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ACYCLIC EDGE COLORING OF PLANAR GRAPHS WITH Δ COLORS

Dávid Hudák František Kardoš
Borut Lužar Roman Soták

Riste Škrekovski

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Acyclic edge coloring of planar graphs with Δ colors

Dávid Hudák* František Kardoš* Borut Lužar† Roman Soták*

Riste Škrekovski‡

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Abstract

An acyclic edge coloring of a graph is a proper edge coloring without bichromatic cycles. In 1978, it was conjectured that $\Delta(G) + 2$ colors suffice for an acyclic edge coloring of every graph G [6]. The conjecture has been verified for several classes of graphs, however, the best known upper bound for as special class as planar graphs are, is $\Delta + 12$ [2]. In this paper, we study simple planar graphs which need only $\Delta(G)$ colors for an acyclic edge coloring. We show that a planar graph with girth g and maximum degree Δ admits such acyclic edge coloring if $g \geq 12$, or $g \geq 8$ and $\Delta \geq 4$, or $g \geq 7$ and $\Delta \geq 5$, or $g \geq 6$ and $\Delta \geq 6$, or $g \geq 5$ and $\Delta \geq 10$. Our results improve some previously known bounds.

Keywords: Acyclic edge coloring, Planar graph, Discharging method

1 Introduction

An *acyclic edge coloring* of a graph G is a proper edge coloring with an additional condition that any pair of colors induces a linear forest (an acyclic graph with maximum degree two); in other words, there are no bichromatic cycles in G. The least number $\chi'_a(G)$ of colors for which G admits an acyclic edge coloring is called *acyclic chromatic index*. Since graphs with parallel edges do not admit acyclic edge colorings, in this paper we consider only simple graphs.

The acyclic coloring was first introduced for vertices of graphs by Grünbaum [8] and has been later extended to edges. Since acyclic edge coloring is also proper, we have the inequality $\chi(G)' \leq \chi'_a(G)$ for every graph G. It is well known that the chromatic index of graphs is at least $\Delta(G)$ and at most $\Delta(G) + 1$, what was proved by Vizing [13]. For an acyclic chromatic index a similar bound is believed to be true.

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‡Department of Mathematics, Faculty of Mathematics and Physics, University of Ljubljana. Partially supported by Ministry of Science and Technology of Slovenia, Research Program P1-0297.

Conjecture 1 ([6, 1]). *For every graph G it holds that*

$$\Delta(G) \leq \chi'_a \leq \Delta(G) + 2\,.$$

An analysis of cycles in graphs is a hard task, thus it is not surprising that the best known upper bound for an acyclic chromatic index is $16\Delta(G)$, which has been proved by Molloy and Reed [10]. However, Conjecture 1 has been verified for several classes of graphs. The first result is due to Burnstein [5], who proved that every graph with maximum degree 4 has an acyclic vertex coloring with 5 colors. Since the maximum degree of a line graph $L(G)$ of a subcubic graph G is at most 4, and since an acyclic edge coloring of a graph G is in fact an acyclic vertex coloring of $L(G)$, it follows that for every subcubic graph G, we have $\chi'_a(G) \leq 5 = \Delta + 2$. Note that $\chi'_a(G) \leq 3$ if $\Delta(G) = 2$.

Furthermore, in [1] Conjecture 1 has been proved for almost all d-regular graphs and for all d-regular graphs with girth at least $c\Delta(G)\log\Delta(G)$, where c is a constant. Recently, Basavaraju and Chandran [3] proved that the conjecture also holds for complete bipartite graphs $K_{p,p}$, where p is an odd prime.

Since 2008, the acyclic edge coloring of planar graphs has received a lot of attention. Fiedorowicz, Hałuszczak, and Narayanan [7] proved that $\chi'_a(G) \leq 2\Delta(G) + 29$ for every planar graph G, and if the girth of G is at least 4, the bound reduces to $\Delta + 6$. In the same year, Sun and Wu [12] verified Conjecture 1 for planar graphs without k-cycles, where $k \in \{4, 5, 6, 7, 8\}$, and planar graphs without 4- and 5-cycles in which no two 3-cycles share a vertex. In 2009, Hou et al. [9] improved the bound for planar graphs to $\max\{2\Delta(G) - 2, \Delta + 22\}$. They also studied planar graphs with specified girth and maximum degree. They showed the following theorem.

Theorem 1 ([9]). *Let G be a planar graph with maximum degree Δ and girth g. Then*

1. $\chi'_a(G) \leq \Delta + 2$ *if* $g \geq 5$;
2. $\chi'_a(G) \leq \Delta + 1$ *if* $g \geq 7$;
3. $\chi'_a(G) = \Delta$ *if* $g \geq 16$ *and* $\Delta \geq 3$.

Moreover, they proved that $\Delta + 1$ color suffice for an acyclic edge coloring of a series-parallel graph G.

Cohen, Havet, and Müller [11] proved another bound for acyclic chromatic index of planar graphs. They showed that $\chi'_a(G) \leq \Delta(G) + 25$. They also posed the following conjecture.

Conjecture 2 ([11]). *There exists an integer Δ for which every planar graph G with maximum degree $\Delta(G) \geq \Delta$ admits an acyclic edge coloring with $\Delta(G)$ colors.*

This is an analogue to the conjecture of Vizing [13] which says that all planar graphs with maximum degree at least 6 are Δ-edge colorable.

The upper bound for acyclic chromatic index of planar graphs has been recently improved by Basavaraju and Chandran [3]. They proved that $\chi'_a(G) \leq \Delta + 12$ for every planar graph G.

Furthermore, the bounds for planar graphs with specified girth were improved by Yu, Hou, Liu, Liu, and Xu [14]. They proved the two theorems below.

Theorem 2 ([14]). *Let G be a planar graph with girth g and maximum degree Δ. Then $\chi'_a(G) \leq \Delta + 1$ if at least one of the conditions below holds:*

1. $g \geq 6$, *or*

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2. $g \geq 5$ *and* $\Delta \geq 11$.

Theorem 3 ([14]). *Let G be a planar graph with girth g and maximum degree Δ. Then $\chi'_a(G) = \Delta$ if at least one of the conditions below holds:*

1. $\Delta \geq 4$ *and* $g \geq 12$, *or*
2. $\Delta \geq 5$ *and* $g \geq 10$, *or*
3. $\Delta \geq 6$ *and* $g \geq 8$, *or*
4. $\Delta \geq 12$ *and* $g \geq 7$.

Finally, in 2010 Borowiecki and Fiedorowicz [4] verified Conjecture 1 for planar graphs with girth at least 5. They also showed that $\chi'_a(G) \leq \Delta(G) + 1$ for every planar graph G with girth at least 6, which improves the previous result of Hou et al. [9].

In this paper we studied the acyclic edge colorings of planar graphs with $\Delta(G)$ colors and improved several previous results. Our results are the following.

Theorem 4. *Let G be a planar graph with girth g and maximum degree Δ. Then $\chi'_a(G) = \Delta$ if one of the following conditions holds:*

1. $\Delta \geq 3$ *and* $g \geq 12$, *or*
2. $\Delta \geq 4$ *and* $g \geq 8$, *or*
3. $\Delta \geq 5$ *and* $g \geq 7$, *or*
4. $\Delta \geq 6$ *and* $g \geq 6$, *or*
5. $\Delta \geq 10$ *and* $g \geq 5$.

In Table 1 we present the best known bounds for planar graphs. Our results are marked with an asterisk.

		$\Delta(G)$							
		3	4	5	6	...	10	11	12
$g(G)$	3	$\Delta+12$							
	4	$\Delta+6$							
	5	$\Delta+2$					Δ^*	$\Delta+1$	
	6	$\Delta+1$			Δ^*				
	7			Δ^*					Δ
	8		Δ^*		Δ				
	10			Δ					
	12	Δ^*	Δ						
	16	Δ							

Table 1: Overview of known and new results.

