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List-Colouring Squares of Sparse Subcubic Graphs

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Abstract

The problem of colouring the square of a graph naturally arises in connection with the distance labelings, which have been studied intensively. We consider this problem for sparse subcubic graphs. We show that the choosability $\chi_\ell(G^2)$ of the square of a subcubic graph G of maximum average degree d is at most four if $d < 24/11$ and G does not contain a 5-cycle, $\chi_\ell(G^2)$ is at most five if $d < 7/3$ and it is at most six if $d < 5/2$. Wegner's conjecture claims that the chromatic number of the square of a subcubic planar graph is at most seven. Let G be a planar subcubic graph of girth g. Our result implies that $\chi_\ell(G^2)$ is at most four if $g \geq 24$, it is at most 5 if $g \geq 14$, and it is at most 6 if $g \geq 10$. For lower bounds, we find a planar subcubic graph G_1 of girth 9 such that $\chi(G_1^2) = 5$ and a planar subcubic graph G_2 of girth five such that $\chi(G_2^2) = 6$. As a consequence, we show that the problem of 4-colouring of the square of a subcubic planar graph of girth $g = 9$ is NP-complete. We conclude the paper by posing few conjectures.

1 Introduction

We study the colouring of squares of graphs, and so consider simple undirected graphs. Let $\chi(G)$ and $\chi_\ell(G)$ be the chromatic number and the choosability of a graph G, respectively. The *square* G^2 of a graph G is the graph with the same

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vertex set in which two vertices are joined by an edge if their distance in G is at most two. It is easy to see that $\Delta + 1 \leq \chi(G^2) \leq \chi_\ell(G^2) \leq \Delta^2 + 1$, where Δ is the maximum degree of G. However, by Brooks theorem, it is not hard to infer that there are only finitely many connected graphs for which the upper bound is attained. On the other hand, the chromatic number of the square of a planar graph is bounded by a function linear in the maximum degree (note that this bound does not follow directly from the 5-degeneracy of planar graphs [7]).

The notion of colouring of the square of a graph, as well as the other variants of a distance colouring, arise from the problem of assigning frequencies to transmitters. We are given a set of locations of transmitters, and we want to assign frequencies to them in such a way that they do not interfere. The transmitters interfere if they are close enough and the frequencies assigned to them are similar. Choosability version of the problem corresponds to the situation where we are not allowed to use all frequencies on all transmitters. The notion of closeness can be often approximated by forming the graph of the neighbouring transmitters, and colouring of the square or a higher power of this graph enables us to take the interference between farther transmitters into account.

Let us briefly survey the rich history of the colouring of the squares of planar graphs. Wegner [15] proved that the squares of cubic planar graphs are 8-colourable. He conjectured that his bound can be improved:

Conjecture 1. *Let G be a planar graph with maximum degree Δ. The chromatic number of G^2 is at most 7, if $\Delta = 3$, at most $\Delta + 5$, if $4 \leq \Delta \leq 7$, and at most $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$, otherwise.*

If this conjecture were true, then the bounds would be the best possible. The reader is welcome to see Section 2.18 in [8] for more details. Though Conjecture 1 has been verified for several special classes of planar graphs, including the outerplanar graphs [10], it remains open for all values of $\Delta \geq 3$. The best known upper bounds are due to Molloy and Salavatipour [11, 12]: $\lceil 5\Delta/3 \rceil + 78$ for all Δ and $\lceil 5\Delta/3 \rceil + 25$ for $\Delta \geq 241$. Sharper results can be obtained for special classes of graphs. Dvořák et al. [4] have proved that the chromatic number of the square of a planar graph G with sufficiently large maximal degree is $\Delta + 1$ if the girth of G is at least seven and it is bounded by $\Delta + 2$ if the girth of G is six.

A graph G is called *subcubic* if $\Delta(G) \leq 3$. We consider (not necessarily planar) subcubic graphs of small maximum average degree d and show the following bounds on the choosability of G^2 (Section 3):

- If $d < 24/11$ and G does not contain a 5-cycle, then $\chi_\ell(G^2) \leq 4$.
- If $d < 7/3$, then $\chi_\ell(G^2) \leq 5$.
- If $d < 5/2$, then $\chi_\ell(G^2) \leq 6$.

For planar subcubic graphs of large girth g, this implies that (Section 4):

- If $g \geq 24$, then $\chi_\ell(G^2) \leq 4$.
- If $g \geq 14$, then $\chi_\ell(G^2) \leq 5$.
- If $g \geq 10$, then $\chi_\ell(G^2) \leq 6$.

Montassier and Raspaud [13] have shown similar results for colouring of the squares of planar subcubic graphs. They have shown that $\chi(G^2) \leq 5$ if $g \geq 14$ and $\chi(G^2) \leq 6$ if $g \geq 10$ and their proof can be easily generalised to work for choosability as well. They use some of the reducible configurations we do, however their proofs of reducibility are simpler, because they use the girth assumptions (not only the sparseness of the graph).

Regarding the lower bounds, we construct subcubic planar graphs G_1 and G_2 of girths $g(G_1) = 9$ and $g(G_2) = 5$ such that $\chi(G_1^2) = 5$ and $\chi(G_2^2) = 6$ (Section 5).

The problem of finding the chromatic number of the square of a planar graph is NP-complete. It is NP-complete even to decide whether the square of a planar graph can be coloured by seven colours [14] – note that this implies that determining the chromatic number of the square of a planar graph is NP-complete when restricted to graphs of maximum degree six. It is also NP-complete to decide whether the square of a cubic (not necessarily planar) graph can be coloured by four colours [6]. On the other hand, the problem can be solved in a polynomial time for partial k-trees [16]. For further complexity and structural results we refer to the survey [2]. We are interested in the complexity of the problem for planar graphs of large girth. The graphs used in the reduction of Ramanathan [14] contain many triangles, thus they do not provide any guidance in this direction.

We show that the problem of determining the chromatic number of the square of a graph is NP-complete even for subcubic planar graphs of girth 9 (Section 6), more precisely, we show that it is NP-complete to decide whether such a graph can be coloured by four colours.

2 Notation

Let $\mathbb{N}$ denote the set of all nonnegative integers. A *d-vertex* is a vertex of degree d. The *maximum average degree* $d(G)$ of a graph G is the maximum of $2|E(H)|/|V(H)|$ over all induced subgraphs H of G.

A *k-cycle* is a cycle of length k. The *girth* g of G is the length of the shortest cycle in G, and it is infinity if G is a forest. A *thread* is an induced path in G whose vertices are all of degree 2 in G. A *k-thread* for $k \geq 1$ is a thread with k vertices. The *length* of a thread is number of its vertices, i.e., a k-thread has length k.

If G is a connected plane graph, then let $F(G)$ be the set of all faces of G. Denote by $\ell(f)$ the length of a face f (we count multiple occurrences of an edge if f is not a simple cycle, which can happen if G is not 2-connected). A *ℓ-face* is a face of length ℓ.

We say that a graph G is *k-minimal*, if G^2 is not k-choosable, but the square of every proper subgraph of G is k-choosable. Obviously, a k-minimal graph is connected. A *configuration* is an induced subgraph of G, and we say that a configuration is *k-reducible*, if it cannot appear in a k-minimal graph. Note that a k-reducible configuration is also k'-reducible for every $k' \geq k$.

We introduce the following notation to simplify the description of reducible configurations: Let $Y_{a,b,c}$ be a 3-vertex v together with an a-thread, a b-thread and a c-thread incident to v, such that the threads are vertex-disjoint and there are no edges between the end-vertices of the threads. Note however that the end-vertices may have a common neighbour. We allow the possibility that $a = 0$ or $b = 0$ or $c = 0$, and in this case v is adjacent to one or more vertices of degree 3.

Let $Y_{a,b}$—j—$Y_{c,d}$ be two 3-vertices v_1 and v_2 joined by a j-thread together with an a-thread and a b-thread incident to v_1 and a c-thread and a d-thread incident to v_2. Again, the threads are vertex-disjoint and there are no edges between the end-vertices of the threads. We also allow that a, b, j, c or d could be 0. Thus, if $j = 0$ then v_1 and v_2 are adjacent.

Let G be a graph and let L be an assignment of lists to the vertices of G. We say that L is a *p-list assignment* for a given function $p : V(G) \to \mathbb{N}$, if $|L(v)| = p(v)$ for each vertex v of G. If p is a constant function with value k, we say that L is an *k-list assignment*.

3 Colouring Sparse Graphs

In this section we show that the squares of sparse subcubic graphs can be coloured by few colours. We proceed by showing that if a subgraph induced by vertices close to any vertex v is too sparse in such a graph, then it is k-reducible for given k. Thus we show that in the k-minimal graph, each such subgraph is dense, and thus that the k-minimal graph itself is dense, which is a contradiction.

We first identify the reducible configurations in Subsection 3.1. Then, in Subsection 3.2, we discuss the possible neighbourhoods of vertices in a k-minimal graph. Finally, we derive the bounds on density of k-minimal graphs, for $k = 4$, 5 and 6.

3.1 Reducible Configurations

In this section, we give k-reducible configurations for $k = 4$, 5 and 6. Let us first show that a k-minimal graph has minimum degree at least two.

Lemma 1. *A 1-vertex is a k-reducible configuration for each $k \geq 4$.*

Proof. Let G be a k-minimal graph and let L be a k-list assignment of G such that G^2 is not L-colourable. Assume for the contradiction that G contains a 1-vertex v. By the k-minimality of G, the graph $(G-v)^2 = G^2 - v$ has an L-colouring c. The vertex v has at most three neighbours in G^2. We let $c(v)$ be a colour that does not appear in the neighbourhood of v in G^2. Thus, we extend c to an L-colouring of G^2, a contradiction. $\square$

Let R be a configuration in a graph G. We define the function $s_R : V(R) \to \mathbb{N}$. The value $s_R(v)$ is the number of neighbours of v in G^2 that are not its neighbours in R^2, i.e., $s_R(v) = \deg_{G^2}(v) - \deg_{R^2}(v)$. Note that some of such neighbours may belong to R.

Suppose that we want to show k-reducibility of a configuration R by the following approach: we assume that R appears in a k-minimal graph G. We remove R from G and colour the graph $G \setminus R$, which is always possible by k-minimality of G. We want extend this colouring to the vertices of R. For each vertex v of R, we remove the colours used at the vertices in distance at most two from the list of v. At most $s_R(v)$ colours are removed in this way. Then we consider colouring of R from these new lists. In the next lemma, we describe this idea more precisely.

Lemma 2. *Let R be a configuration in a graph G and p a function from vertices of R such that $p(v) \leq k - s_R(v)$ for every vertex v of R. Suppose that*

(a) no two vertices of $G \setminus R$ have a common neighbour in R, and

(b) no two vertices u and v of R such that $uv \notin E(R^2)$ have a common neighbour in $G \setminus R$.

