frac{e^{imR}}{R} = \frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(m\varrho) H_{k+\frac{1}{2}}^{(1)}(mr) P_k(\cos \varphi) \quad \text{MO 31}$$

$$2. \quad \frac{e^{-imR}}{R} = -\frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(m\varrho) H_{k+\frac{1}{2}}^{(2)}(mr) P_k(\cos \varphi) \quad \text{MO 31}$$

8.534 A degenerate addition theorem ($r \rightarrow \infty$):

$$e^{im\varrho \cos \varphi} = \sqrt{\frac{\pi}{2m\varrho}} \sum_{k=0}^{\infty} i^k (2k+1) J_{k+\frac{1}{2}}(m\varrho) P_k(\cos \varphi) \quad \text{WA 401(1)}$$

$$= 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) i^k (m\varrho)^{-\nu} J_{\nu+k}(m\varrho) C_k^\nu(\cos \varphi) \quad [\nu \neq 0, -1, -2, \dots] \quad \text{WA 401(2)}$$

8.535 The term “product theorem” is also applied to the formula

$$Z_\nu(\lambda z) = \lambda^\nu \sum_{k=0}^{\infty} \frac{1}{k!} Z_{\nu+k}(z) \left(\frac{1-\lambda^2}{2} z \right)^k \quad [|1-\lambda|^2 < 1]$$

For $Z_\nu = J_\nu$, it is valid for all values of λ and z .

MO 32

8.536

$$1. \quad \sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{n+k}^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2} \right)^{2n} \quad [n > 0] \quad \text{WA 47(1)}$$

$$2. \quad 2 \sum_{k=n}^{\infty} \frac{k \Gamma(n+k)}{\Gamma(k-n+1)} J_k^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2} \right)^{2n} \quad [n > 0] \quad \text{WA 47(2)}$$

$$3. \quad J_0^2(z) + 2 \sum_{k=1}^{\infty} J_k^2(z) = 1 \quad \text{WA 41(3)}$$

8.537

$$1. \quad \sum_{k=-\infty}^{\infty} Z_{\nu-k}(t) J_k(z) = Z_\nu(z+t) \quad [|z| < |t|] \quad \text{WA 158(2)}$$

$$2. \quad \sum_{k=-\infty}^{\infty} J_k(z) J_{n-k}(z) = J_n(2z) \quad \text{WA 41}$$

8.538

$$1. \quad \sum_{k=-\infty}^{\infty} (-1)^k J_{-\nu+k}(t) J_k(z) = J_{-\nu}(z+t) \quad [|z| < |t|] \quad \text{WA 159}$$

$$2. \quad \sum_{k=-\infty}^{\infty} Z_{\nu+k}(t) J_k(z) = Z_\nu(t-z) \quad [|z| < |t|] \quad \text{WA 159(5)}$$

8.54 The zeros of Bessel functions

8.541 For arbitrary real ν , the function $J_\nu(z)$ has infinitely many real zeros. For $\nu > -1$, all its zeros are real. WA 526, 530

A Bessel function $Z_\nu(z)$ has no multiple zeros except possibly the coordinate origin. WA 528

8.542 All zeros of the function $Y_0(z)$ with positive real parts are real. WA 531

8.543 If $-(2s+2) < \nu < -(2s+1)$, where s is a natural number or 0, then $J_\nu(z)$ has exactly $4s+2$ complex roots, two of which are purely imaginary. If $-(2s+1) < \nu < -2s$, where s is a natural number, then the function $J_\nu(z)$ has exactly $4s$ complex zeros, none of which are purely imaginary. WA 532

8.544 If x_ν and x'_ν are, respectively, the smallest positive zeros of the functions $J_\nu(z)$ and $J'_\nu(z)$ for $\nu > 0$, then $x_\nu > \nu$ and $x'_\nu > \nu$. Suppose also that y_ν is the smallest positive zero of the function $Y_\nu(z)$. Then, $x_\nu < y_\nu < x'_\nu$. WA 534, 536

Suppose that $z_{\nu,m}$ (for $m = 1, 2, 3, \dots$) are the zeros of the function $z^{-\nu} J_\nu(z)$, numbered in order of the absolute value of their real parts. Here, we assume that $\nu \neq -1, -2, -3, \dots$. Then, for arbitrary z

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right). \quad \text{WA 550}$$

8.545⁸ The number of zeros of the function $z^{-\nu} J_\nu(z)$ that occur between the imaginary axis and the line on which

$$\operatorname{Re} z = \left(m + \frac{1}{2} \operatorname{Re} \nu + \frac{1}{4}\right) \pi, \quad \text{WA 497}$$

is exactly m .

8.546 For $\nu \geq 0$, the number of zeros of the function $K_\nu(z)$ that occur in the region $\operatorname{Re} z < 0$, $|\arg z| < \pi$ is equal to the even number closest to $\nu - \frac{1}{2}$. WA 562

8.547 Large zeros of the functions $J_\nu(z) \cos \alpha - Y_\nu(z) \sin \alpha$, where ν and α are real numbers, are given by the asymptotic expansion

$$\begin{aligned} x_{\nu,m} \sim & \left(m + \frac{1}{2}\nu - \frac{1}{4}\right) \pi - \alpha - \frac{4\nu^2 - 1}{8 \left[(m + \frac{1}{2}\nu - \frac{1}{4}) \pi - \alpha\right]} \\ & - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384 \left[(m + \frac{1}{2}\nu - \frac{1}{4}) \pi - \alpha\right]^3} - \dots \end{aligned} \quad \text{KU 109(24), WA 558}$$

8.548 In particular, large zeros of the function $J_0(z)$ are given by the expansion

$$x_{0,m} \sim \frac{\pi}{4}(4m-1) + \frac{1}{2\pi(4m-1)} - \frac{31}{6\pi^3(4m-1)^3} + \frac{3779}{15\pi^5(4m-1)^5} - \dots \quad \text{KU 109(25), WA 556}$$

This series is suitable for calculating all (except the smallest x_{01}) zeros of the function $J_0(z)$ correctly to at least five digits.

8.549 To calculate the roots $x_{\nu,m}$ of the function $J_\nu(z)$ of smallest absolute value, we may use the identity

$$\sum_{m=1}^{\infty} \frac{1}{x_{\nu,m}^{16}} = \frac{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738}{2^{16}(\nu+1)^8(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)}. \quad \text{KU 112(27)a, WA 554}$$

8.55 Struve functions

8.550 Definitions:

$$1. \quad \mathbf{H}_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right) \Gamma\left(\nu + m + \frac{3}{2}\right)} \quad \text{WA 358(2)}$$

$$2. \quad \mathbf{L}_\nu(z) = -ie^{-i\nu\frac{\pi}{2}} \mathbf{H}_\nu\left(ze^{i\frac{\pi}{2}}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right) \Gamma\left(\nu + m + \frac{3}{2}\right)} \quad \text{WA 360(11)}$$

8.551 Integral representations:

$$1. \quad \mathbf{H}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin zt \, dt = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sin(z \cos \varphi) (\sin \varphi)^{2\nu} \, d\varphi$$

WA 358(1)

$[\operatorname{Re} \nu > -\frac{1}{2}]$

$$2. \quad \mathbf{L}_\nu(z) = \frac{2 \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sinh(z \cos \varphi) (\sin \varphi)^{2\nu} d\varphi$$

[Re $\nu > -\frac{1}{2}$] WA 360(11)

8.552 Special cases:

$$1.^6 \quad \mathbf{H}_n(z) = \frac{1}{\pi} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma\left(m + \frac{1}{2}\right) \left(\frac{z}{2}\right)^{n-2m-1}}{\Gamma\left(n + \frac{1}{2} - m\right)} - \mathbf{E}_n(z) \quad [n = 1, 2, \dots] \quad \text{EH II 40(66), WA 337(1)}$$

$$2.^6 \quad \mathbf{H}_{-n}(z) = (-1)^{n+1} \frac{1}{\pi} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma\left(n - m - \frac{1}{2}\right) \left(\frac{z}{2}\right)^{-n+2m+1}}{\Gamma\left(m + \frac{3}{2}\right)} - \mathbf{E}_{-n}(z)$$

[$n = 1, 2, \dots$] EH II 40(67), WA 337(2)

$$3. \quad \mathbf{H}_{n+\frac{1}{2}}(z) = Y_{n+\frac{1}{2}}(z) + \frac{1}{\pi} \sum_{m=0}^n \frac{\Gamma\left(m + \frac{1}{2}\right) \left(\frac{z}{2}\right)^{-2m+n-\frac{1}{2}}}{\Gamma(n+1-m)}$$

[$n = 0, 1, \dots$] EH II 39(64)

$$4. \quad \mathbf{H}_{-(n+\frac{1}{2})}(z) = (-1)^n J_{n+\frac{1}{2}}(z) \quad [n = 0, 1, \dots] \quad \text{EH II 39(65)}$$

$$5. \quad \mathbf{L}_{-(n+\frac{1}{2})}(z) = I_{n+\frac{1}{2}}(z) \quad [n = 0, 1, \dots] \quad \text{EH II 39(65)}$$

$$6. \quad \mathbf{H}_{\frac{1}{2}}(z) = \frac{\sqrt{2}}{\sqrt{\pi z}} (1 - \cos z) \quad \text{EH II 39, WA 364(3)}$$

$$7. \quad \mathbf{H}_{\frac{3}{2}}(z) = \left(\frac{z}{2\pi}\right)^{1/2} \left(1 + \frac{2}{z^2}\right) - \left(\frac{2}{\pi z}\right)^{1/2} \left(\sin z + \frac{\cos z}{z}\right) \quad \text{WA 364(3)}$$

8.553 Functional relations:

$$1. \quad \mathbf{H}_\nu(z e^{im\pi}) = e^{i\pi(\nu+1)m} \mathbf{H}_\nu(z) \quad [m = 1, 2, 3, \dots] \quad \text{WA 362(5)}$$

$$2. \quad \frac{d}{dz} [z^\nu \mathbf{H}_\nu(z)] = z^\nu \mathbf{H}_{\nu-1}(z) \quad \text{WA 358}$$

$$3. \quad \frac{d}{dz} [z^{-\nu} \mathbf{H}_\nu(z)] = 2^{-\nu} \pi^{-1/2} [\Gamma(\nu + \frac{3}{2})]^{-1} - z^{-\nu} \mathbf{H}_{\nu+1}(z) \quad \text{WA 359}$$

$$4. \quad \mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{H}_\nu(z) + \pi^{-1/2} \left(\frac{z}{2}\right)^\nu [\Gamma(\nu + \frac{3}{2})]^{-1} \quad \text{WA 359(5)}$$

$$5. \quad \mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}'_\nu(z) - \pi^{-1/2} \left(\frac{z}{2}\right)^\nu [\Gamma(\nu + \frac{3}{2})]^{-1} \quad \text{WA 359(6)}$$

8.554 Asymptotic representations:

$$\mathbf{H}_\nu(\xi) = Y_\nu(\xi) + \frac{1}{\pi} \sum_{m=0}^{p-1} \frac{\Gamma\left(m + \frac{1}{2}\right) \left(\frac{\xi}{2}\right)^{-2m+\nu-1}}{\Gamma\left(\nu + \frac{1}{2} - m\right)} + O\left(|\xi|^{\nu-2p-1}\right)$$

[$|\arg \xi| < \pi$] EH II 39(63), WA 363(2)

For the asymptotic representation of $Y_\nu(\xi)$, see **8.451** 2.

8.555 The differential equation for Struve functions:

$$z^2 y'' + zy' + (z^2 - \nu^2)y = \frac{1}{\sqrt{\pi}} \frac{4 \left(\frac{z}{2}\right)^{\nu+1}}{\Gamma\left(\nu + \frac{1}{2}\right)} \quad \text{WA 359(10)}$$

8.56 Thomson functions and their generalizations

$\text{ber}_\nu(z)$, $\text{bei}_\nu(z)$, $\text{her}_\nu(z)$, $\text{hei}_\nu(z)$, $\text{ker}_\nu(z)$, $\text{kei}_\nu(z)$

8.561

$$1. \quad \text{ber}_\nu(z) + i \text{bei}_\nu(z) = J_\nu \left(z e^{\frac{3}{4}\pi i} \right) \quad \text{WA 96(6)}$$

$$2. \quad \text{ber}_\nu(z) - i \text{bei}_\nu(z) = J_\nu \left(z e^{-\frac{3}{4}\pi i} \right). \quad \text{WA 96(6)}$$

8.562

$$1. \quad \text{her}_\nu(z) + i \text{hei}_\nu(z) = H_{(1)}^\nu \left(z e^{\frac{3}{4}\pi i} \right) \quad (\text{see also } \mathbf{8.567}) \quad \text{WA 96(7)}$$

$$2. \quad \text{her}_\nu(z) - i \text{hei}_\nu(z) = H_{(1)}^\nu \left(z e^{-\frac{3}{4}\pi i} \right) \quad (\text{see also } \mathbf{8.567}) \quad \text{WA 96(7)}$$

8.563

$$1. \quad \text{ber}_0(z) \equiv \text{ber}(z); \quad \text{bei}_0(z) \equiv \text{bei}(z) \quad \text{WA 96(8)}$$

$$2. \quad \text{ker}(z) \equiv -\frac{\pi}{2} \text{hei}_0(z); \quad \text{kei}(z) \equiv \frac{\pi}{2} \text{hei}_0(z) \quad \text{WA 96(8)}$$

For integral representations, see **6.251**, **6.536**, **6.537**, **6.772** 4, **6.777**.

Series representation

8.564

$$1. \quad \text{ber}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{2^{4k} [(2k)!]^2} \quad \text{WA 96(3)}$$

$$2. \quad \text{bei}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{2^{4k+2} [(2k+1)!]^2} \quad \text{WA 96(4)}$$

$$3. \quad \text{ker}(z) = \left(\ln \frac{2}{z} - C \right) \text{ber}(z) + \frac{\pi}{4} \text{bei}(z) + \sum_{k=1}^{\infty} (-1)^k \frac{z^{4k}}{2^{4k} [(2k)!]^2} \sum_{m=1}^{2k} \frac{1}{m} \quad \text{WA 96(9)a, DW}$$

$$4. \quad \text{kei}(z) = \left(\ln \frac{2}{z} - C \right) \text{bei}(z) - \frac{\pi}{4} \text{ber}(z) + \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{2^{4k+2} [(2k+1)!]^2} \sum_{m=1}^{2k+1} \frac{1}{m} \quad \text{WA 96(10)a, DW}$$

$$\mathbf{8.565} \quad \text{ber}_\nu^2(z) + \text{bei}_\nu^2(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2\nu+4k}}{k! \Gamma(\nu+k+1) \Gamma(\nu+2k+1)} \quad \text{WA 163(6)}$$

Asymptotic representation

8.566

1. $\text{ber}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \cos \beta(z) \quad \left[|\arg z| < \frac{\pi}{4} \right] \quad \text{WA 227(1)}$
2. $\text{bei}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \sin \beta(z) \quad \left[|\arg z| < \frac{\pi}{4} \right] \quad \text{WA 227(1)}$
3. $\text{ker}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \cos \beta(-z) \quad \left[|\arg z| < \frac{5}{4} \right] \quad \text{WA 227(2)}$
4. $\text{kei}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \sin \beta(-z) \quad \left[|\arg z| < \frac{5}{4} \pi \right], \quad \text{WA 227(2)}$

where

$$\alpha(z) \sim \frac{z}{\sqrt{2}} + \frac{1}{8z\sqrt{2}} - \frac{25}{384z^3\sqrt{2}} - \frac{13}{128z^4} - \dots,$$

$$\beta(z) \sim \frac{z}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8z\sqrt{2}} - \frac{1}{16z^2} - \frac{25}{384z^3\sqrt{2}} + \dots$$

8.567 Functional relations

1. $\text{ker}(z) + i \text{kei}(z) = K_0(z\sqrt{i}) \quad (\text{see } \mathbf{8.562}) \quad \text{WA 96(5), DW}$
2. $\text{ker}(z) - i \text{kei}(z) = K_0(z\sqrt{-i}) \quad (\text{see } \mathbf{8.562}) \quad \text{WA 96(5), DW}$

For integrals of Thomson's functions, see **6.87**.

8.57 Lommel functions

8.570 Definitions of the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$:

1.
$$s_{\mu,\nu}(z) = \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2 - \nu^2][(\mu+3)^2 - \nu^2] \dots [(\mu+2m+1)^2 - \nu^2]}$$

$$= z^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+2} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + m + \frac{3}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + m + \frac{3}{2}\right)}$$

[$\mu \pm \nu$ is not a negative odd integer] EH II 40(69), WA 377(2)
- 2.¹¹
$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)$$

$$\times \frac{\cos\left[\frac{1}{2}(\mu - \nu)\pi\right] J_{-\nu}(z) - \cos\left[\frac{1}{2}(\mu + \nu)\pi\right] J_{\nu}(z)}{\sin \nu \pi}$$

EH II 40(71), WA 379(2)

$$= s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)$$

$$\times \left\{ \sin\left[\frac{1}{2}(\mu - \nu)\pi\right] J_{\nu}(z) - \cos\left[\frac{1}{2}(\mu - \nu)\pi\right] Y_{\nu}(z) \right\}$$

EH II 41(71), WA 379(3)

Integral representations

$$8.571 \quad s_{\mu,\nu}(z) = \frac{\pi}{2} \left[Y_{\nu}(z) \int_0^z z^{\mu} J_{\nu}(z) dz - J_{\nu}(z) \int_0^z z^{\mu} Y_{\nu}(z) dz \right] \quad \text{WA 378(9)}$$

$$8.572 \quad s_{\mu,\nu}(z) = 2^{\mu} \left(\frac{z}{2} \right)^{\frac{1}{2}(1+\nu+\mu)} \Gamma \left(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu \right) \int_0^{\pi/2} J_{\frac{1}{2}(1+\mu-\nu)}(z \sin \theta) (\sin \theta)^{\frac{1}{2}(1+\nu-\mu)} (\cos \theta)^{\nu+\mu} d\theta$$

$$[\operatorname{Re}(\nu + \mu + 1) > 0] \quad \text{EH II 42(86)}$$

8.573 Special cases:

$$1. \quad S_{1,2n}(z) = z O_{2n}(z) \quad \text{WA 382(1)}$$

$$2. \quad S_{0,2n+1}(z) = \frac{z}{2n+1} O_{2n+1}(z) \quad \text{WA 382(1)}$$

$$3. \quad S_{-1,2n}(z) = \frac{1}{4n} S_{2n}(z) \quad \text{WA 382(2)}$$

$$4. \quad S_{0,2n+1}(z) = \frac{1}{2} S_{2n+1}(z) \quad \text{WA 382(2)}$$

$$5. \quad S_{\nu,\nu}(z) = \Gamma \left(\nu + \frac{1}{2} \right) \sqrt{\pi} 2^{\nu-1} \mathbf{H}_{\nu}(z) \quad \text{EH II 42(84)}$$

$$6. \quad S_{\nu,\nu}(z) = [\mathbf{H}_{\nu}(z) - Y_{\nu}(z)] 2^{\nu-1} \sqrt{\pi} \Gamma \left(\nu + \frac{1}{2} \right) \quad \text{EH II 42(84)}$$

8.574 Connections with other special functions:

$$1. \quad \mathbf{J}_{\nu}(z) = \frac{1}{\pi} \sin(\nu\pi) [s_{0,\nu}(z) - \nu s_{-1,\nu}(z)] \quad \text{EH II 41(82)}$$

$$2. \quad \mathbf{E}_{\nu}(z) = -\frac{1}{\pi} [(1 + \cos \nu\pi) s_{0,\nu}(z) + \nu (1 - \cos \nu\pi) s_{-1,\nu}(z)] \quad \text{EH II 42(83)}$$

A connection with a hypergeometric function

$$3. \quad s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu - \nu + 1)(\mu + \nu + 1)} {}_1F_2 \left(1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4} \right)$$

EH II 40(69), WA 378(10)

8.575 Functional relations:

$$1. \quad s_{\mu+2,\nu}(z) = z^{\mu+1} - [(\mu+1)^2 - \nu^2] s_{\mu,\nu}(z) \quad \text{EH II 41(73), WA 380(1)}$$

$$2.^8 \quad s'_{\mu,\nu}(z) + \left(\frac{\nu}{z} \right) s_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z) \quad \text{EH II 41(74), WA 380(2)}$$

$$3. \quad s'_{\mu,\nu}(z) - \left(\frac{\nu}{z} \right) s_{\mu,\nu}(z) = (\mu - \nu - 1) s_{\mu-1,\nu+1}(z) \quad \text{EH II 41(75), WA 380(3)}$$

$$4. \quad \left(2 \frac{\nu}{z} \right) s_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z) - (\mu - \nu - 1) s_{\mu-1,\nu+1}(z) \quad \text{EH II 41(76), WA 380(4)}$$

$$5.^8 \quad 2 s'_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z) + (\mu - \nu - 1) s_{\mu-1,\nu+1}(z) \quad \text{EH II 41(77), WA 380(5)}$$

In formulas **8.575** 1–5, $s_{\mu,\nu}(z)$ can be replaced with $S_{\mu,\nu}(z)$.

8.576 Asymptotic expansion of $S_{\mu,\nu}(z)$.

In the case in which $\mu \pm \nu$ is not a positive odd integer, $S_{\mu,\nu}(z)$ has the following asymptotic expansion:

$$S_{\mu,\nu}(z) \sim z^{\mu-1} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1-\mu+\nu}{2} \right)_m \left(\frac{1-\mu-\nu}{2} \right)_m \left(\frac{z}{2} \right)^{-2m} \quad [|z| \rightarrow \infty, \quad |\arg z| < \pi] \quad \text{WA 347, 352}$$

The series terminates and is equal to $S_{\mu,\nu}(z)$ when $\mu \pm \nu$ is a positive odd integer.

8.577 Lommel functions satisfy the following differential equation:

$$z^2 w'' + z w' + (z^2 - \nu^2) w = z^{\mu+1} \quad \text{WA 377(1), EH II 40(68)}$$

8.578 Lommel functions of two variables $U_\nu(w, z)$ and $V_\nu(w, z)$:**Definition**

$$1. \quad U_\nu(w, z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z} \right)^{\nu+2m} J_{\nu+2m}(z) \quad \text{EH II 42(87), WA 591(5)}$$

$$2. \quad V_\nu(w, z) = \cos \left[\frac{1}{2} \left(w + \frac{z^2}{w} + \nu\pi \right) \right] + U_{-\nu+2}(w, z) \quad \text{EH II 42(88), WA 591(6)}$$

Particular values:

$$3. \quad U_0(z, z) = V_0(z, z) = \frac{1}{2} \{ J_0(z) + \cos z \} \quad \text{WA 591(9)}$$

$$4. \quad U_1(z, z) = -V_1(z, z) = \frac{1}{2} \sin z \quad \text{WA 591(10)}$$

$$5. \quad U_{2n}(z, z) = \frac{(-1)^n}{2} \left\{ \cos z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m} J_{2m}(z) \right\} \quad [n \geq 1], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases} \quad \text{WA 591(11)}$$

$$6. \quad U_{2n+1}(z, z) = \frac{(-1)^n}{2} \left\{ \sin z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m+1} J_{2m+1}(z) \right\} \quad [n \geq 0], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases} \quad \text{WA 591(12)}$$

$$7. \quad V_n(w, z) = (-1)^n U_n \left(\frac{z^2}{w}, z \right)$$

$$8. \quad U_\nu(w, 0) = \frac{\left(\frac{w}{2} \right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2}, \frac{1}{2}} \left(\frac{w}{2} \right) \quad \text{WA 593(9)}$$

$$9. \quad V_{-\nu+2}(w, 0) = \frac{\left(\frac{w}{2} \right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2}, \frac{1}{2}} \left(\frac{w}{2} \right) \quad \text{WA 593(10)}$$

8.579 Functional relations:

$$1. \quad 2 \frac{\partial}{\partial w} U_\nu(w, z) = U_{\nu-1}(w, z) + \left(\frac{z}{w} \right)^2 U_{\nu+1}(w, z) \quad \text{WA 593(2)}$$

$$2. \quad 2 \frac{\partial}{\partial w} V_\nu(w, z) = V_{\nu+1}(w, z) + \left(\frac{z}{w} \right)^2 V_{\nu-1}(w, z) \quad \text{WA 593(4)}$$

$$3. \quad \text{The function } U_\nu(w, z) \text{ is a particular solution of the differential equation}$$

$$\frac{\partial^2 U}{\partial z^2} - \frac{1}{z} \frac{\partial U}{\partial z} + \frac{z^2 U}{w^2} = \left(\frac{w}{z}\right)^{\nu-2} J_\nu(z) \quad \text{WA 592(2)}$$

4. The function $V_\nu(w, z)$ is a particular solution of the differential equation

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{z} \frac{\partial V}{\partial z} + \frac{z^2 V}{w^2} = \left(\frac{w}{z}\right)^{-\nu} J_{-\nu+2}(z) \quad \text{WA 592(3)}$$

8.58 Anger and Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$

8.580 Definitions:

1. The Anger function $\mathbf{J}_\nu(z)$:

$$\mathbf{J}_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta \quad \text{WA 336(1), EH II 35(32)}$$

2. The Weber function $\mathbf{E}_\nu(z)$:

$$\mathbf{E}_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu\theta - z \sin \theta) d\theta \quad \text{WA 336(2), EH II 35(32)}$$

8.581 Series representations:

$$\begin{aligned} 1. \quad \mathbf{J}_\nu(z) = & \cos \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right) \Gamma\left(n+1-\frac{1}{2}\nu\right)} \\ & + \sin \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right) \Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)} \end{aligned} \quad \text{EH II 36(36), WA 337(3)}$$

$$\begin{aligned} 2. \quad \mathbf{E}_\nu(z) = & \sin \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right) \Gamma\left(n+1-\frac{1}{2}\nu\right)} \\ & - \cos \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right) \Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)} \end{aligned} \quad \text{EH II 36(37), WA 338(4)}$$

8.582 Functional relations:

$$\begin{aligned} 1.^6 \quad & 2\mathbf{J}'_\nu(z) = \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z) & \text{EH II 36(40), WA 340(2)} \\ 2.^6 \quad & 2\mathbf{E}'_\nu(z) = \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z) & \text{EH II 36(41), WA 340(6)} \\ 3.^6 \quad & \mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{J}_\nu(z) - 2(\pi z)^{-1} \sin(\nu\pi) & \text{EH II 36(42), WA 340(1)} \\ 4.^6 \quad & \mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{E}_\nu(z) - 2(\pi z)^{-1} (1 - \cos \nu\pi) & \text{EH II 36(43), WA 340(5)} \end{aligned}$$

8.583 Asymptotic expansions:

$$1.^6 \quad \mathbf{J}_\nu(z) = J_\nu(z) + \frac{\sin \nu\pi}{\pi z} \left[\sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + \frac{1+\nu}{2})}{\Gamma(\frac{1+\nu}{2})} \frac{\Gamma(n + \frac{1-\nu}{2})}{\Gamma(\frac{1-\nu}{2})} z^{-2n} \right. \\ \left. + O(|z|^{-2p}) - \nu \sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + 1 + \frac{1}{2}\nu)}{\Gamma(1 + \frac{1}{2}\nu)} \frac{\Gamma(n + 1 - \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}\nu)} z^{-2n-1} + \nu O(|z|^{-2p-1}) \right] \\ [|\arg z| < \pi] \quad \text{EH II 37(47), WA 344(1)}$$

$$2. \quad \mathbf{E}_\nu(z) = -Y_\nu(z) \\ - \frac{1 + \cos(\nu\pi)}{\pi z} \left[\sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + \frac{1+\nu}{2})}{\Gamma(\frac{1+\nu}{2})} \frac{\Gamma(n + \frac{1-\nu}{2})}{\Gamma(\frac{1-\nu}{2})} z^{-2n} + O(|z|^{-2p}) \right] \\ - \frac{\nu(1 - \cos \nu\pi)}{z\pi} \left[\sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + 1 + \frac{1}{2}\nu)}{\Gamma(1 + \frac{1}{2}\nu)} \frac{\Gamma(n + 1 - \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}\nu)} z^{-2n-1} + O(|z|^{-2p-1}) \right] \\ \text{WA 344(2), EH II 37(48)}$$

For the asymptotic expansion of $J_\nu(z)$ and $Y_\nu(z)$, see **8.451**.

8.584 The Anger and Weber functions satisfy the differential equation

$$y'' + z^{-1}y' + \left(1 - \frac{\nu^2}{z^2}\right)y = f(\nu, z),$$

$$\text{where } f(\nu, z) = \frac{z - \nu}{\pi z^2} \sin \nu\pi \text{ for } \mathbf{J}_\nu(z) \quad \text{WA 341(9), EH II 37(44)}$$

$$\text{and } f(\nu, z) = -\frac{1}{\pi z^2} [z + \nu + (z - \nu) \cos \nu\pi] \text{ for } \mathbf{E}_\nu(z) \quad \text{EH II 37(45), WA 341(10)}$$

8.59 Neumann's and Schlöfli's polynomials: $O_n(z)$ and $S_n(z)$

8.590 Definition of Neumann's polynomials

$$1. \quad O_n(z) = \frac{1}{4} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n(n-m-1)!}{m!} \left(\frac{z}{2}\right)^{2m-n-1} \quad [n \geq 1] \quad \text{WA 299(2), EH II 33(6)}$$

$$2. \quad O_{-n}(z) = (-1)^n O_n(z) \quad [n \geq 1] \quad \text{WA 303(8)}$$

$$3. \quad O_0(z) = \frac{1}{z} \quad \text{WA 299(3), EH II 33(7)}$$

$$4. \quad O_1(z) = \frac{1}{z^2} \quad \text{EH II 33(7)}$$

$$5. \quad O_2(z) = \frac{1}{z} + \frac{4}{z^3} \quad \text{EH II 33(7)}$$

In general, $O_n(z)$ is a polynomial in z^{-1} of degree $n+1$.

8.591 Functional relations:

$$1. \quad O'_0(z) = -O_1(z) \quad \text{EH II 33(9), WA 301(3)}$$

$$2. \quad 2 O'_n(z) = O_{n-1}(z) - O_{n+1}(z) \quad [n \geq 1] \quad \text{EH II 33(10), WA 301(2)}$$

$$3. \quad (n-1) O_{n+1}(z) + (n+1) O_{n-1}(z) - 2z^{-1} (n^2 - 1) O_n(z) = 2nz^{-1} \left(\sin n \frac{\pi}{2} \right)^2 \quad [n \geq 1] \quad \text{EH II 33(11), WA 301(1)}$$

$$4. \quad nz O_{n-2}(z) - (n^2 - 1) O_n(z) = (n-1)z O'_n(z) + n \left(\sin n \frac{\pi}{2} \right)^2 \quad \text{EH II 33(12), WA 303(4)}$$

$$5. \quad nz O_{n+1}(z) - (n^2 - 1) O_n(z) = -(n+1)z O'_n(z) + n \left(\sin n \frac{\pi}{2} \right)^2 \quad \text{EH II 33(13), WA 303(5)a}$$

8.592 The generating function:

$$\frac{1}{z - \xi} = J_0(\xi)z^{-1} + 2 \sum_{n=1}^{\infty} J_n(\xi) O_n(z) \quad [|\xi| < |z|] \quad \text{EH II 32(1), WA 298(1)}$$

8.593 The integral representation:

$$O_n(z) = \int_0^{\infty} \frac{[u + \sqrt{u^2 + z^2}]^n + [u - \sqrt{u^2 + z^2}]^n}{2z^{n+1}} e^{-u} du$$

See also **3.547** 6, 8, **3.549** 1, 2.

EH II 32(3), WA 305(1)

8.594 The inequality

$$|O_n(z)| \leq 2^{n-1} n! |z|^{-n-1} e^{\frac{1}{4}|z|^2} \quad [n > 1] \quad \text{EH II 33(8), WA 300(8)}$$

8.595 Neumann's polynomial $O_n(z)$ satisfies the differential equation

$$z^2 \frac{d^2 y}{dz^2} + 3z \frac{dy}{dz} + (z^2 + 1 - n^2) y = z \left(\cos n \frac{\pi}{2} \right)^2 + n \left(\sin n \frac{\pi}{2} \right)^2 \quad \text{EH II 33(14), WA 303(1)}$$

8.596 Schläfli's polynomials $S_n(z)$. These are the functions that satisfy the formulas

$$1. \quad S_0(z) = 0 \quad \text{EH II 34(18), WA 312(2)}$$

$$2. \quad S_n(z) = \frac{1}{n} \left[2z O_n(z) - 2 \left(\cos n \frac{\pi}{2} \right)^2 \right] \quad [n \geq 1] \quad \text{EH II 34(19), WA 312(3)}$$

$$= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-m-1)!}{m!} \left(\frac{z}{2} \right)^{2m-n} \quad [n \geq 1] \quad \text{EH II 34(18)}$$

$$3. \quad S_{-n}(z) = (-1)^{n+1} S_n(z) \quad \text{WA 313(6)}$$

8.597 Functional relations:

$$1. \quad S_{n-1}(z) + S_{n+1}(z) = 4 O_n(z) \quad \text{WA 313(7)}$$

Other functional relations may be obtained from **8.591** by replacing $O_n(z)$ with the expression for $S_n(z)$ given by **8.596** 2.

8.6 Mathieu Functions

8.60 Mathieu's equation

$$\frac{d^2 y}{dz^2} + (a - 2k^2 \cos 2z) y = 0, \quad k^2 = q$$

MA

8.61 Periodic Mathieu functions

8.610 In general, Mathieu's equation **8.60** does not have periodic solutions. If k is a real number, there exist infinitely many *eigenvalues* a , not identically equal to zero, corresponding to the periodic solutions

$$y(z) = y(2\pi + z).$$

If k is nonzero, there are no other linearly independent periodic solutions. Periodic solutions of Mathieu's equations are called *Mathieu's periodic functions* or *Mathieu functions of the first kind*, or, more simply, *Mathieu functions*.

8.611 Mathieu's equation has four series of distinct periodic solutions:

$$1. \quad \text{ce}_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz \quad \text{MA}$$

$$2. \quad \text{ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z \quad \text{MA}$$

$$3. \quad \text{se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z \quad \text{MA}$$

$$4. \quad \text{se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z \quad \text{MA}$$

5. The coefficients A and B depend on q . The eigenvalues a of the functions ce_{2n} , ce_{2n+1} , se_{2n} , se_{2n+1} are denoted by a_{2n} , a_{2n+1} , b_{2n} , b_{2n+1} .

8.612 The solutions of Mathieu's equation are normalized so that

$$\int_0^{2\pi} y^2 dx = \pi \quad \text{MO 65}$$

8.613

$$1. \quad \lim_{q \rightarrow 0} \text{ce}_0(x) = \frac{1}{\sqrt{2}}$$

$$2. \quad \lim_{q \rightarrow 0} \text{ce}_n(x) = \cos nx \quad [n \neq 0]$$

$$3. \quad \lim_{q \rightarrow 0} \text{se}_n(x) = \sin nx \quad \text{MO 65}$$

8.62 Recursion relations for the coefficients $A_{2r}^{(2n)}$, $A_{2r+1}^{(2n+1)}$, $B_{2r+1}^{(2n+1)}$, $B_{2r+2}^{(2n+2)}$

8.621

$$1. \quad aA_0^{(2n)} - qA_2^{(2n)} = 0 \quad \text{MA}$$

$$2. \quad (a-4)A_2^{(2n)} - q(A_4^{(2n)} + 2A_0^{(2n)}) = 0 \quad \text{MA}$$

$$3. \quad (a-4r^2)A_{2r}^{(2n)} - q(A_{2r+2}^{(2n)} + A_{2r-2}^{(2n)}) = 0 \quad [r \geq 2] \quad \text{MA}$$

8.622

1. $(a - 1 - q)A_1^{(2n+1)} - qA_3^{(2n+1)} = 0$ MA
2. $[a - (2r + 1)^2] A_{2r+1}^{(2n+1)} - q \left(A_{2r+3}^{(2n+1)} + A_{2r-1}^{(2n+1)} \right) = 0 \quad [r \geq 1]$ MA

8.623

1. $(a - 1 + q)B_1^{(2n+1)} - qB_3^{(2n+1)} = 0$ MA
2. $[a - (2r + 1)^2] B_{2r+1}^{(2n+1)} - q \left(B_{2r+3}^{(2n+1)} + B_{2r-1}^{(2n+1)} \right) = 0$
 $[r \geq 1]$ MA

8.624

1. $(a - 4)B_2^{(2n+2)} - qB_4^{(2n+2)} = 0$ MA
- 2.¹¹ $(a - 4r^2) B_{2r}^{(2n+2)} - q \left(B_{2r+2}^{(2n+2)} + B_{2r-2}^{(2n+2)} \right) = 0 \quad [r \geq 2]$ MA

8.625 We can determine the coefficients A and B from equations **8.612**, **8.613** and **8.621-8.624** provided a is known. Suppose, for example, that we need to determine the coefficients $A_{2r}^{(2n)}$ for the function $\text{ce}_{2n}(z, q)$. From the recursion formulas, we have

$$1. \quad \begin{vmatrix} a & -q & 0 & 0 & 0 & \dots \\ -2q & a-4 & -q & 0 & 0 & \dots \\ 0 & -q & a-16 & -q & 0 & \dots \\ 0 & 0 & -q & a-36 & -q & \dots \\ 0 & 0 & 0 & -q & a-64 & \dots \\ \vdots & \vdots & \vdots & & & \ddots \end{vmatrix} = 0 \quad \text{ST}$$

For given q in equation **8.625** 1, we may determine the eigenvalues

$$2. \quad a = A_0, A_2, A_4, \dots \quad [|A_0| \leq |A_2| \leq |A_4| \leq \dots]$$

If we now set $a = A_{2n}$, we can determine the coefficients $A_{2r}^{(2n)}$ from the recursion formulas **8.621** up to a proportionality coefficient. This coefficient is determined from the formula

$$3. \quad 2 \left[A_0^{(2n)} \right]^2 + \sum_{r=1}^{\infty} \left[A_{2r}^{(2n)} \right]^2 = 1, \quad \text{MA}$$

which follows from the conditions of normalization.

8.63 Mathieu functions with a purely imaginary argument

8.630 If, in equation **8.60**, we replace z with iz , we arrive at the differential equation

$$1.^{11} \quad \frac{d^2 y}{dz^2} + (-a + 2q \cosh 2z) y = 0$$

We can find the solutions of this equation if we replace the argument z with iz in the functions $\text{ce}_n(z, q)$ and $\text{se}_n(z, q)$. The functions obtained in this way are called *associated Mathieu functions of the first kind* and are denoted as follows:

$$1. \quad \text{Ce}_{2n}(z, q), \quad \text{Ce}_{2n+1}(z, q), \quad \text{Se}_{2n+1}(z, q), \quad \text{Se}_{2n+2}(z, q)$$

8.631

$$1. \quad \text{Ce}_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2rz \quad \text{MA}$$

$$2. \quad \text{Ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh(2r+1)z \quad \text{MA}$$

$$3. \quad \text{Se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh(2r+1)z \quad \text{MA}$$

$$4. \quad \text{Se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh(2r+2)z \quad \text{MA}$$

8.64 Non-periodic solutions of Mathieu's equation

Along with each periodic solution of equation **8.60**, there exists a second non-periodic solution that is linearly independent. The non-periodic solutions are denoted as follows:

$$\text{fe}_{2n}(z, q), \quad \text{fe}_{2n+1}(z, q), \quad \text{ge}_{2n+1}(z, q), \quad \text{ge}_{2n+2}(z, q).$$

Analogously, the second solutions of equation **8.630 1** are denoted by

$$\text{Fe}_{2n}(z, q), \quad \text{Fe}_{2n+1}(z, q), \quad \text{Ge}_{2n+1}(z, q), \quad \text{Ge}_{2n+2}(z, q).$$

8.65 Mathieu functions for negative q

8.651 If we replace the argument z in equation **8.60** with $\pm \left(\frac{\pi}{2} \pm z \right)$, we get the equation

$$\frac{d^2 y}{dz^2} + (a + 2q \cos 2z) y = 0. \quad \text{MA}$$

This equation has the following solutions:

8.652

$$1. \quad \text{ce}_{2n}(z, -q) = (-1)^n \text{ce}_{2n} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

$$2. \quad \text{ce}_{2n+1}(z, -q) = (-1)^n \text{se}_{2n+1} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

$$3. \quad \text{se}_{2n+1}(z, -q) = (-1)^n \text{ce}_{2n+1} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

$$4. \quad \text{se}_{2n+2}(z, -q) = (-1)^n \text{se}_{2n+2} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

$$5. \quad \text{fe}_{2n}(z, -q) = (-1)^{n+1} \text{fe}_{2n} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

$$6. \quad \text{fe}_{2n+1}(z, -q) = (-1)^n \text{ge}_{2n+1} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

$$7. \quad \text{ge}_{2n+1}(z, -q) = (-1)^n \text{fe}_{2n+1} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

$$8. \quad \text{ge}_{2n+2}(z, -q) = (-1)^n \text{ge}_{2n+2} \left(\frac{1}{2}\pi - z, q \right) \quad \text{MA}$$

8.653 Analogously, if we replace z with $\frac{\pi}{2}i + z$ in equation **8.630** 1, we get the equation

$$\frac{d^2 y}{dz^2} - (a + 2q \cosh z) y = 0.$$

It has the following solutions:

8.654

1. $\text{Ce}_{2n}(z, -q) = (-1)^n \text{Ce}_{2n}\left(\frac{\pi}{2}i + z, q\right)$ MA
2. $\text{Ce}_{2n+1}(z, -q) = (-1)^{n+1}i \text{Se}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
3. $\text{Se}_{2n+1}(z, -q) = (-1)^{n+1}i \text{Ce}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
4. $\text{Se}_{2n+2}(z, -q) = (-1)^{n+1} \text{Se}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$ MA
5. $\text{Fe}_{2n}(z, -q) = (-1)^n \text{Fe}_{2n}\left(\frac{1}{2}\pi i + z, q\right)$ MA
- 6.¹¹ $\text{Fe}_{2n+1}(z, -q) = (-1)^{n+1}i \text{Ge}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
- 7.¹¹ $\text{Ge}_{2n+1}(z, -q) = (-1)^{n+1}i \text{Fe}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
- 8.¹¹ $\text{Ge}_{2n+2}(z, -q) = (-1)^{n+1} \text{Ge}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$ MA

8.66 Representation of Mathieu functions as series of Bessel functions

8.661

1.
$$\begin{aligned} \text{ce}_{2n}(z, q) &= \frac{\text{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_{2r}(2k \cos z) & \text{MA} \\ &= \frac{\text{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(2k \sin z) & \text{MA} \end{aligned}$$
2.
$$\begin{aligned} \text{ce}_{2n+1}(z, q) &= -\frac{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k A_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} J_{2r+1}(2k \cos z) & \text{MA} \\ &= \frac{\text{ce}_{2n+1}(0, q)}{k A_1(2n+1)} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)} I_{2r+1}(2k \sin z) & \text{MA} \end{aligned}$$
3.
$$\begin{aligned} \text{se}_{2n+1}(z, q) &= \frac{\text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{k B_1^{(2n+1)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)} J_{2r+1}(2k \cos z) & \text{MA} \\ &= \frac{\text{se}'_{2n+1}(0, q)}{k B_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} I_{2r+1}(2k \sin z) & \text{MA} \end{aligned}$$
4.
$$\begin{aligned} \text{se}_{2n+2}(z, q) &= \frac{-\text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} J_{2r+2}(2k \cos z) & \text{MA} \\ &= \frac{\text{se}'_{2n+2}(0, q)}{k^2 B_2^{(2n+2)}} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} I_{2r+2}(2k \sin z) & \text{MA} \end{aligned}$$

8.662

1.
$$\text{fe}_{2n}(z, q) = -\frac{\pi \text{fe}'_{2n}(0, q)}{2 \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \text{Im} [J_r(k e^{iz}) Y_r(k e^{-iz})]$$
 MA

$$\begin{aligned}
 2. \quad \text{fe}_{2n+1}(z, q) &= \frac{\pi k \text{fe}'_{2n+1}(0, q)}{2 \text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \\
 &\times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} \text{Im} [J_r(ke^{iz}) Y_{r+1}(ke^{-iz}) + J_{r+1}(ke^{iz}) Y_r(ke^{-iz})]
 \end{aligned}$$

MA

$$\begin{aligned}
 3. \quad \text{ge}_{2n+1}(z, q) &= -\frac{\pi k \text{ge}_{2n+1}(0, q)}{2 \text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \\
 &\times \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \text{Re} [J_r(ke^{iz}) Y_{r+1}(ke^{-iz}) - J_{r+1}(ke^{iz}) Y_r(ke^{-iz})]
 \end{aligned}$$

MA

$$\begin{aligned}
 4. \quad \text{ge}_{2n+2}(z, q) &= -\frac{\pi k^2 \text{ge}_{2n+2}(0, q)}{2 \text{se}'_{2n+2}\left(\frac{1}{2}\pi, q\right)} \\
 &\times \sum_{r=0}^{\infty} (-1)^r \text{Re} [J_k(ke^{iz}) Y_{r+2}(ke^{-iz}) - J_{r+2}(ke^{iz}) Y_r(ke^{-iz})]
 \end{aligned}$$

MA

The expansions of the functions Fe_n and Ge_n as series of the functions Y_ν are denoted, respectively, by Fey_n and Gey_n , and the expansions of these functions as series of the functions K_ν are denoted, respectively, by Fek_n and Gek_n .

8.663

$$\begin{aligned}
 1. \quad \text{Fey}_{2n}(z, q) &= \frac{\text{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} Y_{2r}(2k \sinh z) \\
 &\quad k^2 = q [|\sinh z| > 1, \quad \text{Re } z > 0] \\
 &= \frac{\text{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} Y_{2r}(2k \cosh z) \\
 &\quad [|\cosh z| > 1] \\
 &= \frac{\text{ce}_{2n}(0, q) \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{[A_0^{(2n)}]^2} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_r(ke^{-z}) Y_r(ke^z)
 \end{aligned}$$

MA

$$\begin{aligned}
2. \quad \text{Fey}_{2n+1}(z, q) &= \frac{\text{ce}_{2n+1}(0, q) \coth z}{k A_1(2n+1)} \sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(2n+1)} Y_{2r+1}(2k \sinh z), \\
&\quad k^2 = q, \quad [|\sinh z| > 1, \quad \text{Re } z > 0] \quad \text{MA} \\
&= -\frac{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k A_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} Y_{2r+1}(2k \cosh z) \\
&\quad [|\cosh z| > 1] \quad \text{MA} \\
&= -\frac{\text{ce}_{2n+1}(0, q) \text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k \left[A_1^{(2n+1)}\right]^2} \\
&\quad \times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} \left[J_r(ke^{-z}) Y_{r+1}(ke^z) + J_{r+1}(ke^{-z}) Y_r(ke^z) \right] \\
&\quad \text{MA}
\end{aligned}$$

$$\begin{aligned}
3. \quad \text{Gey}_{2n+1}(z, q) &= \frac{\text{se}'_{2n+1}(0, q)}{k B_1^{(2n+1)}} \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} Y_{2r+1}(2k \sinh z) \\
&\quad [|\sinh z| > 1, \quad \text{Re } z > 0] \quad \text{MA} \\
&= \frac{\text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{k B_1^{(2n+1)}} \tanh z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)} Y_{2r+1}(2k \cosh z) \\
&\quad [|\cosh z| > 1] \quad \text{MA} \\
&= \frac{\text{se}_{2n+1}(0, q) \text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{k \left[B_1^{(2n+1)}\right]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \\
&\quad \times \left[J_r(ke^{-z}) Y_{r+1}(ke^z) \right] J_{r+1}(ke^{-z}) Y_r(ke^z) \\
&\quad \text{MA}
\end{aligned}$$

$$\begin{aligned}
4. \quad \text{Gey}_{2n+2}(z, q) &= \frac{\text{se}'_{2n+2}(0, q)}{k^2 B_2^{(2n+2)}} \coth z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} Y_{2r+2}(2k \sinh z) \\
&\quad [|\sinh z| > 1, \quad \text{Re } z > 0] \quad \text{MA} \\
&= -\frac{\text{se}'_{2n+2}(\frac{\pi}{2}, q)}{k^2 B_2^{(2n+2)}} \tanh z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} Y_{2r+2}(2k \cosh z) \\
&\quad [|\cosh z| > 1] \quad \text{MA} \\
&= \frac{\text{se}'_{2n+2}(0, q) \text{se}'_{2n+2}(\frac{\pi}{2}, q)}{k^2 [B_2^{(2n+2)}]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)} \\
&\quad \times [J_r(ke^{-z}) Y_{r+2}(ke^z)] - J_{r+2}(ke^{-z}) Y_r(ke^z) \\
&\quad \text{MA}
\end{aligned}$$

8.664

$$\begin{aligned}
1. \quad \text{Fek}_{2n}(z, q) &= \frac{\text{ce}_{2n}(0, q)}{\pi A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} K_{2r}(-2ik \sinh z) \\
&\quad k^2 = q, \quad [|\sinh z| > 1, \quad \text{Re } z > 0] \quad \text{MA} \\
2. \quad \text{Fek}_{2n+1}(z, q) &= \frac{\text{ce}_{2n+1}(0, q)}{\pi k A_1^{(2n+1)}} \coth z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)} K_{2r+1}(-2ik \sinh z) \\
&\quad k^2 = q \quad [|\sinh z| > 1, \quad \text{Re } z > 0] \quad \text{MA} \\
3. \quad \text{Gek}_{2n+1}(z, q) &= \frac{\text{se}_{2n+1}(\frac{\pi}{2}, q)}{\pi k B_1^{(2n+1)}} \tanh z \sum_{r=0}^{\infty} (2r+1) B_{2r+1}^{(2n+1)} K_{2r+1}(-2ik \cosh z) \quad \text{MA} \\
4. \quad \text{Gek}_{2n+2}(z, q) &= \frac{\text{se}'_{2n+2}(\frac{\pi}{2}, q)}{\pi k^2 B_2^{(2n+2)}} \tanh z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} K_{2r+2}(-2ik \cosh z) \quad \text{MA}
\end{aligned}$$

8.67 The general theory

If $i\mu$ is not an integer, the general solution of equation **8.60** can be found in the form

8.671

$$1. \quad y = Ae^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rzi} + Be^{-\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rzi} \quad \text{MA}$$

The coefficients c_{2r} can be determined from the homogeneous system of linear algebraic equations

$$\begin{aligned}
2.^{11} \quad c_{2r} + \xi_{2r}(c_{2r+2} + c_{2r-2}) &= 0, \quad r = \dots, -2, -1, 0, 1, 2, \dots, \quad \text{MA} \\
&\text{where}
\end{aligned}$$

$$\xi_{2r} = \frac{q}{(2r - i\mu)^2 - a}$$

The condition that this system be compatible yields an equation that μ must satisfy:

$$3.7 \quad \Delta(i\mu) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \quad \text{MA}$$

This equation can also be written in the form

4. $\cosh \mu\pi = 1 - 2\Delta(0) \sin^2 \left(\frac{\pi\sqrt{a}}{2} \right)$, where $\Delta(0)$ is the value that is assumed by the determinant of the preceding article if we set $\mu = 0$ in the expressions for ξ_{2r} .
5. If the pair (a, q) is such that $|\cosh \mu\pi| < 1$, then $\mu = i\beta$, $\text{Im } \beta = 0$, and the solution **8.671** 1 is bounded on the real axis.
6. If $|\cosh \mu\pi| > 1$, μ may be real or complex, and the solution **8.671** 1 will not be bounded on the real axis.
7. If $\cosh \mu\pi = \pm 1$, then $i\mu$ will be an integer. In this case, one of the solutions will be of period π or 2π (depending on whether n is even or odd). The second solution is non-periodic (see **8.61** and **8.64**).

8.7–8.8 Associated Legendre Functions

8.70 Introduction

8.700 An *associated Legendre function* is a solution of the differential equation

$$1. \quad (1 - z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] u = 0,$$

in which ν and μ are arbitrary complex constants.

This equation is a special case of (Riemann's) hypergeometric equation (see **9.151**). The points

$$+1, -1, \infty$$

are, in general, its *singular points*, specifically, its ordinary branch points.

We are interested, on the one hand, in solutions of the equation that correspond to real values of the independent variable z that lie in the interval $[-1, 1]$ and, on the other hand, in solutions corresponding to an arbitrary complex number z such that $\text{Re } z > 1$. These are multiple-valued in the z -plane. To separate these functions into single-valued branches, we make a cut along the real axis from $-\infty$ to $+1$. We are also interested in those solutions of equation **8.700** 1 for which ν or μ or both are integers. Of special significance is the case in which $\mu = 0$.

8.701 In connection with this, we shall use the following notations:

The letter z will denote an *arbitrary complex variable*; the letter x will denote a *real* variable that varies over the interval $[-1, +1]$. We shall sometimes set $x = \cos \varphi$, where φ is a real number.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation **8.700** 1 that are single-valued and regular for $|z| < 1$ and, in particular, uniquely determined for $z = x$.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation **8.700** 1 that are single-valued and regular for $\operatorname{Re} z > 1$. When these functions cannot be unrestrictedly extended without violating their single-valuedness, we make a cut along the real axis to the left of the point $z = 1$. The values of the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ on the upper and lower boundaries of that portion of the cuts lying between the points -1 and $+1$ are denoted, respectively, by

$$P_\nu^\mu(x \pm i0), \quad Q_\nu^\mu(x \pm i0).$$

The letters n and m denote natural numbers or zero. The letters ν and μ denote arbitrary complex numbers unless the contrary is stated.

The upper index will be omitted when it is equal to zero. That is, we set

$$P_\nu^0(z) = P_\nu(z), \quad Q_\nu^0(z) = Q_\nu(z)$$

The *linearly independent* functions

$$\mathbf{8.702} \quad P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1} \right)^{\frac{\mu}{2}} F \left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2} \right) \left[\arg \frac{z+1}{z-1} = 0, \text{ if } z \text{ is real and greater than } 1 \text{ and} \right] \quad \text{MO 80, WH}$$

$$\mathbf{8.703} \quad Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1) \Gamma(\frac{1}{2})}{2^{\nu+1} \Gamma(\nu + \frac{3}{2})} (z^2 - 1)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F \left(\frac{\nu + \mu + 2}{2}, \frac{\nu + \mu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{z^2} \right)$$

$[\arg(z^2 - 1) = 0 \text{ when } z \text{ is real and greater than } 1; \arg z = 0 \text{ when } z \text{ is real and greater than zero}]$ which are solutions of the differential equation **8.700** 1, are called *associated Legendre functions* (or *spherical functions*) of the *first* and *second kinds*, respectively. They are uniquely defined, respectively, in the intervals $|1 - z| < 2$ and $|z| > 1$, with the portion of the real axis that lies between $-\infty$ and $+1$ excluded. They can be extended by means of hypergeometric series to the entire z -plane where the above-mentioned cut was made. These expressions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ lose their meaning when $1 - \mu$ and $\nu + \frac{3}{2}$ are non-positive integers, respectively. MO 80

When z is a real number lying on the interval $[-1, +1]$, so that ($z = x = \cos \varphi$), we take the following functions as linearly independent solutions of the equation:

$$\mathbf{8.704} \quad P_\nu^\mu(x) = \frac{1}{2} \left[e^{\frac{1}{2}\mu\pi i} P_\nu^\mu(\cos \varphi + i0) + e^{-\frac{1}{2}\mu\pi i} P_\nu^\mu(\cos \varphi - i0) \right] \quad \text{EH I 143(1)}$$

$$= \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x} \right)^{\frac{\mu}{2}} F \left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2} \right) \quad \text{EH I 143(6)}$$

$$\mathbf{8.705} \quad Q_\nu^\mu(x) = \frac{1}{2} e^{-\mu\pi i} \left[e^{-\frac{1}{2}\mu\pi i} Q_\nu^\mu(x + i0) + e^{\frac{1}{2}\mu\pi i} Q_\nu^\mu(x - i0) \right] \quad \text{EH I 143(2)}$$

$$= \frac{\pi}{2 \sin \mu\pi} \left[P_\nu^\mu(x) \cos \mu\pi - \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} P_\nu^{-\mu}(x) \right] \quad (\text{cf. } \mathbf{8.732} \text{ 5})$$

If $\mu = \pm m$ is an integer, the last equation loses its meaning. In this case, we get the following formulas by passing to the limit:

8.706

$$1. \quad Q_\nu^m(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_\nu(x) \quad (\text{cf. } \mathbf{8.752} \text{ 1}) \quad \text{EH I 149(7)}$$

$$2.^{11} \quad Q_\nu^{-m}(x) = \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} Q_\nu^m(x) \quad \text{EH I 144(18)}$$

The functions $Q_\nu^\mu(z)$ are not defined when $\nu + \mu$ is equal to a negative integer. Therefore, we must exclude the cases when $\nu + \mu = -1, -2, -3, \dots$ for these formulas.

The functions

$$P_{\nu}^{\pm\mu}(\pm z), \quad Q_{\nu}^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$$

are *linearly independent solutions* of the differential equation for $\nu + \mu \neq 0, \pm 1, \pm 2, \dots$

8.707 Nonetheless, two linearly independent solutions can always be found. Specifically, for $\nu \pm \mu$ not an integer, the differential equation **8.700** 1 has the following solutions:

$$1. \quad P_{\nu}^{\pm\mu}(\pm z), \quad Q_{\nu}^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$$

respectively, for $z = x = \cos \varphi$,

$$2. \quad P_{\nu}^{\pm\mu}(\pm x), \quad Q_{\nu}^{\pm\mu}(\pm x), \quad P_{-\nu-1}^{\pm\mu}(\pm x), \quad Q_{-\nu-1}^{\pm\mu}(\pm x).$$

If $\nu \pm \mu$ is not an integer, the solutions

$$3. \quad P_{\nu}^{\mu}(z), \quad Q_{\nu}^{\mu}(z), \text{ respectively, and } P_{\nu}^{\mu}(x), \quad Q_{\nu}^{\mu}(x)$$

are linearly independent. If $\nu \pm \mu$ is an integer but μ itself is not an integer, the following functions are linearly independent solutions of equation **8.700** 1:

$$4. \quad P_{\nu}^{\mu}(z), \quad P_{\nu}^{-\mu}(z), \text{ respectively, and } P_{\nu}^{\mu}(x), \quad P_{\nu}^{-\mu}(x).$$

If $\mu = \pm m, \nu = n$, or $\nu = -n - 1$, the following functions are linearly independent solutions of equation **8.700** 1 for $n \geq m$:

$$5. \quad P_n^m(z), \quad Q_n^m(z), \text{ respectively, and } P_n^m(x), \quad Q_n^m(x),$$

and for $n < m$, the following functions will be linearly independent solutions

$$6. \quad P_n^{-m}(z), \quad Q_n^m(z), \text{ respectively, and } P_n^{-m}(x), \quad Q_n^m(x).$$

8.71 Integral representations

8.711

$$1. \quad P_{\nu}^{-\mu}(z) = \frac{(z^2 - 1)^{\frac{\mu}{2}}}{2^{\mu} \sqrt{\pi} \Gamma(\mu + \frac{1}{2})} \int_{-1}^1 \frac{(1 - t^2)^{\mu - \frac{1}{2}}}{(z + t\sqrt{z^2 - 1})^{\mu - \nu}} dt \quad [\operatorname{Re} \mu > -\frac{1}{2}, \quad |\arg(z \pm 1)| < \pi]$$

MO 88

$$\begin{aligned} 2. \quad P_{\nu}^m(z) &= \frac{(\nu + 1)(\nu + 2) \dots (\nu + m)}{\pi} \int_0^{\pi} [z + \sqrt{z^2 - 1} \cos \varphi]^{\nu} \cos m\varphi d\varphi \\ &= (-1)^m \frac{\nu(\nu - 1) \dots (\nu - m + 1)}{\pi} \int_0^{\pi} \frac{\cos m\varphi d\varphi}{[z + \sqrt{z^2 - 1} \cos \varphi]^{\nu + 1}} \\ &\quad \left[|\arg z| < \frac{\pi}{2}, \quad \arg(z + \sqrt{z^2 - 1} \cos \varphi) = \arg z \text{ for } \varphi = \frac{\pi}{2} \right] \quad (\text{cf. } \mathbf{8.822} \text{ 1}) \quad \text{SM 483(15), WH} \end{aligned}$$

$$\begin{aligned} 3. \quad Q_{\nu}^{\mu}(z) &= \sqrt{\pi} \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1)}{2^{\mu} \Gamma(\mu + \frac{1}{2}) \Gamma(\nu - \mu + 1)} (z^2 - 1)^{\frac{\mu}{2}} \int_0^{\infty} \frac{\sinh^2 t dt}{(z + \sqrt{z^2 - 1} \cosh t)^{\nu + \mu + 1}} \\ &\quad [\operatorname{Re}(\nu \pm \mu) > -1, \quad |\arg(z \pm 1)| < \pi] \quad (\text{cf. } \mathbf{8.822} \text{ 2}) \quad \text{MO 88} \end{aligned}$$

$$\begin{aligned} 4. \quad Q_{\nu}^{\mu}(z) &= \frac{e^{\mu\pi i} \Gamma(\nu + 1)}{\Gamma(\nu - \mu + 1)} \int_0^{\infty} \frac{\cosh \mu t dt}{(z + \sqrt{z^2 - 1} \cosh t)^{\nu + 1}} \\ &\quad [\operatorname{Re}(\nu + \mu) > -1, \nu \neq -1, -2, -3, \dots, \quad |\arg(z \pm 1)| < \pi] \quad \text{WH, MO 88} \end{aligned}$$

$$5. \quad \int_{-1}^1 P_l^2(x) P_l^0(x) dx = -\frac{l!}{(l-2)!} \frac{1}{2l+1} = -\frac{l(l-1)}{2l+1}$$

$$8.712 \quad Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1)}{2^{\nu+1} \Gamma(\nu + 1)} (z^2 - 1)^{-\frac{\mu}{2}} \int_{-1}^1 (1-t^2)^\nu (z-t)^{-\nu-\mu-1} dt$$

[Re($\nu + \mu$) > -1, Re μ > -1, |arg($z \pm 1$)| < π] (cf. **8.821** 2) MO 88a, EH I 155(5)a

8.713

$$1. \quad Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{2\pi}} (z^2 - 1)^{\frac{\mu}{2}} \left\{ \int_0^\pi \frac{\cos\left(\nu + \frac{1}{2}\right)t dt}{(z - \cos t)^{\mu+\frac{1}{2}}} - \cos \nu\pi \int_0^\infty \frac{e^{-(\nu+\frac{1}{2})t} dt}{(z + \cosh t)^{\mu+\frac{1}{2}}} \right\}$$

[Re $\mu > -\frac{1}{2}$, Re($\nu + \mu$) > -1, |arg($z \pm 1$)| < π] MO 89

$$2. \quad P_\nu^{-\mu}(z) = \frac{(z^2 - 1)^{\frac{\mu}{2}}}{2^\nu \Gamma(\mu - \nu) \Gamma(\nu + 1)} \int_0^\infty \frac{\sinh^{2\nu+1} t}{(z + \cosh t)^{\nu+\mu+1}} dt$$

[Re $z > -1$, |arg($z \pm 1$)| < π , Re($\nu + 1$) > 0, Re($\mu - \nu$) > 0] MO 89

$$3. \quad P_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\mu + \frac{1}{2}\right) (z^2 - 1)^{\frac{\mu}{2}}}{\Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)} \int_0^\infty \frac{\cosh\left(\nu + \frac{1}{2}\right)t dt}{(z + \cosh t)^{\mu+\frac{1}{2}}}$$

[Re $z > -1$, |arg($z \pm 1$)| < π , Re($\nu + \mu$) > -1, Re($\mu - \nu$) > 0] MO 89

8.714

$$1. \quad P_\nu^\mu(\cos \varphi) = \sqrt{\frac{2}{\pi}} \frac{\sin^\mu \varphi}{\Gamma\left(\frac{1}{2} - \mu\right)} \int_0^\varphi \frac{\cos\left(\nu + \frac{1}{2}\right)t dt}{(\cos t - \cos \varphi)^{\mu+\frac{1}{2}}} \quad \left[0 < \varphi < \pi, \text{ Re } \mu < \frac{1}{2}\right]; \quad (\text{cf. } \mathbf{8.823})$$

MO 87

$$2. \quad P_\nu^{-\mu}(\cos \varphi) = \frac{\Gamma(2\mu + 1) \sin^\mu \varphi}{2^\mu \Gamma(\mu + 1) \Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)} \int_0^\infty \frac{t^{\nu+\mu} dt}{(1 + 2t \cos \varphi + t^2)^{\mu+\frac{1}{2}}}$$

[Re($\nu + \mu$) > -1, Re($\mu - \nu$) > 0]

MO 89

$$3. \quad Q_\nu^\mu(\cos \varphi) = \frac{1}{2^{\mu+1}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{\sin^\mu \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)}$$

$$\times \int_0^\infty \left[\frac{\sinh^{2\mu} t}{(\cos \varphi + i \sin \varphi \cosh t)^{\nu+\mu+1}} + \frac{\sinh^{2\mu} t}{(\cos \varphi - i \sin \varphi \cosh t)^{\nu+\mu+1}} \right] dt$$

[Re($\nu + \mu + 1$) > 0, Re($\nu - \mu + 1$) > 0, Re $\mu > -\frac{1}{2}$] MO 89

$$4. \quad P_\nu^\mu(\cos \varphi) = \frac{i}{2^\mu} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{\sin^\mu \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)}$$

$$\times \int_0^\infty \left[\frac{\sinh^{2\mu} t}{(\cos \varphi + i \sin \varphi \cosh t)^{\nu+\mu+1}} - \frac{\sinh^{2\mu} t}{(\cos \varphi - i \sin \varphi \cosh t)^{\nu+\mu+1}} \right] dt$$

[Re($\nu \pm \mu + 1$) > 0, Re $\mu > -\frac{1}{2}$] MO 89

8.715

$$1. \quad P_{\nu}^{\mu}(\cosh \alpha) = \frac{\sqrt{2} \sinh^{\mu} \alpha}{\sqrt{\pi} \Gamma(\frac{1}{2} - \mu)} \int_0^{\alpha} \frac{\cosh(\nu + \frac{1}{2}) t dt}{(\cosh \alpha - \cosh t)^{\mu + \frac{1}{2}}} \quad [\alpha > 0, \quad \operatorname{Re} \mu < \frac{1}{2}] \quad \text{MO 87}$$

$$2. \quad Q_{\nu}^{\mu}(\cosh \alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{\mu \pi i} \sinh^{\mu} \alpha}{\Gamma(\frac{1}{2} - \mu)} \int_{\alpha}^{\infty} \frac{e^{-(\nu + \frac{1}{2})t} dt}{(\cosh t - \cosh \alpha)^{\mu + \frac{1}{2}}} \quad [\alpha > 0, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re}(\nu + \mu) > -1] \quad \text{MO 87}$$

See also **3.277** 1, 4, 5, 7, **3.318**, **3.516** 3, **3.518** 1, 2, **3.542** 2, **3.663** 1, **3.894**, **3.988** 3, **6.622** 3, **6.628** 1, 4–7, and also **8.742**.

