Graphs with odd cycle lengths 5 and 7 are 3-colorable*

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Abstract

Let $L(G)$ denote the set of all odd cycle lengths of a graph G. Gyárfás gave an upper bound for $\chi(G)$ depending on the size of this set: if $|L(G)| = k \geq 1$, then $\chi(G) \leq 2k+1$ unless some block of G is a K_{2k+2}, in which case $\chi(G) = 2k+2$. This bound is generally tight, but when investigating $L(G)$ of special forms, better results can be obtained. Wang completely analyzed the case $L(G) = \{3, 5\}$; Camacho proved that if $L(G) = \{k, k+2\}$, $k \geq 5$, then $\chi(G) \leq 4$. We show that $L(G) = \{5, 7\}$ implies $\chi(G) = 3$.

Keywords: graph, coloring, 3-colorability, odd cycle, odd cycle lengths, 4-critical, H-bridge

1 Introduction

Let $L(G)$, shortly L, be the set of all odd cycle lengths of a graph G. One may investigate the relation between this set, or its size, and the chromatic number $\chi(G)$ of G. For example, it is well-known that $|L(G)| = 0$ precisely when $\chi(G) \leq 2$. The following result, originally conjectured in a weaker form by Bollobás and Erdős and later, in the version presented here, by Gallai, states that there is a general upper bound for $\chi(G)$ in terms of $|L(G)|$.

Theorem 1.1 (Gyárfás [5, Corollary of Theorem 1])**.** *If $|L(G)| = k \geq 1$, then the chromatic number of G is at most $2k+1$ unless some block of G is a K_{2k+2}. (If there is such a block, then the chromatic number of G is $2k+2$.)*

This bound is globally tight for any k as seen by considering a K_{2k+1}. However, the following results demonstrate that it can be improved for particular cases. Wang [8] proved that if $L(G) = \{3, 5\}$, then $\chi(G) = 3$ unless there is a K_4 or a wheel on six vertices in G, in which case $\chi(G) = 4$ unless G contains a K_5. Camacho [1] showed that $\chi(G) \leq 4$ whenever $L(G) = \{k, k+2\}$, $k \geq 5$.

In this paper, we concentrate on a special class of graphs with $L = \{5, 7\}$, the main result being the following theorem:

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Theorem 1.2. *Every graph with $L = \{5, 7\}$ is 3-colorable.*

It definitively refines the Camacho's result for this case. Although the last step of the proof is closely tailored to the particular class of graphs examined, the rest of the argument works for all graphs with $L = \{k, k+2\}$, $k \geq 5$.

The paper is organized as follows. Section 2 includes a few general, mutually unrelated lemmas used at various places later on. The purpose of Section 3 is to state and prove Theorem 3.1, a slight strengthening of Theorem 1.1 for graphs with $|L| = 1$ that is needed in the subsequent argument. Finally, in Section 4 we first restrict the structure of 4-critical graphs with $L = \{k, k+2\}$, $k \geq 5$, and using that, we prove the main result, Theorem 1.2.

For the convenience of the reader, the text is accompanied by a number of figures, all of which obey the following rules of style. Vertices are shown as small filled discs. Solid lines, usually straight, stand for single edges; dashed lines, either curly or curved, are used for paths and cycles. Subgraphs which are not depicted completely by showing all their vertices and edges are typically represented by light or dark gray regions with thin dotted line as their boundary. Next, boldface is sometimes used to distinguish (important) objects. On the other hand, the possible nonexistence of a particular entity is always indicated by dotting its standard representation, in the case of an edge or a path, or, for a vertex, by replacing the ordinary disc with a circle. Lastly, labels not meaning names of objects, i.e., those describing lengths of paths or colors of vertices, are enclosed in brackets.

In the remainder of this section, we mention several notions and results that are particularly important for the next discussion or are not so well-known. Besides these, we use only standard graph theory concepts and notation, which can be found in Diestel's monograph [2] for example.

Due to the nature of the problem, we confine ourselves to simple graphs, i.e., (undirected) graphs without multiple edges and loops. Let G be a graph with the set of vertices V and the set of edges E; we also write $G = (V, E)$ and refer to V, E as $V(G)$, $E(G)$ respectively. Then by $|G|$ we mean $|V(G)|$, that is, the number of vertices of G; $||G||$ denotes $|E(G)|$, the number of edges of G. Hence, whenever P is a path, $||P||$ is the length of P; if C is a cycle, its length is equal to $|C| = ||C||$. The graph G is called *trivial* if $|G| \leq 1$. For the path P, an *internal vertex* is any of its vertices that is not an end-vertex of P. Two paths P_1 and P_2 are *internally disjoint* if the sets of their internal vertices are disjoint. A *k-cycle* is a cycle of length k. When two vertices $x, y \in V(G)$ are at distance k in G, i.e., k is the length of a shortest path joining x and y in G, we write $d_G(x, y) = k$. Next, the set notation is extended in the following manner. Let G_1, G_2 be graphs. By $G_1 \subseteq G_2$ we mean that G_1 is a subgraph of G_2; $G_1 \cup G_2$ and $G_1 \cap G_2$ denote the graphs $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and $(V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$ respectively. If X is a set, $G_1 \setminus X$ stands for the graph with $V(G_1) \setminus X$ as its set of vertices and having all the edges of G_1 not incident with any element of X as its edges. We put $G_1 \setminus G_2 := G_1 \setminus V(G_2)$. For a vertex x, $G_1 \setminus \{x\}$ is written rather as $G_1 \setminus x$ unless there is a risk of confusion.

We use the notation of Diestel [2] for describing subpaths of paths or cycles and combinations of these. If P is a path, x and y its vertices, then xPy denotes the subpath of P between x and y including these two vertices. Moreover, if x or y is an end-vertex of P, we can optionally omit it writing Py or xP respectively. For the concatenation of paths $P_1, P_2, \ldots$, subpaths $x_1R_1x_2, x_2R_2x_3, \ldots$, and single edges $x_nx_{n+1}, x_{n+1}x_{n+2}, \ldots$ sharing their end-vertices in sequence, such that the resulting graph is a path or a cycle, we write shortly $P_1P_2 \cdots x_1R_1x_2R_2x_3 \cdots x_nx_{n+1}x_{n+2} \cdots$. All this convention is extended in a respective manner also for cycles, provided that the resulting description is unambiguous. For instance, to specify a single subpath of a cycle C delimited by vertices x and y, we need to name a third vertex z lying on that path, writing $xCzCy$.

We adopt a few less common concepts used by, e.g., Diestel [2] and Voss [7]. Let G be a graph, H and I its subgraphs. A nontrivial path in G is an *H-path* if it has precisely its end-vertices in common with H; if this path is a single edge, then it is called a *chord* of H. An *H–I path* is any (possibly trivial) path $P = u \cdots v$ in G for which $V(P \cap H) = \{u\}$ and $V(P \cap I) = \{v\}$. If H or I consists of a single vertex x or y respectively, we also write x–I, H–y, or x–y path instead of H–I path. By an *H-bridge* we mean any subgraph M of G which is either a chord of H or a component of $G \setminus H$ together with all the edges of G linking the component to the vertices of H. The vertices from $V(M \cap H)$ are the *attachment vertices* of M to H. It is clear that any vertex or edge of G not in H belongs to a unique H-bridge, that is, G is the union of all the H-bridges and the subgraph H itself, and two different H-bridges intersect only in their common attachment vertices. Other trivial properties of H-bridges, often used afterwards, we state in the following lemma.

Lemma 1.3. *Let G be a graph, H its subgraph, M an H-bridge. Then:*

(1) *if $M \neq K_2$, then no two of its attachment vertices are adjacent in M;*

(2) *for every two vertices u, v of M there exists a path $P = u \cdots v \subseteq M$ such that $V(P \cap H) \subseteq \{u, v\}$.*

Now we recall the following notion concerning graph coloring, which is crucial in this paper. A graph G is *k-critical* for $k \geq 2$ if it is not $(k-1)$-colorable but each of its proper subgraphs is $(k-1)$-colorable. Clearly, G is then k-chromatic. For example, the 3-critical graphs are precisely the odd cycles. We will deal with 4-critical graphs, so we recollect some of their basic properties here. Directly from the definition, every 4-critical graph G is 2-connected and the minimum degree of G is at least 3. One can easily see that G is even 3-edge-connected, which also follows from Theorem 1 of Dirac [4] as a special case. However, simple examples show that G need not be 3-connected. Dirac [3, page 45, statement (4)] and Toft [6, Theorem 2.1] discuss the presence of $(k-1)$-critical subgraphs in k-critical graphs; we use part of their results formulated as the next lemma.

Lemma 1.4. *Let G be a 4-critical graph. Then for every two distinct vertices x, y of G there is an odd cycle $C \subseteq G \setminus x$ containing y.*

Regarding the graph connectivity, we will often exploit Menger's theorem in the following form.

Theorem 1.5. *Let G be a k-connected graph, A and B two sets of its vertices such that $A \nsubseteq B$ and $B \nsubseteq A$. Then there are k distinct internally disjoint A–B paths in G. If $|A| \geq k$, then these paths can be chosen such that they have pairwise different end-vertices in A. If $|B| \geq k$ as well, then there are k pairwise disjoint A–B paths in G.*

We remark that the assumption $A \nsubseteq B$, $B \nsubseteq A$ is needed only for the case $|A| < k$, $|B| < k$ respectively.

Finally, let us recall several facts and notions related to 2-connected graphs. A *cut-vertex* of a graph G is a vertex whose removal increases the number of components of G. A *block* of G is a maximal connected subgraph of G without a cut-vertex; that is either a maximal 2-connected subgraph, or a cut-edge, or an isolated vertex of G. Two blocks intersect in at most one vertex, which is then a cut-vertex of G. The *block graph* of G, denoted by $B(G)$, is the bipartite graph on the set of the cut-vertices of G and the set of the blocks of G with xB being its edge if and only if $x \in V(B)$. For any graph G, its block graph is a forest having no cut-vertex of G as its leaf. Further, it is connected if and only if G is connected.