If R^2 can be coloured from any p-list assignment L, then G is not k-minimal.

Proof. Let G be a k-minimal graph and let L be a k-list assignment such that G is not L-colourable. Assume for the contradiction that G contains the configuration R satisfying the assumptions of the lemma. Let $G' = G - R$. By the k-minimality of G, the graph G'^2 has an L-colouring c. We show that we can extend this colouring to G^2, which is a contradiction.

We remove the colours used on the neighbours in G^2 from the lists of the vertices of R. The length of the new list of each vertex $v \in V(R)$ is at least $k - s_R(v) \geq p(v)$. We can colour R^2 from these lists, thus obtaining the values of the colouring c on $V(R)$. By the assumptions of the lemma, the subgraph of G^2 induced by $V(R)$ is equal to R^2 and the subgraph induced by $V(G')$ is equal to G'^2. Consequently, c is a proper L-colouring of G^2, a contradiction. $\square$

The condition (a) of the above lemma is equivalent to say that each vertex of R has at most one neighbour outside of R. This is satisfied in all of our applications of this lemma. The condition (b) is more complicated to satisfy. In the case $k = 4$, all the configurations for that the lemma is used satisfy this property trivially. In case $k \geq 5$ we need to consider the cases when two vertices of R share a common neighbour outside of R separately, though.

Using $p(v) = k - s_R(v)$ in the applications of this lemma is not practical, since it requires us to know the exact neighbourhood of R in G. Instead, we use upper bounds on s derived from the knowledge about the degrees of vertices in the neighbourhood of R. In particular, we often use $p(v) = k - s'_R(v)$, where

$$s'_R(v) = 3(\deg_G(v) - \deg_R(v)) + \sum_{uv \in E(R)} (\deg_G(u) - \deg_R(u)).$$

A *block* of a graph G is its maximal 2-connected subgraph. A connected graph G is said to be a *Gallai tree* if each of the blocks of G is a complete graph or an odd cycle. In several proofs of the reducibility of configurations, we use the following theorem, which is proved independently by Borodin [1] and Erdös et al. [5].

Theorem 3. *Let G be a connected graph with a list assignment L such that $|L(v)| \geq \deg(v)$ for each vertex v of G and G is not L-colourable. Then, $|L(v)| = \deg(v)$ for every v and G is a Gallai tree. Moreover, if G is 2-connected, then the lists $L(v)$ of all the vertices v of G are the same.*

Let us show a stronger version of this theorem for a particular graph.

Lemma 4. *Let G be a path on four vertices $v_1v_2v_3v_4$ and let L be a list assignment to the vertices of G such that $|L(v_1)| = |L(v_4)| = 2$ and $|L(v_2)| = |L(v_3)| = 3$. Then G^2 has two L-colourings c_1 and c_2 such that $c_1(v_2) \neq c_2(v_2)$.*

Proof. Since G^2 is not a Gallai tree, at least one colouring c_1 of G^2 exists by Theorem 3.

Let $a_1 = c_1(v_2)$ and $L(v_2) = \{a_1, a_2, a_3\}$. Let us assume for the sake of contradiction that all colourings of G use the colour a_1 on the vertex v_2. This means that if we colour v_2 by a_2, then this colouring cannot be extended. Let $L'(v_1) = L(v_1) \setminus \{a_2\}$, $L'(v_3) = L(v_3) \setminus \{a_2\}$ and $L'(v_4) = L(v_4) \setminus \{a_2\}$. By Theorem 3, if v_1, v_3 and v_4 cannot be coloured from lists L', then $|L'(v_1)| = |L'(v_4)| = 1$ and $|L'(v_3)| = 2$. But then $a_2 \in L(v_1) \cap L(v_4)$. Now observe that the colouring c_2 defined by $c_2(v_1) = c_2(v_4) = a_2$, $c_2(v_2) = a_3$ and $c_2(v_3) \in L(v_3) \setminus \{a_2, a_3\}$ satisfies the requirements of the lemma. □

Let us first consider cycles that contain at most one 3-vertex in k-minimal graphs.

Lemma 5. *Let R be a configuration formed by a cycle $v_1v_2 \cdots v_s$ such that v_2, v_3, $\ldots$, v_s are 2-vertices. Then R is k-reducible for $k \geq 4$ unless $k = 4$ and $s = 5$.*

Proof. Since there is at most one edge joining R and $G \setminus R$, all the assumptions of Lemma 2 are trivially satisfied. It suffices to show 4-reducibility if $s \neq 5$ and 5-reducibility if $s = 5$, and we may assume that v_1 is a 3-vertex.

Suppose first that $s \neq 5$. We use the function p defined by $p(v_1) = 1$, $p(v_2) = p(v_s) = 3$ and $p(v_i) = 4$ for the remaining vertices. Let L be any p-list assignment. We need to show that R^2 can be coloured from L. Observe that this is the case if $s = 3$ or $s = 4$.

In case $s > 5$, let $c(v_1)$ be the single colour in $L(v_1)$. Let $p'(v_2) = p'(v_s) = 2$, $p'(v_3) = p'(v_{s-1}) = 3$ and $p'(v_i) = 4$ for the remaining vertices, and let L' be an arbitrary p'-list assignment such that $L'(v_i) \subseteq L(v_i) \setminus \{c(v_1)\}$ for $i = 2, 3, s-1$ and s and $L'(v_i) = L(v_i)$ for the remaining vertices. We need to find a colouring of vertices $v_2, \ldots, v_s$ from the lists L'.

If $s > 6$, we choose $c(v_2) \in L'(v_2)$ and $c(v_s) \in L'(v_s)$ so that $c(v_2) \neq c(v_s)$. We remove the colour $c(v_2)$ from the lists $L'(v_3)$ and $L'(v_4)$ and the colour $c(v_s)$ from the lists $L'(v_{s-1})$ and $L'(v_{s-2})$ (note that all these vertices are mutually distinct, since $s > 6$). Finally, we colour the subgraph of R^2 induced by the path $v_3 \cdots v_{s-1}$ from the new lists using Theorem 3, thus obtaining a proper colouring c.

Finally, consider the case $s = 6$. If $L'(v_2) \neq L'(v_6)$, we select $c(v_2)$ from $L'(v_2) \setminus L'(v_6)$, remove $c(v_2)$ from the lists of v_3 and v_4 and colour the vertices v_3, v_4, v_5 and v_6 using Lemma 4. Therefore, suppose that $L'(v_2) = L'(v_6) = \{a_1, a_2\}$. If $L'(v_2) \subset L'(v_3)$ and $L'(v_2) \subset L'(v_5)$, then set $c(v_2) = c(v_5) = a_1$, $c(v_3) = c(v_6) = a_2$ and choose $c(v_4)$ from $L'(v_4) \setminus L'(v_2)$ arbitrarily. If this is not the case, we may assume that $a_1 \notin L'(v_3)$. Then we set $c(v_2) = a_1$, $c(v_6) = a_2$, remove the colour a_1 from list of v_4 and the colour a_2 from lists of v_4 and v_5, and then colour v_3, v_4 and v_5 from their lists using Theorem 3.

In the case $s = 5$, we let $p(v_1) = 2$, $p(v_2) = p(v_5) = 4$ and $p(v_3) = p(v_4) = 5$. Note that we can colour the vertices v_1, v_2, v_5, v_3 and v_4 one by one in this order. □

We are now ready to identify several 4-reducible configurations.

Lemma 6. *The following configurations are 4-reducible:*

(1) *a* 6*-thread,*

(2) $Y_{1,4,5}$*,*

(3) $Y_{2,3,4}$*,*

(4) $Y_{5,5}$—0—$Y_{4,5}$*, and*

(5) *a* 7*-cycle* $v_1 v_2 \cdots v_7$ *such that* v_1 *incident to a* 2*-thread* $x_1 x_2$*, the vertex* v_2 *incident to a* 2*-thread* $y_1 y_2$ *(where the* 2*-threads are vertex-disjoint), and each* v_i *is a* 2*-vertex for* $3 \leq i \leq 7$*.*

Proof. Let us prove the reducibility of each configuration separately. Let R be one of the configurations. We use Lemma 2 for the configuration R' obtained from R by removing all 2-vertices that have degree 1 in R, i.e., those that have a neighbour in $G - R$. We remove the vertices x_2 and y_2 in case (5) even if they are adjacent. Note that we then know that the neighbours of R' in $G - R'$ are mutually distinct 2-vertices, and thus the assumptions of Lemma 2 are trivially satisfied for R'. We can also use the function p defined for vertices of R' as

$$p(v) = 4 - 2(\deg_G(v) - \deg_{R'}(v)) - \sum_{uv \in E(R')} (\deg_G(u) - \deg_{R'}(u)).$$

It suffices to show that the configuration R' can be coloured from any p-list assignment L. This will imply that R' with the additional assumptions on the degrees of surrounding vertices cannot appear in a 4-minimal graph, and hence that R is reducible.

(1) Let R be a 6-thread. In this case, R' is a 4-thread $v_1v_2v_3v_4$ with $p(v_1) = p(v_4) = 2$ and $p(v_2) = p(v_3) = 3$. By Theorem 3, R'^2 can be coloured from the lists L.

(2) Let R be $Y_{1,4,5}$. In this case, the configuration R' is a path $u_3u_2u_1vw_1w_2w_3w_4$, with $p(w_4) = p(u_3) = p(v) = 2$, $p(w_1) = p(w_3) = p(u_1) = p(u_2) = 3$, and $p(w_2) = 4$.

First we try to construct an L-colouring of R'^2 in the following way. We fix a colouring c' of the subgraph of R'^2 induced by vertices v, u_1, u_2 and u_3. Notice that such a colouring always exists by Lemma 4. We choose $c'(w_1) \in L(w_1) \setminus \{c'(v), c'(u_1)\}$, and afterwards set $L'(w_2) = L(w_2) \setminus \{c'(v), c'(w_1)\}$, $L'(w_3) = L(w_3) \setminus \{c'(w_1)\}$ and $L'(w_4) = L(w_4)$. Each of the lists L' contains at least two colours. If we can colour w_2, w_3 and w_4 from lists L', then we obtain a proper colouring of R'^2. If this is not the case, then by Theorem 3, $L'(w_2) = L'(w_3) = L'(w_4) = \{a_1, a_2\}$ for some two colours a_1 and a_2. Let $a_3 = c'(w_1)$, $a_4 = c'(v)$ and $a_5 = c'(u_1)$. Then $L(w_3) = \{a_1, a_2, a_3\}$ and $L(w_2) = \{a_1, a_2, a_3, a_4\}$.