In the paper we use the standard notation. The degree of a vertex v (the size of a face f) is denoted by $d(v)$ (resp. $d(f)$). A vertex with degree k (at least k, at most k) is called a *k-vertex* (a *$\geq k$-vertex*, a *$\leq k$-vertex*, respectively). The neighbor u of degree k of a vertex v is called a *k-neighbor* of v.

Given an edge coloring φ of G, we say that the color a is *free at the vertex* v if there is no edge incident to v colored with color a. On the other hand, a color a is *used* at v if there is some edge incident with v which is colored by a. A path induced by colors a and b is called an *$\{a, b\}$-path*.

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2 Proof of Theorem 4.

We prove each claim of Theorem 4 separately. In all proofs we assume that there exists a minimal counterexample G to the claim and show that it cannot exist by studying its properties. First, we show that certain configurations cannot occur in the minimal counterexample. Then we assign charge to the vertices and faces of G. Using Euler's formula we compute that the total charge is negative. However, by redistributing the charge among vertices and faces, we show that it is nonnegative, reaching a contradiction. Hence, the minimal counterexample G does not exist. This approach is the well known discharging method which remains the only technique for proving the Four Color Theorem.

2.1 Reducible configurations

First, we prove some general properties of the minimal counterexample G. Throughout this section, we assume that the girth of G is at least 5 and the maximum degree of G is $\Delta \geq 3$.

Claim 1. *Minimum degree of G is at least 2.*

Proof. Let v be a vertex of degree 1 in G. Then the graph $G' = G - v$ is not a counterexample, thus it has an acyclic edge coloring using at most Δ colors. This coloring can obviously be extended to a desired coloring of G, what is a contradiction with the assumption of G being a counterexample. □

From now on we may assume that there are no 1-vertices in G.

Claim 2. *Let v be a vertex of degree k in G, let d_1, d_2,...,d_k be the degrees of the neighbors of v. Then*

$$\sum_{i=1}^{k} d_i \geq \Delta + k.$$

Proof. Let v be a k-vertex in G with neighbors $v_1, v_2, \ldots, v_k$ of degrees $d_1, d_2, \ldots, d_k$ such that $d_1 + d_2 + \cdots + d_k \leq \Delta + k - 1$.

Let φ be an acyclic edge coloring of $G - v$ using at most Δ colors. There are $d_i - 1$ colors used at v_i, $i = 1, 2, \ldots, k$. Since

$$(d_1 - 1) + (d_2 - 1) + \cdots + (d_k - 1) \leq \Delta - 1,$$

there is a color, say c_1, which is not used at any v_i, $i = 1, 2, \ldots, k$. We color the edge vv_1 with this color. For the edge vv_i $(2 \leq i \leq k)$ we use any color c_i which does not appear on edges vv_1, vv_2,...,vv_{i-1} and which is not used at v_i, v_{i+1},...,v_k. There are at most

$$i - 1 + (d_i - 1) + \cdots + (d_k - 1) \leq \sum_{j=1}^{k} (d_j - 1) \leq \Delta - 1$$

forbidden colors, hence, such color c_i exists. It is clear that we obtained a proper coloring of G using at most Δ colors. To conclude the proof it suffices to prove that no bichromatic cycle could have arisen. Let vv_i and vv_j be a part of a cycle colored with colors c_i and c_j, $1 \leq i < j \leq k$. Then the color c_i must be used at v_j, a contradiction with the choice of c_i. □

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As a special case of Claim 2 we get the following statement:

Claim 3. *There is no* 2*-vertex* v *in* G *incident with vertices* v_1 *and* v_2 *such that* $d(v_1) + d(v_2) \leq \Delta + 1$.

Consider the graph H induced by 2-vertices of G. Let vertices isolated in H be *white*, let the other 2-vertices be *black*. The previous claim immediately implies:

Claim 4. *There is no path of at least 3 black vertices in* G*. Moreover, each black* 2*-vertex is adjacent to a* Δ*-vertex.*

It means that H consists of (isolated) white vertices and (isolated) pairs of black vertices. More detailed analysis yields:

Claim 5. *Every vertex in* G *has at most one black 2-neighbor.*

Proof. Suppose a vertex v has two black 2-neighbors v_1 and v_2 in G. Let u_i be the neighbor of v_i distinct from v; let $e_i = vv_i$, $f_i = v_iu_i$, $i = 1, 2$. Let f_i' be the edge incident with u_i distinct from f_i, $i = 1, 2$. See Figure 1 for illustration.

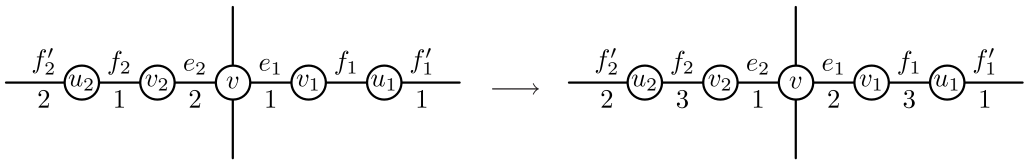

Figure 1: Reducing a vertex v with two black 2-neighbors.

Let φ be an acyclic edge coloring of $G' = G - f_1$ using at most Δ colors. We may assume that $\varphi(e_1) = 1$ and $\varphi(e_2) = 2$. If $\varphi(f_1') \neq 1$, we can extend φ to an acyclic edge coloring of G easily. Hence, we may assume $\varphi(f_1') = 1$.

We may set $\varphi(f_1) = 2$, unless there is a $\{1, 2\}$-path in G' from v_1 to u_1. If this is the case, we have $\varphi(f_2) = 1$ and $\varphi(f_2') = 2$. Now we recolor the edges in the following way: we set $\varphi(f_1) = 3$, $\varphi(e_1) = 2$, $\varphi(e_2) = 1$, $\varphi(f_2) = 3$. It can be easily checked that no bichromatic cycle has been created. □

2.1.1 Neighborhood of Δ-vertices

We say that vertices u and v are *subadjacent*, if there is a 2-vertex adjacent to both u and v.

Claim 6. *Let* v *be a* Δ*-vertex subadjacent to a vertex* u *in* G*. Then the number of* 2*-neighbors of* v *is at most* $d(u)$*.*

Proof. If $d(u) = \Delta$ there is nothing to prove, so we may assume that $d(u) = k < \Delta$. Suppose the number of 2-neighbors of v is at least $k + 1$. Let v_1, v_2,..., v_{k+1} be 2-neighbors of v; let u_i be the neighbor of v_i different from v, let $e_i = vv_i$ and $f_i = v_iu_i$, $i = 1, 2, \ldots, k + 1$. Assume $u = u_1$. See Figure 2 for illustration.

Let φ be an acyclic edge coloring of $G' = G - f_1$ using at most Δ colors. Let $1, 2, \ldots, k - 1$ be the colors used at u. If $\varphi(e_1) \notin \{1, 2, \ldots, k - 1\}$, we color f_1 with any color free at u distinct from $\varphi(e_1)$. This is always possible, since u has only $k - 1 < \Delta - 1$ colored edges. It is clear that we obtain an acyclic edge coloring of G using at most Δ colors. Hence, we may assume that $\varphi(e_1) = 1$.

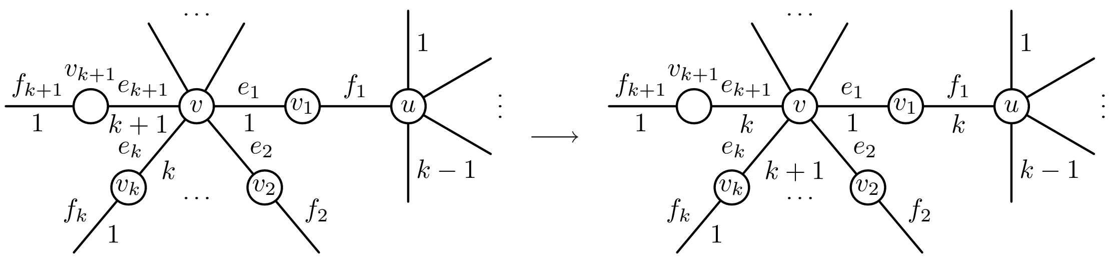

Figure 2: Reducing a Δ-vertex v subadjacent to a k-vertex u and having at least $k+1$ 2-neighbors.