8.72 Asymptotic series for large values of $|\nu|$

8.721⁶ For real values of $\mu, |\nu| \gg 1, |\nu| \gg |\mu|, |\arg \nu| < \pi$, we have:

$$1. \quad P_{\nu}^{\mu}(\cos \varphi) = \frac{2}{\sqrt{\pi}} \Gamma(\nu + \mu + 1) \sum_{k=0}^{\infty} \frac{\Gamma(\mu + k + \frac{1}{2})}{\Gamma(\mu - k + \frac{1}{2})} \frac{\cos \left[(\nu + k + \frac{1}{2}) \varphi + \frac{\pi}{4} (2k - 1) + \frac{\mu \pi}{2} \right]}{k! \Gamma(\nu + k + \frac{3}{2}) (2 \sin \varphi)^{k + \frac{1}{2}}} \quad \left[\nu + \mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, \frac{7}{2}, \dots; \text{ for } \frac{\pi}{6} < \varphi < \frac{5\pi}{6} \right]$$

This series also converges for complex values of ν and μ .

In the remaining cases, it is an asymptotic expansion for

$$|\nu| \gg |\mu|, |\nu| \gg 1, \text{ if } \nu > 0, \mu > 0 \text{ and } 0 < \varepsilon \leq \varphi \leq \pi - \varepsilon$$

MO 92

$$2.^6 \quad Q_{\nu}^{\mu}(\cos \varphi) = \sqrt{\pi} \Gamma(\nu + \mu + 1) \times \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\mu + k + \frac{1}{2})}{\Gamma(\mu - k + \frac{1}{2})} \frac{\cos \left[(\nu + k + \frac{1}{2}) \varphi - \frac{\pi}{4} (2k - 1) + \frac{\mu \pi}{2} \right]}{k! \Gamma(\nu + k + \frac{3}{2}) (2 \sin \varphi)^{k + \frac{1}{2}}} \quad \left[\nu + \mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots; \text{ for } \frac{\pi}{6} < \varphi < \frac{5\pi}{6} \right]$$

This series also converges for complex values of ν and μ .

In the remaining cases, it is an asymptotic expansion for

$$|\nu| \gg |\mu|, \quad |\nu| \gg 1, \text{ if } \nu > 0, \quad \mu > 0, \quad 0 < \varepsilon \leq \varphi \leq \pi - \varphi$$

EH I 147(6), MO 92

$$3. \quad P_{\nu}^{\mu}(\cos \varphi) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \frac{\cos \left[(\nu + \frac{1}{2}) \varphi - \frac{\pi}{4} + \frac{\mu \pi}{2} \right]}{\sqrt{2 \sin \varphi}} \left[1 + O\left(\frac{1}{\nu}\right) \right] \quad \left[0 < \varepsilon \leq \varphi \leq \pi - \varepsilon, \quad |\nu| \gg \frac{1}{\varepsilon} \right] \quad \text{MO 92}$$

For $\nu > 0, \mu > 0$ and $\nu > \mu$, it follows from formulas **8.721** 1 and **8.721** 2 that

$$\begin{aligned}
4. \quad \nu^{-\mu} P_{\nu}^{\mu}(\cos \varphi) &= \sqrt{\frac{2}{\nu \pi \sin \varphi}} \cos \left[\left(\nu + \frac{1}{2} \right) \varphi - \frac{\pi}{4} + \frac{\mu \pi}{2} \right] + O \left(\frac{1}{\sqrt{\nu^3}} \right) \\
5. \quad \nu^{-\mu} Q_{\nu}^{\mu}(\cos \varphi) &= \sqrt{\frac{\pi}{2 \nu \sin \varphi}} \cos \left[\left(\nu + \frac{1}{2} \right) \varphi + \frac{\pi}{4} + \frac{\mu \pi}{2} \right] O \left(\frac{1}{\sqrt{\nu^3}} \right) \\
&\left[0 < \varepsilon \leq \varphi \leq \pi - \varepsilon; \quad \nu \gg \frac{1}{\varepsilon} \right] \quad \text{MO 92}
\end{aligned}$$

8.722 If φ is sufficiently close to 0 or π that $\nu\varphi$ or $\nu(\pi - \varphi)$ is small in comparison with 1, the asymptotic formulas **8.721** become unsuitable. In this case, the following asymptotic representation is applicable for $\mu \leq 0, \nu \gg 1$, and *small* values of φ :

$$1. \quad \left[\left(\nu + \frac{1}{2} \right) \cos \frac{\varphi}{2} \right]^{\mu} P_{\nu}^{-\mu}(\cos \varphi) = J_{\mu}(\eta) + \sin^2 \frac{\varphi}{2} \left[\frac{J_{\mu+1}(\eta)}{2\eta} - J_{\mu+2}(\eta) + \frac{\eta}{6} J_{\mu+3}(\eta) \right] + O \left(\sin^4 \frac{\varphi}{2} \right)$$

where $\eta = (2\nu + 1) \sin \frac{\varphi}{2}$. In particular, it follows that

$$1. \quad \lim_{\nu \rightarrow \infty} \nu^{\mu} P_{\nu}^{-\mu} \left(\cos \frac{x}{\nu} \right) = J_{\mu}(x) \quad [x \geq 0, \mu \geq 0] \quad \text{MO 93}$$

8.723 We can see how the functions $P_{\nu}^{\mu}(z)$ and $Q_{\nu}^{\mu}(z)$ behave for large $|\nu|$ and real values of $z > \frac{3}{2\sqrt{2}}$:

$$\begin{aligned}
1. \quad P_{\nu}^{\mu}(\cosh \alpha) &= \frac{2^{\mu}}{\sqrt{\pi}} \left\{ \frac{\Gamma(-\nu - \frac{1}{2})}{\Gamma(-\nu - \mu)} \frac{e^{(\mu-\nu)\alpha} \sinh^{\mu} \alpha}{(e^{2\alpha} - 1)^{\mu+\frac{1}{2}}} F \left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{1 - e^{2\alpha}} \right) \right. \\
&\quad \left. + \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1)} \frac{e^{(\nu+\mu+1)\alpha} \sinh^{\mu} \alpha}{(e^{2\alpha} - 1)^{\mu+\frac{1}{2}}} F \left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; -\nu + \frac{1}{2}; \frac{1}{1 - e^{2\alpha}} \right) \right\} \\
&\quad [\nu \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots; \quad a > \frac{1}{2} \ln 2] \quad \text{MO 94}
\end{aligned}$$

$$\begin{aligned}
2. \quad Q_{\nu}^{\mu}(\cosh \alpha) &= e^{\mu\pi i} 2^{\mu} \sqrt{\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \frac{e^{-(\nu+\mu+1)\alpha} \sinh^{\mu} \alpha}{(1 - e^{-2\alpha})^{\mu+\frac{1}{2}}} \\
&\quad \times F \left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{1 - e^{2\alpha}} \right) \\
&\quad [\mu + \nu + 1 \neq 0, -1, -2, \dots; \quad \alpha > \frac{1}{2} \ln 2] \quad \text{MO 94}
\end{aligned}$$

See also **8.776**.

8.724 For the inequalities in **8.776** 1–4, ν and μ are arbitrary real numbers satisfying the inequalities $\nu \geq 1, \nu - \mu + 1 > 0$, and $\mu \geq 0$:

$$\begin{aligned}
1. \quad |P_{\nu}^{\pm\mu}(\cos \varphi)| &< \sqrt{\frac{8}{\nu \pi}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi} \quad \text{MO 91-92} \\
2. \quad |Q_{\nu}^{\pm\mu}(\cos \varphi)| &< \sqrt{\frac{2\pi}{\nu}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi} \quad \text{MO 91-92} \\
3. \quad |P_{\nu}^{\pm\mu}(\cos \varphi)| &< \frac{2}{\sqrt{\nu \pi}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi} \quad \text{MO 91-92}
\end{aligned}$$

$$4. \quad |Q_{\nu}^{\pm\mu}(\cos \varphi)| < \sqrt{\frac{\pi}{\nu}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu + \frac{1}{2}} \varphi} \quad \text{MO 91-92}$$

$$5.^8 \quad \left| \sqrt{\sin \varphi} P_n^m(\cos \varphi) \right| < \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n - m + 1)} 2^{(m+n)^2/n} \sup_{0 < t < \infty} \left| \sqrt{t} J_m(t) \right|$$

[uniformly $0 \leq m \leq n$]

8.725¹⁰ For fixed z and ν and $\operatorname{Re} \mu \rightarrow \infty$, with z not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$, the following are asymptotic expansions in which the upper and lower signs are taken according to whether $\operatorname{Im} z$ is greater than or less than 0:

$$1. \quad P_{\nu}^{\mu}(z) = \frac{\Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)}{\pi \Gamma(\mu + 1)} \left(\frac{z+1}{z-1} \right)^{\frac{1}{2}\mu} \sin \mu \pi \left[F \left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z \right) \right. \\ \left. - \frac{\sin \nu \pi}{\sin \mu \pi} e^{\mp i \mu \pi} \left(\frac{z-1}{z+1} \right)^{\mu} F \left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z \right) \right] \quad \text{AS 8.10.1}$$

$$2. \quad Q_{\nu}^{\mu}(z) = \frac{1}{2} e^{i \mu \pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\mu + 1)} \left(\frac{z+1}{z-1} \right)^{\frac{1}{2}\mu} \Gamma(\mu - \nu) \left[F \left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z \right) \right. \\ \left. - e^{\mp i \nu \pi} \left(\frac{z-1}{z+1} \right)^{\mu} F \left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z \right) \right] \quad \text{AS 8.10.2}$$

$$3. \quad Q_{\nu}^{-\mu}(z) = \frac{e^{-i \mu \pi} \operatorname{cosec} [\pi(\nu - \mu)]}{2\pi \Gamma(1 + \mu)} \left[e^{\mp i \nu \pi} \left(\frac{z+1}{z-1} \right)^{-\frac{1}{2}\mu} F \left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z \right) \right. \\ \left. - \left(\frac{z-1}{z+1} \right)^{-\frac{1}{2}\mu} F \left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z \right) \right] \quad \text{AS 8.10.3}$$

8.73–8.74 Functional relations

8.731

$$1. \quad (z^2 - 1) \frac{dP_{\nu}^{\mu}(z)}{dz} = (\nu - \mu + 1) P_{\nu+1}^{\mu}(z) - (\nu + 1) z P_{\nu}^{\mu}(z) \quad (\text{cf. } \mathbf{8.832} \text{ 1, } \mathbf{8.914} \text{ 2})$$

EH I 161(10), MO 81

$$1(1)^9 \quad (z^2 - 1) \frac{dP_{\nu}^{\mu}(z)}{dz} = \nu z P_{\nu}^{\mu}(z) - (\nu + \mu) P_{\nu-1}^{\mu}(z) \quad \text{AS 8.5.4}$$

$$1(2) \quad (z^2 - 1) \frac{dP_{\nu}^{\mu}(z)}{dz} = (\nu + \mu)(\nu - \mu + 1) \sqrt{z^2 - 1} P_{\nu}^{\mu-1}(z) - \mu z P_{\nu}^{\mu}(z) \quad \text{AS 8.5.2}$$

2. $(2\nu + 1)z P_\nu^\mu(z) = (\nu - \mu + 1) P_{\nu+1}^\mu(z) + (\nu + \mu) P_{\nu-1}^\mu(z)$
(cf. **8.832** 2, **8.914** 1) EH I 160(2), MO 81
3. $P_\nu^{\mu+2}(z) + 2(\mu + 1) \frac{z}{\sqrt{z^2 - 1}} P_\nu^{\mu+1}(z) = (\nu - \mu)(\nu + \mu + 1) P_\nu^\mu(z)$ MO 82, EH I 160(1)
- 3(1)⁹ $P_\nu^{\mu+1}(z) = (z^2 - 1)^{-1/2} [(\nu - \mu)z P_\nu^\mu(z) - (\nu + \mu) P_{\nu-1}^\mu(z)]$ AS 8.5.1
4. $P_{\nu+1}^\mu(z) - P_{\nu-1}^\mu(z) = (2\nu + 1)\sqrt{z^2 - 1} P_\nu^{\mu-1}(z)$ EH I 160(3), MO 82
- 4(1)⁹ $(\nu - \mu + 1) P_{\nu+1}^\mu(z) = (2\nu + 1)z P_\nu^\mu(z) - (\nu + \mu) P_{\nu-1}^\mu(z)$ AS 334(8.5.3)
- 4(2)⁹ $P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu + 1)(z^2 - 1)^{1/2} P_\nu^{\mu-1}(z)$ AS 334(8.5.5)
5. $P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$ (cf. **8.820**, **8.832** 4) EH I 140(1), MO 82

8.732

1. $(z^2 - 1) \frac{d Q_\nu^\mu(z)}{dz} = (\nu - \mu + 1) Q_{\nu+1}^\mu(z) - (\nu + 1)z Q_\nu^\mu(z)$
(cf. **8.832** 3) MO 82
- 2.¹⁰ $(2\nu + 1)z Q_\nu^\mu(z) = (\nu - \mu + 1) Q_{\nu+1}^\mu(z) + (\nu + \mu) Q_{\nu-1}^\mu(z)$
(cf. **8.832** 4) MO 82
3. $Q_\nu^{\mu+2}(z) + 2(\mu + 1) \frac{z}{\sqrt{z^2 - 1}} Q_\nu^{\mu+1}(z) = (\nu - \mu)(\nu + \mu + 1) Q_\nu^\mu(z)$ MO 82
4. $Q_{\nu-1}^\mu(z) - Q_{\nu+1}^\mu(z) = -(2\nu + 1)\sqrt{z^2 - 1} Q_\nu^{\mu-1}(z)$ MO 82a
5. $e^{-\mu\pi i} Q_\nu^\mu(x \pm i0) = e^{\pm \frac{1}{2}\mu\pi i} \left[Q_\nu^\mu(x) \mp i \frac{\pi}{2} P_\nu^\mu(x) \right]$ MO 83

8.733

1. $(1 - x^2) \frac{d P_\nu^\mu(x)}{dx} = P_\nu^\mu(x) - (\nu - \mu + 1) P_{\nu+1}^\mu(x)$ (cf. **8.731** 1)
 $= -\nu x P_\nu^\mu(x) + (\nu + \mu) P_{\nu-1}^\mu(x)$
 $= -\sqrt{1 - x^2} P_\nu^{\mu+1}(x) - \mu x P_\nu^\mu(x);$
 $= (\nu - \mu + 1)(\nu + \mu)\sqrt{1 - x^2} P_\nu^{\mu-1}(x) + \mu x P_\nu^\mu(x)$ MO 82
2. $(2\nu + 1)x P_\nu^\mu(x) = (\nu - \mu + 1) P_{\nu+1}^\mu(x) + (\nu + \mu) P_{\nu-1}^\mu(x)$
(cf. **8.731** 2) MO 82
- 3.¹¹ $P_\nu^{\mu+2}(x) + 2(\mu + 1) \frac{x}{\sqrt{1 - x^2}} P_\nu^{\mu+1}(x) + (\nu - \mu)(\nu + \mu + 1) P_\nu^\mu(x) = 0$
(cf. **8.731** 3) MO 82
4. $P_{\nu-1}^\mu(x) - P_{\nu+1}^\mu(x) = (2\nu + 1)\sqrt{1 - x^2} P_\nu^{\mu-1}(x)$ (cf. **8.731** 4) MO 82
5. $P_{-\nu-1}^\mu(x) = P_\nu^\mu(x)$ (cf. **8.731** 5)

8.734

1. $(\nu + \mu + 1)z Q_\nu^\mu(z) + \sqrt{z^2 - 1} Q_\nu^{\mu+1}(z) = (\nu - \mu + 1) Q_{\nu+1}^\mu(z)$ MO 82
2. $(\nu + \mu) Q_{\nu-1}^\mu(z) + \sqrt{z^2 - 1} Q_\nu^{\mu+1}(z) = (\nu - \mu)z Q_\nu^\mu(z)$ MO 82
3. $Q_{\nu-1}^\mu(z) - z Q_\nu^\mu(z) = -(\nu - \mu + 1)\sqrt{z^2 - 1} Q_\nu^{\mu-1}(z)$ MO 82
4. $z Q_\nu^\mu(z) - Q_{\nu+1}^\mu(z) = -(\nu + \mu)\sqrt{z^2 - 1} Q_\nu^{\mu-1}(z)$ MO 82
5. $(\nu + \mu)(\nu + \mu + 1) Q_{\nu-1}^\mu(z) + (2\nu + 1)\sqrt{z^2 - 1} Q_\nu^{\mu+1}(z) = (\nu - \mu)(\nu - \mu + 1) Q_{\nu+1}^\mu(z)$ MO 82

8.735

1. $(\nu + \mu + 1)x P_\nu^\mu(x) + \sqrt{1 - x^2} P_\nu^{\mu+1}(x) = (\nu - \mu + 1) P_{\nu+1}^\mu(x)$ MO 83
2. $(\nu - \mu)x P_\nu^\mu(x) - (\nu + \mu) P_{\nu-1}^\mu(x) = \sqrt{1 - x^2} P_\nu^{\mu+1}(x)$ MO 83
3. $P_{\nu-1}^\mu(x) - x P_\nu^\mu(x) = (\nu - \mu + 1)\sqrt{1 - x^2} P_\nu^{\mu-1}(x)$ MO 83
4. $x P_\nu^\mu(x) - P_{\nu+1}^\mu(x) = (\nu + \mu)\sqrt{1 - x^2} P_\nu^{\mu-1}(x)$ MO 83
5. $(\nu - \mu)(\nu - \mu + 1) P_{\nu+1}^\mu(x) = (\nu + \mu)(\nu + \mu + 1) P_{\nu-1}^\mu(x) + (2\nu + 1)\sqrt{1 - x^2} P_\nu^{\mu+1}(x)$ MO 83

8.736

1. $P_\nu^{-\mu}(z) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[P_\nu^\mu(z) - \frac{2}{\pi} e^{-\mu\pi i} \sin \mu\pi Q_\nu^\mu(z) \right]$ MO 83
2. $P_\nu^\mu(-z) = e^{\nu\pi i} P_\nu^\mu(z) - \frac{2}{\pi} \sin[(\nu + \mu)\pi] e^{-\mu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z < 0] \quad (\text{cf. 8.833 1})$ MO 83
3. $P_\nu^\mu(-z) = e^{-\nu\pi i} P_\nu^\mu(z) - \frac{2}{\pi} \sin[(\nu + \mu)\pi] e^{-\mu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z > 0] \quad (\text{cf. 8.833 2})$ MO 83
4. $Q_\nu^{-\mu}(z) = e^{-2\mu\pi i} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} Q_\nu^\mu(z)$ MO 82
5. $Q_\nu^\mu(-z) = -e^{-\nu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z < 0]$ MO 82
6. $Q_\nu^\mu(-z) = -e^{\nu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z > 0]$ MO 82
- 7.⁶ $Q_\nu^\mu(z) \sin[(\nu + \mu)\pi] - Q_{-\nu-1}^\mu(z) \sin[(\nu - \mu)\pi] = \pi e^{\mu\pi i} \cos \nu\pi P_\nu^\mu(z)$ MO 83

8.737

1. $P_\nu^{-\mu}(x) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[\cos \mu\pi P_\nu^\mu(x) - \frac{2}{\pi} \sin(\mu\pi) Q_\nu^\mu(x) \right]$ MO 84
2. $P_\nu^\mu(-x) = \cos[(\nu + \mu)\pi] P_\nu^\mu(x) - \frac{2}{\pi} \sin[(\nu + \mu)\pi] Q_\nu^\mu(x)$ MO 84
3. $Q_\nu^\mu(-x) = -\cos[(\nu + \mu)\pi] Q_\nu^\mu(x) - \frac{\pi}{2} \sin[(\nu + \mu)\pi] P_\nu^\mu(x)$ MO 83, EH I 144(15)
4. $Q_{-\nu-1}^\mu(x) = \frac{\sin[(\nu + \mu)\pi]}{\sin[(\nu - \mu)\pi]} Q_\nu^\mu(x) - \frac{\pi \cos \nu\pi \cos \mu\pi}{\sin[(\nu - \mu)\pi]} P_\nu^\mu(x)$ MO 84

8.738

$$1.^{11} \quad Q_{\nu}^{\mu}(i \cot \varphi) = \exp \left[i\pi \left(\mu - \frac{\nu+1}{2} \right) \right] \sqrt{\pi} \Gamma(\nu + \mu + 1) \sqrt{\frac{1}{2} \sin \varphi} P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos \varphi) \\ \left[0 < \varphi < \frac{\pi}{2} \right] \quad \text{MO 83}$$

$$2.^6 \quad P_{\nu}^{\mu}(i \cot \varphi) = \sqrt{\frac{2}{\pi}} \exp \left[i\pi \left(\nu + \frac{1}{4} \right) \right] \frac{\sqrt{\sin \varphi}}{\Gamma(-\nu - \mu)} Q_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos \varphi - i0) \\ \left[0 < \varphi < \frac{\pi}{2} \right] \quad \text{MO 83}$$

$$8.739 \quad e^{-\mu\pi i} Q_{\nu}^{\mu}(\cosh \alpha) = \frac{\sqrt{\pi} \Gamma(\nu + \mu + 1)}{\sqrt{2 \sinh \alpha}} P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\coth \alpha) \quad [\operatorname{Re}(\cosh \alpha) > 0] \quad \text{MO 83}$$

8.741

$$1. \quad P_{\nu}^{-\mu}(x) \frac{d P_{\nu}^{\mu}(x)}{dx} - P_{\nu}^{\mu}(x) \frac{d P_{\nu}^{-\mu}(x)}{dx} = \frac{2 \sin \mu \pi}{\pi(1-x^2)} \quad \text{MO 83}$$

$$2. \quad P_{\nu}^{\mu}(x) \frac{d Q_{\nu}^{\mu}(x)}{dx} - Q_{\nu}^{\mu}(x) \frac{d P_{\nu}^{\mu}(x)}{dx} = \frac{2^{2\mu}}{1-x^2} \frac{\Gamma(\frac{\nu+\mu+1}{2}) \Gamma(\frac{\nu+\mu}{2} + 1)}{\Gamma(\frac{\nu-\mu+1}{2}) \Gamma(\frac{\nu-\mu}{2} + 1)} \quad \text{MO 83}$$

8.742

$$1. \quad \frac{\Gamma(\nu - \mu - 1)}{\Gamma(\nu + \mu + 1)} \left\{ \cos \mu \pi P_{\nu}^{\mu}(\cos \varphi) - \frac{2}{\pi} \sin \mu \pi Q_{\nu}^{\mu}(\cos \varphi) \right\} = \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^{\mu} \varphi}{\Gamma(\mu + \frac{1}{2})} \int_0^{\varphi} \frac{\cos(\nu + \frac{1}{2}) t dt}{(\cos t - \cos \varphi)^{\frac{1}{2}-\mu}} \\ [\operatorname{Re} \mu > -\frac{1}{2}] \quad \text{MO 88}$$

$$2. \quad \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left\{ \cos \nu \pi P_{\nu}^{\mu}(\cos \varphi) - \frac{2}{\pi} \sin \nu \pi Q_{\nu}^{\mu}(\cos \varphi) \right\} \\ = \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^{\mu} \varphi}{\Gamma(\mu + \frac{1}{2})} \int_{\varphi}^{\pi} \frac{\cos[(\nu + \frac{1}{2})(t - \pi)] dt}{(\cos \varphi - \cos t)^{\frac{1}{2}-\mu}} \\ [\operatorname{Re} \mu > -\frac{1}{2}] \quad \text{MO 88}$$

$$3. \quad P_{\nu}^{\mu}(\cos \varphi) \cos(\nu + \mu) \pi - \frac{2}{\pi} Q_{\nu}^{\mu}(\cos \varphi) \sin(\nu + \mu) \pi = \sqrt{\frac{2}{\pi}} \frac{\sin^{\mu} \varphi}{\Gamma(\frac{1}{2} - \mu)} \int_{\varphi}^{\pi} \frac{\cos[(\nu + \frac{1}{2})(t - \pi)] dt}{(\cos \varphi - \cos t)^{\mu+\frac{1}{2}}} \\ [\operatorname{Re} \mu < \frac{1}{2}] \quad \text{MO 88}$$

$$4. \quad \cos \mu \pi P_{\nu}^{\mu}(\cos \varphi) - \frac{2}{\pi} \sin \mu \pi Q_{\nu}^{\mu}(\cos \varphi) \\ = \frac{1}{2^{\mu} \sqrt{\pi}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{\sin^{\mu} \varphi}{\Gamma(\mu + \frac{1}{2})} \int_0^{\pi} \frac{\sin^{2\mu} t dt}{(\cos \varphi \pm i \sin \varphi \cos t)^{\nu-\mu}} \\ [\operatorname{Re} \mu > -\frac{1}{2}, \quad 0 < \varphi < \pi] \quad \text{MO 38}$$

For integrals of Legendre functions, see 7.11–7.21.

8.75 Special cases and particular values

8.751

$$1. \quad P_{\nu}^m(x) = (-1)^m \frac{\Gamma(\nu + m + 1) (1 - x^2)^{\frac{m}{2}}}{2^m \Gamma(\nu - m + 1) m!} F\left(m - \nu, m + \nu + 1; m + 1; \frac{1 - x}{2}\right) \quad \text{MO 84}$$

$$2. \quad P_{\nu}^m(z) = \frac{\Gamma(\nu + m + 1) (z^2 - 1)^{\frac{m}{2}}}{2^m m! \Gamma(\nu - m + 1)} F\left(m - \nu, m + \nu + 1; m + 1; \frac{1 - z}{2}\right) \quad \text{MO 84}$$

$$3.^8 \quad Q_{n+\frac{1}{2}}^{\mu}(z) = \frac{e^{\mu\pi i} \Gamma\left(\mu + n + \frac{3}{2}\right)}{2^{n+\frac{3}{2}} (n+1)!} (z^2 - 1)^{\frac{\mu}{2}} \pi^{1/2} z^{-n-\mu-3/2} F\left(\frac{\mu + n + \frac{5}{2}}{2}, \frac{\mu + n + \frac{3}{2}}{2}; n + 2; \frac{1}{z^2}\right) \quad \text{MO 84}$$

8.752

$$1. \quad P_{\nu}^m(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_{\nu}(x) \quad \text{WH, MO 84, EH I 148(6)}$$

$$2. \quad P_{\nu}^{-m}(x) = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_{\nu}^m(x) = (1 - x^2)^{-\frac{m}{2}} \int_x^1 \dots \int_x^1 P_{\nu}(x) (dx)^m \quad [m \geq 1] \quad \text{HO 99a, MO 85, EH I 149(10)a}$$

$$3. \quad P_{\nu}^{-m}(z) = (z^2 - 1)^{-\frac{m}{2}} \int_1^z \dots \int_1^z P_{\nu}(z) (dz)^m \quad [m \geq 1] \quad \text{MO 85, EH I 149(8)}$$

$$4. \quad Q_{\nu}^m(z) = (z^2 - 1)^{\frac{m}{2}} \frac{d^m}{dz^m} Q_{\nu}(z) \quad \text{WH, MO 85, EH I 148(5)}$$

$$5. \quad Q_{\nu}^{-m}(z) = (-1)^m (z^2 - 1)^{-\frac{m}{2}} \int_z^{\infty} \dots \int_z^{\infty} Q_{\nu}(z) (dz)^m \quad [m \geq 1] \quad \text{MO 85, EH I 149(9)}$$

Special values of the indices

8.753

$$1. \quad P_0^{\mu}(\cos \varphi) = \frac{1}{\Gamma(1 - \mu)} \cot^{\mu} \frac{\varphi}{2} \quad \text{MO 84}$$

$$2. \quad P_{\nu}^{-1}(\cos \varphi) = -\frac{1}{\nu(\nu + 1)} \frac{d P_{\nu}(\cos \varphi)}{d \varphi} \quad \text{MO 84}$$

$$3. \quad P_n^m(z) \equiv 0, \quad P_n^m(x) \equiv 0 \quad \text{for } m > n \quad \text{MO 85}$$

8.754

$$1. \quad P_{\nu-\frac{1}{2}}^{1/2}(\cosh \alpha) = \sqrt{\frac{2}{\pi \sinh \alpha}} \cosh \nu \alpha \quad \text{MO 85}$$

$$2. \quad P_{\nu-\frac{1}{2}}^{1/2}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \cos \nu \varphi \quad \text{MO 85}$$

$$3. \quad P_{\nu-\frac{1}{2}}^{-1/2}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \frac{\sin \nu \varphi}{\nu} \quad \text{MO 85}$$

$$4. \quad Q_{\nu-\frac{1}{2}}^{1/2}(\cosh \alpha) = i\sqrt{\frac{\pi}{2\sinh \alpha}} e^{-\nu\alpha} \quad \text{MO 85}$$

8.755

$$1. \quad P_{\nu}^{-\nu}(\cos \varphi) = \frac{1}{\Gamma(1+\nu)} \left(\frac{\sin \varphi}{2}\right)^{\nu} \quad \text{MO 85}$$

$$2. \quad P_{\nu}^{-\nu}(\cosh \alpha) = \frac{1}{\Gamma(1+\nu)} \left(\frac{\sinh \alpha}{2}\right)^{\nu} \quad \text{MO 85}$$

Special values of Legendre functions**8.756**

$$1. \quad P_{\nu}^{\mu}(0) = \frac{2^{\mu}\sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)\Gamma\left(\frac{-\nu-\mu+1}{2}\right)} \quad \text{MO 84}$$

$$2. \quad \frac{d P_{\nu}^{\mu}(0)}{dx} = \frac{2^{\mu+1} \sin \frac{1}{2}(\nu+\mu)\pi \Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu-\mu+1}{2}\right)} \quad \text{MO 84}$$

$$3. \quad Q_{\nu}^{\mu}(0) = -2^{\mu-1}\sqrt{\pi} \sin \frac{1}{2}(\nu+\mu)\pi \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)} \quad \text{MO84}$$

$$4. \quad \frac{d Q_{\nu}^{\mu}(0)}{dx} = 2^{\mu}\sqrt{\pi} \cos \frac{1}{2}(\nu+\mu)\pi \frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \quad \text{MO 84}$$

8.76 Derivatives with respect to the order

$$\begin{aligned} 8.761 \quad \frac{\partial P_{\nu}^{-\mu}(x)}{\partial \nu} &= \frac{1}{\Gamma(\mu+1)} \left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}} \sum_{n=1}^{\infty} \frac{(-\nu)(1-\nu)\dots(n-1-\nu)(\nu+1)(\nu+2)\dots(\nu+n)}{(\mu+1)(\mu+2)\dots(\mu+n)1\cdot 2\dots n} \\ &\quad \times [\psi(\nu+n+1) - \psi(\nu-n+1)] \left(\frac{1-x}{2}\right)^n \\ &\quad [\nu \neq 0, \pm 1, \pm 2, \dots; \quad \operatorname{Re} \mu > -1] \quad \text{MO 94} \end{aligned}$$

8.762

$$1. \quad \left[\frac{\partial P_{\nu}(\cos \varphi)}{\partial \nu}\right]_{\nu=0} = 2 \ln \cos \frac{\varphi}{2} \quad \text{MO 94}$$

$$2. \quad \left[\frac{\partial P_{\nu}^{-1}(\cos \varphi)}{\partial \nu}\right]_{\nu=0} = -\tan \frac{\varphi}{2} - 2 \cot \frac{\varphi}{2} \ln \cos \frac{\varphi}{2} \quad \text{MO 94}$$

$$3. \quad \left[\frac{\partial P_{\nu}^{-1}(\cos \varphi)}{\partial \nu}\right]_{\nu=1} = -\frac{1}{2} \tan \frac{\varphi}{2} \sin^2 \frac{\varphi}{2} + \sin \varphi \ln \cos \frac{\varphi}{2} \quad \text{MO 94}$$

- For a connection with the polynomials $C_n^{\lambda}(x)$, see **8.936**.
- For a connection with a hypergeometric function, see **8.77**.

8.77 Series representation

For a representation in the form of a series, see **8.721**. It is also possible to represent associated Legendre functions in the form of a series by expressing them in terms of a hypergeometric function.

8.771

$$1. \quad P_{\nu}^{\mu}(z) = \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \frac{1}{\Gamma(1-\mu)} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \quad \text{MO 15}$$

$$2.^8 \quad Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \frac{\Gamma(\frac{1}{2})(z^2-1)^{\frac{\mu}{2}}}{z^{\nu+\mu+1}} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) \quad \text{MO 15}$$

See also **8.702**, **8.703**, **8.704**, **8.723**, **8.751**, **8.772**.

The analytic continuation for $|z| > 1$

The formulas are consequences of theorems on the analytic continuation of hypergeometric series (see **9.154** and **9.155**):

8.772

$$1. \quad P_{\nu}^{\mu}(z) = \frac{\sin(\nu+\mu)\pi \Gamma(\nu+\mu+1)}{2^{\nu+1}\sqrt{\pi} \cos \nu\pi \Gamma(\nu+\frac{3}{2})} (z^2-1)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) \\ + \frac{2^{\nu} \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi} \Gamma(\nu-\mu+1)} (z^2-1)^{\frac{\mu}{2}} z^{\nu-\mu} F\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right) \\ [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |z| > 1; \quad |\arg(z \pm 1)| < \pi] \quad \text{MO 85}$$

$$2. \quad P_{\nu}^{\mu}(z) = \frac{\Gamma(-\nu-\frac{1}{2})(z^2-1)^{-\frac{\nu+1}{2}}}{2^{\nu+1}\sqrt{\pi} \Gamma(-\nu-\mu)} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right) \\ + \frac{2^{\nu} \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi} \Gamma(\nu-\mu+1)} (z^2-1)^{\frac{\mu}{2}} F\left(\frac{\mu-\nu}{2}, -\frac{\mu+\nu}{2}; \frac{1}{2}-\nu; \frac{1}{1-z^2}\right) \\ [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |1-z^2| > 1; \quad |\arg(z \pm 1)| < \pi] \quad \text{MO 85}$$

$$3. \quad P_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} \left(\frac{z+1}{2}\right)^{\nu} F\left(-\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1}\right) \\ \left[\left|\frac{z-1}{z+1}\right| < 1\right] \quad \text{MO 86}$$

8.773

$$1. \quad Q_{\nu}^{\mu}(z) = e^{\mu\pi i} \frac{\sqrt{\pi} \Gamma(\nu+\mu+1)}{2^{\nu+1} \Gamma(\nu+\frac{3}{2})} (z^2-1)^{-\frac{\nu+1}{2}} F\left(\frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right) \\ [\nu+\mu \neq -1, -2, -3, \dots; \quad |\arg(z \pm 1)| < \pi; \quad |1-z^2| > 1] \quad \text{MO 86}$$

$$2. \quad Q_{\nu}^{\mu}(z) = \frac{1}{2} e^{\mu\pi i} \left\{ \Gamma(\mu) \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \right. \\ \left. + \frac{\Gamma(-\mu) \Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1+\mu; \frac{1-z}{2}\right) \right\} \\ [|\arg(z \pm 1)| < \pi, \quad |1-z| < 2] \quad \text{MO 86}$$

$$\begin{aligned}
 8.774 \quad P_\nu^\mu(i \cot \varphi) = & \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma(-\nu - \frac{1}{2})}{\Gamma(-\nu - \mu)} e^{-i(\nu+1)\frac{\pi}{2}} \left(\tan \frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; \nu + \frac{3}{2}; \sin^2 \frac{\varphi}{2}\right) \\
 & + \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1)} e^{i\nu\frac{\pi}{2}} \left(\cot \frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; \frac{1}{2} - \nu; \sin^2 \frac{\varphi}{2}\right) \\
 & \left[2\nu \neq \pm 1, \pm 3, \pm 5, \dots, \quad 0 < \varphi < \frac{\pi}{2}\right] \quad \text{MO 86}
 \end{aligned}$$

8.775

$$\begin{aligned}
 1.^6 \quad P_\nu^\mu(x) = & \frac{2^\mu \cos(\frac{1}{2}(\nu + \mu)\pi) \Gamma(\frac{\nu+\mu+1}{2})}{\sqrt{\pi} \Gamma(\frac{\nu-\mu}{2} + 1)} (1-x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu + 1}{2}, \frac{\mu - \nu}{2}; \frac{1}{2}; x^2\right) \\
 & + \frac{2^{\mu+1} \sin(\frac{1}{2}(\nu + \mu)\pi) \Gamma(\frac{\nu+\mu}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\nu-\mu+1}{2})} x (1-x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu}{2} + 1, \frac{-\nu + \mu + 1}{2}; \frac{3}{2}; x^2\right) \\
 & \text{MO 87}
 \end{aligned}$$

$$\begin{aligned}
 2.^6 \quad Q_\nu^\mu(x) = & -\frac{\sqrt{\pi} \sin(\frac{1}{2}(\nu + \mu)\pi) \Gamma(\frac{\nu+\mu+1}{2})}{2^{1-\mu} \Gamma(\frac{\nu-\mu}{2} + 1)} (1-x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu + 1}{2}, \frac{\mu - \nu}{2}; \frac{1}{2}; x^2\right) \\
 & + 2^\mu \sqrt{\pi} \frac{\cos(\frac{1}{2}(\nu + \mu)\pi) \Gamma(\frac{\nu+\mu}{2} + 1)}{\Gamma(\frac{\nu-\mu+1}{2})} x (1-x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu}{2} + 1, \frac{\mu - \nu + 1}{2}; \frac{3}{2}; x^2\right) \\
 & \text{MO 87}
 \end{aligned}$$

8.776 For $|z| \gg 1$

$$\begin{aligned}
 1. \quad P_\nu^\mu(z) = & \left\{ \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} z^\nu + \frac{\Gamma(-\nu - \frac{1}{2})}{2^{\nu+1} \sqrt{\pi} \Gamma(-\nu - \mu)} z^{-\nu-1} \right\} \left(1 + O\left(\frac{1}{z^2}\right)\right) \\
 & [2\nu \neq \pm 1, \pm 3, \pm 5, \dots, \quad |\arg z| < \pi] \quad \text{MO 87}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad Q_\nu^\mu(z) = & \sqrt{\pi} \frac{e^{\mu\pi i}}{2^{\nu+1}} \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\nu + \frac{3}{2})} z^{-\nu-1} \left(1 + O\left(\frac{1}{z^2}\right)\right) \\
 & [2\nu \neq -3, -5, -7, \dots; \quad |\arg z| < \pi] \quad \text{MO 87}
 \end{aligned}$$

8.777 Set $\zeta = z + \sqrt{z^2 - 1}$. The variable ζ is uniquely defined by this equation on the entire z -plane in which a cut is made from $-\infty$ to $+1$. Here, we are considering that branch of the variable ζ for which values of ζ exceeding 1 correspond to real values of z exceeding 1. In this case,

$$\begin{aligned}
 1. \quad P_\nu^\mu(z) = & \frac{2^\mu \Gamma(-\nu - \frac{1}{2})}{\sqrt{\pi} \Gamma(-\nu - \mu)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}} F\left(\frac{1}{2} + \mu, \nu + \mu + 1; \nu + \frac{3}{2}; \frac{1}{\zeta^2}\right) \\
 & + \frac{2^\mu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\mu-\nu}} F\left(\frac{1}{2} + \mu, \mu - \nu; \frac{1}{2} - \nu; \frac{1}{\zeta^2}\right) \\
 & [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |\arg(z - 1)| < \pi] \quad \text{MO 86}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad Q_\nu^\mu(z) = & 2^\mu e^{\mu\pi i} \sqrt{\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}} F\left(\frac{1}{2} + \mu, \nu + \mu + 1; \nu + \frac{3}{2}; \frac{1}{\zeta^2}\right) \\
 & [|\arg(z - 1)| < \pi] \quad \text{MO 86}
 \end{aligned}$$

8.78 The zeros of associated Legendre functions

8.781 The function $P_\nu^{-\mu}(\cos \varphi)$, considered as a function of ν , has infinitely many zeros for $\mu \geq 0$. These are all simple and real. If a number ν_0 is a zero of the function $P_\nu^{-\mu}(\cos \varphi)$, the number $-\nu_0 - 1$ is also a zero of this function. MO 91

8.782 If ν and μ are both real and $\mu \leq 0$, or if ν and μ are integers, the function $P_\nu^\mu(t)$ has no *real* zeros exceeding 1. If ν and μ are both real with $\nu < \mu < 0$, the function $P_\nu^\mu(t)$ has no real zeros exceeding 1 when $\sin \mu \pi \sin(\mu - \nu)\pi > 0$, but does have one such zero when $\sin \mu \pi \sin(\mu - \nu)\pi < 0$. Finally, if $\mu \leq \nu$, the function $P_\nu^\mu(t)$ has no zeros exceeding 1 for $\lfloor \mu \rfloor$ even but does have one zero for $\lfloor \mu \rfloor$ odd.

8.783 If $\nu > -\frac{3}{2}$ and $\nu + \mu + 1 > 0$, the function $Q_\nu^\mu(t)$ has no real zeros exceeding 1. MO 91

8.784 The function $P_{-\frac{1}{2}+i\lambda}(z)$ has infinitely many zeros for real λ . All these zeros are *real* and *greater than unity*.

8.785 For n a natural number, the function $P_n(x)$ has exactly n real zeros which lie in the closed interval $-1, +1$.

8.786 The function $Q_n(z)$ has no zeros for which $|\arg(z - 1)| < \pi$ if n is a natural number. The function $Q_n(\cos \varphi)$ has exactly $n + 1$ zeros in the interval $0 \leq \varphi \leq \pi$. MO 91

8.787 The following approximate formula can be used to calculate the values of ν for which the equation $P_\nu^{-\mu}(\cos \varphi) = 0$ holds for given small values of φ :

$$\nu + \frac{1}{2} = -\frac{j_\mu}{2 \sin \frac{\varphi}{2}} \left\{ 1 - \frac{\sin^2 \frac{\varphi}{2}}{6} \left(1 - \frac{4\mu^2 - 1}{j_\mu^2} \right) + O\left(\sin^4 \frac{\varphi}{2}\right) \right\}. \quad \text{MO 93}$$

Here, j_μ denotes an arbitrary nonzero root of the equation $J_\mu(z) = 0$ (for $\mu \geq 0$). If φ is close to π then, instead of this formula, we can use the following formulas:

$$1. \quad \nu \approx \mu + k + \frac{\Gamma(2\mu + k + 1)}{\Gamma(\mu)\Gamma(\mu + 1)\Gamma(k + 1)} \left(\frac{\pi - \varphi}{3} \right)^{2\mu} \quad [\mu > 0, \quad k = 0, 1, 2, \dots] \quad \text{MO 93}$$

$$2. \quad \nu \approx k + \frac{1}{2 \ln \left(\frac{2}{\pi - \varphi} \right)} \quad [\mu = 0, \quad k = 0, 1, 2, \dots] \quad \text{MO 93}$$

8.79 Series of associated Legendre functions

8.791

$$1. \quad \frac{1}{z - t} = \sum_{k=0}^{\infty} (2k + 1) P_k(t) Q_k(z) \quad \left[\left| t + \sqrt{t^2 - 1} \right| < \left| z + \sqrt{z^2 - 1} \right| \right]$$

Here, t must lie inside an ellipse passing through the point z with foci at the points ± 1 .

$$2. \quad \frac{1}{\sqrt{1 - 2tz + t^2}} \ln \frac{z - t + \sqrt{1 - 2tz + t^2}}{\sqrt{z^2 - 1}} = \sum_{k=0}^{\infty} t^k Q_k(z) \quad [\operatorname{Re} z > 1, \quad |t| < 1] \quad \text{MO 78}$$

$$\begin{aligned} \mathbf{8.792} \quad P_\nu^{-\alpha}(\cos \varphi) P_\nu^{-\beta}(\cos \psi) &= \frac{\sin \nu \pi}{\pi} \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{\nu - k} - \frac{1}{\nu + k + 1} \right] P_k^{-\alpha}(\cos \varphi) P_k^{-\beta}(\cos \psi) \\ &[a \geq 0, \quad \beta \geq 0, \quad \nu \text{ real}, \quad -\pi < \varphi \pm \psi < \pi] \quad \text{MO 94} \end{aligned}$$

$$\mathbf{8.793} \quad P_{\nu}^{-\mu}(\cos \varphi) = \frac{\sin \nu \pi}{\pi} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu - k} - \frac{1}{\nu + k + 1} \right) P_k^{-\mu}(\cos \varphi) \quad [\mu \geq 0, \quad 0 < \varphi < \pi]$$

MO 94

Addition theorems**8.794**

$$\begin{aligned} 1.^{11} \quad P_{\nu}(\cos \psi_1 \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \varphi) \\ &= P_{\nu}(\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^{-k}(\cos \psi_1) P_{\nu}^k(\cos \psi_2) \cos k\varphi \\ &= P_{\nu}(\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} \frac{\Gamma(\nu - k + 1)}{\Gamma(\nu + k + 1)} P_{\nu}^k(\cos \psi_1) P_{\nu}^k(\cos \psi_2) \cos k\varphi \\ &[0 \leq \psi_1 < \pi, \quad 0 \leq \psi_2 < \pi, \quad \psi_1 + \psi_2 < \pi, \quad \varphi \text{ real}] \quad (\text{cf. } \mathbf{8.814}, \mathbf{8.844} \text{ 1}) \quad \text{MO 90} \end{aligned}$$

$$\begin{aligned} 2. \quad Q_{\nu}(\cos \psi_1) \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \varphi \\ &= P_{\nu}(\cos \psi_1) Q_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^{-k}(\cos \psi_1) Q_{\nu}^k(\cos \psi_2) \cos k\varphi \\ &[0 < \psi_1 < \frac{\pi}{2}, \quad 0 < \psi_2 < \pi, \quad 0 < \psi_1 + \psi_2 < \pi; \quad \varphi \text{ real}] \quad (\text{cf. } \mathbf{8.844} \text{ 3}) \quad \text{MO 90} \end{aligned}$$

8.795

$$\begin{aligned} 1. \quad P_{\nu} \left(z_1 z_2 - \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \cos \varphi \right) &= P_{\nu}(z_1) P_{\nu}(z_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^k(z_1) P_{\nu}^{-k}(z_2) \cos k\varphi \\ &[\operatorname{Re} z_1 > 0, \quad \operatorname{Re} z_2 > 0, \quad |\arg(z_1 - 1)| < \pi, \quad |\arg(z_2 - 1)| < \pi] \quad \text{MO 91} \end{aligned}$$

$$\begin{aligned} 2. \quad Q_{\nu} \left(x_1 x_2 - \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} \cos \varphi \right) &= P_{\nu}(x_1) Q_{\nu}(x_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^{-k}(x_1) Q_{\nu}^k(x_2) \cos k\varphi \\ &[1 < x_1 < x_2, \quad \nu \neq -1, -2, -3, \dots, \quad \varphi \text{ real}] \quad \text{MO 91} \end{aligned}$$

$$\begin{aligned} 3. \quad Q_n \left(x_1 x_2 + \sqrt{x_1^2 + 1} \sqrt{x_2^2 + 1} \cosh \alpha \right) &= \sum_{k=n+1}^{\infty} \frac{1}{(k - n - 1)!(k + n)!} Q_n^k(ix_1) Q_n^k(ix_2) e^{-k\alpha} \\ &[x_1 > 0, \quad x_2 > 0, \quad \alpha > 0] \quad \text{MO 91} \end{aligned}$$

$$\begin{aligned} \mathbf{8.796} \quad P_{\nu}(-\cos \psi_1 \cos \psi_2 - \sin \psi_1 \sin \psi_2 \cos \varphi) &= P_{\nu}(-\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(\nu + k + 1)}{\Gamma(\nu - k + 1)} \\ &\times P_{\nu}^{-k}(-\cos \psi_1) P_{\nu}^k(\cos \psi_2) \cos k\varphi \\ &[0 < \psi_2 < \psi_1 < \pi, \quad \varphi \text{ real}] \quad (\text{cf. } \mathbf{8.844} \text{ 2}) \quad \text{MO 91} \end{aligned}$$

See also **8.934** 3.

8.81 Associated Legendre functions with integer indices

8.810 For *integer* values of ν and μ , the differential equation **8.700** 1. (with $|\nu| > |\mu|$) has a simple solution in the real domain, namely:

$$u = P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The functions $P_n^m(x)$ are called *associated Legendre functions* (or *spherical functions*) of the first kind. The number n is called the *degree*, and the number m is called the *order* of the function $P_n^m(x)$. The functions $\{\cos m\vartheta P_n^m(\cos \varphi), \sin m\vartheta P_n^m(\cos \varphi)\}$, which depend on the angles φ and ϑ , are also called Legendre functions of the first kind, or, more specifically, *tesseral harmonics* for $m < n$ and *sectoral harmonics* for $m = n$. These last functions are periodic with respect to the angles φ and ϑ . Their periods are, respectively, π and 2π . They are single-valued and continuous everywhere on the surface of the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ (where $x_1 = \sin \varphi \cos \vartheta$, $x_2 = \sin \varphi \sin \vartheta$, $x_3 = \cos \varphi$), and they are solutions of the differential equation

$$\frac{1}{\sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial Y}{\partial \varphi} \right) + \frac{1}{\sin^2 \varphi} \frac{\partial^2 Y}{\partial \vartheta^2} + n(n+1)Y = 0.$$

8.811 The integral representation

$$P_n^m(\cos \varphi) = \frac{(-1)^m (n+m)!}{\Gamma(m + \frac{1}{2}) (n-m)!} \sqrt{\frac{2}{\pi}} \sin^{-m} \varphi \int_0^\varphi (\cos t - \cos \varphi)^{m-\frac{1}{2}} \cos(n + \frac{1}{2}) t dt \quad \text{MO 75}$$

8.812 The series representation:

$$P_n^m(x) = \frac{(-1)^m (n+m)!}{2^m m! (n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ 1 - \frac{(n-m)(m+n+1)}{1!(m+1)} \frac{1-x}{2} \right. \\ \left. + \frac{(n-m)(n-m+1)(m+n+1)(m+n+2)}{2!(m+1)(m+2)} \left(\frac{1-x}{2} \right)^2 - \dots \right\} \quad \text{MO 73}$$

$$= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)} x^{n-m-2} \right. \\ \left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-m-4} - \dots \right\} \quad \text{MO 73}$$

$$= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} x^{n-m} F\left(\frac{m-n}{2}, \frac{m-n+1}{2}; \frac{1}{2} - n; \frac{1}{x^2}\right) \quad \text{MO 73}$$

8.813 Special cases:

$$1. \quad P_1^1(x) = -(1-x^2)^{1/2} = -\sin \varphi \quad \text{MO 73}$$

$$2. \quad P_2^1(x) = -3(1-x^2)^{1/2} x = -\frac{3}{2} \sin 2\varphi \quad \text{MO 73}$$

$$3. \quad P_2^2(x) = 3(1-x^2) = \frac{3}{2} (1 - \cos 2\varphi) \quad \text{MO 73}$$

$$4. \quad P_3^1(x) = -\frac{3}{2} (1-x^2)^{1/2} (5x^2 - 1) = -\frac{3}{8} (\sin \varphi + 5 \sin 3\varphi) \quad \text{MO 73}$$

$$5. \quad P_3^2(x) = 15(1-x^2)x = \frac{15}{4} (\cos \varphi - \cos 3\varphi) \quad \text{MO 73}$$

$$6. \quad P_3^3(x) = -15(1-x^2)^{3/2} = -\frac{15}{4} (3 \sin \varphi - \sin 3\varphi) \quad \text{MO 73}$$

Functional relations

For recursion formulas, see **8.731**.

$$\mathbf{8.814} \quad P_n(\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2 \cos \Theta)$$

$$= P_n(\cos \varphi_1) P_n(\cos \varphi_2) + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \varphi_1) P_n^m(\cos \varphi_2) \cos m\Theta$$

$$[0 \leq \varphi_1 \leq \pi, \quad 0 \leq \varphi_2 \leq \pi] \quad (\text{"addition theorem"}) \quad \text{MO 74}$$

8.815 If

$$Y_{n_1}(\varphi, \vartheta) = A_0 P_{n_1}(\cos \varphi) + \sum_{m=1}^{n_1} (a_m \cos m\vartheta + b_m \sin m\vartheta) P_{n_1}^m(\cos \varphi),$$

$$Z_{n_2}(\varphi, \vartheta) = \alpha_0 P_{n_2}(\cos \varphi) + \sum_{m=1}^{n_2} (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta) P_{n_2}^m(\cos \varphi),$$

then

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_{n_1}(\varphi, \vartheta) Y_{n_2}(\varphi, \vartheta) = 0,$$

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_n(\varphi, \vartheta) P_n[\cos \varphi \cos \psi + \sin \varphi \sin \psi \cos(\vartheta - \theta)] = \frac{4\pi}{2n+1} Y_n(\psi, \theta) \quad \text{MO 75}$$

$$\mathbf{8.816} \quad (\cos \varphi + i \sin \varphi \cos \vartheta)^n = P_n(\cos \varphi) + 2 \sum_{m=1}^n (-1)^m \frac{n!}{(n+m)!} \cos m\vartheta P_n^m(\cos \varphi) \quad \text{MO 75}$$

For integrals of the functions, $P_n^m(x)$, see **7.112** 1, **7.122** 1.

8.82–8.83 Legendre functions

8.820 The differential equation

$$\frac{d}{dz} \left[(1-z^2) \frac{du}{dz} \right] + \nu(\nu+1)u = 0 \quad (\text{cf. } \mathbf{8.700} \text{ 1}),$$

where the parameter ν can be an arbitrary number, has the following two linearly independent solutions:

1. $P_\nu(z) = F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right)$
2. $Q_\nu(z) = \frac{\Gamma(\nu+1)\Gamma(\frac{1}{2})}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\nu-1} F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{z^2}\right) \quad \text{SM 518(137)}$

The functions $P_\nu(z)$ and $Q_\nu(z)$ are called *Legendre functions of the first and second kind* respectively. If ν is not an integer, the function $P_\nu(z)$ has *singularities* at $z = -1$ and $z = \infty$. However, if $\nu = n = 0, 1, 2, \dots$, the function $P_\nu(z)$ becomes the *Legendre polynomial* $P_n(z)$ (see **8.91**) For $\nu = -n = -1, -2, \dots$, we have

$$P_{-n-1}(z) = P_n(z).$$

3. If $\nu \neq 0, 1, 2, \dots$, the function $Q_\nu(z)$ has singularities at the points $z = \pm 1$ and $z = \infty$. These points are branch points of the function. On the other hand, if $\nu = n = 0, 1, 2, \dots$, the function $Q_n(z)$ is single-valued for $|z| > 1$ and regular for $z = \infty$.

4. In the right half-plane,

$$P_\nu(z) = \left(\frac{1+z}{2}\right)^\nu F\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right) \quad [\operatorname{Re} z > 0]$$

5. The function $P_\nu(z)$ is uniquely determined by equations **8.820** 1 and **8.820** 4 within a circle of radius 2 with its center at the point $z = 1$ in the right half-plane.

For $z = x = \cos \varphi$, a solution of equation **8.820** is the function

$$6. \quad P_\nu(x) = P_\nu(\cos \varphi) = F\left(-\nu, \nu + 1; 1; \sin^2 \frac{\varphi}{2}\right);$$

In general,

$$7. \quad P_\nu(z) = P_{-\nu-1}(z) = P_\nu(x) = P_{-\nu-1}(x), \text{ for } z = x$$

8. The function $Q_\nu(z)$ for $|z| > 1$ is uniquely determined by equation **8.820** 2 everywhere in the z -plane in which a cut is made from the point $z = -\infty$ to the point $z = 1$. By means of a hypergeometric series, the function can be continued analytically inside the unit circle. On the cut ($-1 \leq x \leq +1$) of the real axis, the function $Q_\nu(x)$ is determined by the equation

$$9. \quad Q_\nu(x) = \frac{1}{2} [Q_\nu(x + i0) + Q_\nu(x - i0)] \quad \text{HO 52(53), WH}$$

Integral representations

8.821

$$1. \quad P_\nu(z) = \frac{1}{2\pi i} \int_A^{(1+, z+)} \frac{(t^2 - 1)^\nu}{2^\nu (t - z)^{\nu+1}} dt$$

Here, A is a point on the real axis to the right of the point $t = 1$ and to the right of z if z is real. At the point A , we set

$$\arg(t - 1) = \arg(t + 1) = 0 \text{ and } [|\arg(t - z)| < \pi] \quad \text{WH}$$

$$2. \quad Q_\nu(z) = \frac{1}{4i \sin \nu \pi} \int_A^{(1-, 1+)} \frac{(t^2 - 1)^\nu}{2^\nu (z - t)^{\nu+1}} dt$$

[ν is not an integer; the point A is at the end of the major axis of an ellipse to the right of $t = 1$ drawn in the t -plane with foci at the points ± 1 and with a minor axis sufficiently small that the point z lies outside it. The contour begins at the point A , follows the path $(1-, -1+)$, and returns to A ; $|\arg z| \leq \pi$ and $|\arg(z - t)| \rightarrow \arg z$ as $t \rightarrow 0$ on the contour; $\arg(t + 1) = \arg(t - 1) = 0$ at the point A ; z does not lie on the real axis between -1 and 1 .]

For $\nu = n$ an integer,

$$3. \quad Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^1 (1 - t^2)^n (z - t)^{-n-1} dt \quad \text{SM 517(134), WH}$$

8.822

$$1. \quad P_\nu(z) = \frac{1}{\pi} \int_0^\pi \frac{d\varphi}{(z + \sqrt{z^2 - 1} \cos \varphi)^{\nu+1}} = \frac{1}{\pi} \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos \varphi\right)^\nu d\varphi$$

$$\left[\operatorname{Re} z > 0 \text{ and } \arg \left\{ z + \sqrt{z^2 - 1} \cos \varphi \right\} = \arg z \text{ for } \varphi = \frac{\pi}{2} \right] \quad \text{WH}$$

$$2. \quad Q_\nu(z) = \int_0^\infty \frac{d\varphi}{(z + \sqrt{z^2 - 1} \cosh \varphi)^{\nu+1}},$$

WH

$$\left[\operatorname{Re} \nu > -1; \quad \text{if } \nu \text{ is not an integer, } \left\{ (z + \sqrt{z^2 - 1}) \cosh \varphi \right\} \text{ for } \varphi = 0 \text{ has its principal value} \right]$$

$$8.823 \quad P_\nu(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos(\nu + \frac{1}{2}) \varphi}{\sqrt{2(\cos \varphi - \cos \theta)}} d\varphi \quad \text{WH}$$

$$8.824 \quad Q_n(z) = 2^n n! \int_z^\infty \dots \int_z^\infty \frac{(dz)^{n+1}}{(z^2 - 1)^{n+1}} = 2^n \int_z^\infty \frac{(t - z)^n}{(t^2 - 1)^{n+1}} dt$$

$$= \frac{(-1)^n}{(2n - 1)!!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \int_z^\infty \frac{dt}{(t^2 - 1)^{n+1}} \right] \quad [\operatorname{Re} z > 1]$$

WH, MO 78

$$8.825 \quad Q_n(z) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)}{z - t} dt \quad [|\arg(z - 1)| < \pi] \quad \text{WH, MO 78}$$

See also **6.622** 3, **8.842**.**8.826** Fourier series:

$$1. \quad P_n(\cos \varphi) = \frac{2^{n+2}}{\pi} \frac{n!}{(2n + 1)!!} \left[\sin(n + 1)\varphi + \frac{1}{1} \frac{n + 1}{2n + 3} \sin(n + 3)\varphi \right. \\ \left. + \frac{1 \cdot 3(n + 1)(n + 2)}{1 \cdot 2(2n + 3)(2n + 5)} \sin(n + 5)\varphi + \dots \right] \quad [0 < \varphi < \pi] \quad \text{MO 79}$$

$$2. \quad Q_n(\cos \varphi) = 2^{n+1} \frac{n!}{(2n + 1)!!} \left[\cos(n + 1)\varphi + \frac{1}{1} \frac{n + 1}{2n + 3} \cos(n + 3)\varphi \right. \\ \left. + \frac{1 \cdot 3}{1 \cdot 2} \frac{(n + 1)(n + 2)}{(2n + 3)(2n + 5)} \cos(n + 5)\varphi + \dots \right] \quad [0 < \varphi < \pi] \quad \text{MO 79}$$

The expressions for Legendre functions in terms of a hypergeometric function (see **8.820**) provide other series representations of these functions.

Special cases and particular values

8.827

$$1. \quad Q_0(x) = \frac{1}{2} \ln \frac{1 + x}{1 - x} = \operatorname{arctanh} x \quad \text{JA}$$

$$2. \quad Q_1(x) = \frac{x}{2} \ln \frac{1 + x}{1 - x} - 1 \quad \text{JA}$$

$$3. \quad Q_2(x) = \frac{1}{4} (3x^2 - 1) \ln \frac{1 + x}{1 - x} - \frac{3}{2} x \quad \text{JA}$$

$$4. \quad Q_3(x) = \frac{1}{4} (5x^3 - 3x) \ln \frac{1 + x}{1 - x} - \frac{5}{2} x^2 + \frac{2}{3} \quad \text{JA}$$

$$5. \quad Q_4(x) = \frac{1}{16} (35x^4 - 30x^2 + 3) \ln \frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x \quad \text{JA}$$

$$6. \quad Q_5(x) = \frac{1}{16} (63x^5 - 70x^3 + 15x) \ln \frac{1+x}{1-x} - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15} \quad \text{JA}$$

8.828

$$1. \quad P_\nu(1) = 1 \quad \text{MO 79}$$

$$2. \quad P_\nu(0) = -\frac{1}{2} \frac{\sin \nu\pi}{\sqrt{\pi^3}} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) \quad \text{MO 79}$$

$$8.829 \quad Q_\nu(0) = \frac{1}{4\sqrt{\pi}} (1 - \cos \nu\pi) \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) \quad \text{MO 79}$$

Functional relationships**8.831**

$$1. \quad Q_\nu(x) = \frac{\pi}{2 \sin \nu\pi} [\cos \nu\pi P_\nu(x) - P_\nu(-x)] \quad [\nu \neq 0, \pm 1, \pm 2, \dots] \quad \text{MO 76}$$

$$2. \quad Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x) \quad [n = 0, 1, 2, \dots],$$

where

$$3. \quad W_{n-1}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2(n-2k)-1}{(2k+1)(n-k)} P_{n-2k-1}(x) = \sum_{k=1}^n \frac{1}{k} P_{k-1}(x) P_{n-k}(x)$$

and

$$4. \quad W_{-1}(x) \equiv 0 \quad (\text{see also } 8.839) \quad \text{SM 516(131), MO 76}$$

$$5. \quad \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1} \right) P_k(\cos \varphi) = \frac{\pi}{\sin \nu\pi} P_\nu(\cos \varphi) \quad [\nu \text{ not an integer; } 0 \leq \varphi < \pi] \quad \text{MO 77}$$

$$6. \quad \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1} \right) P_k(\cos \varphi) P_k(\cos \psi) = \frac{\pi}{\sin \nu\pi} P_\nu(\cos \varphi) P_\nu(\cos \psi) \quad [\nu \text{ not an integer, } -\pi < \varphi + \psi < \pi, \quad -\pi < \varphi - \psi < \pi] \quad \text{MO 77}$$

See also 8.521 4.

8.832

$$1. \quad (z^2 - 1) \frac{d}{dz} P_\nu(z) = (\nu + 1) [P_{\nu+1}(z) - z P_\nu(z)] \quad \text{WH}$$

$$2. \quad (2\nu + 1)z P_\nu(z) = (\nu + 1) P_{\nu+1}(z) + \nu P_{\nu-1}(z) \quad \text{WH}$$

$$3. \quad (z^2 - 1) \frac{d}{dz} Q_\nu(z) = (\nu + 1) [Q_{\nu+1}(z) - z Q_\nu(z)] \quad \text{WH}$$

$$4. \quad (2\nu + 1)z Q_\nu(z) Q_{\nu+1}(z) + \nu Q_{\nu-1}(z) \quad \text{WH}$$

8.833

1. $P_\nu(-z) = e^{\nu\pi i} P_\nu(z) - \frac{2}{\pi} \sin \nu\pi Q_\nu(z)$ [Im $z < 0$] MO 77
2. $P_\nu(-z) = e^{-\nu\pi i} P_\nu(z) - \frac{2}{\pi} \sin \nu\pi Q_\nu(z)$ [Im $z > 0$] MO 77
3. $Q_\nu(-z) = -e^{-\nu\pi i} Q_\nu(z)$ [Im $z < 0$] MO 77
4. $Q_\nu(-z) = -e^{\nu\pi i} Q_\nu(z)$ [Im $z > 0$] MO 77

8.834

1. $Q_\nu(x \pm i0) = Q_\nu(x) \mp \frac{\pi i}{2} P_\nu(x)$ MO 77
2. $Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - W_{n-1}(z)$ (see **8.831** 3) MO 77

8.835

1. $Q_\nu(z) - Q_{-\nu-1}(z) = \pi \cot \nu\pi P_\nu(z)$ [sin $\nu\pi \neq 0$] MO 77
2. $Q_{-\nu-1}(\cos \varphi) = Q_\nu(\cos \varphi) - \pi \cot \nu\pi P_\nu(\cos \varphi)$ [sin $\nu\pi \neq 0$] MO 77
3. $Q_\nu(-\cos \varphi) = -\cos \nu\pi Q_\nu(\cos \varphi) - \frac{\pi}{2} \sin \nu\pi P_\nu(\cos \varphi)$ MO 77

8.836

1. $Q_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \ln \frac{z+1}{z-1} \right] - \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1}$ MO 79
2. $Q_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \ln \frac{1+x}{1-x} \right] - \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x}$ MO 79

8.837

1. $P_\nu(x) = P_\nu(\cos \varphi) = F\left(-\nu, \nu+1; 1; \sin^2 \frac{\varphi}{2}\right)$ (cf. **8.820** 6) MO 76
2.
$$P_\nu(z) = \frac{\tan \nu\pi}{2^{\nu+1} \sqrt{\pi}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})} z^{-\nu-1} F\left(\frac{\nu}{2}+1, \frac{\nu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) \\ + \frac{2^\nu}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)} z^\nu F\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right)$$
 MO 78

See also **8.820**.