Note that if we can choose colour for w_1 distinct from a_3, i.e., if $|L(w_1) \setminus \{a_4, a_5\}| > 1$, then we are able to extend the colouring to the rest of the configuration. Therefore, let us assume that $L(w_1) = \{a_3, a_4, a_5\}$. The colours $a_4 = c'(v)$ and $a_5 = c'(u_1)$ for that the colouring c' cannot be extended to the subconfiguration on vertices w_1, w_2, w_3 and w_4 cannot be coloured are determined uniquely. By Lemma 4, we can choose a colouring c of v, u_1, u_2 and u_3 such that $c(u_1) \neq a_5$. Hence, we can extend c to R'^2.

(3) Let R be $Y_{2,3,4}$. In this case, the configuration R' is $Y_{1,2,3}$. Let v be the 3-vertex of R', and let u_1, w_1w_2 and $y_1y_2y_3$ be the threads incident to v,

where u_1, w_1 and y_1 are adjacent to v. The lengths of lists are $p(u_1) = p(w_2) = p(y_3) = 2$, $p(w_1) = p(y_2) = p(v) = 3$ and $p(y_1) = 4$.

Let us first fix a colouring c' of the subgraph of R'^2 induced by u_1, v, w_1 and w_2, which exists by Lemma 4. Let $L'(y_1) = L(y_1) \setminus \{c'(u_1), c'(w_1), c'(v)\}$, $L'(y_2) = L(y_2) \setminus \{c'(v)\}$ and $L'(y_3) = L(y_3)$. Note that $|L'(y_1)| \geq 1$, $|L'(y_2)| \geq 2$ and $|L'(y_3)| = 2$. If y_1, y_2 and y_3 can be coloured from lists L', we obtain a proper colouring of R'^2. Assume this is not the case. Then $|L'(y_2)| = 2$ and $L'(y_1) \subseteq L'(y_2) = L'(y_3)$. This implies that $L(y_2) = L(y_3) \cup \{c'(v)\}$. But by Lemma 4 we may choose a colouring c of subgraph of R'^2 induced by u_1, v, w_1 and w_2 such that $c(v) \neq c'(v)$, and such a colouring can be extended to R'^2.

(4) Let R be $Y_{5,5}$—0—$Y_{4,5}$. The configuration R' is $Y_{4,4}$—0—$Y_{3,4}$. Let v_1 and v_2 be the two adjacent 3-vertices of R', and let $u_1u_2u_3u_4$, $w_1w_2w_3w_4$, $y_1y_2y_3y_4$ and $z_1z_2z_3$ be the threads of R', where u_1 and w_1 are adjacent to v_1 and y_1 and z_1 are adjacent to v_2. Notice that $p(u_4) = p(w_4) = p(y_4) = p(z_3) = 2$, $p(u_3) = p(w_3) = p(y_3) = p(z_2) = 3$ and $p(x) = 4$ for every other vertex x of R'.

Suppose that we have a colouring c' of the square of the path $P = w_4w_3w_2w_1v_1v_2z_1z_2z_3$, and suppose that this colouring cannot be extended to a colouring of R'^2. The colourings of the threads $P_u = u_1u_2u_3u_4$ and $P_y = y_1y_2y_3y_4$ are independent, because the distance of u_1 and y_1 is three. Assume that c' cannot be extended to P_u. Similarly as in the proof of the second claim of this lemma, we conclude that the lists of vertices of P_u uniquely determine the colour $a_u = c'(v_1)$. And, if c' cannot be extended to P_y, then the lists of vertices of P_y uniquely determine the colour $a_y = c'(v_2)$. Let $L'(v_1) = L(v_1) \setminus \{a_u\}$, $L'(v_2) = L(v_2) \setminus \{a_y\}$ and $L'(x) = L(x)$ for the other vertices of P. If we can find an L'-colouring c of P^2, then c can be extended to a proper colouring of R'^2.

Let us choose a colour $a_1 \in L'(z_2) \setminus L'(z_3)$. Using Theorem 3, fix an L'-colouring c of the square of the path $w_4w_3w_2w_1v_1v_2$ such that $c(v_2) \neq a_1$. We extend c to the vertices z_1, z_2 and z_3 in the following way. We assign lists $L''(z_1) = L'(z_1)\setminus\{c(v_1), c(v_2)\}$, $L''(z_2) = L'(z_2)\setminus\{c(v_2)\}$ and $L''(z_3) = L'(z_3)$ to these vertices. Note that each of these lists has size at least 2. If an L''-colouring of the square of $z_1z_2z_3$ exists, it extends c to a proper L'-colouring of P^2. If such a colouring does not exist, then by Theorem 3, $L''(z_1) = L''(z_2) = L''(z_3)$. And hence, $L'(z_2) = L'(z_3) \cup \{c(v_2)\}$. But this is impossible, because $c(v_2) \neq a_1$.

(5) Let R be the last configuration given in the lemma. The configuration R' is induced by vertices $v_1, \ldots, v_7$, x_1 and y_1, and the lengths of the lists are $p(x_1) = p(y_1) = 2$, $p(v_1) = p(v_2) = 3$ and 4 for the remaining vertices. Let us construct the colouring c from the lists L.

We first colour the subgraph induced by vertices x_1, v_1, v_2 and y_1, which is possible by Lemma 4. In case $L(v_4) \setminus L(v_5)$ contains precisely one element, say a, we may also assume that the colour $c(v_2)$ is distinct from a. Let us choose colours $c(v_3)$ and $c(v_7)$ from their lists such that they extend this colouring. Then, we construct the assignment of lists to the rest of the vertices in the following way: $L'(v_6) = L(v_6) \setminus \{c(v_1), c(v_7)\}$, $L'(v_5) = L(v_5) \setminus \{c(v_3), c(v_7)\}$ and $L'(v_4) = L(v_4) \setminus \{c(v_2), c(v_3)\}$. If we can extend the colouring c to these vertices from lists L', we properly colour the whole configuration R'^2. By Theorem 3, this is possible unless $L'(v_4) = L'(v_5) = L'(v_6) = S$ and the length of S is exactly two. This implies that $L(v_4) = S \cup \{c(v_2), c(v_3)\}$ and $L(v_5) = S \cup \{c(v_3), c(v_7)\}$. But this means that $c(v_2)$ is the single colour of $L(v_4) \setminus L(v_5)$, which contradicts the choice of the colouring of vertices x_1, v_1, v_2 and y_1.

$\square$

Now let us focus on 5-reducible configurations.

Lemma 7. *The following two configurations are* 5*-reducible:*

(1) a 3*-thread, and*

(2) $Y_{1,2,2}$.

Proof. Let us prove the reducibility of each configuration R separately. We use Lemma 2. Let L be an arbitrary p-list assignment to the vertices of R, where $p(v) = 5 - s'_R(v)$ for each vertex v of R. In order to be able to apply the lemma, we need to consider the case when two of the vertices of R share a neighbour outside of R, and to show that R can be coloured from any p-list assignment.

(1) The configuration R is a 3-thread $v_1v_2v_3$ with the lengths of lists $p(v_1) = p(v_3) = 2$ and $p(v_2) = 3$. If v_1 and v_3 have a common neighbour outside of R, then R together with this vertex forms a 4-cycle with at least three 2-vertices. Such a configuration is reducible by Lemma 5. And, if v_1 and v_3 do not have a common neighbour outside R, then we can apply Lemma 2, since R can be coloured from any p-list assignment by Theorem 3.

(2) The configuration R is $Y_{1,2,2}$ with the 3-vertex v and threads u_1, w_1w_2 and y_1y_2, where u_1, w_1 and y_1 are the neighbours of v, see Figure 1(a). The lengths of the lists are $p(u_1) = p(w_2) = p(y_2) = 2$ and $p(w_1) = p(y_1) = p(v) = 4$.

Assume first that no two vertices of R have a common neighbour outside of R. In this case we can directly apply Lemma 2, and it is sufficient to show that R^2 can be coloured from the lists L. Let us choose colours in this order: $c(v) \in L(v) \setminus L(w_2)$, $c(u_1) \in L(u_1) \setminus \{c(v)\}$, $c(y_2) \in L(y_2) \setminus \{c(v)\}$,

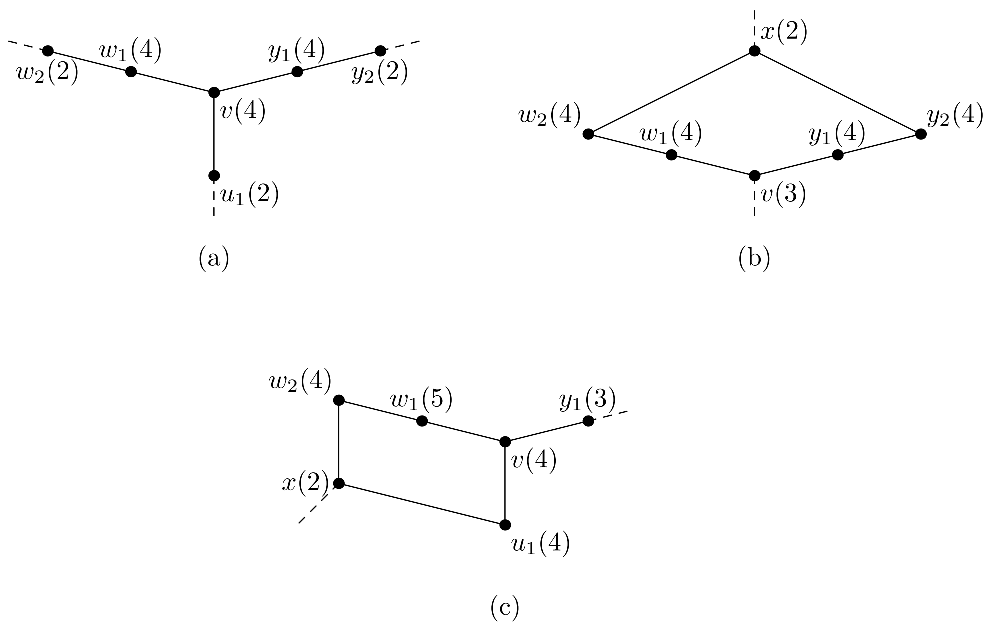

Figure 1: Configurations arising from $Y_{1,2,2}$. The numbers in brackets denote the lengths of the lists assigned to the vertices. The dashed edges join the considered configuration with the rest of the graph.

$c(y_1) \in L(y_1) \setminus \{c(v), c(u_1), c(y_2)\}$, $c(w_1) \in L(w_1) \setminus \{c(v), c(u_1), c(y_1)\}$ and $c(w_2) \in L(w_2) \setminus \{c(w_1)\}$. This is always possible by sizes of the lists, and the constructed colouring is a proper colouring of R^2, since $c(w_2) \neq c(v)$ by the choice of $c(v)$.

Next, consider the case that w_2, y_2 and u_1 have a common neighbour x. Then $V(G) = V(R) \cup \{x\}$, and G^2 is 5-choosable – colour vertices v, u_1 and x in this order, remove their colours from lists of vertices w_1, w_2, y_1 and y_2 and use Theorem 3 to colour them from the restricted lists of size at least two. This is possible, since these vertices induce a 4-cycle in G^2.

