Brooks Theorem for Generalized Dart Graphs*

Martin Kochol†

MÚ SAV, Štefánikova 49, 814 73 Bratislava 1, Slovakia
martin.kochol@mat.savba.sk

Riste Škrekovski ‡

Department of Mathematics, University of Ljubljana,
Jadranska 19, 1111 Ljubljana, Slovenia
skrekovski@gmail.com

Abstract

The well-known Brooks' Theorem says that each graph G of maximum degree $k \geq 3$ is k-colorable unless $G = K_{k+1}$. We generalize this theorem by allowing higher degree vertices with prescribed types of neighborhood.

1 Introduction

A *k-coloring* of a graph is a mapping from the set of vertices to $\{1, \dots, k\}$ such that any two adjacent vertices have different colors. The decision problem whether a given graph G has a k-coloring is a classical NP-complete problem for every fixed $k \geq 3$ (see [3, 4]).

By Brooks' Theorem [1], every graph with maximum vertex degree at most $k \geq 3$ and without a component isomorphic to K_{k+1} (a complete graph on $k + 1$ vertices) has a k-coloring. Furthermore, as follows from [2, 6, 7, 8, 9], there exists a linear-time algorithm that finds a k-coloring for such a graph.

Kochol, Lozin, and Randerath [6, Theorem 4.3] proved that if $\mathcal{D}$ is a class of graphs in which the neighborhood of each 4-degree vertex induces a graph isomorphic to a disjoint union of an isolated vertex and a path of length 2, then every graph from $\mathcal{D}$ is either 3-colorable or has a component isomorphic to K_4. Furthermore, there exists

*These results were presented in IWOCA 2010.

†Supported in part by grant VEGA 2/0118/10.

‡Supported in part by ARRS Research Program P1-0297.

keywords: (k, s)-dart graph, graph algorithms, NP-complete problem, (k, s)-diamond, Brooks' Theorem.

a linear-time algorithm that finds either a 3-coloring or a component isomorphic to K_4 for each graph from $\mathcal{D}$. This generalizes the Brooks' Theorem for the case $k = 3$.

The aim of this paper is to generalize the Brooks' Theorem and the result from [6, Theorem 4.3]. We consider classes of graphs where each vertex of degree at least $k+2$ has a strictly prescribed neighborhood, so called "(k,s)-dart graphs", defined in the following section. Our main result, Theorem 1, is that if G is a (k,s)-dart graph, $k \geq \max\{3,s\}$, and $s \geq 2$, then G is $(k+1)$-colorable if and only if it has no component isomorphic to K_{k+2}. Furthermore, if G is $(k+1)$-colorable, then a $(k+1)$-coloring of G can be constructed in a linear time. We also show that if $s > k \geq 3$, then it is an NP-complete problem to decide whether a (k,s)-dart graph is $(k+1)$-colorable (see Theorem 2).

2 Definitions

In this paper we consider simple graphs, i.e., without multiple edges and loops. If G is a graph, then $V(G)$ and $E(G)$ denote the vertex and the edge sets of G, respectively.

Let G be a graph and x, y two vertices of G. Then $G + xy$ denotes the graph constructed from G by adding an edge xy. Since we consider simple graphs, $G + xy = G$ if x, y are adjacent in G. For a vertex v of G, let $d_G(v)$ denote the degree of v in G. Let H, G be two graphs such that no subgraph of G is isomorphic with H. Then we say that G is a H-*free* graph.

A (k,s)-*diamond* is a join of a clique of size $k \geq 1$ and an independent set of size $s \geq 1$. Notice that these graphs are edge-maximal split graphs. In a (k,s)-diamond D, vertices that belong to the independent set are called *pick* vertices, and the remaining (i.e. those in the k-clique) are called *central* vertices. Denote by $C(D)$ and $P(D)$ the sets of central vertices and pick vertices of D, respectively. An example of a $(4,3)$-diamond D with $C(D) = \{c_1, \ldots, c_4\}$ and $P(D) = \{p_1, p_2, p_3\}$ is in Figure 1.

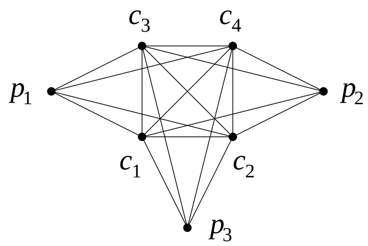

Figure 1: A $(4,3)$-diamond.

