Non-rainbow colorings of 3-, 4- and 5-connected plane graphs*

Zdeněk Dvořák†
DEPARTMENT OF APPLIED MATHEMATICS AND INSTITUTE FOR THEORETICAL COMPUTER SCIENCE (ITI)
FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY
MALOSTRANSKÉ NÁMĚSTí 25, 118 00 PRAGUE, CZECH REPUBLIC
E-MAIL: RAKDVER@KAM.MFF.CUNI.CZ.

Daniel Král'
INSTITUTE FOR THEORETICAL COMPUTER SCIENCE (ITI)
FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY
MALOSTRANSKÉ NÁMĚSTí 25, 118 00 PRAGUE, CZECH REPUBLIC
E-MAIL: KRAL@KAM.MFF.CUNI.CZ.

Riste Škrekovski
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LJUBLJANA
JADRANSKA 19, 1111 LJUBLJANA, SLOVENIA.

ABSTRACT

We study vertex-colorings of plane graphs that do not contain a rainbow face, i.e., a face with vertices of mutually distinct colors. If G is a 3-connected plane graph with n vertices, then the number of colors in such a coloring does not exceed $\left\lfloor\frac{7n-8}{9}\right\rfloor$. If G is 4-connected, then the number of colors is at most $\left\lfloor\frac{5n-6}{8}\right\rfloor$, and for $n \equiv 3 \pmod 8$, it is at most $\left\lfloor\frac{5n-6}{8}\right\rfloor - 1$. Finally, if G is 5-connected, then the number of colors is at most $\left\lfloor\frac{25}{58}n - \frac{22}{29}\right\rfloor$. The bounds for 3-connected and 4-connected plane graphs are the best possible as we exhibit constructions of graphs with colorings matching the bounds.

Journal of Graph Theory Vol. ???, 1–19 (???)
 CCC ???

Keywords: Plane graphs, non-rainbow colorings.

1. INTRODUCTION

Colorings of graphs embedded on surfaces with face-constraints have recently drawn a substantial amount of attention. There are two natural questions derived from hypergraph colorings that one may ask in this setting:

1. *What is the minimal number of colors needed to color an embedded graph in such a way that each of its faces is incident with vertices of at least two different colors, i.e., there is no monochromatic face?*

2. *What is the maximal number of colors that can be used in a coloring of an embedded graph that contains no face with vertices of mutually distinct colors, i.e., that contains no rainbow face?*

The first question can be traced back to work of Zykov [24] and was further explored by Kündgen and Ramamurthi [17]. It can be shown [8] that every graph embedded on a surface of genus ε has a coloring with $O(\sqrt[3]{\varepsilon})$ colors that avoids a monochromatic face. Let us remark that this type of coloring can be also formulated in the terms of colorings of so-called face hypergraphs of embedded graphs. Also let us mention that colorings that avoid both monochromatic and rainbow faces have been also studied, see, e.g., [7, 14, 16]. For instance, the results of Penaud [20] and Diwan [5] imply that each plane graph with at least five vertices has a coloring with two colors as well as a coloring with three colors that avoid both monochromatic and rainbow faces.

In this paper, we focus on the second question. A *non-rainbow coloring* of a plane graph G is a vertex-coloring such that each face of G is incident with at least two vertices with the same color. Unlike in the case of ordinary colorings, the goal is to maximize the number of used colors. The maximum number of colors that can be used in a non-rainbow coloring of a plane graph G is denoted by $\chi_f(G)$. Let us remark at this point that graphs considered in the problems of this type can usually contain parallel edges unless they form a bigon in an embedding, but since graphs appearing in our proofs are 3- or more connected, the graphs in Sections 2.–5. do not have any parallel edges. The following extremal anti-Ramsey problem is equivalent to the second question:

What is the smallest number $k(G)$ of colors such that every vertex-coloring of a plane graph G with $k(G)$ colors contains a rainbow face?

It is easy to see that $\chi_f(G) = k(G) - 1$ and the results obtained in either of the scenarios translate smoothly to the other one.

---

* This work was supported by the Czech-Slovenian bilateral research project 15/2006–2007 and MEB 090805.

† Institute for Theoretical computer science is supported as project 1M0545 by Czech Ministry of Education.

Let us briefly survey some results on the second question. The problem independently appeared in the work of Ramamurthi and West [22] and Negami [19] who considered the problem in the anti-Ramsey framework (see also [1, 2, 18] for some even earlier results of this flavor). Ramamurthi and West [21] noticed that every plane graph G has a non-rainbow coloring with at least $\alpha(G)+1$ colors, in particular, every plane graph G of order n has a coloring with at least $\lceil \frac{n}{4} \rceil + 1$ colors by the Four Color Theorem. Also, Grötzsch's theorem [9, 23] implies that every triangle-free plane graph has a non-rainbow coloring with $\lceil \frac{n}{3} \rceil + 1$ colors. Ramamurthi et al. [21] conjectured that this bound can be improved to $\lceil \frac{n}{2} \rceil + 1$. Partial results on this conjecture were obtained in [15] and the conjecture has been eventually proven in [13]. More generally, Jungić et al. [13] proved that every planar graph of order n with girth $g \geq 5$ has a non-rainbow coloring with at least $\left\lceil \frac{g-3}{g-2}n - \frac{g-7}{2(g-2)} \right\rceil$ colors if g is odd, and $\left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil$ colors if g is even. All these bounds are the best possible.

Negami [19] investigated non-rainbow colorings of plane triangulations G and showed that $\alpha(G)+1 \leq \chi_f(G) \leq 2\alpha(G)$. Jendrol' and Schrötter [12] determined the number $\chi_f(G)$ for all semiregular polyhedra. In addition, Jendrol' [10] established that $\frac{n}{2}+\alpha_1^* - 2 \leq \chi_f(G) \leq n - \alpha_0^*$ for 3-connected cubic plane graphs G where α_0^* is the independence number of the dual graph G^* of G and α_1^* is the size of the largest matching of G^*. Jendrol' also conjectured [10, 11] the following (let us remark that the former conjecture was proven in [6]):

Conjecture 1. Every cubic 3-connected plane graph G of order n has $\chi_f(G) = \frac{n}{2} + \alpha_1^* - 2$.

Conjecture 2. A non-rainbow coloring of a plane 3-connected graph G of order n uses at most $\left\lfloor \frac{3n-1}{4} \right\rfloor$ colors.

Motivated by Conjecture 2, we establish tight upper bounds on the maximal numbers of colors used in non-rainbow colorings of plane 3-connected and 4-connected graphs and close lower and upper bounds on the maximal number of such colors for 5-connected plane graphs. We show that a non-rainbow coloring of a plane 3-connected graph of order n always uses at most $\left\lfloor \frac{7n-8}{9} \right\rfloor$ colors and a non-rainbow coloring of a plane 4-connected graph always uses at most $\left\lfloor \frac{5n-6}{8} \right\rfloor$ colors, and for $n \equiv 3 \pmod 8$, it uses at most $\left\lfloor \frac{5n-6}{8} \right\rfloor - 1$ colors. All these bounds are the best possible. In particular, Conjecture 2 is false. For completeness, let us also remark that the optimal bound for 2-connected plane graphs is $n-1$ and is achieved for a cycle.