Suppose now that w_2 and y_2 have a common neighbour x that is not adjacent to u_1. We apply Lemma 2 on the 6-cycle induced by vertices v, x, w_1, w_2, y_1 and y_2, see Figure 1(b). Since x and u_1 are not adjacent, this is possible. The lengths of lists are $p'(v) = 3$, $p'(x) = 2$ and 4 for the remaining vertices. Colour first x and v from their lists arbitrarily, and then extend this colouring to the vertices w_1, w_2, y_1 and y_2 that induce a 4-cycle in G^2 using Theorem 3 – each of the vertices w_1, w_2, y_1 and y_2 have at least two colours distinct from colours of x and v in their lists.

Finally, consider the case that w_2 and u_1 have a common neighbour x that is not adjacent to y_2. Let us apply Lemma 2 on the graph induced by vertices v, x, w_1, w_2, u_1 and y_1, see Figure 1(c). The list sizes are $p''(x) = 2$,

$p''(y_1) = 3$, $p''(w_2) = p''(u_1) = p''(v) = 4$ and $p''(w_1) = 5$. Let L'' be an arbitrary p''-list assignment and let us find an L''-colouring c''. Choose $c''(u_1) \in L''(u_1) \setminus L''(y_1)$ and colour the remaining vertices one by one in the following order: x, w_2, v, w_1 and y_1. The sizes of the lists ensure that this is always possible.

□

Finally, we identify the 6-reducible configurations.

Lemma 8. *The following configurations are* 6*-reducible in graphs of girth at least* 6*:*

(1) a 2*-thread,*

(2) a 3-cycle containing two 3-vertices and one 2*-vertex,*

(3) a 4-cycle containing two nonadjacent 3*-vertices and two* 2*-vertices, and*

(4) $Y_{1,1}$—1—$Y_{0,1}$.

Proof. Let us prove the reducibility of each configuration R separately. We use Lemma 2. Let L be an arbitrary p-list assignment to the vertices of R, where $p(v) = 6 - s'_R(v)$ for each vertex v of R. For the fourth configuration, we need to discuss the case when two of the vertices of R share a neighbour outside of R in order to be able to apply Lemma 2. For the remaining configurations the assumptions of the same lemma are satisfied trivially.

(1) The configuration R is a 2-thread v_1v_2 such that $p(v_1) = p(v_2) = 2$. The graph R^2 can obviously be coloured from L.

(2) Let v_1, v_2 and v_3 be the vertices of the configuration R, where v_3 is the 2-vertex. Then $p(v_1) = p(v_2) = 2$ and $p(v_3) = 4$. The graph R^2 can be coloured by Theorem 3.

(3) Let v_1, v_2, v_3 and v_4 be the vertices of the configuration R, where v_2 and v_4 are the 2-vertices. Then $p(v_1) = p(v_3) = 3$ and $p(v_2) = p(v_4) = 4$. Again, the graph R^2 can be coloured by Theorem 3.

(4) The 3-vertices of R are v_1 and v_2 with the common neighbour z. The 1-threads are u_1, w_1 and y_1, where u_1 and w_1 are the neighbours of v_1 and y_1 is a neighbour of v_2, see Figure 2(a). The lengths of the lists are $p(u_1) = p(w_1) = 3$, $p(v_1) = 4$, $p(v_2) = p(y_1) = 2$ and $p(z) = 5$.

Suppose first that the assumptions of Lemma 2 are satisfied, and let us show that R^2 is colourable from the lists L. Choose a colour $c(v_1) \in L(v_1) \setminus L(u_1)$ and $c(v_2) \in L(v_2) \setminus \{c(v_1)\}$. Define the lists $L'(u_1) = L(u_1)$, $L'(w_1) =$

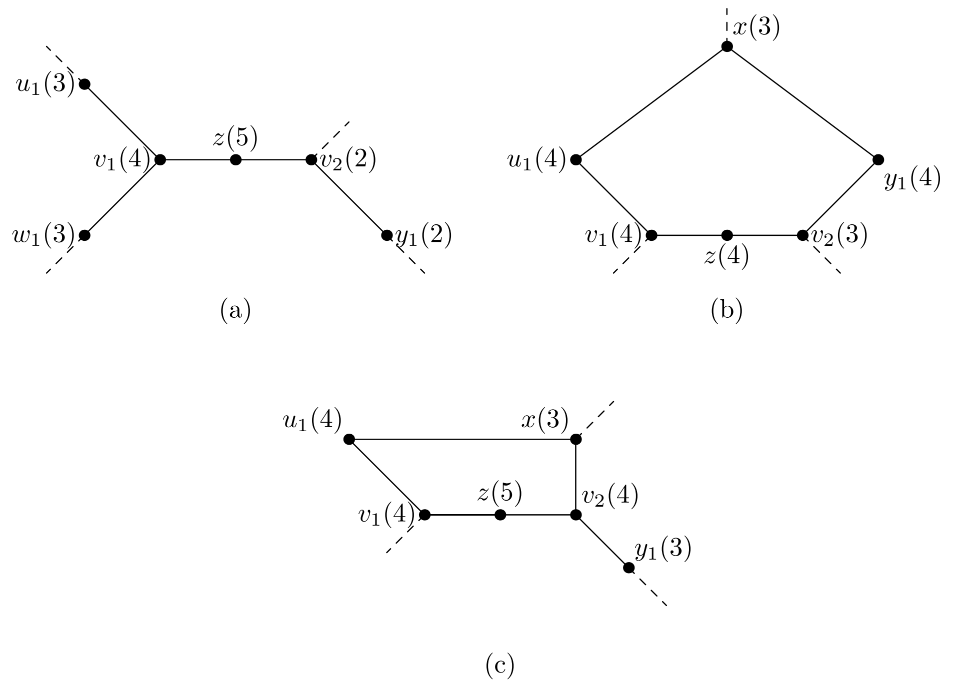

Figure 2: Configurations arising from $Y_{1,1}$—1—$Y_{0,1}$

$L(w_1) \setminus \{c(v_1)\}$, $L'(z) = L(z) \setminus \{c(v_1), c(v_2)\}$ and $L'(y_1) = L(y_1) \setminus \{c(v_2)\}$. We can L'-colour the subgraph induced by vertices u_1, w_1, z and y_1 in R^2 by Theorem 3. In this way, we extend c to a proper colouring of R^2.

Now, consider the case that the assumptions of Lemma 2 are not satisfied. By the previous claims 2 and 3 of this lemma, and by Lemma 5, it suffices to discuss the cases when u_1 and y_1, or u_1 and v_2 have a common neighbour x outside R. Moreover, x has exactly two neighbours in R.

Suppose first that u_1 and y_1 share the neighbour x. Let us consider the 6-cycle induced by vertices u_1, v_1, z, v_2, y_1 and x, see Figure 2(b). We may apply Lemma 2 for this configuration, since distances among v_1, v_2 and x are all equal to two. The sizes of the lists are $p'(x) = p'(v_2) = 3$ and 4 for the rest of the vertices. Let L' be a p'-list assignment. We construct a colouring c of the configuration from these lists. We choose $c(u_1) \in L'(u_1) \setminus L'(x)$ and $c(v_2) \in L'(v_2)$ arbitrarily, and extend the colouring to the remaining four vertices (that induce a 4-cycle in the square of the configuration) using Theorem 3.

Finally, suppose that u_1 and v_2 have the common neighbour x. Let us consider the configuration induced by vertices v_1, v_2, z, u_1, y_1 and x, see Figure 2(c). The distance between x and v_1 is two, and the between x and y_1 is two as well. The vertices v_1 and y_1 cannot share the neighbour w_1, since two 2-vertices cannot be adjacent in a 6-minimal graph by claim 1 of this lemma. Therefore, we may apply Lemma 2. The lengths of lists are $p''(y_1) = p''(x) = 3$, $p''(v_1) = p''(v_2) = p''(u_1) = 4$ and $p''(z) = 5$.

Let L'' be a p''-list assignment and let us construct a colouring c from these lists. If there exists a colour $a \in L''(y_1) \cap L''(v_1)$, we set $c(y_1) = c(v_1) = a$, remove the colour a from the lists of the remaining vertices and colour them in order x, v_2, u_1 and z. If the lists $L''(y_1)$ and $L''(v_1)$ are disjoint, then at least one of them contains a colour a that does not belong to $L''(z)$. Let us colour one of the vertices y_1 and v_1 by a and the other one arbitrarily from its list, and remove the colours $c(y_1)$ and $c(v_1)$ from the lists of vertices in the distance at most two from them. Then again colour the remaining vertices from the new lists in order x, v_2, u_1 and z.

□

3.2 Neighbourhoods of vertices

Let us consider the possible neighbourhoods of 3-vertices in k-minimal graphs. We prove that these neighbourhoods cannot contain too many 2-vertices, which we later use to show that any k-minimal graph must be dense. All of the following lemmata are proved by a straightforward case analysis using the results of the previous subsection.

Given a graph G and a vertex v, let $G_2(v)$ be the subgraph of G induced by v and all 2-vertices reachable from v by paths consisting only from 2-vertices.

Lemma 9. *Let G be a 4-minimal graph different from a 5-cycle and v a 3-vertex in G such that v is not adjacent to any 3-vertex. Then $G_2(v)$ is a subgraph of $Y_{1,3,5}$ or $Y_{1,4,4}$ or $Y_{2,2,5}$ or $Y_{3,3,3}$.*

Proof. Let x, y and z be the 3-vertices (not necessarily distinct) that are joined by threads with v. By the 4-minimality, G cannot contain a 5-cycle, and any other cycle containing at most one 3-vertex is reducible by Lemma 5. Therefore, x, y and z are distinct from v. Let l_x, l_y and l_z be the lengths of the threads joining x, y and z with v, and assume $1 \leq l_x \leq l_y \leq l_z$. By Lemma 6(1), $l_z \leq 5$, and by Lemma 6(3), $l_x \leq 3$. If $l_x = 1$, then $l_z < 5$ or $l_y < 4$ by Lemma 6(2) and $G_2(v)$ is a subgraph of $Y_{1,3,5}$ or $Y_{1,4,4}$. If $2 \leq l_x \leq 3$, then $l_x = l_y = 2$ (and $G_2(v)$ is a subgraph of $Y_{2,2,5}$), or $l_z \leq 3$ (and $G_2(v)$ is a subgraph of $Y_{3,3,3}$). □

Lemma 10. *Let G be a 4-minimal graph different from a 5-cycle and let v and w be two adjacent 3-vertices in G. Then, both $G_2(v)$ and $G_2(w)$ are subgraphs of $Y_{0,5,5}$, and at least one of the following conditions holds:*

- *both $G_2(v)$ and $G_2(w)$ are subgraphs of $Y_{0,4,5}$, or*
- *$G_2(v)$ or $G_2(w)$ is a subgraph of $Y_{0,3,5}$, or*

- *$G_2(v)$ or $G_2(w)$ is a subgraph of $Y_{0,4,4}$.*

Proof. Let x and y be the 3-vertices joined by threads with v and let s and t be the 3-vertices joined by threads with w. Let l_x, l_y, l_s and l_t be the lengths of these threads, and assume $l_x \leq l_y \leq l_t$ and $l_s \leq l_t$. The vertices x and y are distinct from v and the vertices s and t are distinct from w by Lemma 5 and by the minimality of G. By Lemma 6(1), $l_t \leq 5$ and both $G_2(v)$ and $G_2(w)$ are subgraphs of $Y_{0,5,5}$.