Note that a $(k,1)$-diamond is isomorphic to K_{k+1}; in this case the unique pick vertex does not distinguish from the central vertices. This is irrelevant for us, because in this paper we deal only with (k,i)-diamonds where $i \geq 2$.

Definition 1 *A graph G is a (k,s)-*dart *if each vertex of G of degree $\geq k+2$ is a central vertex of some (k,i)-diamond D as an induced subgraph of G with $2 \leq i \leq s$, for which*

(a) $d_D(x) \geq d_G(x) - 1$ *for each* $x \in V(D)$*;*

(b) *no two vertices of* $C(D)$ *have a common neighbor in* $G - D$*.*

The following remarks related to Definition 1 are straightforward:

(1) Inequality $i \geq 2$ can be removed in Definition 1, because it follows from the fact that D contains a vertex of degree $\geq k+2$.

(2) Every graph of maximum degree $\leq k+1$ is a $(k,1)$-dart graph since in Definition 1, we only prescribe the structure on the neighborhood of vertices of higher degree.

(3) Every (k, s_1)-dart graph is a (k, s_2)-dart if $s_1 \leq s_2$.

Notice that $(2,2)$-diamonds and $(2,2)$-dart graphs are called diamonds and dart graphs, respectively, in [6]. By a *generalized dart graph* and *generalized diamond* we mean any (k,s)-dart graphs and any (k,s)-diamond, $k,s \geq 2$, respectively. In this paper we usually omit the word generalized, if it is clear from the context which term we have in mind.

In a (k,s)-dart graph G, every vertex of degree at least $k+2$ belongs to an induced (k,i)-diamond with $2 \leq i \leq s$. Denote by $\mathcal{D}(G)$ the set of all induced maximal (k,i)-diamonds of G with $i \geq 2$.

We say that a vertex of a (k,s)-dart graph G is *central*, if it is a central vertex of a (k,i)-diamond of $\mathcal{D}(G)$, $i \leq s$. Similarly define a *pick* vertex of G. Denote the sets of central vertices and pick vertices by $C(G)$ and $P(G)$, respectively.

Let G be a (k,s)-dart graph and $D \in \mathcal{D}(G)$. Then, each central vertex $x \in C(D)$ is adjacent to at most one vertex v' from $G - D$. In this case, v' is called an *isolated* neighbor of v. The set of all isolated neighbors of the central vertices of D is denoted by $I(D)$. Notice that the possibility that $I(D) = \emptyset$ is not excluded.

We remark that the following observations for a (k,s)-dart graph G hold:

(4) A central vertex v of a (k,s)-dart graph G is not necessarily of degree at least $k+2$. This happens only if v is a central vertex of a $(k,2)$-diamond $D \in \mathcal{D}(G)$ and it has no neighbor in $G - D$. Then, v is of degree $k+1$. The possibility that all central vertices of D are of degree $k+1$ is not excluded.

(5) If K_{k+2} is a subgraph of a (k,s)-dart graph G, then it must be a component of G. Thus a copy of K_{k+2} in G is disjoint from diamonds of $\mathcal{D}(G)$.

(6) No two pick vertices of the same diamond from $\mathcal{D}(G)$ are adjacent.

3 Properties of dart graphs

The next lemma assures that diamonds in a dart graph are vertex disjoint.

Lemma 1 *Let* G *be a* (k,s)*-dart graph with* $k \geq 3$*. Then*

(a) *$V(D_1) \cap V(D_2) = \emptyset$, for every two distinct diamonds $D_1, D_2 \in \mathcal{D}(G)$.*

(b) *$C(G) \cap P(G) = \emptyset$; in particular each pick vertex is of degree k or $k+1$.*

Proof. We prove (a). Suppose that v is a vertex of two distinct diamonds $D_1, D_2 \in \mathcal{D}(G)$.

Assume that $v \in C(D_1) \cap C(D_2)$. If $C(D_1) = C(D_2)$, then by Definition 1(b) we obtain that $P(D_1) = P(D_2)$, whence $D_1 = D_2$. Thus $C(D_1) \neq C(D_2)$.

