Non-linear oscillators and solitons of Maxwell–Bloch equations *)

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Received 7 August 2006

The Maxwell–Bloch equations of non-linear optics are represented as a continuous chain of C. Neumann oscillators on S^3. This representation enables us to find explicitely one-soliton solutions of the Maxwell–Bloch equations. These solutions form a two-parameter family and they correspond to the homoclinic orbits of the magnetic spherical pendula on the two-sphere.

PACS: 02.30.Ik, 42.65.Tg
Key words: Maxwell–Bloch equations, soliton, C. Neumann system

1 Introduction

The Maxwell–Bloch equations (MBE) are a semi-classical description of the interaction between light and an optically active medium consisting of two-level atoms. It this paper our principal aim will be to construct in a simple and geometrically intuitive way a family of one-soliton solutions of the MBE. The essential feature of our approach is to represent the Maxwell–Bloch system as a chain of interacting non-linear oscillators. The model oscillator turns out to be a case of the C. Neumann oscillator on S^3 with two circular symmetries. This representation enables us to find a new Hamiltonian structure for the MBE and also to reprove the integrability of MBE in a simple way. More on this approach to the MBE can be found in [1] and [2].

One of the circular symmetries of our C. Neumann system can be used to perform the symplectic reduction. The reduced system turns out to be the charged spherical pendulum moving in the field of the magnetic monopole situated in the centre of the configuration space S^2. This quotient procedure does not have a reasonable counterpart on the level of chains. The situation is more involved. We shall introduce a system which can be thought of as a chain of the magnetic spherical pendula (CSP). A certain symplectic quotient of the MBE turns out to be a topologically and dynamically non-trivial coupling of the CSP and a natural system on the loop group $\Omega U(1)$. Nevertheless, in our Theorem 1 we shall prove that every d'Alembert type travelling solution $g(x - vt)$ of the MBE gives rise to a travelling solution $q(x-vt)$ of the CSP. Generic solutions of the magnetic spherical pendulum are quasiperiodic and some of them are periodic. These give rise to travelling wave solutions of the CSP, as well as of the MBE. But for every fixed value of the charge, the magnetic spherical pendulum has exactly one *homoclinic* orbit. These yield the

*) Presented at the 15$^{\text{th}}$ International Colloquium on "Integrable Systems and Quantum Symmetries", Prague, 15–17 June 2006.

two-parameter family of the one-soliton solutions of the CSP and therefore also of the MBE. This family is parametrized by the charge of the pendulum and the strength of the gravitational force, or alternatively, by the charge and the velocity of the soliton.

2 A chain of C. Neumann oscillators

In this paper we shall consider the unpumped MBE without losses and broadening. Let $E(t,x)$, $P(t,x)\colon \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ represent the electric field and the polarization, and let $D(t,x)\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the population inversion. Clearly, t is time and x is the single spatial variable. Under the above hypotheses, the MBE take the form

$$E_t + cE_x = P\,, \quad P_t = ED - \beta P\,, \quad D_t = -\tfrac{1}{2}\,(\overline{E}P + E\overline{P})\,. \tag{1}$$

The constant c is the speed of light in the medium and β is the longitudinal relaxation time. If we introduce the Lie algebra-valued functions $\rho(t,x)\colon \mathbb{R}\times\mathbb{R}\to\mathfrak{su}(2)$ and $F(t,x)\colon \mathbb{R}\times\mathbb{R}\to\mathfrak{su}(2)$ given by

$$\rho(t,x) = \begin{pmatrix} \mathrm{i}D(t,x) & \mathrm{i}P(t,x) \\ -\mathrm{i}\overline{P}(t,x) & -\mathrm{i}D(t,x) \end{pmatrix}, \quad F(t,x) = \frac{1}{2}\begin{pmatrix} \mathrm{i}\beta & E(t,x) \\ -\overline{E}(t,x) & -\mathrm{i}\beta \end{pmatrix}, \tag{2}$$