Suppose that neither of the conditions of the lemma is satisfied by v and w. Then $l_y = l_s = l_t = 5$ and $4 \leq l_x \leq 5$. But if s and t are distinct from v, then this induces a copy of $Y_{5,5}$—0—$Y_{4,5}$ in G, which is reducible by Lemma 6(4). Thus we may assume $s = v$. Hence, v and w are a part of a 7-cycle $vwx_1 \cdots x_5$, where x_i are 2-vertices. However, since $l_x \geq 4$, $l_y \geq 4$, $l_t \geq 4$, the vertices v and w are additionally incident to vertex-disjoint 2-threads, and the configuration is reducible by Lemma 6(5). □

Let us now consider neighbourhoods of vertices in 5-minimal graphs.

Lemma 11. *Let G be a 5-minimal graph and v a 3-vertex in G. Then, $G_2(v)$ is a subgraph of $Y_{0,2,2}$ or $Y_{1,1,2}$.*

Proof. Let x, y and z be the 3-vertices (not necessarily distinct) that are joined by threads with v. By Lemma 5, none of x, y and z is equal to v. Let l_x, l_y and l_z be the lengths of the threads joining x, y and z with v, and assume $l_x \leq l_y \leq l_z$. By Lemma 7(1), $l_z \leq 2$, and by Lemma 7(2), $l_x = 0$ (then $G_2(v)$ is a subgraph of $Y_{0,2,2}$), or $l_y \leq 1$ (and $G_2(v)$ is a subgraph of $Y_{1,1,2}$). □

For vertices in 6-minimal graphs, the situation may be a bit more complicated.

Lemma 12. *Let G be a 6-minimal graph and v a 3-vertex in G. Then, either*

- *$G_2(v)$ is a subgraph of $Y_{0,1,1}$, or*
- *$G_2(v)$ is $Y_{1,1,1}$, the 3-vertices x, y and z incident with $G_2(v)$ are mutually distinct, and all the graphs $G_2(x)$, $G_2(y)$ and $G_2(z)$ are equal to $Y_{0,0,1}$.*

Proof. Let x, y and z be the 3-vertices (not necessarily distinct) that are joined by threads with v. By Lemma 5, x, y and z are distinct from v. Let l_x, l_y and l_z be the lengths of the threads joining x, y and z with v, and again assume $l_x \leq l_y \leq l_z$. By Lemma 8(1), $l_z \leq 1$. If $l_x = 0$, then $G_2(v)$ is a subgraph of $Y_{0,1,1}$. Thus assume that $l_x = 1$ and $G_2(v)$ is $Y_{1,1,1}$. By Lemma 8(3), the vertices x, y and z are mutually distinct. If $G_2(x)$ is not equal to $Y_{0,0,1}$, then neighbourhoods of v and x form a supergraph of $Y_{1,1}$—1—$Y_{0,1}$, which is reducible by Lemma 8(4). Thus $G_2(x)$ (and similarly $G_2(y)$ and $G_2(z)$) is equal to $Y_{0,0,1}$. □

3.3 Final Step

In this section we combine the results of the previous subsections and prove the main results.

If G is a subcubic graph, let $n_2(G)$ be the number of 2-vertices of G and let $n_3(G)$ be the number of 3-vertices of G. Let $V_3(G)$ be the set of 3-vertices of G. By Lemmata 9–12, $G_2(v)$ is equal to $Y_{a,b,c}$ for some a, b and c whenever G is a k-minimal graph for $k \in \{4, 5, 6\}$. We let $d_2(v) = a + b + c$. Note that

$$\sum_{v \in V_3} d_2(v) = 2n_2.$$

Let G be a 4-minimal graph and let v be a 3-vertex in G such that $G_2(v)$ is distinct from $Y_{0,5,5}$. Let $P(v)$ be the set of all 3-vertices u adjacent to v such that $G_2(u)$ is $Y_{0,5,5}$. Then, we define

$$d_3(v) = d_2(v) + \sum_{u \in P(v)} d_2(u) = d_2(v) + 10|P(v)|.$$

Let us show some estimates on d_2 and d_3 in the minimal graphs.

Lemma 13. *Let G be a 4-minimal graph and let v be a 3-vertex of G such that $G_2(v)$ is distinct from $Y_{0,5,5}$. Then, $d_3(v) \leq 9(|P(v)| + 1)$.*

Proof. Let $P(v) = \{u_1, \dots, u_s\}$ with $s \leq 3$. Note that by Lemmata 9 and 10, $d_2(x) \leq 10$ for every 3-vertex x of G. Let us consider several cases regarding s:

$s = 0$: If v is not adjacent to any 3-vertex, then $d_2(v) \leq 9$ by Lemma 9. Otherwise, $G_2(v)$ is a subgraph of $Y_{0,5,5}$ by Lemma 10. The graph $G_2(v)$ is distinct from $Y_{0,5,5}$, thus $d_2(v) \leq 9$ in this case as well.

$s = 1$: By Lemma 10, $G_2(v)$ is a subgraph of $Y_{0,3,5}$ or $Y_{0,4,4}$. Therefore, $d_3(v) = d_2(v) + 10 \leq 18$.

$s = 2$: Since $G_2(v)$ is a subgraph of $Y_{0,5,5}$ and v has at least two neighbours that are 3-vertices, $d_2(v) \leq 5$. Consequently, $d_3(v) = d_2(v) + 20 \leq 27$.

$s = 3$: In this case $d_2(v) = 0$ and $d_3(v) = 30 < 36$.

□

Lemma 14. *Let G be a 6-minimal graph. If u and v are two 3-vertices of G joined by a path consisting of 2-vertices, then $d_2(u) + d_2(v) \leq 4$.*

Proof. We use Lemma 12. If $G_2(v)$ is $Y_{1,1,1}$, then $G_2(u)$ is $Y_{0,0,1}$ and $d_2(u)+d_2(v) = 4$. One can apply a similar argument if $G_2(u)$ is $Y_{1,1,1}$. If neither $G_2(u)$ nor $G_2(v)$ is $Y_{1,1,1}$, then both of them are subgraphs of $Y_{0,1,1}$ and $d_2(u) + d_2(v) \leq 4$. □

We are now ready to prove the main results:

Theorem 15. *Let G be a subcubic graph of maximum average degree $d < 24/11$. The graph G^2 is 4-choosable if and only if G does not contain a 5-cycle.*

Proof. If G contains a 5-cycle, then G^2 contains a clique of size 5 and therefore G^2 is not 4-colourable. For the other implication, assume for the contradiction that G does not contain a 5-cycle, but G^2 is not 4-choosable. Hence G contains a 4-minimal subgraph G' whose average degree is at most $d < 24/11$. We show that average degree of any 4-minimal graph is at least $24/11$, thus obtaining a contradiction.

By the 4-minimality, G' is connected and it contains at least one 3-vertex. Let U be the set of all 3-vertices v of G' such that $G'_2(v)$ is not $Y_{0,5,5}$. Let us now calculate the following sum in two ways:

$$S = \sum_{v \in U} d_3(v).$$

Since $Y_{5,5}$—0—$Y_{5,5}$ is reducible,

$$S = \sum_{v \in V_3(G')} d_2(v) = 2n_2.$$

On the other hand, by Lemma 13

$$S \leq \sum_{v \in U} 9(|P(v)| + 1) = 9n_3(G').$$

Therefore, $2n_2 = S \leq 9n_3$ and we infer that $n_2 \leq 4.5n_3$. This means that the average degree of G' is

$$\frac{2n_2 + 3n_3}{n_2 + n_3} = 2 + \frac{n_3}{n_2 + n_3} \geq 2 + \frac{n_3}{5.5n_3} = \frac{24}{11}.$$

□

Theorem 16. *Let G be a subcubic graph of maximum average degree $d < 7/3$. Then, G^2 is 5-choosable.*

Proof. Assume for contradiction that G^2 is not 5-choosable. Then G contains a 5-minimal subgraph G' whose average degree is at most $d < 7/3$. We show that average degree of any 5-minimal graph is at least $7/3$, thus obtaining a contradiction.

By the 5-minimality, G' is connected and contains at least one 3-vertex. Let us now consider the following sum:

$$S = \sum_{v \in V_3(G')} d_2(v) = 2n_2(G').$$

By Lemma 11, $d_2(v) \leq 4$ for each 3-vertex of G'. Thus, $2n_2(G') = S \leq 4n_3(G')$ and we get that $n_2(G') \leq 2n_3(G')$. This means that the average degree of G' is

$$\frac{2n_2 + 3n_3}{n_2 + n_3} = 2 + \frac{n_3}{n_2 + n_3} \geq 2 + \frac{n_3}{3n_3} = \frac{7}{3}.$$

$\square$

Theorem 17. *Let G be a subcubic graph of maximum average degree $d < 5/2$. Then G^2 is 6-choosable.*

Proof. Assume for contradiction that G^2 is not 6-choosable. Then G contains a 6-minimal subgraph G' whose average degree is at most $d < 5/2$. We show that average degree of any 6-minimal graph is at least $5/2$, thus obtaining a contradiction.

By the 6-minimality, G' is connected and contains at least one 3-vertex. Let X be the set of all unordered pairs of 3-vertices $\{u, v\}$ such that u and v are connected by a thread. The length of the thread is at most one by Lemma 8(1). Note that two 3-vertices u and v are connected by at most one such path, since otherwise the configuration formed by u, v and the two paths is reducible by Lemma 8(2) or (3). Let us now calculate the following sum in two ways:

$$S = \sum_{\{u,v\} \in X} [d_2(u) + d_2(v)].$$

Each vertex v appears in exactly three pairs in X (since a cycle with at most one 3-vertex is excluded by Lemma 5), thus the sum is equal to

$$S = 3 \sum_{v \in V_3(G')} d_2(v) = 6n_2(G').$$

On the other hand, consider an arbitrary pair $\{u, v\} \in X$. By Lemma 14, $d_2(u) + d_2(v) \leq 4$. Hence, $S \leq 4|X| = 6n_3(G')$. Thus, $6n_2(G') = S \leq 6n_3(G')$ and we get that $n_2(G') \leq n_3(G')$. This means that the average degree of G' is

$$\frac{2n_2 + 3n_3}{n_2 + n_3} = 2 + \frac{n_3}{n_2 + n_3} \geq 2 + \frac{n_3}{2n_3} = \frac{5}{2}.$$

$\square$

4 Colouring Planar Graphs of Large Girth

In this section we show bounds on choosability of planar subcubic graphs of large girth. A well-known observation is that such graphs have small maximum average degree.

