Distance constrained labelings of planar graphs with no short cycles

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Abstract

Motivated by a conjecture of Wang and Lih, we show that every planar graph of girth at least 7 and maximum degree $\Delta \geq 190 + 2\lceil p/q \rceil$ has an $L(p,q)$-labeling of span at most $2p + q\Delta - 2$. Since the optimal span of an $L(p,1)$-labeling of an infinite Δ-regular tree is $2p + \Delta - 2$, the obtained bound is the best possible for any $p \geq 1$ and $q = 1$.

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1 Introduction

We study $L(p,q)$-labelings of planar graphs with no short cycles. An $L(p,q)$-*labeling* of a graph G is a labeling c of its vertices by non-negative integers such that the labels (colors) assigned to neighboring vertices differ by at least p and the labels of pairs of vertices at distance two differ by at least q. The least integer K such that there exists a proper $L(p,q)$-labeling of G by integers between 0 and K is called the *span* and denoted by $\lambda_{p,q}(G)$. Clearly, if $p = q = 1$, an $L(p,q)$-labeling of G is just a proper coloring of the square of G with numbers between 0 and K and $\chi(G^2) = \lambda_{1,1}(G) + 1$.

The chromatic number of the square of a graph G is between $\Delta+1$ and Δ^2+1 where Δ is the maximum degree of G. However, it is not hard to infer from Brooks' theorem that there are only finitely many connected graphs for which the upper bound is attained (each such graph must be Δ-regular graph of order $\Delta^2 + 1$ and diameter two and there are only finitely many such graphs [17]). On the other hand, the chromatic number of the square of a planar graph is bounded by a function linear in the maximum degree (note that this does not follow directly from the 5-degeneracy of planar graphs [16]). Wang and Lih [27] conjectured that there exists an integer Δ_0 such that if G is a planar graph with maximum degree $\Delta \geq \Delta_0$ and the girth of G is at least seven, then the chromatic number of G^2 is $\Delta + 1$. Borodin et al. [5] proved this conjecture (without being actually aware of the conjecture itself) and also showed that the analogous statement is not true for graphs of girth six by constructing planar graphs G of girth six and arbitrary maximum degree Δ with $\chi(G^2) = \Delta + 2$. On the other hand, the squares of planar graphs of girth six and sufficiently large maximum degree are $(\Delta + 2)$-colorable [8].

Wang and Lih [27] also asked whether similar results could hold for $L(2,1)$-labelings of graphs. In this paper, we show that this is indeed the case. In particular, we prove that if G is a planar graph of maximum degree $\Delta \geq 190 + 2p$, $p \geq 1$, and its girth is at least seven, then $\lambda_{p,1}(G) \leq 2p + \Delta - 2$. At the end of the paper, we show that our upper bound is tight for all pairs of Δ and p and discuss possible extensions of our results to $L(p,q)$-labelings for $q > 1$.

Before we start with the presentation of our results, let us briefly summarize the known results on $L(p,q)$-labelings of graphs. One of the most important open problems in the area is the conjecture of Griggs and Yeh [15] that $\lambda_{2,1}(G) \leq \Delta^2$ for every graph G with maximum degree $\Delta \geq 2$. This conjecture is widely open, verified only for few classes of graphs including graphs of maximum degree two, chordal graphs [25] (see also [6,22]), Hamiltonian cubic graphs [18,19], and planar graphs with maximum degree four and more [2]. For general graphs, the original bound $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$ from [15] was improved to $\lambda_{2,1}(G) \le \Delta^2 + \Delta$ in [7] and a recent more general result of Král' and Škrekovski [21] yields $\lambda_{2,1}(G) \le \Delta^2 + \Delta - 1$. The present record of $\Delta^2 + \Delta - 2$ was established by Gonçalves [13]. Optimal $L(p, q)$-labelings are also intensively studied for the class of planar graphs. The following bounds are known: $\lambda_{p,q}(G) \le (4q - 2)\Delta + 10p - 38q - 23$ due to van den Heuvel and McGuiness [16], $\lambda_{p,q}(G) \le (2q - 1)\lceil 9\Delta/5\rceil + 8p - 8q + 1$ if $\Delta \ge 47$ due to Borodin et al. [4], and $\lambda_{p,q}(G) \le q\lceil 5\Delta/3\rceil + 18p + 77q - 18$ due to Molloy and Salavatipour [24]. Bounds for planar graphs without short cycles were proven by Wang and Lih [27]:

- $\lambda_{p,q}(G) \le (2q - 1)\Delta + 4p + 4q - 4$ if G is a planar graph of girth at least seven,
- $\lambda_{p,q}(G) \le (2q - 1)\Delta + 6p + 12q - 9$ if G is a planar graph of girth at least six, and
- $\lambda_{p,q}(G) \le (2q - 1)\Delta + 6p + 24q - 15$ if G is a planar graph of girth at least five.

