Integrable anharmonic oscillators on spheres and hyperbolic spaces

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Abstract. We construct two new examples of integrable Hamiltonian systems. They describe motion of a particle under the influence of certain quartic potentials. The first system describes such motion on the n-dimensional sphere. The second gives motion on the n-dimensional hyperbolic space. Here n is an arbitrary positive integer. We represent the system on $(2n+1)$-dimensional sphere with an additional $U(1)$-symmetry as a symmplectic reconstruction of a system which has a topologically non-trivial magnetic term and whose configuration space is the n-dimensional complex projective space. We use this description to give an alternative proof of the integrability of the system on the odd-dimensional sphere.

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1. Introduction

Let S^n be the unit n-dimensional sphere in the Euclidean space $\mathbb{R}^{n+1}$ and let $H^n \subset \mathbb{R}^{n+1}$ be the unit sphere with respect to the Minkowskian metric. More explicitly, let $H^n = \{(q_0, q_1, \ldots, q_n); q_0^2 - q_1^2 - \ldots - q_n^2 = 1, q_0 > 0\}$. It is well-known that the Minkowskian metric induces the metric of n-dimensional hyperbolic space on H^n. Let (T^*S^n, ω_{can}) and (T^*H^n, ω_{can}) be the cotangent bundles equipped with their respective canonical cotangent symplectic forms and let the Hamiltonian functions $H_s: T^*S^n \to \mathbb{R}$ and $H_h: T^*H \to \mathbb{R}$ be given by

$$H_s(q, p_q) = \frac{1}{2}\|p_q\|^2 + \sum_{i=0}^{n} a_i^2 q_i^2 - \Big(\sum_{i=0}^{n} a_i q_i^2\Big)^2, \tag{1}$$

$$H_h(q, p_q) = \frac{1}{2}\|p_q\|^2 + \sum_{i=0}^{n} a_i^2 q_i^2 + \Big(\sum_{i=0}^{n} a_i q_i^2\Big)^2, \tag{2}$$

where a_i are arbitrary real constants. The Hamiltonian systems $(T^*S^n, \omega_{can}, H_s)$ and $(T^*H^n, \omega_{can}, H_h)$ describe the motion of a particle under the influence of quartic potentials. In the case of the first system the particle moves on the sphere, while in the second case it moves on the hyperbolic space.

The main result of this paper is a theorem in which we establish the Arnold-Liouville integrability of the systems $(T^*S^n, \omega_{can}, H_s)$ and $(T^*H^n, \omega_{can}, H_h)$. Thus we add new items to the list of finite-dimensional integrable systems.

Known integrable systems with quartic potentials are rare. An example of such a system is the anharmonic oscillator on $\mathbb{R}^n$ with the potential $V(x_0, \ldots, x_n) = \sum_{i=0}^n a_i x_i^2 + (\sum_{i=0}^n x_i^2)^2$. This is a special case of the Garnier system. The Lax pair for this system was found by D.V. Chudnovsky and G.V. Chudnovsky. Their work is summarized in [19]. In their paper [10] Fordy, Wojciechowski and Marshall describe a family of integrable systems with quartic potentials. The key ingredient of their construction are the Cartan decompositions of Lie algebras corresponding to Hermitian symmetric spaces. These systems were also studied by Reyman in [17] and they are described in [22]. The configuration spaces of all these systems are flat real spaces $\mathbb{R}^n$, while the configuration spaces of our systems are S^n and H^n.

In their paper [14] Kalnins, Benenti and Miller find integrable Hamiltonian systems $(T^*S^n, \omega_{can}, H^s_{kbm})$ and $(T^*H^n, \omega_{can}, H^h_{kbm})$ which are similar but *different* from our $(T^*S^n, \omega_{can}, H_s)$ and $(T^*H^n, \omega_{can}, H_h)$. The Hamiltonian H^s_{kbm} in [14] is given by

$$H^s_{kbm}(q, p_q) = \frac{1}{2}\|p_q\|^2 + \sum_{i=0}^{n}\Big(\sum_{\substack{j\neq k\\ j,k\neq i}} b_j b_k\Big)x_i^2 - \Big(\sum_{i=0}^{n}\Big(\sum_{\substack{j=0\\ j\neq i}}^{n} b_j\Big)x_i^2\Big)^2,$$

where $b_0, \ldots, b_n$ are real constants. The function H^h_{kbm} has the same expression. Written in these terms our Hamiltonian H_s has the form

$$H_s(q, p_q) = \frac{1}{2}\|p_q\|^2 + \sum_{i=0}^{n}\Big(\sum_{\substack{j\neq k\\ j,k\neq i}} b_j b_k + \sum_{j\neq i} b_j^2\Big)x_i^2 - \Big(\sum_{i=0}^{n}\Big(\sum_{\substack{j=0\\ j\neq i}}^{n} b_j\Big)x_i^2\Big)^2$$

for a suitable choice of constants b_i. Even though the systems $(T^*S^n, \omega_{can}, H_s)$ and $(T^*S^n, \omega_{can}, H^s_{kbm})$ are similar, the methods used in [14] are quite different from those used in the present paper. In [14] the authors give an exhaustive list of separable Hamiltonian systems on $\mathbb{R}^n$, S^n and H^n whose potential functions are polynomials or rational functions in Cartesian coordinates. The system $(T^*S^n, \omega_{can}, H_{kbm})$ is the only such system on the sphere S^n with quartic potential. It therefore follows that our system $(T^*S^n, \omega_{can}, H_s)$ is Arnold-Liouville integrable but *not* separable. The same is true for $(T^*H^n, \omega_{can}, H_h)$. To the author's knowledge the two systems from [14] and our systems $(T^*S^n, \omega_{can}, H_s)$ and $(T^*H^n, \omega_{can}, H_h)$ are the only known integrable systems with quartic potentials and nonflat configuration spaces.

The starting point of our construction are two particular members of a certain family of integrable Hamiltonian systems $(T^*M, \omega_{can}, H_M)$, where M is an arbitrary Riemannian symmetric space. The systems $(T^*M, \omega_{can}, H_M)$ are generalizations of the well known Neumann system which describes the harmonic oscillator confined to the sphere. These systems were studied in [20]. A similar family, where the configuration spaces are the coadjoint orbits, was described in [18]. Particular examples of these systems are $(T^*\mathbb{RP}^n, \omega_{can}, H_{\mathbb{RP}^n})$ and $(T^*H^n, \omega_{can}, H_{H^n})$ whose configuration spaces are the real projective space and the hyperbolic space. We obtain $(T^*S^n, \omega_{can}, H_s)$ as a pull-back of $(T^*\mathbb{RP}^n, \omega_{can}, H_{\mathbb{RP}^n})$ by a suitably expressed antipodal map $\vartheta_s\colon S^n \to \mathbb{RP}^n$. In a similar way we construct the "quartic system" $(T^*H^n, \omega_{can}, H_h)$ from the "quadratic" $(T^*H^n, \omega_{can}, H_{H^n})$. The integrability of the systems with quartic potentials then follows easily from the integrability of the generalized Neumann systems.

The second topic of this paper is integrability of a particular case of symplectic reconstruction. Symplectic reconstruction and geometric phases were studied e.g. in [15], [6], [8]. We consider the system $(T^*S^{2n+1}, \omega_{can}, H_c)$, where H_c is given by (1) with the addition that $a_{2j} = a_{2j+1}$ for every $j = 0, \ldots, n$. This system is invariant with respect to a certain $U(1)$-action. The symplectic quotient at the zero level of the moment map is $(T^*\mathbb{CP}^n, \omega_{can}, H_{\mathcal{CP}^n})$. This system belongs to the family of integrable systems $(T^*M, \omega_{can}, H_M)$ mentioned above. But one cannot deduce the integrability of the symplectic reconstruction $(T^*S^{2n+1}, \omega_{can}, H_c)$ from the integrability of $(T^*\mathbb{CP}^n, \omega_{can}, H_{\mathcal{CP}^n})$ alone. The map that connects the two systems is the Hopf fibration $\vartheta^c\colon S^{2n+1} \to \mathbb{CP}^n$. The metric on the sphere defines a connection on the principal $U(1)$-bundle $\vartheta^c\colon S^{2n+1} \to \mathbb{CP}^n$ whose horizontal spaces are orthogonal complements of the vertical ones. The curvature of this connection is a 2-form ω_m on $\mathbb{CP}^n$. We will show that in order to establish the integrability of $(T^*S^{2n+1}, \omega_{can}, H_c)$ one needs the integrability of $(T^*\mathbb{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$ for every real constant P. We prove the integrability of $(T^*\mathbb{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$ in the Appendix. For non-zero values of P the 2-form ω_m adds to the system a magnetic force. In the case when the configuration space is $\mathbb{CP} = S^2$, the form ω_m describes the magnetic field of the Dirac monopole. Roughly speaking, the geodesic motion on the fibre S^1 of ϑ^c and the motion of the generalized Neumann system on $\mathbb{CP}^n$ are coupled in $(T^*S^{2n+1}, \omega_{can}, H_c)$ by means of the non-trivial magnetic term ω_m. These considerations yield a family of Poisson-commuting integrals for $(T^*S^{2n+1}, \omega_{can}, H_c)$ which is different from the family obtained by means of the antipodal map and reflects the coupling mentioned above. The advantage of this new set of integrals is that one of its members has a physical interpretation. It is equal to the charge of the particle projected from S^{2n+1} to $\mathbb{CP}^n$ and moving in the magnetic field given by ω_m.

In the second section we collect a few facts about symmetric spaces which we need later in the text. The integrability of $(T^*S^n, \omega_{can}, H_s)$ and $(T^*H^n, \omega_{can}, H_h)$ is proved in the third section. The forth section contains the discussion about the symplectic reconstruction aspect of the system $(T^*S^{2n+1}, \omega_{can}, H_c)$. We prove the integrability of the generalized Neumann systems, including those with the non-exact magnetic terms, in the Appendix.

2. Cartan models of symmetric spaces

A Riemannian manifold M is a symmetric space if for every point $p \in M$ there exists an involutive isometry of M which fixes p and reverses the sense of the geodesics passing through M. Irreducible symmetric spaces are of the form $M = G/U$, where G is a semi-simple Lie group and $U \subset G$ a Lie subgroup. A homogeneous space $M = G/U$ is symmetric if and only if there exists a decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{u} = \mathrm{Lie}(U)$ and

$$[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}, \qquad [\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{u} \, . \tag{3}$$

The vector subspace $\mathfrak{p} \subset \mathfrak{g}$ is orthogonal to $\mathfrak{u}$ with respect to the Killing form on $\mathfrak{g}$.