Lemma 18. *Let G be a planar graph of girth at least g. Then $d(G) < 2+4/(g-2)$.*

Proof. Since every induced subgraph of G is a subcubic planar graph of girth at least g, it suffices to show that $d = 2|E(G)|/|V(G)| < 2g/(g-2)$. We may also assume that G is connected. If G is a tree, then obviously $d(G) < 2$. So assume that G has a finite girth. Let $e = |E(G)|$, $v = |V(G)|$ and $f = |F(G)|$. By Euler formula, $e + 2 = v + f$. The girth of G is at least g and because G is not a tree, every facial walk in G contains a cycle. Therefore, $2e \geq fg$. We obtain

$$\begin{aligned} e &< e + 2 \leq \frac{2e}{d} + \frac{2e}{g} \\ 1 &< \frac{2}{d} + \frac{2}{g} \\ d &< 2 + \frac{4}{g-2}. \end{aligned}$$

□

Together with Theorems 15–17, this implies:

Corollary 19. *Let G be a planar subcubic graph of girth g. Then,*

- *If $g \geq 24$, then $\chi_\ell(G^2) \leq 4$.*
- *If $g \geq 14$, then $\chi_\ell(G^2) \leq 5$.*
- *If $g \geq 10$, then $\chi_\ell(G^2) \leq 6$.*

5 Lower Bounds

In this section, we show lower bounds on the girth of the planar subcubic graphs for that the analogue of Corollary 19 holds. It is easy to find a 5-minimal graph of girth 5:

Proposition 20. *The graph from Figure 3 has girth* 5 *and its square is not* 5*-colourable.*

We also show that there exists a 4-minimal graph of girth 9. First we need to prove several auxiliary lemmata. In their proofs, the following notation is used (where H is the fixed graph we are trying to colour): $(u, v, w) \to c(x) = k$ stands for the statement: "The vertices u, v and w are the neighbours of vertex x in H^2. Let $c : V(H) \to \{1, 2, 3, 4\}$ be a proper colouring of H^2 such that $c(u)$, $c(v)$ and $c(w)$ are three mutually different colours distinct from colour k. Then necessarily $c(x) = k$".

Lemma 21. *Let H_4 be the graph in Figure 4. Then there is no proper colouring $c : V(H_4) \to \{1, 2, 3, 4\}$ of H_4^2 such that $c(x_1) = c(x_2)$ and $\{c(y_1), c(z_1)\} \neq \{c(y_2), c(z_2)\}$.*

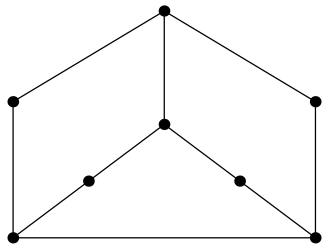

Figure 3: A 5-minimal graph

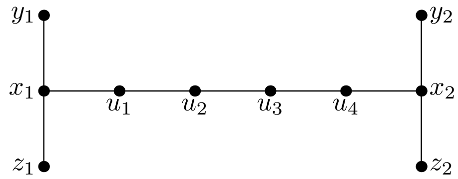

Figure 4: The graph H_4.

Proof. Without loss of generality we may assume that $c(x_1) = c(x_2) = 1$, $c(y_1) = c(y_2) = 2$, $c(z_1) = 3$ and $c(z_2) = 4$. Then $(x_1, y_1, z_1) \to c(u_1) = 4$, similarly $(x_2, y_2, z_2) \to c(u_4) = 3$ and then $(x_1, u_1, u_4) \to c(u_2) = 2$ and $(x_4, u_1, u_4) \to c(u_3) = 2$. But u_2 and u_3 are neighbours in H_4^2, thus the proper colouring does not exist. □

Lemma 22. *The graph $G_{\neq}(x_1, x_2)$ in Figure 5 has the following properties:*

1. *$G_{\neq}$ is a planar subcubic graph.*
2. *The vertices x_1 and x_2 are on distance* 6.
3. *The girth of $G_{\neq}$ is* 9.
4. *Every proper colouring $c : V(G_{\neq}) \to \{1, 2, 3, 4\}$ of $G_{\neq}^2$ satisfies $c(x_1) \neq c(x_2)$.*

Proof. The first three properties are clear from the figure, thus it is sufficient to prove the last property. For the contradiction suppose that $c : V(G_{\neq}) \to \{1, 2, 3, 4\}$ is a proper colouring of $G_{\neq}^2$, such that $c(x_1) = c(x_2)$. We may assume that $c(x_1) = c(x_2) = 1$.

First we show that $c(y_1) \neq c(y_2)$. If $c(y_1) = c(y_2)$ then we may assume that $c(y_1) = c(y_2) = 2$ and $c(z_3) = 3$. Successively we observe that $(x_1, y_1, z_3) \to c(z_1) = 4$, $(x_1, y_1, z_1) \to c(z_7) = 3$, $(x_2, y_2, z_3) \to c(z_4) = 4$, $(y_1, z_4, z_7) \to c(z_6) = 1$, and $(y_2, z_4, z_7) \to c(z_5) = 1$, but z_5 and z_6 are neighbours in $G_{\neq}^2$. That is a contradiction with assumption $c(y_1) = c(y_2)$.

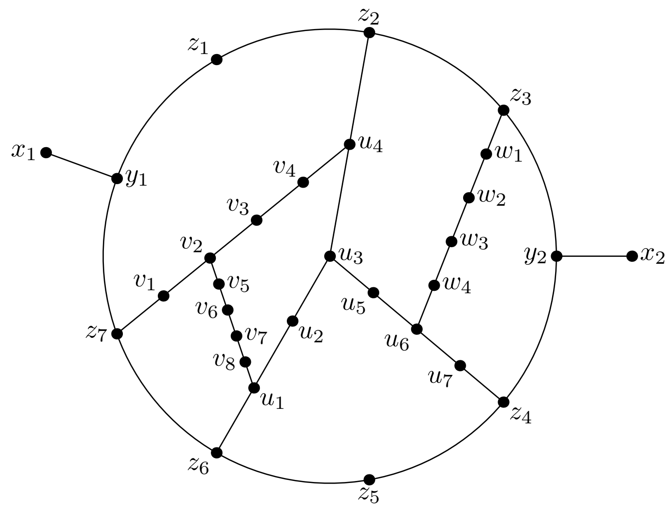

Figure 5: The graph $G_{\neq}(x_1, x_2)$.

Hence, we know that $c(y_1) \neq c(y_2)$. We may assume that $c(y_1) = 2$ and $c(y_2) = 3$. Then, $c(z_1) \in \{3, 4\}$ and $c(z_3) \in \{2, 4\}$. Since z_1 and z_3 are adjacent in $G^2_{\neq}$, they have distinct colours. Consider the following cases:

$c(z_1) = 3$ **and** $c(z_3) = 4$**:** We infer that $(y_1, z_1, z_3) \to c(z_2) = 1$, $(x_1, y_1, z_1) \to c(z_7) = 4$, $(z_1, z_2, z_3) \to c(u_4) = 2$, $(x_2, y_2, z_3) \to c(z_4) = 2$, $(z_4, y_2, z_7) \to c(z_5) = 1$, $(z_5, z_4, z_7) \to c(z_6) = 3$, $(z_5, z_6, z_7) \to c(u_1) = 2$, and $(y_2, z_4, z_5) \to c(u_7) = 4$. Vertex u_2 has neighbours in $G^2_{\neq}$ coloured by 2 and 3, thus $c(u_2) \in \{1, 4\}$. If $c(u_2) = 4$ then all the vertices u_3, u_5 and u_6 have neighbours coloured by 2 and 4, and thus they are coloured by 1 or 3. But it is impossible since they induce a triangle in $G^2_{\neq}$. Therefore, we know that $c(u_2) = 1$. Then $(u_2, u_4, u_7) \to c(u_5) = 3$, $(u_2, u_4, u_5) \to c(u_3) = 4$, $(z_2, u_3, u_4) \to c(v_4) = 3$, $(y_1, z_6, z_7) \to c(v_1) = 1$, $(z_7, v_1, v_4) \to c(v_2) = 2$, and $(v_1, v_2, v_4) \to c(v_3) = 4$. But by Lemma 21 this colouring cannot be extended to vertices v_5, v_6, v_7 and v_8.

$c(z_1) = 4$ **and** $c(z_3) = 2$**:** This case is straightforward: $(y_2, z_1, z_3) \to c(z_2) = 1$, $(x_1, y_1, z_1) \to c(z_7) = 3$, $(x_2, y_2, z_3) \to c(z_4) = 4$, $(y_1, z_4, z_7) \to c(z_6) = 1$, and $(z_4, z_6, z_7) \to c(z_5) = 2$. By Lemma 21, this colouring cannot be extended to vertices u_1, u_2, u_3 and u_4.

$c(z_1) = 3$ **and** $c(z_3) = 2$**:** First, observe that $(x_1, y_1, z_1) \to c(z_7) = 4$ and $(x_2, y_2, z_3) \to c(z_4) = 4$. Both of the vertices z_2 and u_4 have neighbours in $G^2_{\neq}$ coloured by colours 2 and 3, thus they are coloured by 1 or 4. They are adjacent, thus one of them has to be assigned the colour 1 and the second one the colour 4. Hence, $c(u_3) \in \{2, 3\}$. The vertex z_6 has neighbours with colours 2 and 4 so $c(z_6) \in \{1,3\}$. Now, we distinguish cases according to colours of u_3 and z_6.

$c(z_6) = 1$ **and** $c(u_3) = 2$**:** In this case we infer $(z_6, z_7, u_3) \to c(u_1) = 3$, $(z_6, u_1, u_3) \to c(u_2) = 4$, $(z_6, z_7, u_1) \to c(z_5) = 2$, $(y_2, z_4, z_5) \to c(u_7) = 1$, $(z_4, u_3, u_7) \to c(u_6) = 3$, and $(u_2, u_3, u_7) \to c(u_5) = 3$, but that is a contradiction with that colouring is proper since u_5 and u_6 are neighbours.

$c(z_6) = 1$ **and** $c(u_3) = 3$**:** This is an easy case, $(z_7, z_6, u_3) \to c(u_1) = 2$, $(z_6, u_1, u_3) \to c(u_2) = 4$, and $(z_7, z_6, u_1) \to c(z_5) = 3$. But z_5 and y_2 are neighbours in $G^2_{\neq}$ and both of them are coloured by 3. This is a contradiction.

