Gray code compression*

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Abstract. An n-bit (cyclic) Gray code is a (cyclic) sequence of all n-bit strings such that consecutive strings differ in a single bit. We describe an algorithm which for every positive integer n constructs an n-bit cyclic Gray code whose graph of transitions is the d-dimensional hypercube Q_d if $n = 2^d$, or a subgraph of Q_d if $2^{d-1} < n < 2^d$. This allows to compress sequences that follow this code so that only $\Theta(\log \log n)$ bits per n-bit string are needed.

1 Introduction

An n-bit (*cyclic*) *Gray code* $\mathbf{C}_n = (u_1, u_2, \ldots, u_N)$ where $N = 2^n$ is a (cyclic) sequence listing all n-bit strings, so that every two consecutive strings differ in exactly one bit. This corresponds to a Hamiltonian path (cycle) in the n-dimensional hypercube Q_n. A well-known example of such a code [3] is the reflected Gray code $\mathbf{\Gamma}_n$ which may be defined recursively by

$$\mathbf{\Gamma}_1 = (0, 1) \qquad \mathbf{\Gamma}_{n+1} = 0\mathbf{\Gamma}_n, 1\mathbf{\Gamma}_n^R \tag{1.1}$$

where $b\,\mathbf{S}$ denotes the sequence $\mathbf{S}$ with $b \in \{0, 1\}$ prefixed to each string, and $\mathbf{S}^R$ denotes the sequence $\mathbf{S}$ in reverse order.

Gray codes are named after Frank Gray, who in 1953 patented the use of the reflected code $\mathbf{\Gamma}_n$ for shaft encoders: a pattern representing the code, printed on a shaft, determines the angle of shaft rotation. Since then, a considerable attention has been paid to the research on Gray codes satisfying certain additional properties, and applications have been found in such diverse areas as graphics and image processing, information retrieval or signal encoding [8]. Here we are particularly concerned with applications of Gray codes in the field of data compression [7, Section 4.2.1].

The *transitional sequence* $\tau(\mathbf{C}_n) = [t_1, t_2, \ldots, t_N]$ of a code $\mathbf{C}_n$ lists the positions (called *transitions*) $t_i \in [n] = \{1, 2, \ldots, n\}$ for $i \in [N]$ in which u_i and u_{i+1} differ. For simplicity, the indices are always taken cyclically, thus u_{N+1} is

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identified with u_1. A graph G_{C_n} *induced by* $\mathbf{C}_n$ (sometimes called the *graph of transitions* of $\mathbf{C}_n$) is defined by

$$V(G_{C_n}) = [n] \text{ and } E(G_{C_n}) = \{t_i t_{i+1} \mid i \in [N]\}.$$

See Figure 1 for an illustration. Slater [9, 10] and independently Bulbena and Ruskey [1], motivated by applications of Gray codes, asked what graphs can be induced by (cyclic) Gray codes. For example, the star $K_{1,n-1}$ is induced by the reflected Gray code $\mathbf{\Gamma}_n$ defined by (1.1).

The problem to characterize graphs which can be induced by (cyclic) Gray codes is still widely open. By computational search, Bulbena and Ruskey [1] catalogued these graphs for $n \leq 5$, and Ernst and Wilmer [12] extended the list to $n \leq 7$. For general n, there are only some partial results, positive and negative.

Bulbena and Ruskey [1] showed that every tree of diameter 4 can be induced by a cyclic Gray code. On the other hand, no tree of diameter 3 can be induced by such code. Also, they conjectured that all trees induced by cyclic Gray codes have diameter 2 or 4. This was disproved by Ernst and Wilmer [12] who introduced so called supercomposite Gray codes which induce trees of arbitrarily large diameter. Moreover, they answered two questions from [1] by showing that supercomposite Gray codes induce spanning trees of arbitrary 2-dimensional grids, and for a directed version of the problem, that there are cyclic Gray codes that induce digraphs with no bidirectional edge. Futhermore, Suparta and van Zanten [11] showed that the complete graph can also be induced by cyclic Gray codes, which solves a problem in [12]. Among many open problems posed in [1, 9–12], it is particularly interesting whether paths and cycles can be induced by (cyclic) Gray codes.

$$\mathbf{C}_4 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 0, & 0 \end{pmatrix}$$

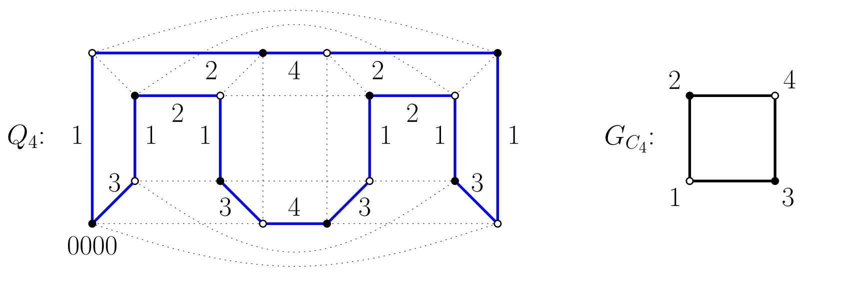

Fig. 1. The cyclic Gray code $\mathbf{C}_4$, the corresponding Hamiltonian cycle of Q_4 and the graph G_{C_4} induced by $\mathbf{C}_4$. The transitional sequence is $\tau(\mathbf{C}_4) = [3, 1, 2, 1, 3, 4, 3, 1, 2, 1, 3, 1, 2, 4, 2, 1]$.