then the system (1) takes the form

$$\rho_t = [\rho, F]\,, \qquad F_t + c\,F_x = [\rho, \sigma]\,, \tag{3}$$

where $\sigma = \frac{1}{2}\,\mathrm{diag}(\mathrm{i}, -\mathrm{i})$. The first of the above equations is a Lax equation, therefore its general solution is of the form $\rho(t,x) = \mathrm{Ad}_{g(t,x)}(\tau(x))$ and $F(t,x) = -g_t(t,x)g^{-1}(t,x)$, where $g(t,x)\colon \mathbb{R}\times\mathbb{R}\to\mathrm{SU}(2)$ and $\tau(x)\colon \mathbb{R}\to\mathfrak{su}(2)$. Putting this in the second equation of (3) gives the second order partial differential equation

$$(g_t g^{-1})_t + c\,(g_t g^{-1})_x = \big[\sigma, \mathrm{Ad}_g(\tau)\big] \tag{4}$$

for the Lie group-valued function $g(t,x)\colon \mathbb{R}\times\mathbb{R}\to\mathrm{SU}(2)$. The Lie algebra-valued function $\tau(x)$ depends only on the spatial variable and can be considered as a part of the initial conditions. The equation (4) together with the constraint $\langle g_t g^{-1}, \sigma\rangle = \mathrm{const.} = -\beta$ is equivalent to the MBE (1).

The rewriting (4) reveals the field-theoretic nature of the MBE. More precisely, the equation (4) is a continuous chain of a particular form of the C. Neumann oscillators. Namely, the ordinary differential equation

$$(g_t g^{-1}) = \big[\sigma, \mathrm{Ad}_g(\tau)\big] \tag{5}$$

for $g(t)\colon \mathbb{R}\to\mathrm{SU}(2)$ is the equation of motion for the C. Neumann oscillator moving on the three-sphere $S^3 = \mathrm{SU}(2)$ under the influence of the quadratic potential $V(g) = \langle\sigma, \mathrm{Ad}_g(\tau)\rangle = -\,\mathrm{Tr}\big(\sigma\cdot\mathrm{Ad}_g(\tau)\big)$. Clearly, τ is constant here. The equation (4) can be considered in the following way. For every x_0 the function $g(t,x_0)\colon \mathbb{R}\to\mathrm{SU}(2)$ describes the motion of the C. Neumann oscillator located at x_0. Its acceleration is influenced by the momenta of the neighbouring oscillators. Thus (4) is a continuous chain of the C. Neumann oscillators which interact among themselves via the interactions of magnetic type.

The above description yields immediately the following two insights into the structure of the MBE.

1) *Hamiltonian structure:* Let the symbol LSU(2) denote the usual loop group $\mathrm{LG} = \{g(x) \colon S^1 \to \mathrm{SU}(2)\}$ or the space of the "rapidly vanishing" maps $g(x)$. For every g from the latter case there exists a constant R such that for $|x| > R$ we have $g(x) = \mathrm{Exp}(\alpha(x))$ for some rapidly vanishing $\alpha(x)$. The first case corresponds to the periodic solutions of the MBE and the second to the rapidly vanishing solutions, e.g. solitons. We shall denote S^1 or $\mathbb{R}$ by the common symbol T. In both cases, LSU(2) is a group equipped with a right-invariant metric whose value at the identity is given by $\langle \alpha, \beta \rangle = -\int_T \mathrm{Tr}(\alpha \cdot \beta)\, \mathrm{d}x$. The following result is proved in [1]:

Let $(T^*\mathrm{LSU}(2), \omega_c + c\,\omega_i, H)$ be the Hamiltonian system whose Hamiltonian function $H \colon T^*\mathrm{LSU}(2) \to \mathbb{R}$ is given by $H(g, p_g) = \frac{1}{2}\|p_g\|^2 + \langle \sigma, \mathrm{Ad}_g(\tau) \rangle$. The symplectic structure of the system is the perturbation of the canonical cotangent structure ω_c by the "interaction term" ω_i. This term is the pull-back $\omega_i = \pi^*(\widetilde{\omega}_i)$ via the natural projection $\pi \colon T^*\mathrm{LSU}(2) \to \mathrm{LSU}(2)$ of the right-invariant form $\widetilde{\omega}_i$ on LSU(2). Its value at the identity is given by $\widetilde{\omega}_i(\alpha, \beta) = -\langle \alpha_x, \beta \rangle$. Then the equation of motion of $(T^*\mathrm{LSU}(2), \omega_c + c\,\omega_i, H)$ is the MBE (4).