Suppose first $|C(D_1) \cap C(D_2)| = 1$, i.e., $C(D_1) \cap C(D_2) = \{v\}$. Then by Definition 1, either $k-2$ or $k-1$ vertices of $C(D_2)$ (resp. $C(D_1)$) are pick vertices of D_1 (resp. D_2). But then for $k \geq 4$, we obtain also two adjacent pick vertices of D_1 (resp. D_2), a contradiction to (6). So we may assume that $k = 3$, $C(D_1) = \{u_1, w_1, v\}$, $C(D_2) = \{u_2, w_2, v\}$, and u_1 (resp. u_2) are pick vertices of D_2 (resp. D_1). By (6), w_1 (resp. w_2) is not a pick vertex of D_2 (resp. D_1). Then $w_1 \in I(D_2)$ (resp. $w_2 \in I(D_1)$) is a common neighbor of $v, u_2 \in C(D_2)$ (resp. $v, u_1 \in C(D_1)$), a contradiction with Definition 1(b).

Suppose now $|C(D_1) \cap C(D_2)| \geq 2$. Then each vertex $u \in C(D_1) \setminus C(D_2)$ is a neighbor of at least two vertices from $C(D_2)$, whence by Definition 1(b), $u \in P(D_2)$ and thus $C(D_1) \setminus C(D_2) \subseteq P(D_2)$. Similarly $C(D_2) \setminus C(D_1) \subseteq P(D_1)$. Thus the subgraph of G induced by $C(D_1) \cup C(D_2)$ is a clique, whence $|C(D_1) \cup C(D_2)| = k+1$, and so $|C(D_1) \cap C(D_2)| = k-1$. By assumption, D_1 is a (k, s_1)-diamond, $s \geq s_1 \geq 2$. Thus there exists $x_1 \in P(D_1) \setminus C(D_2)$. By (6), we infer that $x_1 \in I(D_2)$, but then it is a common neighbor of at least two vertices from $C(D_2)$, a contradiction with Definition 1(b).

By the above two paragraphs, we can assume that $C(D_1) \cap C(D_2) = \emptyset$. If $v \in V(D_1) \cap P(D_2)$, then $d_{D_2}(v) + 1 < d_G(v)$, a contradiction with Definition 1(a). Similarly if $v \in V(D_2) \cap P(D_1)$. This proves (a). Claim (b) is an easy consequence of (a). □

In the next few lemmas, we study properties of a graph G' obtained from G by applying some local modifications.

Lemma 2 *Let G be a K_{k+2}-free (k, s)-dart graph with $k \geq 3$ and let $D \in \mathcal{D}(G)$. Suppose that a_1, a_2 are two central vertices of D and let x_1, x_2 be their isolated neighbors, respectively. Then the graph $G' = G - x_1a_1 - x_2a_2 + x_1x_2$ is a K_{k+2}-free graph unless there exists $D' \in \mathcal{D}(G)$ such that x_1, x_2 are pick vertices of D'.*

Proof. Suppose that G' contains a copy H of K_{k+2}. Then, x_1, x_2 are vertices of H, thus cannot be adjacent in G and there is a set S of k common neighbors of x_1 and x_2 in G, which induce a clique. Notice that $|S| = k$ and $d_G(x_1), d_G(x_2) \geq k+1$.

Suppose that $d_G(x_1) \geq k+2$. Then, x_1 is a central vertex of some diamond $D' \in \mathcal{D}(G)$, whence by Definition 1(b), $S \subseteq V(D')$ and clearly, $|S \cap C(D')| \geq k-1 \geq 2$. Then x_2 has at least 2 neighbors in $C(D')$, whence x_2 belongs to D', and so it is adjacent to x_1 in G, a contradiction.

Thus, by previous paragraph, we may assume that $d(x_1) = k+1$, and analogously $d(x_2) = k+1$. Then x_1, x_2 and S belong to a diamond $D' \in \mathcal{D}(G)$ in which $x_1, x_2 \in P(D')$ and $S = C(D')$. □

Lemma 3 *Let G be a (k,s)-dart graph and let $D \in \mathcal{D}(G)$. Supose that a_1, a_2 are two central vertices of D and let x_1, x_2 be their isolated neighbors, respectively. Then the graph $G' = G - x_1a_1 - a_2x_2 + x_1x_2$ is a (k,s)-dart graph unless one of the following conditions occurs:*

(7) *there exists $D' \in \mathcal{D}(G)$ such that x_1, x_2 are pick vertices of D';*

(8) *there exists $D' \in \mathcal{D}(G)$ and $i \in \{1,2\}$ such that $x_i \in C(D')$ and x_{3-i} is an isolated neighbor of a central vertex from D', which is distinct from x_i.*

Proof. Suppose that G' is not a (k,s)-dart graph. Each vertex preserve its degree from G except a_1, a_2, which belong to D. Notice that D is a diamond in G' as well. If there is some $D' \in \mathcal{D}(G)$ that is not induced diamond of G', then x_1 and x_2 must be pick vertices of D', which gives case (7).