Every symmetric space $M = G/U$ can be represented as a totally geodesic submanifold of G. The involution $\mathrm{d}\theta \colon \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{d}\theta(u, p) = (u, -p)$, where $u \in \mathfrak{u}$ and $p \in \mathfrak{p}$, is a Lie algebra isomorphism if and only if (3) holds. In this case $\mathrm{d}\theta$ is called a Cartan involution of $\mathfrak{g}$. Let $\theta \colon G \to G$ be the involution of the Lie group G such that its derivative $\mathrm{d}_e\theta$ at the identity is equal to $\mathrm{d}\theta$. Then M is diffeomorphic to the fixed-point set of the mapping

$$g \mapsto \theta(g^{-1}) \tag{4}$$

and this fixed-point set is a totally geodesic submanifold of G called the Cartan model of M. For the proofs of the above claims see Helgason's book [12]. In the sequel we shall use the more economical notation $\theta(g) = g^\theta$ and $\mathrm{d}\theta(\alpha) = \alpha^\theta$ for the elements $g \in G$ and $\alpha \in \mathfrak{g}$ alike.

Since M is the fixed-point set of the involutive map (4), we see that M is the image of the map $pr \colon G \to M \subset G$ given by

$$pr(g) = g(g^\theta)^{-1} = h \, .$$

The fibre of pr is clearly U. The derivative $\mathrm{d}(pr)_g \colon T_g G \to T_h M$ is given by

$$\mathrm{d}(pr)_g(X_g) = X_g \cdot h - h \cdot (X_g)^\theta, \qquad X_g \in T_g G \, .$$

By means of the right translations we can trivialize the tangent bundle TG. Restricting the right translations to the tangent bundle $TM \subset TG$ therefore gives the representation of TM as a subbundle of the trivial bundle $M \times \mathfrak{g}$. We get

$$T_h M \cong \mathfrak{p}_h = \{X_h = X - \mathrm{Ad}_h(X^\theta); X \in \mathfrak{g}\} \, . \tag{5}$$

If we define the involution $\mathrm{d}\theta_h \colon \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{d}\theta_h(X) = \mathrm{Ad}_h(X^\theta)$, then $\mathfrak{p}_h$ is precisely the (-1)-eigenspace of $\mathrm{d}\theta_h$. From this we easily see that $\mathfrak{p}_h = \mathrm{Ad}_g(\mathfrak{p})$, where $g \in G$ is any element such that $h = g(g^\theta)^{-1}$.

Let now $X_1, X_2 \in T_h M$ be two tangent vectors and suppose that there exists a central element $m \in \mathfrak{u}$. Define the 2-form ω_m on M by

$$(\omega_m)_h(X_1, X_2) = \langle \mathrm{Ad}_g(m) \; , \; [X_1 \cdot h^{-1}, X_2 \cdot h^{-1}] \rangle, \qquad h = pr(g) = g(g^\theta)^{-1} \, . \tag{6}$$

Two elements g_1, g_2 such that $pr(g_1) = pr(g_2) = h$ differ by an element $u \in U$. That is, $g_2 = g_1 u$. Since the element m is central in $\mathfrak{u}$, the form ω_m is independent of the choice of g in the fibre $pr^{-1}(h)$ and is therefore well defined.

Let $\rho \colon G \to Diff(M)$ be the natural action of G on M. If we represent M as the Cartan model, then ρ is given by the formula

$$\rho_f(h) = f \cdot h \cdot (f^\theta)^{-1} \, .$$

From this we get $\mathrm{d}\rho_f(X_h)\cdot\rho_f(h)^{-1} = X_{\rho_f(h)}\cdot\rho_f(h)^{-1} = \mathrm{Ad}_f(X_h\cdot h^{-1})$. It is then not difficult to check that ω_m is invariant with respect to the action ρ of G on M. For the details see [21]. If M is a compact symmetric space, then every non-zero G-invariant k-form represents a nonzero element in the k-th deRham cohomology group $H^k_{DR}(M)$. For the proof see [9]. The form ω_m therefore represents a non-zero element in $H^2_{DR}(M)$.

3. Systems with quartic potentials on spheres and hyperbolic spaces

As we have already mentioned in the Introduction, we shall construct the systems $(T^*S^n, \omega_{can}, H_s)$ and $(T^*H^n, \omega_{can}, H_h)$ from Hamiltonian systems whose phase spaces are appropriate symmetric spaces. Recall that $\mathbb{RP}^n$ is a symmetric space and that $\mathbb{RP}^n = SO(n+1)/S(O(1)\times O(n))$. The non-compact dual of $\mathbb{RP}^n$ (in the sense explained in [12]) is the hyperbolic space $H^n = SO(1,n)/S(O(1)\times O(n))$. Let $J = \mathrm{diag}(-1,1,\ldots,1)$ be the diagonal $(n+1)\times(n+1)$-matrix whose first entry is equal to -1 and all the others are equal to 1. Let the maps $\theta_p\colon SO(n+1)\to SO(n+1)$ and $\theta_h\colon SO(1,n)\to SO(1,n)$ be given by

$$\begin{aligned}\theta_p(g) &= J\cdot g\cdot J, \qquad g\in SO(n+1)\\ \theta_h(g) &= J\cdot g\cdot J, \qquad g\in SO(1,n)\,.\end{aligned}$$

Clearly, θ_p and θ_h are involutive isomorphisms. The fixed-point set of both maps is the group $S(O(1)\times O(n))$. From this we conclude that the Cartan model of $\mathbb{RP}^n$ is the fixed-point set of the involutive mapping of $SO(n+1)$ given by

$$g\mapsto\theta_p(g^{-1}) = J\cdot g^{-1}\cdot J = J\cdot g^T\cdot J\,,$$

and the Cartan model of the hyperbolic space is the fixed-point set of the involution of $SO(1,n)$ given by

$$g\mapsto\theta_h(g^{-1}) = J\cdot(Jg^TJ)\cdot J = g^T\,.$$

We shall denote the Cartan model of the hyperbolic space by $\mathcal{H}^n$ in order to distinguish it from other representations of the hyperbolic space H^n. The Cartan model of the real projective space $\mathbb{RP}^n$ will be denoted by $\mathcal{RP}^n$.

The expression (5) from the previous section yields the following representations of tangent spaces:

$$\begin{aligned}T_h\mathcal{RP}^n &= (\mathfrak{p}_{\mathbb{RP}^n})_h = \{X_h = X - \mathrm{Ad}_h(JXJ); X\in\mathfrak{so}(n+1)\}\\ T_h\mathcal{H}^n &= (\mathfrak{p}_{\mathcal{H}^n})_h = \{X_h = X - \mathrm{Ad}_h(JXJ); X\in\mathfrak{so}(1,n)\}\,.\end{aligned}$$

The metrics on $\mathcal{RP}^n$ and $\mathcal{H}^n$ are induced by the Killing form $\langle\alpha,\beta\rangle = -\mathrm{Tr}(\alpha\cdot\beta)$ on $\mathfrak{so}(n+1)$ and $\mathfrak{so}(1,n)$. The cotangent spaces $T^*_h\mathcal{RP}^n$ and $T^*_h\mathcal{H}^n$ can be identified with $T_h\mathcal{RP}^n$ and $T_h\mathcal{H}^n$ by means of the Killing form.

This Cartan model representation allows us to construct integrable Hamiltonian systems whose configuration spaces are $\mathcal{RP}^n$ and $\mathcal{H}^n$. Let the Hamiltonians of the systems $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}^n})$ be given by

$$
\begin{aligned}
H_{\mathcal{RP}^n}(h, p_h) &= \frac{1}{2}\|p_h\|^2 + \langle \mathrm{Ad}_h(B), B\rangle, \qquad (h, p_h) \in T^*\mathcal{RP}^n \\
H_{\mathcal{H}^n}(h, p_h) &= \frac{1}{2}\|p_h\|^2 + \langle \mathrm{Ad}_h(B), B\rangle, \qquad (h, p_h) \in T^*\mathcal{H}^n ,
\end{aligned}
$$

where $B = \mathrm{diag}(a_0, a_1, \ldots, a_n)$ is a real diagonal matrix. Denote by $Q_i: \mathfrak{gl}(n+1) \to \mathbb{R}$ the Ad-invariant polynomial functions defined by the characteristic equation

$$
\sum_{i=0}^{n} Q_i(\alpha) \cdot w^i = \det(\alpha - wI) . \tag{7}
$$

Let z be a real indeterminate. We claim that the functions $H^P_{i,j}: T^*\mathcal{RP}^n \to \mathbb{R}$ and $H^H_{i,j}: T^*\mathcal{H}^n \to \mathbb{R}$ given by

$$
Q_i(\mathrm{Ad}^*_{h^{-1}}(B) + zp_h + z^2B) = \begin{cases} \displaystyle\sum_{j=0}^{2\deg(Q_i)} H^P_{i,j}(h, p_h) \cdot z^j & \text{for } (h, p_h) \in T^*\mathcal{RP}^n \\ \displaystyle\sum_{j=0}^{2\deg(Q_i)} H^H_{i,j}(h, p_h) \cdot z^j & \text{for } (h, p_h) \in T^*\mathcal{H}^n \end{cases} \tag{8}
$$

are integrals of the systems $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}^n})$.

Define the pair (L, A) of maps from the time interval I into the loop algebra $\mathfrak{gl}(n+1) \otimes \mathbb{R}(z)$ by

$$
\begin{aligned}
L(t; z) &= \mathrm{Ad}^*_{h(t)^{-1}}(B) + zp_h(t) + z^2B \\
A(t; z) &= p_h(t) + zB .
\end{aligned}
$$

It is then a matter of straightforward checking that the Lax equation

$$
L_t = [A, L]
$$

is equivalent to the equation of motion

$$
(h_t h^{-1})_t = [B, \mathrm{Ad}_h(B)]
$$

of the systems $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}^n})$, which in turn we get from the canonical systems for $H_{\mathcal{RP}^n}$ and for $H_{\mathcal{H}^n}$. Thus the map $L(t)$ is the Lax matrix of our systems and it is well-known that the spectral curve S, given in affine coordinates by

$$
S = \{(z, w); \det(L(z) - Iw) = 0\} ,
$$

is the quantity preserved by our systems. From (7) we then see that this is equivalent to the fact that the functions $\{H^P_{i,j}\}$ and $\{H^H_{i,j}\}$ given by (8) are first integrals of the Hamiltonian systems in question. In the Appendix we shall prove that the element in the families $\{H^P_{i,j}\}$ and $\{H^H_{i,j}\}$ Poisson-commute. These two families actually contain subsets of n functionally independent Poisson-commuting integrals of the systems $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}^n})$.