Consider now the colors of $e_2, \ldots, e_{k+1}$. At most $k-1$ of them are from $\{2, \ldots, k-1\}$, hence, there are (at least) two edges colored with colors free at u. Without loss of generality we may assume that $\varphi(e_k) = k$ and $\varphi(e_{k+1}) = k+1$.

If $\varphi(f_k) \neq 1$, then we set $\varphi(f_1) = k$ and no bichromatic cycle arises. Similarly if $\varphi(f_{k+1}) \neq 1$, then we set $\varphi(f_1) = k+1$. Hence we may assume $\varphi(f_k) = \varphi(f_{k+1}) = 1$. Moreover, we may assume that in the subgraph G'_{1k} of G' induced by the edges of colors 1 and k the vertices v_1 and u are the endvertices of the same $\{1,k\}$-path, another such path starts in v_k. In this case we set $\varphi(e_k) = k+1$ and $\varphi(e_{k+1}) = k$. Clearly, we obtain an acyclic edge coloring of G'. Now, in G'_{1k} the $\{1,k\}$-path from u ends in v_k, hence, we can set $\varphi(f_1) = k$ without introducing a bichromatic cycle. $\square$

2.1.2 Neighborhood of other vertices

Claim 7. *Let u and v be a pair of subadjacent vertices. If $d(v) < \Delta$, then the number of 2-vertices adjacent to v is at most $d(v) + d(u) - \Delta - 1$.*

Proof. Let $d(v) = d$ and $d(u) = k$. Suppose that v has at least $d(v)+d(u)-\Delta = d+k-\Delta$ neighbors of degree 2. It means v has at most $\Delta - k$ neighbors of degree greater than 2. Let $v_1, v_2, \ldots, v_\ell$ be 2-neighbors of v, where $\ell = d + k - \Delta$; let u_i be the neighbor of v_i different from v, let $e_i = vv_i$ and $f_i = v_iu_i$ for $i = 1, 2, \ldots, \ell$. Assume $u = u_1$. See Figure 3 for illustration.

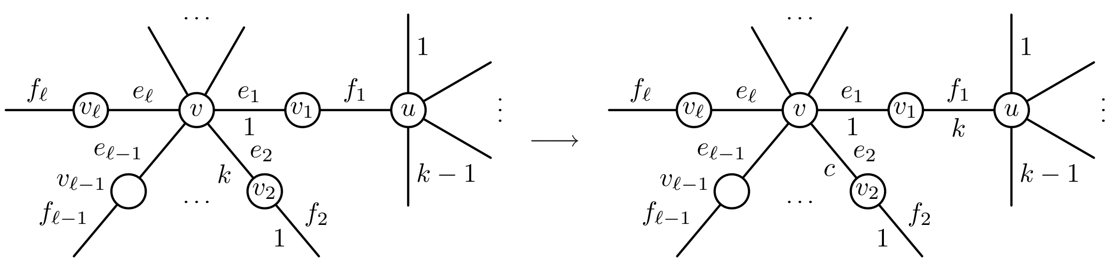

Figure 3: Reducing a d-vertex v subadjacent to a k-vertex u and having at least $d+k-\Delta$ 2-neighbors.

Let φ be an acyclic edge coloring of $G' = G - f_1$ using at most Δ colors. Let $1, 2, \ldots, k-1$ be the colors used at u. If $\varphi(e_1) \notin \{1, 2, \ldots, k-1\}$, then we find a color c which is free at v (this is always possible since $d(v) < \Delta$) and we set $\varphi(f_1) = \varphi(e_1)$ and $\varphi(e_1) = c$. It is easy to see that we obtain an acyclic edge coloring of G.

Hence, without loss of generality we may assume that $\varphi(e_1) = 1$. The colors $k, k+1, \ldots, \Delta$ are free at u. If at least one of them is also free at v, we use this color on f_1

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to extend φ to an acyclic edge coloring of G. Therefore, we may assume all the colors $k, k+1, \ldots, \Delta$ are used at v. Since v has at most $\Delta - k$ neighbors of degree greater than 2, one of the edges $e_2, \ldots, e_\ell$ is colored with one of the colors $k, k+1 \ldots, \Delta$. Without loss of generality assume that $\varphi(e_2) = k$.

Consider the color of f_2. If $\varphi(f_2) \neq 1$, then we set $\varphi(f_1) = k$ and we are done. If $\varphi(f_2) = 1$, then we find a color c which is free at v (recall that $d(v) < \Delta$), and set $\varphi(e_2) = c$ and $\varphi(f_1) = k$. It is clear that no bichromatic cycle was created. □

As a corollary we get the following property of G:

Claim 8. *There is no vertex v in G with $d(v) < \Delta$ adjacent only to 2-vertices.*

2.2 Planar graphs with girth 5

Let us reformulate the first statement of Theorem 4.

Lemma 5. *Let $\Delta \geq 10$. Every planar graph with girth at least 5 and maximum degree at most Δ admits an acyclic edge coloring with Δ colors.*

Let G be a minimal counterexample to Lemma 5 with respect to the number of edges.

2.2.1 Discharging rules

Let the initial charge be set as follows:

- $w(v) = 3d(v) - 10$ for each vertex v of G;
- $w(f) = 2d(f) - 10$ for each face f of G.

Using Euler's formula and handshaking lemma we can derive that the sum of the charge in whole graph is negative:

$$\sum_{v \in V} w(v) + \sum_{f \in F} w(f) = 3 \cdot \sum_{v \in V} d(v) - 10 \cdot |V| + 2 \cdot \sum_{f \in F} d(f) - 10 \cdot |F| =$$
$$= 3 \cdot 2 \cdot |E| - 10 \cdot |V| + 2 \cdot 2 \cdot |E| - 10 \cdot |F| = 10 \cdot (|E| - |V| - |F|) = -20.$$

It is clear that all the faces have nonnegative charge since $g \geq 5$. Vertices of degree at least 4 have positive charge, 3-vertices have charge -1 and 2-vertices have charge -4.

We move the negative charge from 2-vertices and 3-vertices according to the following rules:

(R1) Let v be a 2-vertex with neighbors v_1 and v_2. Let $d = d(v_1) \leq d(v_2)$.

- (R1a) If $d \leq 3$, then v sends no charge to v_1 and -4 of charge to v_2.
- (R1b) If $d = 4$, then v sends $-\frac{1}{2}$ of charge to v_1 and $-\frac{7}{2}$ of charge to v_2.
- (R1c) If $5 \leq d < 9$, then v sends $-\frac{3d-11}{d-1}$ of charge to v_1 and $-\frac{d+7}{d-1}$ of charge to v_2.
- (R1d) If $d \geq 9$, then v sends -2 of charge both to v_1 and v_2.

(R2) Let v be a 3-vertex with neighbors v_1, v_2, and v_3. Let $d(v_1) \leq d(v_2) \leq d(v_3)$. Then v sends -1 of charge to v_3.

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Notice that for $5 \leq d < 9$ we have $\frac{3d-11}{d-1} < 2 < \frac{d+7}{d-1}$ and that for $d = 9$ rules (R1c) and (R1d) coincide. Observe that by Claim 3 in (R1a) we have $d(v_2) \geq \Delta - 1 \geq 9 > 3 \geq d(v_1)$ and in (R1b) we have $d(v_2) \geq \Delta - 2 \geq 8 > 4 = d(v_2)$. In (R1c), if $d(v_1) = d(v_2)$, we choose v_1 arbitrarily.

To make the proof complete we need to show that after the discharging rules are applied, the charge of all vertices is nonnegative. Observe that all 2-vertices send all their negative charge to their neighbors, moreover, only vertices of degree at least 4 receive some negative charge from 2-vertices.