Thus each diamond $D' \in \mathcal{D}(G)$ is an induced diamond of G'. Clearly D' satisfies Definition 1(a) in G'. If D' does not satisfy Definition 1(b) in G', then there are two central vertices u and v of D' with a common neighbor w outside D'. Notice that x_1x_2 is one of the edges uw or vw. Then without loss of generality, we may assume that x_1 is a central vertex in D' and x_2 is an isolated neighbor of a central vertex of D' distinct from x_1, which gives case (8). □

Notice that in the exceptional case (7) of the above lemma, G' may still be a dart graph, when x_1, x_2 are pick vertices of a $(k,2)$-diamond D' with no isolated vertices. Then, D' becomes a copy of K_{k+2} in G'.

4 An extension of Brooks theorem

For a diamond $D \in \mathcal{D}(G)$, a vertex of $I(D)$ could be a central or pick vertex of another diamond of $\mathcal{D}(G)$. Denote by $I_c(D)$ and $I_p(D)$ the subset of all such vertices of $I(D)$, respectively. By Lemma 1(b), sets $I_c(D)$ and $I_p(D)$ are disjoint. Finally, let $I_s(D)$ be the vertices of $I(D)$ that are neither in $I_c(D)$, nor in $I_p(D)$.

Lemma 4 *Suppose that we have a K_{k+2}-free (k,s)-dart graph G, $k \geq \max\{3,s\}$, $s \geq 2$, together with the set $\mathcal{D}(G) \neq \emptyset$. Then we can find $D \in \mathcal{D}(G)$ and construct a K_{k+2}-free (k,s)-dart graph G^* together with $\mathcal{D}(G^*)$ in $O(|E(D)|)$ time such that*

(a) $|\mathcal{D}(G^*)| < |\mathcal{D}(G)|$;

(b) $|E(G^*)| \leq |E(G)| - |E(D)|$;

(c) *From any $(k+1)$-coloring λ of G^* one can construct a $(k+1)$-coloring of G in $O(|E(D)|)$ time.*

Proof. Consider a (k,i)-diamond $D' \in \mathcal{D}(G)$, $2 \leq i \leq s$, and check three cases:

Case 1. $|I(D')| < k$. Thus there exists $v \in C(D')$ having no isolated neighbor. In this case we take $D := D'$ and $G^* := G - D'$. Suppose that u' is an arbitrary vertex of degree $\geq k+2$ in G^*. Then, it is also of degree $\geq k+2$ in G, and hence it belongs to a (k,i)-diamond $D'' \in \mathcal{D}(G)$ with $2 \leq i \leq s$. Diamonds D and D'' are disjoint, by Lemma 1, and hence D'' is an induced (k,s)-diamond in G^*. Furthermore, Lemma 1 assures that $\mathcal{D}(G)$ consists of D and $\mathcal{D}(G^*)$. Thus G^* is a (k,s)-dart graph. Obviously, G^* is a K_{k+2}-free graph and $|E(G^*)| \leq |E(G)| - |E(D)|$.

Let λ^* be a $(k+1)$-coloring of G^*. Since every pick vertex of D has at most one neighbor outside D and since $|P(D)| \leq k$, it follows that there exists a color c that we can assign to all pick vertices of D. Denote by $u_1, \ldots, u_{k-1}$ the vertices from $C(D) \setminus \{v\}$ and take $u_k := v$. For $i = 1, \ldots, k$, take $L(u) = \{1, \ldots, k+1\} \setminus \{c, \lambda^*(x_i)\}$ if u_i has an isolated neighbor x_i, otherwise take $L(u) = \{1, \ldots, k+1\} \setminus \{c\}$. Thus $k \geq |L(u_i)| \geq k-1$ for $i < k$ and $|L(u_k)| = k$ (because $u_k = v$ has no isolated neighbor). For $i = 1, \ldots, k$ we assign u_i a color from $L(u_i)$ and remove this color from all $L(u_j)$ where $j > i$. Clearly, each $L(u_i)$ is nonempty after $i-1$ steps, thus this process gives a coloring λ of G, and can be done in $O(|E(D)|)$ time.

