Lattices of Neumann oscillators and Maxwell-Bloch equations

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Abstract

We introduce a family of new non-linear many-body dynamical systems which we call the Neumann lattices. These are lattices of N interacting Neumann oscillators. The interactions are of magnetic type. We construct large families of conserved quantities for the Neumann lattices. For this purpose we develop a new method of constructing the first integrals which we call the reduced curvature condition. Certain Neumann lattices are natural partial discretizations of the Maxwell-Bloch equations. The Maxwell-Bloch equations have a natural Hamiltonian structure whose discretizations yields a twisted Poisson structures (in the sense of P. Ševera and A. Weinstein) for the Neumann lattices. Thus the Neumann lattices are candidates for integrable systems with twisted Poisson structures.

1 Introduction

The Neumann oscillator is a point particle moving on a sphere S^n under the influence of a force whose potential is quadratic. This is one of the best-known classical non-linear integrable systems. It was first studied by Carl Neumann in the mid 19th

century and more recently by J. Moser, D. Mumford, and by many other authors (See e.g. [1], [2], [3], [4]). In this paper we shall study a certain type of lattices of interacting Neumann oscillators.

Let $g: I \to SU(2)$ be a solution of the ordinary differential equation

$$(g_t g^{-1})_t = [\sigma, \mathrm{Ad}_g(\tau)] \tag{1}$$

where σ and τ are arbitrary elements in the Lie algebra $\mathfrak{su}(2)$. Then $g(t)$ is the evolution of the Neumann oscillator on $S^3 = SU(2)$ with the force potential $V(g) = \langle \sigma, \mathrm{Ad}_g(\tau) \rangle$. Let now N be a positive integer and let us arrange N copies of the three-sphere S^3 on the vertices of the regular N-sided polygon inscribed in the unit circle S^1. The system of $3N$ ordinary non-linear differential equations

$$\left((g_i)_t (g_i)^{-1} \right)_t = \sum_{k=1}^{N} \psi_{i,k} (g_k)_t (g_k)^{-1} + [\sigma, \mathrm{Ad}_{g_i}(\tau_i)], \quad i = 1, \ldots, N \tag{2}$$

describes a lattice of N interacting Neumann oscillators given by the equation (1). The interaction is of *magnetic type* in the sense that the acceleration of each oscillator in the lattice depends on the velocities $(g_i)_t (g_i)^{-1}(t)$ of the other oscillators and not on their positions $g_i(t)$.

As we mentioned above, the Neumann systems are integrable. A natural question is, whether the lattice (2) is an integrable system for some choice of non-zero $\psi_{i,k}$. In this paper we make the first and probably the most important step towards answering this question. We show that the equation (2) has a large number of conserved quantities, if the $N \times N$-matrix $\widetilde{D} = (\psi_{i,k})_{i,k=1,\ldots,N}$ satisfies two conditions:

(i) $\widetilde{D}$ is an element of the special orthogonal Lie algebra $\mathfrak{so}(N)$.
(ii) The kernel of $\widetilde{D}$ contains the vector $w = (1, \ldots, 1)^T$.

Our construction of the conserved quantities of (2) stems from the relation between the system (2) and the Maxwell-Bloch equations.

The Maxwell-Bloch equations are a system of partial differential equations which plays an important role in non-linear optics. It describes the resonant interaction between light and an active optical medium consisting of two-level atoms. The Maxwell-Bloch equations, without broadening and pumping, have the following form:

$$E_t + E_x = P - \alpha E, \quad P_t = E\mathcal{N} - \beta P, \quad \mathcal{N}_t = -\frac{1}{2}(\overline{E}P + E\overline{P}) - \gamma(\mathcal{N} - 1). \tag{3}$$

The independent variables x and t parametrize one spatial dimension and the time, the complex valued functions $E(t,x)$ and $P(t,x)$ describe the slowly varying envelopes of

the electric field and the polarization of the medium, respectively, and the real valued function $\mathcal{N}$ is the level inversion. The constant α represents the losses of the electric field, while β is the longitudinal and γ the transverse relaxation rate in the medium. We shall assume that $\alpha = \gamma = 0$ and will concetrate on the spatialy periodic case of (3).

The Maxwell-Bloch equations can be represented as the equation of motion for a continuous lattice of Neumann oscillators given by (1). The system (3) with $\alpha = \gamma = 0$ is equivalent to the equation

$$(g_t g^{-1})_t + (g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau(x))] \tag{4}$$

for the unknown function $g(t,x) \colon I \times S^1 \to SU(2)$. Above, $\sigma = \mathrm{diag}(i,-i) \in \mathfrak{su}(2)$ and $\tau(x) \colon S^1 \to \mathfrak{su}(2)$ is an arbitrary loop in the Lie algebra $\mathfrak{su}(2)$. The comparison of the equations (4) and (1) suggests that the Maxwell-Bloch equations can be understood as the equation of motion for a continuous lattice of the Neumann oscillators, parametrized by $x \in S^1$. The term $(g_t g^{-1})_x$ in (4) is responsible for the interaction of magnetic type among the oscillators.

For suitable choices of $\widetilde{D} = (\psi_{i,k})$, the system (2) can be considered as a discretization of the equation (4) with respect to the spatial variable x. The circle S^1 is replaced by the N-sided polygon whose vertices are labeled by $i \in \{1, \ldots, N\}$ and the operator $\frac{\partial}{\partial x}$ is replaced by the $N \times N$ matrix $\widetilde{D} = (\psi_{i,k})_{i,k=1,\ldots,N}$. The Maxwell-Bloch system is integrable, it satisfies the zero-curvature condition. One could therefore hope that the lattice (2) will "inherit" the integrability from the Maxwell-Bloch system. The discretization of integrable systems is a very active field of current research. Many different approaches to this subject have been invented by Ablowitz, Ladik, Fadeev, Takhtajan, Nijhoff, Hirota, Suris and others (See [5] for an excellent survey). The existing discretization schemes keep track of some structure that a system in question is endowed with due to its integrability (e.g., Lax equation or R-matrix). Our discretization is a naïve one and takes for the starting point the equation itself rather then some manifestation of the integrability structure. Therefore, the integrability of our discretization is not guaranteed in advance. In particular, the equation (2) does not have a Lax pair.

The construction of the first integrals of the Neumann lattices requires a new approach. The tool that we propose for this purpose will be called *the reduced curvature condition*. The geometric foundation of the reduced curvature condition is given in theorem 1. A differential equation satisfies the zero curvature condition, if there exists a family of connections $A(z)$ such that the vanishing of the curvature $F_{A(z)} = 0$ for every z is equivalent to the equation. An equation satisfies the reduced curvature condition if there exists a family of connections $B(z)$ such that the curvature $F_{B(z)}$

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takes values in a suitable proper subalgebra of the Lie algebra of the structure group. Like the zero curvature condition, the reduced curvature condition can provide integrals of motion in certain cases. One class of such cases are the Neumann lattices of the form (2). The construction of the integrals of a Neumann lattice is given in theorem 2 and the explicit formulae for the calculation of these integrals in theorem 3. We note that the integrals given by the reduced curvature condition are rather natural. In particular, the three physically obvious integrals, the total energy and the two total angular momenta of the lattice appear explicitly.

The Neumann lattices are not Hamiltonian systems. They are endowed in a natural way by the *twisted Poisson structures*. These structures were recently studied by P. Ševera, A. Weinstein, Y. Kosmann-Schwarzbach, and other authors (See [6], [7], [8]). Since, at least in certain cases, the Neumann lattices possess as many integrals of motion as they have degrees of freedom, they provide examples for what finite dimensional integrable systems with twisted Poisson structures should be. This observation gives rise to many further questions. The most immediate one is: what can be said about the twisted Poisson commutativity of the integrals on the one hand and the Lie commutativity of their respective (twisted) Hamiltonian fields on the other. In the context of the twisted Poisson geometry these two notions do not coincide. A related question is, what are the level sets of the integrals. Another important topic which we do not consider here is the question of the number of functionally independent integrals. We intend to address these and some other issues in another paper.

2 Maxwell-Bloch equations and Neumann oscillator

We begin by rewriting the Maxwell-Bloch equations in a form which will reveal their connection with the Neumann system. Consider the Maxwell-Bloch equations

$$E_t + E_x = P, \quad P_t = E\mathcal{N} - \beta P, \quad \mathcal{N}_t = -\frac{1}{2}(\overline{E}P + E\overline{P}) \tag{5}$$

with spatially periodic boundary conditions:

$$E(t, x + 2\pi) = E(t, x), \quad P(t, x + 2\pi) = P(t, x), \quad \mathcal{N}(t, x + 2\pi) = \mathcal{N}(t, x). \tag{6}$$

Let the Lie algebra valued maps $\rho(t,x), F(t,x) \colon I \times S^1 \to \mathfrak{su}(2)$ be defined as

$$\rho(t,x) = \begin{pmatrix} i\mathcal{N}(t,x) & iP(t,x) \\ -i\overline{P}(t,x) & -i\mathcal{N}(t,x) \end{pmatrix}, \quad F(t,x) = \frac{1}{2}\begin{pmatrix} i\beta & E(t,x) \\ -\overline{E}(t,x) & -i\beta \end{pmatrix}. \tag{7}$$

In terms of ρ and F, the system (5) assumes the form

$$\rho_t = [\rho, F], \quad F_t + F_x = [\rho, \sigma] \tag{8}$$

where

$$\sigma = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The first equation in (8) is of Lax form, therefore

$$\rho(t, x) = \mathrm{Ad}_{g(t,x)}(\tau(x)), \quad F(t, x) = -g_t(t, x) \cdot g^{-1}(t, x) \tag{9}$$

where $\tau(x) \colon S^1 \to \mathfrak{su}(2)$ and $g(t, x) \colon \mathbb{R} \times S^1 \to SU(2)$ are smooth matrix-valued functions. By inserting the above into the second equation of (8), we obtain the following second-order partial differential equation for $g(t, x) \colon \mathbb{R} \times S^1 \to SU(2)$:

$$(g_t g^{-1})_t + (g_t g^{-1})_x = [\sigma, \mathrm{Ad}_g(\tau(x))]. \tag{10}$$

This equation, together with the additional stipulation $\langle g_t g^{-1}, \sigma \rangle = const. = -\beta$, is equivalent to the system (5).