2 Preliminary results

In this section, we accumulate basic and general lemmas used throughout the rest of the paper. We start with a trivial statement.

Lemma 2.1. *Let x, y be two distinct vertices of a bipartite graph G. Then any proper coloring of the subgraph of G induced by $\{x, y\}$ can be extended to a proper 3-coloring of G.*

Two ad hoc observations follow.

Lemma 2.2. *Let u, v, w be three distinct vertices of a 2-connected graph G. Then there exist a cycle C such that u, $v \in V(C)$, and a w–C path (possibly trivial) whose end-vertex on C is different from both u and v.*

Proof. Since G is 2-connected, there is a cycle C containing u and v. If $w \in V(C)$, the assertion is true; otherwise we can apply Theorem 1.5 to $\{w\}$ and $V(C)$, obtaining two internally disjoint w–C paths with distinct end-vertices w_1, w_2 on C. If either of these vertices differs from both u and v, taking the respective path we are done. The remaining case is that $\{w_1, w_2\} = \{u, v\}$, but then the union of both the w–C paths and either of the two subpaths of C delimited by u, v is a cycle going through all of u, v, and w, a desired configuration again. □

Lemma 2.3. *Let α_i, β_i, $i \in \mathbb{Z}_3$, be integers satisfying the following conditions:*

(1) $\alpha_i + \alpha_{i+1}$ *equals* $\beta_i + \beta_{i+1}$ *or* β_{i+2}, $i \in \mathbb{Z}_3$,

(2) $\alpha_i \neq 0$, $i \in \mathbb{Z}_3$, *and*

(3) $\beta_0 + \beta_1 + \beta_2$ *is odd.*

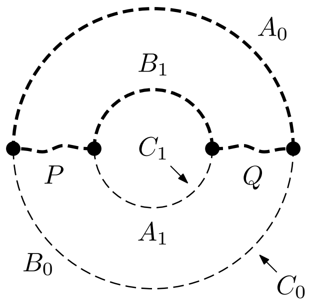
Then $\alpha_i = \beta_i$ for all i.

Proof. Condition (1) yields a certain system of three equations for α_i, β_i. We consider all its possible forms.

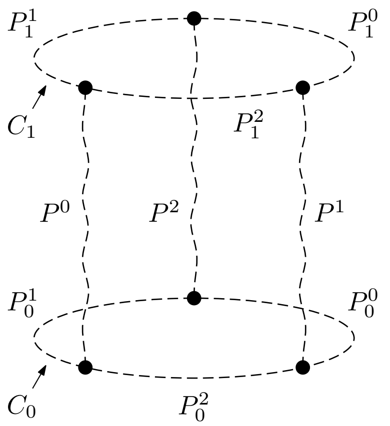
Assume first that an even number of the equations have their right side of the form $\beta_i + \beta_{i+1}$. Then by summing all the three equations we obtain that $2(\alpha_0 + \alpha_1 + \alpha_2)$ equals $\beta_0 + \beta_1 + \beta_2$ or, without loss of generality, $\beta_0 + \beta_1 + 3\beta_2$; this is in both cases impossible by condition (3). If exactly one of the equations has the term $\beta_i + \beta_{i+1}$ as the right side, one can infer that $\alpha_{i+2} = 0$, a contradiction with assumption (2).

The only remaining possibility is that the right sides of all the equations are in the form $\beta_i + \beta_{i+1}$. Then by solving the system we conclude $\alpha_i = \beta_i$ for all i. □

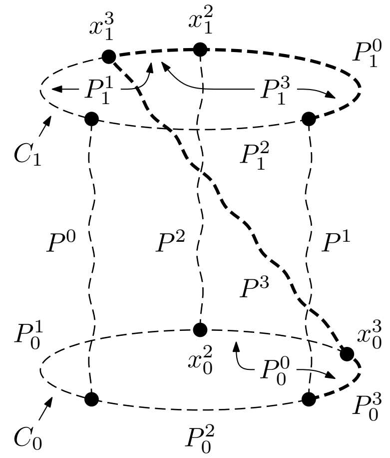
The next lemma, concerning odd cycles and paths joining them in 2-connected graphs with $L \subseteq \{k, k+2\}$, $L \neq \emptyset$, extends the results of Camacho [1, Lemma 3.1] and Gyárfás [5, Lemma 1].



(a) The notation of observation (2.1). One of the two other odd cycles, A_0PB_1Q, is shown in bold.



(b) The situation of statement (2c).



(c) The proof of statement (2d): the case $x_1^3 \in P_1^1$. For clarity, the paths P^3, P_0^3, and P_1^3 are printed in bold.

Figure 2.1. Illustrations for Lemma 2.4.

Lemma 2.4. *Let G be a 2-connected graph with $L(G) \subseteq \{k, k+2\}$; let C_0, C_1 be two odd cycles in G.*

(1) *If $|C_0| \neq |C_1|$, then the cycles are not disjoint. If $|C_0| = |C_1| = \max L(G)$, then $|C_0 \cap C_1| \geq 2$.*

(2) *Suppose that C_0 and C_1 are disjoint. Then:*

  (a) *$L(G) = \{k, k+2\}$, and $|C_0| = |C_1| = k$;*

  (b) *every two disjoint C_0–C_1 paths P^0, P^1 in G are both of length 1. Let x^i, y^i, $i = 0, 1$, be the end-vertex of P^i lying on C_0, C_1 respectively. Then $d_{C_0}(x^0, x^1) = d_{C_1}(y^0, y^1)$;*

  (c) *assume that there are three pairwise disjoint C_0–C_1 paths P^j, $j = 0, 1, 2$, in G. If P_i^j, $i = 0, 1$, denote the three subpaths of C_i delimited by the end-vertices of the paths P^j in such a way that $V(P_i^j \cap P^j) = \emptyset$, then $||P_0^j|| = ||P_1^j||$;*

  (d) *there are no four pairwise disjoint C_0–C_1 paths in G.*

For the situation of part (2c), see Figure 2.1 (b).

Proof. For convenience, let c_i, $i = 0, 1$, denote the length of C_i. We start with a general observation, whose notation is depicted in Figure 2.1 (a).

Suppose that there is a pair of disjoint C_0–C_1 paths P and Q such that $V(C_0 \cap C_1) \subseteq V(P \cup Q)$; let $d := ||P|| + ||Q||$. Next, let A_i, B_i, $i = 0, 1$, be the subpaths of C_i joining the end-vertices of P and Q, and let a_i, b_i refer to their lengths. Then, without loss of generality, G contains two odd cycles of lengths l_j, $j \in \mathbb{Z}_2$, for which (2.1)

$$l_j = a_j + b_{j+1} + d, \quad j \in \mathbb{Z}_2, \tag{2.1.1}$$

and

$$c_0 + c_1 + 2d = l_0 + l_1. \tag{2.1.2}$$

We prove the observation. As C_i is odd, a_i and b_i have different parity. Hence, without loss of generality, $l_j := a_j + b_{j+1} + d$ is odd for both $j \in \mathbb{Z}_2$, and since the paths A_j, B_{j+1} are internally disjoint, $A_j P B_{j+1} Q$ is a cycle of length l_j. Next, summing the equations (2.1.1) and using $c_i = a_i + b_i$, $i = 0, 1$, we obtain the formula (2.1.2).

Let us proceed to statement (1). If $|C_0 \cap C_1| \geq 2$, there is nothing to show, so suppose the contrary. As G is 2-connected, using Theorem 1.5 we can find paths P and Q satisfying the assumptions of observation (2.1). Note that $d > 0$ by the condition on $|C_0 \cap C_1|$. By this, the equation (2.1.2) immediately excludes the possibility that $c_0 = c_1 = \max L(G)$. Now, let $c_0 = k$ and $c_1 = k + 2$ (or vice versa). Then, as the sum $l_0 + l_1$ is at most $2k + 4$, the formula (2.1.2) implies that $d \leq 1$. In other words, at least one of the paths P, Q is trivial; hence C_0 and C_1 are not disjoint.

We continue with part (2). Statement (2a) is implied directly by part (1); we focus on statement (2b). As $V(C_0 \cap C_1) = \emptyset$, we can put the paths P^0 and P^1 in place of P and Q in observation (2.1). By assumption, $d \geq 2$. Then the equation (2.1.2) is satisfied only if $d = 2$ and $l_j = k + 2$. Hence, both P^0 and P^1 are of length 1 and, further, the formulas (2.1.1) turn into the equations $a_j + b_{j+1} = k$, $j \in \mathbb{Z}_2$. As $a_i + b_i = c_i = k$, $i = 0, 1$, by part (2a), we easily derive the wanted equalities $a_0 = a_1$, $b_0 = b_1$.

Now suppose that the assumptions of statement (2c) hold; cf. Figure 2.1 (b). For brevity, let α_j, β_j denote $||P_0^j||$, $||P_1^j||$ respectively. Applying part (2b) to all the three pairs of paths which can be chosen from $\{P^0, P^1, P^2\}$, we obtain that $\alpha_k + \alpha_{k+1}$ equals either $\beta_k + \beta_{k+1}$ or β_{k+2} for all $k \in \mathbb{Z}_3$. As also $\alpha_j \neq 0$ for all j and $\beta_0 + \beta_1 + \beta_2$ is odd, Lemma 2.3 gives the desired conclusion.