We note that the interaction term ω_i accounts for the magnetic type interactions among the C. Neumann oscillators in the chain.

2) *Integrability:* For the sake of simplicity, set $c = 1$. The MBE is integrable. The equation (4) is equivalent to the zero-curvature condition $V_t - U_x + [U, V] = 0$ for the family of connections $U = -(-z\sigma + g_t g^{-1})$, $V = -z\sigma + g_t g^{-1} - z^{-1}\mathrm{Ad}_g(\tau)$ parametrized by the spectral parameter z. This Lax pair can be easily deduced from the known Lax pair for the C. Neumann equation which in turn can be found, e.g. in [3] as well as in many other sources. We note that the integrability and in particular the above Lax pair were known before.

3 A chain of magnetic spherical pendula

The equation (5) is the equation of motion of the system $(T^*\mathrm{SU}(2), \omega_c, H_{\mathrm{cn}})$, where the Hamiltonian is given by $H_{\mathrm{cn}}(g, p_g) = \frac{1}{2}\|p_g\|^2 + \langle \sigma, \mathrm{Ad}_g(\tau) \rangle$. Here $\langle \alpha, \beta \rangle = -\mathrm{Tr}(\alpha \cdot \beta)$. Obviously, this system is not a generic C. Neumann oscillator on S^3. It has two circular symmetries. Let $U_\tau = \{\mathrm{Exp}(s\tau), s \in \mathbb{R}\} \subset \mathrm{SU}(2)$. This copy of U(1) acts on $(T^*\mathrm{SU}(2), \omega_c, H_{\mathrm{cn}})$ by means of the action ρ. In the trivialization of $T^*\mathrm{SU}(2)$ by the right translations we have $\rho_u(g, p_g) = (g \cdot u, p_g)$. This action is clearly Hamiltonian and its moment map $\mu \colon T^*\mathrm{SU}(2) \to \mathrm{i}\mathfrak{u} = \mathbb{R}$ is given by $\mu(g, p_g) = p_g(\mathrm{Ad}_g(\tau))$. The symplectic quotient $(\mu^{-1}(m)/U_\tau, \omega_{\mathrm{sq}}, H_{\mathrm{sq}})$ at the level m of the moment map turns out to be the system $(T^*S^2, \omega_c + m\,\omega_m, H_{\mathrm{sp}})$. The Hamiltonian is given by $H_{\mathrm{sp}}(q, p_q) = \frac{1}{2}\|p_q\|^2 + \langle \sigma, q \rangle$ and the symplectic form is the canonical cotangent form ω_c perturbed by the so-called magnetic term ω_m. This term is the pull-back via the natural projection $\pi \colon T^*S^2 \to S^2$ of the Kostant–

Kirillov form $\widetilde{\omega}_m$ on $S^2 = \{q = \mathrm{Ad}_g(\tau); g \in \mathrm{SU}(2)\} \subset \mathfrak{su}(2)$ which is given by $(\omega_m)_q(X_q, Y_q) = \langle g, [X_q, Y_q]\rangle$, for X_q, $Y_q \in T_qS^2 \subset \mathfrak{su}(2)$. The Hamiltonian system $(T^*S^2, \omega_c + m\,\omega_m, H_{\mathrm{sp}})$ describes the motion of a charged particle on S^2 under the influence of the gravitational potential $\langle \sigma, \mathrm{Ad}_g(\tau)\rangle$ and the magnetic field of the magnetic monopole situated at the centre of the sphere S^2. In other words, this system describes the motion of the charged spherical pendulum in the field of the magnetic monopole. The parameter m measures the strength of the monopole or, alternatively, the charge of the pendulum. The equation of motion is