For integrals of Legendre functions, see **7.1–7.2**.

8.838 Inequalities ($0 \leq \varphi \leq \pi$, $\nu > 1$, and C_0 is a number that does not depend on the values of ν or φ):

1. $|P_\nu(\cos \varphi) - P_{\nu+2}(\cos \varphi)| \leq 2C_0 \sqrt{\frac{1}{\nu\pi}}$ MO 78
2. $|Q_\nu(\cos \varphi) - Q_{\nu+2}(\cos \varphi)| < C_0 \sqrt{\frac{\pi}{\nu}}$ MO 78

With regard to the zeros of Legendre functions of the second kind, see **8.784**, **8.785**, and **8.786**. For the expansion of Legendre functions in series of associated Legendre functions, see **8.794**, **8.795**, and **8.796**.

8.839 A differential equation leading to the functions W_{n-1} (see **8.831** 3):

$$(1-x^2) \frac{d^2 W_{n-1}}{dx^2} - 2x \frac{dW_{n-1}}{dx} + (n+1)nW_{n-1} = 2 \frac{dP_\nu}{dx} \quad \text{MO 76}$$

8.84 Conical functions

8.840 Let us set

$$\nu = -\frac{1}{2} + i\lambda,$$

where λ is a real parameter, in the defining differential equation **8.700** 1 for associated Legendre functions. We then obtain the differential equation of the so-called conical functions. A conical function is a special case of the associated Legendre function. However, the Legendre functions

$$P_{-\frac{1}{2}+i\lambda}(x), \quad Q_{-\frac{1}{2}+i\lambda}(x)$$

have certain peculiarities that make us distinguish them as a special class—the class of conical functions. The most important of these peculiarities is the following

8.841 The functions

$$P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = 1 + \frac{4\lambda^2 + 1^2}{2^2} \sin^2 \frac{\varphi}{2} + \frac{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2)}{2^2 4^2} \sin^4 \frac{\varphi}{2} + \dots$$

are real for real values of φ . Also,

$$P_{-\frac{1}{2}+i\lambda}(x) \equiv P_{-\frac{1}{2}-i\lambda}(x) \quad \text{MO 95}$$

8.842 Integral representations:

$$1. \quad P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = \frac{2}{\pi} \int_0^\varphi \frac{\cosh \lambda u \, du}{\sqrt{2(\cos u - \cos \varphi)}} = \frac{2}{\pi} \cosh \lambda \pi \int_0^\infty \frac{\cos \lambda u \, du}{\sqrt{2(\cos \varphi + \cosh u)}} \quad \text{MO 95}$$

$$2.^6 \quad Q_{-\frac{1}{2} \mp i\lambda}(\cos \varphi) = \pm i \sinh \lambda \pi \int_0^\infty \frac{\cos \lambda u \, du}{\sqrt{2(\cosh u + \cos \varphi)}} + \int_0^\infty \frac{\cos \lambda u \, du}{\sqrt{2(\cosh u - \cos \varphi)}} \quad \text{MO 95}$$

Functional relations

(See also **8.73**)

$$\mathbf{8.843} \quad P_{-\frac{1}{2}+i\lambda}(-\cos \varphi) = \frac{\cosh \lambda \pi}{\pi} [Q_{-\frac{1}{2}+i\lambda}(\cos \varphi) + Q_{-\frac{1}{2}-i\lambda}(\cos \varphi)] \quad \text{MO 95}$$

8.844

$$1. \quad P_{-\frac{1}{2}+i\lambda}(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) \\ = P_{-\frac{1}{2}+i\lambda}(\cos \psi) P_{-\frac{1}{2}+i\lambda}(\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) P_{-\frac{1}{2}+i\lambda}^k(\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2) \cdots [4\lambda^2 + (2k-1)^2]} \\ \left[0 < \vartheta < \frac{\pi}{2}, \quad 0 < \psi < \pi, \quad 0 < \psi + \vartheta < \pi \right] \quad (\text{cf. } \mathbf{8.794} \text{ 1}) \quad \text{MO 95}$$

$$2. \quad P_{-\frac{1}{2}+i\lambda}(-\cos \psi \cos \vartheta - \sin \psi \sin \vartheta \cos \varphi) \\ = P_{-\frac{1}{2}+i\lambda}(\cos \psi) P_{-\frac{1}{2}+i\lambda}(-\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) P_{-\frac{1}{2}+i\lambda}^k(-\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1)(4\lambda^2 + 3^2) \cdots [4\lambda^2 + (2k-1)^2]} \\ \left[0 < \psi < \frac{\pi}{2} < \vartheta, \quad \psi + \vartheta < \pi \right] \quad (\text{cf. } \mathbf{8.796}) \quad \text{MO 95}$$

$$\begin{aligned}
3. \quad & Q_{-\frac{1}{2}+i\lambda}(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) \\
&= P_{-\frac{1}{2}+i\lambda}(\cos \psi) Q_{-\frac{1}{2}+i\lambda}(\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) Q_{-\frac{1}{2}+i\lambda}^k(\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1)(4\lambda^2 + 3^2) \cdots [4\lambda^2 + (2k-1)^2]} \\
&\quad \left[0 < \psi < \frac{\pi}{2} < \vartheta, \quad \psi + \vartheta < \pi \right] \quad (\text{cf. 8.794 2}) \quad \text{MO 96}
\end{aligned}$$

Regarding the zeros of conical functions, see **8.784**.

8.85 Toroidal functions

8.850 Solutions of the differential equation

$$1. \quad \frac{d^2 u}{d\eta^2} + \frac{\cosh \eta}{\sinh \eta} \frac{du}{d\eta} - \left(n^2 - \frac{1}{4} + \frac{m^2}{\sinh^2 \eta} \right) u = 0,$$

are called toroidal functions. They are equivalent (under a coordinate transformation) to associated Legendre functions. In particular, the functions

$$P_{n-\frac{1}{2}}^m(\cosh \eta), \quad Q_{n-\frac{1}{2}}^m(\sinh \eta) \quad \text{MO 96}$$

are solutions of equation **8.850 1**.

The following formulas, obtained from the formulas obtained earlier for associated Legendre functions, are valid for toroidal functions:

8.851 Integral representations:

$$\begin{aligned}
1. \quad P_{n-\frac{1}{2}}^m(\cosh \eta) &= \frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \frac{(\sinh \eta)^m}{2^m \sqrt{\pi} \Gamma(m+\frac{1}{2})} \int_0^\pi \frac{\sin^{2m} \varphi \, d\varphi}{(\cosh \eta + \sinh \eta \cos \varphi)^{n+m+\frac{1}{2}}} \\
&= \frac{(-1)^m}{2\pi} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \int_0^{2\pi} \frac{\cos m\varphi \, d\varphi}{(\cosh \eta + \sinh \eta \cos \varphi)^{n+\frac{1}{2}}} \quad \text{MO 96}
\end{aligned}$$

$$\begin{aligned}
2. \quad Q_{n-\frac{1}{2}}^m(\cosh \eta) &= (-1)^m \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \int_0^\infty \frac{\cosh mt \, dt}{(\cosh \eta + \sinh \eta \cosh t)^{n+\frac{1}{2}}} \quad [n \geq m] \\
&= (-1)^m \frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \int_0^{\ln \coth \frac{\eta}{2}} \frac{\cosh mt \, dt}{(\cosh \eta - \sinh \eta \cosh t)^{n-\frac{1}{2}}} \quad \text{MO 96}
\end{aligned}$$

8.852 Functional relations:

$$\begin{aligned}
1. \quad Q_{n-\frac{1}{2}}^m(\cosh \eta) &= (-1)^m \frac{2^m \Gamma(n+m+\frac{1}{2}) \sqrt{\pi}}{\Gamma(n+1)} \sinh^m \left(\eta e^{-(n+m+\frac{1}{2})\eta} \right) \\
&\quad \times F\left(m+\frac{1}{2}, n+m+\frac{1}{2}; n+1; e^{-2\eta}\right)
\end{aligned}$$

MO 96

*Sometimes called *torus functions*

$$2. \quad P_{n-\frac{1}{2}}^{-m}(\cosh \eta) = \frac{2^{-2m}}{\Gamma(m+1)} (1 - e^{-2\eta})^m e^{-(n+\frac{1}{2})\eta} F\left(m + \frac{1}{2}, n + m + \frac{1}{2}; 2m + 1; 1 - e^{-2\eta}\right)$$

MO 96

8.853 An asymptotic representation $P_{n-\frac{1}{2}}(\cosh \eta)$ for large values of n :

$$P_{n-\frac{1}{2}}(\cosh \eta) = \frac{\Gamma(n)e^{(n-\frac{1}{2})\eta}}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \times \left[\frac{2\Gamma^2(n+\frac{1}{2})}{\pi n! \Gamma(n)} \ln(4e^\eta) e^{-2n\eta} F\left(\frac{1}{2}, n + \frac{1}{2}; n + 1; e^{-2\eta}\right) + A + B \right],$$

where

$$A = 1 + \frac{1}{2^2} \frac{1 \cdot (2n-1)}{1 \cdot (n-1)} e^{-2\eta} + \frac{1}{2^4} \frac{1 \cdot 3 \cdot (2n-1)(2n-3)}{1 \cdot 2 \cdot (n-1)(n-2)} e^{-4\eta} + \cdots + \frac{1}{2^{2n-2}} \left(\frac{(2n-1)!!}{(n-1)!} \right)^2 e^{-2(n-1)\eta}$$

$$B = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi^3}\Gamma(n)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})\Gamma(n+k+\frac{1}{2})}{\Gamma(n+k+1)\Gamma(k+1)} \left(u_{n+k} + u_k - v_{n+k-\frac{1}{2}} - v_{k-\frac{1}{2}} \right) e^{-2(n+k)\eta}$$

Here,

$$u_r = \sum_{s=1}^r \frac{1}{s}, \quad v_{r-\frac{1}{2}} = \sum_{s=1}^r \frac{2}{2s-1} \quad [r \text{ is a natural number}]$$

MO 97

8.9 Orthogonal Polynomials

8.90 Introduction

8.901 Suppose that $w(x)$ is a nonnegative real function of a real variable x . Let (a, b) be a fixed interval on the x -axis. Let us suppose further that, for $n = 0, 1, 2, \dots$, the integral

$$\int_a^b x^n w(x) dx$$

exists and that the integral

$$\int_a^b w(x) dx$$

is positive. In this case, there exists a sequence of polynomials $p_0(x), p_1(x), \dots, p_n(x), \dots$, that is uniquely determined by the following conditions:

1. $p_n(x)$ is a polynomial of degree n and the coefficient of x^n in this polynomial is positive.
2. The polynomials $p_0(x), p_1(x), \dots$ are orthonormal; that is,

$$\int_a^b p_n(x) p_m(x) w(x) dx = \begin{cases} 0 & \text{for } n \neq m, \\ 1 & \text{for } n = m. \end{cases}$$

We say that the polynomials $p_n(x)$ constitute a *system of orthogonal polynomials on the interval (a, b) with the weight function $w(x)$* .

8.902 If q_n is the coefficient of x^n in the polynomial $p_n(x)$, then

$$1. \quad \sum_{k=0}^n p_k(x)p_k(y) = \frac{q_n}{q_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \quad (\text{Darboux-Christoffel formula})$$

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$$2.^{11} \quad \sum_{k=0}^n [p_k(x)]^2 = \frac{q_n}{q_{n+1}} [p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x)]$$

EH II 159(11)

8.903 Between any three consecutive orthogonal polynomials, there is a dependence

$$p_n(x) = (A_n x + B_n) p_{n-1}(x) - C_n p_{n-2}(x) \quad [n = 2, 3, 4, \dots]$$

In this formula, A_n, B_n , and C_n are constants and

$$A_n = \frac{q_n}{q_{n-1}}, \quad C_n = \frac{q_n q_{n-2}}{q_{n-1}^2}$$

MO 102

8.904 Examples of normalized systems of orthogonal polynomials:

Notation and name		Interval	Weight
$(n + \frac{1}{2})^{1/2} P_n(x)$	see 8.91	$(-1, +1)$	1
$2^\lambda \Gamma(\lambda) \left[\frac{(n + \lambda) n!}{2\pi \Gamma(2\lambda + n)} \right]^{1/2} C_n^\lambda(x)$	see 8.93	$(-1, +1)$	$(1 - x^2)^{\lambda - \frac{1}{2}}$
$\sqrt{\frac{\varepsilon_n}{\pi}} T_n(x), \quad \varepsilon_0 = 1, \varepsilon_n = 2 \text{ for } n = 1, 2, 3, \dots$	see 8.94	$(-1, +1)$	$(1 - x^2)^{-1/2}$
$2^{-\frac{n}{2}} \pi^{-1/4} (n!)^{-1/2} H_n(x)$	see 8.95	$(-\infty, \infty)$	e^{-x^2}
$\left[\frac{\Gamma(n+1) \Gamma(\alpha + \beta + 1 + n) (\alpha + \beta + 1 + 2n)}{\Gamma(\alpha + 1 + n) \Gamma(\beta + 1 + n) 2^{\alpha + \beta + 1}} \right]^{1/2} P_n^{(\alpha, \beta)}(x)$	see 8.96	$(-1, +1)$	$(1 - x)^\alpha (1 + x)^\beta$
$\left[\frac{\Gamma(n+1)}{\Gamma(\alpha + n + 1)} \right]^{1/2} (-1)^n L_n^\alpha(x)$	see 8.97	$(0, \infty)$	$x^\alpha e^{-x}$

Cf. **7.221** 1, **7.313**, **7.343**, **7.374** 1, **7.391** 1, **7.414** 3.

8.91 Legendre polynomials

8.910 Definition. The Legendre polynomials $P_n(z)$ are polynomials satisfying equation **8.700** 1 with $\mu = 0$ and $\nu = n$: that is, they satisfy the differential equation

$$1. \quad (1 - z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + n(n+1)u = 0$$

This equation has a polynomial solution if, and only if, n is an integer. Thus, Legendre polynomials constitute a special type of associated Legendre function.

Legendre polynomials of degree n are of the form

$$2. \quad P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n$$

8.911 Legendre polynomials written in expanded form:

$$\begin{aligned}
 1. \quad P_n(z) &= \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} z^{n-2k} \\
 &= \frac{(2n)!}{2^n (n!)^2} \left(z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} z^{n-4} - \dots \right) \\
 &= \frac{(2n-1)!!}{n!} z^n F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2}\right)
 \end{aligned}$$

HO 13, AD (9001), MO 69

$$\begin{aligned}
 2. \quad P_{2n}(z) &= (-1)^n \frac{(2n-1)!!}{2^n n!} \left(1 - \frac{2n(2n+1)}{2!} z^2 + \frac{2n(2n-2)(2n+1)(2n+3)}{4!} z^4 - \dots \right) \\
 &= (-1)^n \frac{(2n-1)!!}{2^n n!} F\left(-n, n + \frac{1}{2}; \frac{1}{2}; z^2\right)
 \end{aligned}$$

AD (9002), MO 69

$$\begin{aligned}
 3. \quad P_{2n+1}(z) &= (-1)^n \frac{(2n+1)!!}{2^n n!} \left(z - \frac{2n(2n+3)}{3!} z^3 + \frac{2n(2n-2)(2n+3)(2n+5)}{5!} z^5 - \dots \right) \\
 &= (-1)^n \frac{(2n+1)!!}{2^n n!} z F\left(-n, n + \frac{3}{2}; \frac{3}{2}; z^2\right)
 \end{aligned}$$

AD (9002), MO 69

$$\begin{aligned}
 4. \quad P_n(\cos \varphi) &= \frac{(2n-1)!!}{2^n n!} \left(\cos n\varphi + \frac{1}{1} \frac{n}{2n-1} \cos(n-2)\varphi \right. \\
 &\quad + \frac{1 \cdot 3}{1 \cdot 2} \frac{n(n-1)}{(2n-1)(2n-3)} \cos(n-4)\varphi \\
 &\quad \left. + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos(n-6)\varphi - \dots \right)
 \end{aligned}$$

WH

$$\begin{aligned}
 5. \quad P_{2n}(\cos \varphi) &= (-1)^n \frac{(2n-1)!!}{2^n n!} \\
 &\quad \times \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{2!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n n!}{(2n-1)!!} \cos^{2n} \varphi \right\}
 \end{aligned}$$

AD (9011)

$$\begin{aligned}
 6. \quad P_{2n+1}(\cos \varphi) &= (-1)^n \frac{(2n+1)!!}{2^n n!} \cos \varphi \\
 &\quad \times \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{3!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n n!}{(2n+1)!!} \cos^{2n} \varphi \right\}
 \end{aligned}$$

AD (9012)

$$7. \quad P_n(z) = \sum_{k=0}^n \frac{(-1)^k (n+k)!}{(n-k)! (k!)^2 2^{k+1}} [(1-z)^k + (-1)^n (1+z)^k]$$

WH

8.912 Special cases:

1. $P_0(x) = 1$ JA
2. $P_1(x) = x = \cos \varphi$ JA
3. $P_2(x) = \frac{1}{2} (3x^2 - 1) = \frac{1}{4} (3 \cos 2\varphi + 1)$ JA
4. $P_3(x) = \frac{1}{2} (5x^3 - 3x) = \frac{1}{8} (5 \cos 3\varphi + 3 \cos \varphi)$ JA
5. $P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) = \frac{1}{64} (35 \cos 4\varphi + 20 \cos 2\varphi + 9)$ JA
6. $P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x) = \frac{1}{128} (63 \cos 5\varphi + 35 \cos 3\varphi + 30 \cos \varphi)$ JA
- 7.¹⁰ $P_6(x) = \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) = \frac{1}{512} (231 \cos 6\varphi + 126 \cos 4\varphi + 105 \cos 2\varphi + 50)$
8. $P_7(x) = \frac{1}{16} (429x^7 - 693x^5 + 315x^3 - 35x)$
 $= \frac{1}{1024} (429 \cos 7\varphi + 231 \cos 5\varphi + 189 \cos 3\varphi + 175 \cos \varphi)$
9. $P_8(x) = \frac{1}{128} (6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$
 $= \frac{1}{16384} (6435 \cos 8\varphi - 3432 \cos 6\varphi + 2772 \cos 4\varphi - 2520 \cos 2\varphi + 1225)$

8.913 Integral representations:

1. $P_n(\cos \varphi) = \frac{2}{\pi} \int_{\varphi}^{\pi} \frac{\sin(n + \frac{1}{2})t}{\sqrt{2(\cos \varphi - \cos t)}} dt$ WH
 See also **3.611** 3, **3.661** 3, 4.
- 2.⁷ Schläfli's integral formula:

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n}{2^n(t - z)^{n+1}} dt,$$

with C a simple contour containing z .

SA 175(9)

- 3.¹⁰ Laplace integral formula:

$$P_n(z) = \frac{1}{\pi} \int_0^{\pi} \left[x + (x^2 - 1)^{1/2} \cos \varphi \right]^n d\varphi \quad [|x| \leq 1] \quad \text{SA 180(19)}$$

Functional relations**8.914** Recurrence formulas:

1. $(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0$ WH

$$2. \quad (z^2 - 1) \frac{dP_n}{dz} = n[zP_n(z) - P_{n-1}(z)] = \frac{n(n+1)}{2n+1} [P_{n+1}(z) - P_{n-1}(z)] \quad \text{WH}$$

8.915

$$1.^{10} \quad \sum_{k=0}^n (2k+1) P_k(x) P_k(y) = (n+1) \frac{P_n(x) P_{n+1}(y) - P_n(y) P_{n+1}(x)}{y-x} \quad \text{(Christoffel summation formula)} \quad \text{MO 70}$$

$$1(1)^{10}. \quad (y-x) \sum_{k=0}^n (2k+1) P_k(x) Q_k(y) = 1 - (n+1) [P_{n+1}(x) Q_n(y) - P_n(x) Q_{n+1}(y)] \quad \text{AS 335(8.9.2)}$$

$$2.^7 \quad \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2n-4k-1) P_{n-2k-1}(z) = P'_n(z) \quad \text{(summation theorem)} \quad \text{MO 70}$$

$$3.^7 \quad \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} (2n-4k-3) P_{n-2k-2}(z) = z P'_n(z) - n P_n(z) \quad \text{SM 491(42), WH}$$

$$4.^{10} \quad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (2n-4k+1)[k(2n-2k+1)-2] P_{n-2k}(z) = z^2 P''_n(z) - n(n-1) P_n(z) \quad \text{WH}$$

$$5.^{11} \quad \sum_{k=0}^m \frac{a_{m-k} a_k a_{n-k}}{a_{n+m-k}} \left(\frac{2n+2m-4k+1}{2n+2m-2k+1} \right) P_{n+m-2k}(z) = P_n(z) P_m(z) \quad \left[a_k = \frac{(2k-1)!!}{k!}, \quad m \leq n \right] \quad \text{AD (9036)}$$

8.916

$$1. \quad P_n(\cos \varphi) = \frac{(2n-1)!!}{2^n n!} e^{\mp i n \varphi} F\left(\frac{1}{2}, -n; \frac{1}{2} - n; e^{\pm 2i \varphi}\right) \quad \text{MO 69}$$

$$2. \quad P_n(\cos \varphi) = F\left(n+1, -n; 1; \sin^2 \frac{\varphi}{2}\right) \quad \text{MO 69}$$

$$3. \quad P_n(\cos \varphi) = (-1)^n F\left(n+1, -n; 1; \cos^2 \frac{\varphi}{2}\right) \quad \text{WH}$$

$$4. \quad P_n(\cos \varphi) = \cos^n \varphi F\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1; -\tan^2 \varphi\right) \quad \text{HO 23}$$

$$5. \quad P_n(\cos \varphi) = \cos^{2n} \frac{\varphi}{2} F\left(-n, -n; 1; -\tan^2 \frac{\varphi}{2}\right) \quad \text{HO 23, 29, WH}$$

See also **8.911** 1, **8.911** 2, **8.911** 3. For a connection with other functions, see **8.936** 3, **8.836**, **8.962** 2.

- For integrals of Legendre polynomials, see **7.22–7.25**.
- For the zeros of Legendre polynomials, see **8.785**.

8.917 Inequalities:

1. $P_0(x) < P_1(x) < P_2(x) < \cdots < P_n(x) < \cdots$ [$x > 1$] MO 71
2. For $x > -1$, $P_0(x) + P_1(x) + \cdots + P_n(x) > 0$. MO 71
3. $[P_n(\cos \varphi)]^2 > \frac{\sin(2n+1)\varphi}{(2n+1)\sin \varphi}$ MO 71
4. $\sqrt{n \sin \varphi} |P_n(\cos \varphi)| \leq 1$. MO 71
5. $|P_n(\cos \varphi)| \leq 1$. WH
- 6.¹⁰ Let $n \geq 2$. The successive relative maxima of $|P_n(x)|$, when x decreases from 1 to 0, form a decreasing sequence. More precisely, if $\mu_1, \mu_2, \dots, \mu_{\lfloor n/2 \rfloor}$ denote these maxima corresponding to decreasing values of x , we have

$$1 > \mu_1 > \mu_2 > \cdots > \mu_{\lfloor n/2 \rfloor} \quad \text{SZ 162(7.3.1)}$$

- 7.¹⁰ Let $n \geq 2$. The successive relative maxima of $(\sin \theta)^{1/2} |P_n(\cos \theta)|$ when θ increases from 0 to $\pi/2$, form an increasing sequence. SZ 163(7.3.2)

- 8.¹⁰ We have

$$(\sin \theta)^{1/2} |P_n(\cos \theta)| < (2/\pi)^{1/2} n^{-1/2} \quad [0 \leq \theta \leq \pi] \quad \text{SZ 163(7.3.8)}$$

Here the constant $(2/\pi)^{1/2}$ cannot be replaced by a smaller one.

- 9.¹⁰ $\max_{0 \leq \theta \leq \pi} (\sin \theta)^{1/2} |P_n(\cos \theta)| \cong (2/\pi)^{1/2} n^{-1/2} \quad [n \rightarrow \infty]$ SZ 164(7.3.12)

- 10.¹⁰ Stieltjes' first theorem:

$$|P_n(\cos \theta)| \leq \left(\frac{2}{\pi}\right)^{1/2} \frac{4}{\sqrt{n \sin \theta}} \quad [n = 1, 2, \dots, 0 < \theta < \pi] \quad \text{SA 197(8)}$$

- 11.¹⁰ Stieltjes' second theorem:

$$|P_n(x) - P_{n+2}(x)| < \frac{4}{\sqrt{\pi} \sqrt{n+2}} \quad [|x| \leq 1] \quad \text{SA 199(15)}$$

- 12.¹⁰ $\left| \frac{dP_n(x)}{dx} \right| < \frac{2}{\sqrt{\pi}} \frac{\sqrt{n}}{1-x^2} \quad [|x| < 1, \quad n = 1, 2, \dots]$ SA 201(18)

- 13.¹⁰ $|P_{n+1}(x) + P_n(x)| < 6 \left(\frac{2}{\pi n}\right)^{\frac{1}{2}} (1-x)^{-1/2} \quad [|x| < 1, \quad n = 0, 1, \dots]$ SA 201(19)

8.918¹⁰ Asymptotic approximations:

1. $P_n(\cos \theta) = \left(\frac{2}{\pi n \sin \varphi}\right)^{1/2} \cos \left[\left(n + \frac{1}{2}\right) \theta - \frac{\pi}{4} \right] + O(n^{-3/2})$
[$\varepsilon \leq \theta \leq \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2m$] (Laplace's formula) SA 208(1)

$$\begin{aligned}
2. \quad P_n(\cos \theta) &= \left(\frac{2}{\pi n \sin \theta} \right)^{1/2} \left\{ \left(1 - \frac{1}{4n} \right) \cos \left[\left(n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right] + \frac{1}{8n} \cos \theta \sin \left[\left(n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right] \right\} \\
&\quad + O\left(n^{-5/2}\right) \\
&\quad [\varepsilon \leq \theta \leq \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2] \quad (\text{Bonnet-Heine formula}) \quad \text{SA 208(2)}
\end{aligned}$$

8.919¹⁰ Series of products of Legendre and Chebyshev polynomials

$$1. \quad 2 \int_{-1}^1 T_n(x) P_n(x) dx = \sum_{i,j=0}^{i+j=n} \int_{-1}^1 P_i(x) P_j(x) P_n(x) dx$$

8.92 Series of Legendre polynomials

8.921 The generating function:

$$\begin{aligned}
\frac{1}{\sqrt{1-2tz+t^2}} &= \sum_{k=0}^{\infty} t^k P_k(z) \quad \left[|t| < \min \left| z \pm \sqrt{z^2-1} \right| \right] && \text{SM 489(31), WH} \\
&= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} P_k(z) \quad \left[|t| > \max \left| z \pm \sqrt{z^2-1} \right| \right] && \text{MO 70}
\end{aligned}$$

8.922

$$1. \quad z^{2n} = \frac{1}{2n+1} P_0(z) + \sum_{k=1}^{\infty} (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} P_{2k}(z) \quad \text{MO 72}$$

$$2. \quad z^{2n+1} = \frac{3}{2n+3} P_1(z) + \sum_{k=1}^{\infty} (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} P_{2k+1}(z) \quad \text{MO 72}$$

$$\begin{aligned}
3. \quad \frac{1}{\sqrt{1-x^2}} &= \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 P_{2k}(x) \quad [|x| < 1, \quad (-1)!! \equiv 1] \\
&&& \text{MO 72, LA 385(15)}
\end{aligned}$$

$$\begin{aligned}
4. \quad \frac{x}{\sqrt{1-x^2}} &= \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+3) \frac{(2k-1)!!(2k+1)!!}{2^{2k+1} k! (k+1)!} P_{2k+1}(x) \\
&&& [|x| < 1, \quad (-1)!! \equiv 1] \quad \text{LA 385(17)}
\end{aligned}$$

$$\begin{aligned}
5. \quad \sqrt{1-x^2} &= \frac{\pi}{2} \left\{ \frac{1}{2} - \sum_{k=1}^{\infty} (4k+1) \frac{(2k-3)!!(2k-1)!!}{2^{2k+1} k! (k+1)!} P_{2k}(x) \right\} \\
&&& [|x| < 1, \quad (-1)!! \equiv 1] \quad \text{LA 385(18)}
\end{aligned}$$

$$6.^{10} \quad \sqrt{\frac{1-x}{2}} = \frac{2}{3} P_0(x) - 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+3)} P_n(x) \quad [-1 \leq x \leq 1]$$

$$7.10 \quad \frac{1 - \rho^2}{(1 - 2\rho x + \rho^2)^{1/2}} = 1 + \sum_{n=0}^{\infty} (2n+1)\rho^n P_n(x), \quad [|\rho| < 1, \quad |x| \leq 1] \quad \text{SA 170(4)}$$

$$8.923 \quad \arcsin x = \frac{\pi}{2} \sum_{k=1}^{\infty} \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 [P_{2k+1}(x) - P_{2k-1}(x)] + \pi x/2$$

$$[|x| < 1, \quad (-1)!! \equiv 1] \quad \text{WH}$$

8.924

$$1. \quad -\frac{1 + \cos n\pi}{2(n^2 - 1)} P_0(\cos \theta) - \frac{1 + \cos n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k+5)n^2(n^2 - 2^2) \dots [n^2 - (2k)^2]}{(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k+3)^2]} P_{2k+2}(\cos \theta)$$

$$- \frac{3(1 - \cos n\pi)}{2(n^2 - 2^2)} P_1(\cos \theta)$$

$$- \frac{1 - \cos n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k+3)(n^2 - 1^2) \dots [n^2 - (2k-1)^2]}{(n^2 - 2^2)(n^2 - 4^2) \dots [n^2 - (2k+2)^2]} P_{2k+1}(\cos \theta) = \cos n\theta$$

AD (9060.1)

$$2. \quad \frac{-\sin n\pi}{2(n^2 - 1)} P_0(\cos \theta) - \frac{\sin n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k+5)n^2(n^2 - 2^2) \dots [n^2 - (2k)^2]}{(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k+3)^2]} P_{2k+2}(\cos \theta)$$

$$+ \frac{3\sin n\pi}{2(n^2 - 2^2)} P_1(\cos \theta)$$

$$+ \frac{\sin n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k+3)(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k-1)^2]}{(n^2 - 2^2)(n^2 - 4^2) \dots [n^2 - (2k+2)^2]} P_{2k+1}(\cos \theta) = \sin n\theta$$

AD (9060.2)

$$3.3 \quad \frac{2^{n-1}n!}{(2n-1)!!} P_n(\cos \theta) - n \sum_{k=1}^{\lfloor n/2 \rfloor} (2n-4k+1) \frac{2^{n-2k-1}(n-k-1)!(2k-3)!!}{(2n-2k+1)!!k!} P_{n-2k}(\cos \theta)$$

$$= \cos n\theta$$

AD (9061.1)

$$4. \quad \frac{(2n-1)!!P_{n-1}(\cos \theta)}{2^{n-1}(n-1)!} - \frac{n}{2^{n+1}} \sum_{k=0}^{\infty} \frac{(2n+2k-1)!!(2k-1)!!(2n+4k+3)}{2^{2k}(n+k+1)!(k+1)!} P_{n+2k+1}(\cos \theta)$$

$$= \frac{4\sin n\theta}{\pi}$$

AD (9061.2)

8.925

$$1. \quad \sum_{k=1}^{\infty} \frac{4k-1}{2^{2k}(2k-1)^2} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = 1 - \frac{2\theta}{\pi}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{4k+1}{2^{2k+1}(2k-1)(k+1)} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{1}{2} - \frac{2\sin \theta}{\pi} \quad \text{AD (9062.2)}$$

$$3. \quad \sum_{k=1}^{\infty} \frac{k(4k-1)}{2^{2k-1}(2k-1)} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = \frac{2\cot \theta}{\pi} \quad \text{AD (9062.3)}$$

$$4. \quad \sum_{k=1}^{\infty} \frac{4k+1}{2^{2k}} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{2}{\pi \sin \theta} - 1 \quad \text{AD (9062.4)}$$

8.926

$$1. \quad \sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) = \ln \frac{2 \tan \frac{\pi-\theta}{4}}{\sin \theta} = -\ln \sin \frac{\theta}{2} - \ln \left(1 + \sin \frac{\theta}{2} \right) \quad \text{AD (9063.2)}$$

$$2. \quad \sum_{n=1}^{\infty} \frac{1}{n+1} P_n(\cos \theta) = \ln \frac{1 + \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} - 1 \quad \text{AD (9063.1)}$$

$$\begin{aligned} \mathbf{8.927} \quad \sum_{k=0}^{\infty} \cos \left(k + \frac{1}{2} \right) \beta P_k(\cos \varphi) &= \frac{1}{\sqrt{2(\cos \beta - \cos \varphi)}} & [0 \leq \beta < \varphi < \pi] \\ &= 0 & [0 < \varphi < \beta < \pi] \end{aligned}$$

MO 72

8.928

$$1. \quad \sum_{n=1}^{\infty} \frac{(-1)^n (4k+1) [(2n-1)!!]^3}{2^{3n} (n!)^3} P_{2n}(\cos \theta) = \frac{4 \mathbf{K}(\sin \theta)}{\pi^2} - 1 \quad \text{AD (9064.1)}$$

$$2. \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4n+1) [(2n-1)!!]^3}{(2n-1)(2n+2)2^{3n} (n!)^3} P_{2n}(\cos \theta) = \frac{4 \mathbf{E}(\sin \theta)}{\pi^2} - \frac{1}{2} \quad \text{AD (9064.2)}$$

- For series of products of Bessel functions and Legendre polynomials, see **8.511** 4, **8.531** 3, **8.533** 1, **8.533** 2, and **8.534**.
- For series of products of Legendre and Chebyshev polynomials, see **8.919**.

8.93 Gegenbauer polynomials $C_n^\lambda(t)$

8.930 Definition. The polynomials $C_n^\lambda(t)$ of degree n are the coefficients of α^n in the power-series expansion of the function

$$(1 - 2t\alpha + \alpha^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t) \alpha^n \quad \text{WH}$$

Thus, the polynomials $C_n^\lambda(t)$ are a *generalization of the Legendre polynomials*.

$$\begin{aligned} 1.^{10} \quad C_0^\lambda(t) &= 1 \\ 2.^{10} \quad C_1^\lambda(t) &= 2\lambda t \\ 3.^{10} \quad C_2^\lambda(t) &= 2\lambda(\lambda+1)t^2 - \lambda \\ 4.^{10} \quad C_3^\lambda(t) &= \frac{1}{3}\lambda(4\lambda^2 + 12\lambda + 8)t^3 - 2\lambda(\lambda+1)t \\ 5.^{11} \quad C_4^\lambda(t) &= \frac{2}{3}\lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)t^4 - 2\lambda(\lambda^2 + 3\lambda + 2)t^2 + \frac{1}{2}\lambda(\lambda+1) \\ 6.^{10} \quad C_5^\lambda(t) &= \frac{1}{15}\lambda(4\lambda^4 + 40\lambda^3 + 140\lambda^2 + 200\lambda + 96)t^5 \\ &\quad - \frac{1}{3}\lambda(4\lambda^3 + 24\lambda^2 + 44\lambda + 24)t^3 + \lambda(\lambda^2 + 3\lambda + 2)t \end{aligned}$$

$$\begin{aligned}
7.10 \quad C_6^\lambda(t) &= \frac{1}{45} \lambda (\lambda^5 + 60\lambda^4 + 340\lambda^3 + 900\lambda^2 + 1096\lambda + 480) t^6 \\
&\quad - \frac{1}{3} \lambda (2\lambda^4 + 20\lambda^3 + 70\lambda^2 + 100\lambda + 48) t^4 \\
&\quad + \lambda (\lambda^3 + 6\lambda^2 + 11\lambda + 6) t^2 + \frac{1}{6} \lambda (\lambda^2 + 3\lambda + 2)
\end{aligned}$$

8.931 Integral representation:

$$C_n^\lambda(t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2\lambda + n)}{n! \Gamma(2\lambda)} \frac{\Gamma(\frac{2\lambda+1}{2})}{\Gamma(\lambda)} \int_0^\pi \left(t + \sqrt{t^2 - 1} \cos \varphi\right)^n \sin^{2\lambda-1} \varphi d\varphi \quad \text{MO 99}$$

See also **3.252** 11, **3.663** 2, **3.664** 4.

Functional relations

8.932 Expressions in terms of hypergeometric functions:

$$1. \quad C_n^\lambda(t) = \frac{\Gamma(2\lambda + n)}{\Gamma(n+1) \Gamma(2\lambda)} F\left(2\lambda + n, -n; \lambda + \frac{1}{2}; \frac{1-t}{2}\right)^* \quad \text{MO 97}$$

$$= \frac{2^n \Gamma(\lambda + n)}{n! \Gamma(\lambda)} t^n F\left(-\frac{n}{2}, \frac{1-n}{2}; 1 - \lambda - n; \frac{1}{t^2}\right) \quad \text{MO 99}$$

$$2. \quad C_{2n}^\lambda(t) = \frac{(-1)^n}{(\lambda + n) B(\lambda, n+1)} F\left(-n, n + \lambda; \frac{1}{2}; t^2\right) \quad \text{MO 99}$$

$$3. \quad C_{2n+1}^\lambda(t) = \frac{(-1)^n 2t}{B(\lambda, n+1)} F\left(-n, n + \lambda + 1; \frac{3}{2}; t^2\right) \quad \text{MO 99}$$

8.933 Recursion formulas:

$$1. \quad (n+2) C_{n+2}^\lambda(t) = 2(\lambda + n+1)t C_{n+1}^\lambda(t) - (2\lambda + n) C_n^\lambda(t) \quad \text{Mo 98}$$

$$2. \quad n C_n^\lambda(t) = 2\lambda \left[t C_{n-1}^{\lambda+1}(t) - C_{n-2}^{\lambda+1}(t)\right] \quad \text{WH}$$

$$3. \quad (2\lambda + n) C_n^\lambda(t) = 2\lambda \left[C_n^{\lambda+1}(t) - t C_{n-1}^{\lambda+1}(t)\right] \quad \text{WH}$$

$$4. \quad n C_n^\lambda(t) = (2\lambda + n - 1)t C_{n-1}^\lambda(t) - 2\lambda (1 - t^2) C_{n-2}^{\lambda+1}(t) \quad \text{WH}$$

8.934

$$1. \quad C_n^\lambda(t) = \frac{(-1)^n}{2^n} \frac{\Gamma(2\lambda + n)}{\Gamma(2\lambda) \Gamma(\frac{2\lambda+1}{2} + n)} \frac{(1-t^2)^{\frac{1}{2}-\lambda}}{n!} \frac{d^n}{dt^n} \left[(1-t^2)^{\lambda+n-\frac{1}{2}}\right] \quad \text{WH}$$

$$2. \quad C_n^\lambda(\cos \varphi) = \sum_{\substack{k, l=0 \\ k+l=n}}^n \frac{\Gamma(\lambda + k) \Gamma(\lambda + l)}{k! l! [\Gamma(\lambda)]^2} \cos(k-l)\varphi \quad \text{MO 99}$$

*Equation 8.932.1 defines the generalized functions $C_n^\lambda(t)$, where the subscript n can be an arbitrary number.

$$\begin{aligned}
3. \quad C_n^\lambda (\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) \\
&= \frac{\Gamma(2\lambda - 1)}{[\Gamma(\lambda)]^2} \sum_{k=0}^n \frac{2^{2k} (n-k)! [\Gamma(\lambda + k)]^2}{\Gamma(2\lambda + n + k)} (2\lambda + 2k - 1) \sin^k \psi \sin^k \vartheta \\
&\quad \times C_{n-k}^{\lambda+k} (\cos \psi) C_{n-k}^{\lambda+k} (\cos \vartheta) C_k^{\lambda-\frac{1}{2}} (\cos \varphi) \\
&\quad [\psi, \vartheta, \varphi \text{ real}; \quad \lambda \neq \tfrac{1}{2}] \quad [\text{"summation theorem"}] \quad (\text{see also } \mathbf{8.794-8.796}) \quad \text{WH}
\end{aligned}$$

$$4. \quad \lim_{\lambda \rightarrow 0} \Gamma(\lambda) C_n^\lambda (\cos \varphi) = \frac{2 \cos n\varphi}{n} \quad \text{MO 98}$$

For orthogonality, see **8.904**, **7.313**.

8.935 Derivatives:

$$1. \quad \frac{d^k}{dt^k} C_n^\lambda(t) = 2^k \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} C_{n-k}^{\lambda+k}(t) \quad \text{MO 99}$$

In particular,

$$2.^{11} \quad \frac{d C_n^\lambda(t)}{dt} = 2\lambda C_{n-1}^{\lambda+1}(t) \quad \text{WH}$$

For integrals of the polynomials $C_n^\lambda(x)$ see **7.31-7.33**.

8.936 Connections with other functions:

$$1. \quad C_n^\lambda(t) = \frac{\Gamma(2\lambda + n) \Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda) \Gamma(n + 1)} \left\{ \frac{1}{4} (t^2 - 1) \right\}^{\frac{1}{4} - \frac{\lambda}{2}} P_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(t) \quad \text{MO 98}$$

$$\begin{aligned}
2. \quad C_{n-m}^{m+\frac{1}{2}}(t) &= \frac{1}{(2m-1)!!} \frac{d^m P_n(t)}{dt^m} = (-1)^m \frac{(1-t^2)^{-\frac{m}{2}} m! 2^m}{(2m)!} P_n^m(t) \\
&\quad [m+1 \text{ a natural number}] \quad \text{MO 98, WH}
\end{aligned}$$

$$3. \quad C_n^{1/2}(t) = P_n(t)$$

$$\begin{aligned}
4. \quad J_{\lambda-\frac{1}{2}}(r \sin \vartheta \sin \alpha) (r \sin \vartheta \sin \alpha)^{-\lambda+\frac{1}{2}} e^{-ir \cos \vartheta \cos \alpha} \\
&= \sqrt{2} \frac{\Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})} \sum_{k=0}^{\infty} (\lambda + k) i^{-k} \frac{\mathbf{J}_{\lambda+k}(r) C_k^\lambda(\cos \vartheta) C_k^\lambda(\cos \alpha)}{r^\lambda C_k^\lambda(1)} \\
&\quad \text{MO 99}
\end{aligned}$$

$$5. \quad \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{n}{2}} C_n^{\frac{\lambda}{2}} \left(t \sqrt{\frac{2}{\lambda}} \right) = \frac{2^{-\frac{n}{2}}}{n!} H_n(t) \quad \text{MO 99a}$$

See also **8.932**.

8.937 Special cases and particular values:

$$1. \quad C_n^1(\cos \varphi) = \frac{\sin(n+1)\varphi}{\sin \varphi} \quad \text{MO 99}$$

$$2. \quad C_0^0(\cos \varphi) = 1 \quad \text{MO 98}$$

$$3. \quad C_0^\lambda(t) \equiv 1 \quad \text{MO 98}$$

$$4. \quad C_n^\lambda(1) \equiv \binom{2\lambda + n - 1}{n} \quad \text{MO 98}$$

8.938 A differential equation leading to the polynomials $C_n^\lambda(t)$:

$$y'' + \frac{(2\lambda + 1)t}{t^2 - 1}y' - \frac{n(2\lambda + n)}{t^2 - 1}y = 0 \quad (\text{cf. } \mathbf{9.174}) \quad \text{WH}$$

For series of products of Bessel functions and the polynomials $C_n^\lambda(x)$, see **8.532**, **8.534**.

8.939¹⁰ Differentiation and Rodrigues' formulas and orthogonality relation

$$1. \quad \frac{d}{dt} C_n^\lambda(t) = 2\lambda C_{n-1}^{\lambda+1}(t) \quad \text{MS 5.3.2}$$

$$2. \quad \frac{d^m}{dt^m} C_n^\lambda(t) = 2^m \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + m - 1) C_{n-m}^{\lambda+m}(t) \quad \text{MS 5.3.2}$$

$$3. \quad \frac{d}{dt} C_{n-1}^\lambda(t) = t \frac{d}{dt} C_n^\lambda(t) - n C_n^\lambda(t) \quad \text{MS 5.3.2}$$

$$4. \quad \frac{d}{dt} C_{n+1}^\lambda(t) = t \frac{d}{dt} C_n^\lambda(t) + (2\lambda + n) C_n^\lambda(t) \quad \text{MS 5.3.2}$$

$$5. \quad (1 - t^2) \frac{d}{dt} C_n^\lambda(t) = (n + 2\lambda - 1) C_{n-1}^\lambda(t) - nt C_n^\lambda(t) = (n + 2\lambda)t C_n^\lambda(t) - (n + 1) C_{n+1}^\lambda(t) \\ = 2\lambda(1 - t^2) C_{n-1}^{\lambda+1}(t) \quad \text{MS 5.3.2}$$

$$6. \quad \frac{d}{dt} [C_{n+1}^\lambda(t) - C_{n-1}^\lambda(t)] = 2(n + \lambda) C_n^\lambda(t) \quad \text{MS 5.3.2}$$

$$7. \quad C_n^\lambda(t) = \frac{(-1)^n 2\lambda(2\lambda + 1)(2\lambda + 2) \dots (2\lambda + n - 1) (1 - t^2)^{\frac{1}{2} - \lambda}}{2^n n! (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) \dots (\lambda + n - \frac{1}{2})} \frac{d^n}{dt^n} [(1 - t^2)^{n + \lambda - \frac{1}{2}}] \\ = \frac{(-1)^n \Gamma(\lambda + \frac{1}{2}) \Gamma(n + 2\lambda) (1 - t^2)^{\frac{1}{2} - \lambda}}{2^n n! \Gamma(2\lambda) \Gamma(n + \lambda + \frac{1}{2})} \frac{d^n}{dt^n} [(1 - t^2)^{n + \lambda - \frac{1}{2}}] \\ \quad \text{[Rodrigues' formula]} \quad \text{MS 5.3.2}$$

$$8. \quad \int_{-1}^1 C_n^\lambda(t) C_m^\lambda(t) (1 - t^2)^{\lambda - \frac{1}{2}} dt = 0 \quad n \neq m \\ = \frac{\pi 2^{1-2\lambda} \Gamma(n + 2\lambda)}{n! (\lambda + n) [\Gamma(\lambda)]^2} \quad n = m \\ \quad [\lambda \neq 0] \quad \text{[Orthogonality relation]} \quad \text{MS 5.3.2}$$

8.94 The Chebyshev polynomials $T_n(x)$ and $U_n(x)$

8.940 Definition

1. Chebyshev's polynomials of the first kind

$$T_n(x) = \cos(n \arccos x) = \frac{1}{2} \left[\left(x + i\sqrt{1 - x^2} \right)^n + \left(x - i\sqrt{1 - x^2} \right)^n \right] \\ = x^n - \binom{n}{2} x^{n-2} (1 - x^2) + \binom{n}{4} x^{n-4} (1 - x^2)^2 - \binom{n}{6} x^{n-6} (1 - x^2)^3 + \dots$$

2. Chebyshev's polynomials of the second kind:

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin[\arccos x]} = \frac{1}{2i\sqrt{1-x^2}} \left[\left(x + i\sqrt{1-x^2}\right)^{n+1} - \left(x - i\sqrt{1-x^2}\right)^{n+1} \right]$$

$$= \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2}(1-x^2) + \binom{n+1}{5}x^{n-4}(1-x^2)^2 - \dots$$

Functional relations

8.941 Recursion formulas:

1. $T_{n+1}(x) - 2x T_n(x) + T_{n-1}(x) = 0$ NA 358
2. $U_{n+1}(x) - 2x U_n(x) + U_{n-1}(x) = 0$
3. $T_n(x) = U_n(x) - x U_{n-1}(x)$ EH II 184(3)
4. $(1-x^2) U_{n-1}(x) = x T_n(x) - T_{n+1}(x)$ EH II 184(4)

For the orthogonality, see **7.343** and **8.904**.

8.942 Relations with other functions:

1. $T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right)$ MO 104
2. $T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}$ MO 104
3. $U_n(x) = \frac{(-1)^n(n+1)}{\sqrt{1-x^2}(2n+1)!!} \frac{d^n}{dx^n} (1-x^2)^{n+\frac{1}{2}}$ EH II 185(15)

See also **8.962** 3.

8.943¹⁰ Special cases

1. $T_0(x) = 1$	10. $U_0(x) = 1$
2. $T_1(x) = x$	11. $U_1(x) = 2x$
3. $T_2(x) = 2x^2 - 1$	12. $U_2(x) = 4x^2 - 1$
4. $T_3(x) = 4x^3 - 3x$	13. $U_3(x) = 8x^3 - 4x$
5. $T_4(x) = 8x^4 - 8x^2 + 1$	14. $U_4(x) = 16x^4 - 12x^2 + 1$
6. $T_5(x) = 16x^5 - 20x^3 + 5x$	15. $U_5(x) = 32x^5 - 32x^3 + 6x$
7. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$	16. $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$
8. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$	17. $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$
9. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$	18. $U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$

8.944 Particular values:

1.	$T_n(1) = 1$	5.	$U_{2n+1}(0) = 0$
2.	$T_n(-1) = (-1)^n$	6.	$U_{2n}(0) = (-1)^n$
3.	$T_{2n}(0) = (-1)^n$		
4.	$T_{2n+1}(0) = 0$		

8.945 The generating function:

1. ¹¹	$\frac{1-t^2}{1-2tx+t^2} = T_0(x) + 2 \sum_{k=1}^{\infty} T_k(x)t^k$	$[t < 1]$	MO 104
2. ¹¹	$\frac{1}{1-2tx+t^2} = \sum_{k=0}^{\infty} U_k(x)t^k$	$[t < 1]$	MO 104a, EH II 186(31)

8.946 Zeros. The polynomials $T_n(x)$ and $U_n(x)$ only have real simple zeros. All these zeros lie in the interval $(-1, +1)$.

8.947 The functions $T_n(x)$ and $\sqrt{1-x^2} U_{n-1}(x)$ are two linearly independent solutions of the differential equation

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0. \quad \text{NA 69(58)}$$

8.948 Of all polynomials of degree n with leading coefficient equal to 1, the one that deviates the least from zero on the interval $[-1, +1]$ is the polynomial $2^{-n+1} T_n(x)$.

8.949¹⁰ Differentiation and Rodrigues' formulas and orthogonality relations

1.	$\frac{d}{dx} T_n(x) = n U_{n-1}(x)$	MS 5.7.2
2.	$\frac{d^m}{dx^m} T_n(x) = 2^{m-1} \Gamma(m) n C_{n-m}^m(x)$	MS 5.7.2
3.	$(1-x^2) \frac{d}{dx} T_n(x) = n [T_{n-1}(x) - x T_n(x)] = n [x T_n(x) - T_{n+1}(x)]$	MS 5.7.2
4.	$\frac{d}{dx} U_n(x) = 2 C_{n-1}^2(x)$	MS 5.7.2
5.	$\frac{d^m}{dx^m} U_n(x) = 2^m m! C_{n-m}^{m+1}(x)$	MS 5.7.2
6.	$(1-x^2) \frac{d}{dx} U_n(x) = (n+1) U_{n-1}(x) - nx U_n(x) = (n+2)x U_n(x) - (n+1) U_{n+1}(x)$	MS 5.7.2
7.	$T_n(x) = \frac{(-1)^n \pi^{1/2} (1-x^2)^{c\frac{1}{2}}}{2^{n+1} \Gamma(n + \frac{1}{2})} \frac{d^n}{dx^n} [(1-x^2)^{n-\frac{1}{2}}]$	[Rodrigues' formula] MS 5.7.2
8.	$U_n(x) = \frac{(-1)^n \pi^{1/2} (n+1) (1-x^2)^{-1/2}}{2^{n+1} \Gamma(n + \frac{3}{2})} \frac{d^n}{dx^n} [(1-x^2)^{n+\frac{1}{2}}]$	[Rodrigues' formula] MS 5.7.2

$$9. \quad \int_{-1}^1 T_m(x) T_n(x) (1-x^2)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

[Orthogonality relation] MS 5.7.2

$$10. \quad \int_{-1}^1 U_m(x) U_n(x) (1-x^2)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/8, & m = n \end{cases}$$

[Orthogonality relation] MS 5.7.2

8.95 The Hermite polynomials $H_n(x)$

8.950 Definition

$$1. \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \quad \text{SM 567(14)}$$

or

$$2. \quad H_n(x) = 2^n x^n - 2^{n-1} \binom{n}{2} x^{n-2} + 2^{n-2} \cdot 1 \cdot 3 \cdot \binom{n}{4} x^{n-4} - 2^{n-3} \cdot 1 \cdot 3 \cdot 5 \cdot \binom{n}{6} x^{n-6} + \dots \quad \text{MO 105a}$$

$$3.^{10} \quad H_0(x) = 1$$

$$4.^{10} \quad H_1(x) = 2x$$

$$5.^{10} \quad H_2(x) = 4x^2 - 2$$

$$6.^{10} \quad H_3(x) = 8x^3 - 12x$$

$$7.^{10} \quad H_4(x) = 16x^4 - 48x^2 + 12$$

$$8.^{10} \quad H_5(x) = 32x^5 - 160x^3 + 120x$$

$$9.^{10} \quad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

$$10.^{10} \quad H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$$

$$11.^{10} \quad H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$$

8.951 The integral representation:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+it)^n e^{-t^2} dt \quad \text{MO 106a}$$

Functional relations

8.952 Recursion formulas:

$$1. \quad \frac{dH_n(x)}{dx} = 2n H_{n-1}(x) \quad \text{SM 569(22)}$$

$$2. \quad H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \quad \text{SM 570(23)}$$

For the orthogonality, see **7.374** 1 and **8.904**.

$$3.^{10} \quad n H_n(x) = -n H'_{n-1}(x) + x H'_n(x) \quad \text{MS 5.6.2}$$

$$4.^{10} \quad H_n(x) = 2x H_{n-1}(x) - H'_{n-1}(x) \quad \text{MS 5.6.2}$$

8.953 The connection with other functions:

$$1. \quad H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} \Phi\left(-n, \frac{1}{2}; x^2\right) \quad \text{MO 106a}$$

$$2. \quad H_{2n+1}(x) = (-1)^n 2 \frac{(2n+1)!}{n!} x \Phi\left(-n, \frac{3}{2}; x^2\right) \quad \text{MO 106a}$$

- For a connection with the polynomials $C_n^\lambda(x)$, see **8.936** 5.
- For a connection with the Laguerre polynomials, see **8.972** 2 and **8.972** 3.
- For a connection with functions of a parabolic cylinder, see **9.253**.

8.954 Inequalities:

$$1.^{10} \quad |H_n(x)| \leq 2^{\frac{n}{2} - \lfloor \frac{n}{2} \rfloor} \frac{n!}{\lfloor n/2 \rfloor!} e^{2x\sqrt{\lfloor n/2 \rfloor}} \quad \text{MO 106a}$$

$$2.^{10} \quad |H_n(x)| < k\sqrt{n!} 2^{n/2} e^{x^2/2}, \quad k \approx 1.086435 \quad \text{SA 324}$$

8.955 Asymptotic representation:

$$1. \quad H_{2n}(x) = (-1)^n 2^n (2n-1)!! e^{x^2/2} \left[\cos(\sqrt{4n+1}x) + O\left(\frac{1}{\sqrt[4]{n}}\right) \right] \quad \text{SM 579}$$

$$2. \quad H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} (2n-1)!! \sqrt{2n+1} e^{x^2/2} \left[\sin(\sqrt{4n+3}x) + O\left(\frac{1}{\sqrt[4]{n}}\right) \right] \quad \text{SM 579}$$

8.956 Special cases and particular values:

- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$
- $H_{2n}(0) = (-1)^n 2^n (2n-1)!! \quad \text{SM 570(24)}$
- $H_{2n+1}(0) = 0$

Series of Hermite polynomials

8.957 The generating function:

$$1. \quad \exp(-t^2 + 2tx) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) \quad \text{SM 569(21)}$$

$$2. \quad \frac{1}{e} \sinh 2x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} H_{2k+1}(x) \quad \text{MO 106a}$$

$$3. \quad \frac{1}{e} \cosh 2x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} H_{2k}(x) \quad \text{MO 106a}$$

$$4. \quad e \sin 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} H_{2k+1}(x) \quad \text{MO 106a}$$

$$5. \quad e \cos 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} H_{2k}(x) \quad \text{MO 106a}$$

8.958 “The summation theorem”:

$$1.^{11} \quad \frac{\left(\sum_{k=1}^r a_k^2\right)^{\frac{n}{2}}}{n!} H_n \left(\frac{\sum_{k=1}^r a_k x_k}{\sqrt{\sum_{k=1}^r a_k^2}} \right) = \sum_{m_1+m_2+\dots+m_r=n} \prod_{k=1}^r \left\{ \frac{a_k^{m_k}}{m_k!} H_{m_k}(x_k) \right\} \quad \text{MO 106a}$$

2. A special case:

$$2^{\frac{n}{2}} H_n(x+y) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(x\sqrt{2}) H_k(y\sqrt{2}) \quad \text{MO 107a}$$

8.959 Hermite polynomials satisfy the differential equation

$$1. \quad \frac{d^2 u_n}{dx^2} - 2x \frac{du_n}{dx} + 2n u_n = 0; \quad \text{SM 566(9)}$$

A second solution of this differential equation is provided by the functions (A and B are arbitrary constants):

$$2. \quad u_{2n} = A x \Phi\left(\frac{1}{2} - n; \frac{3}{2}; x^2\right),$$

$$3. \quad u_{2n+1} = B \Phi\left(-\frac{1}{2} - n; \frac{1}{2}; x^2\right) \quad \text{MO 107}$$

8.959(1)¹⁰ Rodrigues' formula and orthogonality relation

$$1. \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left[e^{-x^2} \right] \quad \text{[Rodrigues' formula]} \quad \text{MS 5.6.2}$$

$$2. \quad \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi^{1/2} 2^n n! & \text{for } m = n \end{cases} \quad \text{MS 5.6.2}$$

8.96 Jacobi's polynomials

8.960 Definition

$$1. \quad P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{\alpha+n} (1+x)^{\beta+n}] \quad \text{EH II 169(10), CO}$$

$$= \frac{1}{2^n} \sum_{m=0}^n \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m \quad \text{EH II 169(2)}$$

8.961 Functional relations:

$$1.^{11} \quad P_n^{(\alpha, \alpha)}(-x) = (-1)^n P_n^{(\alpha, \alpha)}(x) \quad \text{EH II 169(13)}$$

$$\begin{aligned} 2. \quad & 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) P_{n+1}^{(\alpha, \beta)}(x) \\ &= (2n+\alpha+\beta+1) [(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2] P_n^{(\alpha, \beta)}(x) \\ &\quad - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) P_{n-1}^{(\alpha, \beta)}(x) \end{aligned} \quad \text{EH II 169(11)}$$

$$\begin{aligned} 3. \quad & (2n+\alpha+\beta)(1-x^2) \frac{d}{dx} P_n^{(\alpha, \beta)}(x) = n[(\alpha-\beta) - (2n+\alpha+\beta)x] P_n^{(\alpha, \beta)}(x) \\ & \quad + 2(n+\alpha)(n+\beta) P_{n-1}^{(\alpha, \beta)}(x) \end{aligned} \quad \text{EH II 170(15)}$$

$$\begin{aligned} 4.^{11} \quad & \frac{d^m}{dx^m} [P_n^{(\alpha, \beta)}(x)] = \frac{1}{2^m} \frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-m}^{(\alpha+m, \beta+m)}(x) \\ & \quad [m = 1, 2, \dots, n] \end{aligned} \quad \text{EH II 170(17)}$$

$$5. \quad (n + \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + 1)(1-x) P_n^{(\alpha+1, \beta)}(x) = (n+\alpha+1) P_n^{(\alpha, \beta)}(x) - (n+1) P_{n+1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(32)}$$

$$6. \quad (n + \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + 1)(1+x) P_n^{(\alpha, \beta+1)}(x) = (n+\beta+1) P_n^{(\alpha, \beta)}(x) + (n+1) P_{n+1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(33)}$$

$$7. \quad (1-x) P_n^{(\alpha+1, \beta)}(x) + (1+x) P_n^{(\alpha, \beta+1)}(x) = 2 P_n^{(\alpha, \beta)}(x) \quad \text{EH II 173(34)}$$

$$8. \quad (2n+\alpha+\beta) P_n^{(\alpha-1, \beta)}(x) = (n+\alpha+\beta) P_n^{(\alpha, \beta)}(x) - (n+\beta) P_{n-1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(35)}$$

$$9. \quad (2n+\alpha+\beta) P_n^{(\alpha, \beta-1)}(x) = (n+\alpha+\beta) P_n^{(\alpha, \beta)}(x) + (n+\alpha) P_{n-1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(36)}$$

$$10. \quad P_n^{(\alpha, \beta-1)}(x) - P_n^{(\alpha-1, \beta)}(x) = P_{n-1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(37)}$$

8.962 Connections with other functions:

$$\begin{aligned} 1. \quad & P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n \Gamma(n+1+\beta)}{n! \Gamma(1+\beta)} F\left(n+\alpha+\beta+1, -n; 1+\beta; \frac{1+x}{2}\right) \quad \text{CO, EH II 170(16)} \\ &= \frac{\Gamma(n+1+\alpha)}{n! \Gamma(1+\alpha)} F\left(n+\alpha+\beta+1, -n; 1+\alpha; \frac{1-x}{2}\right) \quad \text{EH II 170(16)} \\ &= \frac{\Gamma(n+1+\alpha)}{n! \Gamma(1+\alpha)} \left(\frac{1+x}{2}\right)^n F\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right) \quad \text{EH II 170(16)} \\ &= \frac{\Gamma(n+1+\beta)}{n! \Gamma(1+\beta)} \left(\frac{x-1}{2}\right)^n F\left(-n, -n-\alpha; \beta+1; \frac{x+1}{x-1}\right) \quad \text{EH II 170(16)} \end{aligned}$$

$$2. \quad P_n(x) = P_n^{(0,0)}(x) \quad \text{CO, EH II 179(3)}$$

$$3. \quad T_n(x) = \frac{2^{2n} (n!)^2}{(2n)!} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) \quad \text{CO, EH II 184(5)a}$$

$$4. \quad C_n^\nu(x) = \frac{\Gamma(n+2\nu) \Gamma(\nu + \frac{1}{2})}{\Gamma(2\nu) \Gamma(n+\nu + \frac{1}{2})} P_n^{(\nu-1/2, \nu-1/2)}(x) \quad \text{MO 108a, EH II 174(4)}$$

8.963 The generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta},$$

$$R = \sqrt{1-2xz+z^2} \quad [|z| < 1] \quad \text{EH II 172(29)}$$

8.964 The Jacobi polynomials constitute the *unique* rational solution of the differential (hypergeometric) equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0. \quad \text{EH II 169(14)}$$

8.965 Asymptotic representation

$$P_n^{(\alpha, \beta)}(\cos \theta) = \frac{\cos \left\{ \left[n + \frac{1}{2}(\alpha + \beta + 1) \right] \theta - \left(\frac{1}{2}\alpha + \frac{1}{4} \right) \pi \right\}}{\sqrt{\pi n} \left(\sin \frac{1}{2}\theta \right)^{\alpha+\frac{1}{2}} \left(\cos \frac{1}{2}\theta \right)^{\beta+\frac{1}{2}}} + O\left(n^{-3/2}\right) \quad [\text{Im } \alpha = \text{Im } \beta = 0, \quad 0 < \theta < \pi] \quad \text{EH II 198(10)}$$

8.966 A limit relationship:

$$\lim_{n \rightarrow \infty} \left[n^{-\alpha} P_n^{(\alpha, \beta)} \left(\cos \frac{z}{n} \right) \right] = \left(\frac{z}{2} \right)^{-\alpha} J_{\alpha}(z) \quad \text{EH II 173(41)}$$

8.967 If $\alpha > -1$ and $\beta > -1$, all the zeros of the polynomial $P_n^{(\alpha, \beta)}(x)$ are simple, and they lie in the interval $(-1, 1)$.

