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TOTAL-COLORING OF PLANE GRAPHS WITH MAXIMUM DEGREE NINE*

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Abstract. The central problem of the total-colorings is the total-coloring conjecture, which asserts that every graph of maximum degree Δ admits a $(\Delta+2)$-total-coloring. Similar to edge-colorings—with Vizing's edge-coloring conjecture—this bound can be decreased by 1 for plane graphs of higher maximum degree. More precisely, it is known that if $\Delta \geq 10$, then every plane graph of maximum degree Δ is $(\Delta+1)$-totally-colorable. On the other hand, such a statement does not hold if $\Delta \leq 3$. We prove that every plane graph of maximum degree 9 can be 10-totally-colored.

Key words. total-coloring, planar graph, discharging method

AMS subject classifications. 05C15

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1. Introduction.

Given a graph $G = (V, E)$ and a positive integer k, a *k-total-coloring of G* is a mapping $\lambda : V \cup E \to \{1, 2, \ldots, k\}$ such that

(i) $\lambda(u) \neq \lambda(v)$ for every pair u, v of adjacent vertices,
(ii) $\lambda(v) \neq \lambda(e)$ for every vertex v and every edge e incident to v,
(iii) $\lambda(e) \neq \lambda(e')$ for every pair e, e' of incident edges.

This notion was independently introduced by Behzad [3] in his doctoral thesis, and Vizing [15]. It is now a prominent notion in graph coloring, to which a whole book is devoted [17]. Both Behzad and Vizing made the celebrated *total-coloring conjecture*, stating that every graph of maximum degree Δ admits a $(\Delta+2)$-total-coloring. Notice that every such graph cannot be totally-colored with less than $\Delta+1$ colors, and that a cycle of length 5 cannot be 3-totally-colored. The best general bound so far has been obtained by Molloy and Reed [10], who established that every graph of maximum degree Δ can be $(\Delta+10^{26})$-totally-colored. Moreover, the conjecture has been shown to be true for several special cases, namely for $\Delta = 3$ by Rosenfeld [11] and Vijayaditya [14], and then for $\Delta \in \{4, 5\}$ by Kostochka [9].

Another natural subclass to consider is the one of planar graphs, which has attracted a considerable amount of attention and several results were obtained. First, Borodin [5] proved that if $\Delta \geq 9$, then every plane graph of maximum degree Δ fulfills the conjecture. This result can be extended to the case where $\Delta = 8$ by the use of the four color theorem [1, 2], combined to Vizing's Theorem about edge coloring—the reader can consult the book by Jensen and Toft [8] for more details. Elsewhere, Sanders and Zhao [12] solved the case $\Delta = 7$ of the total-coloring conjecture for plane

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graphs. So the only open case regarding plane graphs is $\Delta = 6$. Interestingly, $\Delta = 6$ is also the only remaining open case for Vizing's edge-coloring conjecture, after Sanders and Zhao [13] resolved the case $\Delta = 7$.

However, plane graphs with high maximum degree allow a stronger assertion. More precisely, Borodin [5] showed that if $\Delta \geq 14$, then every plane graph with maximum degree Δ is $(\Delta+1)$-totally-colorable, and asked whether 14 could be decreased. Borodin, Kostochka, and Woodall extended this result to the case where $\Delta \geq 12$ [6], and later to $\Delta = 11$ [7]. Recently, Wang [16] established the result for $\Delta = 10$. On the other hand, this bound is not true if $\Delta \leq 3$. The complete graphs K_2, K_4, and the cycles of length $3k+2$ with $k \geq 1$ are examples of plane graphs that cannot be $(\Delta+1)$-totally-colored. We continue along those lines and establish the following theorem.

THEOREM 1. *Every plane graph of maximum degree* 9 *is* 10*-totally-colorable.*

So, the values of Δ, for which it is not known whether all plane graphs of maximum degree Δ are $(\Delta+1)$-totally-colorable are now $4, 5, 6, 7$, and 8. Recall that the case where $\Delta = 6$ is even open for the total-coloring conjecture. We also note that if $\Delta \geq 3$, then every outerplane graph with maximum degree Δ can be $(\Delta+1)$-totally-colored [19]. Another result of the same type is that every Halin graph of maximum degree 4 admits a 5-total-coloring [18]. Note also that the complete r-partite balanced graph K_{r*n}, whose maximum degree Δ is $n(r-1)$, admits a $(\Delta+2)$-total-coloring, and the cases where this bound can be decreased by 1 have been characterized [4].

We prove Theorem 1 by contradiction. From now on, we let $G = (V, E)$ be a minimum counter-example to the statement of Theorem 1, in the sense that the quantity $|V| + |E|$ is minimum. In particular, every proper subgraph of G is 10-totally-colorable. First, we establish various structural properties of G in section 2. Then, relying on these properties, we use the discharging method in section 3 to obtain a contradiction.

In what follows, a vertex of degree d is called a *d-vertex*. A vertex is a *$(\leq d)$-vertex* if its degree is at most d; it is a *$(\geq d)$-vertex* if its degree is at least d. If f is a face of G, then the *degree of f* is its length; i.e., the number of its incident vertices. The notions of *d-face*, *$(\leq d)$-face*, and *$(\geq d)$-face* are defined analogously as for the vertices. Moreover, if a vertex v is adjacent to a d-vertex u, then we say that u is a *d-neighbor* of v. A cycle of length 3 is called a *triangle*. For integers a, b, c, an *$(\leq a, \leq b, \leq c)$-triangle* is a triangle xyz of G with $\deg(x) \leq a, \deg(y) \leq b$, and $\deg(z) \leq c$. The notions of $(a, \leq b, \leq c)$-triangles, $(a, b, \geq c)$-triangles, and so on are defined analogously.

2. Reducible configurations. In this section, we establish some structural properties of the graph G. We prove that some planar graphs are *reducible configurations*; i.e., they cannot be subgraphs of G.

For convenience, we sometimes define configurations by depicting them in figures. In all of the figures of this paper, 2-vertices are represented by small black bullets, 3-vertices by black triangles, 4-vertices by black squares, and white bullets represent vertices whose degree is at least the one shown on the figure.

Let λ be a (partial) 10-total-coloring of G. For each element $x \in V \cup E$, we define $\mathcal{C}(x)$ to be the set of colors (with respect to λ) of vertices and edges incident or adjacent to x. Also, we set $\mathcal{F}(x) := \{1, 2, \ldots, 10\} \setminus \mathcal{C}(x)$. If $x \in V$, then we define $\mathcal{E}(x)$ to be the set of colors of the edges incident to x. Moreover, λ is *nice* if only some (≤ 4)-vertices are not colored. Observe that every nice coloring can be greedily extended to a 10-total-coloring of G, since $|\mathcal{C}(v)| \leq 8$ for each (≤ 4)-vertex v; i.e., v has at most 8 forbidden colors. Therefore, in the rest of this paper, we shall always suppose that such vertices are colored at the very end. More precisely, every time we consider a partial coloring of G, we uncolor all ($\leq$4)-vertices, and implicitly color them at the very end of the coloring procedure of G. We make the following observation about nice colorings and use it implicitly throughout this paper.

Observation. Let uv be an edge with $\deg(v) \leq 4$. There exists a nice coloring λ of $G - e$, in which u is colored and v is uncolored. Moreover, it then suffices to properly color the edge e with a color from $\{1, 2, \ldots, 10\}$ to extend λ to a nice coloring of G.

We now study the structural properties of G in a series of lemmas.

Lemma 2. *The graph G has the following properties:*

(i) *the minimum degree is at least* 2*;*
(ii) *if vu is an edge with* $\deg(v) \leq 4$ *then* $\deg(u) \geq 11 - \deg(v)$*;*
(iii) *a* 9*-vertex is adjacent to at most one* 2*-vertex;*
(iv) *a triangle incident to a* 3*-vertex must also contain a* 9*-vertex;*
(v) *there is no* $(4, \leq 7, \leq 8)$*-triangle;*
(vi) *a triangle contains at most one* ($\leq$5)*-vertex.*

Proof. (i) Suppose that v is a 1-vertex, and let u be its neighbor. By the minimality of G, the graph $G - v$ admits a nice coloring in which u is colored. Since the degree of u in $G - v$ is at most 8, we obtain $|\mathcal{C}(vu)| \leq 9$. Thus, the edge vu can be properly colored, which yields a nice coloring of G.

(ii) Suppose that $vu \in E$ with $\deg(v) \leq 4$ and $\deg(u) \leq 10 - \deg(v)$. There exists a nice coloring of $G' := G - vu$, in which u is colored and v is uncolored. Therefore, $|\mathcal{C}(vu)| \leq \deg(v) - 1 + \deg(u) - 1 + 1 \leq 9$. Hence we can color properly the edge vu, thereby obtaining a nice coloring of G.