$$[q, q_{tt}] + m\, q_t = [\sigma, q]\,. \tag{6}$$

Let us now denote by LS^2 the loop space $LS^2 = \{q(x)\colon T \to S^2 \subset \mathfrak{su}(2)\}$, and let us consider the infinite-dimensional system $(T^*LS^2, \omega_c + m\,\omega_m + c\,\omega_i, H_{\mathrm{csp}})$. Here the Hamiltonian is given by $H_{\mathrm{csp}}(q, p_q) = \int_T \left(\frac{1}{2}\|p_q\|^2 + \langle \sigma, q\rangle\right) \mathrm{d}x$. The perturbations of the canonical cotangent form ω_c are the pull-backs of the 2-forms on LS^2 given by $(\widetilde{\omega}_m)_q(X_q, Y_q) = \int_T \langle q, [X_q, Y_g]\rangle \,\mathrm{d}x$ and $(\widetilde{\omega}_i)_q(X_q, Y_q) = -\int_T \langle (X_q)_x, Y_q\rangle \,\mathrm{d}x$. Note that $T_qLS^2 = \{X_q(x)\colon T \to T_qS^2\}$. The equation of motion of the system is

$$[q, q_{tt} + c\, q_{tx}] + m\,(q_t + c\, q_x) = [\sigma, q]\,. \tag{7}$$

This system is a chain of magnetic spherical pendula, all of them with the same charge m, in a similar way as the Maxwell–Bloch system is a chain of C. Neumann oscillators. From the equation (7) we see that whenever the charge of the pendula is non-zero, the interactions among them are magnetic as well as potential. We shall refer to this chain of spherical pendula by the abbreviation CSP.

4 The travelling solutions of the two chains

We have seen above that the magnetic spherical pendulum is the symplectic reduction of our case of the C. Neumann oscillator. But it turns out that the CSP is not a symplectic quotient of the Maxwell–Bloch system as one might expect. The relationship is more involved. It turns out that a certain symplectic quotient of the MBE is a topologically and dynamically nontrivial coupling of CSP and the system $(\Omega U(1), \omega_f, H_f)$. Here $\Omega U(1)$ is the space of based loops, ω_f is given by the formula $(\omega_f)_u(X_u, Y_u) = -\int_T (X_u)_x \cdot Y_u \,\mathrm{d}x$, and the Hamiltonian $H_f\colon \Omega U(1) \to \mathbb{R}$ is given by the formula $H_f\big(u(x)\big) = \int_T \big\| u_x u^{-1} \big\| \,\mathrm{d}x$. This is the simplest instance of a system studied in Chapter 8 of [4]. A more detailed description of this coupled system and the proofs will appear elsewhere. Here we shall concentrate on a more immediate relation between the Maxwell–Bloch system and the CSP. Let a solution $g(t,x)$ of a hyperbolic partial differential equation be of the form $g(t,x) = f(x - vt)$. Such solutions will be called the travelling solutions.

Theorem 1 *Let $g(t,x)\colon T \times \mathbb{R} \to \mathrm{SU}(2)$ be a travelling solution of the Maxwell–Bloch system. Then the map $q(t,x) = \mathrm{Ad}_{g(t,x)}(\tau)\colon T \times \mathbb{R} \to \mathfrak{su}(2)$ is a travelling solution of the CSP. Let the functional $m\colon T^*\mathrm{LSU}(2) \to C^\infty(\mathbb{R})$ be given by the formula $m = \langle g_t g^{-1}, \mathrm{Ad}_g(\tau)\rangle$. Along every travelling solution m is a constant.*

Proof: First, we shall rewrite the equation (4) with respect to the Hopf fibration $U_\tau \hookrightarrow \mathrm{SU}(2) = S^3 \to S^2$, $g \mapsto q = \mathrm{Ad}_g(\tau)$ which lies at the heart of the connection between the magnetic spherical pendulum and the symmetric C. Neumann oscillator. A calculation shows that $g_t g^{-1} = [q, q_t] + \langle g_t g^{-1}, q\rangle\, q$. Denoting $\langle g_t g^{-1}, q\rangle = m$ and inserting this expression into (4) gives us the following form of the MBE:

$$[q, q_{tt} + c\, g_{tx}] + m\,(q_t + c\, q_x) = [\sigma, q]\,, \tag{8}$$

$$(m_t + c\, m_x)\, q = [q_t, q_x]\,. \tag{9}$$

We obtain two equations from the single equation (4) due to the fact that q is perpendicular to q_t, q_x and to $[q,\alpha]$ for any $\alpha \in \mathfrak{su}(2)$.