Case 2. $|I(D')| = k$ *and* $I(D')$ *does not consist of pick vertices of one diamond of* $\mathcal{D}(G)$. Suppose that each pair $x_1, x_2 \in I(D')$ satisfies either (7), or (8). This implies immediately that $|I_c(D')| \leq 1$ and $|I_s(D')| \leq 1$. Thus $|I_p(D')| \geq 1$ (because $k \geq 3$). Each $x_1 \in I_s(D') \cup I_c(D')$ and $x_2 \in I_p(D')$ satisfy neither (7), nor (8), whence $I_s(D') \cup I_c(D') = \emptyset$. Thus all vertices of $I(D')$ must be pick vertices of one diamond of $\mathcal{D}(G)$. This contradicts the assumption of Case 2.

Thus there exist two distinct vertices $x_1, x_2 \in I(D')$ satisfying neither (7), nor (8). To find them is an easy process. Take $x_1 \in I_s(D') \cup I_c(D')$ and $x_2 \in I_p(D')$ if possible. If $I_p(D') = \emptyset$, then either $|I_s(D')| \geq 2$, or $|I_c(D')| \geq 2$, and we can choose x_1, x_2 from one of them. If $I_s(D') \cup I_c(D') = \emptyset$, $I_p(D')$ has at least two vertices from different diamonds of $\mathcal{D}(G)$, and choose them.

After choosing x_1, x_2, take the graph $G' = G - x_1a_1 - x_2a_2 + x_1x_2$. By Lemmas 2 and 3, G' is a K_{k+2}-free (k,s)-dart graph. Moreover, $|E(G')| < |E(G)|$ and $\mathcal{D}(G) = \mathcal{D}(G')$. $D' \in \mathcal{D}(G')$ but the number of isolated vertices of D' in G' is smaller then k. Thus we can apply the construction from Case 1 for G' and D', i.e., we take $D := D'$ and $G^* := G' - D'$. Analogously as in Case 1, G^* is a K_{k+2}-free (k,s)-dart graph, $|E(G^*)| \leq |E(G)| - |E(D)|$ and $\mathcal{D}(G^*) = \mathcal{D}(G) \setminus \{D\}$.

Let λ^* be a $(k+1)$-coloring of G^*. Applying the process described in Case 1, we get a $(k+1)$-coloring λ' of G'. Clearly $\lambda'(a_1) \neq \lambda'(a_2)$ and $\lambda'(x_1) \neq \lambda'(x_2)$. By Definition 1, a_1 and x_2 are non-adjacent, and similarly a_2 and x_1 are non-adjacent. Notice that λ' is not a coloring of G if and only if $\lambda'(a_1) = \lambda'(x_1)$ or $\lambda'(a_2) = \lambda'(x_2)$. But in that case, we can simply interchange the colors of a_1 and a_2, and obtain a proper $(k+1)$-coloring λ of G. Furthermore, λ^* can be transformed to λ in $O(|E(D)|)$ time.

Case 3. $|I(D')| = k$ *and* $I(D')$ *consists of pick vertices of some* $D'' \in \mathcal{D}(G)$. Now D'' is a (k,k)-diamond, because there exists a perfect matching between $C(D')$ and $P(D'')$. Thus $s = k$ and $|E(D')| \leq |E(D'')|$ (because D' is a (k,i)-diamond where $i \leq k = s$). If Cases 1 or 2 are satisfied for D'', we set $D = D''$ and apply the constructions described in these cases for D and obtain G^* with required properties. Otherwise $|I(D'')| = k$ and $I(D'')$ consists of pick vertices of some $D''' \in \mathcal{D}(G)$. We consider two subcases:

Case 3.1. $D''' = D'$. Then vertices of D' and D'' induce a component G' of G. In this case we take $D := D'$ and $G^* := G - G'$. Notice that G^* is a (k,s)-dart graph, $|E(G^*)|+2|E(D')| = |E(G)|$ and $\mathcal{D}(G^*) = \mathcal{D}(G)\setminus\{D', D''\}$. Moreover, we can construct a $(k+1)$-coloring of G' in $O(k)$ time: just color all vertices of $P(D')$ and $P(D'')$ by the color $k+1$, and assign colors $1,\ldots,k$ to the vertices of $C(D')$ and $C(D'')$.