(iii) Suppose that v is a 9-vertex adjacent to two 2-vertices x and y. Let x' be the neighbor of x different from v, and let y' be the neighbor of y different from v. Notice that we may have $x' = y'$. By the previous assertion, x' and y' are 9-vertices. It is enough to consider the following two possibilities.

v is adjacent to neither x' nor y'. Then, we construct the graph G' by first removing x and y, and then adding the edge vx'. If $y' \neq x'$, then we additionally add the edge vy'. Note that G' is a simple plane graph of maximum degree 9 with fewer vertices and edges than G. Therefore, it admits a nice coloring λ by the minimality of G. We easily modify λ to obtain a nice coloring of G. First, put $\lambda(xx') := \lambda(vy) := \lambda'(vx')$. Now, if $x' \neq y'$, then we put $\lambda(vx) := \lambda(yy') := \lambda'(vy')$. See Figure 1(a) for an illustration. And, if $x' = y'$, then we note that each of the edges yy' and vx has at most 9 forbidden colors. Thus, both of them can be colored and the obtained 10-total-coloring of G is nice.

v is adjacent to x'. Thus vxx' is a triangle. Consider a nice coloring of $G - vy$. To extend it to G, it suffices to properly color the edge vy. If this cannot be done greedily, then $|\mathcal{C}(vy)| = 10$, and up to a permutation of the colors, we can assume that the coloring is the one shown in Figure 1(b). If $a \neq 10$, then recolor vx with 10 and color vy with 5 to obtain a nice coloring of G. And if $a = 10$, then we interchange the colors of vx' and xx', and afterwards color vy with 4.

(iv) By 2, a 3-vertex has only ($\geq$8)-neighbors. Thus we may suppose that vwu is a $(3, 8, 8)$-triangle, with u being the 3-vertex. Consider a nice coloring of $G - vu$. To extend it to G, again it suffices to properly color the edge vu. If we cannot do this greedily, then it means that $|\mathcal{C}(vu)| = 10$. Thus, up to a permutation of the colors, the

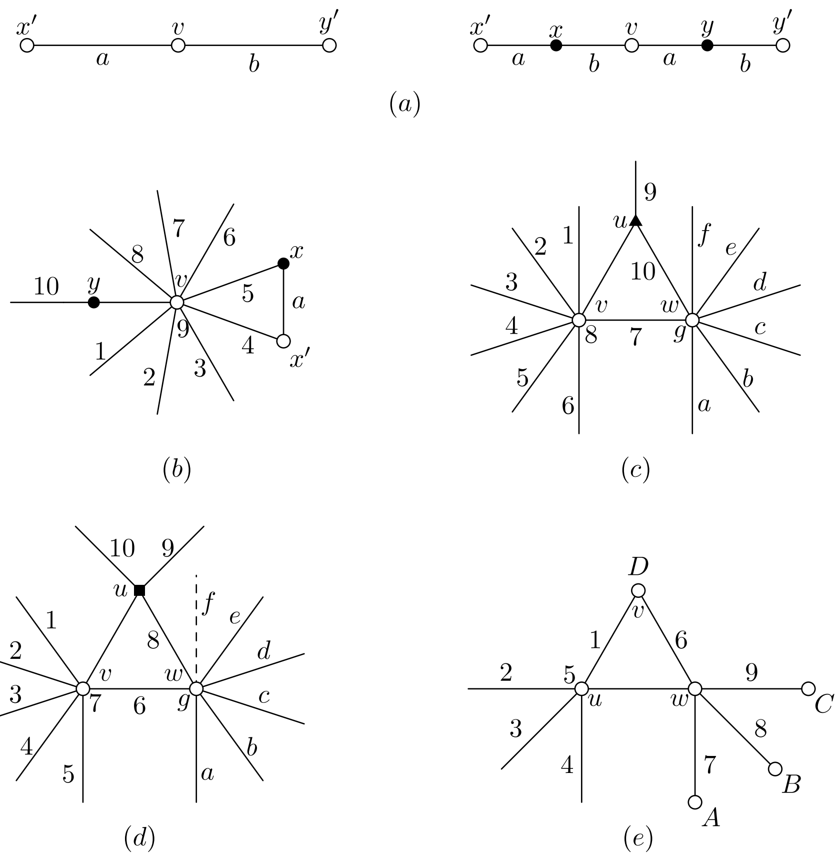

FIG. 1. *Configurations for the proof of Lemma* 2.

coloring is the one shown in Figure 1(c). If the edge wu can be properly recolored, then we do so, and afterwards color the edge vu with 10, which gives a nice coloring of G. So we deduce that $|\mathcal{C}(wu)| = 9$. Consequently, $\{a, b, c, d, e, f, g\} = \{1, 2, 3, 4, 5, 6, 8\}$. Thus we obtain $9 \notin \mathcal{C}(vw)$. So, we can recolor vw with 9 and color vu with 7 to conclude the proof.

(v) By 2, it is enough to prove that there is no $(4, 7, \delta)$-triangle in G for $\delta \in \{7, 8\}$. Suppose that vwu is such a triangle with w having degree δ and u degree 4. Consider a nice coloring of $G - vu$. It is sufficient to properly color the edge vu to obtain a nice coloring of G. Again, $|\mathcal{C}(vu)| = 10$, so up to a permutation of the colors, we assume that the coloring is the one of Figure 1(d). If the edge wu can be properly recolored, then do so, and color vu with 8 to obtain a nice coloring of G. Thus, we deduce that $|\mathcal{C}(wu)| = 9$. Therefore, $\{1, 2, 3, 4, 5, 7\} \subset \{a, b, c, d, e, f, g\}$. From this we infer that $|\mathcal{C}(vw)| \leq 6 + \delta - 6 = \delta \leq 8$. Thus, the edge vw can be properly recolored, and so the edge vu can be colored with 6, yielding a nice coloring of G.

(vi) Let vuw be a triangle with $\deg(u) = \deg(w) = 5$. Consider a total-coloring of $G - uw$, and uncolor the vertex w. Observe that $|\mathcal{F}(uw)| \geq 1$ and $|\mathcal{F}(w)| \geq 1$. Furthermore, these two sets must actually be equal and of size 1, otherwise we can extend the coloring to G. Up to a permutation of the colors, the coloring is the one shown in Figure 1(e), with $\{A, B, C, D\} = \{1, 2, 3, 4\}$. Notice that the colors of the edges vu and vw can be safely interchanged. Now, the vertex w can be properly

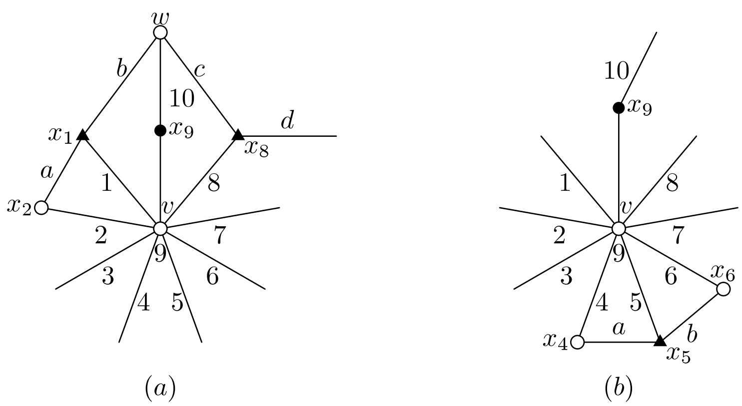

FIG. 2. *Reducible configurations of Lemma* 33 *and* 3.

colored with 6, and the edge uw with 10. ◻

LEMMA 3. *For the graph G, the following assertions hold.*

(i) *There is no* $(5,6,6)$*-triangle.*

(ii) *A* 6*-vertex has at most two* 5*-neighbors.*

(iii) *Suppose that v is a* 7*-vertex, and let x_1 be one of its neighbors. If v and x_1 have at least two common neighbors, then at most one of them has degree* 4*.*

(iv) *Suppose that vwu and vwu' are two triangles with* $\deg(u) = 2$*. Then,* $\deg(u') \geq 4$.

(v) *Suppose that v is a* 9*-vertex incident to a* $(2,9,9)$*-triangle. Then it is not incident to a* $(\leq 3, \geq 8, 9)$*-triangle.*

(vi) *The configuration of Figure* 2(a) *is reducible.*

(vii) *The configuration of Figure* 2(b) *is reducible.*

Proof. (i) Suppose on the contrary that G contains a $(5,6,6)$-triangle uvw with u being of degree 5. The proof is in two steps. In the first step, we prove the existence of a 10-total-coloring of G in which only u is uncolored. And in the second step, we establish that such a coloring can be extended to G. Consider a nice coloring of $G - vu$, and uncolor the vertex u. Our only goal in the first step is to properly color the edge vu. If we cannot do this greedily, then $|\mathcal{C}(vu)| = 10$, and thus we can assume that the coloring is the one of Figure 3(a). We infer that $\{6,7,8,9,10\} = \{a,b,c,d,e\}$, otherwise we can choose a color $\alpha \in \{6,7,8,9,10\} \setminus \{a,b,c,d,e\}$, recolor uw with α, and color vu with 4. Consequently, we have $\mathcal{C}(vw) = \{4,6,7,8,9,10\}$. Thus, we can recolor vw with 1, and color vu with 5.