Consider now the equation

$$(f_t f^{-1})_t = [\sigma, \mathrm{Ad}_f(\tau)] \tag{11}$$

where $f(t) \colon I \to SU(2)$, and $\sigma, \tau \in \mathfrak{su}(2)$ are arbitrary elements. It is shown in [9] that the equation (11) is the equation of motion for the Hamiltonian system $(T^* SU(2), \omega_c, H_n)$, where ω_c is the canonical cotangent symplectic form and the Hamiltonian is given by

$$H_n(g, p_g) = \frac{1}{2} \|p_g\|^2 + \langle \sigma, \mathrm{Ad}_g(\tau) \rangle.$$

Throughout the paper, the bracket $\langle -, - \rangle$ denotes the Killing form on $\mathfrak{su}(2)$. This Hamiltonian system is the Neumann oscillator on the three-sphere $S^3 = SU(2)$ and the quadratic force potential is given by $V(g) = \langle \sigma, \mathrm{Ad}_g(\tau) \rangle$. This is not a generic quadratic potential. If has two circular symmetries. One is given by the left action of the group $U_\sigma(1) = \{s \cdot \sigma; s \in [0, 2\pi]\} \subset SU(2)$ and the other by the right action of $U_\tau(1) = \{s \cdot \tau; s \in [0, 2\pi]\}$. The quadratic form $V(g)$ has two double eigenvalues. In the diagonalizing coordinates $\vec{q} = (q_1, q_2, q_3, q_4)$ on $\mathbb{R}^4$, we have

$$V(g(\vec{q})) = \lambda(q_1^2 + q_2^2) - \lambda(q_3^2 + q_4^2)$$

where $\frac{1}{2}\lambda$ is equal to the norm of $\tau \in \mathfrak{su}(2)$ with respect to the Killing metric. The Hamiltonian system $(T^* SU(2), \omega_c, H_n)$ is a special case of a family of integrable

system which was studied by man authors. (See e.g. [10], [11] and also [12] for the connection with the Nahm's equations of the Yang-Mills theory.)

Let us now write the equation (10) in the form

$$(g_t g^{-1})_t(t,x) = [\sigma, \mathrm{Ad}_{g(t,x)}] + \frac{1}{2\epsilon}\Big((g_t g^{-1})(t, x-\epsilon) - (g_t g^{-1})(t, x+\epsilon)\Big)\,|_{\epsilon\to 0}\,. \quad (12)$$

Comparison of this expression with the equation (11) shows that the equation (10) is the equation of motion for the continuous lattice of interacting Neumann oscillators parametrized by $x \in S^1$. For every $x_0 \in S^1$, the function $g(t, x_0)\colon I \to SU(2)$ gives the evolution of the oscillator at the x-th place. The acceleration of the oscillator $g(t, x_0)$ is influenced by the momenta $g_t g^{-1}(t, x_0 \pm \epsilon)$ of its neighbours. Therefore we say that the interaction between the oscillators is of *magnetic type*.

The above observations suggest a natural Hamiltonian structure for the Maxwell-Bloch equations. If we take into account the periodicity conditions (6), then the configuration space of a continuous lattice of Neumann oscillators is the loop group $LSU(2) = \{g(x)\colon S^2 \to SU(2)\}$. Thus the phase space of the Maxwell-Bloch equations is the cotangent bundle $T^*LSU(2)$. The natural choice for the Hamiltonian is the total energy of all the oscillators in the lattice:

$$H_{mb}(g(x), p_g(x)) = \int_{S^1}\Big(\frac{1}{2}\,\|p_g(x)\|^2 + \langle\sigma, \mathrm{Ad}_{g(x)}(\tau(x))\rangle\Big)\mathrm{d}x. \quad (13)$$

The interaction term $(g_t g^{-1})_x$ is magnetic in nature, therefore it will be encoded in the symplectic structure. The presence of $(g_t g^{-1})_x$ gives rise to the perturbation $\omega_c + \omega_m$ of the canonical cotangent form ω_c on $T^*LSU(2)$. Let $\widetilde{\omega}_m$ be the right-invariant two-form on $LSU(2)$ whose value at the identity is given by

$$(\widetilde{\omega}_m)_e(\xi(x), \eta(x)) = \int_{S^1}\langle\xi'(x), \eta(x)\rangle\,\mathrm{d}x, \quad \xi(x), \eta(x) \in L\mathfrak{su}(2). \quad (14)$$

The magnetic perturbation term ω_m is the pull-back $\pi^*(\widetilde{\omega}_m)$ of $\widetilde{\omega}_m \in \Omega^2(LSU(2))$ with respect to the natural projection $\pi\colon T^*LSU(2) \to LSU(2)$. In [9] the following theorem is proved.

Theorem *The equation of motion for the Hamiltonian system $(T^*LSU(2), \omega_c + \omega_m, H_{mb})$ is the Maxwell-Bloch equation (10).*

The form $(\widetilde{\omega}_m)_e$ is the cocycle of the central extension $\mathbb{R} \to \widetilde{L\mathfrak{su}}(2) \to L\mathfrak{su}(2)$ of the loop Lie algebra $L\mathfrak{su}(2)$. The right-invariant form $\widetilde{\omega}_m \in \Omega^2(LSU(2))$ is closed but not exact, so its deRham class is non-zero. It is the first Chern class of the non-trivial

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$U(1)$-bundle $U(1) \to \widetilde{L}SU(2) \to LSU(2)$, where $\widetilde{L}SU(2)$ is the central extension of the loop group $LSU(2)$ (See [13], [14]).

In more detail the Hamiltonian and the Lagrangian structures of the Maxwell-Bloch equations are treated in [9].

3   Neumann lattices and discretizations of Maxwell-Bloch equations

In this section we shall discretize the equation (10) with respect to the spatial variable $x \in S^1$. This will give us natural examples of the Neumann lattices.

We replace the circle S^1 by the "discrete circle" $\mathbb{Z}_N = \{e^{k\frac{2\pi i}{N}}; k = 1, \ldots, N\}$ and the configutation space $LSU(2)$ by the space of discrete loops $L_N SU(2) = \{g(k): \mathbb{Z}_N \to SU(2)\}$. Clearly, $L_N SU(2)$ is just the Cartesian product $SU(2)^N$. We shall use the following notation:

$$q = (g_1, \ldots, g_N)^T \in \overbrace{SU(2) \times \ldots \times SU(2)}^{N} = SU(2)^N = L_N SU(2).$$

The element $g_k \in SU(2)$ represents the position of the k-th Neumann oscillator in the lattice which consists of N oscillators.

Now we have to discretize the derivation with respect to x. In (10) this derivation acts on $g_t g^{-1}$ which takes values in the Lie algebra $L\mathfrak{su}(2)$. Thus we have to replace the derivation operator $\partial_x: L\mathfrak{su}(2) \to L\mathfrak{su}(2)$ by a suitable operator

$$D : \mathfrak{su}(2)^N \longrightarrow \mathfrak{su}(2)^N$$

where $\mathfrak{su}(2)^N$ is the Lie algebra of $SU(2)^N$. Let the $2N \times 2N$-matrix R be the operator of cyclic permutation on $\mathfrak{su}(2)^N$ given by the block matrix

$$R = \begin{pmatrix} 0 & 0 & \ldots & Id_2 \\ Id_2 & 0 & \ldots & 0 \\ 0 & Id_2 & \ldots & 0 \\ \vdots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & Id_2 & 0 \end{pmatrix}, \quad Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

A possible discretization of the Maxwell-Bloch equations is the system

$$(q_t q^{-1})_t + (R - R^{-1})(q_t q^{-1}) = [\vec{\sigma}, \mathrm{Ad}_q(\vec{\tau})] \tag{16}$$

where $\vec{\sigma} = (\sigma, \ldots, \sigma)^T \in \mathfrak{su}(2)^N$ is a "constant" $\mathfrak{su}(2)$-valued vector and $\vec{\tau} = (\tau_1, \ldots, \tau_N)^T \in \mathfrak{su}(2)^N$ is arbitrary. Here the derivation ∂_x is replaced by the operator

$$D_l = (R - R^{-1}) \colon \mathfrak{su}(2)^N \longrightarrow \mathfrak{su}(2)^N.$$

The equation (16) is the system of N ordinary $\mathfrak{su}(2)$-valued differential equations of the form

$$\left((g_i)_t(g_i)^{-1}\right)_t = (g_{[i+1]})_t(g_{[i+1]})^{-1} - (g_{[i-1]})_t(g_{[i-1]})^{-1} + [\sigma, \mathrm{Ad}_{g_i}(\tau_i)], \quad i = 1, \ldots, N \tag{17}$$

where $[i \pm 1] = (i \pm 1) \bmod N$. This is a Neumann lattice in which the acceleration of the i-th oscillator is influenced by the force $[\sigma, \mathrm{Ad}_{g_i}(\tau_i)]$ and by the velocities of the two neighbouring particles.

Other, more faithful discretizations are possible. Note that the derivation ∂_x is the infinitesimal rotation of the loops from $L\mathfrak{su}(2)$. This is equivalent to the equation $\exp(s \cdot \partial_x)(\alpha(x)) = R_s(\alpha(x)) = \alpha(x + s)$. Thus we can write $\partial_x = \frac{1}{s} \log(R_s)$, and with this in mind we define

$$D_g = \frac{N}{2\pi} \log R$$

where R is given by (15). We note that the operators D_l and D_g both have 2×2-block structure. The blocks are scalar 2×2 matrices. Thus

$$D_l = \widetilde{D}_l \otimes Id_2, \quad D_g = \widetilde{D}_g \otimes Id_2; \quad Id_2 = \mathrm{diag}(1, 1)$$

where $\widetilde{D}_l$ and $\widetilde{D}_g$ are real scallar $N \times N$ matrices.