In this paper, for every positive integer n we construct an n-bit cyclic Gray code $\mathbf{C}_n$ which induces the d-dimensional hypercube Q_d if $n = 2^d$, or a subgraph of Q_d if $2^{d-1} < n < 2^d$. More precisely, since the vertices of G_{C_n} are labeled by the elements of $[n]$, we obtain the graph Q^*_d defined by

$$V(Q^*_d) = [2^d] \text{ and } E(Q^*_d) = \{xy \mid \text{where } |x - y| = 2^i \text{ for some } 0 \leq i < d\}.$$

Clearly, $Q^*_d \cong Q_d$ by the isomorphism that maps $x \in [2^d]$ to the binary representation of $x - 1$.

We conclude the introduction with an explanation of the title of this paper. Note that every Gray code $\mathbf{C}_n = (u_1, u_2, \ldots, u_N)$ is uniquely determined by its first string u_1 and the transitional sequence $\tau(\mathbf{C}_n) = [t_1, t_2, \ldots, t_N]$. Since each transition is an integer from $[n]$, it may be encoded with $d = \lceil \log_2 n \rceil$ bits. This provides a representation of $\mathbf{C}_n$ with $\Theta(\log n)$ bits per one n-bit string.

However, in case that $\mathbf{C}_n$ induces a subgraph of Q^*_d, we may further explore the property that two consecutive transitions of $\tau(\mathbf{C}_n)$ always form an edge of Q^*_d. Indeed, each transition t_{i+1}, $i \in [N-1]$, is then determined by the preceding transition t_i and by the edge $t_i t_{i+1} \in E(Q^*_d)$, which may be represented by its *direction*

$$d(t_i t_{i+1}) = j \text{ such that } |t_i - t_{i+1}| = 2^j.$$

Consequently, the code $\mathbf{C}_n$ may be represented by the sequence

$$u_1, t_1, d(t_1 t_2), d(t_2 t_3), \ldots, d(t_{N-1} t_N).$$

Since edges of Q^*_d occur only in d directions, each $d(t_i t_{i+1})$ for $i \in [N-1]$ may be encoded with $\lceil \log_2 d \rceil$ bits. Hence we obtain a representation of $\mathbf{C}_n$ which requires only $\Theta(\log \log n)$ bits on the average to represent one n-bit string of the code, which outperforms the $\Theta(\log n)$ bits obtained above.

2 Preliminaries

For the rest of the paper, all Gray codes are cyclic. Let $\mathbf{C}_n = (u_1, u_2, \ldots, u_N)$ be a Gray code where n denotes the dimension of the code and $N = 2^n$, and let $\tau(\mathbf{C}_n) = [t_1, t_2, \ldots, t_N]$ be the transitional sequence of $\mathbf{C}_n$. We deal with $\mathbf{C}_n$ as with a Hamiltonian cycle of the n-dimensional hypercube Q_n, which is the graph with $V(Q_n) = \{0, 1\}^n$ and $uv \in E(Q_n)$ if and only if u and v differ in exactly one coordinate. For a vertex $v \in V(Q_n)$ let $Q_n - v$ denote the graph obtained by removing v and incident edges from Q_n.

Let e_i denote the vertex of Q_n with 1 exactly in the i-th coordinate for $i \in [n]$. Thus $u_i \oplus u_{i+1} = e_{t_i}$ for every $i \in [N]$ where $\oplus$ denotes the (coordinatewise) addition modulo 2. Moreover, let $e_{ij} = e_i \oplus e_j$ for distinct $i, j \in [n]$. The elements of $[n]$ are called *directions*.

Let $\mathbf{C}^R_n = (u_N, \ldots, u_2, u_1)$ denote the Gray code $\mathbf{C}_n$ in reverse order. Similarly, for any path $P = (v_1, v_2, \ldots, v_m)$ of Q_n, let $P^R = (v_m, \ldots, v_2, v_1)$ denote the reverse of P. The notion of transitional sequences and induced graphs can be naturally extended to paths as follows. We define $\tau(P) = [p_1, p_2, \ldots, p_{m-1}]$

where p_i for $i \in [m-1]$ is the coordinate in which v_i and v_{i+1} differ, and the graph G_P induced by P by

$$V(G_P) = [n] \text{ and } E(G_P) = \{p_i p_{i+1} \mid i \in [m-2]\}.$$

Note that for cycles, the transitional sequence is considered to be cyclic, whereas for paths it is not.