For the second step, consider a partial 10-total-coloring of G such that only u is not colored. If we cannot greedily extend it to G, then without loss of generality the coloring is the one of Figure 3(b). Note that if $|\mathcal{C}(vu)| \leq 8$, then we can recolor vu, and color u with 5. Thus, we infer that $\{a,b,c,d,e\} \supset \{7,8,9,10\}$. Similarly, $\{e,f,g,h,i\} \supset \{6,8,9,10\}$. Observe that $|\mathcal{C}(v)| = 9$, otherwise we just properly recolor v, and color u with 6.

We assert that we can assume that $e \in \{1,2,3\}$. If it is not the case, then $e \in \{8,9,10\}$, say $e = 10$. By what precedes, $|\mathcal{C}(vw)| \leq 12 - 4 = 8$ and $\{4,5,6,7,8,9\} \subset \mathcal{C}(vw)$. Thus at least one color among $1,2,3$ can be used to recolor vw, which proves the assertion. Therefore, $\{a,b,c,d\} = \{7,8,9,10\}$ and $\{f,g,h,i\} = \{6,8,9,10\}$. Thus vw can be recolored by every color of $\{1,2,3\}$. So, if there exists a color $\alpha \in \{1,2,3\} \setminus \{A,B,C,D\}$, we can recolor vw with a color of $\{1,2,3\}$ different from α, recolor v

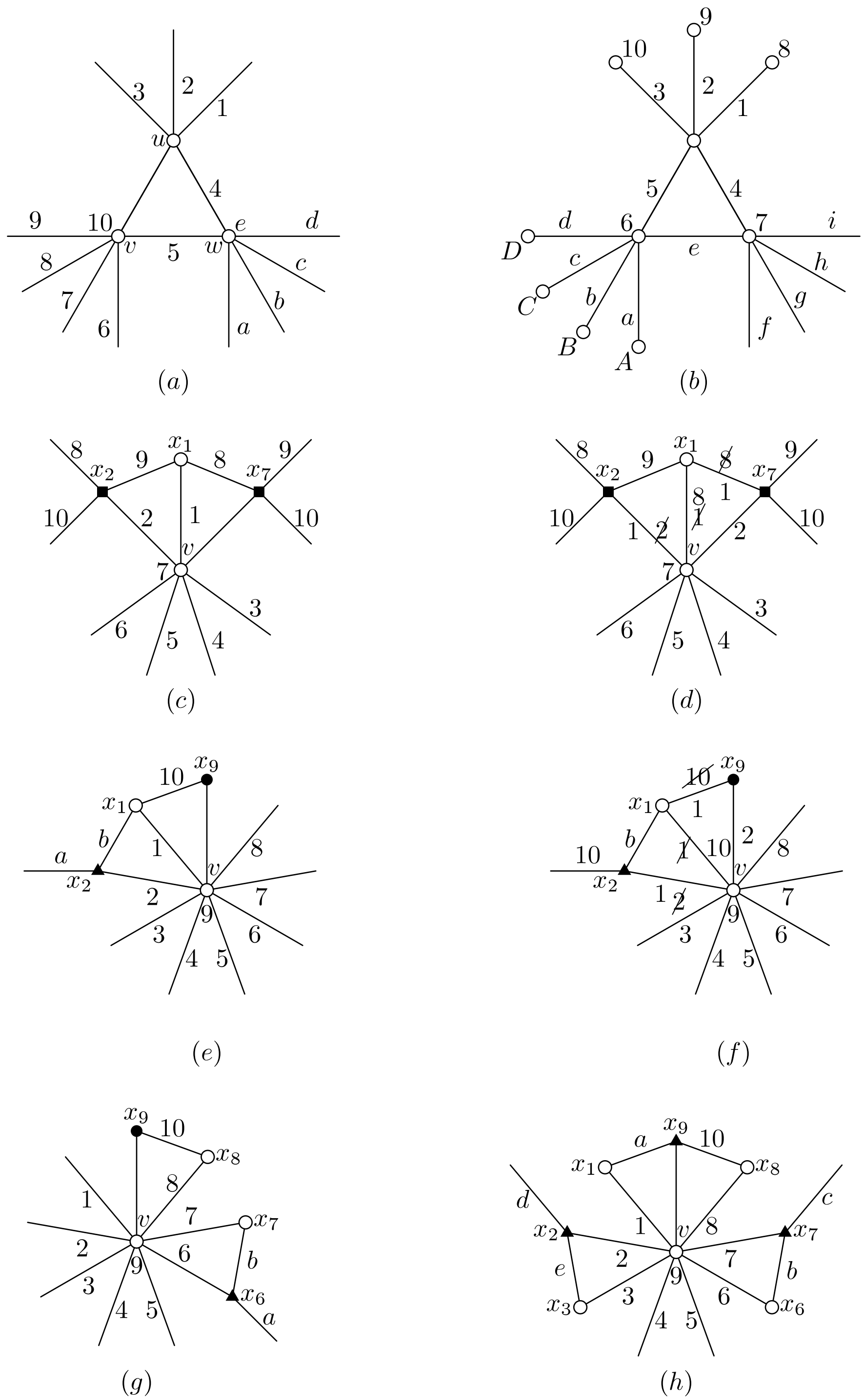

Fig. 3. *Configurations for the proofs of Lemmas* 3 *and* 4.

with α and color u with 6. Hence $\{1,2,3\} \subseteq \{A,B,C,D\}$. Now, recall that $|\mathcal{C}(v)| = 9$, thus $4 \in \{A,B,C,D\}$. Consequently, we can interchange safely the colors of vu and wu, recolor v with 5, and finally color u with 6.

(ii) Suppose that v is a 6-vertex with three 5-neighbors x_1, x_2, x_3. By Lemma 22, these three vertices are pairwise nonadjacent. Let λ be a nice coloring of $G - vx_1$, and uncolor the edges vx_2 and vx_3 as well as the vertices v, x_1, x_2, and x_3. Notice that $|\mathcal{C}(x_i)| \leq 8$ and $|\mathcal{C}(vx_i)| \leq 7$ for each $i \in \{1,2,3\}$. Moreover, $|\mathcal{C}(v)| \leq 6$. Recall that $\mathcal{F}(x) := \{1,2,\ldots,10\} \setminus \mathcal{C}(x)$ for every $x \in V \cup E$. Observe that for each $i \in \{1,2,3\}$, we have $\mathcal{F}(v) \cap \mathcal{F}(x_i) \subseteq \mathcal{F}(vx_i)$. Hence, we infer that $\mathcal{F}(v) \cap (\mathcal{F}(vx_i) \cup \mathcal{F}(x_i)) = \mathcal{F}(v) \cap \mathcal{F}(vx_i)$. Consequently, there exists a color $\alpha \in \mathcal{F}(v)$ such that, after setting $\lambda(v) := \alpha$, it holds that $|\mathcal{F}(x_3)| \geq 2$ and $|\mathcal{F}(vx_3)| \geq 3$. If we color properly x_1, vx_1, x_2, and vx_2, then we will be able to color greedily x_3 and vx_3, and hence the proof would be complete. Observe that if α does not belong to $\mathcal{F}(x_1)$ or to $\mathcal{F}(vx_2)$, then the coloring can be extended greedily to x_1, x_2, vx_1, vx_2—just color x_1 or vx_2 last, respectively. Therefore we assume that α belongs to these two lists. Uncolor v and color x_1 and vx_2 with α. With respect to this coloring, note that $|\mathcal{F}(vx_1)| \geq 2$, $|\mathcal{F}(v)| \geq 3$, $|\mathcal{F}(x_2)| \geq 1$, $|\mathcal{F}(vx_3)| \geq 3$ and $|\mathcal{F}(x_3)| \geq 2$. Hence, we can color x_2. Now, if there exists $\beta \in \mathcal{F}(vx_1) \cap \mathcal{F}(x_3)$, then we let $\lambda(vx_1) := \lambda(x_3) := \beta$, and afterwards greedily color v and vx_3.

So, $\mathcal{F}(vx_1) \cap \mathcal{F}(x_3) = \emptyset$. If there exists $\kappa \in \mathcal{F}(v) \cap \mathcal{F}(x_3)$, then we set $\lambda(v) := \kappa$, and afterwards we greedily color x_3, vx_3, and vx_1 in this order. This is possible since $\kappa \notin \mathcal{F}(vx_1)$. Otherwise, greedily coloring vx_1, v, vx_3, and x_3 in this order yields a nice coloring of G.

(iii) Suppose that the statement is false, so the graph G contains the configuration of Figure 3(c). Consider a nice coloring λ of $G - vx_7$. If it cannot be extended to G, then $|\mathcal{C}(vx_7)| = 10$. Furthermore, $|\mathcal{C}(vx_2)| = 9$, otherwise we can color the edge vx_7 with $\lambda(vx_2)$ and greedily recolor the edge vx_2, thereby obtaining a nice coloring of G. Therefore, we can assume that the coloring is the one shown in Figure 3(c). Then a nice coloring of G is obtained by interchanging the colors of the edges x_7x_1 and vx_1, recoloring vx_2 with 1 and coloring vx_7 with 2, as shown in Figure 3(d).

(iv) Suppose on the contrary that G contains the configuration of Figure 3(e). Consider a nice coloring of $G - vx_9$. If the edge vx_9 cannot be greedily colored, then $|\mathcal{C}(vx_9)| = 10$. Thus we may assume that the coloring is the one shown in Figure 3(e). Notice that $a = 10$, otherwise we recolor vx_2 with 10 and color vx_9 with 2. So, the recoloring in Figure 3(f) is nice.