We finish the proof by considering part (2d); what follows is depicted in Figure 2.1 (c). Assume to the contrary that there are four pairwise disjoint C_0–C_1 paths in G. We adopt all the notation of statement (2c) for the first three of them; let P^3 denote the fourth path. Next, let x_i^k, $i = 0, 1$, $k = 2, 3$, be the end-vertex of P^k lying on C_i. Without loss of generality, x_0^3 belongs to P_0^0; we refer to the subpath of P_0^0 between x_0^3 and an end-vertex of P^1 as P_0^3. Similarly, let P_1^3 be the part of C_1 between x_1^3 and an end-vertex of P^1 such that it is disjoint from P^0.

Now we apply part (2c) twice, to the triples of paths $\{P^0, P^1, P^2\}$ and $\{P^0, P^1, P^3\}$. It yields

$$||P_0^j|| = ||P_1^j||, \quad j = 0, 1, 2, \tag{2.2}$$

in the first case,

$$||P_0^3|| = ||P_1^3|| \tag{2.3}$$

in the second one. If x_1^3 is contained in P_1^1, then clearly $||P_1^0|| < ||P_1^3||$, but as $||P_0^0|| > ||P_0^3||$ by definition, we have a contradiction with the preceding equalities. By symmetry, one could argue the same way that also $x_1^3 \notin V(P_1^2)$.

Thus x_1^3 is on P_1^0, and hence $||x_0^2 P_0^0 x_0^3|| = ||x_1^2 P_1^0 x_1^3||$ by the equations (2.2), (2.3). Further, all the four paths P^i, $i = 0, \ldots, 3$, are single edges, as asserted by statement (2b). Therefore, there is a cycle $P^0 P_1^2 P^1 P_0^3 P^3 x_1^3 P_1^0 P^2 P_0^1$ of length $k+4$ in G, a contradiction. □

Finally, we include a useful condition on the lengths of C-paths in graphs with $L \subseteq \{k, k+2\}$, $L \neq \emptyset$.

Lemma 2.5. *Let C be an odd cycle in a graph G; let $P = u \cdots v$ be a C-path. Then:*

(1) *if $|L(G)| = 1$, then $d_C(u,v) = \min\{||P||, |C| - ||P||\}$. Thus, P is not a chord of C;*

(2) *if $L(G) = \{k, k+2\}$ and $|C| = k+2$, then $d_C(u,v)$ equals either $\min\{||P||, k+2-||P||\}$ or $\min\{||P||+2, k-||P||\}$. That implies:*

   (a) *$||P|| \leq k+1$;*

   (b) *if P is a chord of C, then $d_C(u,v) = \min\{3, k-1\}$.*

Proof. We consider the two subpaths of C delimited by u and v. As the cycle C is odd, the length of one of these paths, say, P', must have parity distinct from that of $||P||$. Then P and P' together form another odd cycle in G; the rest of the argument is straightforward. □

3 Graphs with one odd cycle length

This section is devoted to the study of graphs with $|L| = 1$; we prove auxiliary Theorem 3.1 here, which strengthens Theorem 1.1 for this case.

Theorem 3.1. *Let G be a graph with $|L(G)| = 1$ not containing a K_4; let C be an odd cycle in G. Then any proper 3-coloring of C can be extended to a proper 3-coloring of G.*

Proof. We may clearly assume that G is 2-connected. Most of the proof consists of collecting structural information about G. To keep the argument transparent, we include such observations as Claims 1–4. After restricting the structure of G sufficiently, we will be able to find the desired 3-coloring directly.

At the beginning, we use Lemma 2.5 (1) to constrain the following configuration, depicted in Figure 3.1.

Claim 1. *Let $P_i = x_i \cdots y$, $i \in \mathbb{Z}_3$, be nontrivial paths in G such that $V(P_i \cap C) = \{x_i\}$ and $V(P_i \cap P_j) = \{y\}$ for any $j \in \mathbb{Z}_3$, $j \neq i$. Let P_i' denote the subpath of C between x_{i-1} and x_{i+1} not containing x_i. Then $||P_i|| = ||P_i'||$.*

Proof. For brevity, let $\alpha_i := ||P_i||$, $\beta_i := ||P_i'||$. We consider each of the three C-paths $P_i P_{i+1}$. By Lemma 2.5 (1), its length $\alpha_i + \alpha_{i+1}$ must be equal to either $\beta_i + \beta_{i+1}$ or β_{i+2}. Next, $\alpha_i \neq 0$ by the nontriviality of P_i. Finally, $\beta_0 + \beta_1 + \beta_2 = |C|$, therefore this sum is odd. Hence α_i, β_i satisfy the assumptions of Lemma 2.3, which yields $\alpha_i = \beta_i$ for all i. ■

By Lemma 2.4 (1) we already know that odd cycles in G cannot be disjoint from C. Further restrictions for the position of odd and also even cycles with respect to C are expressed as the two subsequent claims.

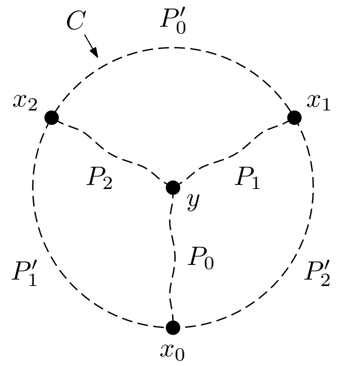


Figure 3.1. The configuration of Claim 1.

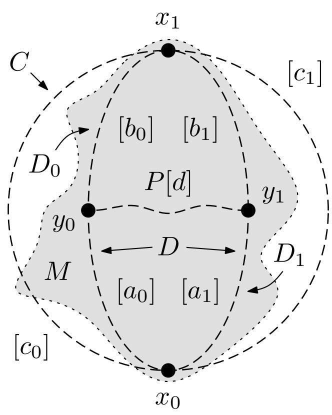


Figure 3.2. The proof of Claim 2. The lengths are enclosed in brackets.

Claim 2. *Every cycle D in G with exactly two vertices x_0, x_1 in common with C, such that $D \subseteq M$ for some C-bridge M, is even. Moreover, $||D_0|| = ||D_1||$, where D_0 and D_1 are the two subpaths of D joining x_0 and x_1.*

Proof. Let c_0, c_1 denote the lengths of the two subpaths of C delimited by x_0 and x_1. As D is a subgraph of the C-bridge M, by Lemma 1.3 (1), (2) each D_i, $i = 0, 1$, has an inner vertex, and there exists a path connecting these two vertices and avoiding both x_0 and x_1. One can take its subpath $P = y_0 \cdots y_1$ such that $V(P \cap D_i) = \{y_i\}$. Note that P is nontrivial. For brevity, let the lengths of the paths P, $x_0 D_i y_i$, and $x_1 D_i y_i$ be denoted by d, a_i, and b_i respectively. Consult Figure 3.2.

As C is odd, c_0 and c_1 have different parity. Suppose first that D is odd. Then $||D_0||$ and $||D_1||$ have different parity as well; by Lemma 2.5 (1) we obtain, without loss of generality, that

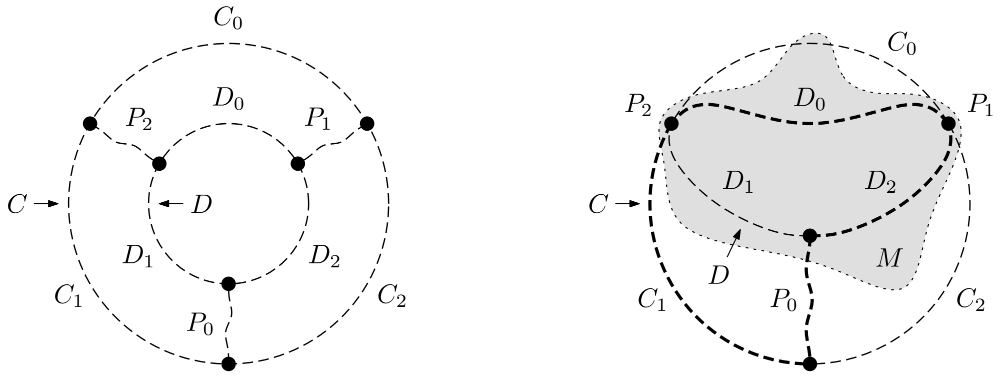
$$a_i + b_i = ||D_i|| = c_i, \quad i = 0, 1. \tag{3.1}$$

We consider two other C-paths, $x_0 D_0 y_0 P y_1 D_1 x_1$ and $x_0 D_1 y_1 P y_0 D_0 x_1$. These must satisfy Lemma 2.5 (1) too, i.e., the following holds:

$$a_0 + b_1 + d = c_{j_0} \quad \text{and} \quad a_1 + b_0 + d = c_{j_1}$$

for some $j_0, j_1 \in \{0, 1\}$. Summing the formulas and using the equalities (3.1), we conclude that

$$c_0 + c_1 + 2d = c_{j_0} + c_{j_1}. \tag{3.2}$$



(a) The general situation.

(b) The proof of statement (2). The cycle $P_0D_2D_0C_1$ is shown in bold.

Figure 3.3. Illustrations for Claim 3.

Now, if $j_0 \neq j_1$, then $d = 0$; that is impossible since P is nontrivial. Hence $j_0 = j_1$, but then the equation (3.2) turns into $c_0 + c_1 = 2(c_{j_0} - d)$, which is a parity contradiction.