Let $T = [t_1, t_2, \ldots, t_m]$ be a (cyclic) transitional sequence of a path $(u_1, u_2, \ldots, u_{m+1})$ (resp. of a cycle $(u_1, u_2, \ldots, u_m)$). We say that T contains a *segment* $S = [s_1, s_2, \ldots, s_k]$ if there is $j \in [m-k]$ (resp. $j \in [m]$) such that

$$s_i = t_{i+j-1} \text{ for all } i \in [k].$$

Furthermore, if k is even, we say that S is *centered at a vertex* $u_{j+k/2}$. For example, $\tau(\mathbf{C}_4)$ on Figure 1 contains a segment $[2,1,3,1]$ centered at $u_1 = 0000$.

We say that a direction t is *repeating* in a transitional sequence T, if T contains a segment $[t, x, t]$ for some x.

Let $\pi : [n] \to [n]$ be a permutation and $w = (w_1 w_2 \cdots w_n) \in \{0,1\}^n$ be a vector called *translation*. It is well known that the mapping $\varrho : V(Q_n) \to V(Q_n)$ given by

$$\varrho(u_1 u_2 \cdots u_n) = (v_1 v_2 \cdots v_n) \text{ such that } v_i = u_{\pi(i)} \oplus w_i \text{ for every } i \in [n] \quad (2.1)$$

is an automorphism of Q_n. Moreover, for every automorphism ϱ of Q_n there exist unique π and w such that ϱ is given by (2.1). That is, every hypercube automorphism is composed of a unique permutation of coordinates and a unique translation. The translation determines where the vertex $\mathbf{0} = (00 \cdots 0)$ is mapped, i.e. $\varrho(\mathbf{0}) = w$.

The hypercube Q_n may be expressed as a Cartesian product $Q_n = Q_k \square Q_{n-k}$ for $1 \leq k < n$. Every vertex $v \in V(Q_n)$ is then represented as a pair $v = (v_1, v_2)$ where $v_1 \in V(Q_k)$ and $v_2 \in V(Q_{n-k})$. The subgraph of Q_n induced on vertices (v_1, v_2) for all $v_1 \in V(Q_k)$ and fixed $v_2 \in V(Q_{n-k})$ is called a *subcube* and denoted by $Q_k(v_2)$. Clearly, $Q_k(v_2)$ is isomorphic to Q_k. Thus, Q_n may be viewed as Q_{n-k} in which every vertex $v_2 \in V(Q_{n-k})$ corresponds to the subcube $Q_k(v_2)$ and every edge $v_2 v_3 \in E(Q_{n-k})$ corresponds to the collection of edges $(v_1, v_2)(v_1, v_3)$ for all $v_1 \in V(Q_k)$.

In particular, the graph Q^*_{d+1} defined in the previous section can be decomposed into two subcubes denoted by Q^A_d and Q^B_d induced on the sets $A = \{1, 2, \ldots, n\}$ and $B = \{n+1, n+2, \ldots, 2n\}$. Note that by the definition, every vertex $i \in A$ of Q^A_d is joined in Q^*_{d+1} only with the vertex $n + i \in B$ of Q^B_d.

Let G_{C_n} be the graph induced by the Gray code $\mathbf{C}_n$. A transition t_j where $j \in [N]$ is *critical* for G_{C_n} if at least one of the edges $t_{j-1}t_j$, $t_j t_{j+1} \in E(G_{C_n})$ is induced by no other pair of consecutive transitions in $\tau(\mathbf{C}_n)$, i.e. $E(G_{C_n}) \neq \{t_i t_{i+1} \mid i \in [N] \setminus \{j-1, j\}\}$. If we view the cycle $\mathbf{C}_n$ in Q_n as a path $\mathbf{P}_n$, then $\tau(\mathbf{C}_n) = [\tau(\mathbf{P}_n), t_N]$. Thus, if t_N is not critical for G_{C_n}, we obtain that $G_{P_n} = G_{C_n}$.

3 Inducing the hypercube

In this section, we construct an n-bit Gray code $\mathbf{C}_n$ for $n = 2^d$ that induces the hypercube Q_d^*. The following lemma shows that under certain conditions, we may modify a Gray code so that the induced Q_d^* is preserved, and at the same time, a given segment of its transitional sequence is replaced by a new prescribed one.

Lemma 3.1. *Let* $\mathbf{C}$ *be an* n*-bit Gray code with* $G_C = Q_d^*$ *such that* $\tau(\mathbf{C}_n)$ *contains two disjoint occurrences of a segment* $[a, b, a, c]$ *where* a, b, c *are distinct and* $n = 2^d$. *Let* S *be a segment* $[x, y, x, z]$ *or* $[z, x, y, x]$ *where* x, y, z *are distinct and* $xy, xz \in E(G_C)$, *and let* v *be a vertex of* Q_n. *Then, there exists a Gray code* $\mathbf{B}$ *such that* $G_B = Q_d^*$, *each occurrence of* $[a, b, a, c]$ *in* $\tau(\mathbf{C})$ *is replaced by* S *in* $\tau(\mathbf{B})$, *and one of them is centered at the vertex* v.