Similarly, all 3-vertices send all their negative charge to some of their neighbors. By Claim 2 for each 3-vertex v with neighbors v_1, v_2, and v_3 such that $d(v_1) \leq d(v_2) \leq d(v_3)$ we have $d_1 + d_2 + d_3 \geq \Delta + 3 \geq 13$. Therefore, $d(v_3) \geq 5$ and only vertices of degree at least 5 receive some negative charge from 3-vertices.

Let v be a 4-vertex. Its initial charge is 2. By (R1b) it only receives $-\frac{1}{2}$ of charge from each 2-neighbor, hence, its charge is at least $2 - 4 \cdot \frac{1}{2} \geq 0$.

2.2.2 Vertices of degree Δ

Let v be a Δ-vertex. Its initial charge is $3\Delta - 10$. It receives at most -1 of charge from each 3-neighbor, thus if it has no 2-neighbor its charge is at least $3\Delta - 10 - \Delta$, which is clearly positive. Assume v has some 2-neighbors. It means it is subadjacent to some vertices; let k be the minimum degree of a vertex subadjacent to v. Then by Claim 6 the number of 2-neighbors of v is at most k.

Let $k \leq 3$. Then v has at most three 2-neighbors which send at most -4 of charge to v by (R1a)–(R1d). The charge of v is at least $3\Delta - 10 - 3 \cdot 4 - (\Delta - 3) = 2\Delta - 19$ which is positive since $\Delta \geq 10$.

Let $k = 4$. Then v has at most four 2-neighbors which send at most $-\frac{7}{2}$ of charge to v by (R1b)–(R1d). The charge of v is at least $3\Delta - 10 - 4 \cdot \frac{7}{2} - (\Delta - 4) = 2\Delta - 20$ which is nonnegative since $\Delta \geq 10$.

Let $5 \leq k \leq 9$. Then each 2-neighbor of v sends at most $-\frac{k+7}{k-1}$ of charge to v by (R1c) or (R1d). The number of 2-neighbors of v is at most k, thus the charge of v is at least

$$3\Delta - 10 - k \cdot \frac{k+7}{k-1} - (\Delta - k) = 2\Delta - 10 - \frac{8k}{k-1} = 2\Delta - 18 - \frac{8}{k-1}.$$

This is nonnegative since $k - 1 \geq 4$ and $\Delta \geq 10$.

Let $k \geq 10$. Then each 2-neighbor of v sends -2 of charge to v. Then the charge of v is at least

$$3\Delta - 10 - k \cdot 2 - (\Delta - k) = 2\Delta - 10 - k.$$

This is nonnegative since $\Delta \geq 10$ and $k \leq \Delta$.

2.2.3 Other vertices of degree at least 5

Let v be a d-vertex, $5 \leq d < \Delta$. Its initial charge is $3d - 10$. It receives at most -1 of charge from each 3-neighbor, thus if it has no 2-neighbor its charge is at least $3d - 10 - d$, which is clearly nonnegative for $d \geq 5$. Assume v has some 2-neighbors. It means it is subadjacent to some vertices; let k be the minimum degree of a vertex subadjacent to v. By Claim 4 we have $k \geq 3$. By Claim 3 for each subadjacent vertex u_i we have $d(u_i) \geq \Delta + 2 - d$. Therefore, $k \geq \Delta + 2 - d$, thus, $d \geq \Delta + 2 - k$. Recall that by Claim 7 the number of 2-neighbors of v is at most $d + k - \Delta - 1 = k - 1 - (\Delta - d) \leq k - 2$. Now, consider several cases regarding k:

If $k = 3$, then $d \geq \Delta - 1 \geq 9$, moreover, v has one 2-neighbor. The charge of v is at least $3d - 10 - 4 - (d - 1) = 2d - 13$ which is positive for $d \geq 9$.

If $k = 4$, then $d \geq \Delta - 2 \geq 8$, moreover, v has at most two 2-neighbors. The charge of v is at least $3d - 10 - 2 \cdot \frac{7}{2} - (d - 2) = 2d - 15$ which is positive for $d \geq 8$.

If $5 \leq k \leq 8$ and $k \leq d$, then each 2-neighbor of v sends at most $-\frac{k+7}{k-1}$ of charge to v by (R1c) or (R1d). So the charge of v is at least

$$3d - 10 - (d + k - \Delta - 1) \cdot \frac{k+7}{k-1} - (\Delta + 1 - k) = \frac{2(kd - 9k - 5d + 4\Delta + 9)}{k-1} \geq$$
$$\geq \frac{2(kd - 9k - 5d + 49)}{k-1} = \frac{2[(k-5)(d-9) + 4]}{k-1}.$$

This is nonnegative since $k - 5 \geq 0$ and $d - 9 \geq -4$.

If $9 \leq k$ and $k \leq d$, then each 2-neighbor of v sends -2 of charge to v by (R1d). Then the charge of v is at least

$$3d - 10 - (d + k - \Delta - 1) \cdot 2 - (\Delta + 1 - k) = d - 9 + \Delta - k.$$

This is nonnegative since $d \geq 9$ and $k \leq \Delta$.

If $5 \leq k$ and $d < k$, then each 2-neighbor of v sends $-\frac{3d-11}{d-1}$ of charge to v by (R1c). By Claim 8 the number of 2-neighbors of v is at most $d - 1$. The charge of v is at least

$$3d - 10 - (d - 1) \cdot \frac{3d - 11}{d - 1} - 1 = 0.$$

Since all the vertices of G have nonnegative charge, we obtain a contradiction which completes the proof.

2.3 Planar graphs with girth 6

Lemma 6. *Let $\Delta \geq 6$. Every planar graph with girth at least 6 and maximum degree at most Δ admits an acyclic edge coloring with Δ colors.*

Suppose G is a minimal counterexample to Lemma 6.

2.3.1 Discharging rules

Let the initial charge be set as follows:

- $w(v) = 2d(v) - 6$ for each vertex v of G;
- $w(f) = d(f) - 6$ for each face f of G.

By Euler's formula we have that the sum of charges of vertices and faces is -12.

It is clear that since $g \geq 6$ all the faces have nonnegative charge. Vertices of degree at least 4 have positive charge, 3-vertices have no charge and 2-vertices have charge -2.

We redistribute the charge among vertices by the following rules:

(R3) Let v be a 2-vertex with neighbors v_1 and v_2 such that $d(v_1) \leq d(v_2)$.

- (R3a) If $d(v_1) \leq 3$, then v sends -2 of charge to v_2.
- (R3b) If $d(v_1) = 4$ and $d(v_2) = 4$, then v sends -1 of charge to both v_1 and v_2.
- (R3c) If $d(v_1) = 4$ and $d(v_2) \geq 5$, then v sends $-\frac{2}{3}$ of charge to v_1 and $-\frac{4}{3}$ of charge to v_2.
- (R3d) If $d(v_1) \geq 5$, then v sends -1 of charge to both v_1 and v_2.

It is easy to see that 2-vertices send all their negative charge to their neighbors. Since $\Delta \geq 6$, by Claim 3 for each 2-vertex with neighbors with degrees d_1 and d_2 we have $d_1 + d_2 \geq \Delta + 2 \geq 8$, therefore, only vertices of degree at least 4 can receive negative charge. Hence, 3-vertices neither send nor receive any charge, so they retain chargeless.

2.3.2 4-vertices

Let v be a 4-vertex in G. Its initial charge is 2. If it has no 2-neighbors, its charge does not change. By Claim 3 it cannot be subadjacent to a 3-vertex. If it is subadjacent to a 4-vertex, by Claim 7 the number of 2-neighbors of v is at most $4+4-\Delta-1 = 7-\Delta \leq 1$, hence, it has only one 2-neighbor from which it receives -1 of charge by (R3b). Its final charge is (at least) $2 - 1 = 1$.