Therefore, D is an even cycle. Then the paths D_0 and D_1 have lengths of the same parity; as the parity of c_0 and c_1 is different, by using Lemma 2.5 (1) once again we obtain $||D_0|| = ||D_1|| = c_j$ for some fixed $j \in \{0, 1\}$. ■

Claim 3. *Let D be a cycle in G; let P_0, P_1, and P_2 be pairwise disjoint C–D paths such that $V(C \cap D) \subseteq V(P_0 \cup P_1 \cup P_2)$. Then none of the following holds:*

(1) *at most one of P_0, P_1, and P_2 is trivial,*

(2) *exactly two of P_0, P_1, and P_2 are trivial, and $D \subseteq M$ for some C-bridge M.*

Proof. The appropriate end-vertices of P_0, P_1, and P_2 split each of the cycles C and D into three subpaths; let these be denoted by C_i, $i = 0, 1, 2$, and D_i respectively in such a way that $V(P_i \cap C_i) = \emptyset$, $V(P_i \cap D_i) = \emptyset$. Further, let $c_i := ||C_i||$. See Figure 3.3 (a).

Assume first part (1) to be true. Then, say, P_0 is the only trivial path if there is one at all. We take the paths P_0D_1, P_1D_0, P_2 and apply Claim 1, obtaining that $||P_2|| = c_2$. Doing the same for the paths P_0D_2, P_2D_0, P_1 yields $||P_2|| + ||D_0|| = c_2$. It follows that $||D_0|| = 0$, a contradiction with the assumptions.

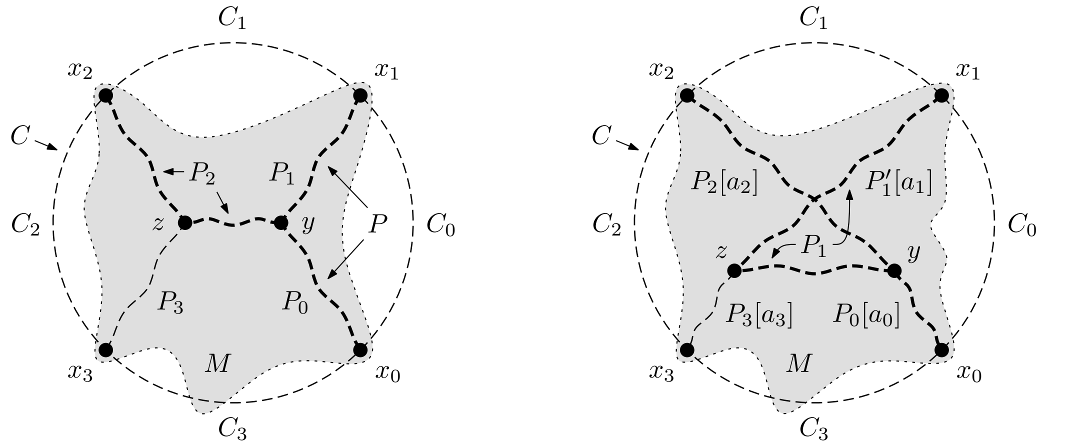
Now, suppose that assertion (2) holds. Without loss of generality, P_0 is the only nontrivial path; the situation is depicted in Figure 3.3 (b). Then by applying Claim 1 to the paths P_0, D_1, and D_2 we obtain that $||P_0|| = c_0$, $||D_1|| = c_2$, and $||D_2|| = c_1$. Next, by Claim 2 we have $||D_0|| = ||D_1|| + ||D_2||$. All this together means that there is a cycle $P_0D_2D_0C_1$ of length $c_0 + 3c_1 + c_2$ in G, an odd cycle longer than C. ■

Now we are able to describe the structure of the C-bridges. We remark that every attachment vertex of an arbitrary C-bridge M belongs to a unique block of M, since by Lemma 1.3 (2) it is not a cut-vertex of M.

Claim 4. *Let M be a C-bridge. Then:*

(1) *M is a bipartite graph;*

(2) *M has at most three attachment vertices;*



(a) The case $z \in V(P_2 \setminus y)$.

(b) The case $z \in V(P_1 \setminus y)$. The lengths are written in brackets next to the respective path labels.

Figure 3.4. The proof of Claim 4 (2). The paths P_0, P_1, and P_2 are printed in bold for clarity.

(3) *if M has precisely three attachment vertices, then each of these vertices lies in a different block of M.*

Proof. We start by proving part (2). Assume to the contrary that M is a C-bridge with more than three attachment vertices; take four of them, x_i, $i \in \mathbb{Z}_4$, located on C in a cyclic order. These vertices split C into four paths denoted by C_i in such a manner that $C_i = x_i \cdots x_{i+1}$. Let $c_i := ||C_i||$.

By Lemma 1.3 (1), (2) there exists a C-path P in M connecting x_0 with x_1 and having an inner vertex. Further, Lemma 1.3 (2) asserts that this vertex is joined to x_2 by a path not containing any vertex of $C \setminus x_2$. Taking its part $P_2 := x_2 \cdots y$ such that $V(P_2 \cap P) = \{y\}$ and putting $P_i := x_i P y$, $i = 0, 1$, we get the situation of Claim 1. However, there is one more attachment vertex x_3. Once again, x_3 must be connected with y by a path avoiding all vertices of $C \setminus x_3$; let $P_3 := x_3 \cdots z$ be its subpath such that $V(P_3) \cap V(P \cup P_2) = \{z\}$. There are four possibilities regarding the position of z in $P \cup P_2$: either $z = y$ or $z \in V(P_i \setminus y)$ for some $i \in \{0, 1, 2\}$.

Let $z = y$ or $z \in V(P_2 \setminus y)$; the latter situation is depicted in Figure 3.4 (a). Applying Claim 1 subsequently to the triples of paths $\{P_0, P_1, P_2\}$ and $\{P_0, P_1, P_3 z P_2 y\}$, we obtain that $||P_0|| = c_1$ and $||P_0|| = c_1 + c_2$ respectively. This implies that $c_2 = 0$, a contradiction. By symmetry, we can deal with the case $z \in V(P_0 \setminus y)$ similarly. Thus z lies on $P_1 \setminus y$. Let first a_i, $i = 0, \ldots, 3$, denote the lengths of the paths P_0, $P_1' := x_1 P_1 z$, P_2, and P_3 respectively. See Figure 3.4 (b) for the notation. Now, using Claim 1 twice for the triples of paths $\{P_0 P_1 z, P_1', P_3\}$ and $\{P_2 P_1 z, P_1', P_3\}$, we obtain in particular that $a_1 = c_3$, $a_3 = c_0$, $a_1 = c_2$, and $a_3 = c_1$; therefore $c_0 = c_1$, $c_2 = c_3$. By symmetry, one can show that also $c_2 = c_1$, $c_0 = c_3$. All together this implies that $c_0 = c_1 = c_2 = c_3$; thereby $|C| = 4c_0$. But this is impossible as C is odd.

Let us focus on statement (3); we refer to the three attachment vertices of M as u, v, and w. First observe that:

there is no cycle in M passing through two of the attachment vertices and avoiding the third one. (3.3)

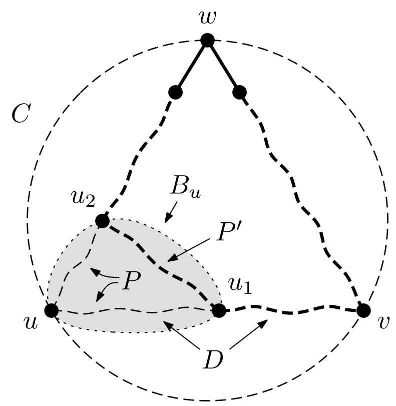


Figure 3.5. The proof of Claim 4 (3). The bold lines and edges represent the cycle contradicting observation (3.3).

If there were such a cycle D with, say, $w \notin V(D)$, then by Lemma 1.3 (2) we could find a w–D path in $M \setminus \{u, v\}$, and thus obtain the configuration excluded by Claim 3 (2).

We proceed by contradiction now: let u, v belong to a single block B of M. By Lemma 1.3 (1), u and v are not adjacent in M. Thus B is not a K_2, it is 2-connected, and there is a cycle D containing both u and v. Observation (3.3) implies that $w \in V(D)$ as well. Consider the graph $M \setminus w$ and the path $D \setminus w$. By Lemma 1.3 (1), (2), u is not an end-vertex of $D \setminus w$ nor a cut-vertex of $M \setminus w$. Hence u belongs to a unique block B_u of $M \setminus w$. Consult Figure 3.5.

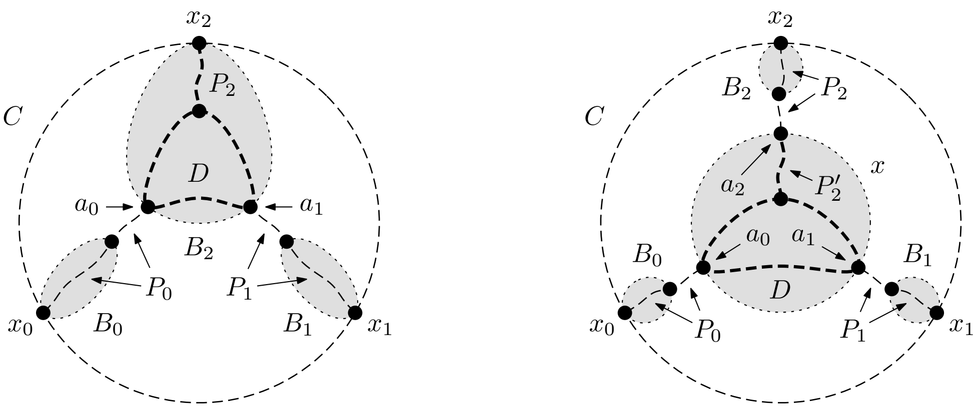
It is easy to see that, in general, every block of an arbitrary graph H and any path P_H in H either are disjoint, or intersect in a path (possibly trivial) whose end-vertices are either cut-vertices of H or end-vertices of P_H. Applying this to the graph $M \setminus w$, its block B_u, and the path $D \setminus w$, we deduce that $P := (D \setminus w) \cap B_u$ is a nontrivial path with end-vertices u_1, u_2 distinct from u. Among other things, this means that B_u is not a K_2; thus it is 2-connected, and by Theorem 1.5 there is a path $P' = u_1 \cdots u_2$ in B_u avoiding u.