Let now $g(t,x) = h(x - vt)\colon T \times \mathbb{R} \to \mathrm{SU}(2)$ be a travelling solution of (4). Then $h(s)\colon \mathbb{R} \to \mathrm{SU}(2)$ must be a solution of the perturbed C. Neumann equation $(h_s h^{-1})_s = [[1/v(v-c)]\,\sigma, \mathrm{Ad}_h(\tau)]$. The quantity $\langle h_s h^{-1}, \mathrm{Ad}_h(\tau)\rangle$ is a constant of motion for the above equation. For the proof, see [5]. Therefore, $m = \langle g_t g^{-1}, \mathrm{Ad}_g(\tau)\rangle = -v\langle h_s h^{-1}, \mathrm{Ad}_h(\tau)\rangle$ is also a constant independent of t and x. This is consistent with the fact that for our solution we have $[q_t, q_x] = [-v h_s, h_s] = 0$. Thus the map given by $q(t,x) = \mathrm{Ad}_{g(t,x)}(\tau)\colon T \times \mathbb{R} \to \mathfrak{su}(2)$ is a solution of the equation (8) in which m is constant. In other words, $q(t,x)$ is a travelling solution of the CSP. □

Note that the rewriting (8) and (9) of the MBE corresponds precisely to the description of the Maxwell–Bloch system as the coupling of the CSP and the system $(\Omega U(1), \omega_f, H_f)$. The essential part of the coupling is provided by the term $[q_t, q_x]$.

We shall now use the above theorem for a simple and geometrically intuitive construction of the single-soliton solutions of the MBE. Single-soliton solutions are travelling solutions, thus by the above theorem for every soliton $g(t,x)$ of the MBE the mapping $q(t,x) = \mathrm{Ad}_{g(t,x)}(\tau)$ is a travelling solution of the CSP.

Let now $q(t,x) = f(x - vt)\colon T \times \mathbb{R} \to \mathfrak{su}(2)$ be a solution of the CSP. Then a straightforward calculation tells us that $f(s)\colon \mathbb{R} \to \mathfrak{su}(2)$ is a solution of the equation

$$[f, f_{ss}] - \frac{m}{v}\, f_s = \big[\widetilde{\sigma}, f\big] = \Big[\frac{1}{v(v-c)}\,\sigma, f\Big].$$

From (6) we see that this is the equation of motion for the magnetic spherical pendulum with the charge $-m/v$ and the gravitational force modified by the factor $1/(v^2 - vc)$. Now, observe that every soliton solution $q(t,x)$ satisfies the condition $\lim_{x\to-\infty} q(t,x) = \lim_{x\to+\infty} q(t,x) = \text{const.}$ and, therefore, $\lim_{s\to-\infty} f(s) = \lim_{s\to+\infty} f(s) =$ const. In other words, for $q(t,x) = f(x - vt)$ to be a soliton solution, $f(s)$ has to be a homoclinic orbit of the magnetic spherical pendulum.