If it is only subadjacent to vertices of degree at least five, it can have at most three 2-neighbors by Claim 8. By (R3c) it receives $-\frac{2}{3}$ of charge from each 2-neighbor, hence, its charge is at least $2 - 3 \cdot \frac{2}{3} = 0$.

2.3.3 5-vertices

Let v be a 5-vertex in G. Its initial charge is 4. If it has no 2-neighbors, its charge does not change. By Claim 4 it cannot have a black 2-neighbor. Hence, it can only be subadjacent to $\geq$3-vertices. If it is subadjacent to a 3-vertex or a 4-vertex, then by Claim 7 it can have at most two 2-neighbors, hence its charge is at least $4 - 2 \cdot 2 \geq 0$.

If it is not subadjacent to any 3- or 4-vertex, then by Claim 8 it can have at most four 2-neighbors, which send -1 of charge each by (R3d); the charge of v is at least $4 - 4 \cdot 1 \geq 0$.

2.3.4 Other vertices

Let v be a d-vertex, where $d \geq 6$. Its initial charge is $2d - 6 \geq 6$. If it has no 2-neighbors, it does not receive any negative charge. Suppose v has some 2-neighbors; let k be a minimum degree of a vertex subadjacent to v.

If $k \leq 3$, then by Claims 6 and 7 the vertex v has at most three 2-neighbors, and each has sent at most -2 of charge. Hence, the charge of v is nonnegative.

If $k = 4$, then v has at most four 2-neighbors, and each has sent at most $-\frac{4}{3}$ of charge. Hence, the charge of v is at least $6 - 4 \cdot \frac{4}{3} = \frac{2}{3} > 0$.

If $k \geq 5$, then v receives at most -1 of charge from each neighbor, hence its charge is at least $2d - 6 - d = d - 6 \geq 0$.

All the vertices of G have nonnegative charge, a contradiction which establishes the lemma.

2.4 Planar graphs with girth 7

Lemma 7. *Let $\Delta \geq 5$. Every planar graph with girth at least 7 and maximum degree at most Δ admits an acyclic edge coloring with Δ colors.*

If $\Delta \geq 6$, then the statement follows from Lemma 6. Therefore, we may assume that $\Delta = 5$ and $\Delta(G) \leq 5$. Suppose G is a minimal counterexample to Lemma 7.

2.4.1 Discharging rules

Let the initial charge be set as follows:

- $w(v) = 5d(v) - 14$ for each vertex v of G;
- $w(f) = 2d(f) - 14$ for each face f of G.

By Euler's formula we have that the sum of charges of vertices and faces is -28.

It is clear that since $g \geq 7$ all the faces have nonnegative charge. Vertices of degree 5 have charge 11, vertices of degree 4 have charge 6, vertices of degree 3 have charge 1, and vertices of degree 2 have charge -4.

We redistribute the charge among vertices by the following rules:

(R4) Let v be a 2-vertex with neighbors v_1 and v_2 such that $d(v_1) \leq d(v_2)$.

- (R4a) If $d(v_1) = 2$, then v sends 0 of charge to v_1 and -4 of charge to v_2.
- (R4b) If $d(v_1) = 3$, then v sends $-\frac{1}{3}$ of charge to v_1 and $-\frac{11}{3}$ of charge to v_2.
- (R4c) If $d(v_1) \geq 4$, then v sends -2 of charge both to v_1 and v_2.

Since $\Delta = 5$, by Claim 3 for each 2-vertex with neighbors with degrees d_1 and d_2 we have $d_1 + d_2 \geq \Delta + 2 = 7$. It is easy to see that 2-vertices send all their negative charge to their neighbors of degree at least 3.

2.4.2 3-vertices

Let v be a 3-vertex in G. Its initial charge is 1. By (R4b) it receives $-\frac{1}{3}$ of charge from each its 2-neighbor, hence its charge is at least $1 - 3 \cdot \frac{1}{3} = 0$.

2.4.3 4-vertices

Let v be a 4-vertex in G. Its initial charge is 6. If it has no 2-neighbors, its charge does not change. By Claim 3 it cannot be subadjacent to a 2-vertex. If it is subadjacent to a 3-vertex, by Claim 7 the number of 2-neighbors of v is at most $3+4-\Delta-1 = 1$, hence, it has only one 2-neighbor from which it receives $-\frac{11}{3}$ of charge by (R4b). Its charge is clearly nonnegative.

If v is not subadjacent to any $\leq$ 3-vertex, then by Claim 8 it can have at most three 2-neighbors, from which it receives -2 of charge by (R4c). Its charge is (at least) $6 - 3 \cdot 2 = 0$.

2.4.4 5-vertices

Let v be a 5-vertex in G. Its initial charge is 11. If it has no 2-neighbors, its charge does not change.

If v is subadjacent to a 2-vertex, then by Claim 6 it has at most two 2-neighbors, which send at most -4 of charge each. The charge of v is at least $11 - 2 \cdot 4 = 3 > 0$.

If v is not subadjacent to any 2-vertex and v is subadjacent to a 3-vertex, by Claim 6 it has at most three 2-neighbors, which send at most $-\frac{11}{3}$ of charge each. The charge of v is at least $11 - 3 \cdot \frac{11}{3} = 0$.

If v is not subadjacent to any $\leq$ 3-vertex, then all its 2-neighbors send -2 of charge by (R4c); the charge of v is at least $11 - 5 \cdot 2 = 1 \geq 0$.

All the vertices of G have nonnegative charge, a contradiction which establishes the lemma.

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2.5 Planar graphs with girth 8

Lemma 8. *Let $\Delta \geq 4$. Every planar graph with girth at least 8 and maximum degree at most Δ admits an acyclic edge coloring with Δ colors.*

If $\Delta \geq 5$, then the statement follows from Lemma 7. Therefore, we may assume that $\Delta = 4$ and $\Delta(G) \leq 4$. Suppose G is a minimal counterexample to Lemma 8.

Before setting discharging rules, we prove several additional properties of G.

2.5.1 More reducible configurations

Let a 3-vertex with two 2-neighbors be *blue*. We focus on the neighborhood of blue vertices.

Claim 9. *Each blue 3-vertex is subadjacent to two 4-vertices.*

Proof. Let v be a blue 3-vertex with 2-neighbors v_1 and v_2; let u_i be the neighbor of v_i distinct from v, $i = 1, 2$. If $d(u_i) \leq 3$, then by Claim 7 there are at most

$$d(v) + d(u_i) - \Delta - 1 \leq 3 + 3 - 4 - 1 = 1$$

2-vertices adjacent to v in G, however, both v_1 and v_2 are 2-vertices, a contradiction. □

Claim 10. *Each blue 3-vertex is adjacent to a 4-vertex.*

Proof. Let v be a blue 3-vertex with neighbors v_1, v_2, v_3. Let $d(v_1) = 2$, $d(v_2) = 2$, $d(v_3) = 3$; let $e_i = vv_i$, $i = 1, 2, 3$. Let f_i be the edge incident with v_i different from e_i, $i = 1, 2$. Let f_3, f_4 be edges incident with v_3 different from e_3. Let G' be the graph obtained from G by deletion of the edges e_1, e_2, and e_3. Let φ be an acyclic edge coloring of G' using at most 4 colors. Consider the colors of f_1, f_2, f_3, and f_4. We distinguish all possible cases up to symmetry and permutation of colors. See Figure 4 for illustration.

Let $\varphi(f_1) = \varphi(f_2) = 1$. If $1 \notin \{\varphi(f_3), \varphi(f_4)\}$, then we set $\varphi(e_1) = \varphi(f_3)$ and $\varphi(e_2) = \varphi(f_4)$; for the edge e_3 we use the fourth color. If $1 = \varphi(f_3)$, then we set $\varphi(e_1) = \varphi(f_4)$; for the edges e_2 and e_3 we use the othe two colors.