Let us now equip the $(n+1)$-dimensional real vector space $V = \{q; q = (q_0, q_1, \dots, q_n)\}$ with two different inner products

$$\langle q^1, q^2\rangle_e = \sum_{i=0}^{n} q_i^1 q_i^2 ,$$
$$\langle q^1, q^2\rangle_m = q_0^1 q_0^2 - \sum_{i=1}^{n} q_i^1 q_i^2 = -q^1 J (q^2)^T .$$

By q^T we denote the column vector which is the transpose of the row vector q. Denote the Euclidean space $(V, \langle -, -\rangle_e)$ by $\mathbb{R}^{(n+1)}$ and the Minkowskian space $(V, \langle -, -\rangle_m)$ by $\mathbb{R}^{(1,n)}$. We shall consider the "unit spheres"

$$S^n = \{q \in \mathbb{R}^{(n+1)}; \|q\|_e = 1\},$$
$$H^n = \{q \in \mathbb{R}^{(1,n)}; \|q\|_m = 1\}$$

of these spaces. It is well known that the Minkowskian metric induces on H^n the Riemannian metric with constant negative scalar curvature. More concretely, H^n is the n-dimensional hyperbolic space.

Proposition 1 *Let the map $\vartheta_s: S^n \to GL(n+1)$ be given by*

$$\vartheta_s(q) = (I - 2q^T q)J . \tag{9}$$

Then ϑ_s is the usual antipodal double-covering map from S^n onto the Cartan model $\mathcal{RP}^n$ of the n-dimensional real projective space in $SO(n+1)$.

Proof. First we shall prove that $\vartheta_s(q)$ is an element of $SO(n+1)$ for every $q \in S^n$. Let $K_q: \mathbb{R}^{(n+1)} \to \mathbb{R}^{(n+1)}$ be the linear map whose matrix in the standard basis is $I - 2q^T q$. Then for every $x \in \mathbb{R}^{(n+1)}$ we have $K_q(x) = x - 2\langle q, x\rangle x$, which means that K_q is the reflection through the hyperplane $H_q \subset \mathbb{R}^{(n+1)}$ orthogonal to q. The map $J: \mathbb{R}^{(n+1)} \to \mathbb{R}^{(n+1)}$ is also a reflection, and since all products of two reflections are elements of $SO(n+1)$, so is $\vartheta_s(q) = K_q \cdot J$.

Next we have to see that $\theta_p[\vartheta_s(q)] = J \cdot \vartheta_s(q) \cdot J = \vartheta_s(q)^{-1}$ or equivalently $J \cdot \vartheta_s(q) \cdot J \cdot \vartheta_s(q) = I$. Indeed,

$$J \cdot \vartheta_s(q) \cdot J \cdot \vartheta_s(q) = J \cdot K_q \cdot K_q \cdot J = J \cdot I \cdot J = I ,$$

since reflections are involutive. At every $q \in S^n$ the derivative $(\mathrm{d}\vartheta_s)_q: T_q S^n \to T_{\vartheta_s(q)} \mathcal{RP}^n$ is an isomorphism. Thus the fact that ϑ_s is the antipodal map, that is, that $\vartheta_s(q_1) = \vartheta_s(q_2)$ if and only if $q_1 = \pm q_2$, is clear from the expression (9) of ϑ_s.

□

Proposition 2 *The map $\vartheta_h: H^n \to GL(n+1)$ defined by*

$$\vartheta_m(q) = -J(I + 2q^T q J) \tag{10}$$

takes values in the Cartan model $\mathcal{H}^n \subset SO(1,n)$ of the n-dimensional hyperbolic space. The restriction $\vartheta_h: H^n \to \mathcal{H}^n$ is a diffeomorphism.

Proof. First we will show that $\vartheta_h(q) \in SO(1,n)$ for every $q \in H^n$. Recall that $h \in SO(1,n)$ if and only if $h^{-1} = Jh^TJ$. Since $J\vartheta_h(q)^TJ = J + 2q^Tq$ and $\langle q, q\rangle_m = 1$ for $q \in H^n$, we have

$$\begin{aligned}((J\vartheta_h(q)^TJ)\cdot\vartheta_h(q))(\psi) &= J(I + 2Jq^Tq)JJ \cdot J(I + 2q^TqJ)(\psi)\\ &= (J + 2q^Tq)J(I + 2q^TqJ)(\psi)\\ &= (I + 2q^TqJ)(\psi - 2\langle q,\psi\rangle_m q)\\ &= \psi - 4\langle q,\psi\rangle_m q + 4\langle q,\psi\rangle_m\langle q,q\rangle_m q = \psi\end{aligned}$$

for an arbitrary element $\psi \in \mathbb{R}^{(1,n)}$. Thus $\vartheta_h(q)^{-1} = J\cdot\vartheta_h(q)^T\cdot J$, which shows that $\vartheta_h(q) \in SO(1,n)$ for every $q \in H^n$. We have seen that an element h of $SO(1,n)$ lies in $\mathcal{H}^n$ if and only if $\theta_h(h^{-1}) = h^T = h$. From (10) we see immediately that $\vartheta_h(q)^T = \vartheta_h(q)$ for an arbitrary $q \in H^n$ and thus we have $\vartheta_h(H^n) \subset \mathcal{H}^n$. For every $q \in H^n$ the derivative $(\mathrm{d}\vartheta_h)_q$ is an isomorphism. Thus, by the inverse function theorem, the map $\vartheta_h \colon H^n \to \mathcal{H}^n$ is a covering map. But since the fundamental group of $\mathcal{H}^n$ is trivial, and since H^n is connected, the map ϑ_h must actually be a diffeomorphism.

□

Let now (M,ω_M,H) and (N,ω_N,K) be two Hamiltonian systems and let $f\colon M \to N$ be a smooth map.

Definition 1 *The system (M,ω_M,H) is a pull-back of (N,ω_N,K) by f if $f^*(\omega_2) = \omega_1$ and $H = c\cdot f^*(K)$ for some constant c.*

Proposition 3 *Let (M,ω_M,H) be the pull-back of (N,ω_N,K) via $f\colon M \to N$ and let (N,ω_N,K) be an integrable system. Then (M,ω_M,H) is also integrable. If $\{G_1,\dots G_n\}$ is a system of functionaly independent of Poisson-commuting first integrals on (N,ω_N), then the pull-backs $\{F_1 = f^*(G_1),\dots,F_n = f^*(G_n)\}$ provide a complete family of Poisson commuting integrals on (M,ω_M). By a complete family we mean a family of the maximum number of functionaly independent integrals.*

Proof. First we see that $f\colon M \to N$ is a local diffeomorphism, or equivalently, that for every $m \in M$ the derivative $\mathrm{d}f_m\colon T_mM \to T_{f(m)}N$ is an isomorphism. If this map had a non-trivial kernel, then the form $f^*(\omega_N) = \omega_M$ would be degenerate and therefore not symplectic, a contradiction. Take an arbitrary function $G\colon N \to \mathbb{R}$, and let $F\colon M \to \mathbb{R}$ be its pull-back $F = f^*(G)$. If X_G is the Hamiltonian vector field on G, then for the Hamiltonian vector field X_F of F we have

$$(X_F)_m = (\mathrm{d}f_m)^{-1}(X_G)_{f(m)}\,. \tag{11}$$

The proof follows immediately from the defining relation $\omega_N(X_G,-) = dG$ and the non-degeneracy of $\mathrm{d}f$. Let now $G_1, G_2\colon N \to \mathbb{R}$ be Poisson-commuting functions. Then their pull-backs $F_1 = f^*(G_1)$ and $F_2 = f^*(G_2)$ Poisson-commute on M. This follows from (11) and from the definition of the Poisson bracket

$$\{F_1,F_2\} = \omega_M(X_{F_1},X_{F_2})\,.$$

The integrals $\{G_1, \ldots, G_n\}$ form a complete system if the n-form $dG_1 \wedge \ldots \wedge dG_n$ is non-degenerate almost everywhere. If F_i are the pull-backs of G_i, then

$$(\mathrm{d}F_1 \wedge \ldots \wedge \mathrm{d}F_n)_m = (\mathrm{d}f_m)^*(\mathrm{d}G_1 \wedge \ldots \wedge \mathrm{d}G_n)_{f(m)} ,$$

and $\mathrm{d}F_1 \wedge \ldots \wedge \mathrm{d}F_n$ is also non-degenerate due to the non-degeneracy of $\mathrm{d}f$.

□

We will use this simple proposition applied to the maps $\vartheta_s\colon S^n \to \mathcal{RP}^n$ and $\vartheta_h\colon H^n \to \mathcal{H}^n$ in the proof of our main theorem bellow.