$c(z_6) = 3$ **and** $c(u_3) = 2$**:** Note that $(z_6, z_7, u_3) \to c(u_1) = 1$, $(z_6, u_1, u_3) \to c(u_2) = 4$, $(z_7, z_6, u_1) \to c(z_5) = 2$, $(y_2, z_4, z_5) \to c(u_7) = 1$, $(z_4, u_3, u_7) \to c(u_6) = 3$, and $(u_2, u_3, u_7) \to c(u_5) = 3$, but this is a contradiction, since u_5 and u_6 are adjacent in $G^2_{\neq}$.

$c(z_6) = 3$ **and** $c(u_3) = 3$**:** If $c(u_4) = 4$ then $(z_1, z_3, u_3) \to c(z_2) = 1$, but by Lemma 21 this colouring cannot be extended to vertices v_1, v_2, v_3 and v_4. Thus necessarily $c(u_4) = 1$. Similarly, if $c(u_6) = 2$ then by Lemma 21 the colouring cannot be extended to vertices w_1, w_2, w_3 and w_4 (note that neither u_5 nor u_7 is coloured by 3, and thus the assumptions of the lemma are satisfied). Hence, it follows that $c(u_6) = 1$. Next, $(y_2, z_4, u_6) \to c(u_7) = 2$, $(u_3, u_6, u_7) \to c(u_5) = 4$, $(u_3, u_4, u_5) \to c(u_2) = 2$, $(z_7, u_2, u_3) \to c(u_1) = 1$, and $(z_6, z_7, u_1) \to c(z_5) = 2$, but z_5 and u_7 are neighbours in $G^2_{\neq}$ coloured by the same colour, a contradiction.

□

Theorem 23. *There exists a planar subcubic graph G with $g(G) = 9$ such that G^2 is not 4-colourable.*

Proof. The graph consists of vertices x, y, z, u and v, edges xy, yz, yu and uv, and copies of $G_{\neq}(x, v)$ and $G_{\neq}(z, v)$ (see Figure 6). By the first three properties of $G_{\neq}$ from Lemma 22, G is a planar graph of girth 9. For the contradiction suppose that G is 4-colourable. Because of the distances of the vertices x, y, z, u and v, and the forth property of $G_{\neq}$ from Lemma 22, all the vertices x, y, z, u and v must have mutually distinct colours. But it is a contradiction with the 4-colourability. □

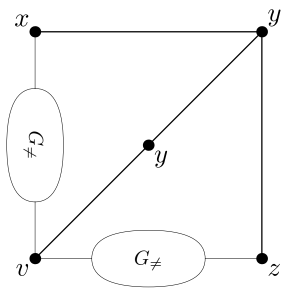

Figure 6: A graph whose square is not 4-colourable

6 NP-completeness

In this section we discuss the complexity aspects of the problem. Let us first show an auxiliary structural lemma.

Lemma 24. *There exists a subcubic planar graph $G_{copy}(x, y)$ of girth 9 with the properties described below. The graph $G_{copy}(x, y)$ contains two 1-vertices x and y on the outer face. Let v_x and v_y be the neighbours of x and y in $G_{copy}(x, y)$, respectively. The graph $G_{copy}(x, y)$ satisfies the following properties:*

- *the distance of x and y in $G_{copy}(x, y)$ is greater than 9, and*
- *$c(x) = c(y)$ in any proper colouring c of the square of $G_{copy}(x, y)$, and*
- *for any $a_1, a_2, a_3 \in \{1, 2, 3, 4\}$ such that $a_1 \neq a_2$ and $a_1 \neq a_3$, there exists a proper colouring c of the square of $G_{copy}(x, y)$ with $c(x) = c(y) = a_1$, $c(v_x) = a_2$ and $c(v_y) = a_3$.*

Proof. We use the graph $G_{\neq}$ from Lemma 22. The graph $G_{\neq}$ has the following colouring c_1: $c_1^{-1}(\{1\}) = \{x_1, y_2, v_1, u_1, u_4, u_6, w_2\}$, $c_1^{-1}(\{2\}) = \{z_1, x_2, z_6, v_2, v_7, u_3, u_7, w_1\}$, $c_1^{-1}(\{3\}) = \{y_1, z_3, z_5, v_4, v_5, v_8, u_5, w_3\}$ and $c_1^{-1}(\{4\}) = \{z_2, z_4, z_7, v_3, v_6, u_2, w_4\}$.

The steps of the construction of $G_{copy}(x, y)$ are depicted in Figure 7. First we construct an auxiliary subcubic planar graph $G_{copy'}(x, y)$ of girth 9 such that the vertices x and y are on the outer face, x is a 1-vertex and y is a 2-vertex, and $c(x) = c(y)$ in any proper colouring c of the square of $G_{copy'}(x, y)$. The graph $G_{copy'}(x, y)$ consists of vertices x, y, u, v and w, edges xu, uv, uw and wy, and a copy of $G_{\neq}(v, y)$. The vertices x, y, u, v and w must have mutually distinct colours, with the exception of vertices x and y that can have the same colour. Since we colour the graph by four colours, the colours of x and y must in fact be the equal in any proper colouring of $G^2_{copy'}(x, y)$. Also note that there exists a proper colouring c_2 of the square of $G_{copy'}(x, y)$ by four colours – let us colour

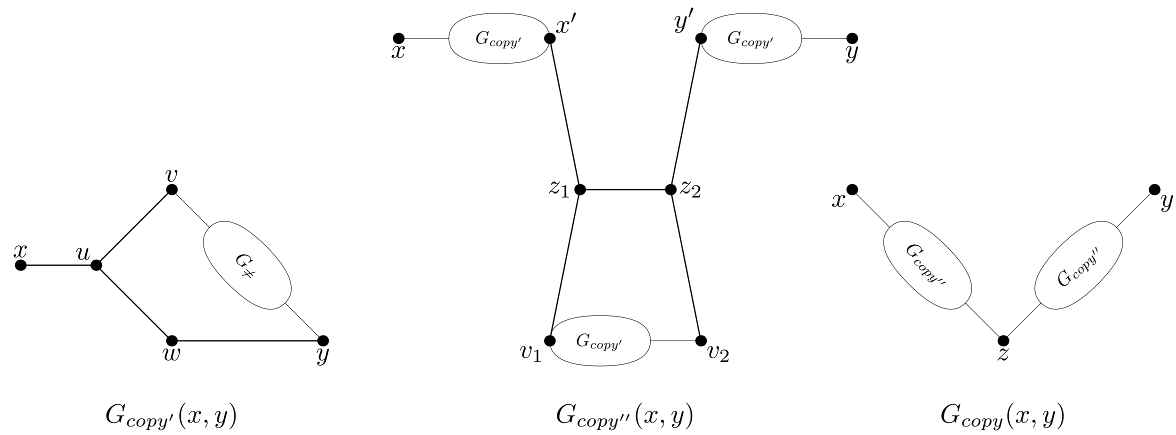

Figure 7: Graphs used in Lemma 24

the copy of $G_{\neq}(v, y)$ using the colouring c_1, and set $c_2(x) = 2$, $c_2(u) = 4$ and $c_2(w) = 3$.

Next we construct the second auxiliary subcubic planar graph $G_{copy''}(x, y)$ of girth 9 such that the 1-vertices x and y are on the outer face, and $c(x) = c(y)$ in any proper colouring c of $G^2_{copy'}(x, y)$. The graph $G_{copy''}(x, y)$ consists of vertices x, x', y, y', z_1, z_2, v_1 and v_2, edges $x'z_1$, $y'z_2$, z_1z_2, z_1v_1, z_2v_2 and the following copies of graph $G_{copy'}$: $G_{copy'}(x, x')$, $G_{copy'}(y, y')$ and $G_{copy'}(v_1, v_2)$.

In any colouring c of $G^2_{copy''}(x, y)$, by properties of $G_{copy'}$ it holds that $c(x) = c(x')$, $c(y) = c(y')$ and $c(v_1) = c(v_2)$. Let us assume that exists such a colouring c with $c(x) \neq c(y)$. We may assume that $c(x) = 1$ and $c(y) = 2$, and also that $c(z_1) = 3$ and $c(z_2) = 4$. But then $c(v_1) = c(y') = c(y) = 2$ and $c(v_2) = c(x') = c(x) = 1$, which is a contradiction with the properties of $G_{copy'}(v_1, v_2)$. On the other hand, there exists the following colouring c_3 of $G^2_{copy''}(x, y)$: colour $G_{copy'}(x, x')$ by c_2 and $G_{copy'}(y, y')$ by c_2 with colours 3 and 4 swapped, so that $c_3(x) = c_3(x') = c_3(y) = c_3(y') = 2$, neighbours of x' have colours 1 and 3 and neighbours of y' have colours 1 and 4. Then set $c_3(z_1) = 4$, $c_3(z_2) = 3$ and finally colour $G_{copy'}(v_1, v_2)$ by the colouring c_2 with colours 1 and 2 and colours 3 and 4 swapped, i.e. $c_3(v_1) = c_3(v_2) = 1$.

The graph $G_{copy''}(x, y)$ satisfies almost all properties of $G_{copy}(x, y)$, except possibly for the last property that any proper colouring of x, y and their neighbours can be extended – we only constructed the colouring c_3 where $c_3(x) = c_3(y) = 2$, the colour of the neighbour of x is 3 and the colour of the neighbour of y is 4. We construct the graph $G_{copy}(x, y)$ by taking vertices x, z and y and adding copies of $G_{copy''}(x, z)$ and $G_{copy''}(z, y)$. Let v_x be the neighbour of x and v_y the neighbour of y in this graph. Let v_{zx} be the neighbour of z in $G_{copy''}(x, z)$ and v_{zy} be the neighbour of z in $G_{copy''}(z, y)$. Up to the permutation of colours, there are only two proper colourings of x, y, v_x and v_y – either $c(x) = c(y) = 1$, $c(v_x) = 2$,

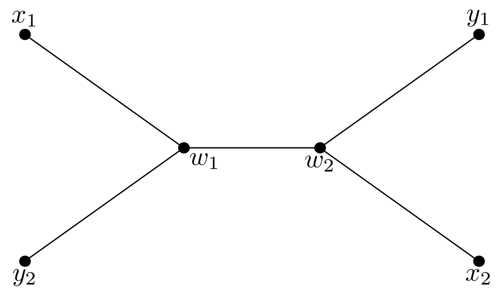

Figure 8: $G_{cross}(x_1, y_1, x_2, y_2)$

$c(v_y) = 3$ or $c(x) = c(y) = 1$, $c(v_x) = 2 = c(v_y) = 2$. In both cases we can colour $G_{copy''}(x, z)$ and $G_{copy''}(z, y)$ by the permutations of c_3 such that in addition to the prescribed colours on x, y, v_x and v_y, we have $c(v_{zx}) = 3$ and $c(v_{zy}) = 4$. This shows that indeed any precolouring of vertices x, y, v_x and v_y can be extended to a proper colouring of $G^2_{copy}(x, y)$ and finishes the proof of the lemma. $\square$

The main result of this section follows:

Theorem 25. *It is NP-complete to decide whether the square of a subcubic planar graph of girth* 9 *is* 4*-colourable.*

Proof. We use the copy gadget from Lemma 24. The special vertices x and y of $G_{copy}(x, y)$ have degree 1. Thus, we can form a binary tree where the edges are replaced by the copies of $G_{copy}(x, y)$, and copy the colour of the root of the tree to an arbitrary number of vertices of degree one. Note that by the properties of $G_{copy}(x, y)$, this graph does not impose any additional constraints on colours of the neighbours of these vertices.