The algorithmic aspects of $L(p, q)$-labelings also attracted a lot of attention of researchers [1,3,10,11,20,23] because of potential applications in radio frequency assignment.

2 Preliminaries

In this section, we introduce notation used throughout the paper. All graphs considered in the paper are simple, i.e., without parallel edges and loops. A *d-vertex* is a vertex of degree exactly d. An $(\le d)$*-vertex* is a vertex of degree at most d. Similarly, an $(\ge d)$*-vertex* is a vertex of degree at least d. A *k-thread* is an induced path comprised of k 2-vertices.

An ℓ*-face* is a face of length ℓ (counting multiple incidences, i.e., bridges incident to the face are counted twice). If the boundary of a face f forms a connected subgraph, then the subgraph formed by the boundary (implicitly equipped with the orientation determined by the embedding) is called the *facial walk*. A face f is said to be *biconnected* if its boundary is formed by a single simple cycle. The neighbors of a vertex v on the facial walk are called f*-neighbors* of v. Note that if f is biconnected, then each vertex incident with f has exactly two f-neighbors.

Let us consider a biconnected face f, and let $v_1, \ldots, v_k$ be (≥ 3)-vertices incident to f listed in the order on the facial walk of f. The *type* of f is a k-tuple $(\ell_1, \ldots, \ell_k)$ if the part of the facial walk between v_i and v_{i+1} is an ℓ_i-thread. In particular, if v_i and v_{i+1} are f-neighbors, then ℓ_i is zero. Two face types are considered to be the same if they can be types of the same face, i.e., they differ only by a cyclic rotation and/or a reflection.

If the face f is biconnected and v is a vertex incident to f, then the neighbors of v that are not its neighbors on the facial walk are said to be *opposite* to the face f. Similarly, if both the faces f_1 and f_2 incident to an edge uv are biconnected, then the faces incident to v distinct from f_1 and f_2 are *opposite* to the vertex u (with respect to the vertex v).

Our main result is that the square of a planar graph of girth seven and sufficiently large maximum degree Δ is $(\Delta+1)$-colorable. In fact, we prove a more general on $L(p,q)$-labelings of such graphs. For an integer $D \geq 192$, a graph G is *D-good* if its maximum degree is at most D and it has an $L(p,1)$-labeling of span at most $D+2p-2$ for every $p \leq (D-190)/2$. A planar graph G of girth at least 7 and maximum degree at most D is said to be *D-minimal* if it is not D-good but every proper subgraph of G is D-good. Clearly, if G is D-minimal, then it is connected. A vertex of G is said to be *small* if its degree is at most 95, and *big* otherwise.

Our proof is based on the discharging method. We show that there is no D-minimal graph, i.e., all planar graphs of girth at least seven and maximum degree at most D are D-good. In order to show this, we first describe configurations that cannot appear in a D-minimal graph (reducible configurations). In the proof, we consider a potential D-minimal graph and assign each vertex and each face a certain amount of charge. The amounts are assigned in such a way that their sum is negative. The charge is then redistributed among the vertices and faces according to the rules described in Section 5. It is shown that if the considered graph is D-minimal, then the final charge of every vertex and every face is non-negative after the redistribution. Since the sum of the initial charges is negative, we obtain a contradiction and conclude that there is no D-minimal graph.

3 Structure of D-minimal graphs

In this section, we identify configurations that cannot appear in D-minimal graphs. The following argument is often used in our considerations: we first assume that there exists a D-minimal graph G that contains a certain configuration. We remove some vertices of G and find a proper $L(p,1)$-labeling of the new graph (the labeling exists because G is D-minimal). We then recolor some of the vertices: at this stage, we state the properties that the new colors of the recolored vertices should have, and recolor the vertices such that the properties are met (and show that it is possible). If the original colors of such vertices already have the desired properties, then the vertices just keep their original colors. Finally, the labeling is extended to the removed vertices.