Theorem 1 *The Hamiltonian systems* $(T^*S^n, \omega_{can}, H_s)$ *and* $(T^*H^n, \omega_{can}, H_h)$*, where*

$$H_s(q, p_q) = \frac{1}{2}\|p_q\|^2 + \sum_{i=0}^{n} a_i^2 q_i^2 - \Big(\sum_{i=0}^{n} a_i q_i^2\Big)^2 \tag{12}$$

and

$$H_h(q, p_q) = \frac{1}{2}\|p_q\|^2 + \sum_{i=0}^{n} a_i^2 q_i^2 + \Big(\sum_{i=0}^{n} a_i q_i^2\Big)^2 \tag{13}$$

are completely integrable. The functions $H^s_{i,j}\colon T^*S^n \to \mathbb{R}$ *and* $H^h_{i,j}\colon T^*H^n \to \mathbb{R}$ *given by the relations*

$$\begin{aligned}\sum_{i=0}^{2\deg(Q_i)} H^s_{i,j}(q, p_q) \cdot z^j = Q_i[B - 2z(p_q^T q + q^T p_q)(I - 2q^T q) \\ + z^2(I - 2q^T q)B(I - 2q^T q)]\end{aligned} \tag{14}$$

$$\begin{aligned}\sum_{j=0}^{2\mathrm{deg}Q_i} H^h_{i,j}(q, p_q) \cdot z^j = Q_i[B + 2zJ(p_q^T q + q^T p_q)(I + 2q^T qJ) \\ + z^2 J(I + 2q^T q)B(J + 2q^T q)]\end{aligned} \tag{15}$$

are Poisson-commuting integrals of $(T^*S^n, \omega_{can}, H_s)$ *and* $(T^*H^n, \omega_{can}, H_h)$ *respectively. Complete systems of n functionally independent integrals can be chosen from the families* $\{H^s_{i,j}\}$ *and* $\{H^h_{i,j}\}$*. If we take* $Q_1(a) = \mathrm{Tr}(a^2)$*, we have*

$$H_s = \frac{1}{8} H^s_{1,2} \quad \text{and} \quad H_h = \frac{1}{8} H^h_{1,2}. \tag{16}$$

Proof. We shall prove that the system $(T^*S^n, \omega_{can}, H_s)$ is a pull-back of the system $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and that $(T^*H^n, \omega_{can} H_h)$ is a pull-back of $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}^n})$. The integrability of the systems $(T^*S^n, \omega_{can}, H)$ and $(T^*H^n, \omega_{can}, H_h)$ will then follow from the integrability of $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}^n})$ and from Proposition 3.

Define the maps $\widetilde{\vartheta}_s\colon T^*S^n \to T^*\mathcal{RP}^n$ and $\widetilde{\vartheta}_h\colon T^*H^n \to T^*\mathcal{H}^n$ by

$$\widetilde{\vartheta}_s(q, p_q) = (\vartheta_s(q), (d\vartheta_s^*)_q^{-1}(p_q)) \quad \text{and} \quad \widetilde{\vartheta}_h(q, p_q) = (\vartheta_h(q), (d\vartheta_h^*)_q^{-1}(p_q)) .$$

We can identify the tangent spaces on manifolds S^n, H^n, $\mathcal{RP}^n$ and $\mathcal{H}^n$ with the appropriate cotangent spaces by means of metrics. Since the derivatives $(\mathrm{d}\vartheta_s)_q$ and $(\mathrm{d}\vartheta_h)_q$ are isomorphisms, the maps $\widetilde{\vartheta}_s$ and $\widetilde{\vartheta}_h$ pull the tautological 1-forms on $T^*\mathcal{RP}^n$

and $T^*\mathcal{H}^n$ to the tautological 1-forms on T^*S^n and T^*H^n. The canonical 2-form on any cotangent bundle is the derivative of the tautological 1-form, therefore the maps $\widetilde{\vartheta}_s$ and $\widetilde{\vartheta}_h$ pull the canonical 2-forms on $T^*\mathcal{RP}^n$ and $T^*\mathcal{H}^n$ back to the canonical 2-forms on T^*S^n and T^*H^n respectively.

Denote $\vartheta_s(q) = h$. We claim that for every $q \in S^n$

$$\|\dot{h_q}\|^2 = -\mathrm{Tr}(\dot{h_q}h_q^{-1} \cdot \dot{h_q}h_q^{-1}) = 4\|\dot{q}\|^2 \,, \tag{17}$$

that is, the map $(\mathrm{d}\vartheta_s)_q\colon T_qS^n \to T_{\vartheta_s(q)}\mathcal{RP}^n$ is equal to twice the isometry. Let $A = q^T \cdot q$. Then $\vartheta_s(q) = (I - 2A) \cdot J$. We have

$$\|\dot{h_q}\|^2 = -\mathrm{Tr}((-2\dot{A} + 4\dot{A}A)^2) = \mathrm{Tr}(-4\dot{A} + 16\dot{A}^2A - 16\dot{A}A\dot{A}A) \,. \tag{18}$$

From $J\vartheta_s(q)J = \vartheta_s(q)^{-1}$ we get $(I - 2A)^2 = I$, and this in turn implies that $A^2 = A$. Differentiation gives $\dot{A} = \dot{A}A + A\dot{A}$. If we put this into (18), we get

$$\|\dot{h_q}\|^2 = -4\mathrm{Tr}(\dot{A}^2) \,. \tag{19}$$

Since $\dot{A} = \dot{q}^T q + q^T \dot{q}$, formula (19) gives

$$\|\dot{h_q}\|^2 = 4(\|\dot{q}\|^2 \cdot \|q\|^2 + \langle \dot{q}, q\rangle^2) \,. \tag{20}$$

On the unit sphere we have $\|q\| = 1$ and $\langle \dot{q}, q\rangle = 0$, therefore (20) indeed yields (17).

Next we calculate the pull-back $V(q) = \widetilde{\vartheta}_s^{\,*}(\langle \mathrm{Ad}_h(B), B\rangle)$ of the potential function. From (9) we get

$$\begin{aligned} V(q) = &- \mathrm{Tr}(\mathrm{Ad}_{(I-2q^Tq)J}(B) \cdot B) \\ = &- \mathrm{Tr}[B^2 - 4q^Tq \cdot B^2 + 4(q^Tq \cdot B)^2] \,. \end{aligned}$$

In the above calculation we have used the fact that $J \cdot \vartheta_s(q) \cdot J = (\vartheta_s(q))^{-1}$ and that, due to the diagonality of B, we have $J \cdot B \cdot J = B$. A straightforward calculation now gives

$$V(q) = -\mathrm{Tr}(B^2) + 4\sum_{i=0}^{n} a_i^2 q_i^2 - 4\Big(\sum_{i=0}^{n} a_i q_i^2\Big)^2 \,. \tag{21}$$

Since the constant $-\mathrm{Tr}(B^2)$ is irrelevant, formulae (17) and (21) show that $\widetilde{\vartheta}_s$ pulls the Hamiltonian $H_{\mathcal{RP}^n}$ back to $4H_s\colon T^*S^n \to \mathbb{R}$, where H_s is the Hamiltonian given by (12).

Calculations similar to those above show that the map $\widetilde{\vartheta}_h\colon T^*H^n \to T^*\mathcal{H}^n$ pulls the Hamiltonian $H_{\mathcal{H}^n}\colon T^*\mathcal{H}^n \to \mathbb{R}$ back to $4H_h\colon T^*H^n \to \mathbb{R}$, where H_h is the Hamiltonian given by (13).

The map $\widetilde{\vartheta}_s\colon T^*S^n \to T^*\mathcal{RP}^n$ pulls the functions $H^P_{i,j}\colon T^*\mathcal{RP}^n \to \mathbb{R}$ given by (8) back to the functions $H^s_{i,j}\colon T^*S^n \to \mathbb{R}$ given by (14). The map $\widetilde{\vartheta}_h\colon T^*H^n \to T^*\mathcal{H}^n$ pulls the functions $H^H_{i,j}\colon T^*\mathcal{H}^n \to \mathbb{R}$ back to $H^h_{i,j}\colon T^*H^n \to \mathbb{R}$ defined by (15). To see this is a matter of trivial checking. One only has to use the fact that the maps $(d\vartheta_s)_q$ and $(d\vartheta_m)_q$ are isometries multiplied by 2. From the families $\{H^P_{i,j}\}$ and $\{H^H_{i,j}\}$ complete sets of Poisson-commuting integrals can be chosen. By Proposition (3) the pull-backs of those will be complete sets of Poisson-commuting integrals on (T^*S^n, ω_{can}) and on (T^*H^n, ω_{can}), which establishes the integrability of the systems $(T^*S^n, \omega_{can}, H_s)$ and $(T^*H^n, \omega_{can}, H_h)$.

If we take $Q_1(a) = \mathrm{Tr}(a^2)$, then (8) gives us $H^P_{1,2} = 2H_{\mathcal{RP}^n}$ and $H^H_{1,2} = 2H_{\mathcal{H}^n}$. On the other hand we have

$$\widetilde{\vartheta}^*_s(H_{\mathcal{RP}^n}) = 4H_s, \quad \widetilde{\vartheta}^*_h(H_{\mathcal{H}^n}) = 4H_h, \quad \widetilde{\vartheta}^*_s(H^P_{1,2}) = H^s_{1,2}, \quad \widetilde{\vartheta}^*_h(H^H_{1,2}) = H^h_{1,2}.$$

Together this gives us (16).

□

4. $U(1)$-invariant case and symplectic reconstruction

In this section we shall consider a special case of the system $(T^*S^n, \omega_{can}, H)$, namely the system $(T^*S^{2n+1}, \omega_{can}, H_c)$, where

$$H_c(q, p_q) = \frac{1}{2}\|p_q\|^2 + \sum_{i=0}^{n} a_{2i}^2(q_{2i}^2 + q_{2i+1}^2) - \Big(\sum_{i=0}^{n} a_{2i}(q_{2i}^2 + q_{2i+1}^2)\Big)^2 .$$

The only difference between this system and the general one considered before is that now we have $a_{2i} = a_{2i+1}$ for every i. We shall describe $(T^*S^{2n+1}, \omega_{can}, H_c)$ as a symplectic reconstruction of a certain system on $T^*\mathbb{CP}^n$. Then we shall use this description to give an alternative proof of the integrability of $(T^*S^{2n+1}, \omega_{can}, H_c)$.

Denote $z_j = q_{2j} + \mathrm{i}q_{2j+1}$ and $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{(n+1)}$. Let the action ρ of $U(1)$ on $S^{2n+1} \subset \mathbb{C}^{(n+1)}$ be given by

$$\rho_{e^{\mathrm{i}\phi}}(z_0, z_1, \ldots, z_n) = e^{\mathrm{i}\phi} \cdot (z_0, z_1, \ldots, z_n) . \tag{22}$$

Denote by ρ^* the natural lift of ρ to T^*S^{2n+1}. Rewriting the Hamiltonian H_c in the form

$$H_c(z, p_z) = \frac{1}{2}\|p_z\|^2 + \sum_{j=0}^{n} a_j^2|z_j|^2 - \Big(\sum_{j+0}^{n} a_j|z_j|^2\Big)^2 \tag{23}$$

clearly shows that the system $(T^*S^{2n+1}, \omega_{can}, H_c)$ is invariant with respect to the action ρ^*.