For 5-connected plane graphs G, we show that the number of colors of a non-rainbow coloring of G does not exceed $\left\lfloor \frac{25}{58}n - \frac{22}{29} \right\rfloor \approx .4310\,n$ where n is the order of G. On the other hand, we construct plane 5-connected graphs G of order n with non-rainbow colorings with almost $\frac{171}{400}n = .4275\,n$ colors. We were not able to close the gap between our lower and upper bounds in this case and conjecture (see Conjecture 3 at the end of the paper) that the correct bound is $\frac{3}{7}n + \text{const.}$

Let us now briefly introduce several definitions that will be useful in our further considerations. Most of them are standard graph theory definitions, but we still like to include. A *color class* of a vertex-coloring is the set of vertices assigned the same color. A *monochromatic* path or cycle is a path or a cycle such that all its vertices have the same color. We often refer to a cycle of length k as a k-*cycle*.

If G is a graph embedded in the plane and C a cycle of G, then $\mathrm{Int}(C)$ is the subgraph of G lying in the closed region bounded by C, in particular, the subgraph $\mathrm{Int}(C)$ includes the cycle C. Similarly, $\mathrm{Ext}(C)$ is the subgraph of G lying outside the open region bounded by C. If both $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ contain more vertices than C, then the cycle C is said to be *separating*. If G and H are two graphs with 3-cycles $u_1u_2u_3$ and $v_1v_2v_3$, then the graph obtained from G and H by identifying the 3-cycles $u_1u_2u_3$ and $v_1v_2v_3$ is called the 3-*sum* of G and H. Observe that if G is a 3-connected plane graph and C is a separating triangle, then both $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ are 3-connected, and if both G and H are 3-connected, then their 3-sum is also 3-connected.

2. COUNTING ARGUMENT

Our upper bounds are proved using counting arguments based on the lemmas established in this section. Let G be a colored plane graph of minimum degree at least d. We define a d-*weight* $w_d(H)$ of a connected monochromatic subgraph H of G as $k - \frac{1}{2}\sum_{v\in V(H)}(\deg_G(v) - d)$, where k is the number of faces of G that share at least one edge with H, and $\deg_G(v)$ is the degree of v in G. The next lemma provides a simple (and in most cases good enough) upper bound on the d-weight of a monochromatic subgraph of G. We say that a subgraph H is a *maximal connected monochromatic subgraph of* G if all the vertices of H have the same color, H is connected and there is no supergraph of H with these two properties.

Lemma 1. Let G be a colored plane graph of minimum degree at least d. If H is a maximal connected monochromatic subgraph of G, then

$$w_d(H) \le \frac{1}{2}\sum_{v\in V(H)} \min\{2\deg_H(v), d\} \le \frac{d}{2}|V(H)| \ .$$

Proof. The second inequality of the statement of the lemma obviously holds and thus we focus on proving the first one. For a vertex v of H, let k_v be the number of faces of G that contain an edge e of H incident with v. By the definition, the d-weight of H is at most

$$\frac{1}{2}\sum_{v\in V(H)} (k_v + d - \deg_G(v))$$

since each face incident with a monochromatic edge of H is counted in at least two variables k_v (note that a single face can be incident with more edges of H).

Observe that $k_v \leq \min\{2\deg_H(v), \deg_G(v)\}$. We infer from $k_v \leq 2\deg_H(v)$ that $k_v + d - \deg_G(v) \leq k_v \leq 2\deg_H(v)$ and from $k_v \leq \deg_G(v)$ that $k_v + d - \deg_G(v) \leq d$. Hence, $k_v + d - \deg_G(v) \leq \min\{2\deg_H(v), d\}$. The assertion of the lemma now follows. ∎

The following lemma provides an upper bound on the number of colors used in a non-rainbow coloring. Note that we prove the lemma under the stronger assumption that each face actually contains a monochromatic edge. As we shall see in the rest of the paper, this does not decrease the generality of our considerations.

Lemma 2. Let G be a plane connected graph of order n and with minimum degree at least d, let c be a vertex-coloring of G such that each face of G contains a monochromatic edge, and let $H_1, \ldots, H_t$ be all maximal connected monochromatic subgraphs of G. If there exist $\alpha > 0$ and $\beta_1, \ldots, \beta_t \geq 0$ such that $w_d(H_i) \leq \alpha(|V(H_i)| - 1) - \beta_i$ for every $i = 1, \ldots, t$, then the coloring c uses at most

$$\left(1 - \frac{d-2}{2\alpha}\right) n - \frac{2 + \sum_{i=1}^{t} \beta_i}{\alpha}$$

colors.

Proof. Let n_i be the number of vertices of H_i and k_i the number of faces of G incident with an edge of H_i.

Since each face of G is incident with a monochromatic edge, the number f of faces of G is at most $\sum_{i=1}^{t} k_i$. By Euler's formula, we have the following:

$$\sum_{i=1}^{t} k_i \geq f = \frac{1}{2} \sum_{v \in V(G)} \deg(v) - n + 2 = \frac{1}{2} \sum_{v \in V(G)} (\deg(v) - d) + \frac{d-2}{2} n + 2 \, .$$

We now plug our assumption that the d-weight of H_i is at most $\alpha(|V(H_i) - 1) - \beta_i$ to the above estimate:

$$\begin{aligned}
\frac{d-2}{2} n + 2 \;&\leq\; \sum_{i=1}^{t} k_i - \frac{1}{2} \sum_{v \in V(G)} (\deg_G(v) - d) \\
&=\; \sum_{i=1}^{t} \left(k_i - \frac{1}{2} \sum_{v \in V(H_i)} (\deg_G(v) - d) \right) \\
&\leq\; \sum_{i=1}^{t} \left(\alpha(n_i - 1) - \beta_i \right) \; .
\end{aligned}$$

We infer from this inequality that

$$\alpha\, t \leq \alpha\, n - \frac{d-2}{2}\, n - 2 - \sum_{i=1}^{t} \beta_i$$

which yields that

$$t \leq \left(1 - \frac{d-2}{2\alpha}\right) n - \frac{2 + \sum_{i=1}^{t} \beta_i}{\alpha} .$$

This finishes the proof of the lemma since the number of colors used by c is at most t. ∎

In Sections 3.–5., we apply Lemma 2 with different values of α and β_i (setting $\beta_i = 0$ in most cases).

3. 3-CONNECTED PLANE GRAPHS

In this section, we prove our lower and upper bounds on the number of colors of non-rainbow colorings of 3-connected plane graphs. The upper bound is rather easy once we have established Lemma 2.