Now, we may assume that $\varphi(f_1) = 1$ and $\varphi(f_2) = 2$. Let the four edges f_1, f_2, f_3, and f_4 be colored by four colors, say $\varphi(f_3) = 3$, $\varphi(f_4) = 4$. Then we set $\varphi(e_1) = 2$, $\varphi(e_2) = 3$, and $\varphi(e_3) = 1$.

Let the four edges f_1, f_2, f_3, and f_4 be colored by three colors, say $\varphi(f_3) = 1$, $\varphi(f_4) = 3$. Then we set $\varphi(e_1) = 2$, $\varphi(e_2) = 3$, and $\varphi(e_3) = 4$.

Let the four edges f_1, f_2, f_3, and f_4 be colored by two colors, say $\varphi(f_3) = 1$, $\varphi(f_4) = 2$. Then we set $\varphi(e_1) = 2$, $\varphi(e_2) = 3$, and $\varphi(e_3) = 4$.

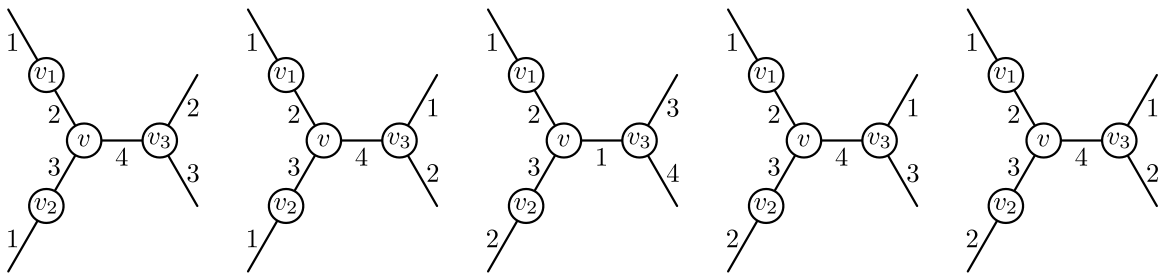

Figure 4: Reducing a 3-vertex with neighbors of degrees $2, 2, 3$, respectively.

It is easy to see that φ is now an acyclic edge coloring of G using at most 4 colors, a contradiction. □

Claim 11. *Let v be a 4-vertex subadjacent to two 3-vertices. Then the number of 2-neighbors of v is two.*

Proof. Let v be a 4-vertex subadjacent to u_1, u_2, u_3 such that $d(u_1) = d(u_2) = 3$. Let v_i be the common neighbor of v and u_i, $i = 1, 2, 3$. Let v_4 be the other neighbor of v. Let $e_i = vv_i$, $i = 1, 2, 3, 4$, $f_i = v_iu_i$, $i = 1, 2, 3$. See Figure 5 for illustration.

Let φ be an acyclic edge coloring of $G' = G - f_1$ using at most 4 colors. Assume $\varphi(e_1) = 1$.

There are two colors free at u_1. If 1 is free at u_1, then we use the other free color for f_1 to extend φ to an acyclic edge coloring of G. Hence, we may assume that 1 is used at u_1. Let 3 and 4 be the colors free at u_1.

We can use the color 3 (or 4) for f_1 unless we introduce a bichromatic cycle. Therefore, we may assume that in G', there is a $\{1, 3\}$-path from v_1 to u_1 and also a $\{1, 4\}$-path from v_1 from u_1.

Consider the color of e_2. Suppose first that $\varphi(e_2) = 3$. Since there is a $\{1, 3\}$-path from v_1 to u_1, we have $\varphi(f_2) = 1$ and we know that 3 is used at u_2. Hence, there is a color $c \in \{2, 4\}$ free at u_2. In this case we set $\varphi(e_1) = 3$, $\varphi(f_1) = 4$, $\varphi(e_2) = 1$, and $\varphi(f_2) = c$, see Figure 5(a). It is easy to see that no bichromatic cycle arises. We can use the same argument if $\varphi(e_2) = 4$. Hence, we may assume that $\varphi(e_2) = 2$; without loss of generality let $\varphi(e_3) = 3$ and $\varphi(e_4) = 4$. Then $\varphi(f_3) = 1$.

Consider the color of f_2. Suppose first that $\varphi(f_2) \neq 1$. In this case, we set $\varphi(f_1) = 3$, $\varphi(e_1) = 2$, and $\varphi(e_2) = 1$. It is easy to see that no bichromatic cycle is created, since the $\{2, 3\}$-path containing v_1 ends at v_3, the $\{1, 2\}$-path and $\{1, 3\}$-path containing v_2 ends at v_1, and $\{1, 4\}$-path containing v_2 ends at u_1, see Figure 5(b).

Finally, suppose that $\varphi(f_2) = 1$. Then there is a color, say $c \neq 1$, free at u_2. In this case, we set $\varphi(f_1) = 3$, $\varphi(e_1) = 2$, $\varphi(e_2) = 1$, and $\varphi(f_2) = c$, see Figure 5(c). Again, no bichromatic cycle arises. □

2.5.2 Discharging rules

Let the initial charge be set as follows:

- $w(v) = 3d(v) - 8$ for each vertex v of G;
- $w(f) = d(f) - 8$ for each face f of G.

By Euler's formula we have that the sum of charges of vertices and faces is -16.

It is clear that since $g \geq 8$ all the faces have nonnegative charge. Vertices of degree 4 have charge 4, vertices of degree 3 have charge 1, and vertices of degree 2 have charge -2.

We redistribute the charge among vertices by the following rules:

(R5a) Each white 2-vertex divides its charge (-2) equally among its two neighbors.

(R5b) Each black 2-vertex sends all its charge (-2) to the neighbor which is not a 2-vertex.

After this phase all 2-vertices have charge 0. However, some other vertices can have become negative.

Consider a 3-vertex v in G. Its initial charge is 1. By Claim 3 it cannot have a black 2-neighbor. By Claim 8 it can have at most two (white) 2-neighbors. If v has at most one 2-neighbor, then it receives at most -1 of charge, so its charge is at least 0. Thus, only 3-vertices with precisely two 2-neighbors – blue 3-vertices – have negative charge.

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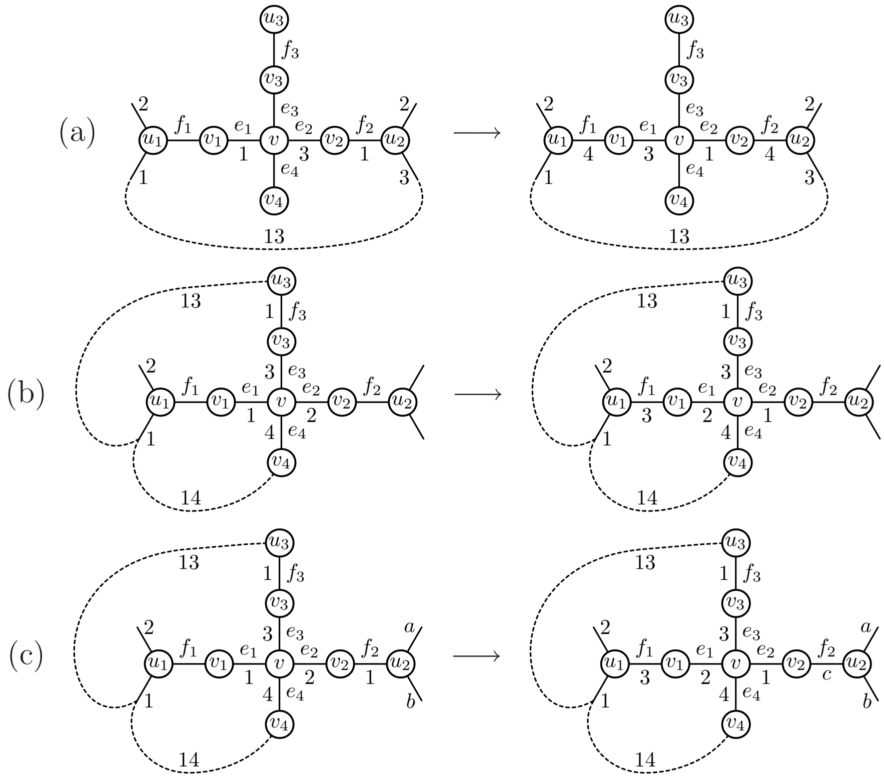

Figure 5: Reducing a 4-vertex v subadjacent to two 3-vertices u_1, u_2 and another vertex u_3.