The complex projective space $\mathbb{CP}^n$ is a symmetric space. Indeed we have $\mathbb{CP}^n = SU(n+1)/S(U(1) \times U(n))$, and the corresponding Cartan involution $\theta\colon SU(n+1) \to SU(n+1)$ is given by $\theta(g) = JgJ$, where J is the diagonal $(n+1)$-matrix $J = \mathrm{diag}(-1, 1, \ldots, 1)$. Thus the Cartan model $\mathcal{CP}^n$ of the complex projective space $\mathbb{CP}^n$ is

$$\mathcal{CP}^n = \{h \in SU(n+1); h = Jh^{-1}J\} .$$

Moreover, $\mathbb{CP}^n$ is a Hermitian symmetric space. The diagonal $(n+1)$-dimensional matrix $m = \mathrm{diag}(-\mathrm{i}/2, \mathrm{i}/2n, \ldots, \mathrm{i}/2n)$ is a central element in the Lie algebra $\mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n))$ of the group $S(U(1) \times U(n))$. Let $X_1, X_2 \in T_h\mathcal{CP}^n \subset T_hSU(n+1)$ be arbitrary tangent vectors. Let $h \in \mathcal{CP}^n$. Denote by $\sqrt{h}$ any element of $SU(n+1)$ such that $\sqrt{h} \cdot [(\sqrt{h})^\theta]^{-1} = h$. Recall that any two such square roots differ by an element of $S(U(1) \times U(n))$. The 2-form ω_m given, as in (6), by

$$(\omega_m)_h(X_1, X_2) = \langle \mathrm{Ad}_{\sqrt{h}}(m), [X_1 \cdot h^{-1}, X_2 \cdot h^{-1}]\rangle, \tag{24}$$

generates the group $H^2_{DR}(\mathbb{CP}^n) \cong \mathbb{R}$.

Let $B = \mathrm{diag}(\mathrm{i}a_0, \mathrm{i}a_1, \ldots, \mathrm{i}a_n)$ be a diagonal $(n+1)\times(n+1)$-matrix. Consider the Hamiltonian system $(T^*\mathcal{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$, where

$$H_{\mathcal{CP}^n}(h, p_h) = \frac{1}{2}\|p_h\|^2 + \langle \mathrm{Ad}_h(B), B\rangle$$

and P is an arbitrary real constant. This system describes the motion on $\mathcal{CP}^n$ of a charged particle under the influence of the magnetic field given by ω_m and of the potential force $\langle \mathrm{Ad}_h(B), B\rangle$. The charge of the particle is equal to P.

In the Appendix we shall prove that the system $(T^*\mathcal{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$ is integrable for every real constant P. A complete system of Poisson-commuting integrals can be chosen from the family of functions $K^P_{i,j}\colon T^*\mathcal{CP} \to \mathbb{R}$ given by

$$\sum_{j=0}^{2\deg Q_i} K^P_{i,j}(h, p_h)\cdot z^j = Q_i[\lambda + z(p_h - P\mathrm{Ad}_{\sqrt{h}}(m)) + z^2\mathrm{Ad}_h(B)]\,.$$

The functions $Q_i\colon \mathfrak{u}(n+1) \to \mathbb{R}$ are the Ad-invariant polynomials defined on the Lie algebra $\mathfrak{u}(n+1)$ of the unitary group $U(n+1)$.

Let us now reprove Theorem 1 for the system $(T^*S^{2n+1}, \omega_{can}, H_c)$ as a corollary of the result stated above. First, consider the map $\vartheta^c\colon S^{2n+1} \to \mathcal{CP}^n \subset SU(n+1)$ given by

$$\vartheta^c(z) = (I - 2z^*z)J\,, \tag{25}$$

where $z = (z_0, z_1, \ldots, z_n) \in S^{2n+1} \subset \mathbb{C}^{(n+1)}$, z^* is the cojugate transpose of this vector, and $J = \mathrm{diag}(-1, 1, \ldots, 1)$ as before. Every fibre of ϑ^c is obviously diffeomorphic to S^1, and the map $\vartheta^c\colon S^{2n+1} \to \mathcal{CP}^n$ is actually the well-known Hopf fibration. Every tangent space T_zS^{2n+1} can be decomposed as

$$T_zS^{2n+1} = \mathrm{Vert}_z \oplus \mathrm{Hor}_z\,,$$

where $\mathrm{Vert}_z = \ker(\mathrm{d}\vartheta^c_z)$ and Hor_z is its orthogonal complement with respect to the natural metric on S^{2n+1}. According to the above decomposition every tangent vector $X_z \in T_zS^{2n+1}$ can be uniquely expressed in the form

$$X_z = X^h_z + X^v_z, \qquad X^v_z \in \mathrm{Vert}_z, \qquad X^h_z \in \mathrm{Hor}_z\,.$$

Similarly we can decompose the cotangent spaces $T^*_zS^{2n+1}$ in the form

$$T^*S^{2n+1} = \mathrm{Vert}^*_z \oplus \mathrm{Hor}^*_z\,,$$

where Hor^*_z is the annihilator of Vert_z and Vert^*_z is the annihilator of Hor_z. Accordingly, every $p_z \in T^*S^{2n+1}$ can be uniquely decomposed as

$$p_z = p^v_z + p^h_z, \qquad p^v_z \in \mathrm{Vert}^*_z, \qquad p^h_z \in \mathrm{Hor}^*_z\,.$$

The distribution Hor_z is invariant with respect to the action ρ of $U(1)$ on S^{2n+1} and is therefore a connection on the principal $U(1)$-bundle $\vartheta^c\colon S^{2n+1} \to \mathcal{CP}^n$. We shall call it the Hopf connection. Let $v_z = \mathrm{i}z \in \mathrm{Vert}_z$ denote the unit vector and let $\widetilde{X}_1, \widetilde{X}_2$ be two vector fields on S^{2n+1} such that they take values X_1 and X_2 at z and let $[\widetilde{X}_1, \widetilde{X}_2]$ be the Lie bracket of these two fields. Then the 2-form

$$(\omega^H)_z(X_1, X_2) = \langle v_z, [\widetilde{X}^h_1, \widetilde{X}^h_2](z)\rangle \tag{26}$$

is the curvature of the Hopf connection.

If in the proof of formula (17) we replace the real q by the complex z and the Euclidean inner product on $\mathbb{R}^n$ by the Hermitian inner product on $\mathbb{C}^n$, we get that for every $z \in S^{2n+1}$ the map

$$(\mathrm{d}\vartheta^c)_z \colon \mathrm{Hor}_z \to T_{\vartheta^c(z)}\mathcal{CP}^n$$

is equal to twice the isometry. From this and from the expressions (26) and (24) it is not difficult to see that

$$(\vartheta^c)^*(\omega_m) = \omega^H ,$$

where ω_m is defined by (24). Let $\pi\colon T^*S^{2n+1} \to S^{2n+1}$ be the natural projection. We shall denote the form $\pi^*(\omega^H)$ on T^*S^{2n+1} briefly by ω^H. In the proposition bellow the canonican form on $T^*\mathcal{CP}^n$ will be denoted by ω^{cp}_{can} to avoid confusion.

Proposition 4 *Let ω_{can} be the canonical symplectic form on T^*S^{2n+1}. Then*

$$(\omega_{can})_{(z,p_z)} = (\omega_{Fib})_{(z,p^v_z)} + p_z(v_z) \cdot \omega^H_{(z,p_z)} + (\vartheta^c)^*(\omega^{cp}_{can})_{(z,p_z)} .$$

By ω_{Fib} we denoted the restriction of ω_{can} to $T^*(U_z)$, where $U_z = \{\rho_u(z),\ u \in U(1)\}$ is the $U(1)$-orbit through z. The above decomposition can also be written in the form

$$(\omega_{can})_{(z,p_z)} = (\omega_{Fib})_{(z,p^v_z)} + (\vartheta^c)^*(p_z(v_z)\omega_m + \omega^{cp}_{can})_{(z,p_z)} . \tag{27}$$

Proof. Let α denote the tautological 1-form on T^*S^{2n+1}. For every tangent vector $Y \in T_{(z,p_z)}(T^*S^{2n+1})$ we have

$$\alpha_{(z,p_z)}(Y) = p_z(X) ,$$

where $X = d\pi_{(z,p_z)}(Y)$ and $\pi\colon T^*S^{2n+1} \to S^{2n+1}$ is the natural projection. The Hopf connection allows us to decompose α into a sum of two 1-forms

$$\alpha_{(z,p_z)}(Y) = \alpha^v_{(z,p_z)}(Y) + \alpha^h_{(z,p_z)}(Y) = p^v_z(X) + p^h_z(X) . \tag{28}$$

Let now $\widetilde{Y}^1$ and $\widetilde{Y}^2$ be two vector fields on T^*S^{2n+1} such that they are invariant with respect to the action ρ^* of $U(1)$ on T^*S^{2n+1}. Let their values at (z,p_z) be $Y^1_{(z,p_z)}$ and $Y^2_{(z,p_z)}$. Then the formula

$$(\mathrm{d}\alpha)_{(z,p_z)}(Y^1,Y^2) = \widetilde{Y}^1(\alpha(\widetilde{Y}^2))(z,p_z) - \widetilde{Y}^2(\alpha(\widetilde{Y}^1))(z,p_z) + \alpha([\widetilde{Y}^1,\widetilde{Y}^2])(z,p_z)$$

which is valid for any 1-form, and the ρ^*-invariance of the vector fields give us

$$\begin{aligned}(\mathrm{d}\alpha^v))_{(z,p_z)}(Y^1,Y^2) = {}& (\omega_{Fib})_{(z,p^v_z)}((Y^1)^v,(Y^2)^v) \\ & + p_z(v_z)(\pi^*\omega^H)_{(z,p_z)}(Y^1,Y^2)\end{aligned}$$

and

$$(\mathrm{d}\alpha^h)_{(z,p_z)}(Y^1,Y^2) = (\widetilde{Y}^1)^h(p^h_z((\widetilde{X}^2)^h) - (\widetilde{Y}^2)^h(p^h_z((\widetilde{X}^1)^h) + p^h_z([(\widetilde{X}^1)^h,(\widetilde{X}^2)^h]) .$$

From the fact that $(\mathrm{d}\vartheta^c)_z\colon \mathrm{Hor}_z \to T_zS^{2n+1}$ is equal to twice the isometry we see that $\mathrm{d}\alpha^h = \vartheta^*(\omega^{cp}_{can})$. This together with the above expression for $\mathrm{d}\alpha^h$ and the sum (28) proves the proposition.