Proof. We assume that $S = [x, y, x, z]$, otherwise we proceed with S^R and obtain $\mathbf{B}^R$, so by changing the direction we get $\mathbf{B}$. Let one occurrence of $[a, b, a, c]$ be centered at a vertex $u \in V(Q_n)$. Since $ab, ac \in E(G_C)$ and $G_C = Q_d^*$, we can extend the mapping $\pi(a) = x$, $\pi(b) = y$, $\pi(c) = z$ to a permutation $\pi : [n] \to [n]$ such that π is an automorphism of G_C. Consider the automorphism ϱ of Q_n given by (2.1) with the permutation π and a translation vector $w = (w_1 w_2 \cdots w_n) \in \{0,1\}^n$ such that $w_i = u_{\pi(i)} \oplus v_i$ for all $i \in [n]$.

It follows directly by (2.1) that $\varrho(u) = v$, and furthermore, ϱ maps the subsequence $(u \oplus e_{ab}, u \oplus e_b, u, u \oplus e_a, u \oplus e_{ac})$ of the code $\mathbf{C}$ to

$$\varrho(u \oplus e_{ab}, u \oplus e_b, u, u \oplus e_a, u \oplus e_{ac}) = (v \oplus e_{xy}, v \oplus e_y, v, v \oplus e_x, v \oplus e_{xz}).$$

Hence, for the n-bit Gray code $\mathbf{B} = \varrho(\mathbf{C})$, each occurrence of $[a, b, a, c]$ in $\tau(\mathbf{C})$ is replaced by S in $\tau(\mathbf{B})$, and one of them is centered at the vertex v. Moreover, for every $p, q \in [n]$,

$$pq \in E(G_B) \text{ if and only if } \pi^{-1}(p)\pi^{-1}(q) \in E(G_C) \text{ if and only if } pq \in E(G_C).$$

The first equivalence holds by the definition of ϱ, the latter holds since π in an automorphism of G_C. It follows that also $\mathbf{B}$ induces $G_C = Q_d^*$. This establishes the lemma. ☐

Now we state one of our main results. Note that the last part of the following theorem (on repeating directions) is only needed in the next section for a general dimension n.

Theorem 3.1. *For every integer* d, *there exists an* n*-bit cyclic Gray code* $\mathbf{C}_n$, $n = 2^d$, *such that* $G_{C_n} = Q_d^*$. *Moreover, for* $d > 1$ *and* $\tau(\mathbf{C}_n) = [T, t_N]$, *the transition* t_N *is not critical for* G_{C_n}, T *contains two disjoint occurrences of some segment* $[a, b, a, c]$, *and every direction from* $[n-1]$ *is repeating in* T.

Proof. We argue by induction on d. For $d = 1$ the statement is trivial. For $d = 2$ consider the 4-bit Gray code $\mathbf{C}_4$ given on Figure 1. Observe that $G_{C_4} = Q_2^*$ and for $\tau(\mathbf{C}_4) = [T, t_N]$, the transition t_N is not critical for G_{C_4}, T contains two disjoint occurrences of the segment $[1, 2, 1, 3]$, and T contains segments $[1, 2, 1]$, $[2, 4, 2]$, and $[3, 4, 3]$, so the directions 1, 2, and 3 are repeating in T.

Now we assume that the statement holds for $d > 1$ and we prove it for $d + 1$. Let $n = 2^d$ and $N = 2^n$.

The idea of the proof is as follows. We view Q_{2n} as a Cartesian product $Q_{2n} = Q_n \square Q_n$. First, we interconnect the copies $(0^n, u)$ of a vertex 0^n in all subcubes $Q_n(u)$ for $u \in V(Q_n)$ by a path P which induces Q_d^B on vertices $B = \{n+1, \dots, 2n\}$. Then, in each subcube $Q_n(u)$ we find a Hamiltonian path $R(u)$ of $Q_n(u) - (0^n, u)$ which induces Q_d^A on vertices $A = \{1, \dots, n\}$. Moreover, by Lemma 3.1 we can choose the path $R(u)$ so that $R(u)$ joins prescribed neighbors of $(0^n, u)$, and its first and last edge are of prescribed directions. This assures that we can interconnect these paths together into a Hamiltonian cycle of Q_{2n}, and when we do so, the newly induced edges are only between $i \in V(Q_d^A)$ and $n + i \in V(Q_d^B)$. See Figure 2 for an illustration. Note that the bold green paths $R(u)$'s are connected by dash-dotted red edges between the subcubes $Q_n(u)$'s, and the dashed blue path P is connected with $R(u_1)$ and $R(u_N)$.