The other gadget we need is a crossover gadget $G_{cross}(x_1, y_1, x_2, y_2)$, which consists of vertices x_1, y_1, x_2, y_2, w_1 and w_2 and edges x_1w_1, w_1y_2, w_1w_2, y_1w_2 and w_2x_2, see Figure 8. In any colouring c of the square of $G_{cross}(x_1, y_1, x_2, y_2)$ with $c(x_1) \neq c(y_1)$, it holds that $c(x_1) = c(x_2)$ and $c(y_1) = c(y_2)$, there exists such a colouring, and a cyclic order of the 1-vertices of the gadget on the outer face is x_1, y_1, x_2 and y_2. This gadget allows us to transfer information across the edges of the graph.

The proof of the NP-completeness proceeds by a reduction from the problem of 3-colouring of a planar graph, that is NP-complete due to Dailey [3]. Given an instance of this problem—a 2-connected planar graph G— we need to create a subcubic planar graph H of girth 9 of a size polynomial in $|V(G)|$ such that H^2 is 4-colourable if and only if G is 3-colourable. The idea of the construction is that we put a vertex into each face of G and force the colour of all these vertices to be the same, say 4. We then join them to all the vertices of G, so the original vertices must have colours 1, 2, or 3. Then we replace the edges by gadgets that ensure

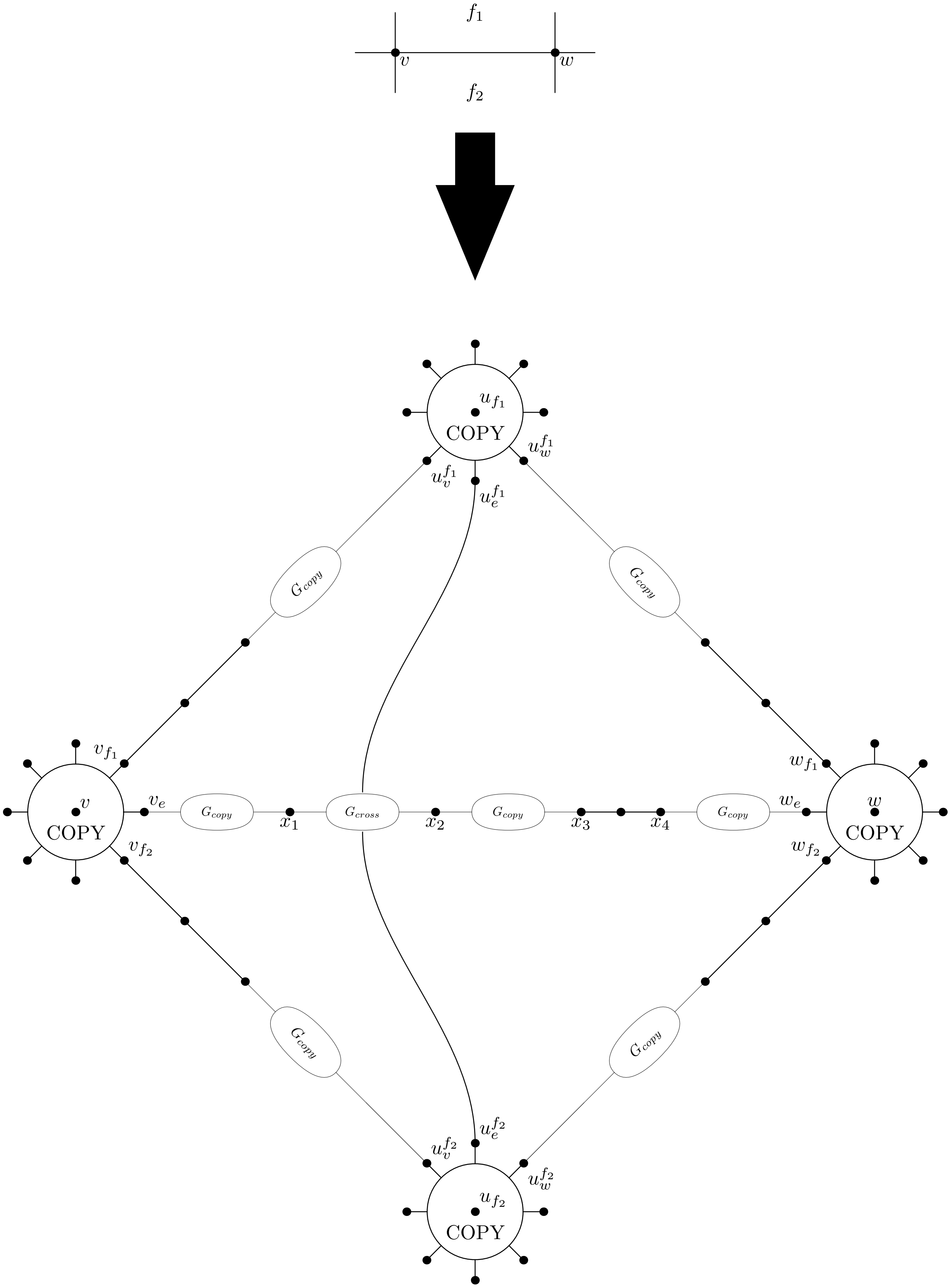

Figure 9: The NP-completeness reduction

large enough girth. A 4-colouring of H^2 then straightforwardly corresponds to a 3-colouring of G.

The graph H is constructed in the following way (see Figure 9): We replace each d-vertex v of G by a tree of copy gadgets that copy the colour of v to $2d$ vertices of degree one. Let e_1, f_1, e_2, f_2, $\ldots$, e_d and f_d be a cyclic order of the edges and the faces incident to v, and let v_{e_1}, v_{f_1}, $\ldots$, v_{e_d} and v_{f_d} be a cyclic order of the copies of v.

We add a vertex u_f for each ℓ-face f of G, and add a tree of copy gadgets that copy its colour to 2ℓ vertices $u^f_{v_1}$, $u^f_{e_1}$, $\ldots$, $u^f_{v_\ell}$ and $u^f_{e_\ell}$, where v_1, e_1, v_2, e_2, $\ldots$, v_ℓ and e_ℓ is a cyclic order of the vertices and the edges around the face f.

If a vertex v of G is incident to a face f, we add a new vertex x and a copy of the graph $G_{copy}(u^f_v, x)$ and join x and v_f by a 1-thread. This ensures that u_f and v have distinct colours.

If $e = vw$ is an edge of G incident to faces f_1 and f_2, we add vertices x_1, x_2, x_3 and x_4, the following copies of gadgets: $G_{copy}(v_e, x_1)$, $G_{cross}(x_1, u^{f_2}_e, x_2, u^{f_1}_e)$, $G_{copy}(x_2, x_3)$ and $G_{copy}(x_4, w_e)$, and join x_3 and x_4 by a 1-thread. Since we already know that u_{f_i} and v have distinct colours, this ensures that the vertices v and w have distinct colours, and that u_{f_1} and u_{f_2} have the same colour.

Given a proper 3-colouring G by the colours $\{1, 2, 3\}$, we can colour all the vertices u_f of H by the colour 4 and preserve the colouring of the copies of the vertices of G inside H. We extend this colouring to whole H^2 by the properties of the gadgets. Conversely, in a proper 4-colouring of H^2, all the vertices u_f have the same colour (say 4) and the colours of the original vertices of G are distinct from it, i.e., belong to $\{1, 2, 3\}$. It follows that this colouring restricted to the vertices of G is a proper 3-colouring of G.

The construction of H can be performed in a polynomial time, and H is a subcubic planar graph of girth 9. This finishes the proof of the NP-completeness of the problem. □

7 Conclusion

The gap between the upper bounds and the examples showing the lower bounds obtained in Sections 4 and 5 is large and it would be of its own interest to find sharper bounds. In particular we believe that the upper bounds can be improved significantly by taking more complicated reducible configurations into account.

We were unable to find a 7-minimal planar subcubic graph of girth larger than three. So we pose the following conjecture:

Conjecture 2. *The square of every triangle-free planar subcubic graph is 6-colourable.*

By subdividing some of the edges of the graph in Figure 3, we can obtain an example showing that the conjecture is tight.

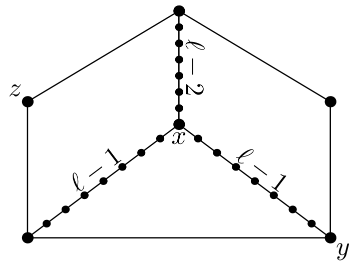

Figure 10: A graph that is not ℓ-facial colourable with $3\ell-1$ colours

In fact we propose the following more general conjecture. A colouring c of a plane graph is an ℓ-facial colouring if any two distinct vertices on a facial walk of length ℓ have distinct colours. Notice that 1-facial colouring is the usual colouring. Also notice that for subcubic graphs 2-facial colouring corresponds to colouring of the square of the graph. This concept is introduced in [9]. They pose also the following conjecture:

$(3\ell+1)$**-Conjecture.** *Let G be a plane graph and $\ell \geq 1$. Then, G is ℓ-facial colourable by $3\ell+1$ colours.*

For $\ell = 1$ the conjecture is equivalent to the Four Colours Theorem. For $\ell = 2$ we obtain Wegner's conjecture for subcubic graphs. If the conjecture is true, it is tight for all values of ℓ. In a same fashion, we propose the following 3ℓ-conjecture:

3ℓ**-Conjecture.** *Let G be a plane triangle-free graph and $\ell \geq 1$. Then G is ℓ-facial colourable by 3ℓ colours.*

For $\ell = 1$ this statement is equivalent to Grötzsch's Theorem. For $\ell = 2$, it implies Conjecture 2. Note that the bound in this conjecture is tight, as witnessed by graphs depicted in Figure 10. All the faces of the graph form cliques for the purposes of ℓ-facial colouring, and at most one of vertices in thread between x and y may have the same colour as vertex z.

If 3ℓ-Conjecture and Conjecture 2 are not false, then they are probably hard. The following nice consequence of these two conjectures might be easier for consideration.

Conjecture 3. *The square of every bipartite planar subcubic graph is 6-colourable.*

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