In the following proposition we collect some properties of the matrix $\widetilde{D}_g$. By $\widetilde{R}$ we shall denote the $N \times N$ cyclic permutation matrix such that $R = \widetilde{R} \otimes Id_2$. We shall assume that $N = 2m + 1$ is an odd number and we shall index the matrix $\widetilde{D}_g$ somewhat more symmetrically: $\widetilde{D}_g = (d_{i,k})_{i,k=-m,\ldots,0,\ldots,m}$.

Proposition 1 *(i) The elements of the matrix $\widetilde{D}_g = (d_{i,k})_{i,k=-m,\ldots,m}$ are given by*

$$d_{k,l} = \sum_{n=-m}^{m} n \sin\left(2\pi \frac{n(k+l)}{N}\right). \tag{18}$$

(ii) Let $x \in \mathbb{R}$ and $\mathcal{R}(x) = \exp(x \cdot \widetilde{D}_g)$. Then for every integer k we have

$$\mathcal{R}(k \frac{2\pi}{N}) = \widetilde{R}^k.$$

(iii) For every k we have

$$\sum_{l=-m}^{m} d_{k,l} = 0. \tag{19}$$

Thus 0 *is an eigenvalue of* $\widetilde{D}_g$ *and* $w = (1, \ldots, 1)^T$ *is the corresponding eigenvector.*

Proof: *(i)* We have to compute the logarithm of $\widetilde{R}$. This matrix is unitary and therefore diagonalizable. We shall first diagonalize $\widetilde{R}$, find the logarithm of the diagonalization and then express the result in the original basis. The eigenvalues of the cyclic permutation $\widetilde{R}$ are clearly the N-th roots of unity. More concretely, we have $\widetilde{R} = Q^{-1} \cdot R_f \cdot Q$, where

$$R_f = \mathrm{diag}(e^{\frac{-2\pi i m}{N}}, e^{\frac{-2\pi i(m-1)}{N}}, \ldots, \ldots, e^{\frac{2\pi i(m-1)}{N}}, e^{\frac{2\pi i m}{N}})$$

and $Q = (e^{2\pi i \frac{j \cdot k}{N}})_{j,k=-m,\ldots,m}$. We note that the unitary matrix Q is the matrix of the discrete Fourier transform [1]. The logarithm of R_f is clearly the diagonal matrix

$$D_f = \mathrm{diag}(-2\pi i \frac{m}{N}, -2\pi i \frac{m-1}{N}, \ldots, 0, \ldots 2\pi i \frac{m-1}{N}, 2\pi i \frac{m}{N}).$$

Then a straightforward calculation shows that the (k, l)-th element $d_{k,l}$ of $\widetilde{D}_g = Q \cdot D_f \cdot Q^{-1}$ is given by the formula

$$d_{k,l} = i \sum_{n=-m}^{m} n e^{2\pi i \frac{n(k+l)}{N}} = \sum_{n=-m}^{m} n \sin 2\pi \frac{n(k+l)}{N} + i \sum_{n=-m}^{m} n \cos 2\pi \frac{n(k+l)}{N}.$$

A moment of inspection reveals that, due to the evenness of the cosine, the imaginary part of the above expression is equal to zero. Thus, we indeed have

$$d_{k,l} = \sum_{n=-m}^{m} n \sin 2\pi \frac{n(k+l)}{N}.$$

(ii) The equation $\mathcal{R}(\frac{2k\pi}{N}) = \mathrm{Exp}(\frac{2k\pi}{N} \cdot \widetilde{D}_g) = \widetilde{R}^k$ is an obvious consequence of the definition $D_g = \frac{N}{2\pi} \log \widetilde{R}$ of the operator D_g.

(iii) It is easily seen that

$$\sum_{l=-m}^{m} d_{k,l} = \sum_{l=-m}^{m} d_{k,0}.$$

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[1]Therein lies the reason for N being odd and for our indexation of $\widetilde{D}_g$. The finite *real* Fourier series are of the form $f(x) = \sum_{n=-m}^{m} a_n e^{inx}$ and $a_{-n} = -a_n^*$.

Indeed, the $k + 1$-st row of $\widetilde{D}_g$ is obtained from the k-th by the cyclic one step permutation. Now the oddness of the sine function and the formula (18) give (19), which concludes the proof of our proposition.

$\square$

In a sense, the operator D_g is the best approximation of ∂_x on the N-sided polygon $\mathbb{Z}_N \subset S^1$. The l-th row of the matrix $\widetilde{D}_g$ can be interpreted as the function $f_l(k) : \mathbb{Z}_N \longrightarrow \mathbb{R}$ given by the formula

$$f_l(k) = \sum_{n=-m}^{m} n \sin \left(\frac{2\pi(k+l)}{N} \, n \right).$$

Recall that the δ function on the circle S^1 is given by $\delta_a(x) = \sum_{n=-\infty}^{\infty} e^{in(x-a)}$. Its derivative is therefore

$$\delta'_a(x) = \sum_{n=-\infty}^{\infty} in e^{in(x-a)} = \sum_{n=-\infty}^{\infty} n \sin n(x - a).$$

We can obtain $f_l(k)$ from the derivative of $\delta_{\frac{2\pi l}{N}}(x)$ by truncating the sum and by evaluating only at the points $(2\pi k)/N$. (The change of sign from $+$ in $f_l(k) = f_0(l+k)$ to to $-$ in $\delta_a(x) = \delta_0(x - a)$ comes from the fact that in a matrix the "vertical coordinate" is measured in the reverse order.)

The above observations show that the Neumann lattice

$$(q_t q^{-1})_t + D_g(q_t q^{-1}) = [\vec{\sigma}, \mathrm{Ad}_q(\vec{\tau})] \tag{20}$$

is the most faithful discretization of the Maxwell-Bloch equation. More explicitely, the above system can be written in the form

$$\left((g_i)_t (g_i)^{-1} \right)_t = - \sum_{k=-m}^{m} d_{i,k}(g_k)_t(g_k)^{-1} + [\sigma, \mathrm{Ad}_{g_i}(\tau_i)], \quad i = -m, \ldots, m \tag{21}$$

where $d_{i,k} = \sum_{n=-m}^{m} n \sin \left(2\pi \frac{n(i+k)}{N} \right)$. In this Neumann lattice the i-th oscillator is influenced by the velocities of all the other oscillators. The influence of the closest pair on the positions $i - 1$ and $i + 1$ is the strongest, the pair one step further away influences roughly half as much, and so on in an approximatelly harmonic succession.

The systems (17) and (21) are only the most natural examples of the Neumann lattices. In section 5 we shall construct integrals of motion for every Neumann lattice

$$\left((g_i)_t (g_i)^{-1} \right)_t = \sum_{k=-m}^{m} \psi_{i,k}(g_k)_t(g_k)^{-1} + [\sigma, \mathrm{Ad}_{g_i}(\tau_i)], \quad i = -m, \ldots, m$$

for which the $N \times N$-matrix $\widetilde{D} = (\psi_{i,k})_{i,k=-m,\dots,m}$ is an element of the special orthogonal Lie algebra $\mathfrak{so}(N)$ and has the vector $w = (1, \dots, 1)^T$ in its kernel. Such general Neumann lattices cannot be considered as sensible discretizations of the Maxwell-Bloch equations. Nevertheless, our construction of the integrals of motion for the general Neumann lattice uses a generalization of the Maxwell-Bloch equations in an essential way.

Remark 1 *Let all the oscillators in a Neumann lattice of N oscillators be of the same type so that the magnetic interaction between any given pair of oscillators depends only on their mutual distance within the lattice. Then the corresponding matrix $\widetilde{D}$ is of the form*

$$\widetilde{D}_t = \sum_{n=0}^{m} a_n (\widetilde{R}^n - \widetilde{R}^{-n}), \quad \widetilde{R} \; - \; \text{the cyclic permutation}$$

where a_n are arbitrary real numbers. Above, $m = \frac{N-2}{2}$, if N is even and $m = \frac{N-1}{2}$, if N is odd. Such matrices are elements of $\mathfrak{so}(N)$ and their kernels contain the vector $w = (1, \dots, 1)^T$. Matrices $\widetilde{D}_t$ are the Toeplitz matrices whose entries are given by

$$\psi_{i,j} = \text{sign}(j - i) \cdot \begin{cases} 0 & j - i = 0 \\ a_{(j-i)} & j - i < N/2 \\ -a_{(N-(j-i))} & j - i > N/2 \\ 0 & j - i = N/2 \end{cases}.$$

The matrices $\widetilde{D}_l$ and $\widetilde{D}_g$ are of this type. In the case of $\widetilde{D}_l$ the only non-zero a_j is $a_1 = 1$. In the case of $\widetilde{D}_g$ all a_j are non-zero, and

$$a_j = \sum_{n=-m}^{m} n \sin \left(2\pi \frac{n \cdot j}{N} \right), \quad j = 0, \dots, m.$$

We conclude this section with a short discussion about the Hamiltonian nature of the Neumann lattices. Recall the theorem cited at the end of the previous section which states that the Maxwell-Bloch equations are Hamiltonian and that their Hamiltonian structure is $(T^*LSU(2), \omega_c + \omega_m, H_{mb})$. The Hamiltonian function H_{mb} is given by the formula (13) and the canonical symplectic structure is perturbed by the right-invariant magnetic term ω_m given by (14). Since certain Neumann lattices arise as discretizations of the Maxwell-Bloch equations, the sensible candidates for their Hamiltonian structures should be the appropriate discretizations of the Hamiltonian system $(T^*LSU(2), \omega_c + \omega_m, H_{mb})$.