By the induction hypothesis, let $\mathbf{C}_n = (u_1, u_2, \dots, u_N)$ be an n-bit Gray code such that $G_{C_n} = Q_d^*$ and for $\tau(\mathbf{C}_n) = [T, t_N]$, t_N is not critical for G_{C_n}, T contains two disjoint occurrences of some segment $S = [a, b, a, c]$, one centered at a vertex u, and every direction from $[n-1]$ is repeating in T.

First, we interconnect the copies of the vertex 0^n in each subcube $Q_n(u_i)$ by a path

$$P = (0^n, u_1), (0^n, u_2), \dots, (0^n, u_N). \tag{3.1}$$

Since P will be a part of $\mathbf{C}_{2n}$, T contains two disjoint occurrences of $S = [a, b, a, c]$, and every direction of $[n-1]$ is repeating in T, it follows that $\tau(\mathbf{C}_{2n})$ will contain two disjoint occurrences of $[a+n, b+n, a+n, c+n]$, and every direction from $\{n+1, n+2, \dots, 2n-1\}$ will be repeating in $\tau(\mathbf{C}_{2n})$.

Second, we claim that there exists a sequence $\sigma(\mathbf{C}_n) = [s_1, s_2, \dots, s_{N-1}]$ such that

(a) $t_i s_i \in E(G_{C_n})$ for every $1 \le i < N$, and
(b) either $t_i = s_{i-1}$ or $s_i = t_{i-1}$ for every $1 < i < N$.

Such a sequence can be found as follows. Note that $\deg_{G_{C_n}}(t_i) = d \ge 2$ for every $i \in [n]$. For $i = 1$, we choose s_i arbitrarily such that $t_i s_i \in E(G_{C_n})$. Now assume $1 < i < N$. If $t_i = s_{i-1}$, then we choose s_i such that $s_i \ne t_{i-1}$ and $t_i s_i \in E(G_{C_n})$. If $t_i \ne s_{i-1}$, then we put $s_i = t_{i-1}$ and observe that $t_i s_i \in E(G_{C_n})$ since $t_{i-1} t_i \in E(G_{C_n})$. Thus both (a) and (b) hold.

The sequence $\sigma(\mathbf{C}_n)$ determines the endvertices of paths $R(u_i)$ as described below. Note that from (a) and (b) we have that $s_{i-1} s_i \in E(G_{C_n})$ for every $1 < i < N$. In each subcube $Q_n(u_i)$ we find a Hamiltonian path $R(u_i)$ of $Q_n(u_i) - (0^n, u_i)$ as follows:

(*i*) For $i = 1$ we apply Lemma 3.1 for a vertex $v = 0^n$ and a segment $S = [t_1, s_1, t_1, z]$ where $z \ne s_1$ such that $t_1 z \in E(G_{C_n})$. Let $\mathbf{B}$ be the obtained