8.97 The Laguerre polynomials

8.970 Definition.

$$1. \quad L_n^{\alpha}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) \quad [\text{Rodrigues' formula}] \quad \text{EH II 188(5), MO 108}$$

$$= \sum_{m=0}^n (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!} \quad \text{MO 109, EH II 188(7)}$$

$$2. \quad L_n^0(x) = L_n(x) \quad \text{ET I 369}$$

$$3.^{10} \quad L_0^{\alpha}(x) = 1$$

$$4.^{10} \quad L_1^{\alpha}(x) = -x + \alpha + 1$$

$$5.^{10} \quad L_2^{\alpha}(x) = \frac{1}{2} [x^2 - 2(\alpha + 2)x + (\alpha + 1)(\alpha + 2)]$$

$$6.^{10} \quad L_3^{\alpha}(x) = -\frac{1}{6} [x^3 - 3(\alpha + 3)x^2 + 3(\alpha + 2)(\alpha + 3)x - (\alpha + 1)(\alpha + 2)(\alpha + 3)]$$

$$7.^{10} \quad L_4^{\alpha}(x) = \frac{1}{24} \left[x^4 - 4(\alpha + 4)x^3 + 6(\alpha + 3)(\alpha + 4)x^2 - 4(\alpha + 2)(\alpha + 3)(\alpha + 4)x + (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4) \right]$$

$$8.^{10} \quad L_5^{\alpha}(x) = -\frac{1}{120} \left[x^5 - 5(\alpha + 5)x^4 + 10(\alpha + 4)(\alpha + 5)x^3 - 10(\alpha + 3)(\alpha + 4)(\alpha + 5)x^2 + 5(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5)x - (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5) \right]$$

8.971 Functional relations:

1. $\frac{d}{dx} [L_n^\alpha(x) - L_{n+1}^\alpha(x)] = L_n^\alpha(x)$ EH II 189(16)
- 2.¹¹ $\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = \frac{n L_n^\alpha(x) - (n + \alpha) L_{n-1}^\alpha(x)}{x}$ EH II 189(15), SM 575(42)a
3. $x \frac{d}{dx} L_n^\alpha(x) = n L_n^\alpha(x) - (n + \alpha) L_{n-1}^\alpha(x)$
 $= (n + 1) L_{n+1}^\alpha(x) - (n + \alpha + 1 - x) L_n^\alpha(x)$ EH II 189(12), MO 109
4. $x L_n^{\alpha+1}(x) = (n + \alpha + 1) L_n^\alpha(x) - (n + 1) L_{n+1}^\alpha(x)$
 $= (n + \alpha) L_{n-1}^\alpha(x) - (n - x) L_n^\alpha(x)$ SM 575(43)a, EH II 190(23)
5. $L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x)$ SM 575(44)a, EH II 190(24)
6. $(n + 1) L_{n+1}^\alpha(x) - (2n + \alpha + 1 - x) L_n^\alpha(x) + (n + \alpha) L_{n-1}^\alpha(x) = 0$
 $[n = 1, 2, \dots]$ MO 109, EH II 190(25, 24)
- 7.¹⁰ $(n + \alpha) L_n^{\alpha-1}(x) = (n + 1) L_{n+1}^\alpha(x) - (n + 1 - x) L_n^\alpha(x)$ MS 5.5.2
- 8.¹⁰ $n L_n^\alpha(x) = (2n + \alpha - 1 - x) L_{n-1}^\alpha(x) - (n + \alpha - 1) L_{n-2}^\alpha(x)$
 $[n = 2, 3, \dots]$ MS 5.5.2

8.972 Connections with other functions:

1. $L_n^\alpha(x) = \binom{n + \alpha}{n} \Phi(-n, \alpha + 1; x)$ MO 109, FI II 189(14)
2. $H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2)$ EH II 193(2), SM 576(47)
3. $H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{1/2}(x^2)$ EH II 193(3), SM 577(48)

8.973 Special cases:

1. $L_0^\alpha(x) = 1$ EH II 188(6)
2. $L_1^\alpha(x) = \alpha + 1 - x$ EH II 188(6)
3. $L_n^\alpha(0) = \binom{n + \alpha}{n}$ EH II 189(13)
4. $L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}$ MO 109
5. $L_1(x) = 1 - x$
6. $L_2(x) = 1 - 2x + \frac{x^2}{2}$ MO 109

8.974 Finite sums:

$$1. \quad \sum_{m=0}^n \frac{m!}{\Gamma(m+\alpha+1)} L_m^\alpha(x) L_m^\alpha(y) = \frac{(n+1)!}{\Gamma(n+\alpha+1)(x-y)} [L_n^\alpha(x) L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x) L_n^\alpha(y)]$$

EH II 188(9)

$$2.^{11} \quad \sum_{m=0}^n \frac{\Gamma(\alpha-\beta+m)}{\Gamma(\alpha-\beta)m!} L_{n-m}^\beta(x) = L_n^\beta(x)$$

MO 110, EH II 192(39)

$$3. \quad \sum_{m=0}^n L_m^\alpha(x) = L_n^{\alpha+1}(x)$$

EH II 192(38)

$$4.^{11} \quad \sum_{m=0}^n L_m^\alpha(x) L_{n-m}^\beta(y) = L_n^{\alpha+\beta+1}(x+y)$$

EH II 192(41)

8.975 Arbitrary functions:

$$1. \quad (1-z)^{-\alpha-1} \exp \frac{xz}{z-1} = \sum_{n=0}^{\infty} L_n^\alpha(x) z^n \quad [|z| < 1] \quad \text{EH II 189(17), MO 109}$$

$$2. \quad e^{-xz}(1+z)^\alpha = \sum_{n=0}^{\infty} L_n^{\alpha-n}(x) z^n \quad [|z| < 1] \quad \text{MO 110, EH II 189(19)}$$

$$3. \quad J_\alpha(2\sqrt{xz}) e^z(xz)^{-\frac{1}{2}\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} L_n^\alpha(x) \quad [\alpha > -1] \quad \text{EH II 189(18), MO 109}$$

8.976 Other series of Laguerre polynomials:

$$1. \quad \sum_{n=0}^{\infty} n! \frac{L_n^\alpha(x) L_n^\alpha(y) z^n}{\Gamma(n+\alpha+1)} = \frac{(xyz)^{-\frac{1}{2}\alpha}}{1-z} \exp\left(-z \frac{x+y}{1-z}\right) I_\alpha\left(2\sqrt{xyz}\right)$$

[|z| < 1] EH II 189(20)

$$2. \quad \sum_{n=0}^{\infty} \frac{L_n^\alpha(x)}{n+1} = e^x x^{-\alpha} \Gamma(\alpha, x) \quad [\alpha > -1, \quad x > 0] \quad \text{EH II 215(19)}$$

$$3.^6 \quad L_n^\alpha(x)^2 = \frac{\Gamma(n+\alpha+1)}{2^{2n}n!} \sum_{k=0}^n \binom{2n-2k}{n-k} \frac{(2k)!}{k!} \frac{1}{\Gamma(\alpha+k+1)} L_{2k}^{2\alpha}(2x)$$

MO 110

$$4.^6 \quad L_n^\alpha(x) L_n^\alpha(y) = \frac{\Gamma(1+\alpha+n)}{n!} \sum_{k=0}^n \frac{L_{n-k}^{\alpha+2k}(x+y)}{\Gamma(1+\alpha+k)} \frac{(xy)^k}{k!}$$

MO 110, EH II 192(42)

8.977 Summation theorems:

$$1. \quad L_n^{\alpha_1+\alpha_2+\dots+\alpha_k+k-1}(x_1+x_2+\dots+x_k) = \sum_{i_1+i_2+\dots+i_k=n} L_{i_1}^{\alpha_1}(x_1) L_{i_2}^{\alpha_2}(x_2) \dots L_{i_k}^{\alpha_k}(x_k) \quad \text{MO 110}$$

$$2. \quad L_n^\alpha(x+y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k L_n^{\alpha+k}(x) \quad \text{MO 110}$$

8.978 Limit relations and asymptotic behavior:

$$1. \quad L_n^\alpha(x) = \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)} \left(1 - \frac{2x}{\beta} \right) \quad \text{EH II 191(35)}$$

$$2. \quad \lim_{n \rightarrow \infty} \left[n^{-\alpha} L_n^\alpha \left(\frac{x}{n} \right) \right] = x^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{x}) \quad \text{EH II 191(36)}$$

$$3. \quad L_n^\alpha(x) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}x} x^{-\frac{1}{2}\alpha - \frac{1}{4}} n^{\frac{1}{2}\alpha - \frac{1}{4}} \cos \left[2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right] + O \left(n^{\frac{1}{2}\alpha - \frac{3}{4}} \right) \\ [\text{Im } \alpha = 0, \quad x > 0] \quad \text{EH II 199(1)}$$

8.979 Laguerre polynomials satisfy the following differential equation:

$$x \frac{d^2 u}{dx^2} + (\alpha - x + 1) \frac{du}{dx} + nu = 0 \quad \text{EH II 188(10), SM 574(34)}$$

8.980¹¹ Orthogonality relation

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = \begin{cases} 0, & m \neq n \\ \Gamma(1 + \alpha) \binom{n + \alpha}{n}, & m = n \end{cases} \quad \text{MS 5.5.2}$$

8.981¹⁰ Behavior of relative maxima of $|L_n^\alpha(x)|$

- Let α be arbitrary and real. The sequence formed by the relative maxima of $|L_n^\alpha(x)|$ and by the value of this function at $x = 0$, is decreasing for $x < \alpha + \frac{1}{2}$, and increasing for $x > \alpha + \frac{1}{2}$. The successive relative maxima of $|L_n^\alpha(x)|$ form a decreasing sequence for $x \leq 0$, and an increasing sequence for $x \geq 0$. SZ 174(7.6.1)

- Let α be an arbitrary real number. The successive relative maxima of

$$e^{-x/2} x^{(\alpha+1)/2} |L_n^\alpha(x)| \quad \text{and} \quad e^{-x/2} x^{\alpha/2 + \frac{1}{4}} |L_n^\alpha(x)|$$

form an increasing sequence, provided $x > x_0$. In the first case

$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq 1, \\ \frac{\alpha^2 - 1}{2n + \alpha + 1} & \text{if } \alpha^2 > 1 \end{cases}$$

In the second case,

$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq q\frac{1}{4}, \\ \left(\alpha^2 - \frac{1}{4} \right)^{\frac{1}{2}} & \text{if } \alpha^2 > \frac{1}{4} \end{cases} \quad \text{SZ 174(7.6.2)}$$

In the first case, we take n so large that $2n + \alpha + 1 > 0$.

8.982¹⁰ Asymptotic and limiting behavior of $L_n^\alpha(x)$

- Let α be arbitrary and real, c and w fixed positive constants, and let $n \rightarrow \infty$. Then

$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2 - \frac{1}{4}} O \left(n^{\alpha/2 - \frac{1}{4}} \right) & \text{if } cn^{-1} \leq qx \leq q\omega \\ O(n^\alpha) & \text{if } 0 \leq qx \leq qcn^{-1} \end{cases}$$

These bounds are precise as regards their orders in n . For $\alpha \geq q - \frac{1}{2}$, both bounds hold in both intervals, that is,

$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2-\frac{1}{4}} O\left(n^{\alpha/2-\frac{1}{4}}\right), & 0 < x \leq q\omega, \quad \alpha \geq q - \frac{1}{2} \\ O(n^\alpha), & \end{cases} \quad \text{SZ 175(7.6.4)}$$

2. Let α be arbitrary and real. Then for an arbitrary complex z

$$\lim_{n \rightarrow \infty} n^{-\alpha} L_n^\alpha(x) = z^{-\alpha/2} J_\alpha\left(2z^{1/2}\right), \quad \text{SZ 191(8.1.3)}$$

uniformly if z is bounded.

9.1 Hypergeometric Functions

9.10 Definition

9.100 A *hypergeometric series* is a series of the form

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 1 \cdot 2} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2) \cdot 1 \cdot 2 \cdot 3} z^3 + \dots$$

9.101 A hypergeometric series terminates if α or β is equal to a negative integer or to zero. For $\gamma = -n$ ($n = 0, 1, 2, \dots$), the hypergeometric series is indeterminate if neither α nor β is equal to $-m$ (where $m < n$ and m is a natural number). However,

$$1. \quad \lim_{\gamma \rightarrow -n} \frac{F(\alpha, \beta; \gamma; z)}{\Gamma(\gamma)} = \frac{\alpha(\alpha+1) \dots (\alpha+n)\beta(\beta+1) \dots (\beta+n)}{(n+1)!} \times z^{n+1} F(\alpha+n+1, \beta+n+1; n+2; z)$$

EH I 62(16)

9.102 If we exclude these values of the parameters α, β, γ , a hypergeometric series converges in the unit circle $|z| < 1$. F then has a branch point at $z = 1$. Then we have the following conditions for convergence on the unit circle:

1. $1 > \operatorname{Re}(\alpha + \beta - \gamma) \geq 0$. The series converges throughout the entire unit circle, except at the point $z = 1$.
2. $\operatorname{Re}(\alpha + \beta - \gamma) < 0$. The series converges (absolutely) throughout the entire unit circle.
3. $\operatorname{Re}(\alpha + \beta - \gamma) \geq 1$. The series diverges on the entire unit circle.

FI II 410, WH

9.11 Integral representations

$$\mathbf{9.111} \quad F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt \quad [\operatorname{Re} \gamma > \operatorname{Re} \beta > 0] \quad \text{WH}$$

$$\mathbf{9.112}^8 \quad F(p, n+p; n+1; z^2) = \frac{z^{-n}}{2\pi} \frac{\Gamma(p)n!}{\Gamma(p+n)} \int_0^{2\pi} \frac{\cos nt \, dt}{(1-2z \cos t + z^2)^p} \\ [n = 0, 1, 2, \dots; \quad p \neq 0, -1, -2, \dots; \quad |z| < 1] \quad \text{WH, MO 16}$$

$$\mathbf{9.113} \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)} (-z)^t dt$$

Here, $|\arg(-z)| < \pi$ and the path of integration are chosen in such a way that the poles of the functions $\Gamma(\alpha+t)$ and $\Gamma(\beta+t)$ lie to the left of the path of integration and the poles of the function $\Gamma(-t)$ lie to the right of it.

$$\mathbf{9.114} \quad F\left(-m, -\frac{p+m}{2}; 1 - \frac{p+m}{2}; -1\right) = \frac{(-2)^m (p+m)}{\sin p\pi} \int_0^\pi \cos^m \varphi \cos p\varphi \, d\varphi \\ [m+1 \text{ is a natural number; } \quad p \neq 0, \pm 1, \dots] \quad \text{EH I 80(8), MO 16}$$

See also **3.194** 1, 2, 5, **3.196** 1, **3.197** 6, 9, **3.259** 3, **3.312** 3, **3.518** 4–6, **3.665** 2, **3.671** 1, 2, **3.681** 1, **3.984** 7.

9.12 Representation of elementary functions in terms of a hypergeometric functions

9.121

$$1.^8 \quad F(-n, \beta; \beta; -z) = (1+z)^n \quad \text{EH I 101(4), GA 127 Ia}$$

$$2. \quad F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n + (t-z)^n}{2t^n} \quad \text{GA 127 II}$$

$$3. \quad \lim_{\omega \rightarrow \infty} F\left(-n, \omega; 2\omega; -\frac{z}{t}\right) = \left(1 + \frac{z}{2t}\right)^n \quad \text{GA 127 IIIa}$$

$$4. \quad F\left(-\frac{n-1}{2}, -\frac{n-2}{2}; \frac{3}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}} \quad \text{GA 127 IV}$$

$$5. \quad F\left(1-n, 1; 2; -\frac{z}{t}\right) = \frac{(t+z)^n - t^n}{nzt^{n-1}} \quad \text{GA 127 V}$$

$$6. \quad F(1, 1; 2; -z) = \frac{\ln(1+z)}{z} \quad \text{GA 127 VI}$$

$$7. \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = \frac{\ln \frac{1+z}{1-z}}{2z} \quad \text{GA 127 VII}$$

$$8. \quad \lim_{k \rightarrow \infty} F\left(1, k; 1; \frac{z}{k}\right) = 1 + z \lim_{k \rightarrow \infty} F\left(1, k; 2; \frac{z}{k}\right) \\ = 1 + z + \frac{z^2}{2} \lim_{k \rightarrow \infty} F\left(1, k; 3; \frac{z}{k}\right) = \cdots = e^z \quad \text{GA 127 VIII}$$

$$9. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z + e^{-z}}{2} = \cosh z \quad \text{GA 127 IX}$$

$$10. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z - e^{-z}}{2z} = \frac{\sinh z}{z} \quad \text{GA 127 X}$$

$$11. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; -\frac{z^2}{4kk'}\right) = \frac{\sin z}{z} \quad \text{GA 127 XI}$$

$$12. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; -\frac{z^2}{4kk'}\right) = \cos z \quad \text{GA 127 XII}$$

$$13. \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z} \quad \text{GA 127 XIII}$$

$$14. \quad F\left(1, 1; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z \cos z} \quad \text{GA 127 XIV}$$

$$15. \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; -\tan^2 z\right) = \frac{z}{\tan z} \quad \text{GA 127 XV}$$

$$16. \quad F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z} \quad \text{GA 127 XVI}$$

$$17. \quad F\left(\frac{n+2}{2}, -\frac{n-2}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z \cos z} \quad \text{GA 127 XVII}$$

18. $F\left(-\frac{n-2}{2}, -\frac{n-1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz}{n \sin z \cos^{n-1} z}$ GA 127 XVIII
19. $F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz \cos^{n+1} z}{n \sin z}$ GA 127 XIX
20. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) = \cos nz$ EH I 101(11), GA 127 XX
21. $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{1}{2}; \sin^2 z\right) = \frac{\cos nz}{\cos z}$ EH I 101(11), GA 127 XXI
22. $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; -\tan^2 z\right) = \frac{\cos nz}{\cos^n z}$ EH I 101(11), GA 127 XXII
23. $F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; -\tan^2 z\right) = \cos nz \cos^n z$ GA 127 XXIII
24. $F\left(\frac{1}{2}, 1; 2; 4z(1-z)\right) = \frac{1}{1-z}$ $[|z| \leq \frac{1}{2}; |z(1-z)| \leq \frac{1}{4}]$
25. $F\left(\frac{1}{2}, 1; 1; \sin^2 z\right) = \sec z$
26. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z}$ (cf. **9.121** 13)
27. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\arctan z}{z}$ (cf. **9.121** 15)
28. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{arcsinh} z}{z}$ (cf. **9.121** 26)
29. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz}$ (cf. **9.121** 16)
30. $F\left(1 + \frac{n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}}$ (cf. **9.121** 17)
31. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z)$ (cf. **9.121** 20)
32. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}}$ (cf. **9.121** 21)

The representation of special functions in terms of a hypergeometric function:

- for complete elliptic integrals, see **8.113** 1 and **8.114** 1;
- for integrals of Bessel functions, see **6.574** 1, 3, **6.576** 2–5, **6.621** 1–3;
- for Legendre polynomials, see **8.911** and **8.916**. (All these hypergeometric series terminate; that is, these series are finite sums);
- for Legendre functions, see **8.820** and **8.837**;
- for associated Legendre functions, see **8.702**, **8.703**, **8.751**, **8.77**, **8.852**, and **8.853**;
- for Chebyshev polynomials, see **8.942** 1;
- for Jacobi's polynomials, see **8.962**;

- for Gegenbauer polynomials, see **8.932**;
- for integrals of parabolic cylinder functions, see **7.725** 6.

9.122 Particular values:

1.
$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \quad [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)]$$

GA 147(48), FI II 793
2.
$$F(\alpha, \beta; \gamma; 1) = F(-\alpha, -\beta; \gamma - \alpha - \beta; 1) \quad [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)] \quad \text{GA 148(49)}$$

$$= \frac{1}{F(-\alpha, \beta; \gamma - \alpha; 1)} \quad [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)] \quad \text{GA 148(50)}$$

$$= \frac{1}{F(\alpha, -\beta; \gamma - \beta; 1)} \quad [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)] \quad \text{GA 148(51)}$$
3.
$$F\left(1, 1; \frac{3}{2}; \frac{1}{2}\right) = \frac{\pi}{2} \quad (\text{cf. } \mathbf{9.121} \text{ 14})$$

9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series

9.130 The series $F(\alpha, \beta; \gamma; z)$ defines an analytic function that, speaking generally, has singularities at the points $z = 0, 1$, and ∞ . (In the general case, there are branch points.) We make a cut in the z -plane along the real axis from $z = 1$ to $z = \infty$; that is, we require that $|\arg(-z)| < \pi$ for $|z| \geq 1$. Then, the series $f(\alpha, \beta; \gamma; z)$ will, in the cut plane, yield a single-valued analytic continuation, which we can obtain by means of the formulas below (provided $\gamma + 1$ is not a natural number and $\alpha - \beta$ and $\gamma - \alpha - \beta$ are not integers). These formulas make it possible to calculate the values of F in the given region, even in the case in which $|z| > 1$. There are other closely related transformation formulas that can also be used to get the analytic continuation when the corresponding relationships hold between α, β, γ .

Transformation formulas

9.131

$$1.^{11} \quad F(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z - 1}\right) \quad \text{GA 218(91)}$$

$$= (1 - z)^{-\beta} F\left(\beta, \gamma - \alpha; \gamma; \frac{z}{z - 1}\right) \quad \text{GA 218(92)}$$

$$= (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta; \gamma; z)$$

$$2. \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z)$$

$$+ (1 - z)^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - z)$$

EH I 94, MO 13

9.132

$$1. \quad F(\alpha, \beta; \gamma; z) = \frac{(1-z)^{-\alpha} \Gamma(\gamma) \Gamma(\beta-\alpha)}{\Gamma(\beta) \Gamma(\gamma-\alpha)} F\left(\alpha, \gamma-\beta; \alpha-\beta+1; \frac{1}{1-z}\right) \\ + (1-z)^{-\beta} \frac{\Gamma(\gamma) \Gamma(\alpha-\beta)}{\Gamma(\alpha) \Gamma(\gamma-\beta)} F\left(\beta, \gamma-\alpha; \beta-\alpha+1; \frac{1}{1-z}\right)$$

MO 13

$$2.^{11} \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma) \Gamma(\beta-\alpha)}{\Gamma(\beta) \Gamma(\gamma-\alpha)} (-z)^{-\alpha} F\left(\alpha, \alpha+1-\gamma; \alpha+1-\beta; \frac{1}{z}\right) \\ + \frac{\Gamma(\gamma) \Gamma(\alpha-\beta)}{\Gamma(\alpha) \Gamma(\gamma-\beta)} (-z)^{-\beta} F\left(\beta, \beta+1-\gamma; \beta+1-\alpha; \frac{1}{z}\right) \\ [|\arg z| < \pi, \quad \alpha-\beta \neq \pm m, \quad m=0, 1, 2, \dots] \quad \text{GA 220(93)}$$

$$\mathbf{9.133} \quad F\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; z\right) = F\left(\alpha, \beta; \alpha+\beta+\frac{1}{2}; 4z(1-z)\right) \\ [|z| \leq \frac{1}{2}, \quad |z(1-z)| \leq \frac{1}{4}]$$

WH

9.134

$$1. \quad F(\alpha, \beta; 2\beta; z) = \left(1 - \frac{z}{2}\right)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \beta + \frac{1}{2}; \left(\frac{z}{2-z}\right)^2\right) \quad \text{MO 13, EH I 111(4)}$$

$$2. \quad F(2\alpha, 2\alpha+1-\gamma; \gamma; z) = (1+z)^{-2\alpha} F\left(\alpha, \alpha+\frac{1}{2}; \gamma; \frac{4z}{(1+z)^2}\right) \quad \text{GA 225(100)}$$

$$3. \quad F\left(\alpha, \alpha+\frac{1}{2}-\beta; \beta+\frac{1}{2}; z^2\right) = (1+z)^{-2\alpha} F\left(\alpha, \beta; 2\beta; \frac{4z}{(1+z)^2}\right) \quad \text{GA 225(101)}$$

$$\mathbf{9.135} \quad F\left(\alpha, \beta; \alpha+\beta+\frac{1}{2}; \sin^2 \varphi\right) = F\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; \sin^2 \frac{\varphi}{2}\right) \\ \left[x = \sin^2 \frac{\varphi}{2} \text{ real; } \frac{1-\sqrt{2}}{2} < x < \frac{1}{2} \right]$$

MO 13

9.136⁸ We set

$$A = \frac{\Gamma(\alpha+\beta+\frac{1}{2}) \sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2}) \Gamma(\beta+\frac{1}{2})}, \quad B = \frac{-\Gamma(\alpha+\beta+\frac{1}{2}) 2\sqrt{\pi}}{\Gamma(\alpha) \Gamma(\beta)};$$

then

$$1. \quad F\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) = AF\left(\alpha, \beta; \frac{1}{2}; z\right) + B\sqrt{z} F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}; \frac{3}{2}; z\right) \quad \text{GA 227(106)}$$

$$2. \quad F\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; \frac{1+\sqrt{z}}{2}\right) = AF\left(\alpha, \beta; \frac{1}{2}; z\right) - B\sqrt{z} F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}; \frac{3}{2}; z\right) \quad \text{GA 227(107)}$$

$$3. \quad \frac{(\alpha-\frac{1}{2})(\beta-\frac{1}{2})}{\alpha+\beta-\frac{1}{2}} A\sqrt{z} F\left(\alpha, \beta; \frac{3}{2}; z\right) = F\left(2\alpha-1, 2\beta-1; \alpha+\beta-\frac{1}{2}; \frac{1+\sqrt{z}}{2}\right) \\ - F\left(2\alpha-1, 2\beta-1; \alpha+\beta-\frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) \quad \text{GA 229(110)}$$

9.137⁷ Gauss' recursion functions:

1. $\gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)z] F(\alpha, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) + \gamma(\gamma - 1)(z - 1) F(\alpha, \beta; \gamma - 1; z) = 0$
2. $(2\alpha - \gamma - \alpha z + \beta z) F(\alpha, \beta; \gamma; z) + (\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) + \alpha(z - 1) F(\alpha + 1, \beta; \gamma; z) = 0$
3. $(2\beta - \gamma - \beta z + \alpha z) F(\alpha, \beta; \gamma; z) + (\gamma - \beta) F(\alpha, \beta - 1; \gamma; z) + \beta(z - 1) F(\alpha, \beta + 1; \gamma; z) = 0$
4. $\gamma F(\alpha, \beta - 1; \gamma; z) - \gamma F(\alpha - 1, \beta; \gamma; z) + (\alpha - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0$
- 5.⁸ $\gamma(\alpha - \beta) F(\alpha, \beta; \gamma; z) - \alpha(\gamma - \beta) F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha) F(\alpha, \beta + 1; \gamma + 1; z) = 0$
6. $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha, \beta; \gamma + 1; z) - \alpha\beta z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
7. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha) F(\alpha, \beta + 1; \gamma + 1; z) - \alpha(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
8. $\gamma F(\alpha, \beta; \gamma; z) + (\beta - \gamma) F(\alpha + 1, \beta; \gamma + 1; z) - \beta(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
9. $\gamma(\gamma - \beta z - \alpha) F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) + \alpha\beta z(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
10. $\gamma(\gamma - \alpha z - \beta) F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \beta) F(\alpha, \beta - 1; \gamma; z) + \alpha\beta z(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
11. $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha, \beta + 1; \gamma; z) + \alpha z F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
- 12.⁸ $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha + 1, \beta; \gamma; z) + \beta z F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
13. $\gamma[\alpha - (\gamma - \beta)z] F(\alpha, \beta; \gamma; z) - \alpha\gamma(1 - z) F(\alpha + 1, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0$
14. $\gamma[\beta - (\gamma - \alpha)z] F(\alpha, \beta; \gamma; z) - \beta\gamma(1 - z) F(\alpha, \beta + 1; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0$
- 15.⁸ $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha, \beta + 1; \gamma + 1; z) + \alpha(\gamma - \beta)z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
16. $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha)z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
17. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \beta) F(\alpha, \beta; \gamma + 1; z) - \beta F(\alpha, \beta + 1; \gamma + 1; z) = 0$
- 18.⁸ $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha) F(\alpha, \beta; \gamma + 1; z) - \alpha F(\alpha + 1, \beta; \gamma + 1; z) = 0$ MO 13-14

9.14 A generalized hypergeometric series

The series

$$1. \quad {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \dots (\alpha_p)_k}{(\beta_1)_k (\beta_2)_k \dots (\beta_q)_k} \frac{z^k}{k!} \quad \text{MO 14}$$

is called a *generalized hypergeometric series* (see also 9.210).

$$2. \quad {}_2F_1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z) \quad \text{MO 15}$$

For integral representations, see **3.254** 2, **3.259** 2, and **3.478** 3.

9.15 The hypergeometric differential equation

9.151 A hypergeometric series is one of the solutions of the differential equation

$$z(1 - z) \frac{d^2 u}{dz^2} + [\gamma - (\alpha + \beta + 1)z] \frac{du}{dz} - \alpha\beta u = 0, \quad \text{WH}$$

which is called the *hypergeometric equation*.

The solution of the hypergeometric differential equation

9.152 The hypergeometric differential equation **9.151** possesses *two linearly independent solutions*. These solutions have analytic continuations to the entire z -plane, except possibly for the three points $0, 1$, and ∞ . Generally speaking, the points $z = 0, 1, \infty$ are branch points of at least one of the branches of each solution of the hypergeometric differential equation. The ratio $w(z)$ of two linearly independent solutions satisfies the differential equation

$$2\frac{w'''}{w'} - 3\left(\frac{w''}{w'}\right)^2 = \frac{1-a_1^2}{z^2} + \frac{1-a_2^2}{(z-1)^2} + \frac{a_1^2+a_2^2-a_3^2-1}{z(z-1)},$$

where

$$a_1^2 = (1-\gamma)^2, \quad a_2^2 = (\gamma-\alpha-\beta)^2, \quad a_3^2 = (\alpha-\beta)^2.$$

If α, β, γ are real, the function $w(z)$ maps the upper ($\text{Im } z > 0$) or the lower ($\text{Im } z < 0$) half-plane onto a curvilinear triangle whose angles are $\pi a_1, \pi a_2, \pi a_3$. The vertices of this triangle are the images of the points $z = 0, z = 1$, and $z = \infty$.

9.153 Within the unit circle $|z| < 1$, the linearly independent solutions $u_1(z)$ and $u_2(z)$ of the hypergeometric differential equation are given by the following formulas:

1. If γ is not an integer,

$$\begin{aligned} u_1 &= F(\alpha, \beta; \gamma; z), \\ u_2 &= z^{1-\gamma} e F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z) \end{aligned}$$

2. If $\gamma = 1$, then

$$\begin{aligned} u_1 &= F(\alpha, \beta; 1; z), \\ u_2 &= F(\alpha, \beta; 1; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(k!)^2} \\ &\quad \times \{\psi(\alpha + k) - \psi(\alpha) + \psi(\beta + k) - \psi(\beta) - 2\psi(k+1) + 2\psi(1)\} \end{aligned}$$

(see **9.14 2**)

3. If $\gamma = m + 1$ (where m is a natural number), and if neither α nor β is a positive number not exceeding m , then

$$\begin{aligned} u_1 &= F(\alpha, \beta; m+1; z), \\ u_2 &= F(\alpha, \beta; m+1; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(1+m)_k} \{h(k) - h(0)\} - \sum_{k=1}^m \frac{(k-1)!(-m)_k}{(1-\alpha)_k(1-\beta)_k} z^{-k} \end{aligned}$$

(see **9.14 2**)

where

$$h(n) = \psi(\alpha + n) + \psi(\beta + n) - \psi(m+1+n) - \psi(n+1) \quad [n+1 \text{ is a natural number}]$$

- 4.¹¹ Suppose that $\gamma = m + 1$ (where m is a natural number) and that α or β is equal to $m' + 1$, where $0 \leq m' < m$. Then, for example, for $\alpha = m' + 1$, we obtain

$$\begin{aligned} u_1 &= F(1+m', \beta; 1+m; z), \\ u_2 &= z^{-m} F(1+m' - m, \beta - m; 1-m; z) \end{aligned}$$

In this case, u_2 is a polynomial in z^{-1} .

5. If $\gamma = 1 - m$ (where m is a natural number) and if α and β are both different from the numbers $0, -1, -2, \dots, 1 - m$, then

$$\begin{aligned} u_1 &= z^m F(\alpha + m, \beta + m; 1 + m; z), \\ u_2 &= z^m F(\alpha + m, \beta + m; 1 + m; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha + m)_k (\beta + m)_k}{(1 + m)_k k!} \{h^*(k) - h^*(0)\} \\ &\quad - \sum_{k=1}^{\infty} \frac{(k-1)!(-m)_k}{(1 - \alpha - m)_k (1 - \beta - m)_k} z^{m-k} \end{aligned}$$

(see 9.14 2)

where

$$h^*(n) = \psi(\alpha + m + n) + \psi(\beta + m + n) - \psi(1 + m + n) - \psi(1 + n)$$

We note that

$$\psi(\alpha + n) - \psi(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha + 1} + \dots + \frac{1}{\alpha + n - 1} \quad (\text{cf. 8.365 3})$$

and that, for $\alpha = -\lambda$, where λ is a natural number or zero and $n = \lambda + 1, \lambda + 2, \dots$ the expression

$$(\alpha)_k [\psi(\alpha + n) - \psi(\alpha)]$$

in formulas 9.153 2–5 should be replaced with the expression

$$(-1)^\lambda \lambda! (n - \lambda - 1)!$$

6. Suppose that $\gamma = 1 - m$ (where m is a natural number) and that α or β is an integer ($-m'$), where m' is one of the following numbers: $0, 1, \dots, m - 1$. Suppose, for example, that $\alpha = -m'$. Then,

$$\begin{aligned} u_1 &= F(-m', \beta; 1 - m; z), \\ u_2 &= F(-m' + m, \beta + m; 1 + m; z) \end{aligned}$$

MO 18

7. For $\gamma = \frac{1}{2}(\alpha + \beta + 1)$

$$\begin{aligned} u_1 &= F\left(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); z\right), \\ u_2 &= F\left(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); 1 - z\right) \end{aligned}$$

are two linearly independent solutions of the hypergeometric differential equation, provided α, β , and γ are not zero or negative integers.

MO 17–19

The analytic continuation of a solution that is regular at the point $z = 0$

9.154 Formulas 9.153 make possible the analytic continuation, by means of the hypergeometric series, of the function $F(\alpha, \beta; \gamma; z)$ defined inside the circle $|z| < 1$ to the region $|z| > 1$, and $|\arg(-z)| < \pi$. Here, it is assumed that $\alpha - \beta$ is not an integer. In the event that $\alpha - \beta$ is an integer (for example, if $\beta = \alpha + m$, where m is a natural number), then, for $|z| > 1$, and $|\arg(-z)| < \pi$ we have:

$$\begin{aligned}
1. \quad & \frac{\Gamma(\alpha)\Gamma(\alpha+m)}{\Gamma(\gamma)} F(\alpha, \alpha+m; \gamma; z) \\
&= \frac{\sin \pi(\gamma-\alpha)}{\pi} \left\{ \sum_{k=0}^{m-1} \frac{\Gamma(\alpha+k)\Gamma(1-\gamma+\alpha+k)\Gamma(m-k)}{k!} (-z)^{-\alpha-k} \right. \\
&\quad \left. + (-z)^{-\alpha-m} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+m+k)\Gamma(1-\gamma+\alpha+m+k)}{k!(k+m)!} g(k) z^{-k} \right\}
\end{aligned}$$

where

$$\begin{aligned}
2. \quad & g(n) = \ln(-z) + \pi \cot \pi(\gamma-\alpha) + \psi(n+1) + \psi(n+m+1) \\
& \quad - \psi(\alpha+m+n) - \psi(1-\gamma+\alpha+m+n)
\end{aligned}$$

For $m=0$, we should set $\sum_{k=0}^{m-1} = 0$.

9.155 This formula loses its meaning when α, γ , or $\alpha-\gamma+1$ is equal to one of the numbers $0, -1, -2, \dots$. In this last case, we have

1. If α is a non-positive integer and γ is not an integer, $F(\alpha, \alpha+m; \gamma; z)$ is a polynomial in z .
2. Suppose that γ is a non-positive integer and that α is not an integer. We then set $\gamma = -\lambda$, where $\lambda = 0, 1, 2, \dots$. Then,

$$\frac{\Gamma(\alpha+\lambda+1)\Gamma(\alpha+\lambda+m+1)}{\Gamma(\lambda+2)} z^{\lambda+1} F(\alpha+\lambda+1, \alpha+\lambda+m+1; \lambda+2; z)$$

is a solution of the hypergeometric equation that is regular at the point $z=0$. This solution is equal to the right-hand member of formula **9.154 1** if we replace γ with λ in this equation and in formula **9.154 2**.

3. If $\alpha-\gamma+1$ is a non-positive integer and if α and γ are not themselves integers, we may use the formula

$$F(\alpha, \alpha+m; \gamma; z) = (1-z)^{\gamma-2\alpha-m} F(\gamma-\alpha-m, \gamma-\alpha; \gamma; z)$$

and apply formula **9.154 1** to its right-hand member, provided $\gamma-\alpha-m > 0$. However, if $\alpha-\gamma-m \leq 0$, the right member of this expression is a polynomial taken to the $(1-z)^{\text{th}}$ power.

4. If α, β , and γ are integers, the hypergeometric differential equation always has a solution that is regular for $z=0$ and that is of the form

$$R_1(z) + \ln(1-z)R_2(z),$$

where $R_1(z)$ and $R_2(z)$ are rational functions of z . To get a solution of this form, we need to apply formulas **9.137 1**–**9.137 3** to the function $F(\alpha, \beta; \gamma; z)$. However, if $\gamma = -\lambda$, where $\lambda+1$ is a natural number, formulas **9.137 1** and **9.137 2** should be applied not to $F(\alpha, \beta; \gamma; z)$ but to the function $z^{\lambda+1} F(\alpha+\lambda+1, \beta+\lambda+1; \lambda+2; z)$.

By successive applications of these formulas, we can reduce the positive values of the parameters to the pair, unity and zero. Furthermore, we can obtain the desired form of the solution from the formulas

$$\begin{aligned}
F(1, 1; 2; z) &= -z^{-1} \ln(1-z), \\
F(0, \beta; \gamma; z) &= F(\alpha, 0; \gamma; z) = 1
\end{aligned}$$

9.16 Riemann's differential equation

9.160 The hypergeometric differential equation is a particular case of Riemann's differential equation

$$1.11 \quad \frac{d^2 u}{dz^2} + \left[\frac{1 - \alpha - \alpha'}{z - a} + \frac{1 - \beta - \beta'}{z - b} + \frac{1 - \gamma - \gamma'}{z - c} \right] \frac{du}{dz} + \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c} \right] \frac{u}{(z-a)(z-b)(z-c)} = 0$$

WH

The coefficients of this equation have poles at the points a , b , and c , and the numbers $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are called the indices corresponding to these poles. The indices $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are related by the following equation:

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' - 1 = 0$$

WH

2. The differential equations **9.160 1** are written diagrammatically as follows:

$$3. \quad u = P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \quad z \right\}$$

The singular points of the equation appear in the first row in this scheme, the indices corresponding to them appear beneath them, and the independent variable appears in the fourth column. WH

9.161 The two following transformation formulas are valid for Riemann's P -equation:

$$1. \quad \left(\frac{z-a}{z-b} \right)^k \left(\frac{z-c}{z-b} \right)^l P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \quad z \right\} = P \left\{ \begin{matrix} a & b & c \\ \alpha+k & \beta-k-1 & \gamma+l \\ \alpha'+k & \beta'-k-l & \gamma'+l \end{matrix} \quad z \right\}$$

WH

$$2. \quad P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \quad z \right\} = P \left\{ \begin{matrix} a_1 & b_1 & c_1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \quad z_1 \right\}$$

WH

The first of these formulas means that if

$$u = P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \quad z \right\},$$

then the function

$$u_1 = \left(\frac{z-a}{z-b} \right)^k \left(\frac{z-c}{z-b} \right)^l u$$

satisfies a second-order differential equation having the same singular points as equation **9.161 2** and indices equal to $\alpha + k, \alpha' + k; \beta - k - l, \beta' - k - l; \gamma + l, \gamma' + l$. The second transformation formula converts a differential equation with singularities at the points a, b , and c , indices $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$, and an independent variable z into a differential equation with the same indices, singular points a_1, b_1 , and c_1 , and independent variable z_1 . The variable z_1 is connected with the variable z by the fractional transformation

$$z = \frac{Az_1 + B}{Cz_1 + D} \quad [AD - BC \neq 0]$$

The same transformation connects the points a_1 , b_1 , and c_1 with the points a , b , and c .

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9.162 By the successive application of the two transformation formulas **9.161** 1 and **9.161** 2, we can convert Riemann's differential equation into the hypergeometric differential equation. Thus, the solution of Riemann's differential equation can be expressed in terms of a hypergeometric function.

For $k = -\alpha$, $l = -\gamma$, and $z_1 = \frac{(z-a)(c-b)}{(z-b)(c-a)}$, we have

$$\begin{aligned} 1. \quad u &= P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \begin{matrix} z \\ z \\ z \end{matrix} \right\} = \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma P \left\{ \begin{matrix} a & b & c \\ 0 & \beta + \alpha + \gamma & 0 \\ \alpha' - \alpha & \beta' + \alpha + \gamma & \gamma' - \gamma \end{matrix} \begin{matrix} z \\ z \\ z \end{matrix} \right\} \\ &= \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & \beta + \alpha + \gamma & 0 \\ \alpha' - \alpha & \beta' + \alpha + \gamma & \gamma' - \gamma \end{matrix} \begin{matrix} z \\ z \\ z \end{matrix} \right\} \end{aligned}$$

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Thus, this solution can be expressed as a hypergeometric series as follows:

$$2. \quad u = \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma F \left(\alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)} \right)$$

If the constants $a, b, c; \alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are permuted in a suitable manner, Riemann's equation remains unchanged. Thus, we obtain a set of 24 solutions of differential equations having the following form (provided none of the differences $\alpha - \alpha', \beta - \beta', \gamma - \gamma'$ is an integer):

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9.163

$$\begin{aligned} 1. \quad u_1 &= \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma F \left\{ \alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \\ 2. \quad u_2 &= \left(\frac{z-a}{z-b} \right)^{\alpha'} \left(\frac{z-c}{z-b} \right)^\gamma F \left\{ \alpha' + \beta + \gamma, \alpha' + \beta' + \gamma; 1 + \alpha' - \alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \\ 3. \quad u_3 &= \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^{\gamma'} F \left\{ \alpha + \beta + \gamma', \alpha + \beta' + \gamma'; 1 + \alpha - \alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \\ 4. \quad u_4 &= \left(\frac{z-a}{z-b} \right)^{\alpha'} \left(\frac{z-c}{z-b} \right)^{\gamma'} F \left\{ \alpha' + \beta + \gamma', \alpha' + \beta' + \gamma; 1 + \alpha' - \alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \end{aligned}$$

9.164

$$\begin{aligned} 1.^{10} \quad u_5 &= \left(\frac{z-b}{z-c} \right)^\beta \left(\frac{z-a}{z-c} \right)^\alpha F \left\{ \beta + \gamma + \alpha, \beta + \gamma' + \alpha; 1 + \beta - \beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\} \\ 2. \quad u_6 &= \left(\frac{z-b}{z-c} \right)^{\beta'} \left(\frac{z-a}{z-c} \right)^\alpha F \left\{ \beta' + \gamma + \alpha, \beta' + \gamma' + \alpha; 1 + \beta' - \beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\} \\ 3. \quad u_7 &= \left(\frac{z-b}{z-c} \right)^\beta \left(\frac{z-a}{z-c} \right)^{\alpha'} F \left\{ \beta + \gamma + \alpha', \beta + \gamma' + \alpha'; 1 + \beta - \beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\} \end{aligned}$$

$$4. \quad u_8 = \left(\frac{z-b}{z-c}\right)^{\beta'} \left(\frac{z-a}{z-c}\right)^{\alpha'} F \left\{ \beta' + \gamma + \alpha', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$$

9.165

$$\begin{aligned} 1. \quad u_9 &= \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta} F \left\{ \gamma + \alpha + \beta, \gamma + \alpha' + \beta; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\} \\ 2. \quad u_{10} &= \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta} F \left\{ \gamma' + \alpha + \beta, \gamma' + \alpha' + \beta; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\} \\ 3. \quad u_{11} &= \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta'} F \left\{ \gamma + \alpha + \beta', \gamma + \alpha' + \beta'; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\} \\ 4. \quad u_{12} &= \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta'} F \left\{ \gamma' + \alpha + \beta', \gamma' + \alpha' + \beta'; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\} \end{aligned}$$

9.166

$$\begin{aligned} 1. \quad u_{13} &= \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta} F \left\{ \alpha + \gamma + \beta, \alpha + \gamma' + \beta; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \\ 2. \quad u_{14} &= \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta} F \left\{ \alpha' + \gamma + \beta, \alpha' + \gamma' + \beta; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \\ 3. \quad u_{15} &= \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta'} F \left\{ \alpha + \gamma + \beta', \alpha + \gamma' + \beta'; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \\ 4. \quad u_{16} &= \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta'} F \left\{ \alpha' + \gamma + \beta', \alpha' + \gamma' + \beta'; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \end{aligned}$$

9.167

$$\begin{aligned} 1. \quad u_{17} &= \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha} F \left\{ \gamma + \beta + \alpha, \gamma + \beta' + \alpha; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \\ 2. \quad u_{18} &= \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha} F \left\{ \gamma' + \beta + \alpha, \gamma' + \beta' + \alpha; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \\ 3. \quad u_{19} &= \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha'} F \left\{ \gamma + \beta + \alpha', \gamma + \beta' + \alpha'; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \\ 4. \quad u_{20} &= \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha'} F \left\{ \gamma' + \beta + \alpha', \gamma' + \beta' + \alpha'; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \end{aligned}$$

9.168

$$\begin{aligned} 1. \quad u_{21} &= \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma} F \left\{ \beta + \alpha + \gamma, \beta + \alpha' + \gamma; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\} \\ 2. \quad u_{22} &= \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma} F \left\{ \beta' + \alpha + \gamma, \beta' + \alpha' + \gamma; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\} \end{aligned}$$

$$3. \quad u_{23} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta + \alpha + \gamma', \beta + \alpha' + \gamma'; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$

$$4. \quad u_{24} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta' + \alpha + \gamma', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\} \quad \text{WH}$$

9.17 Representing the solutions to certain second-order differential equations using a Riemann scheme

9.171 The hypergeometric equation (see **9.151**):

$$u = P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ 0 & \alpha & 0 \\ 1 - \gamma & \beta & \gamma - \alpha - \beta \end{array} \quad z \right\} \quad \text{WH}$$

9.172 The associated Legendre's equation defining the functions $P_n^m(z)$ for n and m integers (see **8.700** 1):

$$1. \quad u = P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ \frac{1}{2}m & n+1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{array} \quad \frac{1-z}{2} \right\} \quad \text{WH}$$

$$2. \quad u = P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ -\frac{1}{2}n & \frac{1}{2}m & 0 \\ \frac{n+1}{2} & -\frac{1}{2}m & \frac{1}{2} \end{array} \quad \frac{1}{1-z^2} \right\} \quad \text{WH}$$

9.173 The function $P_n^m \left(1 - \frac{z^2}{2n^2}\right)$ satisfies the equation

$$u = P \left\{ \begin{array}{ccc} 4n^2 & \infty & 0 \\ \frac{1}{2}m & n+1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{array} \quad z^2 \right\} \quad \text{WH}$$

The function $J_m(z)$ satisfies the limiting form of this equation obtained as $n \rightarrow \infty$.

9.174 The equation defining the Gegenbauer polynomials $C_n^\lambda(z)$ (see **8.938**):

$$u = P \left\{ \begin{array}{ccc} -1 & \infty & 1 \\ \frac{1}{2} - \lambda & n+2\lambda & \frac{1}{2} - \lambda \\ 0 & -n & 0 \end{array} \quad z \right\} \quad \text{WH}$$

9.175 Bessel's equation (see **8.401**) is the limiting form of the equations:

$$1. \quad u = P \left\{ \begin{array}{ccc} 0 & \infty & c \\ n & ic & \frac{1}{2} + ic \\ -n & -ic & \frac{1}{2} - ic \end{array} \quad z \right\} \quad \text{WH}$$

$$2. \quad u = e^{iz} P \left\{ \begin{array}{ccc} 0 & \infty & c \\ n & \frac{1}{2} & 0 \\ -n & \frac{3}{2} - 2ic & 2ic - 1 \end{array} \quad z \right\} \quad \text{WH}$$

$$3. \quad u = P \left\{ \begin{array}{ccc} 0 & \infty & c^2 \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 \end{array} \quad z^2 \right\} \quad \text{WH}$$

as $c \rightarrow \infty$.

9.18 Hypergeometric functions of two variables

9.180

$$1. \quad F_1(\alpha, \beta, \beta', \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n \quad [|x| < 1, \quad |y| < 1] \quad \text{EH I 224(6), AK 14(11)}$$

$$2. \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n m! n!} x^m y^n \quad [|x| + |y| < 1] \quad \text{EH I 224(7), AK 14(12)}$$

$$3. \quad F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n \quad [|x| < 1, \quad |y| < 1] \quad \text{EH I 224(8), AK 14(13)}$$

$$4. \quad F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n m! n!} x^m y^n \quad [|\sqrt{x}| + |\sqrt{y}| < 1] \quad \text{EH I 224(9), AK 14(14)}$$

9.181 The functions F_1 , F_2 , F_3 , and F_4 satisfy the following systems of partial differential equations for z :

$$1. \quad \text{System of equations for } z = F_1: \\ x(1-x) \frac{\partial^2 z}{\partial x^2} + y(1-x) \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha \beta z = 0, \quad \text{EH I 233(9)} \\ y(1-y) \frac{\partial^2 z}{\partial y^2} + x(1-y) \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta' + 1)y] \frac{\partial z}{\partial x} - \beta' x \frac{\partial z}{\partial y} - \alpha \beta' z = 0$$

$$2. \quad \text{System of equations for } z = F_2: \\ x(1-x) \frac{\partial^2 z}{\partial x^2} - xy \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha \beta z = 0, \quad \text{EH I 234(10)} \\ y(1-y) \frac{\partial^2 z}{\partial y^2} - xy \frac{\partial^2 z}{\partial x \partial y} + [\gamma' - (\alpha + \beta' + 1)y] \frac{\partial z}{\partial y} - \beta' x \frac{\partial z}{\partial x} - \alpha \beta' z = 0$$

$$3. \quad \text{System of equations for } z = F_3: \\ x(1-x) \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \alpha \beta z = 0, \\ y(1-y) \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha' + \beta' + 1)y] \frac{\partial z}{\partial y} - \alpha' \beta' z = 0 \quad \text{EH I 234(11)}$$

4. System of equations for $z = F_4$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x]\frac{\partial z}{\partial x} - (\alpha + \beta + 1)y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 234(12)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} - x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + [\gamma' - (\alpha + \beta + 1)y]\frac{\partial z}{\partial y} - (\alpha + \beta + 1)x\frac{\partial z}{\partial x} - \alpha\beta z = 0$$

AK 44

9.182 For certain relationships between the parameters and the argument, hypergeometric functions of two variables can be expressed in terms of hypergeometric functions of a single variable or in terms of elementary functions:

$$1. \quad F_1(\alpha, \beta, \beta', \beta + \beta'; x, y) = (1-y)^{-\alpha} F\left(\alpha, \beta; \beta + \beta'; \frac{x-y}{1-y}\right) \quad \text{EH I 238(1), AK 24(28)}$$

$$2. \quad F_2(\alpha, \beta, \beta', \beta, \gamma'; x, y) = (1-x)^{-\alpha} F\left(\alpha, \beta'; \gamma'; \frac{y}{1-x}\right) \quad \text{EH I 238(2), AK 23}$$

$$3. \quad F_2(\alpha, \beta, \beta', \alpha, \alpha; x, y) = (1-x)^{-\beta}(1-y)^{-\beta'} F\left(\beta, \beta'; \alpha; \frac{xy}{(1-x)(1-y)}\right) \quad \text{EH I 238(3)}$$

$$4. \quad F_3(\alpha, \gamma - \alpha, \beta, \gamma - \beta, \gamma; x, y) = (1-y)^{\alpha+\beta-\gamma} F(\alpha, \beta; \gamma; x+y-xy) \quad \text{EH I 238(4), AK 25(35)}$$

$$5. \quad F_4(\alpha, \gamma + \gamma' - \alpha - 1, \gamma, \gamma'; x(1-y), y(1-x)) \\ = F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma; x) F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma'; y) \quad \text{EH I 238(5)}$$

$$6. \quad F_4\left(\alpha, \beta, \alpha, \beta; -\frac{x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right) = \frac{(1-x)^\beta(1-y)^\alpha}{(1-xy)} \quad \text{EH I 238(6)}$$

$$7. \quad F_4\left(\alpha, \beta, \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) = (1-x)^\alpha(1-y)^\alpha F(\alpha, 1+\alpha-\beta; \beta; xy) \quad \text{EH I 238(7)}$$

$$8. \quad F_4\left(\alpha, \beta, 1+\alpha-\beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) \\ = (1-y)^\alpha F\left[\alpha, \beta; 1+\alpha-\beta; -\frac{x(1-y)}{1-x}\right] \quad \text{EH I 238(8)}$$

$$9. \quad F_4\left(\alpha, \alpha + \frac{1}{2}, \gamma, \frac{1}{2}; x, y\right) = \frac{1}{2}(1+\sqrt{y})^{-2\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1+\sqrt{y})^2}\right) \\ + \frac{1}{2}(1-\sqrt{y})^{-2\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1-\sqrt{y})^2}\right)$$

AK 23

$$10. \quad F_1(\alpha, \beta, \beta', \gamma; x, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')} F(\alpha, \beta; \gamma-\beta'; x) \quad \text{EH I 239(10), AK 22(23)}$$

$$11. \quad F_1(\alpha, \beta, \beta', \gamma; x, x) = F(\alpha, \beta + \beta'; \gamma; x) \quad \text{EH I 239(11), AK 23(25)}$$

9.183 Functional relations between hypergeometric functions of two variables:

$$1. \quad F_1(\alpha, \beta, \beta', \gamma; x, y) = (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma - \alpha, \beta, \beta', \gamma; \frac{x}{x-1}, \frac{y}{y-1}\right) \quad \text{EH I 239(1)}$$

$$= (1-x)^{-\alpha} F_1\left(\alpha, \gamma - \beta - \beta', \beta', \gamma; \frac{x}{x-1}, \frac{y-x}{1-x}\right) \quad \text{EH I 239(2)}$$

$$= (1-y)^{-\alpha} F_1\left(\alpha, \beta, \gamma - \beta - \beta', \gamma; \frac{y-x}{y-1}, \frac{y}{y-1}\right) \quad \text{EH I 239(3)}$$

$$= (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma - \alpha, \gamma - \beta - \beta', \beta', \gamma; x, \frac{x-y}{1-y}\right) \quad \text{EH I 240(4)}$$

$$= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1\left(\gamma - \alpha, \beta, \gamma - \beta - \beta', \gamma; \frac{x-y}{x-1}, y\right) \quad \text{EH I 240(5), AK 30(5)}$$

$$2.^8 \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = (1-x)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \beta', \gamma, \gamma'; \frac{x}{x-1}, \frac{y}{1-x}\right) \quad \text{EH I 240(6)}$$

$$= (1-y)^{-\alpha} F_2\left(\alpha, \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{1-y}, \frac{y}{y-1}\right) \quad \text{EH I 240(7)}$$

$$= (1-x-y)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{x+y-1}, \frac{y}{x+y-1}\right) \quad \text{EH I 240(8), AK 32(6)}$$

$$3.^7 \quad F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \frac{\Gamma(\gamma')\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\Gamma(\beta)}(-y)^{-\alpha} F_4\left(\alpha, \alpha+1-\gamma', \gamma, \alpha+1-\beta; \frac{x}{y}, \frac{1}{y}\right) \\ + \frac{\Gamma(\gamma')\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\Gamma(\alpha)}(-y)^{\beta} F_4\left(\beta+1-\gamma', \beta, \gamma, \beta+1-\alpha; \frac{x}{y}, \frac{1}{y}\right) \quad \text{EH I 240(9), AK 26(37)}$$

9.184 Integral representations: Double integrals of the Euler type

$$\begin{aligned}
1. \quad F_1(\alpha, \beta, \beta', \gamma; x, y) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta - \beta')} \\
&\times \int\limits_{\substack{u \geq 0, v \geq 0 \\ u+v \leq 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha} du dv \\
&[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re}(\gamma - \beta - \beta') > 0] \quad \text{EH I 230(1), AK 28(1)}
\end{aligned}$$

$$\begin{aligned}
2. \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta)\Gamma(\gamma' - \beta')} \\
&\times \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha} du dv \\
&[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re}(\gamma - \beta) > 0, \quad \operatorname{Re}(\gamma' - \beta') > 0] \quad \text{EH I 230(2), AK 28(2)}
\end{aligned}$$

$$\begin{aligned}
3. \quad F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta - \beta')} \\
&\times \int\limits_{\substack{u \geq 0, v \geq 0 \\ u+v \leq 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{-\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'} du dv \\
&[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re}(\gamma - \beta - \beta') > 0] \quad \text{EH I 230(3), AK 28(3)}
\end{aligned}$$

$$\begin{aligned}
4. \quad F_4(\alpha, \beta, \gamma, \gamma'; x(1-y), y(1-x)) &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma - \alpha)\Gamma(\gamma' - \beta)} \int_0^1 \int_0^1 u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1} \\
&\times (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1} du dv \\
&[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\gamma - \alpha) > 0, \quad \operatorname{Re}(\gamma' - \beta) > 0] \quad \text{EH I 230(4)}
\end{aligned}$$

9.185 Integral representations: Integrals of the Mellin–Barnes type

The functions F_1 , F_2 , F_3 , and F_4 can be represented by means of double integrals of the following form:

$$F(x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Psi(s, t) \Gamma(-s) \Gamma(-t) (-x)^s (-y)^t ds dt$$

$\Psi(s, t)$	$F(x, y)$
$\frac{\Gamma(\alpha + s + t) \Gamma(\beta + s) \Gamma(\beta' + t)}{\Gamma(\beta') \Gamma(\gamma + s + t)}$	$F_1(\alpha, \beta, \beta', \gamma; x, y)$
$\frac{\Gamma(\alpha + s + t) \Gamma(\beta + s) \Gamma(\beta' + t) \Gamma(\gamma')}{\Gamma(\beta') \Gamma(\gamma + s) \Gamma(\gamma' + t)}$	$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y)$
$\frac{\Gamma(\alpha + s) \Gamma(\alpha' + t) \Gamma(\beta + s) \Gamma(\beta' + t)}{\Gamma(\alpha') \Gamma(\beta') \Gamma(\gamma + s + t)}$	$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$
$\frac{\Gamma(\alpha + s + t) \Gamma(\beta + s + t) \Gamma(\gamma')}{\Gamma(\gamma + s) \Gamma(\gamma' + t)}$	$F_4(\alpha, \beta, \gamma, \gamma'; x, y)$
[$\alpha, \alpha', \beta, \beta'$ may not be negative integers] EH I 232(9–13), AK 41(33)	

9.19 A hypergeometric function of several variables

$$F_A(\alpha; \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \dots (\gamma_n)_{m_n} m_1! \dots m_n!} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$$

ET I 385

9.2 Confluent Hypergeometric Functions

9.20 Introduction

9.201¹⁰ A *confluent hypergeometric function* is obtained by taking the limit as $c \rightarrow \infty$ in the solution of Riemann's differential equation

$$u = P \begin{Bmatrix} 0 & \infty & c \\ \frac{1}{2} + \mu & -c & c - \lambda \\ \frac{1}{2} - \mu & 0 & \lambda \end{Bmatrix} z$$

WH

9.202 The equation obtained by means of this limiting process is of the form

$$1. \quad \frac{d^2 u}{dz^2} + \frac{du}{dz} + \left(\frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0$$

WH

Equation **9.202** 1 has the following two linearly independent solutions:

$$2. \quad z^{\frac{1}{2} + \mu} e^{-z} \Phi\left(\frac{1}{2} + \mu - \lambda, 2\mu + 1; z\right)$$

$$3. \quad z^{\frac{1}{2} - \mu} e^{-z} \Phi\left(\frac{1}{2} - \mu - \lambda, -2\mu + 1; z\right)$$

which are defined for all values of $\mu \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots$

MO 111

9.21 The functions $\Phi(\alpha, \gamma; z)$ and $\Psi(\alpha, \gamma; z)$

9.210¹⁰ The series

$$1. \quad \Phi(\alpha, \gamma; z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{z^3}{3!} + \dots$$

is also called a *confluent hypergeometric function*.

A second notation: $\Phi(\alpha, \gamma; z) = {}_1F_1(\alpha; \gamma; z)$.

$$2. \quad \Psi(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} \Phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(\alpha-\gamma+1, 2-\gamma; z) \quad \text{EH I 257(7)}$$

3. Bateman's function $k_\nu(x)$ is defined by

$$k_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \tan \theta - \nu \theta) d\theta \quad [x, \nu \text{ real}] \quad \text{EH I 267}$$

9.211 Integral representation:

$$1. \quad \Phi(\alpha, \gamma; z) = \frac{2^{1-\gamma} e^{\frac{1}{2}z}}{B(\alpha, \gamma-\alpha)} \int_{-1}^1 (1-t)^{\gamma-\alpha-1} (1+t)^{\alpha-1} e^{\frac{1}{2}zt} dt \quad [0 < \operatorname{Re} \alpha < \operatorname{Re} \gamma] \quad \text{MO 114}$$

$$2. \quad \Phi(\alpha, \gamma; z) = \frac{1}{B(\alpha, \gamma-\alpha)} z^{1-\gamma} \int_0^z e^t t^{\alpha-1} (z-t)^{\gamma-\alpha-1} dt \quad [0 < \operatorname{Re} \alpha < \operatorname{Re} \gamma] \quad \text{MO 114}$$

$$3. \quad \Phi(-\nu, \alpha+1; z) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} e^z z^{-\frac{\alpha}{2}} \int_0^\infty e^{-t} t^{\nu+\frac{\alpha}{2}} J_\alpha(2\sqrt{zt}) dt \quad \left[\operatorname{Re}(\alpha+\nu+1) > 0, \quad |\arg z| < \frac{\pi}{2} \right] \quad \text{MO 115}$$

$$4.^8 \quad \Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} z > 0] \quad \text{EH I 255(2)}$$

Functional relations

9.212

$$1. \quad \Phi(\alpha, \gamma; z) = e^z \Phi(\gamma-\alpha, \gamma; -z) \quad \text{MO 112}$$

$$2. \quad \frac{z}{\gamma} \Phi(\alpha+1, \gamma+1; z) = \Phi(\alpha+1, \gamma; z) - \Phi(\alpha, \gamma; z) \quad \text{MO 112}$$

$$3. \quad \alpha \Phi(\alpha+1, \gamma+1; z) = (\alpha-\gamma) \Phi(\alpha, \gamma+1; z) + \gamma \Phi(\alpha, \gamma; z) \quad \text{MO 112}$$

$$4. \quad \alpha \Phi(\alpha+1, \gamma; z) = (z+2\alpha-\gamma) \Phi(\alpha, \gamma; z) + (\gamma-\alpha) \Phi(\alpha-1, \gamma; z) \quad \text{MO 112}$$

$$\mathbf{9.213} \quad \frac{d\Phi}{dz} = \frac{\alpha}{\gamma} \Phi(\alpha+1, \gamma+1; z) \quad \text{MO 112}$$

$$\mathbf{9.214} \quad \lim_{\gamma \rightarrow -n} \frac{1}{\Gamma(\gamma)} \Phi(\alpha, \gamma; z) = z^{n+1} \binom{\alpha+n}{n+1} \Phi(\alpha+n+1, n+2; z) \quad [n=0, 1, 2, \dots] \quad \text{MO 112}$$

9.215¹⁰

1. $\Phi(\alpha, \alpha; z) = e^z$ MO 15
2. $\Phi(\alpha, 2\alpha; 2z) = 2^{\alpha-\frac{1}{2}} \exp\left[\frac{1}{4}(1-2\alpha)\pi i\right] \Gamma\left(\alpha + \frac{1}{2}\right) e^z z^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}\left(ze^{\frac{\pi}{2}i}\right)$ MO 112
3. $\Phi\left(p + \frac{1}{2}, 2p + 1; 2iz\right) = \Gamma(p + 1) \left(\frac{z}{2}\right)^{-p} e^{iz} J_p(z)$ MO 15

For a representation of special functions in terms of a confluent hypergeometric function $\Phi(\alpha, \gamma; z)$, see:

- for the probability integral, **9.236**;
- for integrals of Bessel functions, **6.631** 1;
- for Hermite polynomials, **8.953** and **8.959**;
- for Laguerre polynomials, **8.972** 1;
- for parabolic cylinder functions, **9.240**;
- for the Whittaker functions $M_{\lambda, \mu}(z)$, **9.220** 2 and **9.220** 3.

9.216 The function $\Phi(\alpha, \gamma; z)$ is a solution of the differential equation

$$1. \quad z \frac{d^2 F}{dz^2} + (\gamma - z) \frac{dF}{dz} - \alpha F = 0 \quad \text{MO 111}$$

This equation has two linearly independent solutions:

2. $\Phi(\alpha, \gamma; z)$
3. $z^{1-\gamma} \Phi(\alpha - \gamma + 1, 2 - \gamma; z)$ MO 112

9.22–9.23 The Whittaker functions $M_{\lambda, \mu}(z)$ and $W_{\lambda, \mu}(z)$

9.220 If we make the change of variable $u = e^{-\frac{z}{2}} W$ in equation **9.202** 1, we obtain the equation

$$1. \quad \frac{d^2 W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) W = 0 \quad \text{MO 115}$$

Equation **9.220** 1 has the following two linearly independent solutions:

2. $M_{\lambda, \mu}(z) = z^{\mu+\frac{1}{2}} e^{-z/2} \Phi\left(\mu - \lambda + \frac{1}{2}, 2\mu + 1; z\right)$
- 3.¹¹ $M_{\lambda, -\mu}(z) = z^{-\mu+\frac{1}{2}} e^{-z/2} \Phi\left(-\mu - \lambda + \frac{1}{2}, -2\mu + 1; z\right)$ MO 115

To obtain solutions that are also suitable for $2\mu = \pm 1, \pm 2, \dots$, we introduce Whittaker's function

$$4. \quad W_{\lambda, \mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right)} M_{\lambda, \mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} M_{\lambda, -\mu}(z) \quad \text{WH}$$

which, for 2μ approaching an integer, is also a solution of equation **9.220** 1.

For the functions $M_{\lambda, \mu}(z)$ and $W_{\lambda, \mu}(z)$, $z = 0$ is a branch point and $z = \infty$ is an essential singular point. Therefore, we shall examine these functions only for $|\arg z| < \pi$.

These functions $W_{\lambda, \mu}(z)$ and $W_{-\lambda, \mu}(-z)$ are linearly independent solutions of equation **9.220** 1.

Integral representations

9.221 $M_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}}{2^{2\mu} \text{B}(\mu+\lambda+\frac{1}{2}, \mu-\lambda+\frac{1}{2})} \int_{-1}^1 (1+t)^{\mu-\lambda-\frac{1}{2}} (1-t)^{\mu+\lambda-\frac{1}{2}} e^{\frac{1}{2}zt} dt,$ WH
 if the integral converges. See also **6.631** 1 and **7.623** 3.