□

Because for every $z \in S^{2n+1}$ the map $\mathrm{d}\vartheta^c_z\colon \mathrm{Hor}_z \to T_{\vartheta^c(z)}\mathcal{CP}^n$ is an isomorphism, we also have the isomorphism

$$\widetilde{\vartheta^c_z} = [(\mathrm{d}\vartheta^c_z)^{-1}]^*\colon {\mathrm{Hor}_z}^* \to T^*_{\vartheta^c(z)}\mathcal{CP}^n \tag{29}$$

for every $z \in S^{2n+1}$.

Corollary 1 *The system $(T^*\mathcal{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$ is a symplectic quotient of the system $(T^*S^{2n+1}, \omega_{can}, H_s)$, corresponding to the action ρ of $U(1)$ on S^{2n+1} given by (22) and lifted on T^*S^{2n+1}.*

Proof. Let σ be an action of a group G on a manifold N and let $\nu\colon T^*N \to \mathfrak{g}^*$ be the moment map of the natural lifting of σ on T^*N. Then for every $\xi \in \mathfrak{g}$ we have

$$\langle \nu(q, p_q), \xi \rangle = p_q(\xi^N_q) \, ,$$

where ξ^N_q is the infinitesimal action of ξ on N evaluated at $q \in N$. From this we see that the moment map $\mu\colon T^*S^{2n+1} \to i\mathbb{R}$ of the lifted action ρ is given by

$$\mu(z, p_z) = p_z(v_z) = p^v_z \, .$$

Recall that the induced symplectic form ω_Q on the symplectic quotient $\mu^{-1}(iP)/U(1)$ is the 2-form which satisfies the relation

$$i^*(\omega_{can}) = \pi^*(\omega_Q) \, ,$$

where $i\colon \mu^{-1}(iP) \to T^*S^{2n+1}$ is the inclusion and $\pi\colon \mu^{-1}(iP) \to \mu^{-1}(iP)/U(1)$ is the projection. For the proof that such a form is symplectic see e.g. [15]. We have

$$\mu^{-1}(iP) = \{(z, Pv + p^h_z);\ v \in \mathrm{Vert}^*_z, \|v\| = 1,\ p^h_z \in \mathrm{Hor}^*_z\}$$

and the projection $\pi\colon \mu^{-1}(iP) \to \mu^{-1}(iP)/U(1) \cong T^*\mathcal{CP}^n$ is given by

$$\pi(z, Pv + p^h_z) = (\vartheta^c(z), \widetilde{\vartheta^c_z}(p^h_z))$$

From this and from Proposition (4) it is now easily seen that $\omega_Q = \omega_{can} + P\omega_m$. Finally the formula (25) shows that the Hamiltonian H_c descends to the Hamiltonian $H_{\mathcal{CP}^n}$ which completes the proof.

□

Corollary 2 *The Hamiltonian system $(T^*S^{2n+1}, \omega_{can}, H_c)$, where the Hamiltonian H_c is given by (23) is integrable. Let the family*

$$K^P_j(h, p_h)\colon T^*\mathcal{CP}^n \to \mathbb{R} \, , \qquad j = 1, \ldots, 2n$$

be a complete set of commuting integrals of the system $(T^*\mathcal{CP}^n, \omega_{can}+P\omega_m, H_{\mathcal{CP}^n})$. Then the functions $H^c_j\colon T^*S^{2n+1} \to \mathbb{R}$ defined by

$$H^c_j(z, p_z) = K^{p_z(v_z)}_j(\vartheta^c(z), \widetilde{\vartheta^c_z}(p^h_z)) \, , \qquad j = 1, \ldots, 2n \tag{30}$$

together with the function

$$H^c_{2n+1}(z, p_z) = \|p^v_z\|$$

form a complete set of Poisson-commuting integrals of the system $(T^*S^{2n+1}, \omega_{can}, H^c)$.

We note that the value of the integral H^c_{2n+1} is equal to the charge P of the "projected particle" moving on $\mathcal{CP}^n$ in the magnetic field given by ω_m.

Proof. The function H^c_{2n+1} is essentially the moment map of the lifting ρ^* of the $U(1)$-action ρ. Since H_c is invariant with respect to this action, H^c_{2n+1} is indeed an integral. The Hamiltonian vector field of the function H^c_{2n+1} is $Y_{(z,p_z)} = (\mathrm{d}/\mathrm{d}t)|_{t=0}\rho^*_{u(t)}(z,p_z)$, where $u(t) = e^{\mathrm{i}t}$. By their construction all the functions $H^c_j\colon T^*S^{2n+1} \to \mathbb{R}$ are invariant with respect to ρ^*. Therefore

$$\{H^c_j, H^c_{2n+1}\}(z,p_z) = [dH^c_j(Y)](z,p_z) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} H^c[\rho^*_{u(t)}(z,p_z)] = 0 \, .$$

Let now $H^c_j, H^c_k\colon T^*S^{2n+1} \to \mathbb{R}$ be two functions defined by (30). Denote by H^{cf}_j, H^{cf}_k the restrictions of these functions to the subspace T^*U_z. For the Hamiltonian vector field $X^{Fib}_{H_j}$ of H^{cf}_j with respect to ω_{Fib} we have

$$X^{Fib}_{H_j} = (X_{H^c_j})^v \, ,$$

where $(X_{H^c_j})^v$ denotes the vertical part of the Hamiltonian vector field $X_{H^c_j}$ of H^c_j. Due to the invariance of H^c_j with respect to the action ρ the horizontal part $(X_{H^c_j})^h$ of the Hamiltonian field is invariant with respect to $d\rho$. Denote $P = p_z(v_z)$. From (27) we get

$$\{H^c_j, H^c_k\}(z,p_z) = \{H^{cf}_j, H^{cf}_k\}_{Fib}(z, p^v_z) + \{K^P_j, K^P_k\}_P(\vartheta^c(z), \widetilde{\vartheta^c_z}(p_z)) \, , \tag{31}$$

where $\{-,-\}_{Fib}$ is the Poisson bracket on T^*U_z corresponding to ω_{Fib}, and $\{-,-\}_P$ is the Poisson bracket on $T^*\mathcal{CP}^n$ corresponding to the symplectic structure $\omega_{can} + P\omega_m$.

Now the functions $H^{cf}_j\colon T^*U_z \to \mathbb{R}$ are independent on the base space coordinate due to the ρ-invariance of H^c_j. Therefore $\{H^{cf}_j, H^{cf}_k\}_{Fib}(z,p^v_z) = 0$. The functions $K^P_j\colon T^*\mathcal{CP}^n \to \mathbb{R}$ Poisson-commute with respect to the form $\omega_{can} + P\omega_m$. Thus we have

$$\{H^c_j, H^c_k\}(z,p_z) = 0 \, .$$

Finally we have to show that the Hamiltonian H^c can be expressed by means of the functions H^c_j, $j = 1, \ldots, 2n+1$. Suppose that $H_{\mathcal{CP}^n} = K^P_1$, where $H_{\mathcal{CP}^n}$ is the Hamiltonian of the system $(T^*\mathcal{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$. Recall that

$$H_{\mathcal{CP}^n}(h, p_h) = \frac{1}{2}\|p_h\|^2 + \langle \mathrm{Ad}_h(B), B\rangle \, .$$

Above we have seen that the map

$$(\mathrm{d}\vartheta^c)_z\colon \mathrm{Hor}_z \to T_{\vartheta^c(z)}\mathcal{CP}^n$$

is equal to twice the isometry for every $z \in S^{2n+1}$. Therefore we have

$$\|\widetilde{\vartheta^c_z}(p^h_z)\|^2 = \frac{1}{4}\|p^h_z\|^2 \, . \tag{32}$$

The same calculation as in Theorem 1 gives

$$\langle \mathrm{Ad}_{\vartheta^c(z)}(B), B\rangle = \mathrm{Tr}(B^2) - 4\sum_{i=0}^n a_i^2 |z|_i^2 + 4\Big(\sum_{i=0}^n a_i |z|_i^2\Big)^2 \, . \tag{33}$$

We only have to replace the real q_i in (21) by the complex z_i and use the Hermitian inner product on $\mathbb{C}^{(n+1)}$. From (32) and (33) we get

$$4H_1^c(z,p_z) = \frac{1}{2}\|p_z^h\|^2 - \sum_{i=0}^{n} a_i^2|z|_i^2 + \Big(\sum_{i=0}^{n} a_i|z|_i^2\Big)^2 .$$

Thus we finally obtain the expression for our Hamiltonian

$$H_c(z,p_z) = \frac{1}{2}H_{2n+1}^c + 4H_1^c .$$

It is now clear that the Poisson-commuting functions H_j^c are actually first integrals of the system $(T^*S^{2n+1}, \omega_{can}, H_c)$.

□

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Appendix

Let $\mathcal{S}$ denote the vector space of real symmetric $(n+1)\times(n+1)$-matrices and $\mathcal{S}^*$ its dual space. By $\mathrm{Ad}_g^*(A)$ we shall denote the coadjoint action of $g \in GL(n+1)$ on $A \in \mathcal{S}$. Let the functions $Q_i \colon \mathfrak{gl}(n+1) \to \mathbb{R}$ be defined by (7).