We have already seen that every D-minimal graph is connected. Similarly, it is not hard to see that the minimum degree of a D-minimal graph is at least two:

Lemma 1 *If G is a D-minimal graph, then its minimum degree is at least two.*

PROOF. Assume that G contains a vertex v of degree one (since G is connected, it has no vertices of degree zero). Fix an integer $p \leq (D-190)/2$ such that G has no proper $L(p,1)$-labeling of span $D+2p-2$. Let v' be the neighbor of v in G. Remove v from G. Since G is D-minimal, the obtained graph has a proper $L(p,1)$-labeling c of span $D+2p-2$. We extend the labeling c to v: the vertex v cannot be assigned at most $2p-1$ colors whose difference from the color of v' is less than p and it cannot be assigned at most $D-1$ colors which are assigned to the other neighbors of v'. Therefore, there are at most $D+2p-2$ forbidden colors for v. In particular, there exists a color that can be assigned to v, and thus c can be extended to v. This contradicts our assumption that G is D-minimal.

Observe that Lemma 1 implies that every ℓ-face of a D-minimal graph G for $\ell \leq 13$ is biconnected because of the girth assumption and that the facial walk of every ℓ-face with $\ell \leq 11$ induces a chordless cycle of G.

Next, we focus on 2-, 3- and 4-threads contained in D-minimal graphs:

Lemma 2 *If vertices v and w of a D-minimal graph G are joined by a 2-thread, then at least one of the vertices v and w is big.*

PROOF. Fix an integer $p \leq (D-190)/2$ such that G has no proper $L(p,1)$-labeling with span $D+2p-2$. Let $v'w'$ be the 2-thread between v and w in G (where v' is the neighbor of v). Assume for the sake of contradiction that neither v nor w is big. Remove the vertices v' and w' from G. Since G is D-minimal, there exists a proper $L(p,1)$-labeling c of the obtained graph whose span does not exceed $D+2p-2$. We extend the labeling c to the vertices v' and w'.

Let A_v be the set of the colors that differ by at least p from the color of v and are different from the colors of all the neighbors of v and from the color of w. Similarly, let A_w be the set of the colors that differ by at least p from the color of w and are different from the colors of all the neighbors of w and from the color of v. Since w is not a big vertex, the number of these colors is at least $(D+2p-1)-(2p-1)-94-1 \geq 2p$, since $D-95 \geq 2p$. Similarly, we have $|A_v| \geq 2p$.

Color now the vertices v' and w' by colors from A_v and A_w that differ by at least p (observe that such colors always exist). The obtained labeling c is a proper $L(p, 1)$-labeling of G with span at most $D + 2p - 2$.

The following two statements readily follow:

Lemma 3 *No D-minimal graph G contains a 4-thread.*

PROOF. Assume that a D-minimal graph G contains a 4-thread $vv'v''v'''$. By Lemma 2, v or v''' is big and $vv'v''v'''$ is not a 4-thread.

Lemma 4 *If vertices v and w of a D-minimal graph G are joined by a 3-thread, then both v and w are big.*

PROOF. Let $v'v''v'''$ be the 3-thread joining v and w. By Lemma 2, v or v''' is big. Since v''' is a 2-vertex, v is big. Similarly, we infer that w is big.

Next, we focus on cycles of lengths seven and eight contained in D-minimal graphs. Note that the boundary of every 7-face and 8-face is biconnected (because of the girth assumption and Lemma 1), i.e., its boundary is a simple cycle of length seven or eight, and thus the following lemma can always be applied in such cases.

Lemma 5 *Let $v_1v_2v_3v_4v_5v_6v_7$ be a part of a 7-cycle or an 8-cycle contained in a D-minimal graph G. If v_2, v_3, v_5 and v_6 are 2-vertices, then v_1 or v_7 is a big vertex.*

PROOF. Fix an integer $p \leq (D - 190)/2$ such that G has no proper $L(p, 1)$-labeling with span $D + 2p - 2$. Note that the distance between the vertices v_1 and v_7 is at most two. Assume that neither v_1 nor v_7 is big. Remove the vertices v_2, v_3, v_5 and v_6 from G. Since G is D-minimal, the new graph has an $L(p, 1)$-labeling c of span at most $2p + D - 2$. Let A be the set of colors γ that differ from the color of v_4 by at least p and such that no neighbor of v_4 is colored with γ. Since there are $2p + D - 1$ colors available and the degree of v_4 in the new graph does not exceed $D - 2$, we infer that $|A| \geq 2$.