Theorem 1. If G is a plane 3-connected graph with $n \geq 4$ vertices, then the number of colors used by any non-rainbow coloring c of G does not exceed $\lfloor \frac{7n-8}{9} \rfloor$.

Proof. By adding edges to G, we can assume without loss of generality that each face of G is incident with a monochromatic edge. In addition, we can assume that each color class induces a connected subgraph of G; otherwise, we can recolor one of the components to increase the number of used colors. Let H be a subgraph induced by one of the color classes (note that H is a maximal connected monochromatic subgraph of G) and let n' be the number of its vertices.

Since G is 3-connected, it has minimum degree at least three and thus we can apply Lemma 2 with $d = 3$. We now estimate the 3-weight of H. If $n' = 1$, then the 3-weight of H is non-positive. If $n' = 2$, then H is a single edge and thus $w_3(H) \leq 2 \leq \frac{9}{4}(n' - 1)$ by Lemma 1. If $n' \geq 3$, then $w_3(H) \leq \frac{3}{2}n' \leq \frac{9}{4}(n' - 1)$ again by Lemma 1. Therefore, the assumption of Lemma 2 is satisfied for $\alpha = \frac{9}{4}$, $\beta_i = 0$ and $d = 3$. The upper bound $\frac{7n-8}{9}$ on the number of colors used by c easily follows. ∎

In the rest of this section, we show that the bound established in Theorem 1 is the best possible. Let us start with the following lemma that allows us to construct larger examples that match the bound from smaller ones. Notice that in Lemma 3, the monochromatic triangle can be both facial or separating.

Lemma 3. Let G be a plane 3-connected graph with n vertices that has a non-rainbow coloring c with k colors. If G contains a monochromatic triangle, then there exists a plane 3-connected graph G' with $n + 9$ vertices that has a non-rainbow coloring c' with $k + 7$ colors and which also contains a monochromatic triangle.

Proof. Let $v_1v_2v_3$ be a monochromatic triangle contained in G. Split the triangle into two copies, $v'_1v'_2v'_3$ and $v''_1v''_2v''_3$, and keep the rest of the graph (see Figure 1 for illustration). Next, insert a cycle $w_1w_2w_3w_4w_5w_6$ of length six between the cycles $v'_1v'_2v'_3$ and $v''_1v''_2v''_3$ as in the figure, and insert the following edges: v'_1w_1, v''_1w_2, v'_2w_3, v''_2w_4, v'_3w_5, and v''_3w_6. Let G' be the resulting graph. The vertices of $\mathrm{Int}(v''_1v''_2v''_3)$ with the

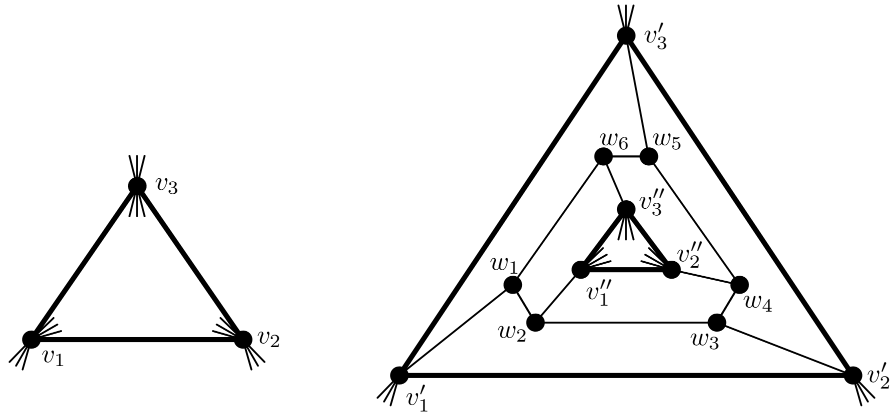

FIGURE 1. A construction presented in Lemma 3. The monochromatic edges in the configurations are drawn bold.

same color as v_1'' are recolored by a new color not previously used by c. Six new colors are also assigned to the vertices $w_1, \ldots, w_6$. We conclude that the resulting coloring c' uses $k+7$ colors. It follows that if c is a non-rainbow coloring of G, then c' is a non-rainbow coloring of G'.

It remains to verify that the graph G' is 3-connected. The 3-connectivity of G implies that both $\mathrm{Int}(v_1v_2v_3)$ and $\mathrm{Ext}(v_1v_2v_3)$ are 3-connected. Since the graph G' can be viewed as the 3-sum of $\mathrm{Int}(v_1v_2v_3)$ and the graph obtained by the 3-sum of $\mathrm{Ext}(v_1v_2v_3)$ and a 12-vertex 3-connected graph, G' is also 3-connected (if $\mathrm{Int}(v_1v_2v_3)$ or $\mathrm{Ext}(v_1v_2v_3)$ is just K_3, we apply the 3-sum only once). ∎

We can now provide constructions of 3-connected graphs that witness that the bound established in Theorem 1 is tight:

Theorem 2. For every $n \geq 4$, there exists a plane 3-connected graph G with $n \geq 4$ vertices that has a non-rainbow coloring with $\left\lfloor \frac{7n-8}{9} \right\rfloor$ colors.

Proof. The reader can find the graphs G for $n = 4, \ldots, 12$ in Figure 2. Since each of the graphs depicted in Figure 2 contains a monochromatic triangle, the existence of graphs G for all $n \geq 13$ follows from Lemma 3. ∎

4. 4-CONNECTED PLANE GRAPHS

In this section, we prove our lower and upper bounds on the number of colors of non-rainbow colorings of 4-connected plane graphs.