9.222

1.¹¹ $W_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}} e^{-z/2}}{\Gamma(\mu-\lambda+\frac{1}{2})} \int_0^\infty e^{-zt} t^{\mu-\lambda-\frac{1}{2}} (1+t)^{\mu+\lambda-\frac{1}{2}} dt$
 $\left[\text{Re}(\mu-\lambda) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right]$ MO 118

2. $W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-z/2}}{\Gamma(\mu-\lambda+\frac{1}{2})} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}} e^{-t} \left(1 + \frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}} dt$
 $\left[\text{Re}(\mu-\lambda) > -\frac{1}{2}, \quad |\arg z| < \pi \right]$ WH

9.223 $W_{\lambda,\mu}(z) = \frac{e^{-\frac{z}{2}}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(u-\lambda) \Gamma(-u-\mu+\frac{1}{2}) \Gamma(-u+\mu+\frac{1}{2})}{\Gamma(-\lambda+\mu+\frac{1}{2}) \Gamma(-\lambda-\mu+\frac{1}{2})} z^u du$
 [the path of integration is chosen in such a way that the poles of the function $\Gamma(u-\lambda)$ are separated from the poles of the functions $\Gamma(-u-\mu+\frac{1}{2})$ and $\Gamma(-u+\mu+\frac{1}{2})$.] See also **7.142**. MO 118

9.224 $W_{\mu, \frac{1}{2}+\mu}(z) = z^{\mu+1} e^{-\frac{1}{2}z} \int_0^\infty (1+t)^{2\mu} e^{-zt} dt = z^{-\mu} e^{\frac{1}{2}z} \int_z^\infty t^{2\mu} e^{-t} dt \quad [\text{Re } z > 0]$ WH

9.225

1. $W_{\lambda,\mu}(x) W_{-\lambda,\mu}(x) = -x \int_0^\infty \tanh^{2\lambda} \frac{t}{2} \{ J_{2\mu}(x \sinh t) \sin(\mu-\lambda)\pi$
 $+ Y_{2\mu}(x \sinh t) \cos(\mu-\lambda)\pi \} dt$
 $\left[|\text{Re } \mu| - \text{Re } \lambda < \frac{1}{2}; \quad x > 0 \right]$ MO 119

2. $W_{\kappa,\mu}(z_1) W_{\lambda,\mu}(z_2) = \frac{(z_1 z_2)^{\mu+\frac{1}{2}} \exp[-\frac{1}{2}(z_1+z_2)]}{\Gamma(1-\kappa-\lambda)}$
 $\times \int_0^\infty e^{-t} t^{-\kappa-\lambda} (z_1+t)^{-\frac{1}{2}+\kappa-\mu} (z_2+t)^{-\frac{1}{2}+\lambda-\mu}$
 $\times F\left(\frac{1}{2}-\kappa+\mu, \frac{1}{2}-\lambda+\mu; 1-\kappa-\lambda; \Theta\right) dt$
 $\Theta = \frac{t(z_1+z_2+t)}{(z_1+t)(z_2+t)}, \quad [z_1 \neq 0, \quad z_2 \neq 0, \quad |\arg z_1| < \pi, \quad |\arg z_2| < \pi, \quad \text{Re}(\kappa+\lambda) < 1]$

MO 119

See also **3.334**, **3.381** 6, **3.382** 3, **3.383** 4, 8, **3.384** 3, **3.471** 2.**9.226** Series representations

$M_{0,\mu}(z) = z^{\frac{1}{2}+\mu} \left\{ 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{2^{4k} k! (\mu+1)(\mu+2) \dots (\mu+k)} \right\}$ WH

Asymptotic representations**9.227**⁷ For large values of $|z|$

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^\lambda \left(1 + \sum_{k=1}^{\infty} \frac{[\mu^2 - (\lambda - \frac{1}{2})^2][\mu^2 - (\lambda - \frac{3}{2})^2] \dots [\mu^2 - (\lambda - k + \frac{1}{2})^2]}{k! z^k} \right) \quad [|\arg z| \leq \pi - \alpha < \pi] \quad \text{WH}$$

9.228 For large values of $|\lambda|$

$$M_{\lambda,\mu}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma(2\mu + 1) \lambda^{-\mu - \frac{1}{4}} z^{1/4} \cos \left(2\sqrt{\lambda z} - \mu\pi - \frac{1}{4}\pi \right) \quad \text{MO 118}$$

9.229

$$1. \quad W_{\lambda,\mu} \sim - \left(\frac{4z}{\lambda} \right)^{\frac{1}{4}} e^{-\lambda + \lambda \ln \lambda} \sin \left(2\sqrt{\lambda z} - \lambda\pi - \frac{\pi}{4} \right) \quad \text{MO 118}$$

$$2. \quad W_{-\lambda,\mu} \sim \left(\frac{z}{4\lambda} \right)^{\frac{1}{4}} e^{\lambda - \lambda \ln \lambda - 2\sqrt{\lambda z}} \quad \text{MO 118}$$

Formulas **9.228** and **9.229** are applicable for

$$|\lambda| \gg 1, \quad |\lambda| \gg |z|, \quad |\lambda| \gg |\mu|, \quad z \neq 0, \quad |\arg \sqrt{z}| < \frac{3\pi}{4} \text{ and } |\arg \lambda| < \frac{\pi}{2}. \quad \text{MO 118}$$

Functional relations**9.231**

$$1. \quad M_{n+\mu+\frac{1}{2},\mu}(z) = \frac{z^{\frac{1}{2}-\mu} e^{\frac{1}{2}z}}{(2\mu+1)(2\mu+2)\dots(2\mu+n)} \frac{d^n}{dz^n} (z^{n+2\mu} e^{-z}) \quad [n = 0, 1, 2, \dots; \quad 2\mu \neq -1, -2, -3, \dots] \quad \text{MO 117}$$

$$2. \quad z^{-\frac{1}{2}-\mu} M_{\lambda,\mu}(z) = (-z)^{-\frac{1}{2}-\mu} M_{-\lambda,\mu}(-z) \quad [2\mu \neq -1, -2, -3, \dots] \quad \text{WH}$$

9.232

$$1. \quad W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z) \quad \text{MO 116}$$

$$2. \quad W_{-\lambda,\mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu+\lambda)} M_{-\lambda,\mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu+\lambda)} M_{-\lambda,-\mu}(-z) \quad [|\arg(-z)| < \frac{3}{2}\pi] \quad \text{WH}$$

9.233

$$1. \quad M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu-\lambda+\frac{1}{2})} e^{i\pi\lambda} W_{-\lambda,\mu}(e^{i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma(\mu+\lambda+\frac{1}{2})} \exp[i\pi(\lambda-\mu-\frac{1}{2})] W_{\lambda,\mu}(z) \quad [-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi; \quad 2\mu \neq -1, -2, \dots] \quad \text{MO 117}$$

$$2. \quad M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu-\lambda+\frac{1}{2})} e^{-i\pi\lambda} W_{-\lambda,\mu}(e^{-i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma(\mu+\lambda+\frac{1}{2})} \exp[-i\pi(\lambda-\mu-\frac{1}{2})] W_{\lambda,\mu}(z) \quad [-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi; \quad 2\mu \neq -1, -2, \dots] \quad \text{MO 117}$$

9.234 Recursion formulas

$$1. \quad W_{\lambda,\mu}(z) = \sqrt{z} W_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(z) + \left(\frac{1}{2} + \mu - \lambda\right) W_{\lambda-1,\mu}(z) \quad \text{WH}$$

$$2.^{11} \quad W_{\lambda,\mu}(z) = \sqrt{z} W_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(z) + \left(\frac{1}{2} - \mu - \lambda\right) W_{\lambda-1,\mu}(z) \quad \text{WH}$$

$$3. \quad z \frac{d}{dz} W_{\lambda,\mu}(z) = \left(\lambda - \frac{1}{2}z\right) W_{\lambda,\mu}(z) - \left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right] W_{\lambda-1,\mu}(z) \quad \text{WH}$$

$$4. \quad \left[\left(\mu + \frac{1-z}{2}\right) W_{\lambda,\mu}(z) - z \frac{d}{dz} W_{\lambda,\mu}(z) \right] \left(\mu + \frac{1}{2} + \lambda\right) \\ = \left[\left(\mu + \frac{1+z}{2}\right) W_{\lambda,\mu+1}(z) + z \frac{d}{dz} W_{\lambda,\mu+1}(z) \right] \left(\mu + \frac{1}{2} - \lambda\right) \quad \text{MO 117}$$

$$5. \quad \left(\frac{3}{2} + \lambda + \mu\right) \left(\frac{1}{2} + \lambda + \mu\right) z W_{\lambda,\mu}(z) = z(z + 2\mu + 1) \frac{d}{dz} W_{\lambda+1,\mu+1}(z) \\ + \left[\frac{1}{2}z^2 + \left(\mu - \lambda - \frac{1}{2}\right)z + 2\mu^2 + 2\mu + \frac{1}{2}\right] W_{\lambda+1,\mu+1}(z) \quad \text{MO 117}$$

Connections with other functions**9.235**

$$1. \quad M_{0,\mu}(z) = 2^{2\mu} \Gamma(\mu + 1) \sqrt{z} I_{\mu}\left(\frac{z}{2}\right) \quad \text{MO 125a}$$

$$2. \quad W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}} K_{\mu}\left(\frac{z}{2}\right) \quad \text{MO 125}$$

9.236

$$1. \quad \Phi(x) = 1 - \frac{e^{\frac{x^2}{2}}}{\sqrt{\pi x}} W_{-\frac{1}{4},\frac{1}{4}}(x^2) = \frac{2x}{\sqrt{\pi}} \Phi\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) \quad \text{WH, MO 126}$$

$$2. \quad \text{li}(z) = -\frac{\sqrt{z}}{\sqrt{\ln \frac{1}{2}}} W_{-\frac{1}{2},0}(-\ln z) \quad \text{WH}$$

$$3. \quad \Gamma(\alpha, x) = e^{-x} \Psi(1 - \alpha, 1 - \alpha; x) \quad \text{EH I 266(21)}$$

$$4. \quad \gamma(\alpha, x) = \frac{x^{\alpha}}{\alpha} \Phi(\alpha, \alpha + 1; -x) \quad \text{EH I 266(22)}$$

9.237

$$1. \quad W_{\lambda,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right) \Gamma\left(\frac{1}{2} + \mu - \lambda\right)} \\ \times \left\{ \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k - \lambda + \frac{1}{2}\right)}{k!(2\mu + k)!} z^k [\Psi(k+1) + \Psi(2\mu + k + 1) - \Psi\left(\mu + k - \lambda + \frac{1}{2}\right) - \ln z] \right. \\ \left. + (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu - k) \Gamma\left(k - \mu - \lambda + \frac{1}{2}\right)}{k!} (-z)^k \right\} \\ \left[|\arg z| < \frac{3}{2}\pi; \quad 2\mu + 1 \text{ is a natural number} \right] \quad \text{MO 116}$$

2. Set $\lambda - \mu - \frac{1}{2} = l$, where $l + 1$ is a natural number. Then
3.
$$W_{l+\mu+\frac{1}{2}, \mu}(z) = (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} (2\mu+1)(2\mu+2) \cdots (2\mu+l) \Phi(-l, 2\mu+1; z)$$

$$= (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} L_l^{2\mu}(z)$$

MO 116

9.238

1. $J_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-ix} \Phi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2ix\right)$ EH I 265(9)
2. $I_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-x} \Phi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right)$ EH I 265(10)
3. $K_\nu(x) = \sqrt{\pi} e^{-x} (2x)^\nu \Psi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right)$ EH I 265(13)

9.24–9.25 Parabolic cylinder functions $D_p(z)$

$$9.240 \quad D_p(z) = 2^{\frac{1}{4}+\frac{p}{2}} W_{\frac{1}{4}+\frac{p}{2}, -\frac{1}{4}}\left(\frac{z^2}{2}\right) z^{-1/2}$$

$$= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \Phi\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma\left(-\frac{p}{2}\right)} \Phi\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right\}$$

MO 120a

are called *parabolic cylinder functions*.**Integral representations****9.241**

$$1. \quad D_p(z) = \frac{1}{\sqrt{\pi}} 2^{p+\frac{1}{2}} e^{-\frac{\pi}{2}pi} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2+2ixz} dx \quad [\operatorname{Re} p > -1; \quad \text{for } x < 0, \quad \arg x^p = p\pi i]$$

MO 122

$$2. \quad D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^{\infty} e^{-xz-\frac{x^2}{2}} x^{-p-1} dx \quad [\operatorname{Re} p < 0] \quad (\text{cf. } \mathbf{3.462} \text{ 1}) \quad \text{MO 122}$$

9.242

$$1.^{10} \quad D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0+)} e^{-zt-\frac{1}{2}t^2} (-t)^{-p-1} dt \quad [|\arg(-t)| \leq \pi] \quad \text{WH}$$

$$2. \quad D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2}+1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4}z^2t} (1+t)^{-\frac{1}{2}p-1} (1-t)^{\frac{1}{2}(p-1)} dt$$

$$[\arg z < \frac{\pi}{4}; \quad |\arg(1+t)| \leq \pi] \quad \text{WH}$$

$$3. \quad D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right) \Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t dt$$

$$[\arg z < \frac{3}{4}\pi; \quad p \text{ is not a positive integer}] \quad \text{WH}$$

$$4. \quad D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0-)} \frac{\Gamma(\frac{1}{2}t - \frac{1}{2}p) \Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t dt$$

[for all values of $\arg z$; also, the contours encircle the poles of the function $\Gamma(-t)$, but they do not encircle the poles of the function $\Gamma(\frac{1}{2}t - \frac{1}{2}p)$]. WH

9.243

$$1. \quad D_n(z) = (-1)^\mu \left(\frac{\pi}{2}\right)^{-1/2} (\sqrt{n})^{n+1} e^{\frac{1}{4}z^2 - \frac{1}{2}n} \left\{ \int_{-\infty}^{\infty} \frac{e^{-n(t-1)^2} \cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt \right. \\ \left. + \int_0^{\infty} \left[e^{\frac{1}{2}n(1-t^2)} t^n - e^{-n(t-1)^2} \right] \frac{\cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt - \int_{-\infty}^0 \frac{e^{-n(t-1)^2} \cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt \right\} \\ [n \text{ is a natural number}] \quad \text{WH}$$

$$2. \quad D_n(z) = (-1)^\mu 2^{n+2} (2\pi)^{-1/2} e^{\frac{1}{4}z^2} \int_0^{\infty} \frac{t^n e^{-2t^2} \cos(2zt)}{\sin(2zt)} dt \\ [n \text{ is a natural number, } \mu = \lfloor \frac{n}{2} \rfloor, \text{ and the cosine or sine is chosen accordingly as } n \text{ is even or odd}] \quad \text{WH}$$

9.244

$$1. \quad D_{-p-1}[(1+i)z] = \frac{e^{-\frac{iz^2}{2}}}{2^{\frac{p-1}{2}} \Gamma(\frac{p+1}{2})} \int_0^{\infty} \frac{e^{-ix^2 z^2} x^p}{(1+x^2)^{1+\frac{p}{2}}} dx \quad [\operatorname{Re} p > -1, \quad \operatorname{Re}(iz^2) \geq 0] \quad \text{MO 122}$$

$$2. \quad D_p[(1+i)z] = \frac{2^{\frac{p+1}{2}}}{\Gamma(-\frac{p}{2})} \int_1^{\infty} e^{-\frac{i}{2}z^2 x} \frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}} dx \quad [\operatorname{Re} p < 0; \quad \operatorname{Re}(iz^2) \geq 0] \quad \text{MO 122}$$

See also **3.383** 6, 7, **3.384** 2, 6, **3.966** 5, 6.

9.245

$$1.^{10} \quad D_p(x) D_{-p-1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^{\infty} \coth^{p+\frac{1}{2}} \left(\frac{t}{2}\right) \frac{1}{\sqrt{\sinh t}} \sin\left(\frac{x^2 \sinh t + p\pi}{2}\right) dt \\ [x \text{ is real, } \operatorname{Re} p < 0] \quad \text{MO 122}$$

$$2. \quad D_p(ze^{\frac{\pi}{4}i}) D_p(ze^{-\frac{\pi}{4}i}) = \frac{1}{\Gamma(-p)} \int_0^{\infty} \coth^p t \exp\left(-\frac{z^2}{2} \sinh 2t\right) \frac{dt}{\sinh t} \\ \left[|\arg z| < \frac{\pi}{4}; \quad \operatorname{Re} p < 0\right] \quad \text{MO 122}$$

See also **6.613**.

9.246 Asymptotic expansions. If $|z| \gg 1$ and $|z| \gg |p|$, then

$$1. \quad D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right) \\ [|\arg z| < \frac{3}{4}\pi] \quad \text{MO 121}$$

$$2.^{11} \quad D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right) \\ - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right) \\ \left[\frac{1}{4}\pi < \arg z < \frac{5}{4}\pi\right] \quad \text{MO 121}$$

$$\begin{aligned}
 3.11 \quad D_p(z) &\sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots \right) \\
 &\quad - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots \right) \\
 &\quad \left[-\frac{1}{4}\pi > \arg z > -\frac{5}{4}\pi \right] \quad \text{MO 121}
 \end{aligned}$$

Functional relations

9.247 Recursion formulas:

1. $D_{p+1}(z) - z D_p(z) + p D_{p-1}(z) = 0$ WH
2. $\frac{d}{dz} D_p(z) + \frac{1}{2} z D_p(z) - p D_{p-1}(z) = 0$ WH
3. $\frac{d}{dz} D_p(z) - \frac{1}{2} z D_p(z) + D_{p+1}(z) = 0$ MO 121

9.248 Linear relations:

1.
$$\begin{aligned}
 D_p(z) &= \frac{\Gamma(p+1)}{\sqrt{2\pi}} \left[e^{\pi/2} D_{-p-1}(iz) + e^{-\pi/2} D_{-p-1}(-iz) \right] \\
 &= e^{-p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-\pi(p+1)i/2} D_{-p-1}(iz) \\
 &= e^{p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{\pi(p+1)i/2} D_{-p-1}(-iz)
 \end{aligned}$$
 MO 121

$$9.249^{10} \quad D_p[(1+i)x] + D_p[-(1+i)x] = \frac{2^{1+p/2}}{\Gamma(-p)} \exp \left[-\frac{i}{2} \left(x^2 + p \frac{\pi}{2} \right) \right] \int_0^\infty \frac{\cos xt}{t^{p+1}} e^{-it^2/4} dt$$

[x real; $-1 < \operatorname{Re} p < 0$] MO 122

$$9.251^{10} \quad D_n(z) = (-1)^n e^{z^2/4} \frac{d^n}{dz^n} \left(e^{-z^2/2} \right) \quad [n = 0, 1, 2, \dots] \quad \text{WH}$$

$$9.252 \quad D_p(ax+by) = \exp \frac{(bx-ay)^2}{4} \left(\frac{a}{\sqrt{a^2+b^2}} \right)^p \sum_{k=0}^\infty \binom{p}{k} D_{p-k} \left(\sqrt{a^2+b^2} x \right) D_k \left(\sqrt{a^2+b^2} y \right) \left(\frac{b}{a} \right)^k$$

[$a > b > 0$, $x > 0$, $y > 0$, $\operatorname{Re} p \geq 0$] "summation theorem" MO 124

Connections with other functions

$$9.253^{11} \quad D_n(z) = 2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n \left(\frac{z}{\sqrt{2}} \right) \quad \text{MO 123a}$$

9.254

$$1. \quad D_{-1}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left[1 - \Phi \left(\frac{z}{\sqrt{2}} \right) \right] \quad \text{MO 123}$$

$$2.11 \quad D_{-2}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left\{ \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} - z \left[1 - \Phi \left(\frac{z}{\sqrt{2}} \right) \right] \right\} \quad \text{MO 123}$$

9.255 Differential equations leading to parabolic cylinder functions:

$$1. \quad \frac{d^2 u}{dz^2} + \left(p + \frac{1}{2} - \frac{z^2}{4} \right) u = 0$$

The solutions are $u = D_p(z)$, $D_p(-z)$, $D_{-p-1}(iz)$, and $D_{-p-1}(-iz)$.

(These four solutions are linearly dependent. See **9.248**.)

$$2. \quad \frac{d^2 u}{dz^2} + (z^2 + \lambda) u = 0,$$

$$u = D_{-\frac{1+i\lambda}{2}} [\pm(1+i)z]$$

EH II 118(12,13)a, MO 123

$$3.^7 \quad \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (p+1)u = 0,$$

$$u = e^{-\frac{z^2}{4}} D_p(z)$$

MO 123

9.26 Confluent hypergeometric series of two variables

9.261

$$1.^6 \quad \Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!} x^m y^n \quad [|x| < 1] \quad \text{EH I 225(20)}$$

$$2. \quad \Phi_2(\beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_m}{(\gamma)_{m+n}m!n!} x^m y^n \quad \text{EH I 225(21)a, ET I 385}$$

$$3. \quad \Phi_3(\beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}m!n!} x^m y^n \quad \text{EH I 225(22)}$$

The functions Φ_1 , Φ_2 , Φ_3 satisfy the following systems of partial differential equations:

9.262

$$1. \quad z = \Phi_1(\alpha, \beta, \gamma, x, y) \quad \text{EH I 235(23)}$$

$$\begin{aligned} x(1-x)\frac{\partial^2 z}{\partial x^2} + y(1-x)\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x]\frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha \beta z &= 0, \\ y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x \partial y} + (\gamma - y)\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x} - \alpha z &= 0 \end{aligned}$$

$$2. \quad z = \Phi_2(\beta, \beta', \gamma, x, y) \quad \text{EH I 235(24)}$$

$$\begin{aligned} x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} + (\gamma - x)\frac{\partial z}{\partial x} - \beta z &= 0, \\ y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x \partial y} + (\gamma - y)\frac{\partial z}{\partial y} - \beta' z &= 0 \end{aligned}$$

$$3. \quad z = \Phi_3(\beta, \gamma, x, y)$$

EH I 235(25)

$$x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + (\gamma - x) \frac{\partial z}{\partial x} - \beta z = 0,$$

$$y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + \gamma \frac{\partial z}{\partial y} - z = 0$$

9.3 Meijer's G -Function

9.30 Definition

$$9.301 \quad G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{1}{2\pi i} \int \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} x^s ds$$

$[0 \leq m \leq q, \quad 0 \leq n \leq p]$, and the poles of $\Gamma(b_j - s)$ must not coincide with the poles of $\Gamma(1 - a_k + s)$ for any j and k (where $j = 1, \dots, m; \quad k = 1, \dots, n$). Besides **9.301**, the following notations are also used:

$$G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right), \quad G_{pq}^{mn}(x), \quad G(x) \quad \text{EH I 207(1)}$$

9.302 Three types of integration paths L in the right member of **9.301** can be exhibited:

1. The path L runs from $-\infty$ to $+\infty$ in such a way that the poles of the functions $\Gamma(1 - a_k + s)$ lie to the left, and the poles of the functions $\Gamma(b_j - s)$ lie to the right of L (for $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, n$). In this case, the conditions under which the integral **9.301** converges are of the form

$$p + q < 2(m + n), \quad |\arg x| < (m + n - \frac{1}{2}p - \frac{1}{2}q) \pi. \quad \text{EH I 207(2)}$$

2. L is a loop, beginning and ending at $+\infty$, that encircles the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \dots, m$) once in the negative direction. All the poles of the functions $\Gamma(1 - a_k + s)$ must remain outside this loop. Then, the conditions under which the integral **9.301** converges are:

$$q \geq 1 \text{ and either } p < q \text{ or } p = q \text{ and } |x| < 1. \quad \text{EH I 207(3)}$$

3. L is a loop, beginning and ending at $-\infty$, that encircles the poles of the functions $\Gamma(1 - a_k + s)$ (for $k = 1, 2, \dots, n$) once in the positive direction. All the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \dots, m$) must remain outside this loop.

The conditions under which the integral in **9.301** converges are

$$p \geq 1 \text{ and either } p > q \text{ or } p = q \text{ and } |x| > 1. \quad \text{EH I 207(4)}$$

The function $G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$ is analytic with respect to x ; it is symmetric with respect to the parameters $a_1, \dots, a_n$ and also with respect to $a_{n+1}, \dots, a_p; b_1, \dots, b_m; b_{m+1}, \dots, b_q$.

EH I 208

9.303¹¹ If no two b_j (for $j = 1, 2, \dots, n$) differ by an integer, then, under the conditions that either $p < q$ or $p = q$ and $|x| < 1$,

$$G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = \sum_{h=1}^m \frac{\prod_{j=1}^m \Gamma(b_j - b_h) \prod_{j=1}^n \Gamma(1 + b_h - a_j)}{\prod_{j=m+1}^q \Gamma(1 + b_h - b_j) \prod_{j=n+1}^p \Gamma(a_j - b_h)} x^{b_h} \\ \times {}_pF_{q-1} \left[1 + b_h - a_1, \dots, 1 + b_h - a_p; \quad 1 + b_h - b_1, \dots \right. \\ \left. \dots, *, \dots, 1 + b_h - b_q; \quad (-1)^{p-m-n} x \right]$$

EH I 208(5)

The prime by the product symbol denotes the omission of the product when $j = h$. The asterisk in the function ${}_pF_{q-1}$ denotes the omission of the h^{th} parameter.

9.304⁷ If no two a_k (for $k = 1, 2, \dots, n$) differ by an integer then, under the conditions that $q < p$ or $q = p$ and $|x| > 1$,

$$G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = \sum_{h=1}^n \frac{\prod_{j=1}^{n'} \Gamma(a_h - a_j) \prod_{j=1}^m \Gamma(b_j - a_h + 1)}{\prod_{j=n+1}^p \Gamma(a_j - a_h + 1) \prod_{j=m+1}^q \Gamma(a_h - b_j)} x^{a_h-1} \\ \times {}_qF_{p-1} \left[1 + b_1 - a_h, \dots, 1 + b_q - a_h; \quad 1 + a_1 - a_h, \dots \right. \\ \left. \dots, *, \dots, 1 + a_p - a_h; \quad (-1)^{q-m-n} x^{-1} \right]$$

EH I 208(6)

9.31 Functional relations

If one of the parameters a_j (for $j = 1, 2, \dots, n$) coincides with one of the parameters b_j (for $j = m + 1, m + 2, \dots, q$), the order of the G -function decreases. For example,

$$1. \quad G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, a_1 \end{matrix} \right. \right) = G_{p-1, q-1}^{m, n-1} \left(x \left| \begin{matrix} a_2, \dots, a_p \\ b_1, \dots, b_{q-1} \end{matrix} \right. \right) \\ [n, p, q \geq 1]$$

An analogous relationship occurs when one of the parameters b_j (for $j = 1, 2, \dots, m$) coincides with one of the a_j (for $j = n + 1, \dots, p$). In this case, it is m and not n that decreases by one unit.

The G -function with $p > q$ can be transformed into the G -function with $p < q$ by means of the relationships:

$$2. \quad G_{pq}^{mn} \left(x^{-1} \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = G_{qp}^{nm} \left(x \left| \begin{matrix} 1 - b_s \\ 1 - a_r \end{matrix} \right. \right) \quad \text{EH I 209(9)}$$

3.
$$x \frac{d}{dx} G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = G_{pq}^{mn} \left(x \left| \begin{matrix} a_1 - 1, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) + (a_1 - 1) G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$$

$$[n \geq 1] \quad \text{EH I 210(13)}$$
4.
$$G_{p+1, q+1}^{m+1, n} \left(z \left| \begin{matrix} \mathbf{a}_p, 1 - r \\ 0, \mathbf{b}_q \end{matrix} \right. \right) = (-1)^r G_{p+1, q+1}^{m, n+1} \left(z \left| \begin{matrix} 1 - r, \mathbf{a}_p \\ \mathbf{b}_q, 1 \end{matrix} \right. \right)$$

$$[r = 0, 1, 2, \dots] \quad \text{MS2 6 (1.2.2)}$$
5.
$$z^k G_{pq}^{mn} \left(z \left| \begin{matrix} \mathbf{a}_p \\ \mathbf{b}_q \end{matrix} \right. \right) = G_{pq}^{mn} \left(z \left| \begin{matrix} \mathbf{a}_p + k \\ \mathbf{b}_q + k \end{matrix} \right. \right)$$

$$\text{MS2 7 (1.2.7)}$$

9.32 A differential equation for the G -function

$G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$ satisfies the following linear q^{th} -order differential equation:

$$\left[(-1)^{p-m-n} x \prod_{j=1}^p \left(x \frac{d}{dx} - a_j + 1 \right) - \prod_{j=1}^q \left(x \frac{d}{dx} - b_j \right) \right] y = 0 \quad [p \leq q] \quad \text{EH I 210(1)}$$

9.33 Series of G -functions

$$\begin{aligned} G_{pq}^{mn} \left(\lambda x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) &= \lambda^{b_1} \sum_{r=0}^{\infty} \frac{1}{r!} (1 - \lambda)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1 + r, b_2, \dots, b_q \end{matrix} \right. \right) \\ &\quad [|\lambda - 1| < 1, \quad m \geq 1, \quad \text{if } m = 1 \text{ and } p < q, \lambda \text{ may be arbitrary}] \\ &\quad \text{EH I 213(1)} \\ &= \lambda^{b_q} \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda - 1)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, b_q + r \end{matrix} \right. \right) \\ &\quad [m < q, \quad |\lambda - 1| < 1] \\ &\quad \text{EH I 213(2)} \\ &= \lambda^{a_1 - 1} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\lambda - \frac{1}{\lambda} \right)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1 - r, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ &\quad [n \geq 1, \quad \text{Re } \lambda > \frac{1}{2}, \quad (\text{if } n = 1 \text{ and } p > q, \text{ then } \lambda \text{ may be arbitrary})] \\ &\quad \text{EH I 213(3)} \\ &= \lambda^{a_p - 1} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\lambda} - 1 \right)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_{p-1}, a_p - r \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ &\quad [n < p, \quad \text{Re } \gamma > \frac{1}{2}] \\ &\quad \text{EH I 213(4)} \end{aligned}$$

For integrals of the G -function, see 7.8.

9.34 Connections with other special functions

1.
$$J_{\nu}(x)x^{\mu} = 2^{\mu} G_{02}^{10} \left(\frac{1}{4}x^2 \left| \begin{matrix} \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}\nu \end{matrix} \right. \right)$$

$$\text{EH I 219(44)}$$
2.
$$Y_{\nu}(x)x^{\mu} = 2^{\mu} G_{13}^{20} \left(\frac{1}{4}x^2 \left| \begin{matrix} \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \\ \frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \end{matrix} \right. \right)$$

$$\text{EH I 219(46)}$$

$$3. \quad K_\nu(x)x^\mu = 2^{\mu-1} G_{02}^{20} \left(\frac{1}{4}x^2 \left| \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu \right. \right) \quad \text{EH I 219(47)}$$

$$4. \quad K_\nu(x) = e^x \sqrt{\pi} G_{12}^{20} \left(2x \left| \frac{1}{2} \right. \right) \quad \text{EH I 219(49)}$$

$$5. \quad \mathbf{H}_\nu(x)x^\mu = 2^\mu G_{13}^{11} \left(\frac{1}{4}x^2 \left| \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu + \frac{1}{2}\nu \right. \right) \quad \text{EH I 220(51)}$$

$$6. \quad S_{\mu,\nu}(x) = 2^{\mu-1} \frac{1}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu}{2}\right)} G_{13}^{31} \left(\frac{1}{4}x^2 \left| \frac{1}{2} + \frac{1}{2}\mu, \frac{1}{2} + \frac{1}{2}\mu, \frac{1}{2}\nu, -\frac{1}{2}\nu \right. \right) \quad \text{EH I 220(55)}$$

$$7.^7 \quad {}_2F_1(a, b; c; -x) = \frac{\Gamma(c)x}{\Gamma(a)\Gamma(b)} G_{22}^{12} \left(x \left| -a, -b \right. \right) \quad \text{EH I 222(74)a}$$

$$8. \quad {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{p,q+1}^{1,p} \left(-x \left| 1-a_1, \dots, 1-a_p \right. \right) \\ = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{q+1,p}^{p,1} \left(-\frac{1}{x} \left| 1, b_1, \dots, b_q \right. \right) \quad \text{EH I 215(1)}$$

$$9. \quad W_{k,m}(x) = \frac{2^k \sqrt{x} e^{\frac{1}{2}x}}{\sqrt{2\pi}} G_{24}^{40} \left(\frac{x^2}{4} \left| \frac{1}{4} - \frac{1}{2}k, \frac{3}{4} - \frac{1}{2}k, \frac{1}{2} + \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \right. \right) \quad \text{EH I 221(70)}$$

9.4 MacRobert's *E*-Function

9.41 Representation by means of multiple integrals

$$E(p; \alpha_r : q; \varrho_s : x) = \frac{\Gamma(\alpha_{q+1})}{\Gamma(\varrho_1 - \alpha_1) \Gamma(\varrho_2 - \alpha_2) \cdots \Gamma(\varrho_q - \alpha_q)} \\ \times \prod_{\mu=1}^q \int_0^\infty \lambda_\mu^{\varrho_\mu - \alpha_\mu - 1} (1 - \lambda_\mu)^{-\varrho_\mu} d\lambda_\mu \prod_{\nu=2}^{p-q-1} \int_0^\infty e^{-\lambda_{q+\nu}} \lambda_{q+\nu}^{\alpha_{q+\nu} - 1} d\lambda_{q+\nu} \\ \times \int_0^\infty e^{-\lambda_p} \lambda_p^{\alpha_p - 1} \left[1 + \frac{\lambda_{q+2} \lambda_{q+3} \cdots \lambda_p}{(1 + \lambda_1) \cdots (1 + \lambda_q) x} \right]^{-\alpha_{q+1}} d\lambda_p$$

$[\arg x] < \pi$, $p \geq q+1$, α_r and ϱ_s are bounded by the condition that the integrals on the right be convergent.] EH I 204(3)

9.42 Functional relations

$$1. \quad \alpha_1 x E(\alpha_1, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x) = x E(\alpha_1 + 1, \alpha_2, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x) \\ + E(\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_p + 1 : \varrho_1 + 1, \dots, \varrho_q + 1 : x) \quad \text{EH I 205(7)}$$

$$2. \quad (\varrho_1 - 1) x E(\alpha_1, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x) = x E(\alpha_1, \dots, \alpha_p : \varrho_1 - 1, \varrho_2, \dots, \varrho_q : x) \\ + E(\alpha_1 + 1, \dots, \alpha_p + 1 : \varrho_1 + 1, \dots, \varrho_q + 1 : x) \quad \text{EH I 205(9)}$$

$$3. \quad \frac{d}{dx} E(\alpha_1, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x) = x^{-2} E(\alpha_1 + 1, \dots, \alpha_p + 1 : \varrho_1 + 1, \dots, \varrho_q + 1 : x) \quad \text{EH I 205(8)}$$

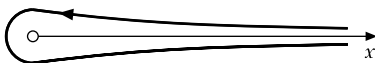
9.5 Riemann's Zeta Functions $\zeta(z, q)$ and $\zeta(z)$, and the Functions $\Phi(z, s, v)$ and $\xi(s)$

9.51 Definition and integral representations

$$9.511 \quad \zeta(z, q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-qt}}{1 - e^{-t}} dt; \quad \text{WH} \\ = \frac{1}{2} q^{-z} + \frac{q^{1-z}}{z-1} + 2 \int_0^\infty (q^2 + t^2)^{-\frac{z}{2}} \left[\sin \left(z \arctan \frac{t}{q} \right) \right] \frac{dt}{e^{2\pi t} - 1} \\ [0 < q < 1, \quad \operatorname{Re} z > 1] \quad \text{WH}$$

$$9.512 \quad \zeta(z, q) = -\frac{\Gamma(1-z)}{2\pi i} \int_{\infty}^{(0+)} \frac{(-\theta)^{z-1} e^{-q\theta}}{1 - e^{-\theta}} d\theta$$

This equation is valid for all values of z , except for $z = 1, 2, 3, \dots$. It is assumed that the path of integration (see drawing below) does not pass through the points $2n\pi i$ (where n is a natural number).



See also **4.251** 4, **4.271** 1, 4, 8, **4.272** 9, 12, **4.294** 11.

9.513

$$1. \quad \zeta(z) = \frac{1}{(1 - 2^{1-z}) \Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1} dt \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

$$2. \quad \zeta(z) = \frac{2^z}{(2^z - 1) \Gamma(z)} \int_0^\infty \frac{t^{z-1} e^t}{e^{2t} - 1} dt \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$3.^{11} \quad \zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma(\frac{z}{2})} \left[\frac{1}{z(z-1)} + \int_1^\infty \left(t^{\frac{1-z}{2}} + t^{\frac{z}{2}} \right) t^{-1} \sum_{k=1}^\infty e^{-k^2 \pi t} dt \right] \quad \text{WH}$$

$$4. \quad \zeta(z) = \frac{2^{z-1}}{z-1} - 2^z \int_0^\infty (1+t^2)^{-\frac{z}{2}} \sin(z \arctan t) \frac{dt}{e^{\pi t} + 1} \quad \text{WH}$$

$$5. \quad \zeta(z) = \frac{2^{z-1}}{2^z - 1} \frac{z}{z-1} + \frac{2}{2^z - 1} \int_0^\infty \left(\frac{1}{4} + t^2 \right)^{-z/2} \sin(z \arctan 2t) \frac{dt}{e^{2\pi t} - 1} \quad \text{WH}$$

See also **3.411** 1, **3.523** 1, **3.527** 1, 3, **4.271** 8.

9.52 Representation as a series or as an infinite product

9.521

$$1. \quad \zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z} \quad [\operatorname{Re} z > 1, \quad q \neq 0, -1, -2, \dots] \quad \text{WH}$$

$$2. \quad \zeta(z, q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin \frac{z\pi}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi qn}{n^{1-z}} + \cos \frac{z\pi}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi qn}{n^{1-z}} \right] \\ [\operatorname{Re} z < 0, \quad 0 < q \leq 1] \quad \text{WH}$$

$$3.^8 \quad \zeta(z, q) = \sum_{n=0}^N \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^{\infty} F_n(z),$$

where

$$F_n(z) = \frac{1}{1-z} \left(\frac{1}{(n+1+q)^{z-1}} - \frac{1}{(n+q)^{z-1}} \right) - \frac{1}{(n+1+q)^z} \\ = z \int_n^{n+1} \frac{(t-n) dt}{(t+q)^{z+1}} \quad \text{WH}$$

9.522

$$1. \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$2. \quad \zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z} \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

9.523 The following product and summation are taken over all primes p :

$$1.^7 \quad \zeta(z) = \prod_p \frac{1}{1-p^{-z}} \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$2. \quad \ln \zeta(z) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{kz}} \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$\mathbf{9.524}^{11} \quad \frac{\zeta'(z)}{\zeta(z)} = - \sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^z}, \quad [\operatorname{Re} z > 1]$$

where $\Lambda(k) = 0$ when k is not a power of a prime and $\Lambda(k) = \ln p$ when k is a power of a prime p . WH

9.53 Functional relations

$$\mathbf{9.531} \quad \zeta(-n, q) = - \frac{B'_{n+2}(q)}{(n+1)(n+2)} = \frac{-B_{n+1}(q)}{n+1} \\ [n \text{ is a nonnegative integer}] \quad \text{see EH I 27 (11)} \quad \text{WH}$$

$$\mathbf{9.532} \quad \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} z^k \zeta(k, q) = \ln \frac{e^{-Cz} \Gamma(q)}{\Gamma(z+q)} - \frac{z}{q} + \sum_{k=1}^{\infty} \frac{qz}{k(q+k)} \quad [|z| < q] \quad \text{WH}$$

9.533

$$1. \quad \lim_{z \rightarrow 1} \frac{\zeta(z, q)}{\Gamma(1-z)} = -1 \quad \text{WH}$$

$$2. \quad \lim_{z \rightarrow 1} \left\{ \zeta(z, q) - \frac{1}{z-1} \right\} = -\Psi(q) \quad \text{WH}$$

$$3. \quad \left\{ \frac{d}{dz} \zeta(z, q) \right\}_{z=0} = \ln \Gamma(q) - \frac{1}{2} \ln 2\pi \quad \text{WH}$$

$$\mathbf{9.534} \quad \zeta(z, 1) = \zeta(z)$$

9.535

$$1. \quad \zeta(z) = \frac{1}{2^z - 1} \zeta\left(z, \frac{1}{2}\right) \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$2.^{11} \quad 2^z \Gamma(1-z) \zeta(1-z) \sin\left(\frac{z\pi}{2}\right) = \pi^{1-z} \zeta(z) \quad \text{WH}$$

$$3. \quad 2^{1-z} \Gamma(z) \zeta(z) \cos \frac{z\pi}{2} = \pi^z \zeta(1-z) \quad \text{WH}$$

$$4. \quad \Gamma\left(\frac{z}{2}\right) \pi^{-\frac{z}{2}} \zeta(z) = \Gamma\left(\frac{1-z}{2}\right) \pi^{\frac{z-1}{2}} \zeta(1-z) \quad \text{WH}$$

$$\mathbf{9.536} \quad \lim_{z \rightarrow 1} \left\{ \zeta(z) - \frac{1}{z-1} \right\} = C$$

9.537 Set $z = \frac{1}{2} + it$. Then, $\Xi(t) = \frac{(z-1)\Gamma(\frac{z}{2}+1)}{\sqrt{\pi^z}} \zeta(z) = \Xi(-t)$ is an even function of t with real coefficients in its expansion in powers of t^2 . JA

9.54 Singular points and zeros**9.541**⁷

$$1. \quad z = 1 \text{ is the only singular point of the function } \zeta(z) \quad \text{WH}$$

$$2. \quad \text{The function } \zeta(z) \text{ has simple zeros at the points } -2n, \text{ where } n \text{ is a natural number. All other zeros of the function } \zeta(z) \text{ lie in the strip } 0 \leq \operatorname{Re} z < 1.$$

$$3.^8 \quad \text{Riemann's hypothesis: All zeros of the function } \zeta(z) \text{ lie on the straight line } \operatorname{Re} z = \frac{1}{2}. \text{ It has been shown that a countably infinite set of zeros of the zeta function lie on this line. The first 1,500,000,001 zeros lying in } 0 < \operatorname{Im} z < 545,439,823.215 \text{ are known to have } \operatorname{Re} z = \frac{1}{2}. \quad \text{WH}$$

9.542 Particular values:

$$1. \quad \zeta(2m) = \frac{2^{2m-1} \pi^{2m} |B_{2m}|}{(2m)!} \quad [m \text{ is a natural number}] \quad \text{WH}$$

$$2. \quad \zeta(1-2m) = -\frac{B_{2m}}{2m} \quad [m \text{ is a natural number}] \quad \text{WH}$$

$$3. \quad \zeta(-2m) = 0 \quad [m \text{ is a natural number}] \quad \text{WH}$$

$$4. \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi \quad \text{WH}$$

9.55 The Lerch function $\Phi(z, s, v)$

9.550 Definition:

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \quad [|z| < 1, \quad v \neq 0, -1, \dots] \quad \text{EH I 27(1)}$$

Functional relations

$$\mathbf{9.551} \quad \Phi(z, s, v) = z^m \Phi(z, s, m+v) + \sum_{n=0}^{m-1} (v+n)^{-s} z^n \quad [m = 1, 2, 3, \dots, \quad v \neq 0, -1, -2, \dots] \quad \text{EH I 27(1)}$$

$$\mathbf{9.552} \quad \Phi(z, s, v) = iz^{-v} (2\pi)^{s-1} \Gamma(1-s) \left[e^{-i\pi \frac{s}{2}} \Phi\left(e^{-2\pi i v}, 1-s, \frac{\ln z}{2\pi i}\right) - e^{i\pi(\frac{s}{2}-2v)} \Phi\left(e^{2\pi i v}, 1-s, 1-\frac{\ln z}{2\pi i}\right) \right] \quad \text{EH I 29(7)}$$

Series representation

$$\mathbf{9.553} \quad \Phi(z, s, v) = z^{-v} \Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\ln z + 2\pi n i)^{s-1} e^{2\pi n v i} \quad [0 < v \leq 1, \quad \operatorname{Re} s < 0, \quad |\arg(-\ln z + 2\pi n i)| \leq \pi] \quad \text{EH I 28(6)}$$

$$\mathbf{9.554} \quad \Phi(z, m, v) = z^{-v} \left\{ \sum_{n=0}^{\infty}{}' \zeta(m-n, v) \frac{(\ln z)^n}{n!} + \frac{(\ln z)^{m-1}}{(m-1)!} \left[\Psi(m) - \Psi(v) - \ln\left(\ln \frac{1}{z}\right) \right] \right\}^* \quad [m = 2, 3, 4, \dots, \quad |\ln z| < 2\pi, \quad v \neq 0, -1, -2, \dots] \quad \text{EH I 30(9)}$$

$$\mathbf{9.555} \quad \Phi(z, -m, v) = \frac{m!}{z^v} \left(\ln \frac{1}{z} \right)^{-m-1} - \frac{1}{z^v} \sum_{r=0}^{\infty} \frac{B_{m+r+1}(v) (\ln z)^r}{r!(m+r+1)} \quad [|\ln z| < 2\pi] \quad \text{EH I 30(11)}$$

Integral representation

$$\mathbf{9.556} \quad \Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt \quad [\operatorname{Re} v > 0, \text{ or } |z| \leq 1, \quad z \neq 1, \quad \operatorname{Re} s > 0, \text{ or } z = 1, \quad \operatorname{Re} s > 1] \quad \text{EH I 27(3)}$$

Limit relationships

$$\mathbf{9.557} \quad \lim_{z \rightarrow 1} (1-z)^{1-s} \Phi(z, s, v) = \Gamma(1-s) \quad [\operatorname{Re} s < 1] \quad \text{EH I 30(12)}$$

$$\mathbf{9.558} \quad \lim_{z \rightarrow 1} \frac{\Phi(z, 1, v)}{-\ln(1-z)} = 1 \quad \text{EH I 30(13)}$$

A connection with a hypergeometric function

$$\mathbf{9.559} \quad \Phi(z, 1, v) = v^{-1} {}_2F_1(1, v; 1+v; z) \quad [|z| < 1] \quad \text{EH I 30(10)}$$

*In 9.554 the prime on the symbol $\sum$ means that the term corresponding to $n = m-1$ is omitted.

9.56 The function $\xi(s)$

$$9.561 \quad \xi(s) = \frac{1}{2}s(s-1) \frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}} \zeta(s) \quad \text{EH III 190(10)}$$

$$9.562 \quad \xi(1-s) = \xi(s) \quad \text{EH III 190(11)}$$

9.6 Bernoulli Numbers and Polynomials, Euler Numbers, the Functions $\nu(x)$, $\nu(x, \alpha)$, $\mu(x, \beta)$, $\mu(x, \beta, \alpha)$, $\lambda(x, y)$ and Euler Polynomials

9.61 Bernoulli numbers

9.610 The numbers B_n , representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad [0 < |t| < 2\pi],$$

are called *Bernoulli* numbers. Thus, the function $\frac{t}{e^t - 1}$ is a generating function for the Bernoulli numbers. GE 48(57), FI II 520

9.611 Integral representations

$$1. \quad B_{2n} = (-1)^{n-1} 4n \int_0^{\infty} \frac{x^{2n-1}}{e^{2\pi x} - 1} dx \quad [n = 1, 2, \dots] \quad (\text{cf. } \mathbf{3.411} \text{ 2, 4})$$

FI II 721a

$$2. \quad B_{2n} = (-1)^{n-1} \pi^{-2n} \int_0^{\infty} \frac{x^{2n}}{\sinh^2 x} dx \quad [n = 1, 2, \dots]$$

$$3. \quad B_{2n} = (-1)^{n-1} \frac{2n(1-2n)}{\pi} \int_0^{\infty} x^{2n-2} \ln(1 - e^{-2\pi x}) dx$$

$$[n = 1, 2, \dots]$$

$$4.* \quad B_n = \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right)$$

See also **3.523** 2, **4.271** 3.

Properties and functional relations

9.612^s A symbolic notation:

$$(B + \alpha)^{[n]} = \sum_{k=0}^n \binom{n}{k} B_k \alpha^{n-k} \quad [n \geq 2]$$

in particular

$$B_n = (B + 1)^{[n]} = \sum_{k=0}^n \binom{n}{k} B_k \quad [n \geq 2]$$

hence by recursion

$$B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!} \quad [n \geq 2]$$

9.613 All the Bernoulli numbers are rational numbers.

9.614 Every number B_n can be represented in the form

$$B_n = C_n - \sum \frac{1}{k+1},$$

where C_n is an integer and the sum is taken over all $k > 0$ such that $k+1$ is a prime and k is a divisor of n . GE 64

9.615¹¹ All the Bernoulli numbers with odd index are equal to zero, except that $B_1 = -\frac{1}{2}$; that is, $B_{2n+1} = 0$ for n a natural number. GE 52, FI II 521

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=1; k \text{ even}}^{n-1} \frac{2n(2n-1) \dots (2n-2k+2)}{(2k)!} B_{k/2} \quad [n \geq 1]$$

9.616 $B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n-1}\pi^{2n}} \zeta(2n) \quad [n \geq 0]$ (cf. **9.542**) GE 56(79), FI II 721a

9.617⁷ $B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{\prod_{p=2}^{\infty} \left(1 - \frac{1}{p^{2n}}\right)} \quad [n \geq 1] \quad (\text{cf. } \mathbf{9.523})$

(where the product is taken over all primes p).

- For a connection with Riemann's zeta function, see **9.542**.
- For a connection with the Euler numbers, see **9.635**.
- For a table of values of the Bernoulli numbers, see **9.71**

9.619 An inequality

$$\left| (B - \theta)^{[n]} \right| \leq |B_n| \quad [0 < \theta < 1]$$

9.62 Bernoulli polynomials

9.620 The Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad \text{GE 51(62)}$$

or symbolically, $B_n(x) = (B + x)^{[n]}$. GE 52(68)

9.621 The generating function

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \quad [0 < |t| < 2\pi] \quad (\text{cf. 1.213}) \quad \text{GE 65(89)a}$$

9.622 Series representation

$$1.^7 \quad B_n(x) = -2 \frac{n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx - \frac{1}{2}\pi n)}{k^n} \quad [n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0] \quad \text{AS 805(23.1.16)}$$

$$2.^7 \quad B_{2n-1}(x) = 2 \frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n-1}} \\ [n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0] \quad \text{AS 805(23.1.17)}$$

$$3.^{10} \quad B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}} \quad [0 \leq x \leq 1, \quad n = 1, 2, \dots] \quad \text{GE 71}$$

9.623 Functional relations and properties:

$$1. \quad B_{m+1}(n) = B_{m+1} + (m+1) \sum_{k=1}^{n-1} k^m \\ [n \text{ and } m \text{ are natural numbers}] \quad (\text{see also } \mathbf{0.121}) \quad \text{GE 51(65)}$$

$$2. \quad B_n(x+1) - B_n(x) = nx^{n-1} \quad \text{GE 65(90)}$$

$$3. \quad B'_n(x) = n B_{n-1}(x) \quad [n = 1, 2, \dots] \quad \text{GE 66}$$

$$4. \quad B_n(1-x) = (-1)^n B_n(x) \quad \text{GE 66}$$

$$5.^{10} \quad (-1)^n B_n(-x) = B_n(x) + nx^{n-1} \quad [n = 0, 1, \dots] \quad \text{AS 804(23.1.9)}$$

$$\mathbf{9.624}^7 \quad B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n \left(x + \frac{k}{m} \right) \\ [m = 1, 2, \dots, n = 0, 1, \dots]; \quad \text{"summation theorem"} \quad \text{GE 67}$$

9.625 For n odd, the differences

$$B_n(x) - B_n$$

vanish on the interval $[0, 1]$ only at the points $0, \frac{1}{2}$, and 1 . They change sign at the point $x = \frac{1}{2}$. For n even, these differences vanish at the end points of the interval $[0, 1]$. Within this interval, they do not change sign, and their greatest absolute value occurs at the point $x = \frac{1}{2}$.

9.626 The polynomials

$$B_{2n}(x) - B_{2n} \text{ and } B_{2n+2}(x) - B_{2n+2}$$

have opposite signs in the interval $(0, 1)$. GE 87

9.627 Special cases:

$$1. \quad B_1(x) = x - \frac{1}{2} \quad \text{GE 70}$$

$$2. \quad B_2(x) = x^2 - x + \frac{1}{6} \quad \text{GE 70}$$

$$3. \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \quad \text{GE 70}$$

$$4. \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \quad \text{GE 70}$$

$$5. \quad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \quad \text{GE 70}$$

9.628 Particular values:

$$1. \quad B_n(0) = B_n$$

$$2. \quad B_1(1) = -B_1 = \frac{1}{2}, \quad B_n(1) = B_n \quad [n \neq 1] \quad \text{GE 76}$$

9.63 Euler numbers

9.630 The numbers E_n , representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \left[|t| < \frac{\pi}{2} \right],$$

are known as the *Euler numbers*. Thus, the function $\frac{1}{\cosh t}$ is a generating function for the Euler numbers.

CE 330

9.631 A recursion formula

$$(E+1)^{[n]} + (E-1)^{[n]} = 0 \quad [n \geq 1], \quad E_0 = 1 \quad \text{CE 329}$$

Properties of the Euler numbers

9.632 The Euler numbers are integers.

9.633 The Euler numbers of odd index are equal to zero; the signs of two adjacent numbers of even indices are opposite; that is,

$$E_{2n+1} = 0, \quad E_{4n} > 0, \quad E_{4n+2} < 0. \quad \text{CE 329}$$

9.634 If $\alpha, \beta, \gamma, \dots$ are the divisors of the number $n - m$, the difference $E_{2n} - E_{2m}$ is divisible by those of the numbers $2\alpha + 1, 2\beta + 1, 2\gamma + 1, \dots$ that are primes.

9.635 A connection with the Bernoulli numbers (symbolic notation):

$$1.^{11} \quad E_{n-1} + 4(-1)^n (3^{n-1} - 1) B_1 = \frac{(4B-1)^{[n]} - (4B-3)^{[n]}}{2n} + 4(-1)^{n+1} (3^{n-1} - 1) B_1 \quad \text{CE 330}$$

$$2. \quad B_n = \frac{n(E+1)^{[n-1]}}{2^n (2^n - 1)} \quad [n \geq 2] \quad \text{CE 330}$$

$$3.^6 \quad \left(B + \frac{1}{4}\right)^{[2n+1]} = -4^{-2n-1} (2n+1) E_{2n} \quad [n \geq 0] \quad \text{CE 341}$$

$$4. \quad E_{n-1} = \frac{(4B+3)^{[n]} - (4B+1)^{[n]}}{2n} \quad [n \geq 1]$$

For a table of values of the Euler numbers, see **9.72**.

9.64 The functions $\nu(x)$, $\nu(x, \alpha)$, $\mu(x, \beta)$, $\mu(x, \beta, \alpha)$, and $\lambda(x, y)$

9.640

$$1. \quad \nu(x) = \int_0^\infty \frac{x^t dt}{\Gamma(t+1)} \quad \text{EH III 217(1)}$$

$$2. \quad \nu(x, \alpha) = \int_0^\infty \frac{x^{\alpha+t} dt}{\Gamma(\alpha+t+1)} \quad \text{EH III 217(1)}$$

$$3. \quad \mu(x, \beta) = \int_0^\infty \frac{x^t t^\beta dt}{\Gamma(\beta+1) \Gamma(t+1)} \quad \text{EH III 217(2)}$$

$$4. \quad \mu(x, \beta, \alpha) = \int_0^\infty \frac{x^{\alpha+t} t^\beta dt}{\Gamma(\beta+1) \Gamma(\alpha+t+1)} \quad \text{EH III 217(2)}$$

$$5. \quad \lambda(x, y) = \int_0^y \frac{\Gamma(u+1) du}{x^u} \quad \text{MI 9}$$

9.65¹⁰ Euler polynomials

9.650 The Euler polynomials are defined by

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k} \quad \text{AS 804 (23.1.7)}$$

9.651 The generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{AS 804 (23.1.1)}$$

9.652 Series representation:

$$1. \quad E_n(x) = 4 \frac{n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x - \frac{1}{2}\pi n)}{(2k+1)^{n+1}} \quad [n > 0, \quad 1 \geq x \geq 0, \quad n = 1, \quad 1 > x > 0] \quad \text{AS 804 (23.1.16)}$$

$$2.^{10} \quad E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}} \quad [n = 1, 2, \dots, \quad 1 \geq x \geq 0] \quad \text{AS 804 (23.1.17)}$$

$$3. \quad E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}} \quad [n > 0, \quad 1 \geq x \geq 0, \quad n = 0, \quad 1 > x > 0] \quad \text{AS 804 (23.1.18)}$$

9.653 Functional relations and properties:

$$1. \quad E_m(n+1) = 2 \sum_{k=1}^n (-1)^{n-k} k^m + (-1)^{n+1} E_m(0), \quad [m \text{ and } n \text{ are natural numbers}] \quad \text{AS 804 (23.1.4)}$$

$$2. \quad E'_n(x) = n E_{n-1}(x). \quad [n = 1, 2, \dots] \quad \text{AS 804 (23.1.5)}$$

$$3. \quad E_n(x+1) + E_n(x) = 2x^n \quad [n = 0, 1, \dots] \quad \text{AS 804 (23.1.6)}$$

$$4.^8 \quad E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(x - \frac{k}{m}\right) \quad [n = 0, 1, \dots, m = 1, 3, \dots] \quad \text{AS 804 (23.1.10)}$$

$$5. \quad E_n(mx) = \frac{-2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(x + \frac{k}{m}\right) \quad [n = 0, 1, \dots, m = 2, 4, \dots] \quad \text{AS 804 (23.1.10)}$$

9.654 Special cases:

$$1. \quad E_1(x) = x - \frac{1}{2}$$

$$2. \quad E_2(x) = x^2 - x$$

3. $E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}$
4. $E_4(x) = x^4 - 2x^3 + x$
5. $E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}$

9.655 Particular values:

1. $E_{2n+1} = 0.$ $[n = 0, 1, \dots]$ AS 805 (23.1.19)
2. $E_n(0) = -E_n(1) = -2(n+1)^{-1} (2^{n+1} - 1) B_{n+1}$ $[n = 1, 2, \dots]$ AS 805 (23.1.20)
3. $E_n\left(\frac{1}{2}\right) = 2^{-n} E_n$ $[n = 0, 1, \dots]$ AS 805 (23.1.21)
4. $E_{2n-1}\left(\frac{1}{3}\right) = -E_{2n-1}\left(\frac{2}{3}\right) = -(2n)^{-1} (1 - 3^{1-2n}) (2^{2n} - 1) B_{2n}$
 $[n = 1, 2, \dots]$ AS 806 (23.1.22)

9.7 Constants

9.71 Bernoulli numbers

- $B_0 = 1$
- $B_1 = -1/2$
- $B_2 = 1/6$
- $B_4 = -1/30$
- $B_6 = 1/42$
- $B_8 = -1/30$
- $B_{10} = 5/66$
- $B_{12} = -691/2730$
- $B_{14} = 7/6$
- $B_{16} = -3617/510$
- $B_{18} = 43867/798$
- $B_{20} = -174611/330$
- $B_{22} = 854513/138$
- $B_{24} = -236364091/2730$
- $B_{26} = 8553103/6$
- $B_{28} = -23749461029/870$
- $B_{30} = 8615841276005/14322$
- $B_{32} = -7709321041217/510$
- $B_{34} = 2577687858367/6$

9.72 Euler numbers

- $E_0 = 1$
- $E_2 = -1$
- $E_4 = 5$
- $E_6 = -61$
- $E_8 = 1385$
- $E_{10} = -50521$
- $E_{12} = 2702765$
- $E_{14} = -199360981$
- $E_{16} = 19391512145$
- $E_{18} = -2404879675441$
- $E_{20} = 370371188237525$

The Bernoulli and Euler numbers of odd index (with the exception of B_1) are equal to zero.

9.73 Euler's and Catalan's constants

Euler's constant

$$C = 0.577\,215\,664\,901\,532\,860\,606\,512\dots \quad (\text{cf. } \mathbf{8.367})$$

Catalan's constant

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915\,965\,594\dots$$

9.74¹⁰ Stirling numbers

9.740 The **Stirling number of the first kind** $S_n^{(m)}$ is defined by the requirement that $(-1)^{n-m} S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles. AS 824 (23.1.3)

9.741 Generating functions:

$$1. \quad x(x-1)\cdots(x-n+1) = \sum_{m=0}^n S_n^{(m)} x^m \quad \text{AS 824 (24.1.3)}$$

$$2. \quad \{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!} \quad [|x| < 1] \quad \text{AS 824 (24.1.3)}$$

9.742 Recurrence relations:

$$1.^8 \quad S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}; \quad S_n^{(0)} = \delta_{0n}; \quad S_n^{(1)} = (-1)^{n-1}(n-1)!; \quad S_n^{(n)} = 1 \\ [n \geq m \geq 1] \quad \text{AS 824 (24.1.3)}$$

$$2. \quad \binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m+r)} \quad [n \geq m \geq r] \quad \text{AS 824 (24.1.3)}$$

9.743 Functional relations and properties

$$1. \quad x(x-h)(x-2h)\cdots(x-mh+h) = \frac{h^m \Gamma\left(\frac{x}{h}+1\right)}{\Gamma\left(\frac{x}{h}-m+1\right)} = h^m \sum_{k=1}^m \left(\frac{x}{h}\right)^k S_k^{(m)}$$

$$2. \quad [(x+1)(x+2)\cdots(x+m)]^{-1} = \left[\binom{x+m}{m} m! \right]^{-1} = \left[\sum_{k=1}^p (x+m)^k S_k^{(m)} \right]^{-1}$$

$$3. \quad [(x+h)(x+2h)\cdots(x+mh)]^{-1} = \frac{\Gamma\left(\frac{x}{h}+1\right)}{h^m \Gamma\left(\frac{x}{h}+m+1\right)} = \left[h^m \sum_{k=1}^m \left(\frac{x}{h}+m\right)^k S_k^{(m)} \right]^{-1}$$

9.744 The Stirling number of the second kind $\mathfrak{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

[illegible]

Stirling numbers of the second kind $\mathfrak{S}_n^{(m)}$

m	$\mathfrak{S}_1^{(m)}$	$\mathfrak{S}_2^{(m)}$	$\mathfrak{S}_3^{(m)}$	$\mathfrak{S}_4^{(m)}$	$\mathfrak{S}_5^{(m)}$	$\mathfrak{S}_6^{(m)}$	$\mathfrak{S}_7^{(m)}$	$\mathfrak{S}_8^{(m)}$	$\mathfrak{S}_9^{(m)}$
1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255
3			1	6	25	90	301	966	3025
4				1	10	65	350	1701	7770
5					1	15	140	1050	6951
6						1	21	266	2646
7							1	28	462
8								1	36
9									1

9.749^s Relationship between Stirling numbers of the first kind and derivatives of $(\ln x)^{-m}$:

1.
$$\frac{d^n}{dx^n} \left(\frac{1}{\ln^m x} \right) = \frac{1}{\ln^m x} \sum_{k=1}^n \frac{(-1)^k (m)_k S_n^{(k)}}{x^n \ln^k x}$$

where $(m)_k = \Gamma(m+k)/\Gamma(m)$, [m, n are positive integers]

10 Vector Field Theory

10.1–10.8 Vectors, Vector Operators, and Integral Theorems

10.11 Products of vectors

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_3)$ be arbitrary vectors, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the set of orthogonal unit vectors in terms of which the components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are expressed. Two different products involving pairs of vectors are defined, namely, the scalar product, written $\mathbf{a} \cdot \mathbf{b}$, and the vector product, written either $\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \wedge \mathbf{b}$. Their properties are as follows:

1. $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ (scalar product)
2. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ (vector product)
3. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (triple scalar product)
4. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ (triple vector product)

10.12 Properties of scalar product

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative)
2. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b} = -\mathbf{a} \times \mathbf{c} \cdot \mathbf{b} = -\mathbf{b} \times \mathbf{a} \cdot \mathbf{c} = -\mathbf{c} \times \mathbf{b} \cdot \mathbf{a}$.
Note: $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is also written $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$; thus (2) may also be written
3. $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}]$

10.13 Properties of vector product

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutative)
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$
3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

10.14 Differentiation of vectors

If $\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$, $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$, $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$, $\phi(t)$ is a scalar and all functions of t are differentiable, then

1. $\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\mathbf{i} + \frac{da_2}{dt}\mathbf{j} + \frac{da_3}{dt}\mathbf{k}$
2. $\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$
3. $\frac{d}{dt}(\phi\mathbf{a}) = \frac{d\phi}{dt}\mathbf{a} + \phi\frac{d\mathbf{a}}{dt}$
4. $\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}$
5. $\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$
6. $\frac{d}{dt}(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \frac{d\mathbf{b}}{dt} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{b} \cdot \frac{d\mathbf{c}}{dt}$
7. $\frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt}\right)$

10.21 Operators grad, div, and curl

In cartesian coordinates $O\{x_1, x_2, x_3\}$, in which system it is convenient to denote the triad of unit vectors by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the vector operator ∇ , called either “del” or “nabla,” has the form

$$1. \quad \nabla \equiv \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}$$

If $\Phi(x, y, z)$ is any differentiable scalar function, the gradient of Φ , written $\text{grad } \Phi$, is

$$2. \quad \text{grad } \Phi \equiv \nabla \Phi = \frac{\partial \Phi}{\partial x_1} \mathbf{e}_1 + \frac{\partial \Phi}{\partial x_2} \mathbf{e}_2 + \frac{\partial \Phi}{\partial x_3} \mathbf{e}_3$$

The divergence of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written $\text{div } \mathbf{f}$, is

$$3. \quad \text{div } \mathbf{f} \equiv \nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

The curl, or rotation, of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written either $\text{curl } \mathbf{f}$ or $\text{rot } \mathbf{f}$, is

$$4. \quad \text{curl } \mathbf{f} \equiv \text{rot } \mathbf{f} \equiv \nabla \times \mathbf{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) \mathbf{e}_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) \mathbf{e}_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \mathbf{e}_3,$$

or equivalently,

$$\text{curl } \mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

10.31 Properties of the operator ∇

Let $\Phi(x_1, x_2, x_3)$, $\Psi(x_1, x_2, x_3)$ be any two differentiable scalar functions, $\mathbf{f}(x_1, x_2, x_3)$, $\mathbf{g}(x_1, x_2, x_3)$ any two differentiable vector functions, and $\mathbf{a}$ an arbitrary vector. Define the scalar operator ∇^2 , called the Laplacian, by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

Then, in terms of the operator ∇ , we have the following:

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1. $\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$
2. $\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi$
3. $\nabla(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{f})$
4. $\nabla \cdot (\Phi\mathbf{f}) = \Phi(\nabla \cdot \mathbf{f}) + \mathbf{f} \cdot \nabla\Phi$
5. $\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot (\nabla \times \mathbf{f}) - \mathbf{f} \cdot (\nabla \times \mathbf{g})$
6. $\nabla \times (\Phi\mathbf{f}) = \Phi(\nabla \times \mathbf{f}) + (\nabla\Phi) \times \mathbf{f}$
7. $\nabla \times (\mathbf{f} \times \mathbf{g}) = \mathbf{f}(\nabla \cdot \mathbf{g}) - \mathbf{g}(\nabla \cdot \mathbf{f}) + (\mathbf{g} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{g}$
8. $\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2\mathbf{f}$
9. $\nabla \times (\nabla\Phi) \equiv \mathbf{0}$
10. $\nabla \cdot (\nabla \times \mathbf{f}) \equiv 0$
- 11.¹⁰ $\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2(\nabla\Phi) \cdot (\nabla\Psi) + \Psi\nabla^2\Phi$

The equivalent results in terms of grad, div, and curl are as follows:

1. $\text{grad}(\Phi + \Psi) = \text{grad}\Phi + \text{grad}\Psi$
2. $\text{grad}(\Phi\Psi) = \Phi\text{grad}\Psi + \Psi\text{grad}\Phi$
3. $\text{grad}(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \text{grad})\mathbf{g} + (\mathbf{g} \cdot \text{grad})\mathbf{f} + \mathbf{f} \times \text{curl}\mathbf{g} + \mathbf{g} \times \text{curl}\mathbf{f}$
4. $\text{div}(\Phi\mathbf{f}) = \Phi\text{div}\mathbf{f} + \mathbf{f} \cdot \text{grad}\Phi$
5. $\text{div}(\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot \text{curl}\mathbf{f} - \mathbf{f} \cdot \text{curl}\mathbf{g}$
6. $\text{curl}(\Phi\mathbf{f}) = \Phi\text{curl}\mathbf{f} + \text{grad}\Phi \times \mathbf{f}$
7. $\text{curl}(\mathbf{f} \times \mathbf{g}) = \mathbf{f}\text{div}\mathbf{g} - \mathbf{g}\text{div}\mathbf{f} + (\mathbf{g} \cdot \text{grad})\mathbf{f} - (\mathbf{f} \cdot \text{grad})\mathbf{g}$
8. $\text{curl}(\text{curl}\mathbf{f}) = \text{grad}(\text{div}\mathbf{f}) - \nabla^2\mathbf{f}$
9. $\text{curl}(\text{grad}\Phi) \equiv \mathbf{0}$
10. $\text{div}(\text{curl}\mathbf{f}) \equiv 0$
11. $\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2\text{grad}\Phi \cdot \text{grad}\Psi + \Psi\nabla^2\Phi$

The expression $(\mathbf{a} \cdot \nabla)$ or, equivalently $(\mathbf{a} \cdot \text{grad})$, defined by

$$(\mathbf{a} \cdot \nabla) \equiv a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3},$$

is the directional derivative operator in the direction of vector $\mathbf{a}$.