(v) Suppose that G contains the configuration of Figure 3(g), and consider a nice coloring λ of $G - vx_9$. Without loss of generality, we may assume that it is the one of Figure 3(g). Observe that $10 \in \{a,b\}$, otherwise we obtain a nice coloring of G by setting $\lambda(vx_6) := 10$ and $\lambda(vx_9) := 6$. Now, we consider two cases regarding b.

$b = 10$. If $a \neq 7$, then we can interchange the colors of the edges x_6x_7 and vx_7, and color vx_9 with 7 to obtain a nice coloring of G. And if $a = 7$, then we interchange the colors of the edges x_9x_8 and vx_8, and then we let $\lambda(vx_6) := 8$ and $\lambda(vx_9) := 6$.

$b \neq 10$. In this case, $a = 10$. We interchange the colors of x_9x_8 and vx_8. Similar to before, we deduce that $b = 8$. Now, the previous case applies with 8 playing the role of color 10.

(vi) Suppose on the contrary that G contains the configuration of Figure 2(a). Up to a permutation of the colors, every nice coloring of $G - vx_9$ is as the one of the figure. Note that $d = 10$, otherwise recolor vx_8 with 10 and color vx_9 with 8.

Similarly, $a = 10$. Now, interchange the colors of the edges x_1x_2 and vx_2. If $b \neq 2$, then the obtained coloring extends to G by coloring vx_9 with 2. If $b = 2$, then interchange the colors of the edges x_9w and x_1w thereby obtaining a nice coloring of $G - vx_9$. Since $d = 10 \neq 2$, observe that we can extend it to G as before; i.e., we recolor vx_8 with 2 and color vx_9 with 8.

(vii) Suppose that G contains the configuration of Figure 2(b). Consider a nice coloring of $G - vx_9$. Without loss of generality, we may assume that it is the one of the figure. Note that $10 \in \{a, b\}$, otherwise recolor vx_5 with 10 and color vx_9 with 5. By symmetry, we can assume that $a = 10$. Interchange the colors of the edges x_5x_4 and vx_4. If $b \neq 4$, then we have a nice coloring of $G - vx_9$, and we extend it to G by coloring vx_9 with 4. Otherwise, $b = 4$, we interchange the colors of the edges x_5x_6 and vx_6, and color vx_9 with 6, which yields a nice coloring of G. □

Lemma 4. *The configuration of Figure* 3(h) *is reducible.*

Proof. Consider a nice coloring of $G - vx_9$. If it cannot be greedily extended to G, then $|\mathcal{C}(vx_9)| = 10$, and so we can assume that the coloring is the one of Figure 3(h). First, we note that if $a \neq 7$, then $10 \in \{b, c\}$; otherwise, we recolor vx_7 by 10, and color vx_9 with 7. Similarly, if $a \neq 2$, then $10 \in \{d, e\}$. We now split the proof into three cases.

$a \notin \{6, 8\}$. Since a is different from either 2 or 7, we may assume that $a \neq 7$. As mentioned above, we must have $10 \in \{b, c\}$. Moreover, if we interchange the colors of the edges x_9x_8 and vx_8, then we deduce as before that $8 \in \{b, c\}$, the color 8 playing the role of color 10. Hence $\{b, c\} = \{8, 10\}$. Now, interchange the colors of the edges x_7x_6 and vx_6, and color vx_9 with 6. If $b = 10$, then the obtained coloring is proper, and if $b = 8$, then we additionally interchange the colors of the edges x_9x_8 and vx_8 to obtain the desired coloring.

$a = 8$. In this case $10 \in \{b, c\}$. By interchanging the colors of the edges x_9x_8 and vx_8, and also of x_9x_1 and vx_1, we infer that $1 \in \{b, c\}$. Hence $\{b, c\} = \{1, 10\}$. Similar to in the previous case, interchange the colors of x_7x_6 and vx_6, and afterwards color vx_9 with 6. If $b = 10$, then the obtained coloring of G is proper, and if $b = 1$, then it suffices to additionally interchange the colors of the edges x_9x_1 and vx_1, and also of x_9x_8 and vx_8 to obtain a nice coloring of G.

$a = 6$. Then, $10 \in \{d, e\}$. Note that the colors of the edges x_9x_8 and vx_8 can be interchanged safely, because $a \neq 8$. Therefore, as $a \neq 2$, we infer that $8 \in \{d, e\}$, and hence $\{d, e\} = \{8, 10\}$. We interchange now the colors of the edges x_2x_3 and vx_3, and color vx_9 with 3. If $e = 10$, then the obtained coloring of G if proper. And, if $e = 8$, then it suffices to interchange the colors of the edges x_9x_8 and vx_8 to obtain the desired coloring. □

Lemma 5. *If uvz is a triangle with an* 8*-vertex v and a* 3*-vertex u, then v has no* 3*-neighbor distinct from u.*

Proof. Suppose that v is an 8-vertex that contradicts the lemma. Let u and w be two 3-neighbors of v, and assume that vuz is a triangle. We consider a nice coloring of $G - vu$. If we cannot extend it to G, then, without loss of generality, we may assume that the coloring is the one shown on Figure 4(a). Observe that $\{a, b\} = \{9, 10\}$, otherwise we obtain the desired coloring by recoloring vw with either 9 or 10, and coloring vu with 2. Now, as depicted in Figure 4(b), we interchange the colors of the edges uz and vz, recolor vw with 1, and color vu with 2 to obtain the sought coloring. □

Lemma 6. *The configuration of Figure* 5(a) *is reducible.*

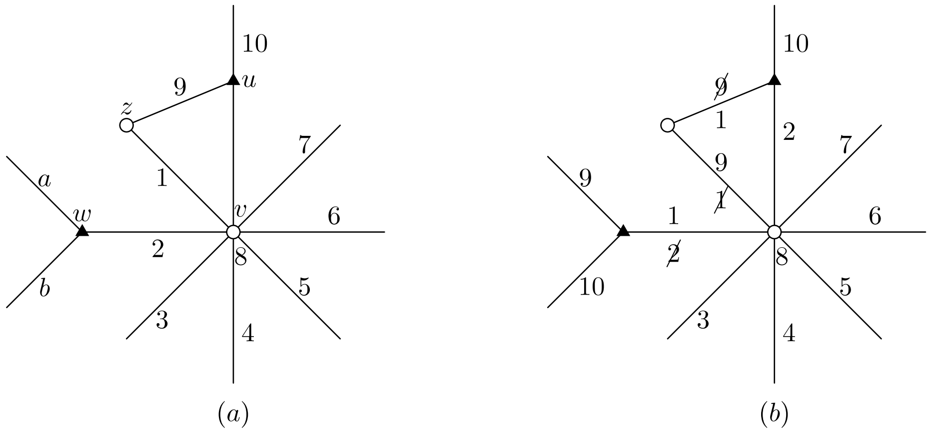

FIG. 4. *Coloring and recoloring for the proof of Lemma* 5.

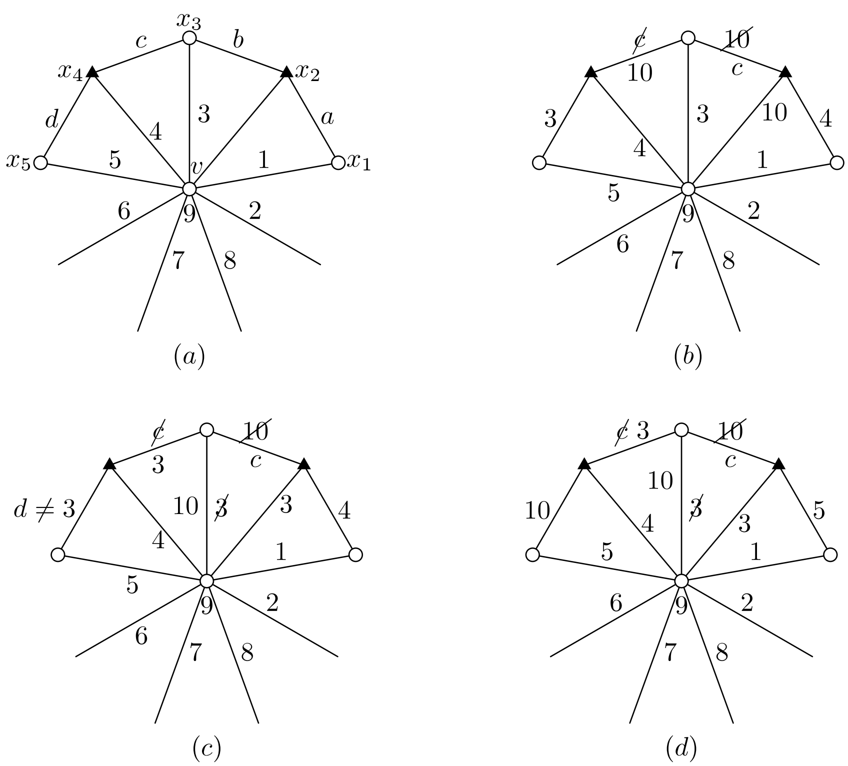

FIG. 5. *Configurations for Lemma* 6.