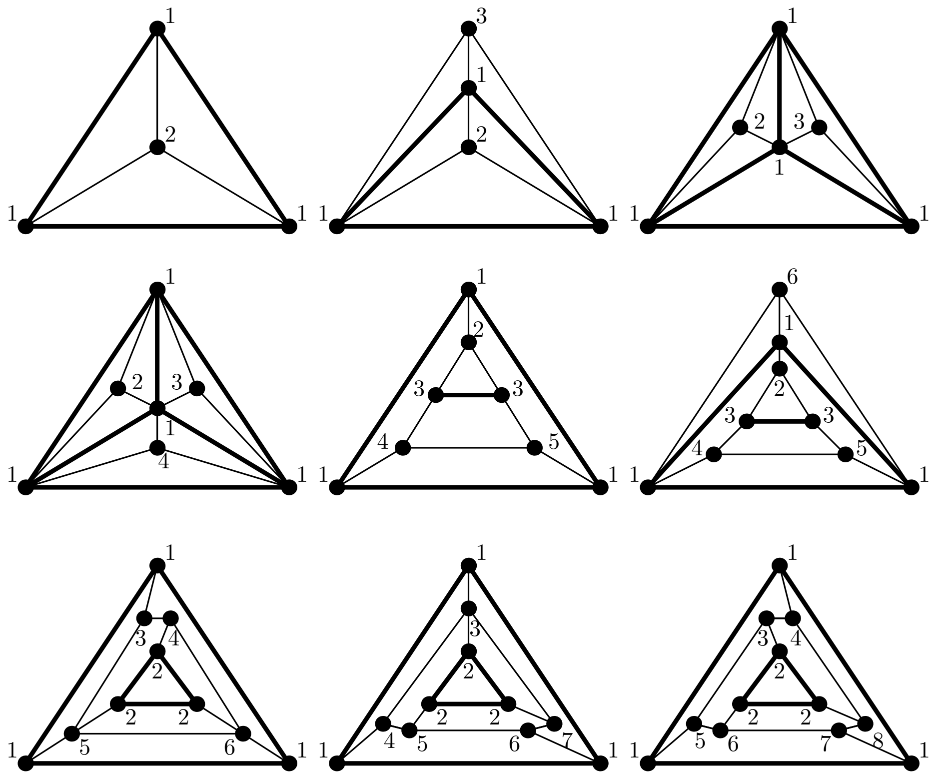

FIGURE 2. 3-connected plane graphs G with n vertices, $n = 4, \ldots, 12$, that have non-rainbow colorings with $\left\lfloor \frac{7n-8}{9} \right\rfloor$ colors. The edges of G that are monochromatic in such a coloring are drawn bold. The colors assigned to the vertices are represented by numbers.

Theorem 3. Let G be a plane 4-connected graph with $n \geq 6$ vertices. The number of colors in a non-rainbow coloring of G does not exceed $\left\lfloor \frac{5n-6}{8} \right\rfloor$. Moreover, if $n \equiv 3 \pmod 8$, then the number of colors does not exceed $\left\lfloor \frac{5n}{8} \right\rfloor - 1$.

Proof. Fix a non-rainbow coloring c of G. Without loss of generality, we can assume by adding edges that each face is incident with at least one monochromatic edge. In addition, we can also assume that the vertices of each color induce a connected subgraph of G; otherwise, recoloring one of the components with a new color yields a non-rainbow coloring of G with more colors.

Let H be a subgraph of G induced by the vertices of one of the colors and n' the number of its vertices. Our aim is to show that the 4-weight $w_4(H)$ of H is at most $\frac{8}{3}(n' - 1)$.

The inequality $w_4(H) \leq \frac{8}{3}(n' - 1)$ clearly holds if $n' = 1$. If $n' \geq 4$, $w_4(H) \leq 2n' \leq \frac{8}{3}(n'-1)$ by Lemma 1. If $n' = 2$, H is a single edge and thus the degree of both the vertices of H is one. Hence, by Lemma 1, we have $w_4(H) \leq 2 < \frac{8}{3}(n'-1)$. If $n' = 3$, H is either a 3-vertex path or a triangle. In the former case, $w_4(H) \leq 1+2+1 \leq \frac{8}{3}(n'-1)$ by Lemma 1. If H is a triangle, then it bounds a 3-face of G since G is 4-connected. Therefore, the edges of H are incident with at most four distinct faces of G, and $w_4(H) \leq 4 < \frac{8}{3}(n'-1)$ by the definition of the d-weight. This finishes the proof of the inequality $w_4(H) \leq \frac{8}{3}(n' - 1)$.

We infer from Lemma 2 applied with $\alpha = \frac{8}{3}$, $\beta_i = 0$, and $d = 4$ that the number of colors used by c is at most $\frac{5n-6}{8}$ which is the bound claimed for $n \not\equiv 3 \pmod 8$ in the statement of the theorem. The case $n \equiv 3 \pmod 8$ is further considered in more detail.

We assume that $n \equiv 3 \pmod 8$ in the remainder of the proof. It is straightforward to verify that unless H is a single vertex or $n' = 4$, the estimates established in the previous paragraph yield $w_4(H) \leq \frac{8}{3}(n'-1) - \frac{2}{3}$. If $n' = 4$, then $w_4(H) \leq 6 \leq \frac{8}{3}(n'-1) - \frac{2}{3}$ unless H is a 4-cycle. Therefore, if there is a maximal connected monochromatic subgraph H_1 of G different from a vertex or a 4-cycle, we can apply Lemma 2 with $\alpha = \frac{8}{3}$, $\beta_1 = \frac{2}{3}$, $\beta_i = 0$ with $i \neq 1$, and $d = 4$ to obtain the desired bound.

We conclude that if the number of colors used by c is greater than $\frac{5n}{8} - 1$, then each color class is either a single vertex or a 4-cycle. We now address this case directly.

Let f be the number of faces of G and s the number of monochromatic 4-cycles of G. Since each face of G is incident with a monochromatic edge, it follows that $f \leq 8s$. On the other hand, since G is 4-connected, its minimum degree is at least four, and thus the number of its edges is at least $2n$. Hence, by Euler's formula, we get the following:

$$8s \geq n + f - n \geq (2n + 2) - n = n + 2 \; .$$

Since $n \equiv 3 \pmod 8$, we infer the following:

$$s \geq \left\lceil \frac{n+2}{8} \right\rceil = \frac{n+5}{8} \; .$$

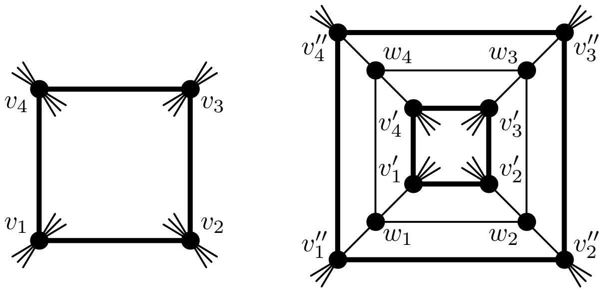

FIGURE 3. A construction presented in Lemma 4. The monochromatic edges in the configurations are drawn bold.

Finally, we can conclude that the number of colors used by c, which is equal to $n - 3s$, is at most

$$n - 3s \leq n - 3 \cdot \frac{n+5}{8} = \frac{5n-15}{8} \leq \frac{5n}{8} - 1 \; .$$

∎

In the rest of this section, we show that the bound established in Theorem 3 is tight. We start with a lemma that allows us to construct larger examples of graphs for which the bound of the theorem is tight from smaller ones.

Lemma 4. Let G be a plane 4-connected graph with n vertices that has a non-rainbow coloring c with k colors. If G contains a separating monochromatic 4-cycle, then there exists a 4-connected plane graph with $n+8$ vertices that has a non-rainbow coloring with $k+5$ colors and with a separating monochromatic 4-cycle.