10.41 Solenoidal fields

A vector field $\mathbf{f}$ is said to be solenoidal if $\operatorname{div} \mathbf{f} \equiv 0$. We have the following representation:

10.411 *Representation theorem for vector Helmholtz equation.* If u is a solution of the scalar Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0,$$

and $\mathbf{m}$ is a constant unit vector, then the vectors

$$\mathbf{X} = \operatorname{curl}(\mathbf{m}u), \quad \mathbf{Y} = \frac{1}{\lambda} \operatorname{curl} \mathbf{X}$$

are independent solutions of the vector Helmholtz equation

$$\nabla^2 \mathbf{H} + \lambda^2 \mathbf{H} = \mathbf{0}$$

involving a solenoidal vector $\mathbf{H}$. The general solution of the equation is

$$\mathbf{H} = \operatorname{curl}(\mathbf{m}u) + \frac{1}{\lambda} \operatorname{curl} \operatorname{curl}(\mathbf{m}u).$$

10.51–10.61 Orthogonal curvilinear coordinates

Consider a transformation from the cartesian coordinates $O\{x_1, x_2, x_3\}$ to the general orthogonal curvilinear coordinates $O\{u_1, u_2, u_3\}$:

$$x_1 = x_1(u_1, u_2, u_3), \quad x_2 = x_2(u_1, u_2, u_3), \quad x_3 = x_3(u_1, u_2, u_3)$$

Then,

$$1. \quad dx_i = \frac{\partial x_i}{\partial u_1} du_1 + \frac{\partial x_i}{\partial u_2} du_2 + \frac{\partial x_i}{\partial u_3} du_3 \quad (i = 1, 2, 3),$$

and the length element dl may be determined from

$$2. \quad dl^2 = g_{11} du_1^2 + g_{22} du_2^2 + g_{33} du_3^2 + 2g_{23} du_2 du_3 + 2g_{31} du_3 du_1 + 2g_{12} du_1 du_2,$$

where

$$3.^3 \quad g_{ij} = \frac{\partial x_1}{\partial u_i} \frac{\partial x_1}{\partial u_j} + \frac{\partial x_2}{\partial u_i} \frac{\partial x_2}{\partial u_j} + \frac{\partial x_3}{\partial u_i} \frac{\partial x_3}{\partial u_j} = g_{ji}, \quad g_{ij} = 0, \quad i \neq j,$$

provided the Jacobian of the transformation

$$4. \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}$$

does not vanish (see **14.313**).

Define the metrical coefficients

$$5. \quad h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}};$$

then the volume element dV in orthogonal curvilinear coordinates is

$$6. \quad dV = h_1 h_2 h_3 du_1 du_2 du_3,$$

and the surface elements of area ds_i on the surfaces $u_i = \text{constant}$, for $i = 1, 2, 3$, are

$$7. \quad ds_1 = h_2 h_3 du_2 du_3, \quad ds_2 = h_1 h_3 du_1 du_3, \quad ds_3 = h_1 h_2 du_1 du_2$$

Denote by $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ the triad of orthogonal unit vectors that are tangent to the u_1, u_2 , and u_3 coordinate lines through any given point P , and choose their sense so that they form a right-handed set in this order. Then in terms of this triad of vectors and the components f_{u_1}, f_{u_2} , and f_{u_3} of $\mathbf{f}$ along the coordinate line,

$$8. \quad \mathbf{f} = f_{u_1} \mathbf{e}_1 + f_{u_2} \mathbf{e}_2 + f_{u_3} \mathbf{e}_3$$

MF I 115

10.611 $\nabla \Phi$, $\operatorname{div} \mathbf{f}$, $\operatorname{curl} \mathbf{f}$, and ∇^2 in general orthogonal curvilinear coordinates.

$$1. \quad \operatorname{grad} \Phi = \frac{\mathbf{e}_1}{h_1} \frac{\partial \Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2} \frac{\partial \Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3} \frac{\partial \Phi}{\partial u_3}$$

$$2.^3 \quad \operatorname{div} \mathbf{f} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 f_{u_1}) + \frac{\partial}{\partial u_2} (h_3 h_1 f_{u_2}) + \frac{\partial}{\partial u_3} (h_1 h_2 f_{u_3}) \right)$$

$$3. \quad \operatorname{curl} \mathbf{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 f_{u_1} & h_2 f_{u_2} & h_3 f_{u_3} \end{vmatrix}$$

$$4. \quad \nabla^2 \equiv \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right)$$

MF I 21-31

10.612 *Cylindrical polar coordinates.* In terms of the coordinates $O\{r, \phi, z\}$, that is, $u_1 = r$, $u_2 = \phi$, $u_3 = z$, where $x_1 = r \cos \phi$, $x_2 = r \sin \phi$, $x_3 = z$ for $-\pi < \phi \leq \pi$, it follows that

$$1. \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

and

$$2. \quad \operatorname{grad} \Phi = \frac{\partial \Phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \mathbf{e}_\phi + \frac{\partial \Phi}{\partial z} \mathbf{e}_z,$$

$$3. \quad \operatorname{div} \mathbf{f} = \frac{1}{r} \frac{\partial}{\partial r} (r f_r) + \frac{1}{r} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z},$$

$$4. \quad \operatorname{curl} \mathbf{f} = \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ f_r & r f_\phi & f_z \end{vmatrix},$$

$$5. \quad \nabla^2 \equiv \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

MF I 116

10.613 *Spherical polar coordinates.* In terms of the coordinates $O\{r, \theta, \phi\}$, that is, $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, where $x_1 = r \sin \theta \cos \phi$, $x_2 = r \sin \theta \sin \phi$, $x_3 = r \cos \theta$, for $0 \leq \theta \leq \pi$, $-\pi < \phi \leq \pi$, we have

$$1. \quad h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta,$$

and

$$2.^{10} \quad \operatorname{grad} \Phi = \frac{\partial \Phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \mathbf{e}_\phi,$$

$$3. \quad \operatorname{div} \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (f_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial f_\phi}{\partial \phi},$$

$$4. \quad \operatorname{curl} \mathbf{f} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_r & r f_\theta & r \sin \theta f_\phi \end{vmatrix},$$

$$5. \quad \nabla^2 \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

MF I 116

Special Orthogonal Curvilinear Coordinates and their Metrical Coefficients h_1, h_2, h_3 **10.614** *Elliptic cylinder coordinates $O\{u_1, u_2, u_3\}$.*

$$1. \quad x_1 = u_1 u_2, \quad x_2 = \sqrt{(u_1^2 - c^2)(1 - u_2^2)}, \quad x_3 = u_3$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{u_1^2 - c^2}}, \quad h_2 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{1 - u_2^2}}, \quad h_3 = 1$$

MF I 657

10.615 *Parabolic cylinder coordinates $O\{u_1, u_2, u_3\}$.*

$$1. \quad x_1 = \frac{1}{2}(u_1^2 - u_2^2), \quad x_2 = u_1 u_2, \quad x_3 = u_3$$

$$2. \quad h_1 = \sqrt{u_1^2 + u_2^2}, \quad h_2 = \sqrt{u_1^2 + u_2^2}, \quad h_3 = 1$$

MF I 658

10.616 *Conical coordinates $O\{u_1, u_2, u_3\}$.*

$$1. \quad x_1 = \frac{u_1}{a} \sqrt{(a^2 - u_2^2)(a^2 + u_3^2)}, \quad x_2 = \frac{u_1}{b} \sqrt{(b^2 + u_2^2)(b^2 - u_3^2)}, \quad x_3 = \frac{u_1 u_2 u_3}{ab}$$

$$\text{with } a^2 + b^2 = 1$$

$$2. \quad h_1 = 1, \quad h_2 = u_1 \sqrt{\frac{u_2^2 + u_3^2}{(a^2 - u_2^2)(b^2 + u_2^2)}}, \quad h_3 = u_1 \sqrt{\frac{u_2^2 + u_3^2}{(a^2 + u_2^2)(b^2 - u_2^2)}}$$

MF I 659

10.617 *Rotational parabolic coordinates $O\{u_1, u_2, u_3\}$.*

$$1. \quad x_1 = u_1 u_2 u_3, \quad x_2 = u_1 u_2 \sqrt{1 - u_3^2}, \quad x_3 = \frac{1}{2}(u_1^2 - u_2^2)$$

$$2. \quad h_1 = \sqrt{u_1^2 + u_2^2}, \quad h_2 = \sqrt{u_1^2 + u_2^2}, \quad h_3 = \frac{u_1 u_2}{\sqrt{1 - u_3^2}}$$

MF I 660

10.618 *Rotational prolate spheroidal coordinates $O\{u_1, u_2, u_3\}$.*

$$1. \quad x_1 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)}, \quad x_2 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)(1 - u_3^2)}, \quad x_3 = u_1 u_2$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{u_1^2 - a^2}}, \quad h_2 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{\frac{(u_1^2 - a^2)(1 - u_2^2)}{1 - u_3^2}}$$

MF I 661

10.619 *Rotational oblate spheroidal coordinates $O\{u_1, u_2, u_3\}$.*

$$1. \quad x_1 = u_3 \sqrt{(u_1^2 + a^2)(1 - u_2^2)}, \quad x_2 = \sqrt{(u_1^2 + a^2)(1 - u_2^2)(1 - u_3^2)}, \quad x_3 = u_1 u_2$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{u_1^2 + a^2}}, \quad h_2 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{\frac{(u_1^2 + a^2)(1 - u_2^2)}{1 - u_3^2}}$$

MF I 662

10.620 Ellipsoidal coordinates $O\{u_1, u_2, u_3\}$.

$$\begin{aligned}
 1. \quad x_1 &= \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2(a^2 - b^2)}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2(b^2 - a^2)}}, \quad x_3 = \frac{u_1 u_2 u_3}{ab} \\
 2. \quad h_1 &= \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \quad h_2 = \sqrt{\frac{(u_2^2 - u_1^2)(u_2^2 - u_3^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}
 \end{aligned}$$

MF I 663

10.621 Paraboloidal coordinates $O\{u_1, u_2, u_3\}$.

$$\begin{aligned}
 1. \quad x_1 &= \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2 - b^2}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2 - a^2}}, \\
 x_3 &= \frac{1}{2}(u_1^2 + u_2^2 + u_3^2 - a^2 - b^2) \\
 2. \quad h_1 &= \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \quad h_2 = u_2 \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = u_3 \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}
 \end{aligned}$$

MF I 664

10.622 Bispherical coordinates $O\{u_1, u_2, u_3\}$.

$$\begin{aligned}
 1. \quad x_1 &= au_3 \frac{\sqrt{1 - u_2^2}}{u_1 - u_2}, \quad x_2 = a \frac{\sqrt{(1 - u_2^2)(1 - u_3^2)}}{u_1 - u_2}, \quad x_3 = \frac{\sqrt{u_1^2 - 1}}{u_1 - u_2} \\
 2. \quad h_1 &= \frac{a}{(u_1 - u_2) \sqrt{u_1^2 - 1}}, \\
 h_2 &= \frac{a}{(u_1 - u_2) \sqrt{1 - u_2^2}}, \quad h_3 = \left(\frac{a}{u_1 - u_2} \right) \sqrt{\frac{1 - u_2^2}{1 - u_3^2}}
 \end{aligned}$$

MF I 665

10.71–10.72 Vector integral theorems

10.711 Gauss's divergence theorem. Let V be a volume bounded by a simple closed surface S and let $\mathbf{f}$ be a continuously differentiable vector field defined in V and on S . Then, if $d\mathbf{S}$ is the outward drawn vector element of area,

$$\int_S \mathbf{f} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{f} \, dV$$

KE 39

10.712 Green's theorems. Let Φ and Ψ be scalar fields which, together with $\nabla^2 \Phi$ and $\nabla^2 \Psi$, are defined both in a volume V and on its surface S , which we assume to be simple and closed. Then, if $\partial/\partial n$ denotes differentiation along the outward drawn normal to S , we have

10.713 Green's first theorem

$$\int_S \Phi \frac{\partial \Psi}{\partial n} \, dS = \int_V (\Phi \nabla^2 \Psi + \operatorname{grad} \Phi \cdot \operatorname{grad} \Psi) \, dV$$

KE 212

10.714 *Green's second theorem*

$$\int_S \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dS = \int_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) dV \quad \text{KE 215}$$

10.715 *Special cases*

$$\begin{aligned} 1. \quad & \int_S (\Phi \operatorname{grad} \Phi) \cdot d\mathbf{S} = \int_V \left(\Phi \nabla^2 \Phi + (\operatorname{grad} \Phi)^2 \right) dV \\ 2. \quad & \int_S \frac{\partial \Phi}{\partial n} dS = \int_V \nabla^2 \Phi dV \end{aligned} \quad \text{MV 81}$$

10.716 *Green's reciprocal theorem.* If Φ and Ψ are harmonic, so that $\nabla^2 \Phi = \nabla^2 \Psi = 0$, then

$$3. \quad \int_S \Phi \frac{\partial \Psi}{\partial n} dS = \int_S \Psi \frac{\partial \Phi}{\partial n} dS \quad \text{MM 105}$$

10.717 *Green's representation theorem.* If Φ and $\nabla^2 \Phi$ are defined within a volume V bounded by a simple closed surface S , and P is an interior point of V , then in three dimensions

$$4. \quad \Phi(P) = -\frac{1}{4\pi} \int_V \frac{1}{r} \nabla^2 \Phi dV + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS \quad \text{KE 219}$$

If Φ is harmonic within V , so that $\nabla^2 \Phi = 0$, then the previous result becomes

$$5. \quad \Phi(P) = \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS$$

In the case of two dimensions, result (4) takes the form

$$\begin{aligned} 6. \quad \Phi(p) = & \frac{1}{2\pi} \int_S \nabla^2 \Phi(q) \ln |p - q| dS \\ & + \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| dq - \frac{1}{2\pi} \int \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) dq \end{aligned} \quad \text{MM 116}$$

where C is the boundary of the planar region S , and result (5) takes the form

$$7. \quad \Phi(p) = \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| dq - \frac{1}{2\pi} \int_C \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) dq \quad \text{VL 280}$$

10.718 *Green's representation theorem in R^n .* If Φ is twice differentiable within a region Ω in R^n bounded by the surface Σ with outward drawn unit normal $\mathbf{n}$, then for $p \notin \Sigma$ and $n > 3$

$$\Phi(p) = \frac{-1}{(n-2)\sigma_n} \int_{\Omega} \frac{\nabla^2 \Phi(q)}{|p - q|^{n-2}} d\Omega_q + \frac{1}{(n-2)\sigma_n} \int_{\Sigma} \left(\frac{1}{|p - q|^{n-2}} \frac{\partial \Phi(q)}{\partial n_q} - \Phi(q) \frac{\partial}{\partial n_q} \frac{1}{|p - q|^{n-2}} \right) d\Sigma_q,$$

where

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{VL 279}$$

is the area of the unit sphere in R^n .

10.719 *Green's theorem of the arithmetic mean.* If Φ is harmonic in a sphere, then the value of Φ at the center of the sphere is the arithmetic mean of its value on the surface. KE 223**10.720** *Poisson's integral in three dimensions.* If Φ is harmonic in the interior of a spherical volume V of radius R and is continuous on the surface of the sphere on which, in terms of the spherical polar coordinates (r, θ, ϕ) , it satisfies the boundary condition $\Phi(R, \theta, \phi) = f(\theta, \phi)$, then

$$\Phi(r, \theta, \phi) = \frac{R(R^2 - r^2)}{4\pi} \int_0^\pi \int_{-\pi}^\pi \frac{f(\theta', \phi') \sin \theta' d\theta' d\phi'}{(r^2 + R^2 - 2rR \cos \gamma)^{3/2}},$$

where

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'). \quad \text{KE 241}$$

10.721 *Poisson's integral in two dimensions.* If Φ is harmonic in the interior of a circular disk S of radius R and is continuous on the boundary of the disk on which, in terms of the polar coordinates (r, θ) , it satisfies the boundary condition $\Phi(R, \theta) = f(\theta)$, then

$$\Phi(r, \theta) = \frac{(R^2 - r^2)}{2\pi} \int_{-\pi}^\pi \frac{f(\phi) d\phi}{r^2 + R^2 - 2rR \cos(\theta - \phi)}.$$

10.722 *Stokes' theorem.* Let a simple closed curve C be spanned by a surface S . Define the positive normal $\mathbf{n}$ to S , and the positive sense of description of the curve C with line element $d\mathbf{r}$, such that the positive sense of the contour C is clockwise when we look through the surface S in the direction of the normal. Then, if $\mathbf{f}$ is continuously differentiable vector field defined on S and C with vector element $\mathbf{S} = \mathbf{n} dS$,

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{f} \cdot d\mathbf{S}, \quad \text{MM 143}$$

where the line integral around C is taken in the positive sense.

10.723 *Planar case of Stokes' theorem.* If a region R in the (x, y) -plane is bounded by a simple closed curve C , and $f_1(x, y), f_2(x, y)$ are any two functions having continuous first derivatives in R and on C , then

$$\oint_C (f_1 dx + f_2 dy) = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy, \quad \text{MM 143}$$

where the line integral is taken in the counterclockwise sense.

10.81 Integral rate of change theorems

10.811 *Rate of change of volume integral bounded by a moving closed surface.* Let f be a continuous scalar function of position and time t defined throughout the volume $V(t)$, which is itself bounded by a simple closed surface $S(t)$ moving with velocity $\mathbf{v}$. Then the rate of change of the volume integral of f is given by

$$\frac{D}{Dt} \int_{V(t)} f dV = \int_{V(t)} \frac{\partial f}{\partial t} dV + \int_{S(t)} f \mathbf{v} \cdot d\mathbf{S},$$

where $d\mathbf{S}$ is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

By virtue of Gauss's theorem, this also takes the form

$$\frac{D}{Dt} \int_{V(t)} f dV = \int_{V(t)} \left(\frac{Df}{Dt} + f \text{div } \mathbf{v} \right) dV. \quad \text{MV 88}$$

10.812 *Rate of change of flux through a surface.* Let $\mathbf{q}$ be a vector function that may also depend on the time t , and $\mathbf{n}$ be the unit outward drawn normal to the surface S that moves with velocity $\mathbf{v}$. Defining the flux of $\mathbf{q}$ through S as

$$m = \int_S \mathbf{q} \cdot \mathbf{n} \, dS,$$

then

$$\frac{Dm}{Dt} = \int_S \left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \operatorname{div} \mathbf{q} + \operatorname{curl}(\mathbf{q} \times \mathbf{v}) \right) \cdot \mathbf{n} \, dS. \quad \text{MV 90}$$

10.813 *Rate of change of the circulation around a given moving curve.* Let C be a closed curve, moving with velocity $\mathbf{v}$, on which is defined a vector field $\mathbf{q}$. Defining the circulation ζ of $\mathbf{q}$ around C by

$$\zeta = \int_C \mathbf{q} \cdot d\mathbf{r},$$

then

$$\frac{D\zeta}{Dt} = \int_C \left(\frac{\partial \mathbf{q}}{\partial t} + (\operatorname{curl} \mathbf{q}) \times \mathbf{v} \right) \cdot d\mathbf{r}. \quad \text{MV 94}$$

11 Algebraic Inequalities

11.1–11.3 General Algebraic Inequalities

11.11 Algebraic inequalities involving real numbers

11.111 *Lagrange's identity.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two sets of real numbers; then

$$\left(\sum_{k=1}^n a_k b_k \right)^2 = \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) - \sum (a_k b_j - a_j b_k)^2 \quad \text{BB 3}$$

11.112 *Cauchy–Schwarz–Buniakowsky inequality.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two arbitrary sets of real numbers; then

$$\left(\sum_{k=1}^n a_k b_k \right)^2 \leq \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right).$$

The equality holds if, and only if, the sequences $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are proportional.

MT 30

11.113 *Minkowski's inequality.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two sets of nonnegative real numbers, and let $p > 1$; then

$$\left(\sum_{k=1}^n (a_k + b_k)^p \right)^{1/p} \leq \left(\sum_{k=1}^n a_k^p \right)^{1/p} + \left(\sum_{k=1}^n b_k^p \right)^{1/p}.$$

The equality holds if, and only if, the sequences $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are proportional.

MT 55

11.114 *Hölder's inequality.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two sets of nonnegative real numbers, and let $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$; then

$$\left(\sum_{k=1}^n a_k^p \right)^{1/p} \left(\sum_{k=1}^n b_k^q \right)^{1/q} \geq \sum_{k=1}^n a_k b_k.$$

The equality holds if, and only if, the sequences $a_1^p, a_2^p, \dots, a_n^p$ and $b_1^q, b_2^q, \dots, b_n^q$ are proportional.

MT 50

11.115 *Chebyshev's inequality.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two arbitrary sets of real numbers such that either $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$, or $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$; then

$$\left(\frac{a_1 + a_2 + \dots + a_n}{n} \right) \left(\frac{b_1 + b_2 + \dots + b_n}{n} \right) \leq \frac{1}{n} \sum_{k=1}^n a_k b_k.$$

The equality holds if, and only if, either $a_1 = a_2 = \dots = a_n$ or $b_1 = b_2 = \dots = b_n$.

11.116 Arithmetic-geometric inequality. Let $a_1, a_2, \dots, a_n$ be any set of positive numbers, with arithmetic mean

$$A_n = \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right)$$

and geometric mean

$$G_n = (a_1 a_2 \dots a_n)^{1/n};$$

then $A_n \geq G_n$ or, equivalently,

$$\left(\frac{a_1 + a_2 + \dots + a_n}{n} \right) \geq (a_1 a_2 \dots a_n)^{1/n}.$$

The equality holds only in the event that all of the numbers a_i are equal.

BB 4

11.117 Carleman's inequality. If $a_1, a_2, \dots, a_n$ is any finite set of non-negative numbers, then

$$\sum_{r=1}^n (a_1 a_2 \dots a_r)^{1/r} \leq e (a_1 + a_2 + \dots + a_n),$$

where e is the best possible constant in this inequality. The inequality is strict except for the trivial case when $a_r = 0$ for $r = 1, 2, \dots, n$.

MT 131

11.118 An inequality involving absolute values. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two arbitrary sets of real numbers; then

$$\sum_{i,j=1}^n \{|a_i - b_j|^p + |b_i - a_j|^p - |a_i - a_j|^p - |b_i - b_j|^p\} \geq 0, \quad 0 < p \leq 2.$$

11.21 Algebraic inequalities involving complex numbers

If α, β are any two real numbers, the complex number $z = \alpha + i\beta$ with real part α and imaginary part β has for its modulus $|z|$ the nonnegative number

$$|z| = \sqrt{\alpha^2 + \beta^2},$$

and for its argument (amplitude) $\arg z$ the angle $\arg z = \theta$ such that

$$\cos \theta = \frac{\alpha}{|z|} \text{ and } \sin \theta = \frac{\beta}{|z|},$$

where $-\pi < \theta \leq \pi$. The complex number $\bar{z} = \alpha - i\beta$ is said to be the **complex conjugate** of $z = \alpha + i\beta$.

$$\text{If } z = re^{i\theta} = r(\cos \theta + i \sin \theta),$$

then

$$z^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta),$$

and, setting $r = 1$, we have **de Moivre's theorem**

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

It follows directly that, if $z = e^{i\theta}$, then

$$\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right), \quad \sin \theta = -\frac{i}{2} \left(z - \frac{1}{z} \right),$$

and

$$\cos r\theta = \frac{1}{2} \left(z^r + \frac{1}{z^r} \right), \quad \sin r\theta = -\frac{i}{2} \left(z^r - \frac{1}{z^r} \right).$$

If $w = z^{p/q}$ with p, q integral, and $z = re^{i\theta}$, then the q roots of $w_0, w_1, \dots, w_{q-1}$ of z are

$$w_k = r^{p/q} \left[\cos \left(\frac{p\theta + 2k\pi}{q} \right) + i \sin \left(\frac{p\theta + 2k\pi}{q} \right) \right],$$

with $k = 0, 1, 2, \dots, q-1$.

11.211⁷ *Simple properties and inequalities involving the modulus and the complex conjugate.* If the real part of z is denoted by $\operatorname{Re} z$ and the imaginary part by $\operatorname{Im} z$, then

$$\begin{aligned} z + \bar{z} &= 2 \operatorname{Re} z = 2\alpha, \\ z - \bar{z} &= 2i \operatorname{Im} z = 2i\beta, \\ z &= \overline{(\bar{z})}, \\ \frac{1}{\bar{z}} &= \overline{\left(\frac{1}{z}\right)}, \\ \overline{(z^n)} &= (\bar{z})^n, \\ \left| \frac{\bar{z}_1}{\bar{z}_2} \right| &= \frac{|\bar{z}_1|}{|\bar{z}_2|}, \\ \overline{(z_1 + z_2 + \dots + z_n)} &= \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n, \\ \overline{z_1 z_2 \dots z_n} &= \bar{z}_1 \bar{z}_2 \dots \bar{z}_n. \end{aligned}$$

11.212 *Inequalities for pairs of complex numbers.* If a, b are any two complex numbers, then

- (i) $|a + b| \leq |a| + |b|$ (triangle inequality),
- (ii) $|a - b| \geq ||a| - |b||$.

11.31 Inequalities for sets of complex numbers

11.311 *Complex Cauchy–Schwarz–Buniakowsky inequality.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two arbitrary sets of complex numbers; then

$$\left| \sum_{k=1}^n a_k b_k \right|^2 \leq \left(\sum_{k=1}^n |a_k|^2 \right) \left(\sum_{k=1}^n |b_k|^2 \right).$$

The equality holds if, and only if, the sequences $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$ and $b_1, b_2, \dots, b_n$ are proportional.

MT 42

11.312 *Complex Minkowski inequality.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two arbitrary sets of complex numbers, and let the real number p be such that $p > 1$; then

$$\left(\sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p \right)^{1/p}.$$

MT 56

11.313 *Complex Hölder inequality.* Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two arbitrary sets of complex numbers, and let the real numbers p, q be such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \left(\sum_{k=1}^n |b_k|^q \right)^{1/q} \geq \left| \sum_{k=1}^n a_k b_k \right|.$$

The equality holds if, and only if, the sequences

$$|a_1|^p, |a_2|^p, \dots, |a_n|^p \text{ and } |b_1|^p, |b_2|^p, \dots, |b_n|^p,$$

are proportional and $\arg a_k b_k$ is independent of k for $k = 1, 2, \dots, n$.

MT 53

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12 Integral Inequalities

12.11 Mean Value Theorems

12.111 First mean value theorem

Let $f(x)$ and $g(x)$ be two bounded functions integrable in $[a, b]$, and let $g(x)$ be of one sign in this interval. Then

$$\int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx, \quad \text{CA 105}$$

with $a \leq \xi \leq b$.

12.112 Second mean value theorem

- (i) Let $f(x)$ be a bounded, monotonic decreasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx,$$

with $a \leq \xi \leq b$.

- (ii) Let $f(x)$ be a bounded, monotonic increasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x) dx = f(b) \int_\eta^b g(x) dx,$$

with $a \leq \eta \leq b$.

- (iii) Let $f(x)$ be bounded and monotonic in $[a, b]$, and let $g(x)$ be a bounded integrable function which experiences only a finite number of sign changes in $[a, b]$. Then,

$$\int_a^b f(x)g(x) dx = f(a+0) \int_a^\xi g(x) dx + f(b-0) \int_\xi^b g(x) dx, \quad \text{CA 107}$$

with $a \leq \xi \leq b$.

12.113 First mean value theorem for infinite integrals

Let $f(x)$ be bounded for $x \geq a$, and integrable in the arbitrary interval $[a, b]$, and let $g(x)$ be of one sign in $x \geq a$ and such that $\int_a^\infty g(x) dx$ is finite. Then,

$$\int_a^\infty f(x)g(x) dx = \mu \int_a^\infty g(x) dx, \quad \text{CA 123}$$

where $m \leq \mu \leq M$ and m, M are, respectively, the lower and upper bounds of $f(x)$ for $x \geq a$.

12.114 Second mean value theorem for infinite integrals

Let $f(x)$ be bounded and monotonic when $x \geq a$, and $g(x)$ be bounded and integrable in the arbitrary interval $[a, b]$ in which it experiences only a finite number of changes of sign. Then, provided $\int_a^\infty g(x) dx$ is finite,

$$\int_a^\infty f(x)g(x) dx = f(a+0) \int_a^\xi g(x) dx + f(\infty) \int_\xi^\infty g(x) dx, \quad \text{CA 123}$$

with $a \leq \xi \leq \infty$.

12.21 Differentiation of Definite Integral Containing a Parameter

12.211 Differentiation when limits are finite

Let $\phi(\alpha)$ and $\psi(\alpha)$ be twice differentiable functions in some interval $c \leq \alpha \leq d$, and let $f(x, \alpha)$ be both integrable with respect to x over the interval $\phi(\alpha) \leq x \leq \psi(\alpha)$ and differentiable with respect to α . Then,

$$\frac{d}{d\alpha} \int_{\phi(\alpha)}^{\psi(\alpha)} f(x, \alpha) dx = \left(\frac{d\psi}{d\alpha} \right) f(\psi(\alpha), \alpha) - \left(\frac{d\phi}{d\alpha} \right) f(\phi(\alpha), \alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial \alpha} dx. \quad \text{FI II 680}$$

12.212 Differentiation when a limit is infinite

Let $f(x, \alpha)$ and $\partial f / \partial \alpha$ both be integrable with respect to x over the semi-infinite region $x \geq a$, $b \leq \alpha < c$. Then, if the integral

$$f(\alpha) = \int_a^\infty f(x, \alpha) dx$$

exists for all $b \leq \alpha \leq c$, and if $\int_a^\infty \frac{\partial f}{\partial \alpha} dx$ is uniformly convergent for α in $[b, c]$, it follows that

$$\frac{d}{d\alpha} \int_a^\infty f(x, \alpha) dx = \int_a^\infty \frac{\partial f}{\partial \alpha} dx$$

12.31 Integral Inequalities

12.311 Cauchy-Schwarz-Buniakowsky inequality for integrals

Let $f(x)$ and $g(x)$ be any two real integrable functions on $[a, b]$. Then,

$$\left(\int_a^b f(x)g(x) dx \right)^2 \leq \left(\int_a^b f^2(x) dx \right) \left(\int_a^b g^2(x) dx \right),$$

and the equality will hold if, and only if, $f(x) = kg(x)$, with k real.

BB 21

12.312 Hölder's inequality for integrals

Let $f(x)$ and $g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^q$ are integrable on $[a, b]$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\int_a^b f(x)g(x) dx \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q}.$$

The equality holds if, and only if, $\alpha|f(x)|^p = \beta|g(x)|^q$, where α and β are positive constants.

BB 21

12.313 Minkowski's inequality for integrals

Let $f(x)$ and $g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^p$ are integrable on $[a, b]$ for $p > 0$; then

$$\left(\int_a^b |f(x) + g(x)|^p dx \right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} + \left(\int_a^b |g(x)|^p dx \right)^{1/p}.$$

The equality holds if, and only if, $f(x) = kg(x)$ for some real $k \geq 0$.

BB 21

12.314 Chebyshev's inequality for integrals

Let $f_1, f_2, \dots, f_n$ be nonnegative integrable functions on $[a, b]$ which are all either monotonic increasing or monotonic decreasing; then

$$\int_a^b f_1(x) dx \int_a^b f_2(x) dx \dots \int_a^b f_n(x) dx \leq (b-a)^{n-1} \int_a^b f_1(x)f_2(x) \dots f_n(x) dx$$

MT 39

12.315 Young's inequality for integrals

Let $f(x)$ be a real-valued continuous strictly monotonic increasing function on the interval $[0, a]$, with $f(0) = 0$ and $b \leq f(a)$. Then

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy,$$

where $f^{-1}(y)$ denotes the function inverse to $f(x)$. The equality holds if, and only if, $b = f(a)$.

BB 15

12.316 Steffensen's inequality for integrals

Let $f(x)$ be nonnegative and monotonic decreasing in $[a, b]$, and $g(x)$ be such that $0 \leq g(x) \leq 1$ in $[a, b]$. Then

$$\int_{b-k}^b f(x) dx \leq \int_a^b f(x)g(x) dx \leq \int_a^{a+k} f(x) dx,$$

where $k = \int_a^b g(x) dx$.

MT 107

12.317 Gram's inequality for integrals

Let $f_1(x), f_2(x), \dots, f_n(x)$ be real square integrable functions on $[a, b]$; then

$$\begin{vmatrix} \int_a^b f_1^2(x) dx & \int_a^b f_1(x)f_2(x) dx & \cdots & \int_a^b f_1(x)f_n(x) dx \\ \int_a^b f_2(x)f_1(x) dx & \int_a^b f_2^2(x) dx & \cdots & \int_a^b f_2(x)f_n(x) dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x)f_1(x) dx & \int_a^b f_n(x)f_2(x) dx & \cdots & \int_a^b f_n^2(x) dx \end{vmatrix} \geq 0.$$

MT 47

12.318 Ostrowski's inequality for integrals

Let $f(x)$ be a monotonic function integrable on $[a, b]$, and let $f(a)f(b) \geq 0, |f(a)| \geq |f(b)|$. Then, if g is a real function integrable on $[a, b]$,

$$\left| \int_a^b f(x)g(x) dx \right| \leq |f(a)| \max_{a \leq \xi \leq b} \left| \int_a^\xi g(x) dx \right|.$$

12.41 Convexity and Jensen's Inequality

A function $f(x)$ is said to be **convex** on an interval $[a, b]$ if for any two points x_1, x_2 in $[a, b]$

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

A function $f(x)$ is said to be **concave** on an interval $[a, b]$ if for any two points x_1, x_2 in $[a, b]$ the function $-f(x)$ is convex in that interval.

If the function $f(x)$ possesses a second derivative in the interval $[a, b]$, then a necessary and sufficient condition for it to be convex on that interval is that $f''(x) \geq 0$ for all x in $[a, b]$.

A function $f(x)$ is said to be **logarithmically convex** on the interval $[a, b]$ if $f > 0$ and $\log f(x)$ is concave on $[a, b]$.

If $f(x)$ and $g(x)$ are logarithmically convex on the interval $[a, b]$, then the functions $f(x) + g(x)$ and $f(x)g(x)$ are also logarithmically convex on $[a, b]$. MT 17

12.411 Jensen's inequality

Let $f(x), p(x)$ be two functions defined for $a \leq x \leq b$ such that $\alpha \leq f(x) \leq \beta$ and $p(x) \geq 0$, with $p(x) \not\equiv 0$. Let $\phi(u)$ be a convex function defined on the interval $\alpha \leq u \leq \beta$; then

$$\phi\left(\frac{\int_a^b f(x)p(x) dx}{\int_a^b p(x) dx}\right) \leq \frac{\int_a^b \phi(f(x))p(x) dx}{\int_a^b p(x) dx}. \quad \text{HL 151}$$

12.412 Carleman's inequality for integrals

If $f(x) \geq 0$ and the integrals exist, then

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x f(t) dt\right) dx \leq e \int_0^\infty f(x) dx.$$

12.51 Fourier Series and Related Inequalities

The trigonometric **Fourier series** representation of the function $f(x)$ integrable on $[-\pi, \pi]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where the **Fourier coefficients** a_n and b_n of $f(x)$ are given by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

(See **0.320–0.328** for convergence of Fourier series on $(-l, l)$.)

TF 1

12.511 Riemann-Lebesgue lemma

If $f(x)$ is integrable on $[-\pi, \pi]$, then

$$\lim_{t \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin tx \, dx \rightarrow 0$$

and

$$\lim_{t \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \cos tx \, dx \rightarrow 0. \quad \text{TF 11}$$

12.512 Dirichlet lemma

$$\int_0^{\pi} \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x} \, dx = \frac{\pi}{2},$$

in which $\sin(n + \frac{1}{2})x / 2 \sin \frac{1}{2}x$ is called the **Dirichlet kernel**. ZY 21

12.513 Parseval's theorem for trigonometric Fourier series

If $f(x)$ is square integrable on $[-\pi, \pi]$, then

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx. \quad \text{Y 10}$$

12.514 Integral representation of the n^{th} partial sum

If $f(x)$ is integrable on $[-\pi, \pi]$, then the n^{th} partial sum

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^n (a_r \cos rx + b_r \sin rx)$$

has the following integral representation in terms of the Dirichlet kernel:

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \, dt. \quad \text{Y 20}$$

12.515 Generalized Fourier series

Let the set of functions $\{\phi_n\}_{n=0}^{\infty}$ form an **orthonormal set** over $[a, b]$, so that

$$\int_a^b \phi_m(x) \phi_n(x) \, dx = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}$$

Then the **generalized Fourier series** representation of an integrable function $f(x)$ on $[a, b]$ is

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x),$$

where the generalized Fourier coefficients of $f(x)$ are given by

$$c_n = \int_a^b f(x) \phi_n(x) \, dx.$$

12.516 Bessel's inequality for generalized Fourier series

For any square integrable function defined on $[a, b]$,

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x) dx,$$

where the c_n are the generalized Fourier coefficients of $f(x)$.

12.517 Parseval's theorem for generalized Fourier series

If $f(x)$ is a square integrable function defined on $[a, b]$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ is a **complete orthonormal** set of continuous functions defined on $[a, b]$, then

$$\sum_{n=0}^{\infty} c_n^2 = \int_a^b f^2(x) dx,$$

where the c_n are generalized Fourier coefficients of $f(x)$.

13 Matrices and Related Results

13.11–13.12 Special Matrices

13.111 Diagonal matrix

A square matrix $\mathbf{A}$ of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix}$$

in which all entries away from the **leading diagonal** are zero.

13.112 Identity matrix and null matrix

The **identity matrix** is a diagonal matrix $\mathbf{I}$ in which all entries in the leading diagonal are unity. The **null matrix** is all zeros.

13.113 Reducible and irreducible matrices

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is said to be **reducible**, if the indices $1, 2, \dots, n$ can be divided into two disjoint non-empty sets $i_1, i_2, \dots, i_\mu; j_1, j_2, \dots, j_\nu$ with $(\mu + \nu = n)$, such that

$$a_{i_\alpha j_\beta} = 0 \quad (\alpha = 1, 2, \dots, \mu; \quad \beta = 1, 2, \dots, \nu).$$

Otherwise, $\mathbf{A}$ will be said to be irreducible.

GA 61

13.114 Equivalent matrices

An $m \times n$ matrix $\mathbf{A}$ is **equivalent** to an $m \times n$ matrix $\mathbf{B}$ if, and only if, $\mathbf{B} = \mathbf{PAQ}$ for suitable non-singular $m \times m$ and $n \times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$, respectively.

13.115 Transpose of a matrix

If $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix with element a_{ij} in the i^{th} row and the j^{th} column, then the transpose $\mathbf{A}^T$ of $\mathbf{A}$ is the $n \times m$ matrix

$$\mathbf{A}^T = [b_{ij}] \quad \text{with} \quad b_{ij} = a_{ji},$$

that is, the matrix derived from $\mathbf{A}$ by interchanging rows and columns.

13.116 Adjoint matrix

If $\mathbf{A}$ is an $n \times n$ matrix, then its **adjoint**, denoted by $\text{adj } \mathbf{A}$, is the transpose of the matrix of cofactors A_{ij} of $\mathbf{A}$, so that

$$\text{adj } \mathbf{A} = [A_{ij}]^T \quad (\text{see } \mathbf{14.13}).$$

13.117 Inverse matrix

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with a nonsingular determinant $|\mathbf{A}|$, then its **inverse** $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}.$$

13.118 Trace of a matrix

The trace of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, written $\text{tr } \mathbf{A}$, is defined to be the sum of the terms on the leading diagonal, so that

$$\text{tr } \mathbf{A} = a_{11} + a_{22} + \dots + a_{nn}.$$

13.119 Symmetric matrix

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **symmetric** if $a_{ij} = a_{ji}$ for $i, j = 1, 2, \dots, n$.

13.120 Skew-symmetric matrix

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **skew-symmetric** if $a_{ij} = -a_{ji}$ for $i, j = 1, 2, \dots, n$.

13.121 Triangular matrices

An $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is of **upper triangular type** if $a_{ij} = 0$ for $i > j$ and of **lower triangular type** if $a_{ij} = 0$ for $j > i$.

13.122 Orthogonal matrices

A real $n \times n$ matrix $\mathbf{A}$ is **orthogonal** if, and only if, $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.

13.123 Hermitian transpose of a matrix

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with complex elements, then its **hermitian transpose** $\mathbf{A}^H$ is defined to be

$$\mathbf{A}^H = [\bar{a}_{ji}],$$

with the bar denoting the complex conjugate operation.

13.124 Hermitian matrix

An $n \times n$ matrix $\mathbf{A}$ is **hermitian** if $\mathbf{A} = \mathbf{A}^H$, or equivalently, if $\mathbf{A} = \overline{\mathbf{A}}^T$, with the bar denoting the complex conjugate operation.

13.125 Unitary matrix

An $n \times n$ matrix $\mathbf{A}$ is **unitary** if $\mathbf{A}\mathbf{A}^H = \mathbf{A}^H\mathbf{A} = \mathbf{I}$.

13.126 Eigenvalues and eigenvectors

If $\mathbf{A}$ is an $n \times n$ matrix, each eigenvector $\mathbf{x}$ corresponding to λ satisfies the equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

while the **eigenvalues** λ satisfy the **characteristic equation**

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (\text{see } 15.61).$$

13.127 Nilpotent matrix

An $n \times n$ matrix $\mathbf{A}$ is **nilpotent** if $\mathbf{A}^k = \mathbf{0}$ for some k .

13.128 Idempotent matrix

An $n \times n$ matrix $\mathbf{A}$ is **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.

13.129 Positive definite

An $n \times n$ matrix $\mathbf{A}$ is **positive definite** if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.130 Non-negative definite

An $n \times n$ matrix $\mathbf{A}$ is **non-negative definite** if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.131 Diagonally dominant

An $n \times n$ matrix $\mathbf{A}$ is **diagonally dominant** if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i .

13.21 Quadratic Forms

A **quadratic form** involving the n real variables $x_1, x_2, \dots, x_n$ that are associated with the real $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the scalar expression

$$Q(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j.$$

In terms of matrix notation, if $\mathbf{x}$ is the $n \times 1$ column vector with real elements $x_1, x_2, \dots, x_n$, and $\mathbf{x}^T$ is the transpose of $\mathbf{x}$, then

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}.$$

Employing the inner product notation, this same quadratic form may also be written

$$Q(\mathbf{x}) \equiv (\mathbf{x}, \mathbf{A}\mathbf{x}).$$

If the $n \times n$ matrix $\mathbf{A}$ is hermitian, so that $\overline{\mathbf{A}}^T = \mathbf{A}$, where the bar denotes the complex conjugate operation, then the quadratic form associated with the hermitian matrix $\mathbf{A}$ and the vector $\mathbf{x}$, which may have complex elements, is the real quadratic form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax}).$$

It is always possible to express an arbitrary quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} x_i x_j$$

in the form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax}),$$

where $\mathbf{A} = [a_{ij}]$ is a symmetric matrix, by defining

$$a_{ii} = \alpha_{ii} \quad \text{for } i = 1, 2, \dots, n$$

and

$$a_{ij} = \frac{1}{2} (\alpha_{ij} + \alpha_{ji}) \quad \text{for } i, j = 1, 2, \dots, n \quad \text{and } i \neq j.$$

13.211 Sylvester's law of inertia

When a quadratic form Q in n variables is reduced by a nonsingular linear transformation to the form

$$Q = y_1^2 + y_2^2 + \dots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \dots - y_r^2,$$

the number p of positive squares appearing in the reduction is an invariant of the quadratic form Q , and it does not depend on the method of reduction itself. ML 377

13.212 Rank

The **rank** of the quadratic form Q in the above canonical form is the total number r of squared terms (both positive and negative) appearing in its reduced form. ML 360

13.213 Signature

The **signature** of the quadratic form Q above is the number s of positive squared terms appearing in its reduced form. It is sometimes also defined to be $2s - r$. ML 378

13.214 Positive definite and semidefinite quadratic form

The quadratic form $Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$ is said to be **positive definite** when $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$. It is said to be **positive semidefinite** if $Q(x) \geq 0$ for $x \neq 0$. ML 394

13.215 Basic theorems on quadratic forms

1. Two real quadratic forms are **equivalent** under the group of linear transformations if, and only if, they have the same rank and the same signature.
2. A real quadratic form in n variables is positive definite if, and only if, its canonical form is

$$Q = z_1^2 + z_2^2 + \dots + z_n^2.$$

3. A real symmetric matrix $\mathbf{A}$ is positive definite if, and only if, there exists a real nonsingular matrix $\mathbf{M}$ such that $\mathbf{A} = \mathbf{MM}^T$.
4. Any real quadratic form in n variables may be reduced to the diagonal form

$$Q = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \dots + \lambda_n z_n^2, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

by a suitable orthogonal point-transformation.

5. The quadratic form $Q = (\mathbf{x}, \mathbf{A}\mathbf{x})$ is positive definite if, and only if, every eigenvalue of $\mathbf{A}$ is positive; it is positive semidefinite if, and only if, all the eigenvalues of $\mathbf{A}$ are nonnegative, and it is indefinite if the eigenvalues of $\mathbf{A}$ are of both signs.
6. The necessary conditions for an hermitian matrix $\mathbf{A}$ to be positive definite are
 - (i) $a_{ii} > 0$ for all i ,
 - (ii) $a_{ii}a_{ij} > |a_{ij}|^2$ for $i \neq j$,
 - (iii) the element of largest modulus must lie on the leading diagonal,
 - (iv) $|\mathbf{A}| > 0$.
7. The quadratic form $Q = (\mathbf{x}, \mathbf{A}\mathbf{x})$ with $\mathbf{A}$ hermitian will be positive definite if all the principal minors in the top left-hand corner of $\mathbf{A}$ are positive, so that

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \dots \quad \text{ML 353-379}$$

13.31 Differentiation of Matrices

If the $n \times m$ matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ have elements that are differentiable functions of t , so that

$$\mathbf{A}(t) = [a_{ij}(t)], \quad \mathbf{B}(t) = [b_{ij}(t)]$$

then

1. $\frac{d}{dt} \mathbf{A}(t) = \left[\frac{d}{dt} a_{ij}(t) \right]$
2. $\frac{d}{dt} [\mathbf{A}(t) \pm \mathbf{B}(t)] = \left[\frac{d}{dt} a_{ij}(t) \pm \frac{d}{dt} b_{ij}(t) \right]$
 $= \frac{d}{dt} \mathbf{A}(t) \pm \frac{d}{dt} \mathbf{B}(t).$
3. If the matrix product $\mathbf{A}(t)\mathbf{B}(t)$ is defined, then

$$\frac{d}{dt} [\mathbf{A}(t)\mathbf{B}(t)] = \left(\frac{d}{dt} \mathbf{A}(t) \right) \mathbf{B}(t) + \mathbf{A}(t) \left(\frac{d}{dt} \mathbf{B}(t) \right).$$
4. If the matrix product $\mathbf{A}(t)\mathbf{B}(t)$ is defined, then

$$\frac{d}{dt} [\mathbf{A}(t)\mathbf{B}(t)]^T = \left(\frac{d}{dt} \mathbf{B}(t) \right)^T \mathbf{A}^T(t) + \mathbf{B}^T(t) \left(\frac{d}{dt} \mathbf{A}(t) \right)^T.$$
5. If the square matrix $\mathbf{A}$ is nonsingular, so that $|\mathbf{A}| \neq 0$, then

$$\frac{d}{dt} [\mathbf{A}^{-1}] = -\mathbf{A}^{-1}(t) \left(\frac{d}{dt} \mathbf{A}(t) \right) \mathbf{A}^{-1}(t)$$
6. $\int_{t_0}^T \mathbf{A}(\tau) d\tau = \left[\int_{t_0}^T a_{ij}(\tau) d\tau \right]$

13.41 The Matrix Exponential

If $\mathbf{A}$ is a square matrix, and z is any complex number, then the matrix exponential $e^{\mathbf{A}z}$ is defined to be

$$e^{\mathbf{A}z} = \mathbf{I} + \mathbf{A}z + \dots + \frac{\mathbf{A}^n z^n}{n!} + \dots = \sum_{r=0}^{\infty} \frac{1}{r!} \mathbf{A}^r z^r.$$

3.411 Basic properties

$$1. \quad e^0 = \mathbf{I}, \quad e^{\mathbf{I}z} = \mathbf{I}e^z, \quad e^{\mathbf{A}(z_1+z_2)} = e^{\mathbf{A}z_1} \cdot e^{\mathbf{A}z_2}, \quad [\text{when } \mathbf{A} + \mathbf{B} \text{ is defined and } \mathbf{AB} = \mathbf{BA}]$$

$$e^{-\mathbf{A}z} = (e^{\mathbf{A}z})^{-1}, \quad e^{\mathbf{A}z} \cdot e^{\mathbf{B}z} = e^{(\mathbf{A}+\mathbf{B})z}$$

$$2. \quad \frac{d^r}{dz^r} (e^{\mathbf{A}z}) = \mathbf{A}^r e^{\mathbf{A}z} = e^{\mathbf{A}z} \mathbf{A}^r.$$

ML 340

$$3. \quad \text{If the square matrix } \mathbf{A} \text{ can be expressed in the form } \mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}, \text{ with } \mathbf{B} \text{ and } \mathbf{C} \text{ square matrices, then}$$

$$e^{\mathbf{A}z} = \begin{bmatrix} e^{\mathbf{B}z} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{C}z} \end{bmatrix}.$$

14 Determinants

14.11 Expansion of Second- and Third-Order Determinants

1. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$
2. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$

14.12 Basic Properties

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $n \times n$ matrices. Then the following results are true:

1. If any two adjacent rows (or columns) of a square matrix are interchanged, then the sign of the associated determinant is changed.
2. If any two rows (or columns) of a determinant are identical, the determinant is zero.
3. A determinant is not changed in value if any multiple of a row (or column) is added to any other row (or column).
4. $|k\mathbf{A}| = k^n|\mathbf{A}|$ for any scalar k .
5. $|\mathbf{A}^T| = |\mathbf{A}|$ where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$.
6. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|.$
7. $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$ when the inverse exists.
8. If the elements a_{ij} of $\mathbf{A}$ are functions of x , then

$$\frac{d|\mathbf{A}|}{dx} = \sum_{i,j=1}^n \frac{da_{ij}}{dx} A_{ij} \quad (\text{see 14.13}).$$

14.13 Minors and Cofactors of a Determinant

The **minor** M_{ij} of the element a_{ij} in the n^{th} -order determinant $|\mathbf{A}|$ associated with the square $n \times n$ matrix $\mathbf{A}$ is the $(n - 1)^{\text{th}}$ -order determinant derived from $\mathbf{A}$ by deletion of the i^{th} row and j^{th} column. The cofactor A_{ij} of the element a_{ij} is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

ML 20

14.14 Principal Minors

A **principal minor** is one whose elements are situated symmetrically with respect to the leading diagonal of $\mathbf{A}$. ML 197

14.15* Laplace Expansion of a Determinant

The n^{th} -order determinant denoted by $|\mathbf{A}|$, or $\det \mathbf{A}$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ may be expanded either by elements of the i^{th} row as

$$|\mathbf{A}| = \sum_{j=1}^n a_{ij} A_{ij},$$

or by elements of the j^{th} column as

$$|\mathbf{A}| = \sum_{i=1}^n a_{ij} A_{ij},$$

where A_{ij} is the cofactor of element a_{ij} . The cofactors A_{ij} satisfy the following n linear equations:

$$\sum_{j=1}^n a_{ij} A_{kj} = \delta_{ik} |\mathbf{A}|, \quad \sum_{i=1}^n a_{ij} A_{ik} = \delta_{jk} |\mathbf{A}|,$$

ML 21

$$\text{for } i, j, k = 1, 2, \dots, n \text{ and } \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

14.16 Jacobi's Theorem

Let M_r be an r -rowed minor of the n^{th} -order determinant $|\mathbf{A}|$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, in which the rows $i_1, i_2, \dots, i_r$ are represented together with the columns $k_1, k_2, \dots, k_r$.

Define the **complementary minor** to M_r to be the $(n-r)$ -rowed minor obtained from $|\mathbf{A}|$ by deleting all the rows and columns associated with M_r , and the **signed complementary minor** $M^{(r)}$ to M_r to be

$$M^{(r)} = (-1)^{i_1+i_2+\dots+i_r+k_1+k_2+\dots+k_r} \times (\text{complementary minor to } M_r).$$

Then, if Δ is the matrix of cofactors given by

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix},$$

and M_r and M'_r are corresponding r -rowed minors of $|\mathbf{A}|$ and Δ , it follows that

$$M'_r = |\mathbf{A}|^{r-1} M^{(r)}.$$

ML 25

Corollary. If $|\mathbf{A}| = 0$, then

$$A_{pk} A_{nq} = A_{nk} A_{pq}.$$

14.17 Hadamard's Theorem

If $|\mathbf{A}|$ is an $n \times n$ determinant with elements a_{ij} that may be complex, then $|\mathbf{A}| \neq 0$ if

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|.$$

14.18 Hadamard's Inequality

Let $\mathbf{A} = [a_{ij}]$ be an arbitrary $n \times n$ nonsingular matrix with real elements and determinant $|\mathbf{A}|$. Then

$$|\mathbf{A}|^2 \leq \prod_{i=1}^n \left(\sum_{k=1}^n a_{ik}^2 \right).$$

This result is also true when $\mathbf{A}$ is hermitian.

ML 418

Deductions.

1. If $M = \max |a_{ij}|$, then

$$|\mathbf{A}| \leq M^n n^{n/2}.$$

ML 419

2. If the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is positive definite, then

$$|\mathbf{A}| \leq a_{11}a_{22} \dots a_{nn}.$$

BL 126

3. If the real $n \times n$ matrix $\mathbf{A}$ is diagonally dominant, so that $\sum_{j \neq i}^n |a_{ij}| < |a_{ii}|$ for $i = 1, 2, \dots, n$, then $|\mathbf{A}| \neq 0$.

14.21 Cramer's Rule

If the n linear equations

$$\begin{array}{cccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n, \end{array}$$

have a nonsingular coefficient matrix $\mathbf{A} = [a_{ij}]$, so that $|\mathbf{A}| \neq 0$, then there is a unique solution

$$x_j = \frac{A_{1j}b_1 + A_{2j}b_2 + \cdots + A_{nj}b_n}{|\mathbf{A}|}$$

for $j = 1, 2, \dots, n$, where A_{ij} is the cofactor of element a_{ij} in the coefficient matrix $\mathbf{A}$.

ML 134

14.31 Some Special Determinants

14.311 Vandermonde's determinant (alternant)

Third order.

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

and, in general, the n^{th} -order Vandermonde's determinant is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

where the right-hand side is the continued product of all the differences that can be formed from the $\frac{1}{2}n(n-1)$ pairs of numbers taken from $x_1, x_2, \dots, x_n$, with the order of the differences taken in the reverse order of the suffixes that are involved. ML 17

14.312 Circulants

Second order.

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix} = (x_1 + x_2)(x_1 - x_2).$$

Third order.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2 x_3)(x_1 + \omega^2 x_2 + \omega x_3),$$

where ω and ω^2 are the complex cube roots of 1. In general, the n^{th} -order circulant determinant is

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{vmatrix} = \prod_{j=1}^n (x_1 + x_2 \omega_j + x_3 \omega_j^2 + \cdots + x_n \omega_j^{n-1}),$$

where ω_j is an n^{th} root of 1. The eigenvalues λ (see **15.61**) of an $n \times n$ circulant matrix are

$$\lambda_j = x_1 + x_2 \omega_j + x_3 \omega_j^2 + \cdots + x_n \omega_j^{n-1},$$

where ω_j is again an n^{th} root of 1. ML 36

14.313 Jacobian determinant

If $f_1, f_2, \dots, f_n$ are n real-valued functions which are differentiable with respect to $x_1, x_2, \dots, x_n$, then the Jacobian $J_f(x)$ of the f_i with respect to the x_j is the determinant

$$J_f(x) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

The notation

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}$$

is also used to denote the Jacobian $J_f(x)$.

14.314 Hessian determinants

The Jacobian of the derivatives $\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \dots, \frac{\partial\phi}{\partial x_n}$ of a function $\phi(x_1, x_2, \dots, x_n)$ with respect to $x_1, x_2, \dots, x_n$ is called the Hessian H of ϕ , so that

$$H = \begin{vmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1\partial x_2} & \frac{\partial^2\phi}{\partial x_1\partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_1\partial x_n} \\ \frac{\partial^2\phi}{\partial x_2\partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} & \frac{\partial^2\phi}{\partial x_2\partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_2\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\phi}{\partial x_n\partial x_1} & \frac{\partial^2\phi}{\partial x_n\partial x_2} & \frac{\partial^2\phi}{\partial x_n\partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_n^2} \end{vmatrix}.$$

14.315 Wronskian determinants

Let $f_1, f_2, \dots, f_n$ be n functions each n times differentiable with respect to x in some open interval (a, b) . Then the Wronskian $W(x)$ of $f_1, f_2, \dots, f_n$ is defined by

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_n^{(1)} \\ f_1^{(2)} & f_2^{(2)} & \cdots & f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where $f_i^{(r)} = \frac{d^r f_i}{dx^r}$.

14.316 Properties

1. $\frac{dW}{dx}$ follows from $W(x)$ by replacing the last row of the determinant defining $W(x)$ by the n^{th} derivatives $f_1^{(n)}, f_2^{(n)}, \dots, f_n^{(n)}$.
2. If constants $k_1, k_2, \dots, k_n$ exist, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$$

for all x in (a, b) , then $W(x) = 0$ for all x in (a, b) .

3. The vanishing of the Wronskian throughout (a, b) is necessary, but not sufficient, for the linear dependence of $f_1, f_2, \dots, f_n$.

14.317 Gram-Kowalewski theorem on linear dependence

A necessary and sufficient condition for n functions $f_1, f_2, \dots, f_n$ square integrable over $a \leq x \leq b$ to be linearly dependent in this interval is the vanishing of the Gram determinant

$$G(f_1, f_2, \dots, f_n) = \begin{vmatrix} \int_a^b f_1^2(x) dx & \int_a^b f_1(x)f_2(x) dx & \cdots & \int_a^b f_1(x)f_n(x) dx \\ \int_a^b f_2(x)f_1(x) dx & \int_a^b f_2^2(x) dx & \cdots & \int_a^b f_2(x)f_n(x) dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x)f_1(x) dx & \int_a^b f_n(x)f_2(x) dx & \cdots & \int_a^b f_n^2(x) dx \end{vmatrix}. \quad \text{SA 2 (Theorem 3)}$$

14.318 If the n functions $f_1, f_2, \dots, f_n$ are square integrable over $a \leq x \leq b$, then the Gram determinant

$$G(f_1, f_2, \dots, f_n) \geq 0,$$

and the equality sign holds only when the functions are linearly dependent in $a \leq x \leq b$.

SA 4 (Corollary 1)

14.319 The rank of the matrix corresponding to the Gram determinant $G(f_1, f_2, \dots, f_n)$ gives the maximum number of linearly independent functions $f_1, f_2, \dots, f_n$ in $a \leq x \leq b$. If the rank is r , then r of the functions are linearly independent, and the other $n - r$ functions are linearly dependent on these.

SA 3 (Theorem 4)

15 Norms

15.1–15.9 Vector Norms

15.11 General Properties

The **vector norm** $||\mathbf{x}||$ of an $n \times 1$ column vector $\mathbf{x}$ is a nonnegative number having the property that

1. $||\mathbf{x}|| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $||\mathbf{x}|| = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$;
2. $||k\mathbf{x}|| = |k| ||\mathbf{x}||$ for any scalar k ;
3. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

15.21 Principal Vector Norms

15.211 The norm $||\mathbf{x}||_1$

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \dots, x_n$, then

$$||\mathbf{x}||_1 = \sum_{r=1}^n |x_r|. \quad \text{VA 15}$$

15.212 The norm $||\mathbf{x}||_2$ (Euclidean or L_2 norm)

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \dots, x_n$, then

$$||\mathbf{x}||_2 = \left(\sum_{r=1}^n |x_r|^2 \right)^{1/2}. \quad \text{VA 8}$$

15.213 The norm $||\mathbf{x}||_\infty$

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \dots, x_n$, then

$$||\mathbf{x}||_\infty = \max_i |x_i|. \quad \text{VA 15}$$

15.31 Matrix Norms

15.311 General properties

The **matrix norm** $\|\mathbf{A}\|$ of a square matrix $\mathbf{A}$ is a nonnegative number associated with $\mathbf{A}$ having the properties that

1. $\|\mathbf{A}\| > 0$ when $\mathbf{A} \neq \mathbf{0}$ and $\|\mathbf{A}\| = 0$ if, and only if, $\mathbf{A} = \mathbf{0}$;
2. $\|k\mathbf{A}\| = |k| \|\mathbf{A}\|$ for any scalar k ;
3. $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$;
4. $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$. VA 9

The matrix norm $\|\mathbf{A}\|$ associated with $\mathbf{A} = [a_{ij}]$, and the vector norm $\|\mathbf{x}\|$ associated with the column vector $\mathbf{x}$ for which the matrix product $\mathbf{Ax}$ is defined, are said to be **compatible** if

$$\|\mathbf{Ax}\| \leq \|\mathbf{A}\| \|\mathbf{x}\|.$$

15.312 Induced norms

When a vector $\mathbf{z}$ with norm $\|\mathbf{z}\|$ exists such that the maximum is attained in the expression

$$\|\mathbf{A}\| = \max_{\|\mathbf{z}\|=1} \|\mathbf{Az}\|,$$

then $\|\mathbf{A}\|$ is a matrix norm and is said to be the **natural norm induced** by, or **subordinate** to, the vector norm $\|\mathbf{z}\|$. NO 428

15.313 Natural norm of unit matrix

If $\mathbf{I}$ is the unit matrix, then for any natural norm

$$\|\mathbf{I}\| = 1. \quad \text{NO 429}$$

15.41 Principal Natural Norms

The natural matrix norms induced on matrix $\mathbf{A} = [a_{ij}]$ by the 1, 2, and ∞ vector norms are as follows:

15.411 Maximum absolute column sum norm

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}| \quad \text{NO 429}$$

15.412 Spectral norm

If $\mathbf{A}^H$ denotes the Hermitian transpose of the square matrix $\mathbf{A} = [a_{ij}]$, so that $\mathbf{A}^H = [\overline{a_{ji}}]$ with a bar denoting the complex conjugate operation, then

$$\|\mathbf{A}\|_2 = \sqrt{\text{maximum eigenvalue of } \mathbf{A}^H \mathbf{A}},$$

or, equivalently,

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2 \neq 0} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2}. \quad \text{NO 429}$$

15.413 Maximum absolute row sum norm

$$\|\mathbf{A}\|_{\infty} = \max_i \sum_{j=1}^n |a_{ij}| \quad \text{NO 429}$$

15.51 Spectral Radius of a Square Matrix

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix with elements that may be complex, and with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the **spectral radius** $\rho(\mathbf{A})$ of $\mathbf{A}$ is the number

$$\rho(\mathbf{A}) = \max_{1 \leq i \leq n} |\lambda_i|. \quad \text{VA 9}$$

15.511 Inequalities concerning matrix norms and the spectral radius

1. $\|\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_1 \|\mathbf{A}\|_{\infty}$. NO 431
2. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, and the $n \times n$ matrix $\mathbf{U}$ is unitary, so that $\mathbf{U}^H = \mathbf{U}^{-1}$, with ^{H} denoting the Hermitian transpose of $\mathbf{A}$ (see 13.123), then

$$\|\mathbf{A}\mathbf{U}\| = \|\mathbf{U}\mathbf{A}\| = \|\mathbf{A}\|. \quad \text{VA 15}$$

3. If $\mathbf{A}$ is any nonsingular $n \times n$ matrix with elements that may be complex with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$\frac{1}{\|\mathbf{A}^{-1}\|} \leq |\lambda| \leq \|\mathbf{A}\|. \quad \text{VA 16}$$

4. For any square matrix $\mathbf{A}$ with spectral radius $\rho(\mathbf{A})$ and any natural norm $\|\mathbf{A}\|$,

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|. \quad \text{NO 430}$$

5. If the square matrix $\mathbf{A}$ is Hermitian, then

$$\rho(\mathbf{A}) = \|\mathbf{A}\|.$$

6. If the square matrix $\mathbf{A}$ is Hermitian and $P_m(x)$ is any polynomial of degree m with real coefficients, then

$$\|P_m(\mathbf{A})\| = \rho(P_m(\mathbf{A})).$$

7. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, then the sequence of matrices $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \dots$ converges to the null matrix as $n \rightarrow \infty$ if, and only if, $\rho(\mathbf{A}) < 1$.

NO 303

15.512 Deductions from Gerschgorin's theorem (see 15.814)

1. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex; then $\rho(\mathbf{A}) \leq \min \left(\max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|, \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \right)$. VA 17

2. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex, and $x_1, x_2, \dots, x_n$ be any set of n positive numbers; then $\rho(\mathbf{A}) \leq \min \left(\max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^n |a_{ij}| x_j}{x_i} \right), \max_{1 \leq j \leq n} \left(x_j \sum_{i=1}^n \frac{|a_{ij}|}{x_i} \right) \right)$.