Proof. Consider a nice coloring of $G - vx_2$. Up to a permutation of the colors, it is the one of Figure 5(a). Note that $10 \in \{a, b\}$, otherwise we obtain a nice coloring of G by coloring vx_2 with 10. We split the proof into two cases, regarding the value of b.

Case 1. $b = 10$. If $a = 4$, then apply the recolorings of Figures 5(b) and (c), regarding whether d is 3.

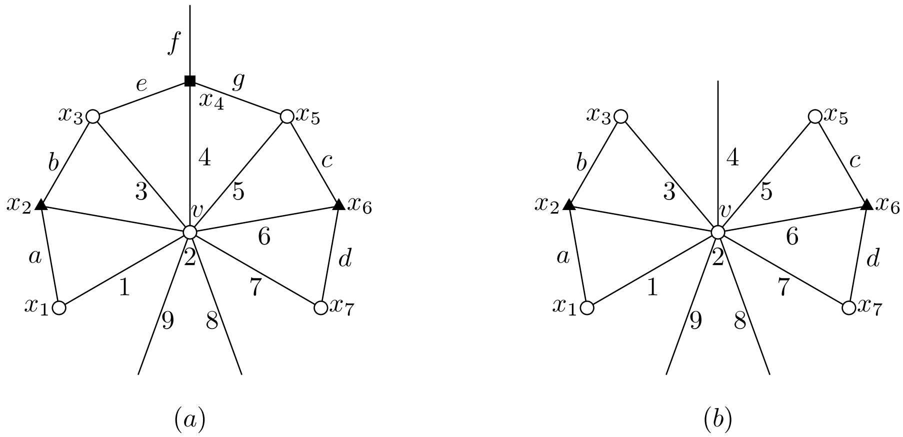

FIG. 6. *Configurations for Lemmas* 7 *and* 8.

Suppose now that $a \neq 4$. In this case, we deduce that $d = 10$; otherwise, we can recolor vx_4 with 10, and color vx_2 with 4. If $c \neq 5$, then the desired coloring can be obtained as follows. If $a \neq 5$, then interchange the colors of the edges x_4x_5 and vx_5, and color vx_2 with 5, and if $a = 5$, then the recoloring of Figure 5(d) is nice.

We may assume now that $c = 5$. Interchange the colors of the edges x_4x_5 and vx_5, and also of the edges x_4x_3 and vx_3. If $a \neq 3$, then it suffices to color vx_2 with 3. And, if $a = 3$, then additionally interchange the colors of the edges x_2x_1 and vx_1, recolor vx_4 with 1, and color vx_2 with 4 to obtain the sought coloring.

Case 2. $b \neq 10$. Therefore, $a = 10$. First, note that $10 \in \{c, d\}$; otherwise, we recolor vx_4 with 10, and color vx_2 with 4. Either the obtained coloring of G is nice, or $b = 4$. In the latter case, we additionally interchange the colors of x_2x_3 and x_4x_3 to obtain the desired coloring.

Suppose now that $c = 10$. Then, $b = 4$; otherwise, we uncolor vx_4, color vx_2 with 4, and apply Case 1 to the obtained coloring with x_4 playing the role of the vertex x_2. Now, interchange the colors of x_4x_3 and vx_3. The obtained coloring is nice if $d \neq 3$, and we extend it to G by coloring vx_2 with 3. And, if $d = 3$, then we additionally interchange the colors of x_4x_5 and vx_5, and color vx_2 with 5.

Finally, assume that $c \neq 10$, and hence $d = 10$. Up to interchanging the colors of x_2x_3 and x_4x_3, we may assume that $b \neq 5$. Interchange the colors of x_4x_5 and vx_5. If $c \neq 5$, the obtained coloring is nice and we extend it to G by coloring vx_2 with 5. And, if $c = 5$, then we additionally interchange the colors of x_4x_3 and vx_3, and color vx_2 with 3. ∎

LEMMA 7. *The configuration of Figure* 6(a) *is reducible.*

Our proof of Lemma 7 uses the following result. Given a coloring λ and a vertex v, recall that $\mathcal{E}(v)$ is the set of colors assigned to the edges incident to v. Let $\mathcal{E}'(v) := \{1, 2, \ldots, 10\} \setminus (\mathcal{E}(v) \cup \{\lambda(v)\})$.

LEMMA 8. *Suppose that G contains the configuration of Figure* 6(b)*. Then, for every nice coloring λ of $G - vx_2$, it holds that $\mathcal{E}'(v) \cup \{\lambda(vx_6)\} \subseteq \mathcal{E}(x_2)$.*

Proof. Up to a permutation of the colors, the coloring λ is the one of Figure 6(b). Notice that $\mathcal{E}'(v) = \{10\}$, $\lambda(vx_6) = 6$, and $\mathcal{E}(x_2) = \{a, b\}$. First, $10 \in \{a, b\}$; otherwise, we just color vx_2 with 10. By symmetry, we may assume that $a = 10$. Thus, to finish the proof, it only remains to prove that $b = 6$. Suppose on the contrary that $b \neq 6$. Note that $10 \in \{c, d\}$; otherwise, we recolor vx_6 with 10 and color vx_2 with 6. By symmetry, we may assume that $d = 10$. We consider two possibilities regarding the value of b.

$b = 1$. Interchange the colors of the edges x_6x_7 and vx_7. The obtained coloring of G is nice if $c \neq 7$, and if $c = 7$ we additionally interchange the colors of x_6x_5 and vx_5. Now, coloring vx_2 with 7 or 5 yields a nice coloring of G, a contradiction.

$b \neq 1$. In this case, $c = 1$. Indeed, if $c \neq 1$, then we recolor vx_6 with 1, interchange the colors of x_2x_1 and vx_1, and color vx_2 with 6 to obtain a nice coloring of G. Now, if $b \neq 7$, then interchange the colors of x_6x_7 and vx_7 and color vx_2 with 7. And, if $b = 7$, then interchange the colors of x_6x_5 and vx_5, and also of x_2x_1 and vx_1, and color vx_2 with 5. ∎

Proof of Lemma 7. Consider a nice coloring λ of $G - vx_2$. Up to a permutation of the colors, we assume that the coloring is the one of Figure 6(a). By Lemma 8, we have $\{a, b\} = \{6, 10\}$. We consider two cases.

$a = 10$ and $b = 6$. If there exists a color $\alpha \in \{1, 10\} \setminus \{e, f, g\}$, then recolor vx_4 with α, and color vx_2 with 4. The obtained coloring is nice if $\alpha = 10$. And, if $\alpha = 1$, then it suffices to additionally interchange the colors of x_2x_1 and vx_1. Thus, $\{1, 10\} \subset \{e, f, g\}$.
Suppose that $6 \notin \{e, f, g\}$. We start by interchanging the colors of the edges x_2x_3 and x_4x_3. If $e = 10$, then we additionally interchange the colors of x_2x_1 and vx_1. Observe that the obtained coloring does not fulfill the conclusion of Lemma 8, a contradiction. Hence, $\{e, f, g\} = \{1, 6, 10\}$ and so $e \in \{1, 10\}$. We interchange the colors of x_4x_3 and vx_3, and color vx_2 with 3. Either this coloring of G is nice, or $e = 1$ and hence additionally interchanging the colors of x_2x_1 and vx_1 yields a nice coloring of G.

$a = 6$ and $b = 10$. If there exists $\alpha \in \{3, 10\} \setminus \{f, g\}$, then recolor vx_4 with α, and color vx_2 with 4. If the obtained coloring is not nice, then $\alpha = 3$ and hence interchanging the colors of x_2x_3 and vx_3 yields a nice coloring of G, a contradiction. Observe that we may assume that $f = 3$ and $g = 10$. Indeed, if it is not the case, then we interchange the colors of x_2x_3 and vx_3 and obtain the desired condition, with 3 playing the role of color 10.
Furthermore, $e = 5$; otherwise, we interchange the colors of x_4x_5 and vx_5, and color vx_2 with 5. Now, observe that $d = 10$; otherwise we recolor vx_6 with 10, vx_4 with 6, and color vx_2 with 4 to obtain a nice coloring of G. Finally, we interchange the colors of x_6x_7 and vx_7. If $c = 7$, then we additionally interchange the colors of x_6x_5 and vx_5. Now, coloring vx_2 with 7 or 5 yields a nice coloring of G, a contradiction. ∎

LEMMA 9. *The configurations of Figure* 7 *are reducible.*

Proof. Consider a nice coloring of $G - vu$. We may assume that the coloring is the one of Figure 7. Let $\alpha \in \{1, 7, 9, 10\} \setminus \{a, b, c\}$. We recolor vx_3 with α and color vu with 3. The obtained coloring of G is nice unless $\alpha \in \{1, 7\}$. If $\alpha = 1$ then we additionally interchange the colors of uw and vw. And if $\alpha = 7$, then we interchange the colors of ut and vt. ∎

LEMMA 10. *A* 6*-vertex incident to* 6 *triangles is not adjacent to two* 5*-vertices.*