Proof. Let $v_1v_2v_3v_4$ be a monochromatic 4-cycle of G. Split the cycle to two cycles $v_1'v_2'v_3'v_4'$ and $v_1''v_2''v_3''v_4''$. Each vertex of G adjacent to v_i is adjacent to v_i' or to v_i'' in such a way that the resulting graph is still plane, see Figure 3. In addition, add a new cycle $w_1w_2w_3w_4$ between the two cycles $v_1'v_2'v_3'v_4'$ and $v_1''v_2''v_3''v_4''$, and add edges $v_i'w_i$ and $v_i''w_i$, $i = 1, 2, 3, 4$. Let G' be the obtained graph. The vertices v_i' keep the color of the vertices v_i, the vertices v_i'' and all the vertices with the same color in $\mathrm{Int}(v_1''v_2''v_3''v_4'')$ are recolored by a new color and each of the vertices w_i also receives a new color. Hence, G' is a graph of order $n + 8$ and the constructed coloring is a non-rainbow coloring of G' with $k + 5$ colors.

It remains to verify that G' is 4-connected. Let A' be a vertex cut of G' formed by at most three vertices. Note that each component of $G' \setminus A'$ contains at least one original vertex of G, i.e., a vertex different from v_i', v_i'' and w_i, $i = 1, 2, 3$. Let A be the set obtained from A' by replacing v_i', v_i'' or w_i by the vertex v_i. Note that A contains at most three vertices. If $G' \setminus A'$ is disconnected, then the graph $G \setminus A$ is also disconnected (any path of $G \setminus A$ can be turned to a path of $G' \setminus A'$ and the subgraph of G induced by the vertices v_i', v_i'' and w_i, $i = 1, 2, 3, 4$, is 4-connected). This contradicts our assumption that G is 4-connected since $|A'| \leq |A|$. ∎

We finish this section with a construction of graphs for which the bound proven in Theorem 3 is tight.

Theorem 4. For every $n \geq 6$, $n \not\equiv 3 \pmod 8$, there exists a plane 4-connected graph G with n vertices that has a non-rainbow coloring with $\left\lfloor \frac{5n-6}{8} \right\rfloor$ colors. Moreover, for $n \geq 6$ and $n \equiv 3 \pmod 8$, there exists such a graph G that has a non-rainbow coloring with $\left\lfloor \frac{5n}{8} \right\rfloor - 1$ colors.

Proof. The construction of graphs G for $n = 6, 7, \ldots, 13$ can be found in Figure 4, and the existence of the graphs G for $n \geq 14$ follows from Lemma 4 by induction on n. ∎

5. 5-CONNECTED PLANE GRAPHS

In the last section of the paper, we focus on bounds on 5-connected plane graphs. We start with showing the upper bound on the number of the colors.

Theorem 5. Let G be a plane 5-connected graph with n vertices. The number of colors in a non-rainbow coloring of G does not exceed $\left\lfloor \frac{25}{58}n - \frac{22}{29} \right\rfloor$.

Proof. By adding edges if necessary, we can assume that each face of G is incident with a monochromatic edge. Our aim is to apply Lemma 2 with $\alpha = \frac{29}{11}$ and $d = 5$. In particular, we have to show that $w_5(H) \leq \frac{29}{11}(n' - 1)$ for each maximal connected monochromatic subgraph H of G, where n' is the number of vertices of H.

If H is a tree, then its edges are incident with at most $2(n' - 1)$ faces of G, and consequently $w_5(H) \leq 2(n' - 1) \leq \frac{29}{11}(n' - 1)$. Therefore, we assume in what follows that H is not a tree, in particular, $n' \geq 3$.

If $H = C_{n'}$ for $n' = 3, 4$, then there are at most $n' + 1$ faces of G that contain an edge of H (as G is 5-connected, H forms a boundary of a face of G). It follows that $w_5(H) \leq n' + 1 \leq \frac{29}{11}(n' - 1)$. We may assume that $H \neq C_3, C_4$ (and thus $n' \geq 4$).

Suppose that $n' = 4$. If $H = K_4$, then the faces of H coincide with the faces of G, and $w_5(H) \leq 4$. Similarly, if $H = K_4 \setminus e$, i.e., H is the complement of an edge, then $w_5(H) \leq 6$, and if H is the complement of a path of length two, then $w_5(H) \leq 6$. As $\frac{29}{11}(4 - 1) > 7.9$, the claimed inequality holds for H.

We have analyzed all cases with $n' \leq 4$ and we focus on graphs H with five or more vertices in the rest. Let k_v be the number of faces of G that contain an edge of H incident with a vertex v. Let s_d, $d = 3, 4$, be the number of faces of H of size d. Since G is 5-connected, every such face f of H must also be a face of G. Consequently, the edges of H are incident with at most $\left(\frac{1}{2}\sum_{v \in V(H)} k_v\right) - \frac{1}{2}s_3 - s_4$ faces of G. An argument