Theorem 2 *Let $B = \mathrm{diag}(a_0, a_1, \ldots, a_n)$ be a real diagonal matrix.*
(a) The Hamiltonian systems $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}})$, where

$$\begin{aligned} H_{\mathcal{RP}^n}(h,p_h) &= \frac{1}{2}\|p_h\|^2 + \langle \mathrm{Ad}_h(B), B\rangle, \\ H_{\mathcal{H}}(h,p_h) &= \frac{1}{2}\|p_h\|^2 + \langle \mathrm{Ad}_h(B), B\rangle \end{aligned}$$

are integrable. Define the functions $H_{i,j}^P \colon T^*\mathcal{RP}^n \to \mathbb{R}$ and $H_{i,j}^H \colon T^*\mathcal{H}^n \to \mathbb{R}$ by

$$Q[\mathrm{Ad}_{h^{-1}}^*(\lambda) + zp_h + z^2\lambda] = \begin{cases} \displaystyle\sum_{j=0}^{2\deg(Q_i)} H_{i,j}^P(h,p_h)\cdot z^j & \text{for } (h,p_h) \in T^*\mathcal{RP}^n \\ \displaystyle\sum_{j=0}^{2\deg(Q_i)} H_{i,j}^H(h,p_h)\cdot z^j & \text{for } (h,p_h) \in T^*H^n \end{cases} . \qquad \text{(A.1)}$$

Here z is a real indeterminate and $\lambda \in \mathcal{S}^*$ is given by $\lambda(A) = \mathrm{Tr}(BA)$ for every $A \in \mathcal{S}$. Then the families $\{H_{i,j}^P\}$ and $\{H_{i,j}^H\}$ contain complete systems of n functionally independent Poisson-commuting integrals of $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and of $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}})$ respectively. If we take $Q_1(a) = \mathrm{Tr}(a^2)$, we have

$$H_{\mathcal{RP}^n} = \frac{1}{2}H_{1,2}^P \quad \text{and} \quad H_{\mathcal{H}^n} = \frac{1}{2}H_{1,2}^P .$$

(b) Let $B = \operatorname{diag}(\mathrm{i}a_0, \mathrm{i}a_1, \ldots, \mathrm{i}a_n)$*. For every real constant* P *the Hamiltonian system* $(T^*\mathcal{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$*, where*

$$H_{\mathcal{CP}^n}(h, p_h) = \frac{1}{2}\|p_h\|^2 + \langle \operatorname{Ad}_h(B), B\rangle$$

is integrable. A complete system of Poisson-commuting integrals can be chosen from the family of functions $K^P_{i,j}: T^*\mathcal{CP}^n \to \mathbb{R}$ *given by*

$$\sum_{j=0}^{2\deg Q_i} K^P_{i,j}(h, p_h) \cdot z^j = Q_i[\lambda + z(p_h - P\operatorname{Ad}^*_{(\sqrt{h})^{-1}}(m_d)) + z^2 \operatorname{Ad}^*_{h^{-1}}(\lambda)]. \tag{A.2}$$

Above $\lambda(A) = \operatorname{Tr}(BA)$ *and* $m_d(A) = \operatorname{Tr}(m^*A)$ *for every* $A \in \mathfrak{u}(n+1)$*. The matrix* $m = \operatorname{diag}(-\mathrm{i}/2, \mathrm{i}/2n, \ldots, \mathrm{i}/2n)$ *is a non-trivial central element in* $\mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n))$*. If* $Q_1(q) = \operatorname{Tr}(a^2)$*, we have* $H_{\mathcal{CP}^n} = \frac{1}{2}K^P_{1,2}$ *up to an irrelevant additive constant.*

We have already mentioned in the beginning of Section 3 that the equations of motion of the systems $(T^*\mathcal{RP}^n, \omega_{can}, H_{\mathcal{RP}^n})$ and $(T^*\mathcal{H}^n, \omega_{can}, H_{\mathcal{H}})$ can be written in the form of Lax equation

$$L_t = [A, L]\,,$$

where

$$\begin{aligned} L &= \operatorname{Ad}^*_{h^{-1}}(\lambda) + zp_h + z^2\lambda \\ A &= p_h + z\lambda\,. \end{aligned} \tag{A.3}$$

Let the indeterminate z above be complex. The integrability of these systems can be proved by means of the theory of spectral curves and associated flows on their Jacobian tori developed by Adler, van Moerbeke, Mumford, Adams, Harnad, Hurtubise, Previato and others, see e.g. [4], [5], [16], [2], [3]. Indeed the Lax pair (A.3) satisfies the condition formulated by Griffiths in [11] (see also [7] and [13]) for the Lax equation to yield a linear flow on the Jacobian torus of the spectral curve S given by

$$S = \{(z, w); \det(L(z) - wI) = 0\}\,.$$

Lax pair of the form (A.3) can be defined for every Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ of an arbitrary semi-simple Lie algebra $\mathfrak{g}$. Let M be the symmetric space given by the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$. Then we take $L(h, p_h) = \operatorname{Ad}^*_{h^{-1}}(\lambda) + zp_h + \mu$, where $h \in M$ and $p_h \in T^*_hM \cong \operatorname{Ad}^*_{g^{-1}}(\mathfrak{p})$ with $g^2 = h$. Elements λ and μ are constants in $\mathfrak{g}^*$. As above we take $A = p_h + z\mu$. Such pairs give integrable motions on symmetric spaces M under the influence of the potentials of the form $V(h) = \langle \operatorname{Ad}_h(\lambda), \mu\rangle$, where $h \in M$ and $\lambda, \mu \in \mathfrak{g}$ are suitably chosen constants. These systems which we call generalized Neumann systems are described in [20].

We note that in [19] the authors study a different type of Lax pairs related to Cartan decompositions. In their case the Lax matrix L is of the form $L = s + zl + z^2a$, where $l \in \mathfrak{u}^*$ and $s, a \in \mathfrak{p}^*$ and a is constant. Even though these Lax pairs are different from those mentioned above, the resulting family of integrable systems is similar (but not the same) as that in [20].

It is however not so easy to find the Lax pair for the system $(T^*\mathcal{CP}^n, \omega_{can} + P\omega_m, H_{\mathcal{CP}^n})$ with the magnetic term ω_m. Below we give a proof of Theorem 2 which works for all our systems and is conceptually simple. It relies only on the notions of semi-direct product and symplectic quotient.

In the proof of Theorem 2 we shall be concerned with the semi-direct products $SO(n+1) \ltimes \mathcal{S}$, $SO(1,n) \ltimes \mathcal{S}$ and $SU(n) \ltimes \mathfrak{u}(n+1)$. We will denote them by the common symbol $G \ltimes \mathcal{V}$. The G-action ρ on $\mathcal{V}$ will be given by $\rho_g(\alpha) = \mathrm{Ad}_g(\alpha)$ in both cases. The product in $G \ltimes \mathcal{V}$ is defined by

$$(g_1, A_1) \cdot (g_2, A_2) = (g_1 g_2, A_1 + \mathrm{Ad}_{g_1}(A_2)) .$$

The adjoint action of $G \ltimes \mathcal{V}$ on $\mathrm{Lie}(G \ltimes \mathcal{V}) = \mathfrak{g} \ltimes \mathcal{V}$ is then

$$\mathrm{Ad}_{(g,A)}(\xi, \gamma) = (\mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\gamma) - [\mathrm{Ad}_g(\xi), A]) .$$

Let us equip the Lie algebra $\mathfrak{g} \ltimes \mathcal{V} = \mathrm{Lie}(G \ltimes \mathcal{V})$ with a non-degenerate inner product $\langle -, - \rangle_{\ltimes}$. We take

$$\langle (\xi, A), (\eta, B) \rangle_{\ltimes} = \mathrm{Tr}(\xi^T \eta) + \mathrm{Tr}(AB), \qquad \langle (\xi, A), (\eta, B) \rangle_{\ltimes} = \mathrm{Tr}(\xi^* \eta) + \mathrm{Tr}(AB) ,$$

where the formula on the left is defined on $\mathfrak{so}(n+1) \ltimes \mathcal{S}$ and $\mathfrak{so}(1,n) \ltimes \mathcal{S}$, and the one on the right on $\mathfrak{su}(n+1) \ltimes \mathfrak{u}(n+1)$. If we identify the dual space $(\mathfrak{g} \ltimes \mathcal{V})^*$ of $\mathfrak{g} \ltimes \mathcal{V}$ via this inner product, then the coadjoint action of $G \ltimes \mathcal{V}$ on $(\mathfrak{g} \ltimes \mathcal{V})^*$ is given by

$$\mathrm{Ad}^*_{(g,A)}(p, \lambda) = (\mathrm{Ad}^*_g(p) + \mathrm{Ad}^*_g([A, \lambda]), \mathrm{Ad}^*_g(\lambda)) .$$

By $\theta: G \to G$ we shall again denote the Cartan involution corresponding to the appropriate symmetric space, that is to $\mathcal{RP}^n$, $\mathcal{H}^n$ or to $\mathcal{CP}^n$. These symmetric spaces will be denoted by the common symbol M.

Proof of Theorem 2. Let $F \subset G \times G$ be the subgroup of the form $F = \{(g, g^\theta)\}$ and let $\mathcal{F} = F \ltimes (\mathcal{V} \oplus \mathcal{V})$ be the semi-direct product with respect to the diagonal adjoint action. The group $\mathcal{F}$ consists of elements of the form $((g, A_1), (g^\theta, A_2))$. Let $\lambda \in \mathcal{V}$ and define $\lambda^\theta = J\lambda J$. Define the functions $K_{i,j}: T^*\mathcal{F} \to \mathbb{R}$ by

$$\sum_{j=0}^{2\deg(Q_i)} K_{i,j}((g, A_1), (g^\theta, A_2), (p, \lambda_1), (p^\theta, \lambda_2)) \cdot z^j = Q_i(\lambda_1 + zp + z^2 \lambda_2^\theta) .$$

The above functions Poisson-commute because they are independent on the base-space variables. We shall construct the Poisson-commuting functions on T^*M from the functions $K_{i,j}$ by means of a two-stage symplectic quotient.