Let v be a blue 3-vertex. It is subadjacent to two vertices u_1 and u_2 via 2-vertices v_1 and v_2. By Claim 9 both u_1 and u_2 are 4-vertices. Moreover, by Claim 10 the third neighbor v_3 of v is also a 4-vertex. The charge of v is now $1 + 2 \cdot (-1) = -1$.

(R6) Each blue 3-vertex v with two 2-neighbors v_1 and v_2 sends $-\frac{1}{2}$ of charge to the face incident both with v_1 and v_2; it sends $-\frac{1}{4}$ of charge to the other two incident faces.

After this phase all 2- and 3-vertices have nonnegative charge. Some negative charge was sent to 4-vertices and faces.

Consider a 4-vertex v in G. Its initial charge is 4. If v has a black 2-neighbor, then by Claim 6 it has at most two 2-neighbors. Moreover, by Claim 5 at most one of them is black, thus, in this case it receives at most -3 units of charge, so its charge is at least 1. If v has only white 2-neighbors, its charge is at least $4 - 4 \cdot 1 \geq 0$. However, if has at most three white 2-neighbors, its charge is at least 1. Let 4-vertices with four 2-neighbors be called *red*. Observe that by Claim 6 a red 4-vertex cannot be subadjacent to a ≤ 3-vertex, hence, each red 4-vertex is subadjacent to four 4-vertices.

(R7) Each 4-vertex v divides all its charge equally to the four faces it is incident with.

Now all vertices have nonnegative charge. Some of the negative charge can have been moved from blue 3-vertices to faces. On the other hand, observe that each face receives at least $\frac{1}{4}$ of charge from each incident 4-vertex which is not red.

2.5.3 Big faces

Let f be a face of size k. Its initial charge is $k - 8$; it receives $-\frac{1}{4}$ or $-\frac{1}{2}$ of charge from each blue 3-vertex it is incident with. Let $v_1, \ldots, v_k$ be the vertices incident with f in a cyclic order. Let v_1 be a blue 3-vertex which sends $-\frac{1}{4}$ of charge to f. According to (R6), we may assume $d(v_k) = 4$ and $d(v_2) = 2$, and $d(u_1) = 2$, where u_1 is the neighbor of v_1 not incident with f. By Claim 9 we have $d(v_3) = 4$. It means there is a facial path of length 3 beginning and ending in a 4-vertex, containing the blue 3-vertex v_1.

Let v_1 be a blue 3-vertex which sends $-\frac{1}{2}$ of charge to f. According to (R6), we may assume $d(v_k) = d(v_2) = 2$. Again, by Claim 9 we have $d(v_{k-1}) = d(v_3) = 4$. It means there is a facial path of length 4 beginning and ending in a 4-vertex, containing the blue vertex v_1.

Altogether, there can be at most $\frac{k}{3}$ blue vertices incident with f. The charge of f is at least $k - 8 - \frac{k}{3} \cdot \frac{1}{2} = \frac{5k}{6} - 8 \geq \frac{5k}{6} - \frac{50}{6}$, which is nonnegative for $k \geq 10$.

2.5.4 9-faces

Let f be a 9-face. Its initial charge is $9 - 8 = 1$. If it is incident with at most two blue 3-vertices, its charge is at least $1 - 2 \cdot \frac{1}{2} = 0$. Therefore we may assume it is incident with three blue 3-vertices. This can only happen if all the three blue 3-vertices are contained in paths of length 3. Hence, they send $3 \cdot (-\frac{1}{4})$ of charge to f, and so the final charge of f is clearly nonnegative.

2.5.5 8-faces

Let f be an 8-face. Its initial charge is 0. If it is incident with no blue 3-vertices, it does not receive negative charge.

Let f have received $-\frac{1}{4}$ of charge from a blue 3-vertex v_1; let v_8 be a 4-vertex, v_2 be a white 2-vertex and let v_3 be a 4-vertex. Since the 4-vertex v_8 is adjacent to a 3-vertex v_1, it is not red; by (R7) it sends at least $\frac{1}{4}$ of charge to f. The charge of f is at least $-\frac{1}{4} + \frac{1}{4} = 0$.

Let f have received $-\frac{1}{2}$ of charge from a blue 3-vertex v_1; let v_8 and v_2 be white 2-vertices and let v_7 and v_3 be 4-vertices. Since v_3 and v_7 are subadjacent to a 3-vertex v_1, they are not red; by (R7) the vertices v_3 and v_7 send at least $\frac{1}{4}$ of charge to f. The charge of f is at least $-\frac{1}{2} + 2 \cdot \frac{1}{4} = 0$.

Let f have received $-\frac{1}{4}$ of charge from two blue 3-vertices. There are five possibilities for their position up to symmetry, see Figure 6, the first five images. In all thee cases there are at least two 4-vertices which are not red (recall that a red 4-vertex cannot be adjacent or subadjacent to a $\leq$ 3-vertex); hence they send at least $\frac{1}{4}$ of charge to f each. It means the charge of f is nonnegative.

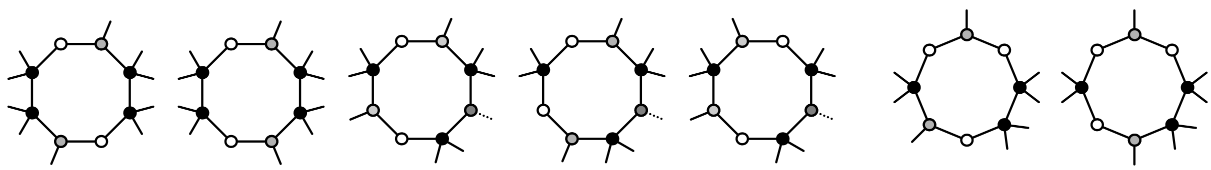

Figure 6: Possible positions of blue vertices incident with an 8-face. Empty circles represent white 2-vertices, full circles represent 4-vertices, light grey circles represent blue 3-vertices, dark grey circles represent unspecified vertices.

Let f have received $-\frac{1}{2}$ of charge from a blue 3-vertex v_1 and $-\frac{1}{4}$ of charge from another blue 3-vertex. There are two possibilities for its position up to symmetry, see Figure 6, the last two images. In both cases there are three 4-vertices which send at least $\frac{1}{4}$ of charge to f, hence, its charge is nonnegative.

Let f have received $-\frac{1}{2}$ of charge from two blue 3-vertices v_1 and v_5. Then v_2, v_4, v_6, v_8 are white 2-vertices and v_3 and v_7 are 4-vertices. The 4-vertex v_3 is subadjacent to two 3-vertices v_1 and v_5, thus by Claim 11 it is not subadjacent to any other vertex. Hence, it only receives -2 of charge from its 2-neighbors by (R5a), and then it sends $\frac{1}{2}$ of charge to all incident faces by (R7). Since the same holds for v_7, the face f receives $\frac{1}{2}$ of charge from both v_3 and v_7, thus its charge in nonnegative.

All the vertices and faces of G have nonnegative charge, a contradiction which establishes the lemma.

2.6 Planar graphs with girth 12

Lemma 9. *Let $\Delta \geq 3$. Every planar graph with girth at least 12 and maximum degree at most Δ admits an acyclic edge coloring with Δ colors.*

If $\Delta \geq 4$, then the statement follows from Lemma 8. Therefore, we may assume that $\Delta = 3$ and $\Delta(G) \leq 3$. Suppose G is a minimal counterexample to Lemma 9.

Before setting discharging rules, we prove one more structural property of G.