VA 18

15.61 Inequalities Involving Eigenvalues of Matrices

The **eigenvalues** (**characteristic values** or **latent roots**) λ of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ are the solutions to the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0.$$

When expanded, the determinant $|\mathbf{A} - \lambda \mathbf{I}|$ is called the **characteristic polynomial**, and it has the form

$$|\mathbf{A} - \lambda \mathbf{I}| = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + c_{n-2} \lambda^{n-2} + \dots + c_1 \lambda + c_0.$$

The zeros of this polynomial satisfy the characteristic equation and so are the eigenvalues of $\mathbf{A}$. In the characteristic polynomial the coefficients have the form

$$c_{n-r} = (-1)^{n-r} \quad (\text{sum of all principal minors of } |\mathbf{A}| \text{ of order } r).$$

It then follows that

$$b_{n-1} = (-1)^n (a_{11} + a_{22} + \dots + a_{nn}),$$

$$b_{n-2} = (-1)^n \sum_{i < j} (a_{ii} a_{jj} - a_{ij} a_{ji}),$$

$$b_0 = |\mathbf{A}|.$$

Since the sum of the elements of the leading diagonal of $\mathbf{A}$ is called the **trace** of $\mathbf{A}$, written $\text{tr } \mathbf{A}$, it follows that $b_{n-1} = (-1)^n \text{tr } \mathbf{A}$.

ML 198

15.611 Cayley-Hamilton theorem

Every square matrix $\mathbf{A}$ satisfies its characteristic equation, so that

$$(-1)^n \mathbf{A}^n + c_{n-1} \mathbf{A}^{n-1} + c_{n-2} \mathbf{A}^{n-2} + \dots + c_1 \mathbf{A} + c_0 \mathbf{I} = \mathbf{0}.$$

ML 206

15.612 Corollaries

1. If $\mathbf{A}$ is nonsingular, then its adjoint, denoted by $\text{adj } \mathbf{A}$, is

$$\text{adj } \mathbf{A} = - \left[(-1)^n \mathbf{A}^{n-1} + c_{n-1} \mathbf{A}^{n-2} + c_{n-2} \mathbf{A}^{n-3} + \dots + c_2 \mathbf{A} + c_1 \mathbf{I} \right].$$

2. If $\mathbf{A}$ is nonsingular, then the characteristic polynomial of $\mathbf{A}^{-1}$ is

$$(-1)^n \left(\lambda^n + \frac{c_1}{|\mathbf{A}|} \lambda^{n-1} + \frac{c_2}{|\mathbf{A}|} \lambda^{n-2} + \dots + \frac{(-1)^n}{|\mathbf{A}|} \right).$$

15.71 Inequalities for the Characteristic Polynomial

The first group of inequalities that follow, which relate to the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ whose elements may be complex, refer directly to the coefficients of the polynomial when written in the form

$P(\lambda) \equiv |\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \cdots + b_{n-1} \lambda + b_n$,
and only implicitly to the coefficients a_{ij} of $\mathbf{A}$ that give rise to the b_i .

15.711 Named and unnamed inequalities

The first group of inequalities relating to the eigenvalues λ satisfying $P(\lambda) = 0$ are unnamed and are as follows:

1. All the eigenvalues λ lie within or on the circle $||z|| \leq r$, where r is the positive root of .

$$|b_n| + |b_{n-1}|z + |b_{n-2}|z^2 + \cdots + |b_1|z^{n-1} - z^n = 0 \quad \text{MG 122}$$

2. All the eigenvalues λ lie within the circle

$$|z| < 1 + \max_i |b_i|. \quad \text{MG 123}$$

3. When $b_n \neq 0$ the eigenvalue λ of smallest modulus lies in the annulus $R \leq |z| \leq \frac{R}{2^{1/n} - 1}$, where R is the positive root of

$$|b_n| - |b_{n-1}|z - |b_{n-2}|z^2 - \cdots - z^n = 0. \quad \text{MG 126}$$

4. All the eigenvalues λ lie on or outside the circle

$$|z| = \min_k \left[\frac{|b_n|}{(|b_n| + |b_k|)} \right]. \quad \text{MG 126}$$

5. If the eigenvalues λ are ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > 1 \geq |\lambda_{p+1}| \geq \cdots \geq |\lambda_n|,$$

then

$$|z_1 z_2 \cdots z_p| \leq N, \quad |z_p| \leq N^{\frac{1}{p}},$$

where

$$N^2 = 1 + |b_1|^2 + |b_2|^2 + \cdots + |b_n|^2. \quad \text{MG 129}$$

6. All the eigenvalues λ lie in or on the circle

$$|z| \leq \sum_{j=1}^n |b_j|^{1/j}. \quad \text{MG 126}$$

7. All the eigenvalues λ lie on the disk

$$\left| z + \frac{b_1}{2} \right| \leq \left| \frac{b_1}{2} \right| + |b_2|^{1/2} + |b_3|^{1/3} + \cdots + |b_n|^{1/n}. \quad \text{MG 145}$$

8. All the eigenvalues λ lie in the annulus $m \leq ||z|| \leq M$, where

$$m^2 = \max \left\{ 0, \min_{1 \leq j \leq n-1} \left[1 - |b_j|, |b_n|^2 \right] \right\}$$

and

$$M^2 = \max \left\{ 1 + |b_j|, |b_n|^2 + 2 \sum_{j=1}^{n-1} |b_j|^2 \right\}.$$

The next group of inequalities are named theorems that apply to the explicit form of the characteristic polynomial $P(\lambda)$. MG 145

15.712 Parodi's theorem

The eigenvalues λ satisfying $P(\lambda) = 0$ lie in the union of the disks

$$|z| \leq 1, \quad |z + b_1| \leq \sum_{j=1}^n |b_j|. \quad \text{MG 143}$$

15.713 Corollary of Brauer's theorem

If

$$|b_1| > 1 + \sum_{j=2}^n |b_j|,$$

then one and only one eigenvalue satisfying $P(\lambda) = 0$ lies on the disk

$$|z + b_1| \leq \sum_{j=2}^n |b_j|. \quad \text{MG 141}$$

15.714 Ballieu's theorem

For any set $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ of positive numbers, let $\mu_0 = 0$ and

$$M_\mu = \max_{0 \leq k \leq n-1} \left[\frac{\mu_k + \mu_n |b_{n-k}|}{\mu_{k+1}} \right].$$

Then all the eigenvalues satisfying $P(\lambda) = 0$ lie on the disk $||z|| \leq M_\mu$. MG 144

15.715 Routh-Hurwitz theorem

Consider the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1 \lambda^{n-1} + \dots + b_{n-1} \lambda + b_n = 0$$

determining the n eigenvalues λ of the real $n \times n$ matrix $\mathbf{A}$. Then the eigenvalues λ all have negative real parts if

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \dots, \quad \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \dots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \dots & b_k \end{vmatrix}. \quad \text{GM 230}$$

15.81–15.82 Named Theorems on Eigenvalues

In the following theorems involving eigenvalue inequalities the elements a_{ij} of matrix $\mathbf{A}$ enter directly, and not in the form of the coefficients of the characteristic polynomial.

15.811 Schur's inequalities

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with elements that may be complex, and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$\begin{aligned} 1. \quad & \sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2 \\ 2. \quad & \sum_{i=1}^n |\operatorname{Re} \lambda_i|^2 \leq \sum_{i,j=1}^n \left| \frac{a_{ij} + \overline{a_{ji}}}{2} \right|^2 \\ 3. \quad & \sum_{i=1}^n |\operatorname{Im} \lambda_i|^2 \leq \sum_{i,j=1}^n \left| \frac{a_{ij} - \overline{a_{ji}}}{2} \right|^2 \end{aligned} \quad \text{ML 309}$$

15.812 Sturmian separation theorem

Let $\mathbf{A}_r = [a_{ij}]$ with $i, j = 1, 2, \dots, r$ and $r = 1, 2, \dots, N$ be a sequence of N symmetric matrices of increasing order. Then if $\lambda_k(\mathbf{A}_r)$ for $k = 1, 2, \dots, r$ denotes the k^{th} eigenvalue of $\mathbf{A}_r$, where the ordering is such that

$$\lambda_1(\mathbf{A}_r) \geq \lambda_2(\mathbf{A}_r) \geq \dots \geq \lambda_r(\mathbf{A}_r),$$

it follows that

$$\lambda_{k+1}(\mathbf{A}_{i+1}) \leq \lambda_k(\mathbf{A}_i) \leq \lambda_k(\mathbf{A}_{i+1}). \quad \text{BL 115}$$

15.813 Poincare's separation theorem

Let $\{\mathbf{y}^k\}$, with $k = 1, 2, \dots, K$, be a set of orthonormal vectors so that the inner product $(\mathbf{y}^k, \mathbf{y}^k) = 1$. Set

$$\mathbf{x} = \sum_{k=1}^K u_k \mathbf{y}^k,$$

so that for any square matrix $\mathbf{A}$ for which the product $\mathbf{A}\mathbf{x}$ is defined, the quadratic form

$$(\mathbf{x}, \mathbf{A}\mathbf{x}) = \sum_{k,l=1}^K u_k u_l (\mathbf{y}^k, \mathbf{A}\mathbf{y}^l).$$

Then if

$$\mathbf{b}_K = (\mathbf{y}^k, \mathbf{A}\mathbf{y}^l) \text{ for } k, l = 1, 2, \dots, K,$$

it follows that

$$\begin{aligned} \lambda_i(\mathbf{b}_K) &\leq \lambda_i(\mathbf{A}) && \text{for } i = 1, 2, \dots, K, \\ \lambda_{K-j}(\mathbf{b}_K) &\geq \lambda_{N-j}(\mathbf{A}) && \text{for } j = 0, 1, 2, \dots, K-1. \end{aligned} \quad \text{BL 117}$$

15.814 Gerschgorin's theorem

Let $\mathbf{A} = [a_{ij}]$ be any arbitrary $n \times n$ matrix with elements that may be complex, and let

$$\Lambda_i \equiv \sum_{j=1, i \neq j}^n |a_{ij}| \text{ for } i = 1, 2, \dots, n.$$

Then all of the eigenvalues λ_i of $\mathbf{A}$ lie in the union of the n disks Γ_i , where

$$\Gamma_i : |z - a_{ii}| \leq \Lambda_i \text{ for } i = 1, 2, \dots, n. \quad \text{VA 16}$$

15.815 Brauer's theorem

If in Gerschgorin's theorem for a given m

$$|a_{jj} - a_{mm}| \geq \Lambda_j + \Lambda_m$$

for all $j \neq m$, then one and only one eigenvalue of $\mathbf{A}$ lies in the disk Γ_m .

MG 141

15.816 Perron's theorem

If $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$ is an arbitrary set of positive numbers, then all the eigenvalues λ of the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ lie on the disk $|z| \leq \mathbf{M}_{\boldsymbol{\mu}}$, where

$$\mathbf{M}_{\boldsymbol{\mu}} = \max_{1 \leq i \leq n} \sum_{j=1}^n \frac{\mu_j}{\mu_i} |a_{ij}|. \quad \text{MG 141}$$

15.817 Frobenius theorem

If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients, so that $a_{ij} > 0$ for all $i, j = 1, 2, \dots, n$, then $\mathbf{A}$ has a positive eigenvalue λ_0 , and all its eigenvalues lie on the disk

$$|z| \leq \lambda_0. \quad \text{MG 142}$$

15.818 Perron–Frobenius theorem

If all elements a_{ij} of an irreducible matrix $\mathbf{A}$ are nonnegative, then $R = \min M_{\boldsymbol{\lambda}}$ is a simple eigenvalue of $\mathbf{A}$, and all the eigenvalues of $\mathbf{A}$ lie on the disk $|z| \leq R$, where, if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a set of nonnegative numbers, not all zero,

$$M_{\boldsymbol{\lambda}} = \inf \left\{ \mu : \mu \lambda_i > \sum_{j=1}^n |a_{ij}| \lambda_j, 1 \leq i \leq n \right\}$$

and $R = \min M_{\boldsymbol{\lambda}}$.

Furthermore, if $\mathbf{A}$ has exactly p eigenvalues ($p \leq n$) on the circle $|z| = R$, then the set of all its eigenvalues is invariant under rotations $2\pi/p$ about the origin.

GM 69

15.819 Wielandt's theorem

If the $n \times n$ matrix $\mathbf{A}$ satisfies the conditions of the Perron–Frobenius theorem and if in the $n \times n$ matrix $\mathbf{C} = [c_{ij}]$

$$|c_{ij}| \leq a_{ij}, \quad i, j = 1, 2, \dots, n,$$

then any eigenvalue λ_0 of $\mathbf{C}$ satisfies the inequality $|\lambda_0| \leq R$. The equality sign holds only when there exists an $n \times n$ matrix $\mathbf{D} = [\pm \delta_{ij}]$ such that $\delta_{ii} = 1$ for all i , $\delta_{ij} = 0$ for all $i \neq j$, and

$$\mathbf{C} = (\lambda_0/R) \mathbf{D} \mathbf{A} \mathbf{D}^{-1}.$$

GM 69

15.820 Ostrowski's theorem

If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients and λ_0 is the positive eigenvalue in Frobenius' theorem, then the $n - 1$ eigenvalues $\lambda_j \neq \lambda_0$ satisfy the inequality

$$|\lambda_j| \leq \lambda_0 \frac{M^2 - m^2}{M^2 + m^2},$$

where

$$M = \max a_{ij}, \quad m = \min a_{ij} \quad \text{for } i, j = 1, 2, \dots, n.$$

MG 145

15.821 First theorem due to Lyapunov

In order that all the eigenvalues of the real $n \times n$ matrix $\mathbf{A}$ have negative real parts, it is necessary and sufficient that if $\mathbf{V}$ is an $n \times n$ matrix, the equation

$$\mathbf{A}^T \mathbf{V} + \mathbf{V} \mathbf{A} = -\mathbf{I}$$

has as a solution the matrix of coefficients $\mathbf{V}$ of some positive-definite quadratic form $(\mathbf{x}, \mathbf{V}\mathbf{x})$ (see 13.21).

GM 224

15.822 Second theorem due to Lyapunov

If all the eigenvalues of the real matrix $\mathbf{A}$ have negative real parts, then to an arbitrary negative-definite quadratic form $(\mathbf{x}, \mathbf{W}\mathbf{x})$ with $\mathbf{x} = \mathbf{x}(t)$ there corresponds a positive-definite quadratic form $(\mathbf{x}, \mathbf{V}\mathbf{x})$ such that if one takes

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

then $(\mathbf{x}, \mathbf{V}\mathbf{x})$ and $(\mathbf{x}, \mathbf{W}\mathbf{x})$ satisfy

$$\frac{d}{dt} (\mathbf{x}, \mathbf{V}\mathbf{x}) = (\mathbf{x}, \mathbf{W}\mathbf{x}).$$

Conversely, if for some negative-definite form $(\mathbf{x}, \mathbf{W}\mathbf{x})$ there exists a positive-definite form $(\mathbf{x}, \mathbf{V}\mathbf{x})$ connected to $(\mathbf{x}, \mathbf{W}\mathbf{x})$ by the preceding two equations, then all the eigenvalues of $\mathbf{A}$ have negative real parts (see 13.21, 13.31).

GM 222

15.823 Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov

1. The off-diagonal Hermitian matrix $\mathbf{A}$ of rank n whose elements are given by

$$a_{jk} = (1 - \delta_{jk}) \left\{ 1 + i \cot \left[\frac{(j-k)\pi}{n} \right] \right\},$$

has the integer eigenvalues

$$\lambda_s^{(a)} = 2s - n - 1 \text{ for } s = 1, 2, \dots, n,$$

and the corresponding eigenvectors $v^{(s)}$ have the components

$$v_j^{(s)} = \exp\left(-\frac{2\pi i s j}{n}\right) \quad \text{for } j = 1, 2, \dots, n.$$

2. The two off-diagonal Hermitian matrices $\mathbf{B}$ and $\mathbf{C}$ whose elements are defined by the formulas

$$b_{jk} = (1 - \delta_{jk}) \sin^{-2} \left[\frac{(j-k)\pi}{n} \right],$$

$$c_{jk} = (1 - \delta_{jk}) \sin^{-4} \left[\frac{(j-k)\pi}{n} \right],$$

are related to the matrix $\mathbf{A}$ in (1) by the equations

$$\mathbf{B} = \frac{1}{2} \left(\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)} \mathbf{I} \right),$$

$$\mathbf{C} = -\frac{1}{6} \left(\mathbf{B}^2 - 2 \left(2 + \sigma_n^{(1)} \right) \mathbf{B} - \sigma_n^{(2)} \mathbf{I} \right),$$

where $\mathbf{I}$ is the unit matrix and

$$\sigma_n^{(1)} = \frac{1}{3} (n^2 - 1), \quad \sigma_n^{(2)} = \frac{1}{45} (n^2 - 1) (n^2 + 11).$$

The eigenvalues of $\mathbf{B}$ and $\mathbf{C}$ corresponding to the eigenvector $v_j^{(s)}$ in (1) have the form

$$\lambda_s^{(b)} = \sigma_n^{(1)} - 2s(n-s) \quad \text{for } s = 1, 2, \dots, n,$$

$$\lambda_s^{(c)} = \sigma_n^{(2)} - 2s(n-s) \frac{s(n-s) + 2}{3} \quad \text{for } s = 1, 2, \dots, n.$$

3. Together, the above two results imply the following diophantine summation rules:

$$(a) \quad \sum_{k=1}^{n-1} \cot\left(\frac{k\pi}{n}\right) \sin\left(\frac{2sk\pi}{n}\right) = n - 2s \quad \text{for } s = 1, 2, \dots, n-1$$

$$(b) \quad \sum_{k=1}^{n-1} \sin^{-2}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = b_s \quad \text{for } s = 1, 2, \dots, n-1,$$

$$(c) \quad \sum_{k=1}^{n-1} \sin^{-4}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = c_s \quad \text{for } s = 1, 2, \dots, n-1,$$

$$(d) \quad \sum_{k=1}^{n-1} \sin^{-2p}\left(\frac{k\pi}{n}\right) = \sigma_n^{(p)},$$

with $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ as defined in (2), and

$$\sigma_n^{(3)} = \sigma_n^{(1)} \frac{2n^4 + 23n^2 + 191}{315}, \quad \sigma_n^{(4)} = \sigma_n^{(2)} \frac{3n^4 + 10n^2 + 227}{315}$$

$$b_s = \sigma_n^{(1)} - 2s(n-s), \quad c_s = \sigma_n^{(2)} - \frac{2}{3}s(n-s)[s(n-s) + 2].$$

15.91 Variational Principles

15.911 Rayleigh quotient

If $\mathbf{A}$ is an Hermitian matrix, the Rayleigh quotient $\rho(\mathbf{x})$ is the expression

$$\rho(\mathbf{x}) = \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}. \quad \text{NO 407}$$

15.912 Basic theorems

1. If the $n \times n$ matrix A is Hermitian and has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then

$$\lambda_1 \leq \rho \leq \lambda_n,$$

where ρ is the Rayleigh quotient for any $\mathbf{x} \neq \mathbf{0}$, and

$$\lambda_1 = \min_{\mathbf{x} \neq \mathbf{0}} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \quad \text{and} \quad \lambda_n = \max_{\mathbf{x} \neq \mathbf{0}} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}. \quad \text{NO 407}$$

2. If the $n \times n$ matrix $\mathbf{A}$ is Hermitian and has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ corresponding to the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, respectively, and $\mathbf{x} \neq \mathbf{0}$ is such that

$$(\mathbf{x}, \mathbf{x}_1) = (\mathbf{x}, \mathbf{x}_2) = \cdots = (\mathbf{x}, \mathbf{x}_n) = 0,$$

then

$$\lambda_j = \min_{\mathbf{x}} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})},$$

and

$$\lambda_j \leq \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \leq \lambda_n. \quad \text{NO 410}$$

3. If the $n \times n$ matrix $\mathbf{A}$ is Hermitian, then the eigenvalue

$$\lambda_r = \max \left(\min \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \right),$$

where first the minimum over $\mathbf{x}$ is taken subject to $(\mathbf{b}_i, \mathbf{x}) = 0, i = 1, 2, \dots, r-1$, with the $\mathbf{b}_i$ regarded as fixed vectors, and then the maximum over all possible $\mathbf{b}_i$. Also, the eigenvalue

$$\lambda_r = \min \left(\max \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \right),$$

where now the maximum over $\mathbf{x}$ is taken first subject to $(\mathbf{b}_i, \mathbf{x}) = 0, i = r+1, r+2, \dots, n$ for fixed $\mathbf{b}_i$, and then the minimum over all possible $\mathbf{b}_i$. NO 414

4. The $(n-1)$ eigenvalues $\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}$ obtained from the $(n-1) \times (n-1)$ matrix derived from an Hermitian matrix $\mathbf{A}$ from which the last row and column have been omitted separate the n eigenvalues of $\mathbf{A}$, so that

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \cdots < \lambda'_{n-1} < \lambda_n \quad (\text{see } \mathbf{15.812}).$$

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16 Ordinary Differential Equations

16.1–16.9 Results Relating to the Solution of Ordinary Differential Equations

16.11 First-Order Equations

16.111 Solution of a first-order equation

Consider the real function $f(t, x)$ that is defined and continuous in an open set $D \subset R^2$. Then a **solution** to the first-order differential equation

$$\frac{dx}{dt} = f(t, x)$$

in the open interval $I \subset R$ is a real function $u(t)$ that is defined and is both continuous and differentiable in I , with the property that

- (i) $(t, u(t)) \in D$ for $t \in I$,
- (ii) $\frac{du}{dt} = f(t, u(t))$ for $t \in I$.

16.112 Cauchy problem

The **Cauchy problem** for the differential equation

$$\frac{dx}{dt} = f(t, x)$$

is the problem of existence and uniqueness of the solution to this equation satisfying the initial condition

$$u(t_0) = x_0,$$

where $(t_0, u(t_0)) \in D$, the open set defined above. The solution to the initial value problem may be expressed in the form of the integral equation

$$u(t) = x_0 + \int_{t_0}^t f(\tau, u(\tau)) d\tau \quad (\text{see } \mathbf{16.316}).$$

16.113 Approximate solution to an equation

The real function $\phi(t)$ is said to be an **approximate solution**, to within the error ϵ , of the differential equation

$$\frac{dx}{dt} = f(t, x)$$

if ϕ' is piecewise continuous, and for a given $\epsilon > 0$ and an open interval $I \subset R$,

$$|\phi'(t) - f(t, \phi(t))| \leq \epsilon,$$

except at points of discontinuity of the derivative.

HU 3

16.114 Lipschitz continuity of a function

The real function $f(t, x)$ defined and continuous in some open set $D \subset R^2$ is said to be **Lipschitz continuous** with respect to x for some constant $k > 0$ if, for all points (t, x_1) and (t, x_2) belonging to D

$$|f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|.$$

HU 5

16.21 Fundamental Inequalities and Related Results

16.211 Gronwall's lemma

Let the three piecewise continuous, non-negative functions u, v , and w be defined in the interval $[0, a]$ and satisfy the inequality

$$w(t) \leq u(t) + \int_0^t v(\tau)w(\tau) d\tau,$$

except at points of discontinuity of the functions. Then, except at these same points,

$$w(t) \leq u(t) + \int_0^t u(\tau)v(\tau) \exp\left(\int_\tau^t v(\sigma) d\sigma\right) d\tau.$$

BB 135

16.212 Comparison of approximate solutions of a differential equation

Let f be a real function that is defined in an open set $D \subset R^2$, in which it is both continuous and Lipschitz continuous. In addition, let u_1 and u_2 be two approximate solutions of

$$\frac{dx}{dt} = f(t, x)$$

in an open set $I \subset R$ in the sense already defined, with

$$|u_1'(t) - f(t, u_1(t))| \leq \epsilon_1, \quad |u_2'(t) - f(t, u_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$|u_1(t_0) - u_2(t_0)| \leq \delta,$$

it follows that

$$|u_1(t) - u_2(t)| \leq \delta \exp\{|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right) [\exp\{k|t - t_0|\} - 1].$$

HU 6

16.31 First-Order Systems

16.311 Solution of a system of equations

The **system** of n first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(t, x_1, x_2, \dots, x_n), \\ \frac{dx_2}{dt} &= f_2(t, x_1, x_2, \dots, x_n), \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, x_2, \dots, x_n),\end{aligned}$$

in which the functions $f_1, f_2, \dots, f_n$ are real and continuous in an open set $D \subset R^{n+1}$, may be written in the concise matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

where $\mathbf{x}$ and $\mathbf{f}$ are $n \times 1$ column vectors. Its solution in the open interval $I \subset R$ is the vector $\mathbf{u}(t)$ with elements $u_1(t), u_2(t), \dots, u_n(t)$ with the property that

$$(i) \quad (t, \mathbf{u}(t)) \in D \text{ for } t \in I,$$

$$(ii) \quad \frac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}(t)) \text{ for } t \in I.$$

HU 24

16.312 Cauchy problem for a system

The **Cauchy problem** for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

is the problem of existence and uniqueness of the solution to this system satisfying the **initial vector condition**

$$\mathbf{u}(t_0) = \mathbf{x}_0,$$

where $(t_0, \mathbf{u}(t_0)) \in D$, the open set defined above in connection with the system. The solution to the initial value problem may be expressed in the form of the **vector integral equation**

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(\tau, \mathbf{u}(\tau)) d\tau.$$

16.313 Approximate solution to a system

The real vector $\phi(t)$ is said to be an **approximate vector solution**, to within the order ϵ , of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

if the elements of ϕ' are piecewise continuous, and for a given $\epsilon > 0$ and open interval $I \subset R$,

$$\|\phi'(t) - \mathbf{f}(t, \phi(t))\| \leq \epsilon,$$

except at points of discontinuity of the derivative, where $\|\mathbf{w}\|$ denotes the supremum norm

$$\|\mathbf{w}\| = \sup(|w_1|, |w_2|, \dots, |w_n|).$$

HU 25

16.314 Lipschitz continuity of a vector

The real vector $\mathbf{f}(t, x)$ defined and continuous in some open set $D \subset R^n$ is said to be **Lipschitz continuous** with respect to x for some constant $k > 0$ if, for all points $(t, \mathbf{x}_1), (t, \mathbf{x}_2)$ belonging to D ,

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq k\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

HU 26

16.315 Comparison of approximate solutions of a system

Let $\mathbf{f}$ be a real vector defined in an open set $D \subset R \times R^n$ in which it is both continuous and Lipschitz continuous. In addition, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two approximate solutions of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

in an open set $I \subset R$ in the sense already defined, with

$$|\mathbf{u}'_1(t) - \mathbf{f}(t, \mathbf{u}_1(t))| \leq \epsilon_1, \quad |\mathbf{u}'_2(t) - \mathbf{f}(t, \mathbf{u}_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$\|\mathbf{u}_1(t_0) - \mathbf{u}_2(t_0)\| \leq \delta,$$

it follows that

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \leq \delta \exp\{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right) [\exp\{k|t - t_0|\} - 1]. \quad \text{HU 27}$$

16.316 First-order linear differential equation

The **first-order linear differential equation** when expressed in the canonical form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

has an integrating factor

$$\mu(t) = \exp\left(\int P(t) dt\right),$$

and a general solution

$$y(t) = \frac{1}{\mu(t)} \left(\mu(t_0) y_0 + \int_{t_0}^t \mu(\xi) Q(\xi) d\xi \right),$$

where $y_0 = y(t_0)$.

16.317 Linear systems of differential equations

Consider the **homogeneous system** of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

where $\mathbf{x}$ is an $n \times 1$ column vector and $\mathbf{A}(t)$ an $n \times n$ matrix. Then a **fundamental system** of solutions of this system is a set of n linearly independent solution vectors $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$. The square matrix $\mathbf{K}(t)$ whose columns comprise the vectors $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$ is called the **fundamental matrix** of the differential equation, and we have the representation

$$|\mathbf{K}(t)| = |\mathbf{K}(t_0)| \exp\left(\int_{t_0}^t \text{tr } \mathbf{A}(\tau) d\tau\right).$$

Using the fundamental matrix $\mathbf{K}(t)$ defined in terms of the homogeneous system, the unique solution to the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t),$$

assuming the initial value $\mathbf{x}(t_0) = \mathbf{x}_0$, is

$$\phi(t) = \mathbf{K}(t)[\mathbf{K}(t_0)]^{-1}\mathbf{x}_0 + \mathbf{K}(t) \int_{t_0}^t [\mathbf{K}(\tau)]^{-1}\mathbf{b}(\tau) d\tau, \quad \text{HU 43}$$

where $\mathbf{b}(t)$ is an $n \times 1$ column vector. CL 69

16.41 Some Special Types of Elementary Differential Equations

16.411 Variables separable

A first-order differential equation is said to be **variables separable** if it is of the form

$$\frac{dy}{dx} = M(x)N(y),$$

or

$$P(x)Q(y) dx + R(x)S(y) dy = 0.$$

It may then be written in the form

$$M(x) dx - \frac{1}{N(y)} dy = 0,$$

or

$$\frac{P(x)}{R(x)} dx + \frac{S(y)}{Q(y)} dy = 0,$$

provided $R(x)Q(y) \neq 0$.

16.412 Exact differential equations

A differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **exact** if there exists a function $h(x, y)$ such that

$$d[h(x, y)] = M(x, y) dx + N(x, y) dy.$$

IN 16

16.413 Conditions for an exact equation

A necessary and sufficient condition that an equation of this form is exact is that the functions $M(x, y)$ and $N(x, y)$ together with their partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ exist and are continuous in a region in which

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

IN 16

16.414 Homogeneous differential equations

A differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **algebraically homogeneous** if, for arbitrary k ,

$$\frac{M(kx, ky)}{N(kx, ky)} = \frac{M(x, y)}{N(x, y)}.$$

Setting $y = sx$, it may then be expressed in the form

$$[M(1, s) + sN(1, s)] dx + xN(1, s) ds = 0,$$

in which the variables s and x are separable.

IN 18

16.51 Second-Order Equations

16.511 Adjoint and self-adjoint equations

The linear second-order differential equation

$$L(u) \equiv a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x)u = 0$$

has associated with it the adjoint equation

$$M(v) \equiv \frac{d^2}{dx^2} [a(x)v] - \frac{d}{dx} [b(x)v] + c(x)v = 0.$$

The equation $L(u) = 0$ is said to be **self-adjoint** if $L(u) \equiv M(u)$.

A linear self-adjoint second-order differential equation defined on $[\alpha, \beta]$ can always be expressed in the form

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = 0,$$

where $p(x)$ and $q(x)$ are continuous on $[\alpha, \beta]$ and $p(x) > 0$. The general equation $L(u) = 0$ can always be made self-adjoint and written in this form by multiplication by the factor

$$\frac{1}{a(x)} \left[\exp \int \frac{b(x)}{a(x)} dx \right],$$

when

$$p(x) = \exp \int \frac{b(x)}{a(x)} dx \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)} \left[\exp \int \frac{b(x)}{a(x)} dx \right].$$

In general, if

$$L(u) = p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u,$$

then its adjoint is

$$M(v) = (-1)^n \frac{d^n}{dx^n} [p_0 v] + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} [p_1 v] + \dots - \frac{d}{dx} [p_{n-1} v] + p_n v. \quad \text{HI 391}$$

16.512 Abel's identity

If $p(x)$ and $q(x)$ are continuous in $[\alpha, \beta]$ in which $p(x) > 0$, and $u(x)$ and $v(x)$ are suitably differentiable with

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = 0,$$

then the result

$$p(x) \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \equiv \text{const.}$$

is known as **Abel's identity**.

More generally, if we consider the linear n^{th} -order equation

$$p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u = 0,$$

and Δ is the Wronskian of a (fundamental) set of linearly independent solutions $u_1, u_2, \dots, u_n$, the Abel identity takes the form

$$\Delta = \Delta_0 \exp \left(- \int_{x_0}^x \frac{p_1(x)}{p_0(x)} dx \right),$$

where Δ_0 is the value of Δ at $x = x_0$.

IN 119

16.513 Lagrange identity

If the linear n^{th} -order equation $L(u) = 0$ is defined by

$$L(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u,$$

then the expression

$$vL(u) - uM(v) = \frac{d}{dx} \{P(u, v)\},$$

where $M(v)$ is the adjoint of $L(u)$, is called the **Lagrange identity**. The expression $P(u, v)$, which is linear and homogeneous in

$$u, \frac{du}{dx}, \dots, \frac{d^{n-1} u}{dx^{n-1}} \quad \text{and} \quad v, \frac{dv}{dx}, \dots, \frac{d^{n-1} v}{dx^{n-1}},$$

is then known as the **bilinear concomitant**. In the case of the second-order equation

$$L(u) = a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x)u = 0,$$

with adjoint $M(v)$, the Lagrange identity becomes

$$vL(u) - uM(v) = \frac{d}{dx} \left(a(x)v \frac{du}{dx} - \frac{d}{dx}(a(x)v)u + b(x)uv \right). \quad \text{IN 124}$$

16.514 The Riccati equation

The general **Riccati equation** has the form

$$\frac{dz}{dx} + a(x)z + b(x)z^2 + c(x) = 0,$$

and an equation of this form results from the substitution

$$z = \frac{(p(x) \frac{du}{dx})}{u}$$

in the general self-adjoint equation

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = 0.$$

The further substitution $v = u (\exp \int_{\alpha}^x a(x) dx)$ in the Riccati equation then gives the more convenient form

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

with

$$r(x) = b(x) \exp \left(- \int_{\alpha}^x a(x) dx \right) \quad \text{and} \quad s(x) = c(x) \exp \left(\int_{\alpha}^x a(x) dx \right). \quad \text{HI 273}$$

16.515 Solutions of the Riccati equation

If in the Riccati equation

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

$r(x) \neq 0$, while $r(x)$ and $s(x)$ are continuous on the interval $[\alpha, \beta]$, then every solution $v(x)$ may be expressed in the form

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)},$$

with A, B arbitrary constants, not both zero, and the prime denoting differentiation, while u and v are linearly independent solutions of

$$\frac{d}{dx} \left(\frac{1}{r(x)} \frac{dz}{dx} \right) + s(x)z = 0.$$

Conversely, if $u(x)$ and $v(x)$ are linearly independent solutions of this last equation and A and B are arbitrary constants, not both zero, the function

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)}$$

is a solution of the Riccati equation wherever $Au(x) + Bv(x) \neq 0$.

IN 24

16.516 Solution of a second-order linear differential equation

A **fundamental system** of solutions of a homogeneous second-order linear differential equation in the canonical form

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = 0$$

is a system of two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$. The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix} = \phi_1(t)\phi_2'(t) - \phi_2(t)\phi_1'(t),$$

and the solution to the inhomogeneous equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = f(t),$$

subject to the initial conditions $x(t_0) = x_0$ and $x'(t_0) = x_1$ may be written

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \int_{t_0}^t \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)} f(\xi) d\xi,$$

where the constants c_1 and c_2 are chosen such that $x(t)$ satisfies the initial conditions.

The linear combination $c_1\phi_1(t) + c_2\phi_2(t)$ is known as the **complementary function** where c_1 and c_2 are arbitrary constants.

16.61–16.62 Oscillation and Non-Oscillation Theorems for Second-Order Equations

Equations whose solutions possess an infinite number of zeros in the interval $(0, \infty)$ are said to have **oscillatory** solutions. The following theorems relate to such properties:

16.611 First basic comparison theorem

If all solutions of the equation

$$\frac{d^2u}{dx^2} + \phi(x)u = 0$$

are oscillatory, and if

$$\psi(x) \geq \phi(x),$$

then all the solutions of

$$\frac{d^2v}{dx^2} + \psi(x)v = 0$$

are oscillatory, and conversely. That is, if $\psi(x) \geq \phi(x)$ and some solutions v are non-oscillatory, then so also must some solutions u be non-oscillatory.

BS 119

16.622 Second basic comparison theorem

If all the solutions of the self-adjoint equation

$$\frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0$$

are oscillatory as $x \rightarrow \infty$, and if

$$q_2(x) \geq q_1(x),$$

$$p_2(x) \geq p_1(x) > 0,$$

then all the solutions of the self-adjoint equation

$$\frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0$$

are oscillatory.

BS 120

16.623 Interlacing of zeros

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$\frac{d^2 y}{dx^2} + F(x)y = 0,$$

and suppose that $y_1(x)$ has at least two zeros in the interval (a, b) . Then if x_1 and x_2 are two consecutive zeros of $y_1(x)$, the function $y_2(x)$ has one, and only one, zero in the interval (x_1, x_2) .

HI 374

16.624 Sturm separation theorem

Let $u(x)$ and $v(x)$ be two linearly independent solutions of the self-adjoint equation

$$\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y = 0,$$

in which $p(x) > 0$ and $p(x), q(x)$ are continuous on $[a, b]$. Then, between any two consecutive zeros of $u(x)$ there will be one, and only one, zero of $v(x)$.

IN 224

16.625 Sturm comparison theorem

Let $p_1(x) \geq p_2(x) > 0$ and $q_1(x) \geq q_2(x)$ be continuous functions in the differential equations

$$\frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0,$$

$$\frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0.$$

Then between any two zeros of a non-trivial solution $u(x)$ of the first equation there will be at least one zero of every non-trivial solution $v(x)$ of the second equation.

IN 228

16.626 Szegő's comparison theorem

Suppose, under the conditions of the Sturm comparison theorem, that $p_1(x) \equiv p_2(x)$, $q_1(x) \not\equiv q_2(x)$, and $u(x) > 0, v(x) > 0$ for $a < x < b$, together with

$$\lim_{x \rightarrow a} p_1(x) \left(\frac{du}{dx} v - \frac{dv}{dx} u \right) = 0.$$

Then, if $u(b) = 0$, there is a point ξ in (a, b) such that $v(\xi) = 0$.

HI 379

16.627 Picone's identity

Consider the equations

$$\begin{aligned}\frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u &= 0, \\ \frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v &= 0,\end{aligned}$$

with p_1, p_2, q_1 , and q_2 positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then with $a < \alpha < \beta < b$, Picone's identity is

$$\left(\frac{u}{v} \left(p_1 \frac{du}{dx} v - p_2 \frac{dv}{dx} u \right) \right)_\alpha^\beta = \int_\alpha^\beta (q_2 - q_1) u^2 ds + \int_\alpha^\beta (p_1 - p_2) \left(\frac{du}{ds} \right)^2 ds + \int_\alpha^\beta \frac{p_2}{v^2} \left(v \frac{du}{ds} - u \frac{dv}{ds} \right)^2 ds.$$

IN 226

16.628 Sturm-Picone theorem

Consider the self-adjoint equations

$$\frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0$$

and

$$\frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0.$$

Let p_1, p_2, q_1 , and q_2 be positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then, if x_1 and x_2 is a pair of consecutive zeros of $u(x)$ in (a, b) , $v(x)$ has at least one zero in the open interval (a, b) .

IN 225

16.629 Oscillation on the half line

Consider the self-adjoint equation

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = 0.$$

We then have the following results:

- (i) Let $p(x) > 0$ and p, q be continuous on $[0, \infty)$. If the two improper integrals

$$\int_1^\infty \frac{dx}{p(x)} \quad \text{and} \quad \int_1^\infty q(x) dx$$

diverge, then every solution $u(x)$ has infinitely many zeros on the interval $[1, \infty)$. Also, if the two integrals

$$\int_0^1 \frac{dx}{p(x)} = +\infty \quad \text{and} \quad \int_0^1 q(x) dx = +\infty,$$

then every solution $u(x)$ has infinitely many zeros on the interval $(0, 1)$.

- (ii) (Moore's theorem). Every non-trivial solution $u(x)$ has at most a finite number of zeros on the interval $[a, \infty)$ if the improper integral

$$\int_a^\infty \frac{dx}{p(x)}$$

converges, and if

$$\left| \int_a^x q(s) ds \right| < M \quad \text{for} \quad a \leq x < \infty$$

with $M > 0$ a finite constant.

16.71 Two Related Comparison Theorems

16.711 Theorem 1

Consider the equations in the Sturm comparison theorem with the same assumptions on $p(x)$ and $q(x)$, and let $u(x), v(x)$ be solutions such that

$$u(x_1) = v(x_1) = 0, \quad u'(x) = v'(x_1) > 0.$$

Then if $u(x)$ is increasing in $[x_1, x_2]$ and reaches a maximum at x_2 , the function $v(x)$ reaches a maximum at some point x_3 such that $x_1 < x_3 < x_2$. HI 376

16.712 Theorem 2

Consider the equation

$$\frac{d^2 y}{dx^2} + F(x)y = 0,$$

in which $F(x)$ is continuous in (a, b) and such that

$$0 < m \leq F(x) \leq M.$$

Then, if the solution $y(x)$ has two successive zeros x_1, x_2 , it follows that

$$\pi M^{-1/2} \leq x_2 - x_1 \leq \pi m^{-1/2}.$$

16.81–16.82 Non-Oscillatory Solutions

The real solution $y(x)$ of

$$\frac{d^2 y}{dx^2} + F(x)y = 0$$

is said to be **non-oscillatory** in the wide sense in $(0, \infty)$ if there exists a finite number c such that the solution has no zeros in $[c, \infty)$. HI 376

16.811 Kneser's non-oscillation theorem

Consider the equation

$$\frac{d^2 y}{dx^2} + F(x)y = 0,$$

and let

$$\limsup [x^2 F(x)] = \gamma^*,$$

$$\liminf [x^2 F(x)] = \gamma_*.$$

Then the solution $y(x)$ is non-oscillatory if $\gamma^* < \frac{1}{4}$, oscillatory if $\frac{1}{4} < \gamma_*$ and no conclusion can be drawn if either γ^* or γ_* equals $\frac{1}{4}$. HI 461

16.822 Comparison theorem for non-oscillation

Consider the differential equations

$$\frac{d^2 y}{dx^2} + F(x)y = 0, \quad f(x) = x \int_x^\infty F(s) ds,$$

$$\frac{d^2 y}{dx^2} + G(x)y = 0, \quad g(x) = x \int_x^\infty G(s) ds,$$

where $0 < g(x) < f(x)$. Then if the first equation is non-oscillatory in the wide sense, so also is the second. HI 460

16.823 Necessary and sufficient conditions for non-oscillation

Consider the equation

$$\frac{d^2 y}{dx^2} + F(x)y = 0.$$

Then, if

$$\lim_{x \rightarrow \infty} \sup \left(x \int_x^\infty F(s) ds \right) = F^*,$$

$$\lim_{x \rightarrow \infty} \inf \left(x \int_x^\infty F(s) ds \right) = F_*,$$

it follows that:

- (i) a necessary condition that the solution $y(x)$ be non-oscillatory is that $F_* \leq \frac{1}{4}$ and $F^* \leq 1$;
- (ii) a sufficient condition that the solution $y(x)$ be non-oscillatory is that $F^* < \frac{1}{4}$.

16.91 Some Growth Estimates for Solutions of Second-Order Equations

16.911 Strictly increasing and decreasing solutions

Suppose that $G(x) > 0$ be continuous in $(-\infty, \infty)$ and such that $xG(x) \notin L(0, \infty)$. Then the equation $\frac{d^2 y}{dx^2} - G(x)y = 0$ has one, and only one, solution $y_+(x)$ passing through the point $(0, 1)$, which is positive and strictly monotonic decreasing for all x , and one and only one solution $y_-(x)$ through the point $(0, 1)$, which is positive and strictly increasing for all x . The solution $y_+(x)$ has the property that

$$[G(x)]^{1/2} y_+(x) \in L_2(0, \infty) \text{ and } \frac{dy_+(x)}{dx} \in L_2(0, \infty).$$

If, in addition, $0 < \alpha^2 \leq G(x) \leq \beta^2 < \infty$, then

$$e^{-\beta x} \leq y_+(x) \leq e^{-\alpha x} \quad \text{for } x > 0.$$

HI 359

16.912 General result on dominant and subdominant solutions

Consider the equations

$$\frac{d^2 y}{dx^2} - g(x)y = 0, \quad \frac{d^2 Y}{dx^2} - G(x)Y = 0,$$

where g and G are continuous on $(0, \infty)$ with $0 < g(x) < G(x)$, and $xg(x) \notin L(0, \infty)$. In addition, let y_α and Y_α be the solutions of these respective equations corresponding to

$$y_\alpha(0) = Y_\alpha(0) = 1, \quad y'_\alpha(0) = Y'_\alpha(0) = \alpha \text{ for } -\infty < \alpha < \infty.$$

Let y_ω and Y_ω be determined, respectively, by

$$y_\omega(0) = Y_\omega(0) = 0, \quad y'_\omega(0) = Y'_\omega(0) = 1,$$

and let y_+ and Y_+ be the **subdominant solutions** for which

$$y_+(0) = Y_+(0) = 1$$

while $[y'_+(x)]^2, g(x)[y_+(x)]^2, [Y'_+(x)]^2$, and $G(x)[Y'_+(x)]^2$ belong to $L(0, \infty)$. Then, if β and γ are such that $y_{-\beta} = y_+$ and $Y_{-\gamma} = Y_+$, it follows that $\beta < \gamma$ and

$$y_\alpha(x) < Y_\alpha(x), \quad 0 < x < \infty, \quad -\gamma \leq \alpha,$$

$$y_\omega(x) < Y_\omega(x),$$

$$y_+(x) > Y_+(x).$$

HI 440

16.913 Estimate of dominant solution

Let $G(x)$ be positive and continuous with continuous first- and second-order derivatives satisfying

$$G(x)G'(x) < \frac{5}{4} [G'(x)]^2.$$

Then there exists a **dominant solution** $y(x)$ of the fundamental solutions $Y_0(x)$ and $Y_1(x)$ of

$$\frac{d^2 y}{dx^2} - G(x)y = 0,$$

determined by the initial conditions

$$2Y_0(0) = 0, \quad Y_1(0) = 1,$$

$$Y'_0(0) = 1, \quad Y'_1(0) = 0,$$

such that

$$y(x) < [G(x)]^{-1/4} \exp \left(\int_0^x [G(\xi)]^{1/2} d\xi \right),$$

and a positive constant C such that the normalized subdominant solution $y_+(x)$, for which $y_+(0) = 1$ and $[y'_+(x)]^2 \in L(0, \infty)$, $G(x)[y_+(x)]^2 \in L(0, \infty)$, satisfies

$$y_+(x) > CG(x)^{-1/4} \exp \left(- \int_0^x [G(\xi)]^{1/2} d\xi \right).$$

HI 443

16.914 A theorem due to Lyapunov

Let $y(x)$ be any solution of

$$\frac{d^2 y}{dx^2} - G(x)y = 0$$

with $G(x)$ positive and continuous in $(0, \infty)$ with $xG(x) \in L(0, \infty)$. Then

$$\exp \left(- \int_0^x [G(\xi) + 1] d\xi \right) < [y(x)]^2 + [y'(x)]^2$$

$$< C \exp \left(\int_0^x [G(\xi) + 1] d\xi \right),$$

HI 446

where $C = [y(0)]^2 + [y'(0)]^2$.

16.92 Boundedness Theorems

16.921⁶ All solutions of the equation

$$\frac{d^2u}{dx^2} + (1 + \phi(x) + \psi(x))u = 0$$

are bounded, provided that

$$(i) \quad \int^\infty |\phi(x)| dx < \infty,$$

$$(ii) \quad \int^\infty |\psi(x)| dx < \infty \quad \text{and} \quad \psi(x) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

BS 112

16.922 If all solutions of the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded, then all solutions of

$$\frac{d^2u}{dx^2} + (a(x) + b(x))u = 0$$

are also bounded if

$$\int^\infty |b(x)| dx < \infty.$$

BS 112

16.923 If $a(x) \rightarrow \infty$ monotonically as $x \rightarrow \infty$, then all solutions of

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded as $x \rightarrow \infty$.

BS 113

16.924 Consider the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

in which

$$\int^\infty x|a(x)| dx < \infty.$$

Then $\lim_{x \rightarrow \infty} \left(\frac{du}{dx} \right)$ exists, and the general solution is asymptotic to $d_0 + d_1x$ as $x \rightarrow \infty$, where d_0 and d_1 may be zero, but not simultaneously.

BS 114

16.93¹⁰ Growth of maxima of $|y|$

Sonin's theorem generalized by Pólya may be stated as follows: *Let $y(x)$ satisfy the differential equation*

$$\{k(x)y'\}' + \phi(x)y = 0,$$

where $k(x) > 0$, $\phi(x) > 0$, and both functions $k(x)$, $\phi(x)$ have a continuous derivative. Then the relative maxima of $|y|$ form an increasing or decreasing sequence according as $k(x)\phi(x)$ is decreasing or increasing.

SZ 164

17 Fourier, Laplace, and Mellin Transforms

17.1–17.4 Integral Transforms

17.11 Laplace transform

The **Laplace transform** of the function $f(x)$, denoted by $F(s)$, is defined by the integral

$$F(s) = \int_0^{\infty} f(x)e^{-sx} dx, \quad \text{Re } s > 0.$$

The functions $f(x)$ and $F(s)$ are called a **Laplace transform pair**, and knowledge of either one enables the other to be recovered.

If f is summable over all finite intervals, and there is a constant c for which

$$\int_0^{\infty} |f(x)|e^{-c|x|} dx$$

is finite, then the Laplace transform exists when $s = \sigma + i\tau$ is such that $\sigma \geq c$.

Setting

$$F(s) = \mathcal{L}[f(x); s]$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{L}^{-1}[F(s); x].$$

The inversion of the Laplace transform is accomplished for analytic functions $F(s)$ of order $O(s^{-k})$ with $k > 1$ by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{sx} ds,$$

where γ is a real constant that exceeds the real part of all the singularities of $F(s)$.

SN 30

17.12 Basic properties of the Laplace transform

- 1.⁸ For a and b arbitrary constants,

$$\mathcal{L}[af(x) + bg(x)] = aF(s) + bG(s) \quad (\text{linearity})$$

2. If $n > 0$ is an integer and $\lim_{x \rightarrow \infty} f(x)e^{-sx} = 0$, then for $x > 0$,

$$\mathcal{L}[f^{(n)}(x); s] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \dots - f^{(n-1)}(0) \quad (\text{transform of a derivative})$$

SN 32

3.¹¹ If $\lim_{x \rightarrow \infty} (e^{-sx} \int_0^x f(\zeta) d\zeta) = 0$, then

$$\mathcal{L} \left[\int_0^x f(\xi) d\xi; s \right] = \frac{1}{s} F(s) \quad (\text{transform of an integral}) \quad \text{SN 37}$$

4. $\mathcal{L} [e^{-ax} f(x); s] = F(s + a)$ (shift theorem) SU 143

5. The **Laplace convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \int_0^x f(x - \xi) g(\xi) d\xi,$$

and it has the property that $f * g = g * f$ and $f * (g * h) = (f * g) * h$. In terms of the convolution operation

$$\mathcal{L} [f * g(x); s] = F(s)G(s) \quad (\text{convolution (Faltung) theorem}). \quad \text{SN 30}$$

17.13 Table of Laplace transform pairs

$f(x)$		$F(s)$	
1	1	$1/s$	
2	$x^n, \quad n = 0, 1, 2, \dots$	$\frac{n!}{s^{n+1}}, \quad \text{Re } s > 0$	ET I 133(3)
3	$x^\nu, \quad \nu > -1$	$\frac{\Gamma(\nu + 1)}{s^{\nu+1}}, \quad \text{Re } s > 0$	ET I 137(1)
4	$x^{n-\frac{1}{2}}$	$\frac{\Gamma(n + \frac{1}{2})}{s^{n+\frac{1}{2}}}, \quad \text{Re } s > 0$	ET I 135(17)
5	$x^{-1/2}(x+a)^{-1}, \quad \arg a < \pi$	$\pi a^{-1/2} e^{as} \operatorname{erfc}(a^{1/2} s^{1/2}), \quad \text{Re } s \geq 0$	ET I 136(25)
6	$\begin{cases} x & \text{for } 0 < x < 1 \\ 1 & \text{for } x > 1 \end{cases}$	$\frac{1 - e^{-s}}{s^2}, \quad \text{Re } s > 0$	ET I 142(14)
7	e^{-ax}	$\frac{1}{s+a}, \quad \text{Re } s > -\operatorname{Re} a$	ET I 143(1)
8	$x e^{-ax}$	$\frac{1}{(s+a)^2}, \quad \text{Re } s > -\operatorname{Re} a$	ET I 144(2)
9a	$\frac{e^{-ax} - e^{-bx}}{b-a}$	$(s+a)^{-1}(s+b)^{-1}, \quad \text{Re } s > \{-\operatorname{Re} a, -\operatorname{Re} b\}$	AS 1022(29.3.12)
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	$f(x)$	$F(s)$
9b¹¹	$\frac{\alpha e^{-ax} + \beta e^{-bx} + \gamma e^{-cx}}{(a-b)(b-c)(c-a)}$ a, b, c distinct, $\alpha = c - b$, $\beta = a - c$, $\gamma = b - a$	$(s+a)^{-1}(s+b)^{-1}(s+c)^{-1}$, $\operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b, -\operatorname{Re} c\}$
10¹¹	$\frac{ae^{-ax} - be^{-bx}}{b-a}$	$s(s+a)^{-1}(s+b)^{-1}$, $\operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b\}$ AS 1022(29.3.13)
11	$\frac{e^{ax} - 1}{a}$	$s^{-1}(s-a)^{-1}$, $\operatorname{Re} s > \operatorname{Re} a$
12	$\frac{e^{ax} - ax - 1}{a^2}$	$s^{-2}(s-a)^{-1}$, $\operatorname{Re} s > \operatorname{Re} a$
13	$\frac{(e^{ax} - \frac{1}{2}a^2x^2 - ax - 1)}{a^3}$	$s^{-3}(s-a)^{-1}$, $\operatorname{Re} s > \operatorname{Re} a$
14	$(1+ax)e^{ax}$	$\frac{s}{(s-a)^2}$, $\operatorname{Re} s > \operatorname{Re} a$
15	$\frac{1 + (ax-1)e^{ax}}{a^2}$	$s^{-1}(s-a)^{-2}$, $\operatorname{Re} s > \operatorname{Re} a$
16	$\frac{2 + ax + (ax-2)e^{ax}}{a^3}$	$s^{-2}(s-a)^{-2}$, $\operatorname{Re} s > \operatorname{Re} a$
17	$x^n e^{ax}$, $n = 0, 1, 2, \dots$	$n!(s-a)^{-(n+1)}$, $\operatorname{Re} s > \operatorname{Re} a$
18	$(x + \frac{1}{2}ax^2)e^{ax}$	$\frac{s}{(s-a)^3}$, $\operatorname{Re} s > \operatorname{Re} a$
19	$(1 + 2ax + \frac{1}{2}a^2x^2)e^{ax}$	$\frac{s^2}{(s-a)^3}$, $\operatorname{Re} s > \operatorname{Re} a$
20	$\frac{1}{6}x^3e^{ax}$	$(s-a)^{-4}$, $\operatorname{Re} s > \operatorname{Re} a$
21	$(\frac{1}{2}x^2 + \frac{1}{6}ax^3)e^{ax}$	$\frac{s}{(s-a)^4}$, $\operatorname{Re} s > \operatorname{Re} a$
22	$(x + ax^2 + \frac{1}{6}a^2x^3)e^{ax}$	$s^2(s-a)^{-4}$, $\operatorname{Re} s > \operatorname{Re} a$
<i>continued on next page</i>		

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$f(x)$	$F(s)$	
23 $(1 + 3ax + \frac{3}{2}a^2x^2 + \frac{1}{6}a^3x^3)e^{ax}$	$s^3(s-a)^{-4},$	$\operatorname{Re} s > \operatorname{Re} a$
24 $\frac{ae^{ax} - be^{bx}}{a-b}$	$s(s-a)^{-1}(s-b)^{-1},$	$\operatorname{Re} s > \{\operatorname{Re} a, \operatorname{Re} b\}$
25 $\frac{(\frac{1}{a}e^{ax} - \frac{1}{b}e^{bx} + \frac{1}{b} - \frac{1}{a})}{a-b}$	$s^{-1}(s-a)^{-1}(s-b)^{-1},$	$\operatorname{Re} s > \{\operatorname{Re} a, \operatorname{Re} b\}$
26 $x^{\nu-1}e^{-ax},$	$\Gamma(\nu)(s+a)^{-\nu},$	$\operatorname{Re} s > -\operatorname{Re} a \quad \text{ET I 144(3)}$
27 $xe^{-x^2/(4a)},$	$2a - 2\pi^{1/2}a^{3/2}se^{as^2}\operatorname{erfc}(sa^{1/2})$	$\operatorname{Re} a > 0 \quad \text{ET I 146(22)}$
28 $\exp(-ae^x),$	$a^s\Gamma(-s, a)$	$\operatorname{Re} a > 0 \quad \text{ET I 147(37)}$
29^s $x^{1/2}e^{-a/(4x)},$	$\frac{1}{2}\pi^{1/2}s^{-3/2}\left(1 + a^{1/2}s^{1/2}\right)\exp\left[(-as)^{1/2}\right],$	$\operatorname{Re} a \geq 0 \quad \operatorname{Re} s > 0 \quad \text{ET I 146(26)}$
30^s $x^{-1/2}e^{-a/(4x)},$	$\pi^{1/2}s^{-1/2}\exp\left[(-as)^{1/2}\right],$	$\operatorname{Re} a \geq 0 \quad \operatorname{Re} s > 0 \quad \text{ET I 146(27)}$
31^s $x^{-3/2}e^{-a/(4x)},$	$2\pi^{1/2}a^{-1/2}\exp\left[(-as)^{1/2}\right],$	$\operatorname{Re} a > 0 \quad \operatorname{Re} s \geq 0 \quad \text{ET I 146(28)}$
32 $\sin(ax)$	$a(s^2 + a^2)^{-1},$	$\operatorname{Re} s > \operatorname{Im} a \quad \text{ET I 150(1)}$
33 $\cos(ax)$	$s(s^2 + a^2)^{-1},$	$\operatorname{Re} s > \operatorname{Im} a \quad \text{ET I 154(3)}$
34 $ \sin(ax) ,$	$a(s^2 + a^2)^{-1}\coth\left(\frac{\pi s}{2a}\right),$	$a > 0 \quad \operatorname{Re} s > 0 \quad \text{ET I 150(2)}$
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$f(x)$	$F(s)$
35¹¹ $ \cos(ax) , \quad a > 0$	$(s^2 + a^2)^{-1} \left[s + a \operatorname{cosech} \left(\frac{\pi s}{2a} \right) \right],$ $\operatorname{Re} s > 0 \quad \text{ET I 155(44)}$
36 $\frac{1 - \cos(ax)}{a^2}$	$s^{-1} (s^2 + a^2)^{-1},$ $\operatorname{Re} s > \operatorname{Im} a \quad \text{AS 1022(29.3.19)}$
37 $\frac{ax - \sin(ax)}{a^3}$	$s^{-2} (s^2 + a^2)^{-1},$ $\operatorname{Re} s > \operatorname{Im} a \quad \text{AS 1022(29.3.20)}$
38 $\frac{\sin(ax) - ax \cos(ax)}{2a^3}$	$(s^2 + a^2)^{-2},$ $\operatorname{Re} s > \operatorname{Im} a \quad \text{AS 1022(29.3.21)}$
39 $\frac{x \sin(ax)}{2a}$	$s (s^2 + a^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Im} a \quad \text{ET I 152(14)}$
40 $\frac{\sin(ax) + ax \cos(ax)}{2a}$	$s^2 (s^2 + a^2)^{-2},$ $\operatorname{Re} s > \operatorname{Im} a \quad \text{AS 1023(29.3.23)}$
41 $x \cos(ax)$	$(s^2 - a^2) (s^2 + a^2)^{-2},$ $\operatorname{Re} s > \operatorname{Im} a \quad \text{ET I 157(57)}$
42 $\frac{\cos(ax) - \cos(bx)}{b^2 - a^2}$	$s (s^2 + a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a , \operatorname{Im} b \} \quad \text{AS 1023(29.3.25)}$
43 $\frac{[\frac{1}{2}a^2x^2 - 1 + \cos(ax)]}{a^4}$	$s^{-3} (s^2 + a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Im} a $
44 $\frac{[1 - \cos(ax) - \frac{1}{2}ax \sin(ax)]}{a^4}$	$s^{-1} (s^2 + a^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Im} a $
45 $\frac{[\frac{1}{b} \sin(bx) - \frac{1}{a} \sin(ax)]}{a^2 - b^2}$	$(s^2 + a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a , \operatorname{Im} b \}$
46¹¹ $\frac{[1 - \cos(ax) + \frac{1}{2}ax \sin(ax)]}{a^2}$	$s^{-1} (s^2 + a^2)^{-2} (2s^2 + a^2), \quad \operatorname{Re} s > \operatorname{Im} a $
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$f(x)$	$F(s)$
47 $\frac{a \sin(ax) - b \sin(bx)}{a^2 - b^2}$	$s^2 (s^2 + a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a , \operatorname{Im} b \}$
48 $\sin(a + bx)$	$(s \sin a + b \cos a) (s^2 + b^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Im} b $
49 $\cos(a + bx)$	$(s \cos a - b \sin a) (s^2 + b^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Im} b $
50 $\frac{[\frac{1}{a} \sinh(ax) - \frac{1}{b} \sin(bx)]}{a^2 + b^2}$	$(s^2 - a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Im} b \}$
51 $\frac{\cosh(ax) - \cos(bx)}{a^2 + b^2}$	$s (s^2 - a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Im} b \}$
52 $\frac{a \sinh(ax) + b \sin(bx)}{a^2 + b^2}$	$s^2 (s^2 - a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Im} b \}$
53 $\sin(ax) \sin(bx)$	$2abs [s^2 + (a - b)^2]^{-1} [s^2 + (a + b)^2]^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a , \operatorname{Im} b \}$
54 $\cos(ax) \cos(bx)$	$s (s^2 + a^2 + b^2) [s^2 + (a - b)^2]^{-1} [s^2 + (a + b)^2]^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a , \operatorname{Im} b \}$
55 $\sin(ax) \cos(bx)$	$a (s^2 + a^2 - b^2) [s^2 + (a - b)^2]^{-1} [s^2 + (a + b)^2]^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a , \operatorname{Im} b \}$
56 $\sin^2(ax)$	$2a^2 s^{-1} (s^2 + 4a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Im} a $
57 $\cos^2(ax)$	$(s^2 + 2a^2) s^{-1} (s^2 + 4a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Im} a $
58 $\sin(ax) \cos(ax)$	$a (s^2 + 4a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Im} a $
59 $e^{-ax} \sin(bx)$	$b [(s + a)^2 + b^2]^{-1}, \quad \operatorname{Re} s > \{-\operatorname{Re} a, \operatorname{Im} b \}$
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$f(x)$	$F(s)$
60 $e^{-ax} \cos(bx)$	$(s + a) [(s + a)^2 + b^2]^{-1},$ $\operatorname{Re} s > \{-\operatorname{Re} a, \operatorname{Im} b \}$
61 $x^{-1} \sin(ax)$	$\arctan(a/s), \quad \operatorname{Re} s > \operatorname{Im} a \quad \text{ET I 152(16)}$
62 $x^{-1} [1 - \cos(ax)]$	$\frac{1}{2} \ln(1 + a^2/s^2),$ $\operatorname{Re} s > \operatorname{Im} a \quad \text{ET I 157(59)}$
63 $\sinh(ax)$	$a(s^2 - a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a \quad \text{ET I 162(1)}$
64 $\cosh(ax)$	$s(s^2 - a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a \quad \text{ET I 162(2)}$
65 $x^{\nu-1} \sinh(ax), \quad \operatorname{Re} \nu > -1$	$\frac{1}{2} \Gamma(\nu) [(s - a)^{-\nu} - (s + a)^{-\nu}],$ $\operatorname{Re} s > \operatorname{Re} a \quad \text{ET I 164(18)}$
66 $x^{\nu-1} \cosh(ax), \quad \operatorname{Re} \nu > 0$	$\frac{1}{2} \Gamma(\nu) [(s - a)^{-\nu} + (s + a)^{-\nu}],$ $\operatorname{Re} s > \operatorname{Re} a \quad \text{ET I 164(19)}$
67 $x \sinh(ax)$	$2as(s^2 - a^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a $
68 $x \cosh(ax)$	$(s^2 + a^2)(s^2 - a^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a $
69 $\sinh(ax) - \sin(ax)$	$2a^3(s^4 - a^4)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Im} a \} \quad \text{AS 1023(29.3.31)}$
70 $\cosh(ax) - \cos(ax)$	$2a^2s(s^4 - a^4)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Im} a \} \quad \text{AS 1023(29.3.32)}$
71 $\sinh(ax) + ax \cosh(ax)$	$2as^2(a^2 - s^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a $
72 $ax \cosh(ax) - \sinh(ax)$	$2a^3(a^2 - s^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a $
<i>continued on next page</i>	