Let the subgroup $\mathcal{V} \subset G \ltimes \mathcal{V}$ act on $G \ltimes \mathcal{V}$ by the action

$$\sigma_B((g, A_1), (g^\theta, A_2)) = ((g, A_1) \cdot (e, B), (g^\theta, A_2) \cdot (e, B)) .$$

Let us trivialize the cotangent bundle $T^*(G \ltimes \mathcal{V})$ by right translations. Then the action σ lifts to the action σ^* on $T^*(G \ltimes \mathcal{V})$ which is given by

$$\sigma^*((g, A_1), (g^\theta, A_2), (p, \lambda_1), (p^\theta, \lambda_2)) = (\sigma((g, A_1), (g^\theta, A_2)), (p, \lambda_1), (p^\theta, \lambda_2)) .$$

Any subgroup $\mathcal{U}$ of a Lie group $\mathcal{G}$ acts on $\mathcal{G}$ by right translations. Let $\widetilde{\rho}$ be the natural lifting of this action on the cotangent bundle $T^*\mathcal{G}$. Then the moment map $M\colon T^*\mathcal{G} \to \mathrm{Lie}(\mathcal{U})^*$ of this action is given by

$$\langle M(g, p_g), \xi\rangle = p_g(\mathrm{Ad}_g(\xi)) = \mathrm{Ad}_g^*(p_g)(\xi), \qquad \xi \in \mathrm{Lie}(\mathcal{U}) .$$

For the proof see [1] or [21]. Let $\nu\colon T^*\mathcal{F} \to \mathcal{V}^*$ be the moment map of the action σ^*. From the above formula we get

$$\nu((g, A_1), (g^\theta, A_2), (p, \lambda_1), (p^\theta, \lambda_2)) = \mathrm{Ad}_g^*(\lambda_1) + \mathrm{Ad}_{g^\theta}^*(\lambda_2) .$$

Let ω_R denote the induced symplectic form on a symplectic quotient. The symplectic quotient $(\nu^{-1}(0)/\mathcal{V}, \omega_R)$ is equal to $(T^*(F \ltimes \mathcal{V}), \omega_{can})$. Since the polynomials Q_i are Ad-invariant, we can express the functions $\widetilde{K}_{i,j}\colon T^*(F \ltimes \mathcal{V}) \to \mathbb{R}$ induced by the functions $K_{i,j}\colon T^*\mathcal{F} \to \mathbb{R}$ in the form

$$\sum_{j=0}^{2\deg(Q)_i} \widetilde{K}_{i,j}(((g, g^\theta), A), ((p, p^\theta), \lambda)) = Q_i(\mathrm{Ad}_g^*(\lambda) + z\mathrm{Ad}_g^*(p) + z^2\mathrm{Ad}_{g^\theta}^*(\lambda^\theta)) . \tag{A.4}$$

The symplectic spaces $(T^*(F \ltimes \mathcal{V}), \omega_{can})$ and $(T^*F, \omega_{can}) \times (T^*\mathcal{V}, \omega_{can})$ are symplectomorphic. Denote the points in $T^*F \times T^*\mathcal{V}$ by (x, y), where $x \in T^*F$ and $y \in T^*\mathcal{V}$. For any pair of functions $H, K\colon T^*(F \ltimes \mathcal{V}) = T^*F \times T^*\mathcal{V} \to \mathbb{R}$ we have

$$\{H, K\}(x_0, y_0) = \{H(x_0, y), K(x_0, y)\}_1(y_0) + \{H(x, y_0), K(x, y_0)\}_2(x_0) , \tag{A.5}$$

where $\{-,-\}$ is the Poisson bracket on $T^*(F \ltimes \mathcal{V})$,while $\{-,-\}_1$ and $\{-,-\}_2$ are the Poisson brackets on T^*F and on $T^*\mathcal{V}$ respectively. Since they are induced by the Poisson-commuting functions $K_{i,j}$, the functions $\widetilde{K}_{i,j}$ Poisson-commute on $T^*(F \ltimes \mathcal{V})$. In addition, the functions $\widetilde{K}_{i,j}$ are independent on the base space variables of the bundle $T^*\mathcal{V}$, therefore

$$\{\widetilde{K}_{i,j}(((g_0, g_0^\theta), A), ((p_0, p_0^\theta), \lambda)), \widetilde{K}_{k,l}(((g_0, g_0^\theta), A), ((p_0, p_0^\theta), \lambda))\}_2 = 0$$

for every fixed point $((g_0, g_0^\theta), (p, p^\theta)) \in T^*F$. Fix an element $\lambda \in \mathcal{V}^*$ and define the functions $\widehat{K}\colon T^*F \to \mathbb{R}$ by

$$\widehat{K}_{i,j}((g, g^\theta), (p, p^\theta)) = \widetilde{K}_{i,j}(((g, g^\theta), A), ((p, p^\theta), \lambda)) .$$

The choice of A in the above definition is of course irrelevant. It follows from (A.5) that the functions $\widehat{K}_{i,j}\colon T^*F \to \mathbb{R}$ Poisson-commute with respect to ω_{can} on T^*F and this concludes the first stage of our construction.

Let now $U \subset G$ be the subgroup which is the fixed-point set of Cartan involution $\theta\colon G \to G$. Let ρ denote the diagonal right translation action of U on F and denote the lifting of this action to T^*F by ρ^*. If we again trivialize T^*F by the right translations, then ρ^* is given by

$$\rho_u^*((g, g^\theta), (p, p^\theta)) = ((gu^{-1}, (gu^{-1})^\theta), (p, p^\theta)) .$$

It is clear from their definition by means of the Ad-invariant polynomials Q_i that the functions $\widehat{K}_{i,j}\colon T^*F \to \mathbb{R}$ are invariant with respect to the action ρ^*. Let $\mu\colon T^*F \to \mathfrak{u}^*$

be the moment map of ρ^*. Here $\mathfrak{u}^*$ is the dual of the Lie algebra $\mathfrak{u} = \mathrm{Lie}(U)$. In a similar way as in the case of the moment map ν above, we see that μ is given by

$$\mu((g, g^\theta), (p, p^\theta)) = \mathrm{Ad}_g^*(p) + \mathrm{Ad}_{g^\theta}^*(p^\theta) .$$

Let $m \in \mathfrak{u}$ be a central element and let $m_d = \langle m, - \rangle \in \mathfrak{u}^*$. Consider the symplectic quotient $(\mu^{-1}(Pm_d)/U, \omega_R)$. As a manifold, the space $\mu^{-1}(Pm_d)/U$ is diffeomorphic to $T^*(G/U) = T^*M$. The induced form ω_R is equal to $\omega_{can} + P\omega_m$, where ω_m is defined by (6). To prove this claim, recall that $i^*(\omega_R) = \pi^*(\omega_{can})$, where $i\colon \mu^{-1}(Pm_d) \to \mathfrak{u}^*$ is the inclusion and $\pi\colon T^*F \to F$ the natural projection. The canonical form ω_{can} on $T^*\mathcal{G}$ for any Lie group $\mathcal{G}$ is given by

$$(\omega_{can})_{(g,p_g)}((X^b, X^t), (Y^b, Y^t)) = \langle X^b, Y^t \rangle - \langle X^t, Y^b \rangle + \langle p_g,\ [X^b, Y^b] \rangle ,$$

and $(X^b, X^t), (Y^b, Y^t) \in T_{(g,p_g)}(T^*\mathcal{G}) \cong \mathfrak{g} \times \mathfrak{g}^*$ via the right translations. If we restrict this formula to

$$\mu^{-1}(Pm_d) = \{p_h + \mathrm{Ad}_{g^{-1}}^*(Pm_d);\ p_h \in \mu^{-1}(0), \text{ and } h = g(g^\theta)^{-1}\}$$

and then pass to the quotient by U, we get indeed $\omega_R = \omega_{can} + P\omega_m$, as desired.

Finally we restrict the functions $\widehat{K}_{i,j}\colon T^*F \to \mathbb{R}$ to $\mu^{-1}(Pm_d)$ and again pass to the quotient. This gives us the family of Poisson-commuting functions $H_{i,j}\colon T^*M \to \mathbb{R}$ which in terms of the Cartan model of M are given by the relations

$$\sum_{j=0}^{2\deg(Q_i)} H_{i,j}(h, p_h) \cdot z^j = Q_i(\lambda + z(p_h - \mathrm{Ad}_{g^{-1}}^*(Pm_d)) + z^2 \mathrm{Ad}_h^*(\lambda^\theta)), \qquad g = \sqrt{h} .$$

This follows from (A.4). In the part *a)* of our theorem the element m is simply equal to zero, while in the part *b)* it is the non-trivial central element $m = \mathrm{diag}(-\mathrm{i}/2, \mathrm{i}/2n, \ldots, \mathrm{i}/2n)$ in $\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$. If we take $Q_1(a) = \mathrm{Tr}(a^2)$, then the equalities $H_{\mathcal{RP}^n} = \frac{1}{2} H_{1,2}^P$ and $H_{\mathcal{H}^n} = \frac{1}{2} H_{1,2}^P$ follow immediately from (A.1).

From (A.2) we get

$$K_{1,2}^P(h, p_h) = \mathrm{Tr}(p_h^2) - 2\,\mathrm{Tr}[p_h \cdot \mathrm{Ad}_{h^{-1}}^*(m_d)] + P^2\,\mathrm{Tr}(m_d^2) + 2\,\mathrm{Tr}[\lambda \cdot \mathrm{Ad}_{h^{-1}}^*(\lambda)] .$$

In Section 2 we have seen that the metric on $\mathcal{CP}^n$ is induced by the Killing form on $\mathfrak{su}(n+1)$ which in turn is given by $\langle a, b \rangle = \mathrm{Tr}(ab)$. Taking into account the definitions of λ and m_d we get

$$K_{1,2}^P(h, p_h) = \|p_h^2\| + 2\langle B, \mathrm{Ad}_h(B) \rangle - 2\langle p_h, \mathrm{Ad}_{h^{-1}}^*(m_d) \rangle + P^2\,\mathrm{Tr}(m^2) .$$

Let $\mathfrak{su}(n+1) = \mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n)) \oplus \mathfrak{cp}$ be the Cartan decomposition corresponding to $\mathcal{CP}^n$. As a consequence of the expression (5) from Section 2 we get $p_h \in \mathrm{Ad}_{h^{-1}}^*(\mathfrak{cp}^*)$. But $\mathrm{Ad}_{h^{-1}}(m_d) \in \mathrm{Ad}_{h^{-1}}(\mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n))^*)$. The spaces $\mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n))$ and $\mathfrak{cp}$ are orthogonal with respect to the Killing form, therefore we have

$$\langle p_h, \mathrm{Ad}_{h^{-1}}^*(m_d) \rangle = 0 .$$

Thus

$$K_{1,2}^P(h, p_h) = 2H_{\mathcal{CP}^n}(h, p_h) + P^2\,\mathrm{Tr}(m^2) ,$$

where $P^2\,\mathrm{Tr}(m^2)$ is constant.

The fact that one can choose a complete family of integrals from the set $\{H_{i,j}\}$ is proved in [20] and we shall not repeat the proof here.

$\square$

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