Let

$$(q_t q^{-1})_t + D(q_t q^{-1}) = [\vec{\sigma}, \text{Ad}_q(\vec{\tau})], \quad D = \widetilde{D} \oplus Id_2 \in \mathfrak{so}(2N) \tag{22}$$

be an arbitrary Neumann lattice. Its phase space is the cotangent bundle $T^*SU(2)^N$. Let us choose the Hamiltonian $H_{nl}\colon T^*SU(2)^N \to \mathbb{R}$ to be the total energy of all the oscillators in the lattice

$$H_{nl}(q, p_q) = \frac{1}{2} \sum_{i=-m}^{m} \|(p_q)_i\|^2 + \sum_{i=-m}^{m} \langle \sigma, \mathrm{Ad}_{q_i}(\tau_i) \rangle. \qquad (23)$$

The magnetic term responsible for the interaction among the oscillators will be the two form ω_{dm} on $T^*SU(2)^N$ given as follows. Let $\widetilde{\omega}_{dm}$ be the right-invariant two-form on $SU(2)^N$ whose value at the identity is given by the formula

$$\widetilde{\omega}_{dm}(\vec{\xi}, \vec{\eta}) = \sum_{i=1}^{N} \langle (D\vec{\xi})_i, \eta_i \rangle, \quad \vec{\xi}, \vec{\eta} \in \mathfrak{su}(2)^N. \qquad (24)$$

Then ω_{dm} is the pull-back $\omega_{dm} = \pi^*(\widetilde{\omega}_{dm})$, where $\pi\colon T^*SU(2)^N \to SU(2)^N$ is the natural projection. Since $D \in \mathfrak{so}(2N)$, the tensors $\widetilde{\omega}_{dm}$ and ω_{dm} are indeed antisymmetric and are therefore well-defined differential two-forms. But the form ω_{dm} is not closed for any sensible choice of $D \in \mathfrak{so}(2N)$. For example, in the case $D = D_l$ we get

$$\mathrm{d}\widetilde{\omega}_{dm}(\vec{X}, \vec{Y}, \vec{Z}) = \sum_{i=1}^{N} \langle (X_i - X_{i+1}), [(Y_i - Y_{i+i}), (Z_i - Z_{i+1})] \rangle$$

where $\vec{X}, \vec{Y}, \vec{Z} \in \Gamma(SU(2)^N)$ are given in the right trivializaton. Thus $\omega_c + \omega_{dm}$ is not a symplectic structure. The form $\omega_c + \omega_{dm}$ equips the space $T^*SU(2)^N$ with a so-called twisted Poisson structure. Let M be a smooth manifold and let $C^\infty(M)$ be the space of smooth functions of M. Then a bracket $\{-, -\}$ on $C^\infty(M)$ is a twisted Poisson structure, if it is anti-commutative and if

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = \Omega(X_f, X_g, X_h)$$

where Ω is a closed three-form on M. Here X_f, X_g, X_h are the (twisted) Hamiltonian vector fields of f, g, h, respectively. The form Ω is called the background form. In our case we have

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = \mathrm{d}\omega_{dm}(X_f, X_g, X_h).$$

For more information on the twisted Poisson structures, see [6], [7], [8] and references therein.

Now we will show that $(T^*SU(2)^N, \omega_c + \omega_{dm}, H_{nl})$ can be thought of as a *twisted* Hamiltonian structure of the Neumann lattice (22).

Proposition 2 *Let the vector fileld $X_{H_{cn}}$ on $T^*SU(2)^N$ be given by the condition*

$$\mathrm{d}H_{nl} = i(X_{H_{nl}})(\omega_c + \omega_{dm})$$

and let $\gamma(t) = (q(t), p_g(t)): I \to T^*LSU(2)$ be an integral curve of the field $X_{H_{cn}}$. Then the curve $q(t)$ is a solution of the Neumann lattice (22) and $(p_g)_i(t) = \langle (q_t q^{-1})_i(t), - \rangle$ for every $i = 1, \dots, N$.

Proof: Let $\vec{\xi}, \vec{\eta} \in \mathfrak{su}(2)^N$ be arbitrary. The formula

$$\langle\!\langle \vec{\xi}, \vec{\eta} \rangle\!\rangle = \sum_{i=-m}^{m} \langle \xi_i, \eta_i \rangle$$

defines a natural Ad-invariant inner product on $\mathfrak{su}(2)^N$. By the same symbol we shall denote the induced inner product on the dual space $(\mathfrak{su}(2)^N)^*$ and also the evaluation $\langle\!\langle \vec{\alpha}, \vec{\xi} \rangle\!\rangle$ of an element $\vec{\alpha} \in (\mathfrak{su}(2)^N)^*$ at $\vec{\xi} \in \mathfrak{su}(2)^N$, and this should cause no confusion. Thus, we can rewrite the Hamiltonian (23) in the form

$$H_{nl}(q, p_q) = \frac{1}{2}\|p_q\|^2 + \langle\!\langle \vec{\sigma}, \mathrm{Ad}_q(\vec{\tau}) \rangle\!\rangle$$

where $\|p_q\|^2 = \langle\!\langle p_q, p_q \rangle\!\rangle$. The canonical cotangent form on $T^*SU(2)^N$ is given by

$$(\omega_c)_{(q,p_q)}\Big((X_b, X_{ct}), (Y_b, Y_{cn}) \Big) = -\langle\!\langle X_{ct}, Y_b \rangle\!\rangle + \langle\!\langle Y_{ct}, X_b \rangle\!\rangle + \langle\!\langle p_g, [X_b, Y_b] \rangle\!\rangle$$

where the tangent vectors $(X_b, X_{ct}), (Y_b, Y_{ct}) \in T_{(q,p_q)}(T^*SU(2)^N)$ are expressed in the right trivialization. For the proof see [15]. Thus for the form $\omega_c + \omega_{dm}$ we have

$$
\begin{aligned}
(\omega_c + \omega_m)_{(g,p_g)}\Big((X_b, X_{ct}), (Y_b, Y_{ct}) \Big) &= -\langle\!\langle X_{ct}, Y_b \rangle\!\rangle + \langle\!\langle Y_{ct}, X_b \rangle\!\rangle \\
&\quad + \langle\!\langle p_g, [X_b, Y_b] \rangle\!\rangle - \langle\!\langle D(X_b), Y_b \rangle\!\rangle.
\end{aligned}
$$

Derivation of the Hamiltonian in the direction (δ_b, δ_{ct}) gives

$$\langle\!\langle \mathrm{d}H_{nl}, (\delta_b, \delta_{ct}) \rangle\!\rangle = -\langle\!\langle [\vec{\sigma}, \mathrm{Ad}_q(\vec{\tau})]^{at}, \delta_b \rangle\!\rangle + \langle\!\langle \delta_{ct}, p_q^\sharp \rangle\!\rangle.$$

Putting $(Y_b, Y_{ct}) = (\delta_b, \delta_{ct})$ and comparing the above two equations gives the following expression for the Hamiltonian field $X_{H_{nl}} = (X_b, X_{ct})$ given in the right trivialization:

$$X_b = p_q^\sharp, \qquad X_{ct} + D(X_b)^{at} = [\vec{\sigma}, \mathrm{Ad}_q(\vec{\tau})]^{at}. \tag{25}$$

Let $\vec{\gamma}(t) = (q(t), p_q(t)): I \to T^*SU(2)^N$ be an integral curve of the field $X_{H_{nl}} = (X_b, X_{ct})$. Then $(q_t q^{-1}, (p_q)_t) = (X_b, X_{ct})$. If we insert this into (25), we finally see that the curve $q(t)$ is a solution of the equation (22), which concludes the proof.

$\square$

Next two sections are devoted to the construction of the integrals of motion of the Neumann lattice (22) or, equivalently, of the dynamical system with the *twisted* Hamiltonian structure $(T^*SU(2)^N, \omega_c + \omega_{dm}, H_{nl})$.

4 Reduced curvature condition

Above we have seen that the Neumann lattices and the Maxwell-Bloch equations are closely related. Maxwell-Bloch equations are integable, they satisfy the zero-curvature condition. The zero-curvature condition does not survive our discretization in any obvious way. It does not induce, for instance, a Lax equation of some sort for the Neumann lattices. It seems that the Neumann lattices do not have Lax equations at all, which might be a consequence of the twistedness of their Poisson structures. Therefore a new method is needed for the construction of the sought-for integrals. We shall call this method the reduced curvature condition. This name should reflect the relation of our method to the zero-curvature condition.

The geometric foundation of the zero-curvature condition is the well-known fact that the holonomies around any two homotopic curves are conjugate. The setting usually encountered in the theory of integrable systems is the following. Let $I \subset \mathbb{R}$ be an interval and let $E \to I \times S^1$ be a trivial $\mathbb{C}^n$-bundle over the cylinder $I \times S^1 = \{(t, x)\}$. Denote by A a $GL(n, \mathbb{C})$ connection $A(t, x) = U(t, x) \, \mathrm{d}t + V(t, x) \, \mathrm{d}x$ on E. Suppose that this connection is flat,

$$F_A = V_t - U_x + [U, V].$$

Denote by $M(\gamma) \in GL(n, \mathbb{C})$ the holonomy of A around a closed loop $\gamma \subset I \times S^1$, and in particular by $M(t)$ the holonomy of A around the loop $\beta(s) = (t, s)$. Let now the closed curve $\kappa(s)$ be the rectangle given by

$$\kappa(s) = \begin{cases} \kappa_1(s) = (s, 0) & s \in [0, t] \\ \kappa_2(s) = (t, s - t) & s \in [t, t + 2\pi] \\ \kappa_3(s) = (2t + 2\pi - s, 2\pi) & s \in [t + 2\pi, 2t + 2\pi] \\ \kappa_4(s) = (0, 2t + 4\pi - s) & s \in [2t + 2\pi, 2t + 4\pi] \end{cases}. \tag{26}$$

Then we have $M(\gamma) = M^{-1}(0) \cdot N^{-1} \cdot M(t) \cdot N$. Here $N \in GL(n, \mathbb{C}) = N(t_1)$, and $N(s)$ is a lifting of $\kappa_2(s)$ onto E. The lifting $N(s)$ is horizontal with respect to A. Since the connection A is flat, we have $M(\gamma) = Id$ and therefore

$$M(t) = N \cdot M(0) \cdot N^{-1}.$$

The *spectrum* of the monodromy $M(t)$ around the curve $\gamma(s) = (s, t)$ is independent of $t \in I$. This fact enables us to construct the integrals of motion for the systems which satisfy the zero-curvature condition.