Proof. Suppose that v is a 6-vertex. We let $x_1, x_2, \ldots, x_6$ be its neighbors, such that x_i is adjacent to x_{i+1} if $i \in \{1, 2, \ldots, 5\}$ and x_6 is adjacent to x_1. We also assume that x_6 is a 5-vertex, and we let w be the other 5-vertex. By symmetry and Lemma 22, we may assume that $w \in \{x_2, x_3\}$. The proof is in two steps. In the first step, we show that there exists a partial 10-total-coloring of G in which only x_6 is

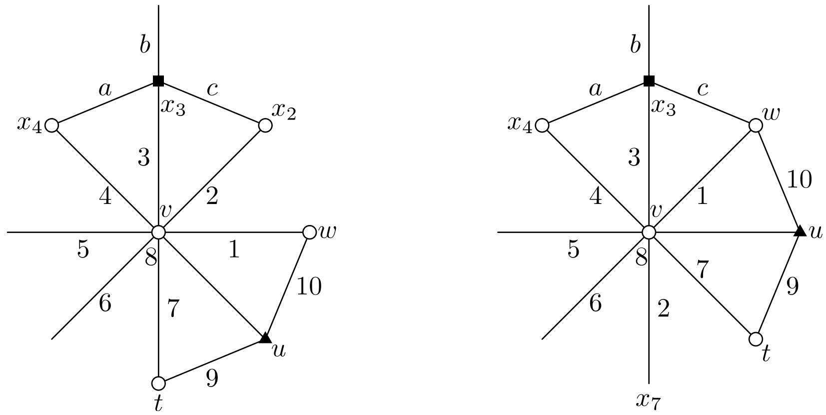

FIG. 7. *Reducible configurations of Lemma* 9. *We assume that the degree of* v *in* G *is* 8.

uncolored. In the second step, we show how to extend it to a 10-total-coloring of G.

Given a total-coloring and an element $x \in V \cup E$, recall that $\mathcal{C}(x)$ is the set of colors of all the elements of $V \cup E$ incident or adjacent to x. Recall also that if $x \in V$, then $\mathcal{E}(x)$ is the set of colors of all the edges incident to x.

Let λ be a total-coloring of $G - vx_6$, in which, furthermore, we uncolor the vertex x_6. Our goal is to properly color the edge vx_6. Note that $|\mathcal{C}(vx_6)| = 10$; otherwise, the edge vx_6 can be greedily colored. Without loss of generality, we may assume that the coloring is the one shown in Figure 8(a).

We want to color vx_6 with $\lambda(vw)$. Recall that w is either x_2 or x_3. We set $\mathcal{E} := \mathcal{E}(w) \cup \{\lambda(w)\}$. If there exists a color $\alpha \in \{7, 8, 9, 10\} \setminus \mathcal{E}$, then we set $\lambda(vx_6) := \lambda(vw)$ and $\lambda(vw) := \alpha$. Furthermore, if $1 \notin \mathcal{E}$, then we interchange the colors of x_6x_1 and vx_1, color vx_6 with $\lambda(vw)$, and recolor vw with 1. Thus, $1 \in \mathcal{E}$. Similarly, we deduce that $5 \in \mathcal{E}$. Finally, note that either 2 or 3 belongs to $\mathcal{E}$, according to whether w is x_2 or x_3. Consequently, this shows that $|\mathcal{E}| \geq 7$. But w has degree 5, thus $|\mathcal{E}| = 6$, a contradiction. This concludes the first step.

Suppose now that we are given a partial 10-total-coloring of G in which only x_6 is not colored. If we cannot extend it to G, then, without loss of generality, we may assume that the coloring is the one shown in Figure 8(b). If there exists a color $\alpha \in \{2, 4, 6, 10\} \setminus \{a, b, c, d, e\}$, then recolor vx_6 with α and color x_6 with 7 to obtain a 10-total-coloring of G. Hence, $\{2, 4, 6, 10\} \subset \{a, b, c, d, e\}$. Suppose that $a \notin \{2, 4, 6\}$. In this case, $\{b, c, d, e\} = \{2, 4, 6, 10\}$, and thus $e \in \{2, 4, 10\}$. Interchange the colors of the edges x_6x_5 and vx_5. Now, if $a \neq 5$, then the obtained coloring is proper, and we extend it to G by coloring x_6 with 5. And, if $a = 5$, then we additionally interchange the colors of x_6x_1 and vx_1, and color v with 9. Consequently, we obtain $a \in \{2, 4, 6\}$.

If $9 \notin \{b, c, d, e\}$, then we can apply a similar recoloring. More precisely, we can interchange the colors of the edges x_6x_1 and vx_1. The obtained coloring is proper and can be extended to G by coloring x_6 with 9. So $9 \in \{b, c, d, e\}$, and hence $5 \notin \mathcal{E}(v)$. We interchange the colors of x_6x_5 and vx_5, and color x_6 with 5. Either the obtained 10-total-coloring of G is proper, or $e = 9$. In the latter case, we additionally interchange the colors of x_6x_1 and vx_1 to obtain the sought contradiction. ∎

LEMMA 11. *The configuration of Figure* 9(a) *is reducible.*

Proof. Consider a nice coloring of $G - vx_9$. Without loss of generality, it is the one of Figure 9(a). First, note that $a = 10$; otherwise, we can recolor the edge vx_8 with 10, and color vx_9 with 8. Next, we infer that $b = 7$; otherwise, we can interchange the

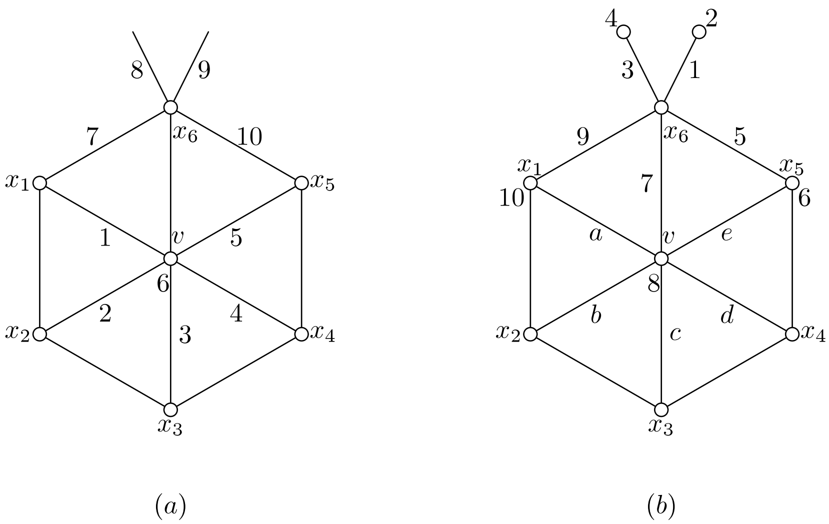

FIG. 8. *Proof of Lemma* 10*:* (a) *coloring of* $G - vx_6$*,* (b) *partial coloring of* G *in which* x_6 *is not colored.*

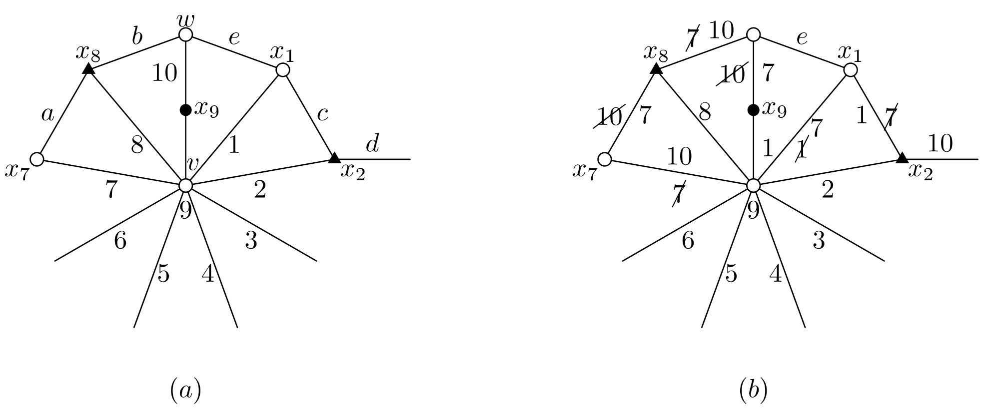

FIG. 9. *Precoloring and recoloring for the proof of Lemma* 11.

colors of x_8x_7 and vx_7, and color vx_9 with 7. Now, observe that $10 \in \{c, d\}$; otherwise, we recolor vx_2 with 10, and color vx_9 with 2. Furthermore, $7 \in \{c, d\}$; otherwise, we interchange the colors of x_8w and x_9w, and also of x_8x_7 and vx_7, recolor vx_2 with 7, and color vx_9 with 10. Thus, $\{c, d\} = \{7, 10\}$. If $d = 7$ and $c = 10$, then we just interchange the colors of the edges x_2x_1 and vx_1, and color vx_9 with 1. And, if $d = 10$ and $c = 7$, then the recoloring shown in Figure 9(b) is a nice coloring of G. ∎

3. Discharging part. Recall that $G = (V, E)$ is a minimum counter-example to the statement of Theorem 1, in the sense that $|V| + |E|$ is minimum. We obtain a contradiction by using the discharging method. Here is an overview of the proof. We fix a planar embedding of G. Each vertex and face of G is assigned an initial charge. The total sum of the charges is negative by Euler's formula. Then, some redistribution rules are applied, and vertices and faces send or receive some charge according to these rules. The total sum of the charges is not changed during this step, but at the end we infer that the charge of each vertex and face is nonnegative, a contradiction.