Note now that $v \notin V(B_u)$. Otherwise, as B_u is 2-connected, we would easily obtain a contradiction with observation (3.3). So we can replace P with P' in the cycle D and obtain a cycle passing through v and w but not u, a contradiction with observation (3.3) again.

We prove assertion (1). So far we know by parts (2) and (3) that every block of M contains at most two attachment vertices of M. Consequently, every cycle D in M has at most two vertices in common with C. If $|D \cap C| = 2$, Claim 2 implies that D is even; if $|D \cap C| \leq 1$, it is even as well by Lemma 2.4 (1). Therefore M contains no odd cycle; it is bipartite. ■

We are finally ready to finish the proof of Theorem 3.1. Let c_C denote a prescribed proper 3-coloring of C. Clearly, it suffices to show that we can extend c_C to the graph $C \cup M$ for every C-bridge M. By Claim 4 (1), (2) and the 2-connectedness of G, any such M is bipartite, it has two or three attachment vertices, and these are pairwise nonadjacent by Lemma 2.5 (1) and Lemma 1.3 (1). Thus, if M has exactly two attachment vertices, then using Lemma 2.1 we are done. The other case is, when M has precisely three attachment vertices x_i, $i = 0, 1, 2$. We recall that none of these vertices is a cut-vertex of M. Therefore, each x_i lies in a unique block B_i of M; by Claim 4 (3) these blocks are pairwise different.

Consider the block graph $B(M)$ of M. Observe that every leaf of $B(M)$ contains x_i



(a) $B(M)$ having two leaves.　　(b) $B(M)$ having three leaves.

Figure 3.6. The proof of Theorem 3.1. The configurations given by Lemma 2.2 are depicted in bold.

for some i, otherwise G would have a cut-vertex. Hence, as $B(M)$ is nontrivial, it has two or three leaves. Assume first that the former case holds; what follows is depicted in Figure 3.6 (a). Then $B(M)$ is a path with, say, B_0 and B_1 as its end-vertices. The block B_2 contains exactly two cut-vertices of M, denoted by a_j, $j = 0, 1$, in such a way that a_j lies on the path $B_j \cdots B_2$ in $B(M)$. Note that both a_0 and a_1 are distinct from x_2. Thus we can apply Lemma 2.2 to the vertices a_0, a_1, x_2, in this order, and the block B_2, thereby obtaining a cycle $D \subseteq B_2$ passing through both a_0 and a_1 along with an x_2–D path P_2 in B_2 (possibly trivial). Further, there clearly exists a nontrivial C–B_2 path P_j, $j = 0, 1$, connecting x_j with a_j in M. But then all the three paths P_i, $i = 0, 1, 2$, together with the cycles C and D form the configuration excluded by Claim 3 (1).

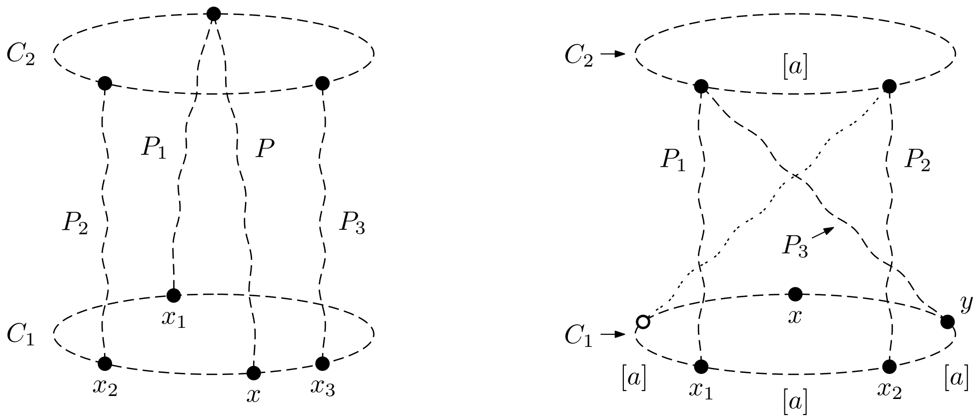
Therefore $B(M)$ has precisely three leaves; it consists of three (nontrivial) paths meeting in a single vertex x. The following discussion is illustrated in Figure 3.6 (b). Suppose that x is a block of M. Then it contains exactly three cut-vertices denoted by a_i in such a manner that a_i lies on the path $x_i \cdots x$ in $B(M)$. Applying Lemma 2.2 to a_0, a_1, a_2, and the block x, we get a cycle D containing a_0 and a_1 together with an a_2–D path P_2', both subgraphs of x. Next we can find three pairwise disjoint nontrivial C–x paths P_i, each connecting x_i with a_i. Then Claim 3 (1) used for the paths P_0, P_1, P_2P_2' and the cycles C, D yields a contradiction again.

Hence x is a cut-vertex of M. Let M_i denote the subgraph of M corresponding to the path $x_i \cdots x$ in $B(M)$. If $x_i x \in E(G)$ for all i, Claim 1 would force that $C = K_3$; thus there would be a K_4 in G. That is excluded by assumption. Therefore, without loss of generality, $x_0 x$ is not an edge of G. But this means that we can extend properly the 3-coloring c_C to the graph $C \cup M$ by first coloring x with a color of c_C distinct from both $c_C(x_1)$ and $c_C(x_2)$, and then using Lemma 2.1 for all M_i. □

4 Graphs with two odd cycle lengths

The aim of this section is to prove the main result, Theorem 1.2. As a preliminary step, we focus on the class of 4-critical graphs with $L = \{k, k+2\}$, $k \geq 5$, obtaining useful structural constraints in the form of Proposition 4.1 and Corollary 4.2.

Clearly, Lemma 2.4 (1) applies to the graphs in question. With the stronger assumption of 4-criticality and the use of Theorem 3.1, we are able to strengthen it to the following



(a) The case of three pairwise disjoint C_1–C_2 paths in G.

(b) The case of two disjoint C_1–C_2 paths in G. The dotted line shows the last possible position of a C_1–C_2 path besides those of P_1, P_2, and P_3.

Figure 4.1. The proof of Proposition 4.1: finding the vertex x.

statement.

Proposition 4.1. *Let G be a 4-critical graph with $L(G) = \{k, k+2\}$, $k \geq 5$. Then no two odd cycles in G are disjoint.*

Proof. Assume to the contrary that there are two such cycles C_1, C_2 in G. Then, by Lemma 2.4 (2a), $|C_1| = |C_2| = k$. We show first that there is a vertex x of C_1 which lies on no C_1–C_2 path. For this, we discuss two cases.

If G contains three pairwise disjoint C_1–C_2 paths P_i, $i = 1, 2, 3$, we choose x as one of the vertices in $V(C_1) \setminus \{x_1, x_2, x_3\}$, where $x_i := V(P_i \cap C_1)$. See Figure 4.1 (a). Let us prove that x indeed has the desired property. Assume to the contrary that there is a C_1–C_2 path P in G containing the vertex x. By Lemma 2.4 (2d), P is not disjoint from all P_i. Note that each P_i is just an edge, as forced by Lemma 2.4 (2b). Consequently, without loss of generality, P is disjoint from P_2 and P_3, and its other end-vertex lying on C_2 coincides with that of P_1. Now we apply Lemma 2.4 (2c) subsequently to the triples of paths $\{P_1, P_2, P_3\}$ and $\{P, P_2, P_3\}$; it implies that $d_{C_1}(x_2, x_1) = d_{C_1}(x_2, x)$ and $d_{C_1}(x_3, x_1) = d_{C_1}(x_3, x)$. As C_1 is odd, it follows that $x = x_1$, a contradiction with the choice of x.

On the other hand, let there be no three pairwise disjoint C_1–C_2 paths in G. As G is 4-critical, it is 2-connected; hence by Theorem 1.5 there are two disjoint C_1–C_2 paths P_1 and P_2. Let x_i, $i = 1, 2$, denote the end-vertex of P_i lying on C_1; let a be the distance of the other end-vertices of P_i in C_2. Consult Figure 4.1 (b). By Lemma 2.4 (2b), both P_i have length 1, and $d_{C_1}(x_1, x_2) = a$. Consider now any C_1–C_2 path P_3 with an end-vertex y on C_1 distinct from both x_i. By assumption, P_3 intersects, say, P_1 in a vertex on C_2, and it is disjoint from P_2. Then by Lemma 2.4 (2b) again, applied to P_2 and P_3, we obtain that $d_{C_1}(x_2, y) = a$ as well. This reasoning implies that there are at most four end-vertices of C_1–C_2 paths on C_1, as shown in Figure 4.1 (b). But $|C_1| \geq 5$; we choose x as any of the remaining vertices.

So, in either case, the vertex x exists. As G is 4-critical, the degree of x is at least three. Hence there is an edge incident with x not belonging to C_1 in G, thus contained in some $(C_1 \cup C_2)$-bridge H. By Lemma 1.3 (2) any two attachment vertices of H are joined by a path in H. Therefore, since x lies on no C_1–C_2 path, all the attachment vertices of

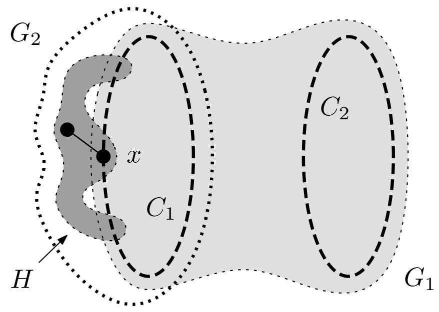


Figure 4.2. The proof of Proposition 4.1: considering the bridge H. The light, dark gray region represents G_1, H respectively; the bold dotted line bounds G_2.