We extend the labeling c to the removed vertices. Color the vertices v_5 and v_3 by distinct colors from A in such a way that the colors of v_5 and v_7 are different, and the colors of v_3 and v_1 are also different. Since the colors of v_7 and v_1 are different (the distance of v_7 and v_1 in G is at most two), this is always possible.

Color now the vertex v_6 by a color that differs by at least p from the colors of v_5 and v_7 and that differ from the colors of v_4 and (at most 94) neighbors of v_7. Since there are at most $95 + 4p - 2 \leq 2p + D - 2$ forbidden colors for v_6, the vertex v_6 can be colored. Similarly, it is possible to color the vertex v_2. Since the obtained labeling is a proper $L(p, 1)$-labeling with span at most $2p + D - 2$, the graph G is not D-minimal.

The following result is an easy consequence of Lemma 5:

Lemma 6 *No D-minimal graph G contains a pair of vertices joined by two 3-threads.*

PROOF. Assume for the sake of contradiction that G contains two vertices v and w joined by two 3-threads. The vertices v, w and the two 3-threads joining them comprise an 8-cycle in G. By Lemma 5, at least one of the neighbors of w in the 3-threads is big, but both the neighbors are 2-vertices.

We now focus on 3-vertices in D-minimal graphs:

Lemma 7 *Let $v_1 v_2 v_3 v_4$ be a path of a D-minimal graph G where v_2 is a 3-vertex. If neither v_1 nor v_4 is big and v_3 is a 2-vertex, then the remaining neighbor w of v_2 is big.*

PROOF. Fix an integer $p \leq (D - 190)/2$ such that G has no $L(p, 1)$-labeling of span $2p + D - 2$. Assume that w is not big. Remove the vertex v_3 from G. Since G is D-minimal, there exists a proper $L(p, 1)$-labeling of the obtained graph with span at most $2p + D - 2$. We first change the color of v_2 and then we extend the labeling c to the vertex v_3.

Recolor the vertex v_2 by a color that differs from the colors of v_1 and w by at least p, and that is different from the colors of all the neighbors of v_1 and w and from the color of v_4. Since neither v_1 nor w is big, there are at most $2(2p - 1) + 2 \cdot 94 + 1 \leq 2p + D - 2$ forbidden colors for v_2. Hence, the vertex v_2 can be recolored.

Finally, color the vertex v_3 by a color that differs from the colors of v_2 and v_4 by at least p, and that is different from the colors of all the neighbors of v_2 and v_4. Since v_2 is a 3-vertex and v_4 is not big, there are at most $2(2p-1)+94+2 \leq 2p + D - 2$ forbidden colors and v_3 can be colored.

We finish this section by establishing a lemma on the structure of faces of type $(2, 1, 1)$:

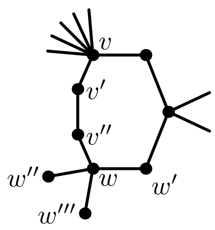

Fig. 1. Notation used in the proof of Lemma 8.

Lemma 8 *The following configuration does not appear in a D-minimal graph G: a 7-face f of type $(2, 1, 1)$ with one big and two 4-vertices such that both the 4-vertices of f are adjacent only to small vertices.*

PROOF. By Lemma 2, the big vertex incident to f delimits the 2-thread. Let v be the big vertex and w the other vertex delimiting the 2-thread and let $v'v''$ be the 2-thread (the 2-vertex v' is an f-neighbor of v). Let w', w'' and w''' be the neighbors of w different from v'' (see Figure 1) and assume that w' is an f-neighbor of w.

Fix an integer $p \leq (D - 190)/2$ such that G has no proper $L(p, 1)$-labeling with span $2p + D - 2$. Remove the vertices v'' and w' from G. Since G is D-minimal, there exists a proper $L(p, 1)$-labeling c of the new graph whose span is at most $2p + D - 2$. Next, we change the color of w and we extend the labeling c to the vertices v'' and w'.