Initial charge. We assign a charge to each vertex and face. For every $x \in V \cup F$, we define the initial charge $\mathrm{ch}(x)$ to be $\deg(x) - 4$, where $\deg(x)$ is the degree of x in G. By Euler's formula the total sum is

$$\sum_{v \in V} \mathrm{ch}(v) + \sum_{f \in F} \mathrm{ch}(f) = -8\,.$$

Rules. We need the following definitions to state the discharging rules. A 2-vertex is *bad* if it is not incident to a ($\geq$5)-face. A triangle is *bad* if it contains a vertex of degree at most 4. Recall that a triangle with vertices x, y, and z, is a $(\deg(x), \deg(y), \deg(z))$-triangle.

*Rule R*0. A ($\geq$5)-face sends 1 to each incident 2-vertex.

*Rule R*1. A 5-vertex v sends $1/5$ to each incident triangle.

*Rule R*2. A 6-vertex sends $13/35$ to each incident $(5, 6, \geq 7)$-triangle, $1/3$ to each incident $(6, 6, 6)$-triangle, and $2/7$ to each incident $(6, \geq 6, \geq 7)$-triangle.

*Rule R*3. A 7-vertex sends $1/2$ to each incident bad triangle, $3/7$ to each incident nonbad $(\leq 7, \leq 7, 7)$-triangle, and $1/3$ to each incident nonbad triangle containing a ($\geq$8)-vertex.

*Rule R*4. A 8-vertex sends

(i) $1/3$ *to each adjacent* 3*-vertex,*

(ii) $1/2$ *to each incident bad triangle,*

(iii) $7/15$ *to each incident* $(5, \leq 7, 8)$*-triangle and each incident* $(6, 6, 8)$*-triangle,*

(iv) $2/5$ *to each incident* $(5, \geq 8, 8)$*-triangle, each incident* $(6, 7, 8)$*-triangle, and each incident* $(6, 8, 8)$*-triangle,*

(v) $1/3$ *to each incident* $(6, 8, 9)$*-triangle and each incident* $(\geq 7, \geq 7, 8)$*-triangle.*

*Rule R*5. A 9-vertex sends

(i) 1 to each adjacent bad 2-vertex and $1/2$ to each adjacent nonbad 2-vertex,

(ii) $1/3$ to each adjacent 3-vertex,

(iii) $1/2$ to each incident bad triangle and each incident $(5, \leq 7, 9)$-triangle,

(iv) $3/7$ to each incident $(6, 6, 9)$-triangle,

(v) $2/5$ to each incident $(5, \geq 8, 9)$-triangle and each incident $(6, \geq 7, 9)$-triangle,

(vi) $1/3$ to each incident $(\geq 7, \geq 7, 9)$-triangle.

In what follows, we prove that the final charge $\mathrm{ch}^*(x)$ of every $x \in V \cup F$ is non-negative. Hence, we obtain

$$-8 = \sum_{v \in V} \mathrm{ch}(v) + \sum_{f \in F} \mathrm{ch}(f) = \sum_{v \in V} \mathrm{ch}^*(v) + \sum_{f \in F} \mathrm{ch}^*(f) \geq 0\,,$$

a contradiction. This contradiction establishes the theorem.

Final charge of faces. Let f be a d-face. Our goal is to show that $\mathrm{ch}^*(f) \geq 0$. By Lemma 22 and 2, f is incident to at most $\lfloor \frac{d}{3} \rfloor$ vertices of degree 2. Therefore, if $d \geq 5$, then by Rule R0 we obtain $\mathrm{ch}^*(f) \geq d - 4 - \lfloor \frac{d}{3} \rfloor = \lceil \frac{2d}{3} \rceil - 4 \geq 0$. A 4-face neither sends nor receives any charge, so its charge stays 0.

Finally, let $f = xyz$ be a triangle with $\deg(x) \leq \deg(y) \leq \deg(z)$. The initial charge of f is -1, and we assert that its final charge $\mathrm{ch}^*(f)$ is at least 0. We consider several cases and subcases according to the degrees of x, y, and z.

$\deg(x) = 2$. Then both y and z have degree 9 by Lemma 22, and hence f receives $1/2$ from each of y and z by Rule R5(iii).

$\deg(x) = 3$. In this case, by Lemma 22 and 2, we infer that $\deg(y) \geq 8$ and $\deg(z) = 9$. Thus, f receives $\frac{1}{2} + \frac{1}{2} = 1$ by Rules R4(ii) and R5(iii).

$\deg(x) = 4$. Then, by Lemma 22 and 2, $\deg(y) \geq 7$ and $\deg(z) \geq 8$. Hence, by Rules R3, R4(ii), and R5(iii), f receives $\frac{1}{2} + \frac{1}{2} = 1$ from y and z.

$\deg(x) = 5$. According to Lemma 22, $\deg(y) \geq 6$ and by Lemma 33, $\deg(z) \geq 7$. By Rule R1, f receives $1/5$ from x, so we only need to show that it receives at least $4/5$ from y and z together. Consider the following subcases.

$\deg(z) = 7$. By Rule R3, z sends $3/7$ to f, and by Rules R2 and R3, y sends at least $13/35$. Thus, f receives at least $\frac{13}{35} + \frac{3}{7} = \frac{4}{5}$ from y and z, as needed.

$\deg(z) = 8$. If $\deg(y) \leq 7$, then z sends $7/15$ to f by Rule R4(iii), and y sends at least $1/3$ by Rules R2 and R3. And, if $\deg(y) = 8$, then both y and z send $2/5$ to f by Rule R4(iv). So, in both cases f receives $4/5$ from y and z together.

$\deg(z) = 9$. Suppose first that $\deg(y) \leq 7$. Then, by Rule R5(iii), z sends $1/2$ to f. Moreover, by Rules R2 and R3, y sends at least $1/3$ to f, which proves the assertion. Now, if $\deg(y) \geq 8$, then according to Rules R4(iv) and R5(v) f receives $2/5$ from each of y and z, as needed.

$\deg(x) = 6$. First, if $\deg(z) = 6$, then f receives $1/3$ from each of its vertices by Rule R2. So we assume that $\deg(z) \geq 7$. In this case, f receives $2/7$ from x by Rule R2. Hence, we only need to show that y and z send at least $5/7$ to f in total. We consider several cases, regarding the degree of z.

$\deg(z) = 7$. Then f receives $3/7$ from z by Rule R3, and at least $2/7$ from y by Rules R2 and R3, as desired.

$\deg(z) = 8$. If $\deg(y) = 6$, then z sends $7/15$ by Rule R4(iii) and y sends $2/7$ by Rule R2. And, if $\deg(y) \geq 7$, then y sends at least $1/3$ by Rules R3 and R4(iv), and z sends at least $2/5$ by Rule R4(iv).

$\deg(z) = 9$. If $\deg(y) = 6$, then f receives $2/7$ from y by Rule R2 and $3/7$ from z by Rule R5(iv). And, if $\deg(y) \geq 7$, then f receives at least $1/3$ from y by Rules R3, R4(v), and R5(v), and at least $2/5$ from z by Rule R5(v), which yields the result.

$\deg(x) \geq 7$. The assertion follows from Rules R3, R4(v), and R5(vi).

Final charge of vertices. Let v be an arbitrary vertex of G. We have $\deg(v) \geq 2$ by Lemma 22. For every positive integer d, we define v_d to be the number of d-neighbors of v, and f_d to be the number of d-faces incident to v. Let $x_1, x_2, \ldots, x_{\deg(v)}$ be the neighbors of v in clockwise order. We prove that the final charge of v is nonnegative. To do so, we consider several cases, regarding the degree of v.

If $\deg(v) = 2$, then its two neighbors are 9-vertices by Lemma 22. If v is bad, then it receives 1 from each of its two 9-neighbors by Rule R5(i), while otherwise it receives at least 1 from its incident faces by Rule R0, and $1/2$ from each of its two 9-neighbors by Rule R5(i). Thus, in both cases its final charge is at least 0.

If $\deg(v) = 3$, then all of its neighbors have degree at least 8, so by Rules R4(i) and R5(ii) it receives $1/3$ from each of its neighbors, setting its final charge to 0. If $\deg(v) = 4$, then it neither sends nor receives anything, so its charge stays 0. If v is a 5-vertex, then by Rule R1 it sends $1/5$ to each of its at most five incident triangles; therefore, its final charge is nonnegative.