Let us find the homoclinic orbit of the magnetic pendulum with the charge k and the gravitational potential equal to 1. A necessary condition for an orbit to be homoclinic is that the values of the conserved quantities along this orbit are equal to the values at a non-stable equilibrium. The magnetic spherical pendulum $(T^*S^2, \omega_c + k\omega_m, H_{\mathrm{sp}})$ is an integrable system. In [5] it is shown that a set of two independent integrals is provided by the Hamiltonian $H_{\mathrm{sp}}(q, p_q)$ and the perturbed angular moment around the axis of gravitation $G(q, p_q) = p_q\big([q,\sigma]\big) + k\langle \sigma, q\rangle$. The only unstable equilibrium is the point $(q, p_q) = (\sigma, 0)$, thus along a homoclinic orbit we have $H_{\rm sp}(q, p_q) = 1$ and $G(q, p_q) = k$. If we introduce the spherical coordinates on the configuration space $S^2 \subset \mathfrak{su}(2)$

$$q(\varphi(t), \vartheta(t)) = \begin{pmatrix} \mathrm{i}\cos\vartheta(t) & \mathrm{e}^{\mathrm{i}\varphi(t)}\sin\vartheta(t) \\ -\mathrm{e}^{-\mathrm{i}\varphi(t)}\sin\vartheta(t) & -\mathrm{i}\cos\vartheta(t) \end{pmatrix},$$

we get the system

$$\tfrac{1}{2}\left(\dot\vartheta^2 + \dot\varphi^2\sin^2\vartheta\right) + \cos\vartheta = 1\,, \qquad 2\dot\varphi\sin^2\vartheta + k\cos\vartheta = k\,. \tag{10}$$

This system of ordinary differential equations for $\varphi(t)$ and $\vartheta(t)$ is rather easily solved. For a chosen value of the charge k the solution is

$$\vartheta(t) = \arccos\Big(1 - \frac{\gamma^2}{4\cosh(\gamma t)}\Big), \quad \varphi(t) = \frac{k}{64}\Big(t - \frac{2\gamma}{\delta}\arctan\Big(\frac{\epsilon}{\delta}\tanh\big(\tfrac{1}{2}\gamma t\big)\Big)\Big), \tag{11}$$

where the constants γ, δ, ϵ are determined by m:

$$\gamma = \sqrt{1 - \Big(\frac{k}{4}\Big)^2}\,, \qquad \delta = \sqrt{63 + 2\Big(\frac{k}{4}\Big)^2 - \Big(\frac{k}{4}\Big)^4}\,, \qquad \epsilon = \Big(\frac{k}{4}\Big)^2 - 9\,.$$

Theorem 2 *Let $E(t,x)$, $P(t,x)$, $D(t,x)$ be a single-soliton solution of the MBE (1) travelling with the velocity v. Then the quantity*

$$m = \frac{E + 2(D_t P - D P_t)}{\mathrm{i}P} \tag{12}$$

is a constant independent of t and x, and the solution is given by

$$\begin{aligned} E(t,x) &= \mathrm{i}\kappa\mathrm{e}^{\mathrm{i}\varphi(s)}\big(\vartheta'(s) + \varphi'(s)\cos\vartheta(s)\sin\vartheta(s)\big) - \mathrm{i}(m/v)\cos\vartheta(s)\,, \quad s = \kappa(x - vt)\,, \\ P(t,x) &= \mathrm{e}^{\mathrm{i}\varphi(\kappa x - \kappa v t)}\sin\vartheta(\kappa x - \kappa v t)\,, \\ D(t,x) &= \cos\vartheta(\kappa x - \kappa v t)\,, \end{aligned}$$

where the functions $\vartheta(s)$ and $\varphi(s)$ are given by (11). The value of the charge parameter k has to be set to $k = \kappa m/v$ and $\kappa = 1/\sqrt{v(v-c)}$.

Proof: Recall the identities $q(t,x) = \rho(t,x)$ and $g_t g^{-1} = [q, q_t] + mq$ as well as the expressions (2) for the quantities ρ and $F = -g_t g^{-1}$. From these the formula (12) follows immediately. The fact that m is constant was proved in the Theorem 1.

Next, we observe that $f(s)$ is a solution of $[f, f_{ss}] + kf_s = [\sigma, f]$ if and only if $\widetilde{f}(s) = f(\kappa t)$ is a solution of $[\widetilde{f}, \widetilde{f}_{ss}] + \kappa k\widetilde{f}_s = \kappa^2[\sigma, \widetilde{f}]$. From this and from the discussion on the previous page our theorem now follows immediately. $\square$

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