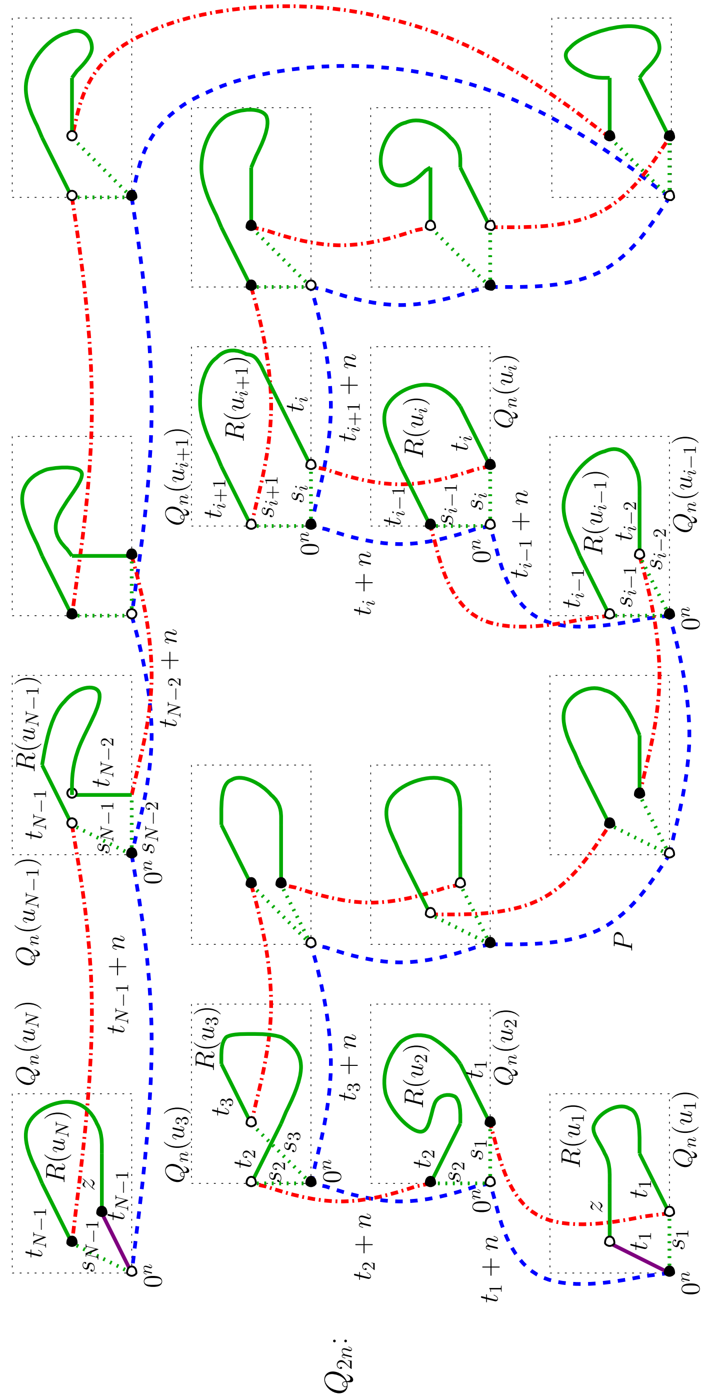

Fig. 2. The example for $d = 2$ illustrating the construction of the code $\mathbf{C}_{2n}$ in Q_{2n}. For the transitional sequence $\tau(\mathbf{C}_4) = [3, 1, 2, 1, 3, 4, 3, 1, 2, 1, 3, 1, 2, 4, 2, 1]$ we may choose $\sigma(\mathbf{C}_4) = [1, 2, 4, 2, 1, 3, 1, 2, 4, 2, 1, 2, 4, 3, 4]$.

Gray code containing S centered at v. By removing v from $\mathbf{B}$ we get a Hamiltonian path $R(u_1)$ of $Q_n(u_1) - (0^n, u_1)$

$$R(u_1) = (e_{t_1}, u_1), (e_{t_1 z}, u_1), \ldots, (e_{t_1 s_1}, u_1), (e_{s_1}, u_1). \tag{3.2}$$

(*ii*) For $1 < i < N$ we proceed similarly, but we apply Lemma 3.1 for $v = 0^n$ and $S = [t_i, s_i, s_{i-1}, t_{i-1}]$. Note that by (a) and (b), the conditions of the lemma are satisfied. Again, let $\mathbf{B}$ be the obtained Gray code containing S centered at v. By removing v from $\mathbf{B}$ we get a Hamiltonian path $R(u_i)$ of $Q_n(u_i) - (0^n, u_i)$

$$R(u_i) = (e_{s_{i-1}}, u_i), (e_{s_{i-1} t_{i-1}}, u_i), \ldots, (e_{s_i t_i}, u_i), (e_{s_i}, u_i). \tag{3.3}$$

(*iii*) For $i = N$ we apply Lemma 3.1 for $v = 0^n$ and $S = [z, t_{N-1}, s_{N-1}, t_{N-1}]$ where $z \neq s_{N-1}$ and $t_{N-1} z \in E(G_{C_n})$. Similarly as above, we get a Hamiltonian path $R(u_N)$ of $Q_n(u_N) - (0^n, u_N)$

$$R(u_N) = (e_{s_{N-1}}, u_N), (e_{s_{N-1} t_{N-1}}, u_N), \ldots, (e_{z t_{N-1}}, u_N), (e_{t_{N-1}}, u_N). \tag{3.4}$$

Recall that R^R denotes the reverted R. Clearly, the following sequence is a $2n$-bit Gray code:

$$\mathbf{C}_{2n} = P, R^R(u_N), R^R(u_{N-1}), \ldots, R^R(u_2), R^R(u_1).$$

Next, we verify that $\mathbf{C}_{2n}$ induces Q^*_{d+1}. We have

$$\tau(\mathbf{C}_{2n}) = [\tau(P),\ t_{N-1},\ \tau(R^R(u_N)),\ t_{N-1}+n,\ \tau(R^R(u_{N-1})),\ t_{N-2}+n, \\ \ldots, t_2+n,\ \tau(R^R(u_2)),\ t_1+n,\ \tau(R^R(u_1)),\ t_1].$$

Since t_N is not critical for G_{C_n}, we have by (3.1) that $\tau(P)$ induces the subcube $Q^B_d \cong Q^*_d$ of $G_{C_{2n}}$ on vertices $B = \{n+1, n+2, \ldots, 2n\}$. Furthermore, no other edge is induced between two vertices of B since $\tau(\mathbf{C}_{2n})$ contains no two consecutive transitions from B other than those in $\tau(P)$.

Moreover, we show that $\tau(R^R(u_i))$ for every $i \in [N]$ induces the subcube $Q^A_d \cong Q^*_d$ of $G_{C_{2n}}$ on vertices $A = \{1, 2, \ldots, n\}$. This follows from the fact that in each of the cases (*i*)–(*iii*) above, $G_B = Q^*_d$ and $\tau(\mathbf{B})$ contains two occurrences of the segment S. In addition, no other edge is induced between two vertices of A since $\tau(\mathbf{C}_{2n})$ contains no two consecutive transitions from A other than those in $\tau(R^R(u_i))$ for some $i \in [N]$.