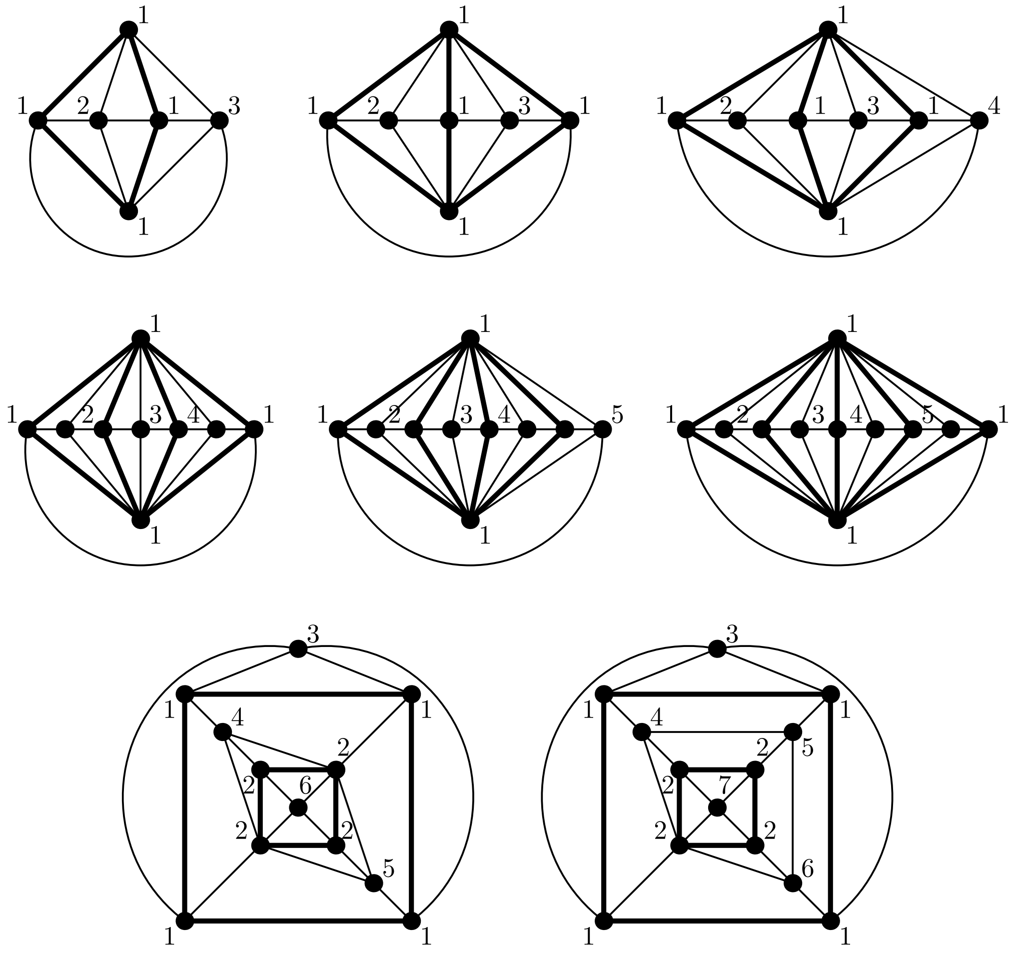

FIGURE 4. 4-connected plane graphs G with n vertices, $n = 6, 7, \ldots, 13$, that have non-rainbow colorings with $\left\lfloor \frac{5n-6}{8} \right\rfloor$ colors if $n \neq 11$, i.e., $n \not\equiv 3 \pmod 8$, and with $\left\lfloor \frac{5n}{8} \right\rfloor - 1 = 5$ colors, if $n = 11$. The edges of G that are monochromatic in such a coloring are drawn bold. The colors assigned to the vertices are represented by numbers (some vertices are not labeled with numbers; those vertices are colored with 1).

analogous to that used in the proof of Lemma 1 yields that

$$w_5(H) \leq \left[\sum_{v\in V(H)} \frac{1}{2}\left(k_v + 5 - \deg_G(v)\right)\right] - \frac{1}{2}s_3 - s_4 \quad \text{where} \tag{1}$$

$$k_v + 5 - \deg_G(v) \leq \min\{2\deg_H(v), 5\} \quad \text{for every } v \in V(H). \tag{2}$$

In particular, a vertex v contributes at most one to the sum in (1) if $\deg_H(v) = 1$, at most two if $\deg_H(v) = 2$, and it contributes at most $\frac{5}{2}$ otherwise. Let n'_d, $d = 1, 2$, denote the number of vertices of H with degree d. We can infer from (1) and (2) the following inequality:

$$w_5(H) \leq \frac{5}{2}n' - \frac{3}{2}n'_1 - \frac{1}{2}n'_2 - \frac{1}{2}s_3 - s_4 \tag{3}$$

We now apply Euler's formula to H. Plugging the inequalities $3n' - 2n'_1 - n'_2 \leq 2m$ and $5s - 2s_3 - s_4 \leq 2m$ where where m is the number of edges of H and s is the number of its faces into Euler's formula, we obtain the following:

$$\begin{aligned} n' + s &= m + 2 \\ n' + s &\geq (s - 2s_3/5 - s_4/5) + (9n'/10 - 6n'_1/10 - 3n'_2/10) + 2 \\ 6n'_1 + 3n'_2 + 4s_3 + 2s_4 &\geq 20 - n' \end{aligned} \tag{4}$$

Combining (3) and (4) yields:

$$w_5(H) \leq \frac{5}{2}n' - \frac{1}{8}(20 - n') = \frac{21n' - 20}{8} \tag{5}$$

If $n' \geq 12$, then $\frac{21n'-20}{8} \leq \frac{29}{11}(n' - 1)$ which yields the claimed estimate on $w_5(H)$.

As $w_5(H)$ is half-integral, it is possible to rewrite the inequality (5) to the form

$$w_5(H) \leq \frac{1}{2}\left\lfloor \frac{21n' - 20}{4} \right\rfloor . \tag{6}$$

Let us compare the values given by the estimate (6) and the claimed bound $\frac{29}{11}(n' - 1)$ for $5 \leq n' \leq 11$:

n'	$\frac{1}{2}\left\lfloor\frac{21n'-20}{4}\right\rfloor$	$\frac{29}{11}(n'-1)$
5	10.5	≈ 10.545
6	13	≈ 13.182
7	15.5	≈ 15.818
8	18.5	≈ 18.455
9	21	≈ 21.091
10	23.5	≈ 23.727
11	26	≈ 26.363

We conclude that it holds $w_5(H) \leq \frac{29}{11}(n'-1)$ with a possible exception of $n' = 8$. The case $n' = 8$ needs to be analyzed in more detail. The inequality (5) gives $w_5(H) \leq 18.5$. As $w_5(H)$ is half-integral, in order to establish $w_5(H) \leq \frac{29}{11}(n'-1)$, it is enough to show that $w_5(H) \neq 18.5$. If (5) were an equality, then all the inequalities used to derive (5) would also be equalities. In particular, it would hold that $n'_1 = n'_2 = s_4 = 0$ in (4) and H would be a cubic graph that contained three faces of size three and its remaining faces have size five. Euler's formula implies that H has six faces. Since there is no planar cubic graph with three faces of size three and three faces of size five, the inequality (5) must be strict which yields the claimed bound on $w_5(H)$ for $n' = 8$.

We have established that $w_5(H) \leq \frac{29}{11}(n'-1)$ for every n'-vertex monochromatic subgraph H of G. Finally, the upper bound of $\frac{25}{58}n - \frac{22}{29}$ on the number of colors follows from Lemma 2 applied with $\alpha = \frac{29}{11}$, $\beta_i = 0$ for all i, and $d = 5$. ∎

Unfortunately, we were not able to find matching lower and upper bounds on the number of colors in non-rainbow colorings of 5-connected plane graphs. The best lower bound construction that we have found is given in the next theorem (note that we only sketch some straightforward but technical steps needed to verify that the constructed graph is 5-connected at the end of the proof).

Theorem 6. For every real number $\varepsilon > 0$, there exists a plane 5-connected graph G with n vertices that has a non-rainbow coloring with $\left(\frac{171}{400} - \varepsilon\right) n$ colors.

Proof. We construct plane 5-connected graphs by combining plane gadgets of two different types, A-gadgets and B-gadgets.

An A-gadget is obtained as follows (the gadget is depicted in Figure 5): consider three concentric cycles in the plane, the inner and outer ones of length 10 and the middle one of length 20, and join the vertices of the middle cycle in an alternating way to the vertices of the inner and outer cycles. The graph obtained in this way is the graph formed by bold edges in the left part of Figure 5. Into each of the 20 pentagonal faces of the obtained graph, add a single vertex and join it with all the five vertices on its boundary. This completes the construction of the gadget. Note that an A-gadget has 60 vertices.