Case 3.2. $D''' \neq D'$. In this case we take $D := D''$ and set G^* to be the graph we obtain by removing the vertices of D'' and inserting a perfect matching between $C(D')$ and $P(D''')$. Then G^* is a (k,s)-dart graph, $|E(G^*)| + |E(D)| = |E(G)|$ and $\mathcal{D}(G^*) = \mathcal{D}(G) \setminus \{D\}$. Let λ^* be a $(k+1)$-coloring of G^*. Then λ^* assigns the same color c to all vertices of $P(D''')$. Assign c also to all vertices of $P(D'')$ and to each of the vertices of $C(D'')$ an unique color from $\{1,\ldots,k+1\} \setminus \{c\}$. This gives a required coloring of G.

Clearly, we can check in $O(k)$ time whether $I(D')$ has cardinality k or satisfies the conditions required in Cases 1, 2, 3.1, and 3.2. Thus all reductions from G to G^* and transformations of $k+1$-colorings of G^* to $k+1$-colorings of G can be done in $O(|E(D)|)$ time. This implies the statement. □

Notice that G^* from Lemma 4 also satisfy $|V(G^*)| \leq |V(G)| - |V(D)|$.

Now we are ready to prove the main result.

Theorem 1 *Let G be a (k,s)-dart graph where $s \geq 2$ and $k \geq \max\{3,s\}$ are arbitrary but fixed integers. Then G is $(k+1)$-colorable if and only if it has no component isomorphic to K_{k+2}. Furthermore, if G is $(k+1)$-colorable, then a $(k+1)$-coloring of G can be constructed in $O(|E(G)|)$ time.*

Proof. The necessity of the first part of the theorem is trivial. We prove sufficiency and the second part of the theorem. Let G be a (k,s)-dart graph. We can check in $O(|E(G)|)$ (linear) time whether G is K_{k+2}-free. Analogously, we can find the set $\mathcal{D}(G)$ in linear time. Consequently, by means of Lemma 4 we can create in linear time a K_{k+2}-free graph G' without vertices of degree more than $k+1$ such that any $(k+1)$-coloring of G' can be transformed into a $(k+1)$-coloring of G in linear time. By [7] (see also [9, 6]), a $(k+1)$-coloring of G' can be found in linear time. □

Notice that if v is a vertex of a (k,s)-dart graph G of degree at least $k+2$ and $N(v)$ is the set of its incident vertices, then the graph induced by $N(v) \cup \{v\}$ is a (k,i)-diamond $(2 \leq i \leq s)$ with a possible pending edge. A similar property have central vertices of G of degree $k+1$. Thus the problem to find $\mathcal{D}(G)$ in G is much easier then to find a maximal clique in a graph (a known NP-hard problem, see [3]). Also it is a trivial problem to determine in time $O(|E(G)|)$ whether a graph G is a (k,s)-dart graph (where k is arbitrary but fixed integer $\leq |V(G)|$).

5 NP-Completeness

In this section we show that Theorem 1 cannot be extended for (k,s)-dart graphs where $s > k \geq 2$ unless P = NP.

We need some more notation. Take n vertex disjoint copies of $(k, k+1)$-diamonds $D_1, \ldots, D_n$, $k, n \geq 2$. For $i = 1, \ldots, n$, denote by $v_{i,1}, \ldots, v_{i,k}$ and $u_{i,1}, \ldots, u_{i,k+1}$ the central and pick vertices of D_i, respectively. Add nk new edges $v_{i,j}u_{i+1,j}$, $i = 1 \ldots, n$, $j = 1, \ldots, k$ (considering the sum $i+1$ mod n). Then the resulting graph is called a $(n, k+1)$-*bracelet* and vertices $u_{1,k+1}, \ldots, u_{n,k+1}$ are called its *connectors*. An example of a $(4, 3)$-bracelet with connectors $u_{1,3}, \ldots, u_{4,3}$ is in Figure 2.