H are on C_1.

We show that the existence of such a bridge leads to a contradiction. Take the graphs G_1, G_2 such that $V(G_1) = (V(G) \setminus V(H)) \cup V(C_1)$, $E(G_1) = E(G) \setminus E(H)$, and $G_2 = H \cup C_1$. The notation is depicted in Figure 4.2. G_1 is a proper subgraph of G; therefore, by the 4-criticality of G, it has a proper 3-coloring c_1. If G_2 contained a cycle of length $k+2$, this cycle together with C_2 would be a pair of disjoint odd cycles contradicting Lemma 2.4 (1). Thus $|L(G_2)| = 1$, and by Theorem 3.1 we can extend the coloring c_1 of C_1 to a proper 3-coloring c_2 of G_2. Clearly, $c_1 \cup c_2$ is then a proper 3-coloring of G. □

By a combination of Proposition 4.1 and Lemma 1.4, we easily infer the next assertion.

Corollary 4.2. *Every 4-critical graph with $L = \{k, k+2\}$, $k \geq 5$, is 3-connected.*

Proof. Suppose that, on the contrary, there exists a graph G satisfying the assumptions but containing a 2-cut $\{u, v\}$, which splits G into two graphs G_1 and G_2. Then Lemma 1.4 applied first to u and any of the vertices of G_1, next to v and any of the vertices of G_2 gives two odd cycles C_1, C_2 such that $V(C_1) \subseteq V(G_1) \cup \{v\}$, $V(C_2) \subseteq V(G_2) \cup \{u\}$. These cycles are disjoint, which contradicts Proposition 4.1. □

Now we are ready to prove the main result.

Proof of Theorem 1.2. We proceed by contradiction. Assuming that the theorem is false, we take a minimal counterexample with respect to inclusion, which clearly is a 4-critical graph G with $L(G) = \{5, 7\}$. Similarly to the proof of Theorem 3.1, we first investigate the structure of G by establishing Claims 1–6. Using these results, we finish by showing that G is in fact 3-colorable.

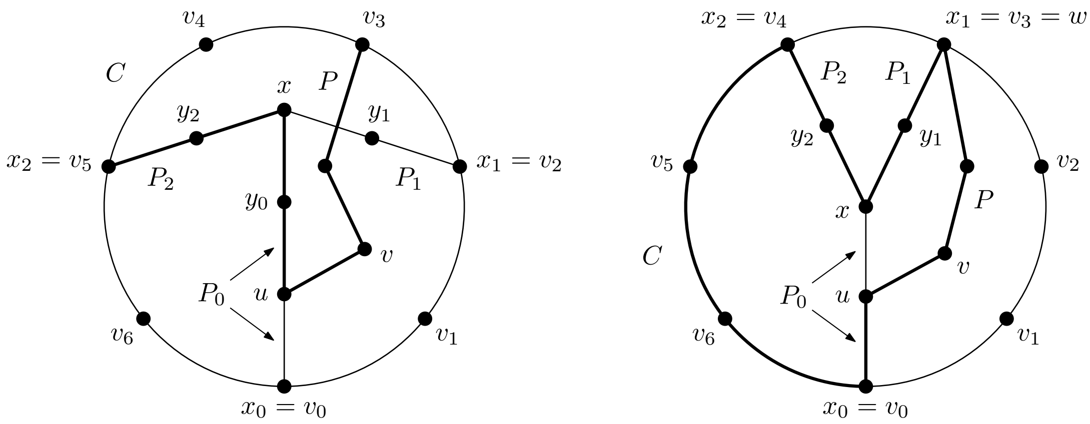
To start, fix $C = v_0 v_1 \cdots v_6$ as one of the 7-cycles in G. For convenience, we define the following notions. Let x be a vertex of $G \setminus C$. Then each of its neighbors lying on C is a *friend* of x; the set of all friends of x is denoted by $F(x)$. We make an easy observation now.

Claim 1. *The following holds:*

(1) *for any vertex x of $G \setminus C$, $F(x)$ is an independent set in G;*

(2) *let x, y be two adjacent vertices of $G \setminus C$. Then $F(x) \cap F(y) = \emptyset$, and $F(x) \cup F(y)$ is an independent set in C.*

Proof. If x and y in part (2) have friends f_1 and f_2 respectively which are adjacent in C, then there is a C-path $f_1 x y f_2$ of length 3 with its end-vertices at distance 1 in C, a contradiction with Lemma 2.5 (2). The rest is clear as there are no triangles in G. ■



(a) The case $\mathcal{P} = \{3, 2, 2\}$. The bold edges comprise the C-path P_2P_0uvP.

(b) The case $\mathcal{P} = \{2, 2, 2\}$. Boldface emphasizes the cycle $v_4P_2P_1Pvuv_0v_6v_5v_4$.

Figure 4.3. The proof of Claim 2.

By Corollary 4.2, G is 3-connected. Using this and the fact that $|C| = 7$, we are able to prove the following statement, crucial for the rest of the argument.

Claim 2. *Every vertex of $G \setminus C$ has at least one friend.*

Proof. Suppose the contrary, i.e., there exists a vertex x of G at distance at least 2 from C. As G is 3-connected, by Theorem 1.5 there are three internally disjoint x–C paths P_i, $i \in \mathbb{Z}_3$, with distinct end-vertices x_i on C. These paths constitute three C-paths $P'_i := P_{i+1}P_{i+2}$, all of which have length at most 6 by Lemma 2.5 (2a). By assumption, $||P_i|| \geq 2$; thereby also $||P_i|| \leq 4$.

We discuss all the possible forms of the multiset $\mathcal{P} = \{||P_i||, i \in \mathbb{Z}_3\}$. First, without loss of generality, let P_0 be of length 4. It follows that $||P_1|| = ||P_2|| = 2$, and P'_i have lengths 4, 6, and 6 respectively. But then Lemma 2.5 (2) applied to each of P'_i forces that $d_C(x_0, x_1) = d_C(x_0, x_2) = 1$ and $d_C(x_1, x_2) = 1$ or 3, which is clearly not possible. By similar reasoning we exclude the cases $\mathcal{P} = \{3, 3, 3\}$ and $\mathcal{P} = \{3, 3, 2\}$; hence $\mathcal{P}$ must equal either $\{3, 2, 2\}$ or $\{2, 2, 2\}$. Thus we have proven the following assertion so far:

for any vertex y of G at distance at least 2 from C, there are at least two internally disjoint y–C paths of length 2 in G. (4.1)

We continue by showing that the two remaining cases for $\mathcal{P}$ cannot occur either. Most of the time, the subsequent proof consists of applying Lemma 2.5 (2) or (2a) to various C-paths; we omit explicit reference to these particular assertions for brevity.

Focus on the case $\mathcal{P} = \{3, 2, 2\}$. Without loss of generality, let $||P_0|| = 3$, $||P_1|| = ||P_2|| = 2$, and $x_0 = v_0$. Then, considering the C-paths P'_1 and P'_2, both of length 5, we see that $d_C(x_0, x_1) = d_C(x_0, x_2) = 2$. Hence, say, $x_1 = v_2$, and $x_2 = v_5$. Let us label the inner vertices of P_0, P_1, and P_2 in such a way that $P_0 = v_0uy_0x$, $P_1 = v_2y_1x$, and $P_2 = v_5y_2x$. Figure 4.3 (a) depicts the following argument.

By the 4-criticality of G, the degree of u is greater than 2, and therefore there is at least one neighbor v of u other than v_0 and y_0. This vertex is distinct from both x and v_1, otherwise there would be a triangle in G. Next, it does not equal v_2, or else we would have a C-path $P_2P_0uv_2$ of length 5 with its end-vertices at distance 3 in C. If $v = v_3$, there would be a C-path $P_1P_0uv_3$ of length 5 with its end-vertices at distance 1 in C. Finally, v cannot be y_1, otherwise the C-path $P_2P_0uy_1v_2$ would have length 6, but its end-vertices are not adjacent in C. As the vertices v_4, v_5, v_6, and y_2 can be treated in a corresponding manner by symmetry, we conclude that $v \notin V(C \cup P_0 \cup P_1 \cup P_2)$.

We analyze which vertices of $C \cup P_0 \cup P_1 \cup P_2 \setminus u$ can be adjacent to v. First we show that v has no friends. Assume the contrary; without loss of generality, we can choose a friend w of v different from v_5. Hence, there is a C-path P_2P_0uvw of length 6 in G. Consequently, $w = v_4$, or $w = v_6$. But then, considering the C-path P_1P_0uvw also of length 6, we obtain a contradiction in both the cases. Next, $vy_1 \notin E(G)$ as otherwise we would get a C-path $P_2P_0uvy_1v_2$ of length 7. By symmetry, the vertex y_2 is not adjacent to v either. Finally, $vy_0 \in E(G)$ would create a triangle in G. Thus the only possible neighbor of v in the set $V(C \cup P_0 \cup P_1 \cup P_2) \setminus \{u\}$ is x.

That means, v is at distance 2 from C. Therefore we may use statement (4.1) and obtain two internally disjoint v–C paths of length 2. At least one of them, P, does not pass through u. It also avoids x because the distance between x and C is greater than 1 by assumption. Hence, P is disjoint from $P_0 \cup P_1 \cup P_2 \setminus C$ by the previous discussion. Without loss of generality, its other end-vertex is distinct from v_5. But then there is a C-path P_2P_0uvP of length 7, a contradiction.

It remains to discuss the case $\mathcal{P} = \{2, 2, 2\}$; we proceed similarly as above. Consult Figure 4.3 (b) for the following. This time, the C-path P'_i of length 4 forces that $d_C(x_{i+1}, x_{i+2}) = 1$ or 3 for every i. One can easily check that this implies, up to symmetry, $x_0 = v_0$, $x_1 = v_3$, and $x_2 = v_4$. Let u, y_1, and y_2 denote the inner vertex of P_0, P_1, and P_2 respectively.