The construction of a B-gadget is more complex. First, start with a copy of the dodecahedron and subdivide each edge of two antipodal faces. Next, place a copy of the dodecahedron into each of the ten hexagonal faces of the obtained graph and join it

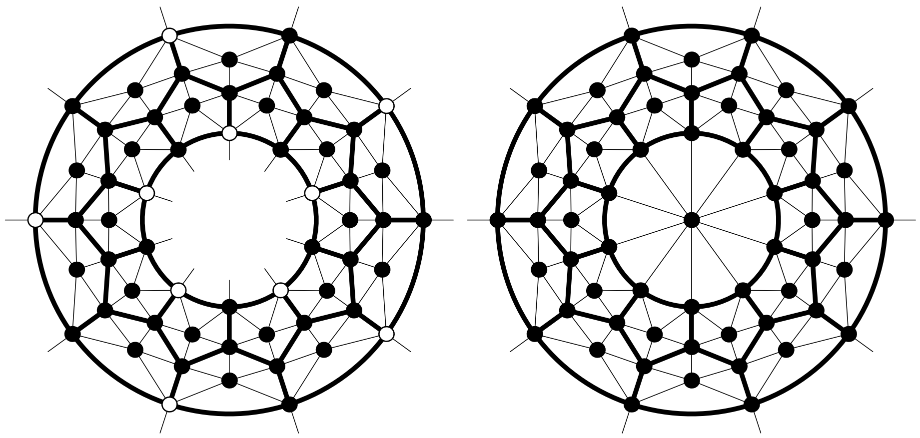

FIGURE 5. An A-gadget and its coloring (the monochromatic edges are bold). The interconnecting edges between an A-gadget and two B-gadgets are depicted in the left part of the figure and the interconnection between an A-gadget with an additional vertex of degree 10 and a B-gadget surrounding it in the right part of the figure. The vertices used to find five vertex-disjoint paths in the proof of 5-connectivity are drawn with empty circles.

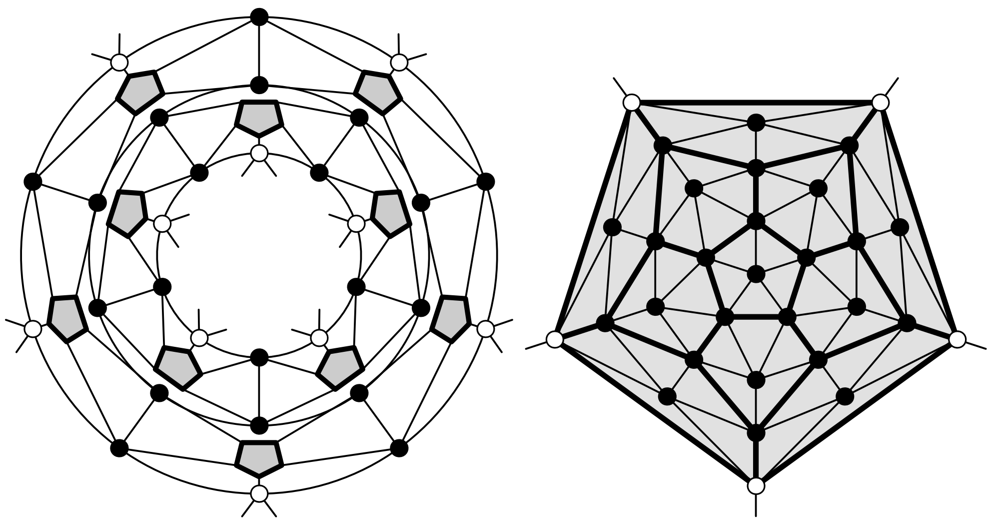

FIGURE 6. A B-gadget (depicted in the left part of the figure) and its coloring where the monochromatic edges are bold. Each of the gray parts of the gadget is a copy of the drawing depicted in the right. The vertices used to find five vertex-disjoint paths in the proof of 5-connectivity are drawn with empty circles.

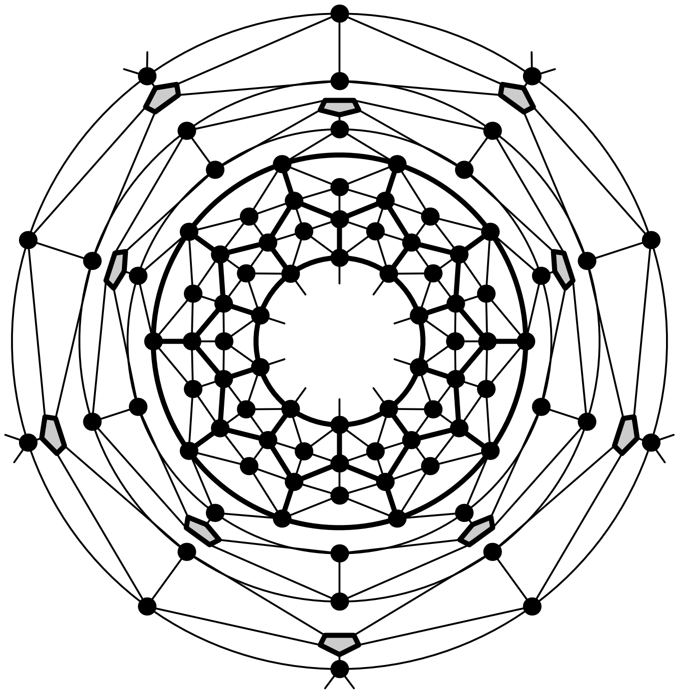

FIGURE 7. Edges joining an A-gadget and a B-gadget.

by five edges to the rest of the graph as depicted in the left part of Figure 6 (the gray pentagons represent the copies of the dodecahedron). Finally, add a vertex to each of the eleven faces of each copy of the dodecahedron and join this vertex to all the five vertices on the boundary of the face (see Figure 6 for illustration). The obtained graph has $30 + 10 \cdot 20 + 10 \cdot 11 = 340$ vertices.

We are now ready to construct plane graphs G_k. The graph G_k is obtained from $k+1$ A-gadgets and k B-gadgets by placing the gadgets concentrically in an alternating way, i.e., each B-gadget is surrounded by two A-gadgets. Next, ten edges between each pair of two neighboring gadgets are added in such a way that each vertex of the A-gadget is incident with one such edge and the vertices of the B-gadget have degree at least five, say this is done in the way drawn in Figure 7. A new vertex is placed in the most inner face and joined by ten edges to the vertices in its boundary as depicted in the right part of Figure 5. Similarly, a new vertex is added to the outer face. Hence, the graph G_k has $n = 60(k+1) + 340k + 2 = 400k + 62$ vertices in total.

Let us now turn to a construction of a non-rainbow coloring of G_k with many colors. We first describe the coloring of an A-gadget. The 40 vertices of the three original cycles are colored with the same color and the 20 new vertices with mutually distinct colors.

In this way, a coloring of the gadget with 21 colors avoiding a rainbow face is obtained (see Figure 5 for illustration).