Finally, observe by (3.1)–(3.4) that the remaining edges of $G_{C_{2n}}$ are joining vertices i and $n+i$ for some $i \in [n]$, and for every $i \in [n]$ there exists such edge since $\tau(\mathbf{C}_n)$ contains all $i \in [n]$. Altogether, we obtain that $G_{C_{2n}} = Q^*_{d+1}$.

To conclude the proof, it remains to verify the second part of the statement. Let $\tau(\mathbf{C}_{2n}) = [t'_1, \ldots, t'_{N^2}] = [T', t'_{N^2}]$. Since $t'_{N^2-1} = z$, $t'_{N^2} = t_1$, and $t'_1 = t_1+n$, observe that the transition t'_{N^2} is not critical for $G_{C_{2n}}$, since the edge $zt_1 \in E(Q^B_d)$ is induced by $\tau(R^R(u_i))$ for any $i \in [n]$, and the edge of $G_{C_{2n}}$ joining t_1 and t_1+n is induced also by transitions $t'_{N^2-N} = t_1$ and $t'_{N^2-N+1} = t_1+n$.

Furthermore, T' contains $\tau(P)$. Consequently, T' contains two disjoint occurrences of a segment $[a+n, b+n, a+n, c+n]$, and every direction from $\{n+1, n+2, \ldots, 2n-1\}$ is repeating in T'. In addition, T' contains the segments $[t_1, t_1+n, t_1]$, $[t_2, t_2+n, t_2]$, $\ldots$, $[t_{N-1}, t_{N-1}+n, t_{N-1}]$. Hence, the directions $D = \{t_1, \ldots, t_{N-1}\}$ are repeating in T'. Clearly $D = [n]$ since every direction from $[n]$ appears at least twice in $\tau(\mathbf{C}_n) = [t_1, \ldots, t_{N-1}, t_N]$. Therefore, every direction from $[2n-1]$ is repeating in T'. □

4 General dimension

In this section, we generalize Theorem 3.1 for an arbitrary dimension n. More precisely, we construct a Gray code inducing a subgraph of Q_d^* for the smallest d possible.

Theorem 4.1. *For every integer $n \geq 1$, there exists an n-bit cyclic Gray code $\mathbf{C}_n$ such that $G_{C_n} \subseteq Q^*_{\lceil \log_2 n \rceil}$. Moreover, if $n \geq 4$ and $n = 2^d + k$ where $0 \leq k \leq 2^d - 2$, then every direction from $\{k+1, \ldots, 2^d - 1\}$ is repeating in $\tau(\mathbf{C}_n)$.*

Proof. We argue by induction on k. By Theorem 3.1, the statement holds if $n = 2^d$ for some integer d. If $n = 1$ or $n = 3$, observe that the reflected codes $\mathbf{\Gamma}_1 = (0, 1)$ and $\mathbf{\Gamma}_3 = (000, 001, 011, 010, 110, 111, 101, 100)$ from (1.1) induce a subgraph of Q_0^* and Q_2^*, respectively.

Now we have $n = 2^d + k$ where $d > 1$ and $1 < k < 2^d$, so $\lceil \log_2 n \rceil = d + 1$. By the induction hypothesis, there is an $(n-1)$-bit Gray code $\mathbf{C}_{n-1}$ inducing a subgraph of Q^*_{d+1} such that every direction from $D = \{k, \ldots, 2^d - 1\}$ is repeating in $\tau(\mathbf{C}_{n-1})$. That is, for every $t \in D$ the transitional sequence $\tau(\mathbf{C}_{n-1}) = [t_1, \ldots, t_{N/2}]$ where $N = 2^n$ contains a segment $[t, x, t]$ for some $x \in [n-1]$. We may assume that

$$t_{N/2-1} = k,\ t_{N/2} = x,\ t_1 = k, \tag{4.1}$$

otherwise we shift the code $\mathbf{C}_{n-1}$ so that the segment $[k, x, k]$ appears at this position.

We define the Gray code $\mathbf{C}_n$ schematically as in (1.1),

$$\mathbf{C}_n = 0\mathbf{C}_{n-1}, 1\mathbf{C}^R_{n-1}. \tag{4.2}$$

From (4.1) and (4.2) it follows that

$$\tau(\mathbf{C}_n) = [k = t_1, \ldots, t_{N/2-1} = k, n, t_{N/2-1} = k, \ldots, t_1 = k, n].$$

Hence, for the graph G_{C_n} induced by $\mathbf{C}_n$ we have that

$$E(G_{C_n}) \subseteq E(G_{C_{n-1}}) \cup \{kn\}.$$

Consequently, $G_{C_n} \subseteq Q^*_{d+1}$ since $G_{C_{n-1}} \subseteq Q^*_{d+1}$ and $kn \in E(Q^*_{d+1})$ because $n - k = 2^d$.