We shall now consider a *unitary* connection A on the $\mathbb{C}^n$-bundle $E \to I \times S^1$ which satisfies a condition weaker than the flatness. The following theorem will be the geometric foundation of our construction of the integrals of the Neumann lattices.

Theorem 1 *Let the bundle $E \to I \times S^1$ be endowed with a Hermitian metric and let A be a unitary connection on E. Let $w_i \in \Omega^0(E)$, $i = 1, \ldots, k$ be an arbitrary orthonormal system of smooth sections of the bundle E. Choose a trivialization of E. Suppose that in this (and hence in any) trivialization we have*

$$(F_A)_m(\xi_m, \eta_m) \cdot w_i(m) = 0 \tag{27}$$

for every $m \in I \times S^1$ and for every pair of tangent vectors $\xi_m, \eta_m \in T_m(I \times S^1)$. Let the $k \times n$-matrix valued function $W(m)$ on $I \times S^1$ be given by

$$W(m) = (w_1(m), \ldots, w_k(m))$$

where w_i are the column-vectors which correspond to our sections in the chosen gauge. Let $M(t)$ be the holonomy matrix of the connection A along the path $\beta(s) = (t, s)$. Then the eigenvalues of the $k \times k$ unitary matrix

$$M_k(t) = W^* \cdot M(t) \cdot W$$

are independent of t.

Proof: First we observe that there exists a gauge in which A is a skew-hermitian matrix and

$$w_i(m) = (a_{i,1}(m), \ldots, a_{i,k}(m), \ldots, 0)^T, \qquad i = 1, \ldots k.$$

In this gauge every matrix $(F_A)_m(\xi_m, \eta_m)$ is also skew-hermitian and it is of the form

$$(F_A)_m(\xi_m, \eta_m) = \begin{pmatrix} 0 & 0 \\ 0 & F_{n-k} \end{pmatrix}$$

where F_{n-k} is an element of the Lie algebra $\mathfrak{su}(n-k)$. The condition (27), together with the unitarity of A, implies that the curvarture takes values in a proper subalgebra $\mathfrak{su}(n-k)$ of the structure algebra $\mathfrak{su}(n)$.

Let $\Phi_0(A)$ be the restricted holonomy group of the connection A. (The restricted holonomy group is generated by the holonomies along all null-homotopic paths in $I \times S^1$.) The holonomy reduction theorem says that the Lie algebra $\mathfrak{h}$ of $\Phi_0(A)$ is equal to the subalgebra of $\mathfrak{su}(n)$ generated by all the values $(F_A)_m(\xi_m, \eta_m)$ of the curvature. For the proof see [16] or [17]. In our case the Lie subalgebra $\mathfrak{h}$ is the a copy of $\mathfrak{su}(n-k)$ lying in $\mathfrak{su}(2)$. In our gauge the holonomy $M(\gamma)$ of the connection A along any null-homotopic loop γ is an element of $SU(n)$ of the form

$$M(\gamma) = \begin{pmatrix} Id_k & 0 \\ 0 & M_{n-k} \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix and M_{n-k} is an element of $SU(n-k)$.

The above form of the holonomy group implies that, in our gauge, the connection A is of the form

$$A(x,t) = \begin{pmatrix} \alpha_1(t,x)\mathrm{d}t + \alpha_2(t,x)\mathrm{d}x & 0 \\ 0 & \beta_1(t,x)\mathrm{d}t + \beta_2(t,x)\mathrm{d}x \end{pmatrix} \qquad (28)$$

where $\alpha_i(t,x) \in \mathfrak{su}(k)$ and $\beta_i(t,x) \in \mathfrak{su}(n-k)$. This is the contents of the reduction theorem which says the following: If A is a connection on a bundle with the structure group G, and if the restricted holonomy group of A is a proper subgroup $H \subset G$, then the connection A is reducible to a connection with the structure group H. For the precise formulation and for the proof see again [16] or [17].

Let $F \to I \times S^1$ be the subbundle of E spanned by the sections w_i, $i = 1 \ldots k$, and let $F^\perp$ be its orthogonal complement with respect to our Hermitian metric. Then we have $F \oplus F^\perp = E$. Formula (28) shows that this decomposition of E is actually a geometric one in the sense that the connection A has a decomposition $A = A_k \oplus B_{n-k}$ into a connection A_k on F and the connection B_{n-k} on $F^\perp$.

Since in our case the upper blocks of all the elements of $\Phi_0(A)$ are equal to I_k, the connection
$$A_k = \alpha_1(t,x)\mathrm{d}t + \alpha_2(t,x)\mathrm{d}x$$
on the rank-k subbundle $F \subset E$ is *flat*. It is also clear that

$$A_k = W^* \cdot A \cdot W$$

where $W = (w_1, \ldots, w_k)$. Note that this expression is independent of the choice of gauge.

Let now the path $\kappa(s)$ again be given by the formula (26). Again we will denote the horizontal lift of $\kappa_1(s) = (s,0)$ by N, and the holonomies around the curves $\kappa_2(s) = (t,s)$ and $\kappa_4(s) = (0,s)$ by $M(t)$ and $M(0)$, respectively. As before, we have $M(\kappa) = M^{-1}(0) \cdot N^{-1} \cdot M(t) \cdot N$, but this time, in the gauge used above, we also have

$$M(\tau) = \begin{pmatrix} M_k(\tau) & 0 \\ 0 & M_{n-k}(\tau) \end{pmatrix}, \quad \tau = 0, t \quad \text{and} \quad N = \begin{pmatrix} N_k & 0 \\ 0 & N_{n-k} \end{pmatrix}$$

due to the fact that $A = A_k \oplus B_{n-k}$ is reducible, and

$$M(\kappa) = \begin{pmatrix} Id_k & 0 \\ 0 & M_{n-k} \end{pmatrix}$$

due to the fact that A_k is flat. From this we get

$$M_k(t) = N_k \cdot M_k(0) \cdot N_k^{-1}. \tag{29}$$

This means that the eigenvalues of the matrix $M_k(t)$ are independent of t.

The matrix $M_k(t)$ is the holonomy of the connection A_k on the subbundle $F \subset E$ around the curve $\gamma_t(x) = (t, x)$. Since $A_k(\dot{\gamma}_t(x)) = A_k(\frac{\partial}{\partial x}) = \alpha_2$, the matrix $M_k(t)$ is equal to $H_k(t, 2\pi)$, and $H_k(t, x)$ is the solution of the initial value problem

$$\frac{\partial}{\partial x} H_k(t, x) = A_k(t, x)(\frac{\partial}{\partial x}) \cdot H_k(t, x), \qquad H_k(t, 0) = Id_k.$$

The bundle E is trivial, therefore there exists a gauge in which the sections

$$w_i(m) = (a_{i,1}(m), \dots, a_{i,k}(m), 0, \dots, 0)$$

are constant with respect to the x-variable. If $H\colon I \times S^1 \to SU(n)$ is the solution of the initial problem

$$\frac{\partial}{\partial x} H(t, x) = A(t, x)(\frac{\partial}{\partial x}) \cdot H(t, x), \qquad H(t, 0) = Id$$

then in this gauge we have

$$\frac{\partial}{\partial x}(W^* \cdot H \cdot W) = W^* \cdot \frac{\partial}{\partial x} H \cdot W = W^* \cdot A(\frac{\partial}{\partial x}) \cdot H \cdot W.$$

Since the sections w_i are an orthonormal system, we have

$$W \cdot W^* = \begin{pmatrix} Id_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}.$$

Recall also that in our gauge the block structure of the connection A is given by the formula (28). Thus

$$\frac{\partial}{\partial x}(W^* \cdot H \cdot W) = (W^* \cdot A(\frac{\partial}{\partial x}) \cdot W) \cdot (W^* \cdot H \cdot W).$$

Since $W^* \cdot A \cdot W = A_k$, this means that $W^* \cdot H(t, x) \cdot H$ is the lift of the loop $\beta(x) = (t, x)$, horizontal with respect to the connection A_k on the subbundle $F \subset E$. Since also $H(t, 0) = Id$, we have

$$W^* \cdot M(t) \cdot W = M_k(t).$$

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We have proved the above formula in a gauge in which the sections $w_i(m)$ are independent on x, but the formula is obviously gauge independent and therefore valid in any gauge. Since the eigenvalues of $M_k(t)$ are independent of t, the above formula completes the proof of our theorem.

$\square$

Note that the statement of our theorem is independent of the choice of gauge and could be therefore easily expressed in a coordinate-free way. But in practice the condition (27) will usually be given in coordinates.

5   Conserved quantities of Neumann lattices

Consider now the equation for a lattice of N Neumann oscillators:

$$(q_t q^{-1})_t + D(q_t q^{-1}) = [\vec{\sigma}, \mathrm{Ad}_q(\vec{\tau})] \tag{30}$$

which satisfies the conditions

$$(a)\ \ D = \widetilde{D} \otimes Id_2, \quad \widetilde{D} \in \mathfrak{so}(N), \qquad (b)\ \ \widetilde{D} \cdot w = 0,\ \ for\ \ w = (1, \ldots, 1)^T. \tag{31}$$

In this section we shall construct a family of integrals for this system. The key element of our construction will be the reduced curvature condition described in theorem 1.

We shall first rewrite the system (30) in a suitable form. Let us replace the vector q by the block-diagonal element G in $SU(2N)$:

$$q = (g_1, \ldots, g_N)^T \quad \leadsto \quad G = \begin{pmatrix} g_1 & 0 & \ldots & 0 \\ 0 & g_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g_N \end{pmatrix}. \tag{32}$$

Entries g_i of the above block-matrix are elements of $SU(2)$. Similarly, we shall replace the $\mathfrak{su}(2)$-valued vectors $\vec{\sigma}$ and $\vec{\tau}$ by the block-diagonal matrices

$$\Sigma = \begin{pmatrix} \sigma & 0 & \ldots & 0 \\ 0 & \sigma & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma \end{pmatrix}, \quad T = \begin{pmatrix} \tau_1 & 0 & \ldots & 0 \\ 0 & \tau_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tau_N \end{pmatrix}.$$

Note that $\Sigma, T \in \mathfrak{su}(2N)$. Let $W = (Id_2, \ldots, Id_2)^T$ be the $2 \times 2N$ block matrix whose blocks are 2×2 identity matrices Id_2.