We now describe the coloring of the vertices of the B-gadget. The vertices of each copy of the dodecahedron are colored with the same color but vertices in different copies receive different colors. The remaining vertices are colored with mutually distinct colors. In this way, a coloring with $30 + 10 + 10 \cdot 11 = 150$ colors that avoids a rainbow face is obtained. The coloring is also depicted in Figure 6.

Finally, a coloring of G_k with non-rainbow faces is obtained from the colorings of the gadgets and vertices of different gadgets are colored with distinct colors. The two vertices not contained in any of the gadgets are assigned colors different from the colors of all the other vertices. In this way, a coloring of G_k with $21(k + 1) + 150k + 2 = 171k + 23$ colors is obtained. Hence, for a sufficiently large integer k, the coloring uses more than $(\frac{171}{400} - \varepsilon)n$ colors.

In order to finish the proof, it remains to verify that the graph G_k is 5-connected. Since a complete proof of this fact is very technical, we sketch only the main idea and the reader is invited to check the missing details. In order to verify that the graph G_k is 5-connected, it is enough to construct five vertex-disjoint paths between any pair of vertices u and v of G_k. Assume first that u and v are contained in different gadgets. If u is contained in an A-gadget, find five vertex-disjoint paths from u to the five vertices depicted with empty cycles in right part of Figure 5 that surrounds the part of G_k containing the vertex v. If u is contained in a copy of the dodecahedron in a B-gadget, find first five vertex disjoint paths to the five vertices on the outer face of the dodecahedron and then extend them to vertex-disjoint paths to the vertices on the boundary of the gadget. Since the five vertices on the inner boundary and the five vertices on the outer boundary that are drawn with empty cycles in Figures 5 and 6 can be joined by five vertex-disjoint paths, there exist five vertex-disjoint paths between u and v. We leave the remaining details to the reader. Note that it is also necessary to verify the existence of five vertex-disjoint paths between u and v if u and v are in the same gadget, or if one or both are the vertices of degree 10. ∎

We were not able to close the gap between the multiplicative constants in the bounds that we provide in Theorems 5 and 6. There are examples of graphs (such as the dodecahedron appearing in the B-gadget) for which the bound on their 5-weight is tight but they seem very hard to combine together and thus a more involved argument may be needed to settle this case. We leave determining the optimal multiplicative constant in the bounds as an open problem.

Conjecture 3. There exists a constant C such that a rainbow coloring of a 5-connected plane graph with n vertices uses at most $\frac{3}{7}n + C$ colors and there exist 5-connected plane graphs with n vertices (for arbitrarily large n) with non-rainbow colorings with at least $\frac{3}{7}n - C$ colors.

Note that the conjectured multiplicative constant of 3/7 is sandwiched between the bounds given in Theorems 5 and 6.

ACKNOWLEDGEMENT

Riste Škrekovski would like to thank Department of Applied Mathematics of Charles University for hospitality and support during this research. We would also like to thank one of the referees for pointing out an error in the original proof of Theorem 5.

References

[1] J. L. Arocha, J. Bracho, V. Neumann-Laura: On the minimum size of tight hypergraphs, J. Graph Theory 16 (1992), pp. 319–326.

[2] J. L. Arocha, J. Bracho, V. Neumann-Laura: Tight and untight triangulated surfaces, J. Combin. Theory Ser. B 63 (1995), pp. 185–199.

[3] D. Barnette: Trees in polyhedral graphs, Canad. J. Math. 18 (1966), pp. 731–736.

[4] D. Biebighauser, M. Ellingham: Prism-hamiltonicity of triangulations, J. Graph Theory 57 (2008), pp. 181–197.

[5] A. A. Diwan: Disconnected 2-factors in planar cubic bridgeless graphs, J. Combin. Theory Ser. B 84 (2002), pp. 249–259.

[6] Z. Dvořák, S. Jendrol', D. Král', G. Pap: Matchings and non-rainbow colorings, submitted.

[7] Z. Dvořák, D. Král': On planar mixed hypergraphs, Electron. J. Combin. 8(1) (2001) #R35.

[8] Z. Dvořák, D. Král', R. Škrekovski: Coloring face hypergraphs on surfaces, European J. Combin. 26(1) (2004), pp. 95–110.

[9] H. Grötzsch: Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Wiss. Z. Martin-Luther-Universität, Halle, Wittenberg, Math.-Nat. Reihe 8 (1959), pp. 109–120.

[10] S. Jendrol': Note—rainbowness of cubic polyhedral graphs, IM Preprint A2-2006, Institute of Mathematics, Pavol Jozef Šafárik University in Košice, 2006, 7 pp.

[11] S. Jendrol': Rainbowness of cubic polyhedral graphs, Discrete Math. 306(24) (2006), pp. 3321–3326.

[12] S. Jendrol', Š. Schrötter: On rainbowness of semiregular polyhedra, Czechoslovak Math. J. 58 (2008), pp. 359–380.

[13] V. Jungić, D. Král', R. Škrekovski: Coloring of plane graphs with no rainbow faces, Combinatorica 26(2) (2006), pp. 169–182.

[14] D. Kobler, A. Kündgen: Gaps in the chromatic spectrum of face-constrained plane graphs, Electron. J. Combin. 8(1) (2001) #N3.

[15] D. Král': On maximum face-constrained coloring of plane graphs with no short face cycles, Discrete Math. 277 (2004), pp. 301–307.

[16] A. Kündgen, E. Mendelsohn, V. Voloshin: Colouring planar mixed hypergraphs, Electron. J. Combin. 7 (2000) #R60.

[17] A. Kündgen, R. Ramamurthi: Coloring face-hypergraphs of graphs on surfaces, J. Combin. Theory Ser. B 85 (2002), pp. 307–337.

[18] S. Negami, T. Midorikawa: Loosely-tightness of triangulations of closed surfaces, Sci. Rep. Yokohama Nat. Univ., Sect. I, Math. Phys. Chem. 43 (1996), pp. 25–41.

[19] S. Negami: Looseness ranges of triangulations on closed surfaces, Discrete Math. 303 (2005), pp. 167–174.

[20] J. G. Penaud: Une propriété de bicoloration des hypergraphes planaires, Colloque sur la Théorie des graphes, Cahiers Centre Études Recherche Opér. 17 (1975), pp. 345–349.

[21] R. Ramamurthi, D. B. West: Maximum face-constrained coloring of plane graphs, Discrete Math. 274(1–3) (2004), pp. 233–240.

[22] R. Ramamurthi, D. B. West: Maximum face-constrained coloring of plane graphs, Electronic Notes in Discrete Math. 11 (2002).

[23] C. Thomassen: Grötzsch's 3-Color Theorem, J. Combin. Theory Ser. B 62 (1994), pp. 268–279.

[24] A. A. Zykov: Hypergraphs, Uspekhi Mat. Nauk 29 (1974), pp. 89–154.