It remains to verify the second part of the statement. Observe that if $S = [s, x, s]$ and $T = [t, y, t]$ are segments of $\tau(\mathbf{C}_{n-1})$ for some $x, y \in [n-1]$ and distinct repeating transitions $s, t \in D$, then S and T must be disjoint. Therefore, since every direction from D is repeating in $\tau(\mathbf{C}_{n-1})$ and by (4.1), it follows that every direction from $D \setminus \{k\}$ is repeating in $[t_1, \ldots, t_{N/2} - 1]$, which is a segment of $\tau(\mathbf{C}_n)$. □

5 Concluding remarks

In this paper we have described a construction of a cyclic n-bit Gray code whose graph of transitions is a subgraph of the d-dimensional hypercube, $d = \lceil \log_2 n \rceil$. Note that the proofs of Theorem 3.1, which covers the case $n = 2^d$, and of Theorem 4.1, which covers the case $2^{d-1} < n < 2^d$, actually provide a description of a recursive algorithm which, given a positive integer n, constructs an n-bit code with the desired property. The running time of the algorithm may be bounded by $O(N \log N)$ where N is the output size.

As mentioned in the introduction, our variant of Gray codes allows for a more space-saving representation compared to Gray codes in general. This suggests that it may be reasonable to inspect other data compression applications where the reflected Gray code $\mathbf{\Gamma}_n$ is traditionally used.

In particular, consider the problem of compressing a set $\{s_1, s_2, \ldots, s_k\}$ of n-bit strings whose order is irrelevant, which arises in the context of compressing bitmap indices of large databases. Efficient methods developed for this purpose [13], which allow performing logical operations on uncompressed bitmaps and therefore faster query processing, are based on a technique known as *run length encoding*: putting strings into rows of a matrix, replace repeated runs of consecutive 0's and 1's in the columns by their lengths.

In order to improve the compression rate, one needs to minimize the number of runs, which leads to the problem of finding a permutation $\pi : [n] \to [n]$ which minimizes the sum

$$\sum_{i=1}^{k-1} d_H(s_{\pi(i)}, s_{\pi(i+1)}) \tag{5.1}$$

where d_H stands for the Hamming distance. In the extremal case that $k = 2^n$, the optimal solution to (5.1) is provided by an n-bit Gray code. Unfortunately, the problem (5.1) in the general case $k < 2^n$ is known to be NP-complete [2]. A popular heuristic that has been used to find an approximate solution [5, 6] orders the strings in such a way that $\pi(i) < \pi(j)$ if $s_{\pi(i)}$ precedes $s_{\pi(j)}$ in the reflected Gray code $\mathbf{\Gamma}_n$. However, in the experimental results performed on real-life datasets [4], the $\mathbf{\Gamma}_n$ code reordering was outperformed by a simple lexicographic order. We suggest that it is conceivable to replace $\mathbf{\Gamma}_n$ with our variant of Gray code.

References

1. B. Bultena, F. Ruskey, *Transition restricted Gray codes*, Electron. J. Combin. 3 (1996) #R11.
2. J. Ernvall, J. Katajainen, M. Penttonen, *NP-completeness of the Hamming salesman problem*, BIT 25 (1985), 289–292.
3. D. E. Knuth, The Art of Computer Programming, Volume 4, Fascicle 2: Generating All Tuples and Permutations, Addison-Wesley Professional, 2005.
4. H. H. Malik, J R. Kender, *Optimizing Frequency Queries for Data Mining Applications*, Proc. 7th Int. Conf. Data Mining (ICDM 2007), IEEE Computer Society, 2007, 595–600.
5. A. Pinar, T. Tao, H. Ferhatosmanoglu, *Compressing bitmap indices by data reorganization*, Proc. 21st Int. Conf. Data Engineering (ICDE 2005), IEEE Computer Society, 2005, 310–321.
6. D. Richards, *Data compression and Gray-code sorting*, Inf. Process. Lett. 22(1986), 201–205.
7. D. Salomon, *Data Compression: The Complete Reference*, 4th ed., Springer, Berlin, 2006.
8. C. Savage, *A survey of combinatorial Gray codes*, SIAM Rev. 39 (1997), 605–629.
9. P. J. Slater, *Open problem*, in: Proceedings of the 10th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Vol. II, Utilitas Mathematica Publishing, Winnipeg, 1979, pp. 918–919.
10. P. J. Slater, *Research Problems 109 and 110*, Discrete Math. 76 (1989) 293–294.
11. I. N. Suparta, A. J. van Zanten, *A construction of Gray codes inducing complete graphs*, Discrete Math. 308 (2008) 4124–4132.
12. E. L. Wilmer, M. D. Ernst, *Graphs induced by Gray codes*, Discrete Math. 257 (2002) 585–598.
13. K. Wu, E. J. Otoo, A. Shoshani, *Optimizing bitmap indices with efficient compression*, ACM Trans. Database Syst. 31 (2006), 1–38.