<i>continued from previous page</i>	
$f(x)$	$F(s)$
73 $x \sinh(ax) - \cosh(ax)$	$s (a^2 + 2a - s^2) (a^2 - s^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a $
74 $\frac{[\frac{1}{a} \sinh(ax) - \frac{1}{b} \sinh(bx)]}{a^2 - b^2}$	$(a^2 - s^2)^{-1} (b^2 - s^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
75 $\frac{\cosh(ax) - \cosh(bx)}{a^2 - b^2}$	$s (s^2 - a^2)^{-1} (s^2 - b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
76 $\frac{a \sinh(ax) - b \sinh(bx)}{a^2 - b^2}$	$s^2 (s^2 - a^2)^{-1} (s^2 - b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
77 $\sinh(a + bx)$	$(b \cosh a + s \sinh a) (s^2 - b^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} b $
78 $\cosh(a + bx)$	$(s \cosh a + b \sinh a) (s^2 - b^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} b $
79 $\sinh(ax) \sinh(bx)$	$2abs [s^2 - (a + b)^2]^{-1} [s^2 - (a - b)^2]^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
80⁸ $\cosh(ax) \cosh(bx)$	$s (s^2 - a^2 - b^2) [s^2 - (a + b)^2]^{-1} [s^2 - (a - b)^2]^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
81 $\sinh(ax) \cosh(bx)$	$a (s^2 - a^2 + b^2) [s^2 - (a + b)^2]^{-1} [s^2 - (a - b)^2]^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
82 $\sinh^2(ax)$	$2a^2 s^{-1} (s^2 - 4a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a $
83 $\cosh^2(ax)$	$(s^2 - 2a^2) s^{-1} (s^2 - 4a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a $
84 $\sinh(ax) \cosh(ax)$	$a (s^2 - 4a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a $
85 $\frac{\cosh(ax) - 1}{a^2}$	$s^{-1} (s^2 - a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a $
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<i>continued from previous page</i>	
$f(x)$	$F(s)$
86 $\frac{\sinh(ax) - ax}{a^3}$	$s^{-2} (s^2 - a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a $
87 $\frac{[\cosh(ax) - \frac{1}{2}a^2x^2 - 1]}{a^4}$	$s^{-3} (s^2 - a^2)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a $
88 $\frac{[1 - \cosh(ax) + \frac{1}{2}ax \sinh(ax)]}{a^4}$	$s^{-1} (s^2 - a^2)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a $
89 $x^{1/2} \sinh(ax)$	$\left(\pi^{1/2}/4\right) [(s-a)^{3/2} - (s+a)^{3/2}], \quad \operatorname{Re} s > \operatorname{Re} a $
90 $\ln x$	$-s^{-1} \ln(\mathbf{C}s), \quad \operatorname{Re} s > 0 \quad \text{ET I 148(1)}$
91 $\ln(1+ax), \quad \arg a < \pi$	$s^{-1} e^{s/a} \operatorname{Ei}(-s/a), \quad \operatorname{Re} s > 0 \quad \text{ET I 148(4)}$
92 $x^{-1/2} \ln x$	$-(\pi/s)^{1/2} \ln(4\mathbf{C}s), \quad \operatorname{Re} s > 0 \quad \text{ET I 148(9)}$
93 $H(x-a) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x > a \end{cases}$ (Heaviside step function)	$s^{-1} e^{-as}, \quad a \geq 0$
94 $\delta(x)$ (Dirac delta function)	1
95 $\delta(x-a)$	$e^{-as}, \quad a \geq 0$
96 $\delta'(x-a)$	$se^{-as}, \quad a \geq 0$
97 $\operatorname{Si}(x) \equiv \int_0^x \frac{\sin \xi}{\xi} d\xi \equiv \frac{1}{2}\pi + \operatorname{si}(x)$	$s^{-1} \operatorname{arccot} s, \quad \operatorname{Re} s > 0 \quad \text{ET I 177(17)}$
98 $\operatorname{Ci}(x) \equiv \operatorname{ci}(x) \equiv -\int_x^\infty \frac{\cos \xi}{\xi} d\xi$	$-\frac{1}{2}s^{-1} \ln(1+s^2), \quad \operatorname{Re} s > 0 \quad \text{ET I 178(19)}$
99^s $\operatorname{erf}\left(\frac{x}{2a}\right)$	$s^{-1} e^{a^2s^2} \operatorname{erfc}(as), \quad \operatorname{Re} s > 0, \arg a < \pi/4 \quad \text{ET I 176(2)}$
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$f(x)$		$F(s)$	
100	$\operatorname{erf}(a\sqrt{x})$	$as^{-1}(s+a^2)^{-1/2},$ $\operatorname{Re} s > \{0, -\operatorname{Re} a^2\}$	ET I 176(4)
101	$\operatorname{erfc}(a\sqrt{x})$	$s^{-1}(s+a^2)^{-\frac{1}{2}} \left[(s+a^2)^{1/2} - a \right],$ $\operatorname{Re} s > 0$	ET I 177(9)
102⁸	$\operatorname{erfc}\left(\frac{a}{\sqrt{x}}\right)$	$s^{-1}e^{-2a\sqrt{s}},$ $\operatorname{Re} s > 0, \quad \operatorname{Re} a > 0$	ET I 177(11)
103⁸	$J_\nu(ax),$ $\operatorname{Re} \nu > -1$	$a^{-\nu} \left(\sqrt{s^2 + a^2} - s \right)^\nu (s^2 + a^2)^{-1/2},$ $\operatorname{Re} s > \operatorname{Im} a ,$	ET I 182(1)
104	$x J_\nu(ax),$ $\operatorname{Re} \nu > -2$	$a^\nu \left[s + \nu (s^2 + a^2)^{1/2} \right] \left[s + (s^2 + a^2)^{1/2} \right]^{-\nu}$ $\times (s^2 + a^2)^{-3/2},$ $\operatorname{Re} s > \operatorname{Im} a ,$	ET I 182(2)
105	$\frac{J_\nu(ax)}{x}$	$a^\nu \nu^{-1} \left[s + (s^2 + a^2)^{1/2} \right]^{-\nu},$ $\operatorname{Re} s \geq \operatorname{Im} a $	ET I 182(5)
106	$x^n J_n(ax)$	$1 \cdot 3 \cdot 5 \cdots (2n-1) a^n (s^2 + a^2)^{-(n+\frac{1}{2})},$ $\operatorname{Re} s > \operatorname{Im} a $	ET I 182(4)
107	$x^\nu J_\nu(ax),$ $\operatorname{Re} \nu > -\frac{1}{2}$	$2^\nu \pi^{-1/2} \Gamma\left(\nu + \frac{1}{2}\right) a^\nu (s^2 + a^2)^{-(\nu+\frac{1}{2})},$ $\operatorname{Re} s > \operatorname{Im} a ,$	ET I 182(7)
108	$x^{\nu+1} J_\nu(ax),$ $\operatorname{Re} \nu > -1$	$2^{\nu+1} \pi^{-1/2} \Gamma\left(\nu + \frac{3}{2}\right) a^\nu s (s^2 + a^2)^{-(\nu+\frac{3}{2})},$ $\operatorname{Re} s > \operatorname{Im} a $	ET I 182(8)
109⁸	$I_\nu(ax),$ $\operatorname{Re} \nu > -1$	$a^{-\nu} \left[s - \sqrt{s^2 - a^2} \right]^\nu (s^2 - a^2)^{-1/2},$ $\operatorname{Re} s > \operatorname{Re} a $	ET I 195(1)
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	$f(x)$		$F(s)$
110	$x^\nu I_\nu(ax), \quad \operatorname{Re} \nu > -\frac{1}{2}$		$2^\nu \pi^{-1/2} \Gamma\left(\nu + \frac{1}{2}\right) a^\nu (s^2 - a^2)^{-(\nu + \frac{1}{2})},$ $\operatorname{Re} s > \operatorname{Re} a \quad \text{ET I 195(6)}$
111	$x^{\nu+1} I_\nu(ax), \quad \operatorname{Re} \nu > -1$		$2^{\nu+1} \pi^{-1/2} \Gamma\left(\nu + \frac{3}{2}\right) a^\nu (s^2 - a^2)^{-(\nu + \frac{3}{2})},$ $\operatorname{Re} s > \operatorname{Re} a \quad \text{ET I 196(7)}$
112	$x^{-1} I_\nu(ax), \quad \operatorname{Re} \nu > 0$		$\nu^{-1} a^\nu \left[s + (s^2 - a^2)^{1/2}\right]^{-\nu},$ $\operatorname{Re} s > \operatorname{Re} a \quad \text{ET I 195(4)}$
113	$\sin\left(2a^{1/2}x^{1/2}\right)$		$(\pi a)^{1/2} s^{-3/2} e^{-a/s}, \quad \operatorname{Re} s > 0 \quad \text{ET I 153(32)}$
114	$x^{-1/2} \cos\left(2a^{1/2}x^{1/2}\right)$		$\pi^{1/2} s^{-1/2} e^{-a/s}, \quad \operatorname{Re} s > 0 \quad \text{ET I 158(67)}$
115	$x^{-1} e^{-ax} I_1(ax)$		$\left[(s+2a)^{1/2} - s^{1/2}\right] \left[(s+2a)^{1/2} + s^{1/2}\right]^{-1},$ $\operatorname{Re} s > \operatorname{Re} a \quad \text{AS 1024(29.3.52)}$
116	$\frac{J_k(ax)}{x}$		$k^{-1} a^{-k} \left[(s^2 + a^2)^{1/2} - s\right]^k,$ $\operatorname{Re} s > \operatorname{Im} a , k > -1 \quad \text{AS 1025(29.3.58)}$
117	$\left(\frac{x}{2a}\right)^{k-\frac{1}{2}} J_{k-\frac{1}{2}}(ax)$		$\Gamma(k) \pi^{-1/2} (s^2 + a^2)^k,$ $\operatorname{Re} s > \operatorname{Im} a , \quad k > 0 \quad \text{AS 1024(29.3.57)}$
118	$J_0(ax) - ax J_1(ax)$		$s^2 (s^2 + a^2)^{-3/2}, \quad \operatorname{Re} s > \operatorname{Im} a $
119	$I_0(ax) + ax I_1(ax)$		$s^2 (s^2 - a^2)^{-3/2}, \quad \operatorname{Re} s > \operatorname{Im} a $

17.21 Fourier transform

The **Fourier transform**, also called the **exponential** or **complex Fourier transform**, of the function $f(x)$, denoted by $F(\xi)$, is defined by the integral

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx.$$

The functions $f(x)$ and $F(\xi)$ are called a **Fourier transform pair**, and knowledge of either one enables the other to be recovered. Setting $F(\xi) = \mathcal{F}[f(x); \xi]$, to emphasize the nature of the transform, we have

the symbolic inverse result $f(x) = \mathcal{F}^{-1}[F(\xi); x]$. The inversion of the Fourier transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x} d\xi.$$

17.22 Basic properties of the Fourier transform

1. For a and b arbitrary constants,

$$\mathcal{F}[af(x) + bg(x)] = aF(\xi) + bG(\xi) \quad (\text{linearity})$$

2. If $n > 0$ is an integer, and $\lim_{|x| \rightarrow \infty} f^{(r)}(x) = 0$ for $r = 0, 1, \dots, n-1$ with $f^{(0)}(x) \equiv f(x)$, then

$$\mathcal{F}[f^{(n)}(x); \xi] = (-i\xi)^n F(\xi) \quad (\text{transform of a derivative}) \quad \text{SN 27}$$

3. The **Fourier convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi,$$

and it has the property $f * g = g * f$, and $f * (g * h) = (f * g) * h$. In terms of the convolution operation,

$$\mathcal{F}[f * g(x); \xi] = F(\xi)G(\xi) \quad (\text{convolution [Faltung] theorem}). \quad \text{SN 24}$$

17.23 Table of Fourier transform pairs

$f(x)$		$F(\xi)$	
1	1	$(2\pi)^{1/2} \delta(\xi)$	SU 496
2⁷	$\frac{1}{x}$	$(\pi/2)^{1/2} i \operatorname{sign} \xi$	SU 50
3	$\delta(x)$	$(2\pi)^{-1/2}$	SU 496
4⁸	$\delta(ax + b), \quad a, b \in \mathbb{R}, \quad a \neq 0$	$(2\pi)^{-1/2} e^{ib\xi/a}$	SU 517
5	$\begin{cases} 1 & x < a \\ 0 & x > a \end{cases}, \quad a > 0$	$(2/\pi)^{1/2} \xi^{-1} \sin(a\xi)$	
6⁸	$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$	$-\frac{1}{i\xi\sqrt{2\pi}} + \sqrt{\frac{\pi}{2}} \delta(\xi)$	SN 523

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	$f(x)$	$F(\xi)$	
7	$\frac{1}{ x ^a}, \quad 0 < \operatorname{Re} a < 1$	$\frac{(2/\pi)^{1/2} \Gamma(1-a) \sin(\frac{1}{2}a\pi)}{ \xi ^{1-a}}$	SN 523
8	$e^{iax}, \quad a \in \mathbb{R}$	$(2\pi)^{1/2} \delta(\xi + a)$	SU 50
9	$e^{-a x }, \quad a > 0$	$\frac{a(2/\pi)^{1/2}}{a^2 + \xi^2}$	SU 50
10⁷	$xe^{-a x }, \quad a > 0$	$\frac{2ai\xi(2/\pi)^{1/2}}{(a^2 + \xi^2)^2}, \quad \xi > 0$	SU 50
11	$ x e^{-a x }, \quad a > 0$	$\frac{(2/\pi)^{1/2} (a^2 - \xi^2)}{(a^2 + \xi^2)^2}$	SU 50
12	$\frac{e^{-a x }}{ x ^{1/2}}, \quad a > 0$	$\frac{[a + (a^2 + \xi^2)^{1/2}]^{1/2}}{x(a^2 + \xi^2)^{1/2}}$	SN 523
13	$e^{-a^2 x^2}, \quad a > 0$	$(a\sqrt{2})^{-1} e^{-\xi^2/4a^2}$	SU 51
14	$\frac{1}{a^2 + x^2}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{1/2} e^{-a \xi }}{a}$	SU 51
15⁷	$\frac{x}{a^2 + x^2}, \quad \operatorname{Re} a > 0$	$i \operatorname{sign} \xi (\pi/2)^{1/2} e^{-a \xi }$	
16⁹	$\sin(ax^2)$	$\frac{1}{(2a)^{1/2}} \cos\left(\frac{\xi^2}{4a} + \frac{\pi}{4}\right)$	SN 523
17	$\cos(ax^2)$	$\frac{1}{(2a)^{1/2}} \cos\left(\frac{\xi^2}{4a} - \frac{\pi}{4}\right)$	SN 523
18	$e^{-a x } \cos(bx), \quad a > 0, \quad b > 0$	$a(2\pi)^{-1/2} \left[\frac{1}{a^2 + (b + \xi)^2} + \frac{1}{a^2 + (b - \xi)^2} \right]$	
19	$e^{-\frac{1}{2}ax^2} \sin(bx), \quad a > 0, \quad b > 0$	$\frac{1}{2}ia^{-1/2} \left\{ \exp\left[-\frac{1}{2}\frac{(\xi - b)^2}{a}\right] - \exp\left[-\frac{1}{2}\frac{(\xi + b)^2}{a}\right] \right\}$	
20⁹	$\frac{\sinh(ax)}{\sinh(bx)}, \quad a < b $	$\frac{(\pi/2)^{1/2} \sin(\pi a/b)}{b[\cosh(\pi\xi/b) + \cos(\pi a/b)]}$	SU 123
21⁹	$\frac{\cosh(ax)}{\sinh(bx)}, \quad a < b $	$\frac{i(\pi/2)^{1/2} \sinh(\pi\xi/b)}{b[\cosh(\pi\xi/b) + \cos(\pi a/b)]}$	SU 123

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	$f(x)$	$F(\xi)$
22	$\frac{\sin(ax)}{x}$	$\begin{cases} (\pi/2)^{1/2} & \xi < a, \\ 0 & \xi > a \end{cases}$ SN 523
23¹¹	$\frac{x}{\sinh x}$	$\frac{(2\pi^3)^{1/2} e^{\pi\xi}}{(1 + e^{\pi\xi})^2}$ SU 123
24⁷	$x^n \operatorname{sign} x, \quad n = 1, 2, \dots$	$(2/\pi)^{1/2} (-i\xi)^{-(1+n)} n!$ SU 506
25⁷	$ x ^\nu,$ $-1 < \nu < 0$, but not integral	$(2/\pi)^{1/2} \Gamma(\nu + 1) \xi ^{-\nu-1} \cos [\pi(\nu + 1)/2]$ SU506
26⁷	$ x ^\nu \operatorname{sign} x,$ $-1 < \nu < 0$, but not integral	$\frac{i \operatorname{sign} \xi (2/\pi)^{1/2} \sin [(\pi/2)(\nu + 1)] \Gamma(\nu + 1)}{ \xi ^{\nu+1}}$ SU 506
27	$e^{-ax} \ln 1 - e^{-x} ,$ $-1 < \operatorname{Re} a < 0$	$\left(\frac{\pi}{2}\right)^{1/2} \frac{\cot(\pi a - i\xi\pi)}{a - i\xi}$ ET I 121(26)
28	$e^{-ax} \ln(1 + e^{-x}),$ $-1 < \operatorname{Re} a < 0$	$\left(\frac{\pi}{2}\right)^{1/2} \frac{\csc(\pi a - i\xi\pi)}{a - i\xi}$ ET I 121 (27)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $1/(2\pi)^{1/2}$ employed in our definition of F has not been used in those tables, and that there is a difference of sign between the exponents used in the definitions of the exponential Fourier transform.

17.24 Table of Fourier transform pairs for spherically symmetric functions

	$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \iiint E(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}$	$E(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \iiint f(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}$
1	$f(r) = \sqrt{\frac{2}{\pi}} \frac{1}{r} \int_0^\infty E(k) \sin(kr) k dk$	$E(k) = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^\infty f(r) \sin(kr) r dr$
2	e^{-ar}	$\sqrt{\frac{2}{\pi}} \frac{2a}{(a^2 + k^2)^2}$
3 ¹¹	$\frac{e^{-ar}}{r}$	$\sqrt{\frac{2}{\pi}} \frac{1}{(a^2 + k^2)^2}$
4 ¹¹	1	$(2\pi)^{3/2} \delta(\mathbf{k})$

17.31 Fourier sine and cosine transforms

The **Fourier sine** and **cosine transforms** of the function $f(x)$, denoted by $F_s(\xi)$ and $F_c(\xi)$, respectively, are defined by the integrals

$$F_s(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\xi x) dx \quad \text{and} \quad F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\xi x) dx.$$

The functions $f(x)$ and $F_s(\xi)$ are called a **Fourier sine transform pair**, and the functions $f(x)$ and $F_c(\xi)$ a **Fourier cosine transform pair**, and knowledge of either $F_s(\xi)$ or $F_c(\xi)$ enables $f(x)$ to be recovered.

Setting

$$F_s(\xi) = \mathcal{F}_s[f(x); \xi] \quad \text{and} \quad F_c(\xi) = \mathcal{F}_c[f(x); \xi],$$

to emphasize the nature of the transforms, we have the symbolic inverses

$$f(x) = \mathcal{F}_s^{-1}[F_s(\xi); x] \quad \text{and} \quad f(x) = \mathcal{F}_c^{-1}[F_c(\xi); x].$$

The inversion of the Fourier sine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\xi) \sin(\xi x) d\xi \quad [x \geq 0]$$

and the inversion of the Fourier cosine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\xi) \cos(\xi x) d\xi \quad [x \geq 0]. \quad \text{SN 17}$$

17.32 Basic properties of the Fourier sine and cosine transforms

1. For a and b arbitrary constants,

$$\mathcal{F}_s[af(x) + bg(x)] = aF_s(\xi) + bG_s(\xi)$$

and

$$\mathcal{F}_c[af(x) + bg(x)] = aF_c(\xi) + bG_c(\xi) \quad (\text{linearity})$$

2. If $\lim_{x \rightarrow \infty} f^{(r-1)}(x) = 0$ and $\lim_{x \rightarrow \infty} \sqrt{\frac{2}{\pi}} f^{(r-1)}(x) = a_{r-1}$, then denoting the Fourier sine and cosine transforms of $f^{(r)}(x)$ by $F_s^{(r)}$ and $F_c^{(r)}$, respectively,

$$(i) \quad F_c^{(r)}(\xi) = -a_{r-1} + \xi F_s^{(r-1)}.$$

$$(ii) \quad F_s^{(r)}(\xi) = -\xi F_c^{(r-1)}(\xi),$$

$$(iii) \quad F_c^{(2r)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \xi^{2n} + (-1)^r \xi^{2r} F_c(\xi),$$

$$(iv) \quad F_c^{(2r+1)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n} \xi^{2n} + (-1)^r \xi^{2r+1} F_s(\xi),$$

$$(v) \quad F_s^{(r)}(\xi) = \xi a_{r-2} - \xi^2 F_s^{(r-2)}(\xi),$$

$$(vi)^6 \quad F_s^{(2r)}(\xi) = -\sum_{n=1}^r (-1)^n \xi^{2n-1} a_{2r-2n} + (-1)^r \xi^{2r} F_s(\xi),$$

$$(vii) \quad F_s^{(2r+1)}(\xi) = -\sum_{n=1}^r (-1)^n \xi^{2n-1} a_{2r-2n+1} + (-1)^{r+1} \xi^{2r+1} F_c(\xi). \quad \text{SN 28}$$

3. (i) $\int_0^\infty F_s(\xi) G_s(\xi) \cos(\xi x) d\xi = \frac{1}{2} \int_0^\infty g(s) [f(s+x) + f(s-x)] ds,$
(ii) $\int_0^\infty F_c(\xi) G_c(\xi) \cos(\xi x) d\xi = \frac{1}{2} \int_0^\infty g(s) [f(s+x) + f(|x-s|)] ds$
(convolution (Faltung) theorem) SN 24
4. (i) If $F_s(\xi)$ is the Fourier sine transform of $f(x)$, then the Fourier sine transform of $F_s(x)$ is $f(\xi)$.
(ii) If $F_c(\xi)$ is the Fourier cosine transform of $f(x)$, then the Fourier cosine transform of $F_c(x)$ is $f(\xi)$.
(iii) If $f(x)$ is an odd function in $(-\infty, \infty)$, then the Fourier sine transform of $f(x)$ in $(0, \infty)$ is $-iF(\xi)$.
(iv) If $f(x)$ is an even function in $(-\infty, \infty)$, then the Fourier cosine transform of $f(x)$ in $(0, \infty)$ is $F(\xi)$.
(v) The Fourier sine transform of $f(x/a)$ is $aF_s(a\xi)$.
(vi) The Fourier cosine transform of $f(x/a)$ is $aF_c(a\xi)$.
(vii) $\mathcal{F}_s[f(x); \xi] = F_s(|\xi|) \text{sign } \xi$ SU 45

17.33 Table of Fourier sine transforms

	$f(x)$	$F_s(\xi)$ ($\xi > 0$)
1	x^{-1}	$(\pi/2)^{1/2}, \quad \xi > 0 \quad \text{ET I 64(3)}$
2	$x^{-\nu}, \quad 0 < \text{Re } \nu < 2$	$(2/\pi)^{1/2} \xi^{\nu-1} \Gamma(1-\nu) \cos(\nu\pi/2), \quad \xi > 0 \quad \text{ET I 68(1)}$
3	$x^{-1/2}$	$\xi^{-1/2}, \quad \xi > 0 \quad \text{ET I 64(6)}$
4	$x^{-3/2}$	$2\xi^{1/2}, \quad \xi > 0 \quad \text{ET I 64(9)}$
5	$\begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$	$(2/\pi)^{1/2} \xi^{-1} [1 - \cos(a\xi)], \quad \xi > 0 \quad \text{ET I 63(1)}$
6	$\begin{cases} x^{-1} & 0 < x < a \\ 0 & x > a \end{cases}$	$(2/\pi)^{1/2} \text{Si}(a\xi), \quad \xi > 0 \quad \text{ET I 64(4)}$
7	$\frac{1}{a-x}, \quad a > 0$	$(2/\pi)^{1/2} \{ \sin(a\xi) \text{Ci}(a\xi) - \cos(a\xi) [\frac{1}{2}\pi + \text{Si}(a\xi)] \}, \quad \xi > 0 \quad \text{ET I 64(11)}$

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$f(x)$	$F_s(\xi) \quad (\xi > 0)$	
8⁷ $\frac{1}{x^2 + a^2}, \quad a > 0$	$(2\pi)^{-1/2} a^{-1} [e^{-a\xi} \text{Ei}(a\xi) - e^{a\xi} \text{Ei}(-a\xi)],$ $\xi > 0$	ET I 65(14)
9 $x (x^2 + a^2)^{-3/2}, \quad \text{Re } a > 0$	$(2/\pi)^{1/2} \xi K_0(a\xi), \quad \xi > 0$	ET I 66(27)
10 $x^{-1/2} (x^2 + a^2)^{-1/2}, \quad \text{Re } a > 0$	$\xi^{1/2} I_{\frac{1}{4}}(\frac{1}{2}a\xi) K_{\frac{1}{4}}(\frac{1}{2}a\xi), \quad \xi > 0$	ET I 66(28)
11⁷ $x (x^2 + a^2)^{-\nu-\frac{3}{2}},$ $\text{Re } \nu > -1, \quad \text{Re } a > 0$	$\frac{\xi^{\nu+1}}{\sqrt{2}(2a)^\nu \Gamma(\nu + \frac{3}{2})} K_\nu(a\xi),$	
12 $\frac{x}{a^2 + x^2}, \quad \text{Re } a > 0$	$\left(\frac{\pi}{2}\right)^{1/2} e^{-a\xi}, \quad \xi > 0$	ET I 65(15)
13 $\frac{x}{(a^2 + x^2)^2}$	$\sqrt{\pi/8} a^{-1} \xi e^{-a\xi}, \quad \xi > 0$	ET I 67(35)
14 $x^{-1} (x^2 + a^2)^{-1}, \quad \text{Re } a > 0$	$\frac{\sqrt{\pi/2}}{a^2} (1 - e^{-a\xi}), \quad \xi > 0$	ET I 65(20)
15 $x^{-1} e^{-ax}, \quad \text{Re } a > 0$	$(2/\pi)^{1/2} \tan^{-1}\left(\frac{\xi}{a}\right), \quad \xi > 0$	ET I 72(2)
16 $x^{\nu-1} e^{-ax},$ $\text{Re } \nu > -1, \quad \text{Re } a > 0$	$(2/\pi)^{1/2} \Gamma(\nu) (a^2 + \xi^2)^{-\nu/2} \sin \left[\nu \tan^{-1}\left(\frac{\xi}{a}\right) \right],$ $\xi > 0$	ET I 72(7)
17 $e^{-ax}, \quad \text{Re } a > 0$	$\frac{\sqrt{2/\pi} \xi}{a^2 + \xi^2}, \quad \xi > 0$	ET I 72(1)
18 $x e^{-ax}, \quad \text{Re } a > 0$	$\frac{(2/\pi)^{1/2} 2a\xi}{(a^2 + \xi^2)^2}, \quad \xi > 0$	ET I 72(3)
19 $x e^{-ax^2}, \quad \arg a < \pi/2$	$(2a)^{-3/2} \xi \exp\left(\frac{-\xi^2}{4a}\right), \quad \xi > 0$	ET I 73(19)
20 $\frac{\sin ax}{x}, \quad a > 0$	$\frac{1}{(2\pi)^{1/2}} \ln \left \frac{\xi + a}{\xi - a} \right , \quad \xi > 0$	ET I 78(1)
21 $\frac{\sin ax}{x^2}, \quad a > 0$	$\begin{cases} \xi \left(\frac{\pi}{2}\right)^{1/2} & 0 < \xi < a \\ a \left(\frac{\pi}{2}\right)^{1/2} & a < \xi < \infty \end{cases}, \quad \xi > 0$	ET I 78(2)
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$f(x)$		$F_s(\xi) \quad (\xi > 0)$
22 $\sin\left(\frac{a^2}{x}\right),$	$a > 0$	$a\left(\frac{\pi}{2}\right)^{1/2} \xi^{-1/2} J_1\left(2a\xi^{1/2}\right),$ $\xi > 0$ ET I 83(6)
23 $x^{-1} \sin\left(\frac{a^2}{x}\right),$	$a > 0$	$\left(\frac{\pi}{2}\right)^{1/2} Y_0\left(2a\xi^{1/2}\right) + \left(\frac{2}{\pi}\right)^{1/2} K_0\left(2a\xi^{1/2}\right)$ ET I 83(7)
24 $x^{-2} \sin\left(\frac{a^2}{x}\right),$	$a > 0$	$\left(\frac{\pi}{2}\right)^{1/2} a^{-1} \xi^{1/2} J_1\left(2a\xi^{1/2}\right),$ $\xi > 0$ ET I 83(8)
25¹⁰ $\operatorname{cosech}(ax),$	$\operatorname{Re} a > 0$	$(\pi/2)^{1/2} a^{-1} \tanh\left(\frac{1}{2}\pi a^{-1}\xi\right),$ $\xi > 0$ ET I 88(2)
26 $\coth\left(\frac{1}{2}ax\right) - 1,$	$\operatorname{Re} a > 0$	$(2\pi)^{1/2} a^{-1} \coth\left(\pi a^{-1}\xi\right) - \xi,$ $\xi > 0$ ET I 88(3)
27 $(1-x^2)^{-1} \sin(\pi x)$		$\begin{cases} (2/\pi)^{1/2} \sin \xi & 0 \leq \xi \leq \pi \\ 0 & \pi < \xi \end{cases}$ ET I 78(4)
28 $e^{-ax^2} \sin(bx),$	$\operatorname{Re} a > 0$	$(2a)^{-1/2} \exp\left[-(\xi^2 + b^2)/(4a)\right] \sinh(b\xi/2a),$ $\xi > 0$ ET I 78(7)
29 $\frac{\sin^2(ax)}{x},$	$a > 0$	$\begin{cases} \pi^{1/2} 2^{-3/2} & 0 < \xi < 2a \\ \pi^{1/2} 2^{-5/2} & \xi = 2a \\ 0 & 2a < \xi \end{cases}$ ET I 78(8)
30 $\sin(ax^2),$	$a > 0$	$a^{-1/2} \left\{ \cos(\xi^2/4a) C\left[(2\pi a)^{-1/2}\xi\right] \right\}$ $+ \sin(\xi^2/4a) S\left[(2\pi a)^{-1/2}\xi\right],$ $\xi > 0$ ET I 82(1)
31 $\cos(ax^2),$	$a > 0$	$a^{-1/2} \left\{ \sin(\xi^2/4a) C\left[(2\pi a)^{-1/2}\xi\right] \right\}$ $- \cos(\xi^2/4a) S\left[(2\pi a)^{-1/2}\xi\right],$ $\xi > 0$
<i>continued on next page</i>		

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	$f(x)$	$F_s(\xi)$	$(\xi > 0)$
32	$\arctan\left(\frac{x}{a}\right), \quad a > 0$	$(\pi/2)^{1/2} \xi^{-1} e^{-a\xi},$	$\xi > 0 \quad \text{ET I 87(3)}$
33⁷	$\arctan\left(\frac{2a}{x}\right), \quad \operatorname{Re} a > 0$	$(2\pi)^{-1/2} e^{-a\xi} \sinh(a\xi),$	$\xi > 0 \quad \text{ET I 87(8)}$
34	$\frac{\ln x}{x}$	$-(\pi/2)^{1/2} (C + \ln \xi),$	$\xi > 0 \quad \text{ET I 76(2)}$
35	$\ln \left \frac{x+a}{x-a} \right , \quad a > 0$	$(2\pi)^{1/2} \xi^{-1} \sin(a\xi),$	$\xi > 0 \quad \text{ET I 77(11)}$
36⁷	$\frac{\ln(1+a^2x^2)}{x}, \quad a > 0$	$-(2\pi)^{1/2} \operatorname{Ei}(-\xi/a),$	$\xi > 0 \quad \text{ET I 77(14)}$
37	$J_0(ax), \quad a > 0$	$\begin{cases} 0 & 0 < \xi < a \\ (2/\pi)^{1/2} (\xi^2 - a^2)^{-1/2} & a < \xi < \infty \end{cases}$	ET I 99(1)
38	$J_\nu(ax), \quad \operatorname{Re} \nu > -2, \quad a > 0$	$(2/\pi)^{1/2} (a^2 - \xi^2)^{-1/2} \sin \left[\nu \sin^{-1} \left(\frac{\xi}{a} \right) \right]$ for $0 < \xi < a$ $\frac{a^\nu \cos(\frac{1}{2}\nu\pi)}{(\xi^2 - a^2)^{1/2} [\xi + (\xi^2 - a^2)^{1/2}]^\nu}$ for $a < \xi < \infty$	ET I 99(3)
39	$\frac{J_0(ax)}{x}, \quad a > 0$	$\begin{cases} (2/\pi)^{1/2} \sin^{-1} \left(\frac{\xi}{a} \right) & 0 < \xi < a \\ (\pi/2)^{1/2} & a < \xi < \infty \end{cases}$	ET I 99(4)
40⁷	$(x^2 + b^2)^{-1} J_0(ax),$ $a > 0, \quad \operatorname{Re} b > 0$	$(2/\pi)^{1/2} \sinh(b\xi) K_0(ab)/b,$ $0 < \xi < a$	ET I 100(12)
41	$x(x^2 + b^2)^{-1} J_0(ax),$ $a > 0, \quad \operatorname{Re} b > 0$	$(\pi/2)^{1/2} e^{-b\xi} I_0(ab),$ $a < \xi < \infty$	ET I 100(13)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_s has not been used in those tables.

17.34 Table of Fourier cosine transforms

$f(x)$		$F_c(\xi)$	
1	$x^{-\nu}, \quad 0 < \operatorname{Re} \nu < 1$	$(\pi/2)^{1/2} [\Gamma(\nu)]^{-1} \sec(\frac{1}{2}\nu\pi) \xi^{\nu-1},$ $\xi > 0$	ET I 10(1)
2	$\begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$	$(2/\pi)^{1/2} \frac{\sin(a\xi)}{\xi},$ $\xi > 0$	ET I 7(1)
3	$\begin{cases} 0 & 0 < x < a \\ 1/x & x > a \end{cases}$	$-(2/\pi)^{1/2} \operatorname{Ci}(a\xi),$ $\xi > 0$	ET I 8(3)
4	$\begin{cases} x^{-1/2} & 0 < x < a \\ 0 & x > a \end{cases}$	$2\xi^{-1/2} C(a\xi),$ $\xi > 0$	ET I 8(5)
5	$\begin{cases} 0 & 0 < x < a \\ x^{-1/2} & x > a \end{cases}$	$2\xi^{-1/2} [\frac{1}{2} - C(a\xi)],$ $\xi > 0$	ET I 8(6)
6⁹	$x^{\nu-1}, \quad 0 < \nu < 1$	$(2/\pi)^{1/2} \Gamma(\nu) \xi^{-\nu} \cos(\frac{1}{2}\nu\pi),$ $0 < \nu < 1$	ET I 10(1)
7	$\frac{1}{x^2 + a^2}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{1/2} e^{-a\xi}}{a},$ $\xi > 0$	ET I 11(7)
8¹¹	$\frac{1}{(x^2 + a^2)^2}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{\frac{1}{2}} (1 + a\xi) e^{-a\xi}}{2a^3},$ $\xi > 0$	ET I 11(7)
9	$(x^2 + a^2)^{-\nu-\frac{1}{2}},$ $\operatorname{Re} a > 0, \operatorname{Re} \nu > -\frac{1}{2}$	$\sqrt{2} \left(\frac{\xi}{2a}\right)^{\nu} \frac{K_{\nu}(a\xi)}{\Gamma(\nu + \frac{1}{2})},$ $\xi > 0$	ET I 11(7)
10	$\begin{cases} (a^2 - x^2)^{\nu} & 0 < x < a \\ 0 & x > a \end{cases},$ $\operatorname{Re} \nu > -1$	$2^{\nu} \Gamma(\nu + 1) (a/\xi)^{\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}(a\xi),$ $\xi > 0$	ET I 11(8)
11	$\begin{cases} 0 & 0 < x < a \\ (x^2 - a^2)^{-\nu-\frac{1}{2}} & x > a \end{cases},$ $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$	$-2^{-(\nu+\frac{1}{2})} \Gamma(\frac{1}{2} - \nu) (\xi/a)^{\nu} Y_{\nu}(a\xi),$ $\xi > 0$	ET I 11(9)
12	$e^{-ax}, \quad \operatorname{Re} a > 0$	$(2/\pi)^{1/2} a (a^2 + \xi^2)^{-1},$ $\xi > 0$	ET I 14(1)

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$f(x)$		$F_c(\xi)$
13 $x e^{-ax},$	$\operatorname{Re} a > 0$	$(2/\pi)^{1/2} (a^2 - \xi^2) (a^2 + \xi^2)^{-2},$ $\xi > 0$ ET I 15(7)
14⁷ $x^{\nu-1} e^{-ax},$	$\operatorname{Re} a > 0, \operatorname{Re} \nu > a$	$(2/\pi)^{1/2} \Gamma(\nu) (a^2 + \xi^2)^{-\nu/2} \cos \left[\nu \tan^{-1} \left(\frac{\xi}{a} \right) \right],$ $\xi > 0$ ET I 15(7)
15 $x^{-1/2} e^{-ax},$	$\operatorname{Re} a > 0$	$(a^2 + \xi^2)^{-1/2} \left[(a^2 + \xi^2)^{1/2} + a \right]^{1/2},$ $\xi > 0$ ET I 14(4)
16⁷ $e^{-a^2 x^2},$	$\operatorname{Re} a > 0$	$2^{-1/2} a ^{-1} e^{-\xi^2/4a^2},$ $\xi > 0$ ET I 15(11)
17 $x^{-1} e^{-x} \sin x$		$(2\pi)^{-1/2} \tan^{-1} \left(\frac{2}{\xi^2} \right),$ $\xi > 0$ ET I 19(7)
18 $\sin(ax^2),$	$a > 0$	$\frac{1}{2\sqrt{a}} \left[\cos \left(\frac{\xi^2}{4a} \right) - \sin \left(\frac{\xi^2}{4a} \right) \right],$ $\xi > 0$ ET I 23(1)
19 $\cos(ax^2),$	$a > 0$	$\frac{1}{2\sqrt{a}} \left[\cos \left(\frac{\xi^2}{4a} \right) + \sin \left(\frac{\xi^2}{4a} \right) \right],$ $\xi > 0$ ET I 24(7)
20 $\frac{\sin(ax)}{x},$	$a > 0$	$\begin{cases} (\pi/2)^{1/2} & \xi < a \\ \frac{1}{2} (\pi/2)^{1/2} & \xi = a \\ 0 & \xi > a \end{cases}$ ET I 18(1)
21⁷ $\frac{\sin^2(ax)}{x^2},$	$a > 0$	$\begin{cases} (\pi/2)^{1/2} (a - \frac{1}{2}\xi) & \xi < 2a \\ 0 & 2a < \xi \end{cases}$ ET I 19(8)
22⁷ $e^{-bx} \sin(ax),$	$a > 0, \operatorname{Re} b > 0$	$(2\pi)^{-1/2} \left[\frac{a + \xi}{b^2 + (a + \xi)^2} + \frac{a - \xi}{b^2 + (a - \xi)^2} \right],$ $\xi > 0$ ET I 19(6)
23 $\frac{\sin \left[b (x^2 + a^2)^{1/2} \right]}{(x^2 + a^2)^2}$	$a > 0$	$(b/a) (\pi/2)^{1/2} e^{-a\xi},$ $\xi > 0$ ET I 26(29)
24 $(x^2 + a^2)^{-1/2} \sin \left[b (x^2 + a^2)^{1/2} \right],$	$a > 0$	$\begin{cases} (\pi/2)^{1/2} J_0 \left[a (b^2 - \xi^2)^{1/2} \right] & 0 < \xi < b \\ 0 & b < \xi \end{cases}$ ET I 26(30)

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	$f(x)$	$F_c(\xi)$
25	$\frac{1 - \cos(ax)}{x^2}, \quad a > 0$	$\begin{cases} (\pi/2)^{1/2} (a - \xi) & \xi < a \\ 0 & a < \xi \end{cases} \quad \text{ET I 20(16)}$
26	$e^{-ax^2} \sin(bx^2), \quad \operatorname{Re} a > \operatorname{Im} b $	$2^{-1/2} (a^2 + b^2)^{-1/4} \exp \{-a\xi^2 / [4(a^2 + b^2)]\} \\ \times \sin \left[\frac{1}{2} \arctan(b/a) - \frac{1}{4} b\xi^2 (a^2 + b^2)^{-1} \right],$ $\xi > 0 \quad \text{ET I 23(5)}$
27	$e^{-ax^2} \cos(bx^2), \quad \operatorname{Re} a > \operatorname{Im} b $	$2^{-1/2} (a^2 + b^2)^{-1/4} \exp \{-a\xi^2 / [4(a^2 + b^2)]\} \\ \times \cos \left[\frac{1}{4} b\xi^2 (a^2 + b^2)^{-1} - \frac{1}{2} \arctan(b/a) \right],$ $\xi > 0 \quad \text{ET I 24(6)}$
28	$\frac{\sinh(ax)}{\sinh(bx)} \quad \operatorname{Re} a < \operatorname{Re} b$	$\left(\frac{\pi}{2}\right)^{1/2} \frac{\sin(\pi a/b)}{b [\cosh(\pi \xi/b) + \cos(\pi a/b)]},$ $\xi > 0 \quad \text{ET I 31(14)}$
29	$\frac{\cosh(ax)}{\cosh(bx)}, \quad \operatorname{Re} a < \operatorname{Re} b$	$\frac{(2\pi)^{1/2} \cos(\pi a/2b) \cosh(\pi \xi/2b)}{b [\cosh(\pi \xi/b) + \cos(\pi a/b)]},$ $\xi > 0 \quad \text{ET I 31(12)}$
30	$\operatorname{sech}(ax), \quad \operatorname{Re} a > 0$	$a^{-1} (\pi/2)^{1/2} \operatorname{sech}(\pi \xi/2a),$ $\xi > 0 \quad \text{ET I 30(1)}$
31	$(x^2 + a^2) \operatorname{sech}\left(\frac{\pi x}{2a}\right), \quad \operatorname{Re} a > 0$	$2(2/\pi)^{1/2} a^3 \operatorname{sech}^3(a\xi), \quad \xi > 0 \quad \text{ET I 32(19)}$
32	$\ln\left(1 + \frac{a^2}{x^2}\right), \quad \operatorname{Re} a > 0$	$(2\pi)^{1/2} \xi^{-1} (1 - e^{-a\xi}), \quad \xi > 0 \quad \text{ET I 18(10)}$
33⁷	$\ln\left(\frac{a^2 + x^2}{b^2 + x^2}\right),$ $\operatorname{Re} a > 0, \operatorname{Re} b > 0$	$(2\pi)^{1/2} (e^{-b\xi} - e^{-a\xi}), \quad \xi > 0 \quad \text{ET I 18(12)}$
34	$(x^2 + b^2)^{-1} J_0(ax),$ $a > 0, \operatorname{Re} b > 0$	$(\pi/2)^{1/2} b^{-1} e^{-b\xi} I_0(ab),$ $a < \xi < \infty \quad \text{ET I 45(14)}$
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$f(x)$	$F_c(\xi)$
35 $x (x^2 + b^2)^{-1} J_0(ax),$ $a > 0, \quad \operatorname{Re} b > 0$	$(2/\pi)^{1/2} \cosh(b\xi) K_0(ab),$ $0 < \xi < a$ ET I 45(15)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_c has not been used in those tables.

17.35 Relationships between transforms

The following relationships exist between transforms, and they may be used to derive further transform pairs from among the results given in Sections 17.13–17.34. The appropriate sections of the main body of the tables may also be used to extend the list of transform pairs.

17.351

Fourier cosine transform and Laplace transform relationship

$$\mathcal{F}_c[f(x); \xi] = \frac{1}{\sqrt{2\pi}} \mathcal{L}[f(x); i\xi] + \frac{1}{\sqrt{2\pi}} \mathcal{L}[f(x); -i\xi].$$

17.352

Fourier sine transform and Laplace transform relationship

$$\mathcal{F}_s[f(x); \xi] = \frac{i}{\sqrt{2\pi}} \mathcal{L}[f(x); i\xi] - \frac{i}{\sqrt{2\pi}} \mathcal{L}[f(x); -i\xi].$$

17.353

Exponential Fourier transform and Laplace transform relationship

$$\mathcal{F}[f(x); \xi] = \sqrt{2\pi} \mathcal{L}[f(x); -i\xi] + \sqrt{2\pi} \mathcal{L}[f(-x); i\xi].$$

17.41¹⁰ Mellin transform

The **Mellin transform** of the function $f(x)$, denoted by $f^*(s)$, is defined by the integral

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx.$$

The functions $f(x)$ and $f^*(s)$ are called a **Mellin transform pair**, and knowledge of either one enables the other to be recovered.

The transform exists, provided the integral

$$\int_0^\infty |f(x)| x^{k-1} dx$$

is bounded for some $k > 0$, and then the inversion of the Mellin transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds,$$

where $c > k$.

Setting

$$f^*(s) = \mathcal{M}[f(x); s]$$

to denote the Mellin transform, we have the symbolic expression for the inverse result

$$f(x) = \mathcal{M}^{-1}[f^*(s); x]. \quad \text{MS 397(6)}$$

17.42 Basic properties of the Mellin transform

1. For a and b arbitrary constants,

$$\mathcal{M}[af(x) + bg(x)] = af^*(s) + bf^*(s) \quad (\text{linearity})$$

2. If $\lim_{x \rightarrow 0} x^{s-r-1} f^{(r)}(x) = 0$, $r = 0, 1, \dots, n-1$,

$$(i) \quad \mathcal{M}[f^{(n)}(x); s] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n) \quad (\text{transform of a derivative}) \quad \text{SU 267 (4.2.3)}$$

$$(ii) \quad \mathcal{M}[x^n f^{(n)}(x); s] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} f^*(s) \quad (\text{transform of a derivative}) \quad \text{SU 267 (4.2.5)}$$

3. Denoting the n^{th} repeated integral of $f(x)$ by $I_n[f(x)]$, where

$$I_n[f(x)] = \int_0^x I_{n-1}[f(u)] du,$$

$$(i) \quad \mathcal{M}[I_n[f(x)]; s] = (-1)^n \frac{\Gamma(s)}{\Gamma(n+s)} f^*(s+n) \quad (\text{transform of an integral}) \quad \text{SU 269 (4.2.15)}$$

$$(ii) \quad \mathcal{M}[I_n^\infty[f(x)]; s] = \frac{\Gamma(s)}{\Gamma(s+n)} f^*(s+n),$$

where

$$I_n^\infty[f(x)] = \int_x^\infty I_{n-1}^\infty[f(u)] du \quad (\text{transform of an integral}) \quad \text{SU 269 (4.2.18)}$$

$$4. \quad \mathcal{M}[f(x)g(x); s] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(u)g^*(s-u) du \quad (\text{Mellin convolution theorem}) \quad \text{SU 275(4.4.1)}$$

17.43 Table of Mellin transforms

	$f(x)$	$f^*(s)$	
1	e^{-x}	$\Gamma(s),$	$\operatorname{Re} s > 0$ SU 521(M13)
2	e^{-x^2}	$\frac{1}{2} \Gamma\left(\frac{1}{2}s\right),$	$\operatorname{Re} s > 0$ SU 521(M14)
3	$\cos x$	$\Gamma(s) \cos\left(\frac{1}{2}\pi s\right),$	$0 < \operatorname{Re} s < 1$ SU 521(M15)
4	$\sin x$	$\Gamma(s) \sin\left(\frac{1}{2}\pi s\right),$	$0 < \operatorname{Re} s < 1$ SU 521(M16)
5	$\frac{1}{1-x}$	$\pi \cot(\pi s),$	$0 < \operatorname{Re} s < 1$ SU 521(M1)
6	$\frac{1}{1+x}$	$\pi \operatorname{cosec}(\pi s),$	$0 < \operatorname{Re} s < 1$ SU 521(M2)
7	$(1+x^a)^{-b}$	$\frac{\Gamma(s/a) \Gamma(b-s/a)}{a \Gamma(b)},$	$0 < \operatorname{Re} s < ab$ SU 521(M3)
8	$\frac{T_n(x) \operatorname{H}(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{-s} \pi \Gamma(s)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}s + \frac{1}{2}n\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}s - \frac{1}{2}n\right)},$	$\operatorname{Re} s > 0$ SU 521(M4)
9	$\frac{T_n(x^{-1}) \operatorname{H}(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{s-2} \Gamma\left(\frac{1}{2}n + \frac{1}{2}s\right) \Gamma\left(\frac{1}{2}s - \frac{1}{2}n\right)}{\Gamma(s)},$	$\operatorname{Re} s > n$ SU 521(M5)
10	$P_n(x) \operatorname{H}(1-x)$	$\frac{\Gamma\left(\frac{1}{2}s\right) \Gamma\left(\frac{1}{2}s + \frac{1}{2}\right)}{2 \Gamma\left(\frac{1}{2}s - \frac{1}{2}n + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}s + \frac{1}{2}n + 1\right)},$	$\operatorname{Re} s > 0$ SU 521(M6)
11	$P_n(x^{-1}) \operatorname{H}(1-x)$	$\frac{2^{s-1} \Gamma\left(\frac{1}{2}s + \frac{1}{2}n + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}s - \frac{1}{2}n\right)}{\sqrt{\pi} \Gamma(s+1)},$	$\operatorname{Re} s > n$ SU 521(M7)
12	$\frac{1+x \cos \phi}{1-2x \cos \phi + x^2}$	$\frac{\pi \cos(s\phi)}{\sin(s\pi)},$	$0 < \operatorname{Re} s < 1$ SU 521(M11)
13	$\frac{x \sin \phi}{1-2x \cos \phi + x^2},$	$\frac{\pi \sin(s\phi)}{\sin(s\pi)},$	$0 < \operatorname{Re} s < 1$ SU 521(M12)
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$f(x)$	$f^*(s)$	
14 $e^{-x \cos \phi} \cos(x \sin \phi),$ $\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s) \cos(s\phi),$	$\operatorname{Re} s > 0$ SU 522(M17)
15 $e^{-x \sin \phi} \sin(x \sin \phi),$ $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s) \sin(s\phi),$	$\operatorname{Re} s > -1$ SU 522(M18)
16 $x^{-\nu} J_{\nu}(x),$ $\nu > -\frac{1}{2}$	$\frac{2^{s-\nu-1} \Gamma(\frac{1}{2}s)}{\Gamma(\nu - \frac{1}{2}s + 1)},$	$0 < \operatorname{Re} s < 1$ SU 522(M19)
17 $Y_{\nu}(x),$ $\nu \in \mathbb{R}$	$-2^{s-1} \pi^{-1} \Gamma(\frac{1}{2}s + \frac{1}{2}\nu) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu)$ $\times \cos(\frac{1}{2}s - \frac{1}{2}\nu) \pi,$ $ \nu < \operatorname{Re} s < \frac{3}{2}$	SU 522(M20)
18 $K_{\nu}(x),$ $\nu \in \mathbb{R}$	$2^{s-2} \Gamma(\frac{1}{2}s + \frac{1}{2}\nu) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu),$ $\operatorname{Re} s > \nu > 0$	SU 522(M21)
19 $\mathbf{H}_{\nu}(x),$ $\nu \in \mathbb{R}$	$\frac{2^{s-1} \tan(\frac{1}{2}\pi s + \frac{1}{2}\pi\nu) \Gamma(\frac{1}{2}s + \frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s + 1)},$ $-1 - \nu < \operatorname{Re} s < \min(\frac{3}{2}, 1 - \nu)$	SU 522(M22)
20 $\frac{1}{a + x^n},$ $ \arg a < \pi, \quad n = 1, 2, 3, \dots,$	$\pi n^{-1} \operatorname{cosec}\left(\frac{\pi s}{n}\right) a^{(s/n)-1},$	$0 < \operatorname{Re} s < n$ MS 453
21 $(1 + ax^h)^{-\nu},$ $h > 0, \quad \arg a < \pi$	$h^{-1} a^{-s/h} \mathbf{B}(s/h, \nu - (s/h))$ $0 < \operatorname{Re} s < h \operatorname{Re} \nu$	MS 454
22 $\begin{cases} (1 - x^h)^{\nu-1} & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1 \end{cases},$ $h > 0, \quad \operatorname{Re} \nu > 0$	$h^{-1} \mathbf{B}(\nu, s/h)$	MS 454
23 $\ln(1 + ax),$ $ \arg a < \pi$	$\pi s^{-1} a^{-s} \operatorname{cosec}(\pi s),$	$-1 < \operatorname{Re} s < 0$ MS 454
24 $\arctan x$	$-\frac{1}{2} \pi s^{-1} \sec(\pi s/2),$	$-1 < \operatorname{Re} s < 0$ MS 454
<i>continued on next page</i>		

<i>continued from previous page</i>			
$f(x)$		$f^*(s)$	
25	$\operatorname{arccot} x$	$\frac{1}{2}\pi s^{-1} \sec(\pi s/2), \quad 0 < \operatorname{Re} s < 1$	MS 454
26	$\operatorname{cosech}(ax) \quad \operatorname{Re} a > 0$	$a^{-s} 2(1 - 2^{-s}) \Gamma(s) \zeta(s), \quad \operatorname{Re} s > 1$	MS 454
27	$\operatorname{sech}^2(ax), \quad \operatorname{Re} a > 0$	$4a^{-s}(1 - 2^{2-s}) \Gamma(s) 2^{-s} \zeta(s-1),$ $\operatorname{Re} s > 2$	MS 454
28	$\operatorname{cosech}^2(ax), \quad \operatorname{Re} a > 0$	$4a^{-s} \Gamma(s) 2^{-s} \zeta(s-1), \quad \operatorname{Re} s > 2$	MS 454
29¹¹	$(x^2 + b^2)^{-\frac{1}{2}\nu} J_\nu \left[a(x^2 + b^2)^{1/2} \right]$	$2^{\frac{1}{2}s-1} a^{-\frac{1}{2}s} b^{\frac{1}{2}s-\nu} \Gamma\left(\frac{1}{2}s\right) J_{\nu-s/2}(ab),$ $0 < \operatorname{Re} s < \frac{3}{2} + \operatorname{Re} \nu$	ET I 328
30	$\begin{cases} (a^2 - x^2)^{\frac{1}{2}\nu} J_\nu \left[a(b^2 - x^2)^{1/2} \right] & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$ $\operatorname{Re} \nu > -1$	$2^{\frac{1}{2}s-1} \Gamma\left(\frac{1}{2}s\right) b^{-\frac{1}{2}s} a^{\nu+\frac{1}{2}s} J_{\nu+\frac{1}{2}s}(ab),$ $\operatorname{Re} s > 0$	MS 455
31	$\begin{cases} (a^2 - x^2)^{-\frac{1}{2}\nu} J_\nu \left[b(a^2 - x^2)^{1/2} \right] & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$	$2^{1-\nu} [\Gamma(\nu)]^{-1} a^{\frac{1}{2}s-\nu} b^{-\frac{1}{2}\nu} s_{\nu-1+\frac{1}{2}s, \frac{1}{2}s-\nu}(ab),$ $\operatorname{Re} s > 0$	MS 455
32	$K_\nu(\alpha x)$	$\alpha^{-s} 2^{s-2} \Gamma\left(\frac{1}{2}s - \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}s + \frac{1}{2}\nu\right),$ $\operatorname{Re} s > \operatorname{Re} \nu $	MS 455
33	$(\beta a^2 + x^2)^{-\frac{1}{2}\nu}$ $\times K_\nu \left[\alpha(\beta a^2 + x^2)^{1/2} \right]$ $\operatorname{Re}(\alpha, \beta) > 0$	$\alpha^{-\frac{1}{2}s} 2^{\frac{1}{2}s-1} \beta^{\frac{1}{2}s-\nu} \Gamma\left(\frac{1}{2}s\right) K_{\nu-\frac{1}{2}s}(\alpha\beta),$ $\operatorname{Re} s > 0$	MS 455

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18 The z-Transform

18.1–18.3 Definition, Bilateral, and Unilateral z-Transforms

18.1 Definitions

The **z-transform** converts a numerical sequence $x[n]$ into a function of the complex variable z , and it takes two different forms. The **bilateral** or **two-sided z-transform**, denoted here by $Z_b\{x[n]\}$, is used mainly in signal and image processing, while the **unilateral** or **one-sided z-transform**, denoted here by $Z_u\{x[n]\}$, is used mainly in the analysis of discrete time systems and the solution of linear difference equations.

The **bilateral z-transform**, $X_b(z)$ of the sequence $x[n] = \{x_n\}_{n=-\infty}^{\infty}$ is defined as

$$Z_b\{x[n]\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} = X_b(z),$$

and the **unilateral z-transform** $X_u(z)$ of the sequence $x[n] = \{x_n\}_{n=0}^{\infty}$ is defined as

$$Z_b\{x[n]\} = \sum_{n=0}^{\infty} x_n z^{-n} = X_u(z),$$

where each has its own domain of convergence (DOC). The series $X_b(z)$ is a Laurent series, and $X_u(z)$ is the principal part of the Laurent series for $X_b(z)$. When $x_n = 0$ for $n < 0$, the two z -transforms $X_b(z)$ and $X_u(z)$ are identical. In each case the sequence $x[n]$ and its associated z -transform is called a **z-transform pair**.

The inverse z -transformation $x[n] = Z^{-1}\{X(z)\}$ is given by

$$x[n] = \frac{1}{2\pi i} \int_{\Gamma} X(z) z^{n-1} dz,$$

where $X(z)$ is either $X_b(z)$ or $X_u(z)$, and Γ is a simple closed contour containing the origin and lying entirely within the domain of convergence of $X(z)$. In many practical situations, the z -transform is either found by using a series expansion of $X(z)$ in the inversion integral or, if $X(z) = N(z)/D(z)$ where $N(z)$ and $D(z)$ are polynomials in z , by means of partial fractions and the use of an appropriate table of z -transform pairs. In order for the inverse z -transform to be unique, it is necessary to specify the domain of convergence, as can be seen by comparison of entries 3 and 4 of Table 18.2. Table 18.1 lists general properties of the bilateral z -transform, and Table 18.2 lists some bilateral z -transform pairs. In what follows, use is made of the **unit integer function** $h(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$, that is, a generalization of

the Heaviside step function, and the **unit integer pulse function** $\Delta(n - k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$, that is, a generalization of the delta function.

18.2 Bilateral z -transform

Table 18.1 General properties of the bilateral z -transform $X_b(n) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$.

	Term in sequence	z -Transform $X_b(z)$	Domain of Convergence
1	$\alpha x_n + \beta y_n$	$\alpha X_b(z) + \beta Y_b(z)$	Intersection of DOC's of $X_b(z)$ and $Y_b(z)$ with α, β constants
2	x_{n-N}	$z^{-N} X_b(z)$	DOC of $X_b(z)$, to which it may be necessary to add or delete the origin or the point at infinity
3	$n x_n$	$-z \frac{dX_b(z)}{dz}$	DOC of $X_b(z)$, to which it may be necessary to add or delete the origin and the point at infinity
4	$z_0^n x_n$	$X_b\left(\frac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $ z_0 $
5	$n z_0^n x_n$	$-z \frac{dX_b(z/z_0)}{dz}$	DOC of $X_b(z)$ scaled by $ z_0 $ to which it may be necessary to add or delete the origin and the point at infinity
6	x_{-n}	$X_b(1/z)$	DOC of radius $1/R$, where R is the radius of convergence of DOC of $X_b(z)$
7	$n x_{-n}$	$-z \frac{dX_b(1/z)}{dz}$	DOC of radius $1/R$, where R is the radius of convergence of DOC of $X_b(z)$
8	$\bar{x}_n$	$\overline{X_b(z)}$	The same DOC as x_n
9	$\operatorname{Re} x_n$	$\frac{1}{2} [X_b(z) + \overline{X_b(z)}]$	DOC contains the DOC of x_n
10	$\operatorname{Im} x_n$	$\frac{1}{2i} [X_b(z) - \overline{X_b(z)}]$	DOC contains the DOC of x_n
11	$\sum_{k=-\infty}^{\infty} x_k y_{n-k}$	$X_b(z) Y_b(z)$	DOC contains the intersection of the DOCs of $X_b(z)$ and $Y_b(z)$ (convolution theorem)
12	$x_n y_n$	$\frac{1}{2\pi i} \int_{\Gamma} X_b(\xi) Y_b\left(\frac{z}{\xi}\right) \xi^{-1} d\xi$	DOC contains the DOCs of $X_b(z)$ and $Y_b(z)$, with Γ inside the DOC and containing the origin (convolution theorem)
13	Parseval formula	$\sum_{n=-\infty}^{\infty} x_n \bar{y}_n = \frac{1}{2\pi i} \int_{\Gamma} X_b(\xi) \overline{Y_b\left(\frac{z}{\xi}\right)} \xi^{-1} d\xi$	DOC contains the intersection of DOCs of $X_b(z)$ and $Y_b(z)$, with Γ inside the DOC and containing the origin
14	Initial value theorem for $x_n h(n)$	$x_0 = \lim_{z \rightarrow \infty} X_b(z)$	

Table 18.2 Basic bilateral z -transforms

	Term in sequence	z -Transform $X_b(z)$	Domain of Convergence
1	$\Delta(n)$	1	Converges for all z
2	$\Delta(n - N)$	z^{-N}	When $N > 0$ convergence is for all z except at the origin. When $N < 0$ convergence is for all z except at ∞
3	$a^n h(n)$	$\frac{z}{z - a}$	$ z > a $
4	$a^n h(-n - 1)$	$\frac{z}{z - a}$	$ z < a $
5	$na^n h(n)$	$\frac{az}{(z - a)^2}$	$ z > a > 0$
6	$na^n h(-n - 1)$	$\frac{az}{(z - a)^2}$	$ z < a, \quad a > 0$
7	$n^2 a^n h(n)$	$\frac{az(z + a)}{(z - a)^3}$	$ z > a > 0$
8	$\left(\frac{1}{a^n} + \frac{1}{b^n}\right) h(n)$	$\frac{az}{az - 1} + \frac{bz}{bz - 1}$	$ z > \max\left(\frac{1}{ a }, \frac{1}{ b }\right)$
9	$a^n h(n - N)$	$\frac{z(1 - (a/z)^N)}{z - a}$	$ z > 0$
10	$a^n h(n) \sin \Omega n$	$\frac{az \sin \Omega}{z^2 - 2az \cos \Omega + a^2}$	$ z > a > 0$
11	$a^n h(n) \cos \Omega n$	$\frac{z(z - a \cos \Omega)}{z^2 - 2az \cos \Omega + a^2}$	$ z > a > 0$
12	$e^{an} h(n)$	$\frac{z}{z - e^a}$	$ z > e^{-a}$
13	$e^{-an} h(n) \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z > e^{-a}$
14	$e^{-an} h(n) \cos \Omega n$	$\frac{ze^a(z e^a - \cos \Omega)}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z > e^{-a}$

18.3 Unilateral z -transform

The relationship between the Laplace transform of a continuous function $x(t)$ sampled at $t = 0, T, 2T, \dots$ and the unilateral z -transform of the function $\hat{x}(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT)$ follows from the result

$$\begin{aligned}\mathcal{L}\{\hat{x}(t)\} &= \int_0^{\infty} \left[\sum_{k=0}^{\infty} x(kT)\delta(t - kT) \right] e^{-st} dt \\ &= \sum_{k=0}^{\infty} x(kT)e^{-ksT}.\end{aligned}$$

Setting $z = e^{sT}$, this becomes:

$$\mathcal{L}\{\hat{x}(t)\} = \sum_{k=0}^{\infty} x(kT)z^{-k} = X(z),$$

showing that the unilateral z -transform $X_u(z)$ can be considered to be the Laplace transform of a continuous function $x(t)$ for $t \geq 0$ sampled at $t = 0, T, 2T, \dots$.

Table 18.3 lists some general properties of the unilateral z -transform, and Table 18.4 lists some unilateral z -transform pairs.

Table 18.3 General properties of the unilateral z -transform

	Term in sequence	z -Transform $X_u(z)$	Domain of Convergence
1	$\alpha x_n + \beta y_n$	$\alpha X_u(z) + \beta Y_u(z)$	Intersection of DOC's of $X_u(z)$ and $Y_u(z)$ with α, β constants
2	x_{n+k}	$z^k X_u(z) - z^k x_0 - z^{k-1} x_1 - z^{k-2} x_2 - \dots - z x_{k-1}$	
3	$n x_n$	$-z \frac{dX_u(z)}{dz}$	DOC of $X_u(z)$, to which it may be necessary to add or delete the origin and the point at infinity
4	$z_0^n x_n$	$X_u\left(\frac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $ z_0 $, to which it may be necessary to add or delete the origin and the point at infinity
5	$n z_0^n x_n$	$-z \frac{dX_u(z/z_0)}{dz}$	DOC of $X_u(z)$ scaled by $ z_0 $, to which it may be necessary to add or delete the origin and the point at infinity
6	$\bar{x}_n$	$\overline{X_u(\bar{z})}$	The same DOC as x_n
7	$\operatorname{Re} x_n$	$\frac{1}{2} [X_u(z) + \overline{X_u(\bar{z})}]$	DOC contains the DOC of x_n
8	$\frac{\partial}{\partial \alpha} x_n(\alpha)$	$\frac{\partial}{\partial \alpha} X_u(z, \alpha)$	Same DOC as $x_n(\alpha)$
9	Initial value theorem	$x_0 = \lim_{z \rightarrow \infty} X_u(z)$	
10	Final value theorem	$\lim_{n \rightarrow \infty} x_n = \lim_{z \rightarrow 1} \left[\left(\frac{z-1}{z} \right) X_u(z) \right]$	When $X_u(z) = N(z)/D(z)$ with $N(z), D(z)$ polynomials in z and the zeros of $D(z)$ inside the unit circle $ z = 1$ or at $z = 1$

Table 18.4 Basic unilateral z -transforms

	Term in sequence	z -Transform $X_u(z)$	Domain of Convergence
1	$\Delta(n)$	1	Converges for all z
2	$\Delta(n - k)$	z^{-k}	Convergence for all $z \neq 0$
3	$a^n h(n)$	$\frac{z}{z - a}$	$ z > a $
4	$na^n h(n)$	$\frac{az}{(z - az)^2}$	$ z > a > 0$
5	$n^2 a^n h(n)$	$\frac{az(z + a)}{(z - a)^3}$	$ z > a > 0$
6	$na^{n-1} h(n)$	$\frac{z}{(z - a)^2}$	$ z > a > 0$
7	$(n - 1)a^n h(n)$	$\frac{z(2a - z)}{(z - a)^2}$	$ z > a > 0$
8	$e^{-an} h(n)$	$\frac{ze^a}{ze^a - 1}$	$ z > e^{-a}$
9	$ne^{-an} h(n)$	$\frac{ze^a}{(ze^a - 1)^2}$	$ z > e^{-a}$
10	$n^2 e^{-an} h(n)$	$\frac{ze^a(1 + ze^a)}{(ze^a - 1)^3}$	$ z > e^{-a}$
11	$e^{-an} h(n) \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z > e^{-a}$
12	$e^{-an} h(n) \cos \Omega n$	$\frac{ze^a(ze^a - \cos \Omega)}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z > e^{-a}$
13	$h(n) \sinh an$	$\frac{z \sinh a}{z^2 - 2z \cosh a + 1}$	$ z > e^{-a}$
14	$h(n) \cosh an$	$\frac{z(z - \cosh a)}{z^2 - 2z \cosh a + 1}$	$ z > e^{-a}$
15	$h(n)a^{n-1}e^{-an} \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2zae^a \cos \Omega + a^2}$	$ z > e^{-a}$
16	$h(n)a^n e^{-an} \cos \Omega n$	$\frac{ze^a(ze^a - a \cos \Omega)}{z^2 - 2zae^a \cos \Omega + a^2}$	$ z > e^{-a}$

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This index shows the occurrence of functions and constants used in the expressions within the text. The numbers refer to pages on which the function or constant appears.

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 $\mathfrak{J}_n^{(m)}$ *see* Bessel functions, **3**
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