Then the equation (30) is equivalent to the equation

$$\left((G_tG^{-1})_t + [D,(G_tG^{-1})] + [\mathrm{Ad}_G(T),\Sigma]\right) \cdot W = 0. \tag{33}$$

To see this, one only has to observe that $D \cdot W = 0$. This follows from part (b) of the condition (31). We have

$$[D, G_tG^{-1}] \cdot W = D \cdot (G_tG^{-1} \cdot W) - G_tG^{-1} \cdot (D \cdot W) = D(q_tq^{-1}).$$

Let now the loop $\mathcal{R}(x) \colon S^1 \to SU(2N)$ be given by

$$\mathcal{R}(x) = \mathrm{Exp}(x \cdot D)$$

and let the path $Q \colon \mathbb{R} \to LSU(2N)$ in the loop group $LSU(2N)$ be defined by the formula

$$Q(t,x) = \mathrm{Ad}_{\mathcal{R}(x)}(G(t)) = \mathcal{R}(x) \cdot G(t) \cdot \mathcal{R}^{-1}(x).$$

Proposition 3 *The equation (30) for the Neumann lattice of N oscillators is equivalent to the system of partial differential equations*

$$\left((Q_tQ^{-1})_t + (Q_tQ^{-1})_x + [\mathrm{Ad}_Q(\hat{T}),\Sigma]\right) \cdot W = 0 \tag{34}$$

where Q is the path in $LSU(2N)$ defined above, and $\hat{T} = \mathrm{Ad}_{\mathcal{R}(x)}(T)$.

Proof: First we observe

$$\begin{aligned}(Q_tQ^{-1})_x &= (\mathcal{R}(x)G_tG^{-1}\mathcal{R}(x)^{-1})_x = [\mathcal{R}(x)_x\mathcal{R}(x)^{-1}, \mathcal{R}(x)G_tG^{-1}\mathcal{R}(x)^{-1}]\\ &= \mathcal{R}(x) \cdot [D, G_tG^{-1}] \cdot \mathcal{R}(x)^{-1}.\end{aligned}$$

The $SU(2N)$ matrix $\mathcal{R}(x)$ is of the form $\mathcal{R}(x) = \widetilde{\mathcal{R}}(x) \otimes Id_2$. This means that it is composed of 2×2-blocks and that these blocks are scalar matrices. Therefore the blocks commute with all 2×2-matrices. On the other hand, $\Sigma = \mathrm{diag}(\sigma, \ldots, \sigma)$ has a block structure of a scalar matrix. Thus we have

$$\mathcal{R}(x) \cdot \Sigma \cdot \mathcal{R}(x)^{-1} = \Sigma.$$

Finally, from $D \cdot W = 0$ it follows that $\mathcal{R}(x) \cdot W = \mathcal{R}^{-1}(x) \cdot W = W$. It is now clear that the equation (34) is equivalent to the equation (33), which in turn is equivalent to (30).

$$\square$$

We note that even though the equation (34) is reminiscent of the Maxwell-Bloch equation, there are two important differences between the two. In a more compact way, the equation (34) can be written as $\mathcal{M}(Q) \cdot W = 0$, where $\mathcal{M}(Q) = 0$ is the "generalized" Maxwell-Bloch equation (10). The first difference lies in the fact that in $\mathcal{M}(Q)$ the path $g \colon I \to LSU(2)$ is replaced by the path $Q \colon I \to LSU(2N)$ in the larger group $LSU(2N)$. The second difference is the appearance of the matrix W at the end of the right-hand part of (34). However, the similarity between the Maxwell-Bloch equation and the equation (34) will enable us to find the reduced curvature condition for the Neumann lattice (30).

It has been known for some time that the Maxwell-Bloch equations are integrable (See [18], [19], [20], [21]). In terms of our rewriting (10), the zero-curvature condition has the following form. Let

$$\mathcal{A}(z) = \mathcal{U}(t, x; z) \, \mathrm{d}t + \mathcal{V}(t, x; z) \, \mathrm{d}x$$

be the family (parametrized by $z \in \mathbb{R}$) of unitary connections on the trivial $\mathbb{C}^2$-bundle $E \to I \times S^1$, where

$$\begin{aligned}
\mathcal{U}(t, x; z) &= -(-z\sigma + g_t g^{-1}) \\
\mathcal{V}(t, x; z) &= -z\sigma + g_t g^{-1} - \frac{1}{z}\mathrm{Ad}_g(\tau).
\end{aligned}$$

The equation (10) is equivalent to the flatness condition

$$F_{\mathcal{A}(z)} = \mathcal{V}(z)_t - \mathcal{U}(z)_x + [\mathcal{U}(z), \mathcal{V}(z)] = 0$$

for the curvature $F_{\mathcal{A}(z)}$ of the connection $\mathcal{A}(z)$ for every $z \in \mathbb{R}$. The proof of this claim is a matter of a straightforward check. We shall use the form of this Lax pair and the rewriting (34) to show that the Neumann lattice (30) satisfies the reduced curvature condition.

For the sake of simplicity and clarity, we shall express the conserved quantities as functions defined on the tangent bundle $TSU(2)^N$ and not on the cotangent bundle $T^*SU(2)$. Since we have the right-invariant metric $\langle\!\langle X_g, Y_g \rangle\!\rangle_g = \langle\!\langle X_g g^{-1}, Y_g g^{-1} \rangle\!\rangle$ on $SU(2)^N$, all our claims involving $TSU(2)$ can easily be translated into claims involving the cotangent bundle $T^*SU(2)^N$.

The space $SU(2)^N$ is a group, therefore we can trivialize the tangent bundle $TSU(2)^N$ by the right translation to obtain the identification $TSU(2)^N \cong SU(2)^N \times \mathfrak{su}(2)^N$. As before, we assign to every element $q \in SU(2)^N$ a loop $Q(x) \in LSU(2N)$ given by $Q(x) = \mathrm{Ad}_{\mathcal{R}(x)}(G)$, where $G = \mathrm{diag}(g_1, \ldots, g_N)$ is defined by (32). Let

$t_q \in T_q L_N SU(2)$ be a tangent vector. Then $t_q q^{-1} \in \mathfrak{su}(2)^N$. Taking the derivative of the map $q \mapsto Q(x)$ at q and performing the right translation yields the map

$$t_q q^{-1} \mapsto T_Q Q^{-1} = \mathrm{Ad}_{\mathcal{R}(x)}(\mathrm{diag}(t_1 g_1^{-1}, \ldots, t_N g_N^{-1})) \in L\mathfrak{su}(2N).$$

Let us now denote by

$$\widetilde{L}\mathfrak{su}(2N) = \{zA + B + \frac{1}{z}C; \quad A, B, C \in L\mathfrak{su}(2N)\}$$

the space of "short" Laurent series in the indeterminate $z \in \mathbb{R}$ with the coefficients in the loop algebra $L\mathfrak{su}(2N)$. Define the maps

$$\Phi, \Theta \colon TSU(2)^N \longrightarrow \widetilde{L}\mathfrak{su}(2N)$$

by the formulae

$$\Phi_{(q,t_q)}(x) = -z\Sigma + T_Q Q^{-1} - \frac{1}{z}\mathrm{Ad}_Q(\hat{T}), \qquad \Theta_{(q,t_q)}(x) = -(-z\Sigma + T_Q Q^{-1})$$

where $\hat{T}$ is defined in proposition 3. Let now $\beta(t) = (q(t), t_q(t))$ be an arbitrary path in $TSU(2)^N$ and let

$$\widetilde{\beta} \colon t \longmapsto \beta(t) = (q(t), t_q(t)) \longmapsto \widetilde{\beta}(t)(x) = (\Phi_{\beta(t)}(x), \Theta_{\beta(t)}(x))$$

be the corresponding path in $(\widetilde{L}\mathfrak{su}(2N))^2$. To the path $\beta \colon I \to TSU(2)^N$ we can assign a family of connections $A_\beta(z)$ on the trivial $\mathbb{C}^{2N}$-bundle $E \to I \times S^1$ by the formula

$$A_\beta(z) = \Phi_\beta(t, x)\,\mathrm{d}t + \Theta_\beta(t, x)\,\mathrm{d}x \tag{35}$$

where we write $\Phi_\beta(t, x)$, $\Theta_\beta(t, x)$ for $\Phi_{\beta(t)}(x)$, $\Theta_{\beta(t)}(x)$.

Proposition 4 *Let $\gamma(t) \colon I \to TL_N SU(2)$ be a solution of the equation (30) for the Neumann lattice, let the conditions (31) be fulfilled, and let*

$$A_\gamma(z) = \Phi_\gamma(t, x)\mathrm{d}t + \Theta_\gamma(t, x)\mathrm{d}x$$

be the family of connections on the trivial $\mathbb{C}^{2N}$-bundle $E \to I \times S^1$ assigned to the curve $\gamma(t)$ in the manner described above. Then for every $z \in \mathbb{R}$ the unitary connection $A_\gamma(z)$ satisfies the restricted curvature condition with respect to the sections w_1, w_2 of E given by $w_1 = (1, 0, \ldots, 1, 0)^T$ and $w_2 = (0, 1, \ldots, 0, 1)^T$ in the gauge at hand. In other words, for every $z \in \mathbb{R}$ we have

$$(F_{A_\gamma(z)})_m(\xi_m, \eta_m) \cdot W = 0, \quad m \in I \times S^1, \quad \xi_m, \eta_m \in T_m(I \times S^1) \tag{36}$$

where $F_{A_\gamma(z)}$ is the curvature of $A_\gamma(z)$ and W is the $2 \times 2N$-matrix with the block structure given by $W = \mathrm{diag}(Id_2, \ldots, Id_2)^T$.