Since the degree of u is at least 3, there is a neighbor v of u distinct from both v_0 and x. It cannot equal v_1 or y_1 because there are no triangles in G. If $uv_2 \in E(G)$, then P_2xuv_2 would be a C-path of length 4 with its end-vertices at distance 2 in C. Also, $v \neq v_3$; in such a case we would have a 9-cycle $v_4P_2P_1v_3uv_0v_6v_5v_4$ in G. By symmetry, we can exclude also the vertices y_2, v_4, v_5, and v_6. Hence $v \notin V(C \cup P_0 \cup P_1 \cup P_2)$.

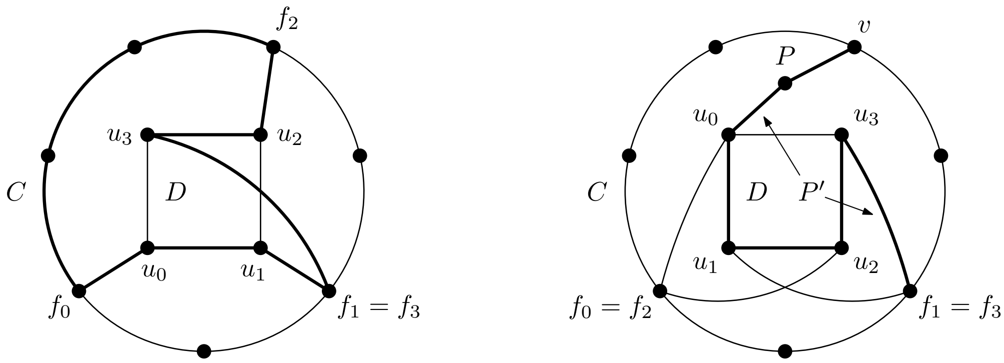
We investigate which vertices of $C \cup P_0 \cup P_1 \cup P_2 \setminus u$ can be neighbors of v. First, let there be a friend w of v. Without loss of generality, $w \neq v_4$. Then the path P_2xuvw of length 5 is a C-path; therefore $d_C(v_4, w) = 2$. It follows that $d_C(v_3, w) = 1$ or 3, which is impossible due to the C-path P_1xuvw also of length 5. Hence v has no friends. Next, if $vy_1 \in E(G)$, we would have a C-path $P_2xy_1vuv_0$ of length 6 with end-vertices not adjacent in C. The vertex y_2 is excluded as well by symmetry. Finally, $vx \in E(G)$ would imply a triangle in G. Thus, v has no neighbor in $C \cup P_1 \cup P_2 \cup P_3$ besides u.

As we have just shown, the distance between v and C is 2; hence statement (4.1) can be applied to v. Using this and the preceding analysis, we see that there exists a v–C path $P = v \cdots w$ of length 2 which is disjoint from $P_1 \cup P_2 \cup P_3 \setminus C$. We may assume that $w \neq v_4$. The C-path P_2xuvP of length 6 then forces that $w = v_3$ or $w = v_5$. But the latter case cannot occur, as seen by considering the C-path P_1xuvP also of length 6. Therefore w equals v_3. Then $v_4P_2P_1Pvuv_0v_6v_5v_4$ is a 11-cycle, a contradiction with the assumption about $L(G)$. ■

Having established the preceding claim, we can restrict significantly the structure of $G \setminus C$.

Claim 3. *$G \setminus C$ is a forest.*

Proof. Assume that the claim is false; let D be a cycle in $G \setminus C$. We take two arbitrary vertices w_0, w_1 consecutive on D. By Claim 2, both the vertices have friends; let g_0, g_1 be a friend of w_0, w_1 respectively. The vertices g_0 and g_1 are distinct by Claim 1 (2); therefore the edges g_0w_0, g_1w_1 together with the nontrivial part of D between w_0 and



(a) The case $f_0 \neq f_2$. The cycle leading to contradiction is shown in bold.

(b) The case $f_j = f_{j+2}$, $j = 0, 1$. The C-path P' is printed in bold.

Figure 4.4. The proof of Claim 3.

w_1 constitute a C-path P of length $|D| + 1$. Lemma 2.5 (2a) then implies that $|D| \leq 5$. Thus, by Lemma 2.4 (1) and the assumption about $L(G)$, $|D| = 4$. Hence P is of length 5, and Lemma 2.5 (2) forces that

$$d_C(g_0, g_1) = 2. \tag{4.2}$$

Now, let u_i, $i \in \mathbb{Z}_4$, denote the vertices of D in a cyclic order. Suppose first that there are two vertices opposite on D, say, u_0 and u_2, with distinct friends f_0 and f_2 respectively; we pick a friend f_j of u_j, $j = 1, 3$. Then, by the equation (4.2), we see that $d_C(f_i, f_{i+1}) = 2$ for all i. Hence $d_C(f_0, f_2) = 3$, and, consequently, the vertices f_1 and f_3 coincide. The situation is shown in Figure 4.4 (a). But then the cycle consisting of the path $f_0u_0u_1f_1u_3u_2f_2$ and the appropriate part of C has length 9, a contradiction.

Thus we must have the following arrangement: every u_i has exactly one friend f_i, it is $f_j = f_{j+2}$, $j = 0, 1$, and $d_C(f_0, f_1) = 2$. See Figure 4.4 (b). However, as G is 3-connected, by Theorem 1.5 there exist three pairwise disjoint D–C paths in G. At least one of them, P, has its end-vertex v on C distinct from both f_0 and f_1. We may assume that the other end-vertex of P is u_0 by symmetry. P is not a single edge; considering the C-path $P' := Pu_0u_1u_2u_3f_1$ and Lemma 2.5 (2a), we see that $||P|| = 2$. Then Lemma 2.5 (2) applied subsequently to the C-paths P' of length 6 and $Pu_0u_1u_2f_0$ of length 5 asserts that $d_C(v, f_1) = 1$ and $d_C(v, f_0) = 2$ respectively, which is impossible. ■

We proceed to a detailed analysis of the possible structure and mutual position of the C-bridges. Let us start with an auxiliary observation.

Claim 4. *Let w_i, $i = 1, \ldots, 4$, be four consecutive vertices of C; let x, y be two distinct vertices of $G \setminus C$. Then it cannot be $w_1, w_3 \in F(x)$ and $w_2, w_4 \in F(y)$.*

Proof. If the contrary held, $w_1xw_3w_2yw_4Cw_1$ would be a 9-cycle in G. ■

Claim 1 (1) and the fact that $|C| = 7$ imply that any vertex x of $G \setminus C$ has at most three friends. Moreover, if it has three friends f_0, f_1, and f_2, the multiset $\{d_C(f_i, f_{i+1}), i \in \mathbb{Z}_3\}$ equals $\{2, 2, 3\}$. Extending this, the subsequent claim describes completely the C-bridges involving at least one vertex with three friends.

Claim 5. *The following holds:*

(1) *every C-bridge containing at least one vertex with three friends is a $K_{1,3}$;*

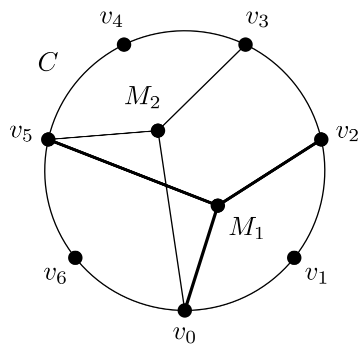


Figure 4.5. The proof of Claim 5 (2). The edges of M_1 are distinguished with boldface.

(2) *there exists an $i \in \mathbb{Z}_7$ such that every C-bridge involving at least one vertex with three friends intersects C in either $\{v_i, v_{i+2}, v_{i+5}\}$ or $\{v_i, v_{i+3}, v_{i+5}\}$.*

Proof. We begin with part (1). To the contrary, let M be a C-bridge which contains a vertex x with three friends, and $M \neq K_{1,3}$. Then $|M \setminus C| > 1$, and hence, by Lemma 1.3 (2), there is a vertex in $M \setminus C$ adjacent to x. This vertex has a friend f, as forced by Claim 2; f together with the three friends of x constitute by Claim 1 (2) a 4-element independent set in C, a contradiction.

We proceed to part (2). Combining statement (1) and the discussion immediately preceding Claim 5, one easily obtains that every C-bridge involving a vertex with three friends intersects C in $\{v_i, v_{i+2}, v_{i+5}\}$ for some $i \in \mathbb{Z}_7$. Thus, if all the C-bridges under consideration have the same attachment vertices, we are done.

On the other hand, let there be two C-bridges M_1 and M_2, each with a vertex having three friends, such that $V(M_1 \cap C) \neq V(M_2 \cap C)$. Without loss of generality, the attachment vertices of M_1 are v_0, v_2, and v_5. By assumption, the attachment vertices of M_2 must be v_i, v_{i+2}, and v_{i+5} for some $i \in \mathbb{Z}_7 \setminus \{0\}$. Now, if $i \in \{1, 3, 4, 6\}$, we would get a configuration forbidden by Claim 4. Hence there remain two cases, $i = 2$ or $i = 5$, identical up to rotation; say, $i = 5$. The situation is shown in Figure 4.5.