Claim 12. *There is no path $v_1v_2v_3v_4v_5v_6v_7$ with degrees $2,2,3,2,3,2,2$ in G.*

Proof. Let $v_1v_2v_3v_4v_5v_6v_7$ be a path with degrees $2,2,3,2,3,2,2$ in G. Let $e_i = v_iv_{i+1}$, $i = 1,\dots,6$; let e_0 be the edge incident with v_1 distinct from e_1, let e_7 be the edge incident with v_7 distinct from e_6; let f_3 (resp. f_5) be the edge incident with v_3 (resp. v_5) distinct from e_2 and e_3 (resp. e_4 and e_5). Since we assume $g \geq 12$ all considered edges are pairwise distinct.

Let φ be an acyclic edge coloring of $G' = G - e_1$ using colors $1,2,3$. Let $\varphi(e_2) = 1$, $\varphi(e_3) = 2$, $\varphi(f_3) = 3$. We may assume $\varphi(e_0) = 1$, otherwise we can extend the coloring easily. We also may assume $\varphi(e_4) = 1$, otherwise we can set $\varphi(e_1) = 2$. See Figure 7(a) for illustration.

Let G_{ij} be a subgraph of G' induced by edges colored i and j, $\{i,j\} \subset \{1,2,3\}$. If v_1 and v_2 are not endvertices of the same path in G_{13}, we set $\varphi(e_1) = 3$. Hence, we may assume there is a $\{1,3\}$-path from v_1 to v_2 in G'. If v_1 and v_2 are not endvertices of one path in G_{12}, we set $\varphi(e_1) = 2$. Hence, we may assume there is a $\{1,2\}$-path from v_1 to v_2 in G'. We set $\varphi(e_1) = 3$, $\varphi(e_2) = 2$ and $\varphi(e_3) = 1$. Now the edges e_3 and e_4 both have color 1. We now look at the end of the $\{1,2\}$-path from v_1 to v_2:

- Let $\varphi(e_5) = 2$, $\varphi(e_6) = 1$, and $\varphi(e_7) = 2$. Then $\varphi(f_5) = 3$. In this case we recolor the path in the following way: $\varphi(e_4) = 2$, $\varphi(e_5) = 1$, and $\varphi(e_6) = 3$, see Figure 7(b).
- Let $\varphi(f_5) = 2$. Then $\varphi(e_5) = 3$. If $\varphi(e_7) \neq 2$, then we set $\varphi(e_4) = 3$, $\varphi(e_5) = 1$, and $\varphi(e_6) = 2$, see Figure 7(c). If $\varphi(e_7) = 2$, then we set $\varphi(e_4) = 3$, $\varphi(e_5) = 1$, and $\varphi(e_6) = 3$, see Figure 7(d).

It can be checked easily that in all the cases no bichromatic cycle can arise. ☐

2.6.1 Discharging rules

Let the initial charge be set as follows:

- $w(v) = 4d(v) - 10$ for each vertex v of G;
- $w(f) = d(f) - 10$ for each face f of G.

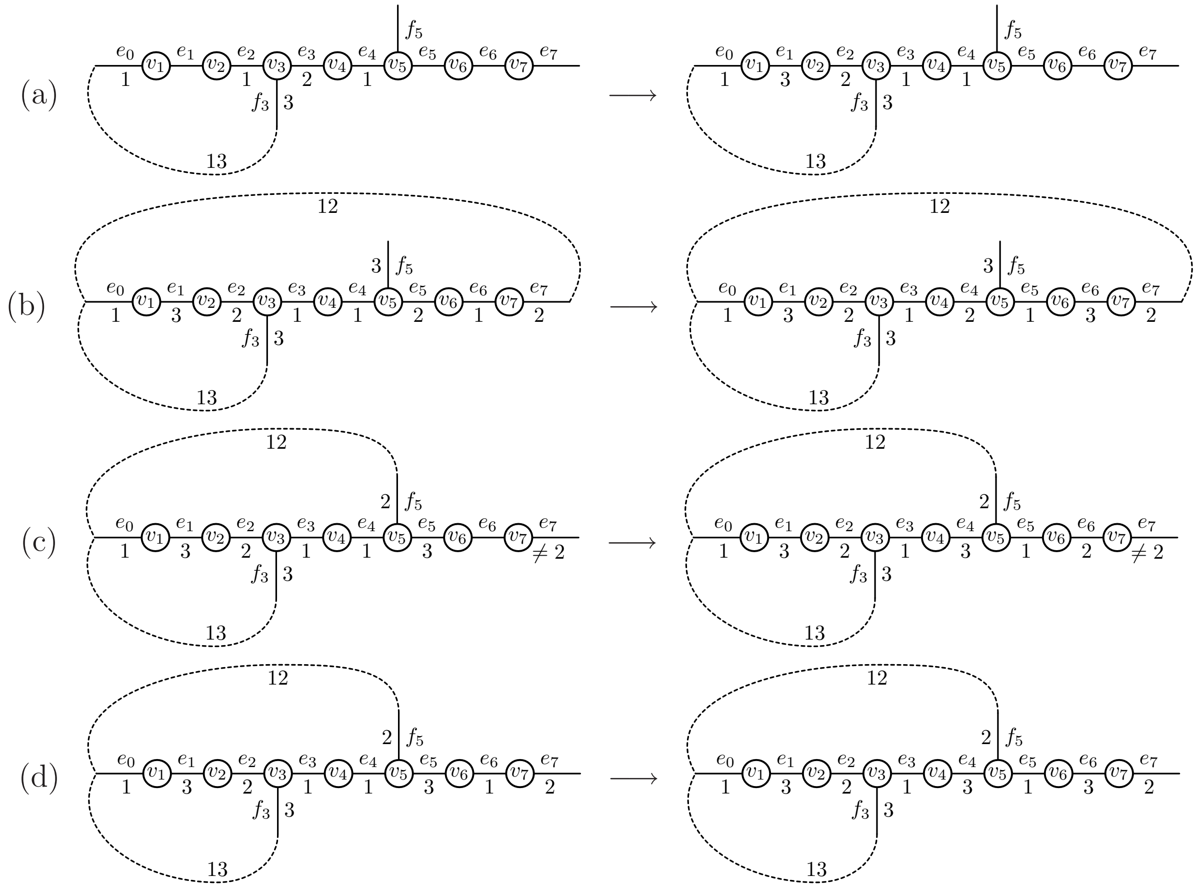

Figure 7: Reducing a path with degrees $2, 2, 3, 2, 3, 2, 2$.

By Euler's formula we have that the sum of charges of vertices and faces is -20.

It is clear that vertices of degree 3 have charge 2, vertices of degree 2 have charge -2. We redistribute the charge from vertices to faces by the following rules:

(R8a) Each 2-vertex sends -1 of charge to each face it is incident with.

(R8b) Each 3-vertex sends $\frac{2}{3}$ of charge to each face it is incident with.

It is clear that all vertices have zero final charge. It suffices to consider how the charge is distributed among the faces.

Let f be a face of size d. By Claims 4 and 5 among any five consecutive vertices incident with f at least two are 3-vertices and at most three are 2-vertices. The charge of f is therefore at least $d - 10 + \frac{2d}{5} \cdot \frac{2}{3} - \frac{3d}{5} \cdot 1 = \frac{2d}{3} - 10$ which is nonnegative for $d \geq 15$.

Let f be a face of size d, $d \leq 14$. If it is incident with at least six 3-vertices, its charge is at least $d - 10 + 6 \cdot \frac{2}{3} - (d - 6) \cdot 1 = 0$. Assume f is incident with (at most) five 3-vertices. By Claim 5 there are at most two pairs of black 2-vertices incident with f. Since $g \geq 12$, there are at least seven 2-vertices incident with f. Hence $d = 12$, there are two pairs of black 2-vertices and three other white 2-vertices. But then there must be a reducible configuration from Claim 12 in G, a contradiction.

All the vertices and faces of G have nonnegative charge, a contradiction which establishes the lemma.

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