Proof: Let $\widetilde{\gamma}\colon t \longmapsto \left(q(t), q_t q^{-1}(t)\right) \longmapsto \left(Q(t), Q_t Q^{-1}(t)\right)$ be the curve in $L\mathfrak{su}(2N)$ associated to the solution $\gamma(t)$. Then the components of the connection

$$A_\gamma(z) = \Phi_\gamma(t,x)\mathrm{d}t + \Theta_\gamma(t,x)\mathrm{d}x$$

have the form

$$
\begin{aligned}
\Phi_\gamma(t,x) = U(t,x;z) &= -(-z\Sigma + Q_t Q^{-1}) \\
\Theta_\gamma(t,x) = V(t,x;z) &= -z\Sigma + Q_t Q^{-1} - \frac{1}{z}\mathrm{Ad}_Q(\hat{T}).
\end{aligned}
$$

The condition (36) can be written in the form

$$\left(V_t - U_x + [U,V]\right)\cdot W = 0, \quad \text{for every } z \in \mathbb{R} \tag{37}$$

where U and V are given above. It is now a matter of a straightforward check that the equation (37) is equivalent to the equation (34), which in turn is equivalent to the Neumann lattice equation (30).

$$\square$$

Note that the reduced curvature condition (37) can be easily guessed from the rewriting (34) of the Neumann lattice equation and the zero-curvature condition for the Maxwell-Bloch equations.

Consider now the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}N_{(q,t_q)}(x) = \Phi_{(q,t_q)}(x)\cdot N_{(q,t_q)}(x), \qquad N_{(q,t_q)}(0) = Id_{2N} \tag{38}$$

associated to every point (q,t_q) from the tangent bundle $TSU(2)^N$. Let $L_z\mathfrak{su}(2N)$ denote the space of all finite Laurent series $z \mapsto \alpha(z)$ with the coefficients $\mathfrak{su}(2N)$. Define the monodromy map $M\colon TSU(2)^N \longrightarrow L_z\mathfrak{su}(2N)$ by

$$M(q,t_q) = N_{(q,t_q)}(2\pi). \tag{39}$$

In the following theorem, which is a corollary of proposition 4, the symbol $\mathcal{C}_z$ will denote the space of the Laurent series with coefficients in $\mathbb{R}$.

Theorem 2 *Let $t \mapsto g(t)$ be a solution of the Neumann lattice equation (30), let the conditions (31) be satisfied, and let the curve $\gamma\colon I \to TSU(2)^N$ be given by*

$$\gamma(t) = \left(g(t), g_t g^{-1}(t)\right).$$

Define the map $\mathbf{F}\colon TSU(2)^N \to \mathcal{C}_z$ *by the formula*

$$\mathbf{F}(q, t_q) = \mathrm{Tr}(W^T \cdot M(q, t_q) \cdot W)$$

where $M(q, t_q)$ *is given by (38) and (39). The map* $\mathbf{F}$ *is constant along the solution path* $\gamma(t)$, $\mathbf{F}(\gamma(t)) \equiv const$.

Let the functions $F_k\colon TSU(2)^N \to \mathbb{R}$ *be given by the formula*

$$\mathbf{F}(q, t_q) = \sum_{k=-\infty}^{\infty} F_k(q, t_q) \cdot z^k.$$

Then the functions F_k *are first integrals of the Neumann lattice (30).*

Proof: Let the family A_γ of connections on the trivial $\mathbb{C}^{2N}$-bundle $E \to I \times S^1$ be associated to the solution curve $\gamma(t)$ by the formula (35). Then by proposition 4 the family A_γ satisfies the reduced curvature condition with respect to the sections w_1, w_2 of E which we collect together into the matrix W. It follows then from theorem 1 that the eigenvalues of the matrix $W^T \cdot M(\gamma(t)) \cdot W$ are independent of time t. By theorem 1 this matrix is an element of the Lie group $SU(2)$. Therefore its eigenvalues appear in pairs of the form $(e^{ia(t)}, e^{-ia(t)})$. But the function $t \mapsto e^{ia(t)}$ is constant if and only if the function

$$t \longmapsto \mathrm{Tr}(W^T \cdot M(\gamma(t)) \cdot W) = e^{ia(t)} + e^{-ia(t)}$$

is constant. Note that the trace is the only non-trivial coefficient in the characteristic equation of an $SU(2)$-matrix. The second part of the theorem now follows immediately.

$\square$

In order to calculate the integrals F_k of the Neumann lattice one has to solve the system of linear differential equations (39) for every $(q, t_q) \in TSU(2)^N$. A linear system is easily solved only when the matrix of coefficients is constant. In our case the matrix of coefficients $\Phi_{(q,t_q)}(x)$ is not constant. But it is not difficult to find the gauge transformation of the bundle $E \to I \times S^1$ which will transform the coefficient matrix $\Phi_{(q, t_q)}(x)$ into a constant matrix for every (q, t_q). Recall that

$$\Phi_{(q,t_q)}(x) = -z\hat{\Sigma} + T_Q Q^{-1} - \frac{1}{z}\mathrm{Ad}_Q(\hat{T}) = \mathrm{Ad}_{\mathcal{R}(x)}\left(K_{(q,t_q)}\right)$$

where

$$K_{(q,t_q)} = -z\Sigma + T_G G^{-1} - \frac{1}{z}\mathrm{Ad}_G(T)$$

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is constant with respect to x. Let $N_{(q,t_q)}(x)$ be a solution of the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}x} N_{(q,t_q)}(x) = \Phi_{(q,t_q)}(x) \cdot N_{(q,t_q)}(x)$$

and let $H_{(q,t_q)}(x) = \mathcal{R}^{-1}(x) N_{(q,t_q)}(x)$. Then $H_{(q,t_q)}(x)$ is a solution of the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} H_{(q,t_q)}(x) = (K_{(q,t_q)} - D) \cdot H_{(q,t_q)}(x) \tag{40}$$

whose coefficient matrix $(K_{(q,t_q)} - D)$ is constant with respect to x. If we set

$$J(q, t_q) = H_{(q,t_q)}(2\pi)$$

then we have $J(q, t_q) = M(q, t_q)$, since $\mathcal{R}(2\pi) = \mathcal{R}(0) = Id_{2N}$. It now follows that

$$M(q, t_q) = \mathrm{Exp}(2\pi(K_{(q,t_q)} - D)).$$

The above observations give the proof of the following proposition.

Proposition 5 *The conserved quantity*

$$\mathbf{F} \colon TSU(2)^N \longrightarrow \mathcal{C}_z, \qquad \mathbf{F}(q, t_q) = \mathrm{Tr}(W^T \cdot M(q, t_q) \cdot W)$$

of the Neumann lattice can be explicitely calculated by means of the formula

$$\mathbf{F}(q, t_q) = \mathrm{Tr}(W^T \cdot \mathrm{Exp}(2\pi(K_{(q,t_q)} - D)) \cdot W). \tag{41}$$

$\square$

Clearly, the formula (41) for $\mathbf{F}$ also yields an explicit way to compute the first integrals $F_k \colon TSU(2)^N \to \mathbb{R}$ of the Neumann lattice. But the integrals obtained in this way are in a sense not the most natural ones. For example, the total energy (23) of the Neumann lattice does not appear among the integrals F_k. Now we shall describe a set of simpler and more natural first integrals which will include the total energy (23) and also two other obvious conserved quantities. The construction of this set will rely essentialy on the fact that the coefficient matrix of the linear differential equation (40) is independent of x.

We shall prove the following theorem.

Theorem 3 *Let the Neumann lattice (30) satisfy the conditions (31). Define the map*

$$\mathbf{H}_k : TSU(2)^N \longrightarrow \mathcal{C}_z$$

by the fomula

$$\mathbf{H}_k(q, t_q) = \mathrm{Tr}(W^T \cdot (K_{(q,t_q)} - D)^k \cdot W).$$

Then for every $k \in \mathbb{N}$ the map $\mathbf{H}_k$ is a conserved quantity of the Neumann lattice (30). The functions

$$H_{k,j} : TSU(2)^N \longrightarrow \mathbb{R} \tag{42}$$

given by the relation

$$\mathbf{H}_k(q, t_q) = \sum_{j=-k}^{k} H_{k,j}(q, t_q) \cdot z^j$$

are the first integrals of the Neumann lattice (30).

Proof: Let $x \in S^1$ be an arbitrary real number from the interval $[0, 2\pi]$ considered as a point on S^1. We shall prove that the map $\mathbf{F}_x : TSU(2)^N \to \mathcal{C}_z$ given by the formula

$$\mathbf{F}_x(q, t_q) = \mathrm{Tr}(W^T \cdot \mathrm{Exp}(x \cdot (K_{(q,t_q)} - D)) \cdot W) \tag{43}$$

is a conserved quantity of the Neumann lattice. This implies that for every positive integer k and for every $x_0 \in [0, 2\pi]$ the map $\frac{\partial^k}{\partial x^k} \mathbf{F}_x(q, t_q)|_{x=x_0}$ is also a conserved quantity. If, in particular, we set $x_0 = 0$, the formula (41) gives

$$\frac{\partial^k}{\partial x^k} \mathbf{F}_x(q, t_q)|_{x=0} = \mathrm{Tr}(W^T \cdot (K_{(q,t_q)} - D)^k \cdot W) = \mathbf{H}_k(q, t_q)$$

which proves the theorem.