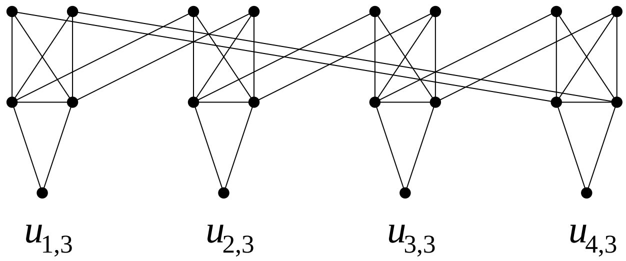

Figure 2: A $(4, 3)$-bracelet.

Lemma 5 *Let G be a $(n, k+1)$-bracelet, $n, k \geq 2$. Then in any $(k+1)$-coloring of G, all connectors of G have the same color.*

Proof. By the above construction, G is composed from n vertex disjoint copies of $(k, k+1)$-diamonds $D_1, \ldots, D_n$. Consider a $(k+1)$-coloring of G. For every $i \in \{1, \ldots, n\}$, the central vertices of D_i form a clique of order k, whence must be colored by k different colors, and thus all pick vertices of D_i have the same color. Furthermore, each central vertex of D_i is adjacent with a pick vertex of D_{i+1}. Therefore all vertices from $P(D_1) \cup \ldots \cup P(D_n)$ have the same color, thus also the connectors of G. □

We study complexity of the following problem.

(k, s)-**DART-**$(k+1)$**-COL**
Instance: A (k, s)-dart graph G.
Question: Is G $k+1$-colorable?

Theorem 2 *The problem* (k, s)-DART-$(k+1)$-COL, $k \geq 2$, *is*

(a) NP-*complete for* $s > k$,

(b) *solvable in linear time for* $2 \leq s \leq k$.

Claim (b) holds true by Theorem 1 for $k \geq 3$ and by [6, Theorem 4.3] for $k = 2$.

We prove (a). Let G be a graph. Replace each vertex v of G of degree ≥ 2 by a $(d_G(v), k+1)$-bracelet H_v. Let H_v be an isolated vertex if $d_G(v) = 1$. Each edge uv of G replace by an edge joining a connector of H_v with a connector of H_u so that each connector is attached to at most one new edge. Denote the resulting graph by G'. Clearly, G' is a $(k, k+1)$-dart graph. From Lemma 5 it follows that by any $(k+1)$-coloring G', all connectors of H_v, $v \in V(G)$, must be colored by the same color. Hence

G' is $(k+1)$-colorable if and only if G is so. Thus the problem whether a $(k, k+1)$-dart graph is $k+1$-colorable can be polynomially reduced to the problem of $(k+1)$-coloring. This problem is NP-complete for every fixed $k \geq 2$ by Garey and Johnson [3, GT4]. This proves claim (a). □

Acknowledgement. Authors thank unknown referees for helpful comments.

References

[1] R. L. Brooks, *On coloring the nodes of a network*, Proc. Cambrigde Phil. Soc. 37 (1941) 194–197.

[2] V. Bryant, *A characterisation of some 2-connected graphs and a comment on an algorithmic proof of Brooks' theorem*, Discrete Math. 158 (1996) 279–281.

[3] M. R. Garey and D. S. Johnson, *Computers and Intractability*, W.H. Freeman, San Francisco, 1979.

[4] M. R. Garey, D. S. Johnson, and L. Stockmeyer, *Some simplified NP-complete graph problems*, Theor. Comput. Sci. 1 (1976), 237–267.

[5] R. Diestel, *Graph Theory,* 3rd ed., Springer, Heidelberg, 2005.

[6] M. Kochol, V. Lozin, and B. Randerath, *The 3-colorability problem on graphs with maximum degree four*, SIAM J. Comput. 32 (2003) 1128–1139.

[7] L. Lovász, *Three short proofs in graph theory*, J. Combin. Theory Ser. B 19 (1975) 269–271.

[8] B. Randerath and I. Schiermeyer, *A note on Brooks theorem for triangle-free graphs*, Australasian J. Combin. 26 (2002), 3–10.

[9] S. Skulrattanakulchai, *Δ-list vertex coloring in linear time*, in Algorithm Theory – SWAT 2002, M. Penttonen and E. Meineche Schmidt, eds., Lecture Notes in Comput. Sci., Vol. 2368, Springer-Verlag, New York, 2003, 240–248.