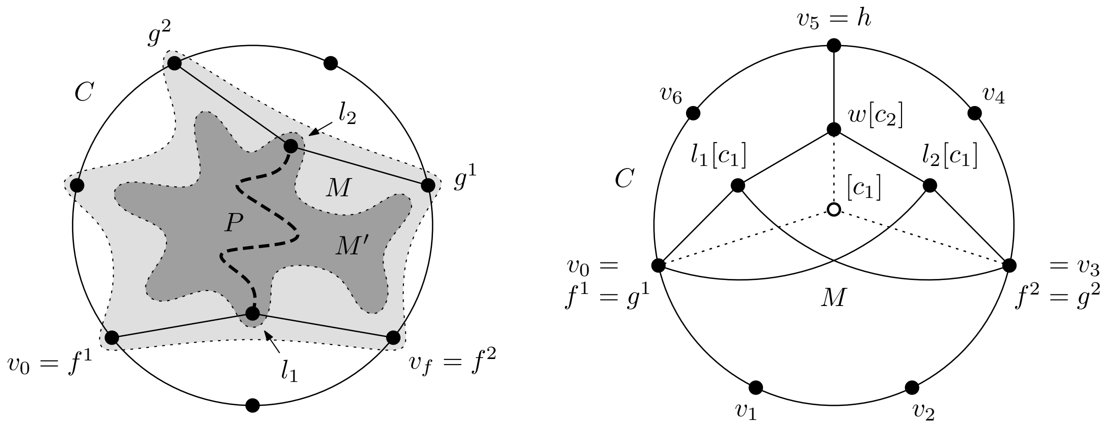
Consider a C-bridge M involving a vertex with three friends such that $V(M \cap C) \neq V(M_j \cap C)$ for both $j = 1, 2$. The preceding analysis repeated for M and M_j, j fixed, implies that M has exactly two attachment vertices in common with M_j, and these are at distance 2 in C. But that obviously cannot occur for both j at the same time, as seen in Figure 4.5; this is a contradiction. Therefore, all the C-bridges in question have the same attachment vertices as either of M_j, and the claim is proven. ∎

Now we characterize the other C-bridges.

Claim 6. *Every C-bridge containing no vertex with three friends is a K_2.*

Proof. Assume to the contrary that there is a C-bridge M without vertices having three friends, such that the graph $M' := M \setminus C$ is nonempty. By Claim 3 and the connectedness of a C-bridge, M' is a tree. Next, every vertex of M' has degree at least 3 in M by the 4-criticality of G; since it has at most two friends by assumption, its degree in M' is nonzero. Therefore, M' is not trivial, and hence has at least two leaves. Note that all leaves of M' have precisely two friends.

We take an arbitrary pair of distinct leaves l_1, l_2 of M'; let $F(l_1) = \{f^1, f^2\}$, $F(l_2) = \{g^1, g^2\}$. The leaves are joined by a unique path P in M'; we refer to its length as d. Further, let $f := d_C(f^1, f^2)$, $g := d_C(g^1, g^2)$, and D be the set $\{d_C(f^i, g^j), i, j =$



(a) The notation. The light, dark gray region depicts M, M' respectively; boldface emphasizes the path P.

(b) The final contradiction. The colors are shown in brackets next to the corresponding vertex labels or vertices; the dotted edges and the white vertex represent the possible remainder of M.

Figure 4.6. Illustrations for the proof of Claim 6.

$1, 2\}$. Without loss of generality, we may assume that $f^1 = v_0$ and $f^2 = v_f$. Consult Figure 4.6 (a) for the situation and notation.

Considering Claim 1 (1) and the fact that $|C| = 7$, we see that $f, g \in \{2, 3\}$. Next, observe the basic properties of the set D:

$$0 \in D \;\Rightarrow\; \{f, g\} \subseteq D, \tag{4.3}$$

$$|D| \;\geq\; 2. \tag{4.4}$$

Statement (4.3) is obvious. To prove assertion (4.4), suppose the contrary, $D = \{k\}$. Clearly, $k \neq 0$ by statement (4.3), and hence the friends of l_1 and l_2 are pairwise distinct. But then both g^1 and g^2 must be at distance k from both f^1 and f^2 in C, which cannot occur as there is at most one vertex with this property.

Now, every two distinct friends f^i and g^j are joined by a C-path P^{ij} of length $d + 2$ consisting of the edges $f^i l_1$, $g^j l_2$, and the path P. Thus, first, Lemma 2.5 (2) lists the possible nonzero elements of D when d is known; for brevity, we use this fact without explicit reference in the following. Second, since by statement (4.3) such a pair of distinct friends indeed exists, it follows that $d \leq 4$ by Lemma 2.5 (2a). Let us consider all the possible values of d.

First, assume that $d = 4$. Then $D \subseteq \{0, 1\}$. Hence, $f \notin D$ and, by assertion (4.3), $D = \{1\}$. That, however, contradicts statement (4.4).

If $d = 3$, $D \subseteq \{0, 2\}$; thus $D = \{0, 2\}$ by statement (4.4). Claim (4.3) then forces that $f = g = 2$. As $0 \in D$, it is, say, $f^1 = g^1$. If f^2 and g^2 were distinct, the vertex g^2 would coincide with v_5, but then $d_C(f^2, g^2) = 3$, a contradiction. Thus $f^2 = g^2$. Now, we consider the inner vertex w of P adjacent to l_1. By Claim 2, this vertex has a friend h; Claim 1 (2) applied to l_1 and w implies that, without loss of generality, $h = v_5$. Consequently, there is a C-path $f^1 l_2 P w h$ of length 4 with its end-vertices at distance 2 on C, which contradicts Lemma 2.5 (2).

Obviously, $d \neq 1$. Otherwise, by Claim 1 (2), the friends of l_1 and l_2 would form a set of four independent vertices in C, a contradiction.

Therefore d equals 2, and hence $D \subseteq \{0, 1, 3\}$. Focus on the unique inner vertex w of P; it has a friend h by Claim 2. As w is adjacent to both l_1 and l_2, we can use Claim 1 (2)

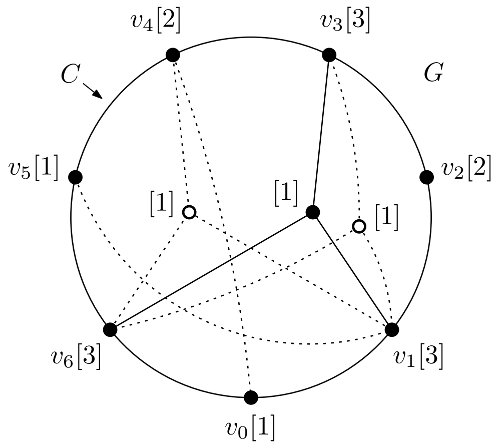


Figure 4.7. The proof of Theorem 1.2: the 3-colorability of G when $\mathcal{S}_1 \neq \emptyset$. The colors are enclosed in brackets; the dotted edges and white vertices represent the possibly nonexistent parts of G.

twice to see that h is the unique friend of w, it is distinct from the friends of l_1 and l_2, and the sets $F_1 := \{f^1, f^2, h\}$, $F_2 := \{g^1, g^2, h\}$ are both independent in C. Suppose first that either of l_1, l_2 has its friends at distance 2 in C, say, it is $f = 2$. Then, by statement (4.3), $D \subseteq \{1, 3\}$, which means that $\{g^1, g^2\} \subseteq \{v_1, v_3, v_6\}$. Claim 4 forces in turn that $\{g^1, g^2\} = \{v_3, v_6\}$, but then there is no vertex on C satisfying the conditions imposed on h, a contradiction. Thus $f = g = 3$. By the independence of F_1 and F_2 in C again, it follows that $\{f^1, f^2\} = \{g^1, g^2\} = \{v_0, v_3\}$ and $h = v_5$.

We sum up all the facts inferred so far. Every two leaves of M' are at distance 2 in M', which means that M' is a star. Next, any two, hence all, leaves of M' have the same (two) friends, and these are at distance 3 in C; without loss of generality, they coincide with v_0 and v_3. Finally, the central vertex w of M' has a unique friend v_5. See Figure 4.6 (b).

But the structure of M then contradicts the 4-criticality of G because any proper 3-coloring c of $G \setminus M'$ can be extended to G as follows. We color all the leaves of M' with a color c_1 of c distinct from both $c(v_0)$ and $c(v_3)$, and assign a color c_2 of c equal to neither c_1 nor $c(v_5)$ to w. ■

We can finally derive a contradiction by proving that G is 3-colorable. Without loss of generality, by Claims 5 and 6 we can partition all the C-bridges into three sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{K}$ such that $\mathcal{S}_1$ and $\mathcal{S}_2$ contain only $K_{1,3}$'s intersecting C in $\{v_1, v_3, v_6\}$, $\{v_1, v_4, v_6\}$, respectively, and $\mathcal{K}$ consists of single edges. Next, we take a proper coloring c of C with colors 1, 2, and 3 such that the color classes are $\{v_0, v_5\}$, $\{v_2, v_4\}$, and $\{v_1, v_3, v_6\}$ respectively. Consult Figure 4.7. The coloring c has the property that there is only one pair of equally colored vertices at distance 3 in C, the vertices v_3 and v_6; using Lemma 2.5 (2b) one observes that:

if $v_3v_6 \notin \mathcal{K}$, *then* c *is a proper coloring of* $C \cup \bigcup \mathcal{K}$. (4.5)

Now we discuss several cases. If $\mathcal{S}_1 \neq \emptyset$, then the assumption of statement (4.5) is satisfied, otherwise there would be a triangle in G. As c can be easily extended to any element of $\mathcal{S}_1 \cup \mathcal{S}_2$ by coloring its central vertex with 1, we conclude that G is indeed 3-colorable. The case $\mathcal{S}_2 \neq \emptyset$ is handled analogously by symmetry. Finally, there remains the possibility that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are empty sets. If $\mathcal{K}$ is empty as well, there is nothing to prove. Otherwise, without loss of generality, $v_2v_6 \in \mathcal{K}$. Then $v_3v_6 \notin \mathcal{K}$ clearly, and we are done by observation (4.5) again. □

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