We thus have to prove that $\mathbf{F}_x$ given by the fomula (43) is indeed a conserved quantity for every $x \in S^1$. Take again a solution $t \mapsto q(t)$ of the Neumann lattice and let $\gamma(t) = (q(t), q_t q^{-1}(t))$ be the corresponding path in the tangent bundle $TSU(2)^N$. As before, we associate to γ the path

$$\widetilde{\gamma} : t \longmapsto (q(t), q_t q^{-1}(t)) \longmapsto (Q(t), Q_t Q^{-1}(t))$$

in $L\mathfrak{su}(2N)$, and finally the family $A(z)$ of connections on $E \to I \times S^1$,

$$A(z) = \Phi_\gamma \mathrm{d}t + \Theta_\gamma \mathrm{d}x$$

where

$$\begin{aligned}
\Phi_\gamma(t, x) = U(t, x; z) &= -(-z\Sigma + Q_t Q^{-1}) \\
\Theta_\gamma(t, x) = V(t, x; z) &= -z\Sigma + Q_t Q^{-1} - \frac{1}{z}\mathrm{Ad}_Q(\hat{T}).
\end{aligned}$$

We shall now change the gauge by the gauge transformation $\mathcal{R}^{-1}(x)$. In the new gauge the components of $A(z)$ are given by

$$
\begin{aligned}
U_R(t; z) &= -(-z\Sigma + G_t G^{-1}) \\
V_R(t; z) &= -z\Sigma + (G_t G^{-1} - D) - \frac{1}{z}\mathrm{Ad}_G(T) = K(t) - D
\end{aligned}
$$

where $K(t) = K_{(q(t), q_t q^{-1}(t))}$. The important point is that in the new gauge the connections $A(z)$ are independent of x and therefore invariant with respect to the translations in the x-direction. The curvature matrices of $A(z)$ in the two gauges are related by

$$
V_t - U_x + [U, V] = \mathrm{Ad}_{\mathcal{R}^{-1}(x)}\Big((V_R)_t - (U_R)_x + [U_R, V_R]\Big).
$$

We have already seen that $\mathcal{R}(x) \cdot W = W$, therefore the formula (37) implies

$$
\Big((V_R)_t - (U_R)_x + [U_R, V_R]\Big) \cdot W = 0. \tag{44}
$$

If we consider

$$
A_R(z) = U_R \, \mathrm{d}t + V_R \, \mathrm{d}x
$$

to be a new family of connections on the bundle E expressed in the original gauge, then (44) shows that $A_R(z)$ satisfies the reduced curvature condition with respect to the sections $w_1 = (1, 0, \ldots 1, 0)$ and $w_2 = (0, 1, \ldots, 0, 1)$ of E which are constant in our gauge. The connection $A_R(z)$ and the sections w_1, w_2 are invariant with respect to the translations in the x-direction. We shall see that in such cases the restricted curvature condition yields a simpler family of conserved quantities.

In the proof of theorem 1 we have seen that the condition (44) ensures the existence of a gauge transformation

$$
\mathcal{G} : I \times S^1 \longrightarrow SU(2N)
$$

of the bundle $E \to I \times S^1$ such that in the new gauge the gauge connection matrix

$$
\widetilde{A}_R(z) = \mathrm{Ad}_{\mathcal{G}}(A_R(z)) + \mathrm{d}\mathcal{G} \cdot \mathcal{G}^{-1}
$$

takes values in the subalgebra $\mathfrak{su}(2) \times \mathfrak{su}(2N - 2) \subset \mathfrak{su}(2N)$. Since $A_R(z)$ and W are invariant with respect to x, we can take $\mathcal{G}$ also to be constant with respect to x. The connection matrix $\widetilde{A}_R(z)$ therefore has the same invariance property,

$$
\widetilde{A}_R(t, x_1; z) = \widetilde{A}_R(t, x_2; z), \quad \text{for every pair } x_1, x_2 \in S^1.
$$

To summarize, $\widetilde{A}_R(z)$ is of the form

$$\widetilde{A}_R(t,x;z) = \widetilde{U}_R \mathrm{d}t + \widetilde{V}_R \mathrm{d}x = \begin{pmatrix} a_1(t;z)\mathrm{d}t + a_2(t;z)\mathrm{d}x & 0 \\ 0 & b_1(t;z)\mathrm{d}t + b_2(t;z)\mathrm{d}x \end{pmatrix} \quad (45)$$

where $a_i \colon I \to \mathfrak{su}(2)$ and $b_i \colon I \to \mathfrak{su}(2N-2)$ for $i = 1, 2$.

Let the closed curve $\kappa_{x_0}(s)$ be the rectangle in $I \times S^1$ given by

$$\kappa_{x_0}(s) = \begin{cases} \kappa_1(s) = (s, 0) & s \in [0, t_0] \\ \mu_1(s) = (t, s - t_0) & s \in [t_0, t_0 + x_0] \\ \kappa_2(s) = (2t_0 + x_0 - s, x_0) & s \in [t_0 + x_0, 2t_0 + x_0] \\ \mu_2(s) = (0, 2t_0 + 2x_0 - s) & s \in [2t_0 + x_0, 2t_0 + 2x_0] \end{cases} \quad (46)$$

Denote by $\widetilde{N}_1$ and $\widetilde{N}_2$ the endpoints of the horizontal lifts of the paths κ_1 and κ_2, respectively. Then $\widetilde{N}_1 = \widetilde{N}(t_0)$, where $\widetilde{N}_1(t)$ is the solution of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{N}_i = \widetilde{U}_R(t,0) \cdot \widetilde{N}, \quad \widetilde{N}_i(0) = Id_{2N}.$$

But $\widetilde{U}_R(t, x_0) = \widetilde{U}_R(t, 0)$, therefore $\widetilde{N}_2 = \widetilde{N}_1^{-1}$. The endpoints of the horizontal lifts of $\mu_1(s)$ and $\mu_2(-s)$ will be denoted by $\widetilde{M}(0)$ and $\widetilde{M}(t_0)$, respectively. The matrices $\widetilde{M}(\tau)$ for $\tau = 0, t_0$ are given by $\widetilde{M}(\tau) = \widetilde{M}(\tau, x_0)$ and $\widetilde{M}(\tau, x)$ are the solutions of the initial value problems

$$\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{M}(\tau, x) = \widetilde{V}_R(\tau) \cdot \widetilde{M}(\tau, x), \qquad \widetilde{M}(\tau, 0) = Id_{2N}.$$

Let $\widetilde{W}$ be the $2N \times 2$ matrix whose block structure is given by $\widetilde{W} = (Id_2, 0_2, \ldots, 0_2)^T$, and 0_2 is the 2×2 zero matrix. From (44) and (45) we see that

$$\widetilde{M}_2(t_0) = \widetilde{N}_2 \cdot \widetilde{M}_2(0) \cdot \widetilde{N}_2^{-1} \quad (47)$$

where

$$\widetilde{M}_2(\tau) = \widetilde{W}^T \cdot M(\tau) \cdot \widetilde{W}, \quad \tau = 0, t_0 \quad \text{and} \quad \widetilde{N}_2 = \widetilde{W}^T \cdot \widetilde{N} \cdot \widetilde{W}.$$

Returning to the original gauge we have

$$W = \mathcal{G}^{-1} \cdot \widetilde{W}, \quad \text{and} \quad M = \mathrm{Ad}_{\mathcal{G}^{-1}}\widetilde{M}$$

and thus (47) tells us that the spectrum of $W^T \cdot M(t) \cdot W$ is independent of time $t \in I$. Finally, we recall that $M(t) = M(t, x_0)$, where $M(t, x)$ is the solution of the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}M(t, x) = \Big(K(t) - D\Big) \cdot M(t, x), \quad M(t, 0) = Id_{2N}$$

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or, explicitely, $M(t) = \mathrm{Exp}(x_0(K(t) - D))$. Thus we have proved that the spectrum of the matrix $SU(2)$- matrix

$$W^T \cdot \mathrm{Exp}\Big(x_0(K(t) - D)\Big) \cdot W = W^T \cdot \mathrm{Exp}\Big(x_0(K_{(q(t),q_t q^{-1}(t))} - D)\Big) \cdot W$$

is constant along every solution $t \mapsto q(t)$ of the Neumann lattice equation. In other words, the map

$$\mathbf{F}_{x_0} : TSU(2)^N \longrightarrow \mathcal{C}_z$$

given by

$$\mathbf{F}_{x_0}(q, t_q) = \mathrm{Tr}\Big(W^T \cdot \mathrm{Exp}\Big(x_0(K_{(q,t_q)} - D)\Big) \cdot W\Big)$$

is indeed constant along the solutions of the Neumann lattice equation as claimed.

$\square$

Let us illustrate the above theorem by calculating the simplest integrals $H_{k,i}$ given by (42). The first nontrivial integrals occur, when $k = 2$. Then

$$W^T \cdot (K - D)^2 \cdot W = W^T(K^2 - KD - DK + D^2) \cdot W = W^T \cdot K^2 \cdot W$$

since by the condition *(b)* of (31) we have $K \cdot W = W^T \cdot K = 0$. The non-trivial coefficients of the Laurent polynomial

$$\mathbf{H}_2(z) = \mathrm{Tr}(W^T \cdot K^2_{(q,t_q)}(z) \cdot W)$$

are the functions

$$H_{2,0}(q, t_q) = 2 \sum_{i=-m}^{m} \left(\frac{1}{2} \|(t_q)_i\|^2 + \langle \sigma, \mathrm{Ad}_{g_i}(\tau_i) \rangle \right)$$

$$H_{2,-1}(q, t_q) = \sum_{i=-m}^{m} \langle \sigma, (t_q)_i \rangle$$

$$H_{2,1}(q, t_q) = \sum_{i=-m}^{m} \langle \mathrm{Ad}_{g_i}(\tau_i), (t_q)_i \rangle.$$

The function $H_{2,0}$ is the total energy of all the oscillators in our lattice. It is also the Hamiltonian of our twisted Hamiltonian structure. Recall that the Neumann oscillator $(T^*SU(2), \omega_c.H_n)$ with

$$H_n(g, p_g) = \frac{1}{2} \|p_g\|^2 + \langle \sigma, \mathrm{Ad}_g(\tau) \rangle$$

has two rotational symmetries: one with respect to the lifted left action of $\varrho_u(g)u \cdot g$ of $U_\sigma(1) = \{\exp(s \cdot \sigma)\} \subset SU(2)$ and the other with respect to the lifted right action $\rho_u(g) = g \cdot u$ of $U_\tau(1) = \{\exp(s \cdot \tau)\} \subset SU(2)$. The corresponding momenta are

$$M_\sigma(g, p_g) = \langle \sigma, p_g \rangle, \quad \text{and} \quad M_\tau(g, p_g) = \langle \mathrm{Ad}_g(\tau), p_g \rangle$$

respectively. Thus the inegrals $H_{2,-1}$ and $H_{2,1}$ are the two total rotational momenta of our Neumann lattice.

The integrals at higher values of k involve the "discrete derivative" D. We intend to discuss some cases in another paper.

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