Suppose now that v is a 6-vertex. All of its neighbors have degree at least 5 by Lemma 22. Note that if $f_3 \leq 5$, then, according to Rule R2, $\mathrm{ch}^*(v) \geq 2 - 5 \cdot \frac{13}{35} > 0$. So, we assume now that $f_3 = 6$; i.e., v is incident to 6 triangles. Thus, we infer from Lemma 10 that $v_5 \leq 1$. If $v_5 = 0$, then following Rule R2, the vertex v sends at most $6 \cdot \frac{1}{3} = 2$; therefore, its final charge is at least 0. And, if $v_5 = 1$, then let

x_1 be the unique 5-neighbor of v. By Lemma 33, it holds that $\deg(x_2) \geq 7$ and $\deg(x_6) \geq 7$. Consequently, vx_3x_2 and vx_5x_6 are two $(6,\geq 6,\geq 7)$-triangles. Thus, $\mathrm{ch}^*(v) \geq 2 - 2\cdot\frac{13}{35} - 2\cdot\frac{1}{3} - 2\cdot\frac{2}{7} = \frac{2}{105} > 0$ by Rule R2.

Suppose that v is a 7-vertex. If $f_3 \leq 6$, then $\mathrm{ch}^*(v) \geq 3 - 6\cdot\frac{1}{2} = 0$ by Rule R3. So, we assume now that $f_3 = 7$. We consider several cases, according to the number of 4-neighbors of v. Note that, by Lemma 22 and Lemma 33, the vertex v has at most two such neighbors; i.e., $v_4 \leq 2$.

$v_4 = 0$. According to Rule R3, we have $\mathrm{ch}^*(v) \geq 3 - 7\cdot\frac{3}{7} = 0$.

$v_4 = 1$. Let x_1 be this 4-neighbor. So, x_2 and x_7 both are 9-vertices by Lemma 22. According to Rule R3, the vertex v sends $1/3$ to each of vx_2x_3 and vx_5x_6. Furthermore, v is incident to exactly two bad triangles and sends at most $3/7$ to each nonbad triangle. Therefore, we obtain $\mathrm{ch}^*(v) \geq 3-2\cdot\frac{1}{2}-3\cdot\frac{3}{7}-2\cdot\frac{1}{3} = \frac{1}{21} > 0$.

$v_4 = 2$. Without loss of generality, we assume that x_1 has degree 4. According to Lemmas 22 and 33, the other 4-neighbor of v must be x_4 or x_5, say x_4 by symmetry. By Lemma 22, the vertices x_2, x_7, x_3, and x_5 all have degree 9. Note that x_6 has degree at least 5. Consequently, $\mathrm{ch}^*(v) = 3-4\cdot\frac{1}{2}-3\cdot\frac{1}{3} = 0$.

Suppose now that v is an 8-vertex. If $v_3 = 0$, then $\mathrm{ch}^*(v) \geq 4 - 8\cdot\frac{1}{2} = 0$ by Rule R4. Thus, we assume now that x_1 is a 3-vertex. Notice that, by Lemma 5, if a 3-neighbor of v is on a triangle, then $v_3 = 1$. Therefore, $v_3 + f_3 \leq 9$. If $f_3 \leq 6$, then we obtain $\mathrm{ch}^*(v) \geq 4-6\cdot\frac{1}{2}-3\cdot\frac{1}{3} = 0$. If $f_3 = 7$, then we infer from Lemma 5 that $v_3 \leq 1$, and so $\mathrm{ch}^*(v) \geq 4 - 7\cdot\frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$. Now we suppose that $f_3 = 8$, and thus $v_3 = 1$. According to Lemma 22, it holds that $\deg(x_2) = 9$ and $\deg(x_9) = 9$. Moreover, by Lemma 9, all the vertices but x_1 have degree at least 5. Thus, by Rule R4, we infer that $\mathrm{ch}^*(v) \geq 4 - \frac{1}{3} - 2\cdot\frac{1}{2} - 4\cdot\frac{7}{15} - 2\cdot\frac{2}{5} = 0$.

Finally, suppose that v is a 9-vertex. By Lemma 22, v is adjacent to at most one 2-vertex. We consider two cases.

Case 1. $v_2 = 0$. Suppose first that v is incident to a (≥ 4)-face; i.e., $f_3 \leq 8$. If $f_3 = 8$, then $v_3 \leq 3$ by Lemmas 4 and 6, and hence $\mathrm{ch}^*(v) \geq 5 - 8\cdot\frac{1}{2} - 3\cdot\frac{1}{3} = 0$. If $f_3 \leq 7$, then we assert that $f_3 + v_3 \leq 12$. Indeed, if $v_3 \geq 6$, then, as two 3-vertices are not adjacent, we infer that $f_3 \leq 2(9 - v_3)$, which yields the assertion. So if $f_3 \leq 6$, then we obtain $\mathrm{ch}^*(v) \geq 5 - 6\cdot\frac{1}{2} - 6\cdot\frac{1}{3} = 0$. If $f_3 = 7$, then we can see that $v_3 \leq 4$ by Lemmas 4 and 6. Consequently, $\mathrm{ch}^*(v) \geq 5 - 7\cdot\frac{1}{2} - 4\cdot\frac{1}{3} = \frac{1}{6} > 0$. Now assume that $f_3 = 9$. Then, $v_3 \leq 2$ according to Lemma 4. If $v_3 \leq 1$, then $\mathrm{ch}^*(v) \geq 5 - 9\cdot\frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$. Assume now that $v_3 = 2$. Without loss of generality, say that x_1 is a 3-vertex, thus both x_2 and x_9 are (≥ 8)-vertices. Note that by Lemma 6, both x_3 and x_8 are (≥ 4)-vertices. Therefore, up to symmetry, it suffices to consider the following two cases.

x_4 *is the second* 3*-neighbor.* Then $\deg(x_3) \geq 8$, so by Rule R5(vi) the vertex v sends $\frac{1}{3}$ to vx_2x_3. Hence, we infer that $\mathrm{ch}^*(v) \geq 5 - 2\cdot\frac{1}{3} - 8\cdot\frac{1}{2} - \frac{1}{3} = 0$.

x_5 *is the second* 3*-neighbor.* In this case, $\deg(x_4) \geq 8$ and $\deg(x_6) \geq 8$. Furthermore, $\deg(x_3) \geq 5$ by Lemma 7. Consequently, x_3x_2v and x_3x_4v are both $(\geq 5, \geq 8, 9)$-triangles. Hence, v sends at most $\frac{2}{5}$ to each of them by Rule R5(v). So, $\mathrm{ch}^*(v) \geq 5 - 2\cdot\frac{1}{3} - 7\cdot\frac{1}{2} - 2\cdot\frac{2}{5} = \frac{1}{30} > 0$.

Case 2. $v_2 = 1$. Let x_1 be the 2-neighbor. Observe that by Lemma 33, v cannot have a 3-neighbor on two triangles. Moreover, x_1 cannot be incident to two triangles, so $f_3 \leq 8$. We consider the following possibilities.

x_1 *is on a triangle.* Let this triangle be vx_1x_2. From Lemma 33 and 3, we infer that $f_3 + v_3 \leq 8$. So, $\mathrm{ch}^*(v) \geq 5 - 1 - 8\cdot\frac{1}{2} = 0$.

x_1 *is bad but not on a triangle.* In this case, x_1 is on two 4-faces; therefore, in particular, $f_3 \le 7$. Note that by Lemma 33, either one vertex among x_2, x_9 has degree at least 4, or $f_3 \le 5$. Besides, according to Lemma 33 there is no 3-neighbor on two triangles. Observe also that if both vx_2x_3 and vx_8x_9 are triangles, then Lemma 11 implies that $v_3 \le 6$. Let us consider several cases regarding the value of f_3.

$f_3 \le 4$. Then $f_3 + v_3 \le 10$; otherwise, we obtain a contradiction by Lemma 33 and 3. Thus, $\mathrm{ch}^*(v) \ge 5 - 1 - 4 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} = 0$.

$f_3 = 5$. Using Lemma 33 and 3, a small case-analysis shows that $v_3 \le 5$. Moreover, if $v_3 = 5$, then the obtained configuration is the one of Lemma 11, which is reducible. And, if $v_3 \le 4$, then we obtain $\mathrm{ch}^*(v) \ge 5 - 1 - 5 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0$.

$f_3 = 6$. In this case, $v_3 \le 3$ by Lemma 33 and 3. Thus, $\mathrm{ch}^*(v) \ge 5 - 1 - 6 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0$.

$f_3 = 7$. By Lemma 33 and 3, v has at most one 3-neighbor, namely x_2 or x_9. Thus, $\mathrm{ch}^*(v) \ge 5 - 1 - 7 \cdot \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$.

x_1 *is neither bad nor on a triangle.* Notice that $f_3 \le 7$. Let us again consider several cases regarding the value of f_3.

$f_3 \le 5$. In this case, $f_3 + v_3 \le 11$ by Lemma 33. So, $\mathrm{ch}^*(v) \ge 5 - \frac{1}{2} - 5 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} = 0$.

$f_3 = 6$. Similarly as before, we infer that $v_3 \le 4$, and hence $\mathrm{ch}^*(v) \ge 5 - \frac{1}{2} - 6 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0$.

$f_3 = 7$. By Lemma 33 the vertex v has at most two 3-neighbors, namely x_2 and x_9. Thus, $\mathrm{ch}^*(v) \ge 5 - \frac{1}{2} - 7 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} = \frac{1}{3} > 0$.

This establishes that the final charge of every vertex is nonnegative; therefore, the proof of Theorem 1 is now complete. ∎

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