Recolor the vertex w by a color that differs by at least p from the colors w'' and w''', and that is different from the colors of all the neighbors of w'' and w''' and that is also different from the color of v' and the other 4-vertex incident to f. Since none of the vertices w'' and w''' is big, the number of colors forbidden for w does not exceed $2(2p - 1) + 2 \cdot 94 + 2 \leq D + 2p - 2$. Hence, the vertex w can be recolored.

Next, color the vertex w' by a color that differs from the colors of both the 4-vertices incident with f by at least p and that is also different from the colors of all the six neighbors of the 4-vertices. Since the number of such forbidden colors does not exceed $2(2p-1)+6 \leq D+2p-2$, the vertex w' can be colored.

Finally, we color the vertex v'' by a color that differs from the colors of v' and w by at least p and that is different from the colors of the vertices v, w', w'' and w'''. Since there are at most $4p + 2 \leq D + 2p - 2$ forbidden colors, the labeling c can be also extended to the vertex v''.

4 Initial charge

We now describe the amounts of initial charge of vertices. The initial charge of a d-vertex v is set to

$$\mathrm{ch}(v) = d - 3,$$

and the initial charge of an ℓ-face f to

$$\mathrm{ch}(f) = \ell/2 - 3.$$

It is easy to verify that the sum of initial charges is negative:

Proposition 9 *If G is a connected planar graph, then the sum of all initial charges of the vertices and faces of G is -6.*

PROOF. Since G is connected, Euler's formula yields that $n + f = m + 2$ where n is the number of the vertices of G, m is the number of its edges and f is the number of its faces. The sum of initial charges of the vertices of G is equal to

$$\sum_{v \in V(G)} (d(v) - 3) = 2m - 3n.$$

The sum of initial charges of the faces of G is equal to

$$\sum_{f \in F(G)} \left(\frac{\ell(f)}{2} - 3 \right) = m - 3f.$$

Therefore, the sum of initial charges of all the vertices and faces is $3m - 3n - 3f = -6$.

Note that the amounts of initial charge were chosen such that each face of size at least 6 (consequently, each face of a D-minimal graph) has non-negative charge, the charge of 6-faces is zero and only 2-vertices have negative charge of -1 unit.

5 Discharging rules

Next, the charge is redistributed among the vertices and faces of a (potential) D-minimal graph by the following rules:

R1 Each face f sends a charge of $1/2$ to every incident 2-vertex.
R2 Each 4-vertex sends a charge of $1/4$ to every incident face.
R3 Each small (≥ 5)-vertex sends a charge of $5/16$ to every incident face.

R4 Each big vertex adjacent to a 3-vertex w sends a charge of $5/16$ to the opposite face through w.

R5 Each big vertex adjacent to a 4-vertex w sends a charge of $1/16$ to each of the two opposite faces through w.

R6 If v is a big vertex incident to a face f and v_1 and v_2 are its f-neighbors, then v sends the following charge to f:

$$\begin{array}{cl} 1/2 & \text{if } k = 0, \\ 3/4 & \text{if } k = 1, \\ 15/16 & \text{if } k = 2 \text{ and the type of } f \text{ is not } (3,2), \text{ and} \\ 1 & \text{if the type of } f \text{ is } (3,2), \end{array}$$

where k is the number of 2-vertices in set $\{v_1, v_2\}$.

If there are multiple incidences, the charge is sent according to the appropriate rule(s) several times, e.g., if a 2-vertex v is incident to a bridge, then it is incident to a single face f and f sends a charge of $1/2$ to v twice by Rule R1.

6 Final charge of vertices

In this section, we analyze the final amounts of charge of vertices.

Lemma 10 *If a graph G is D-minimal, then the final charge of every (≤ 4)-vertex is zero.*

PROOF. The initial charge of a 2-vertex v is -1 and it receives a charge of $1/2$ from each of the two incident faces by Rule R1. Therefore, its final charge is zero. Since a 3-vertex does not receive or send out any charge, its final charge is zero. Similarly, a 4-vertex sends a charge of $1/4$ to each of the four incident faces by Rule R2. Since its initial charge is 1, its final charge is also zero.

Lemma 11 *If a graph G is D-minimal, then the final charge of every small (≥ 5)-vertex is non-negative.*

PROOF. Consider a small vertex v of degree $d \geq 5$. The vertex v sends a charge of $5/16$ to each of the d incident faces by Rule R3. Hence, it sends out a charge of at most $5d/16$. Since the initial charge of v is $d - 3 \geq 5d/16$, the final charge of v is non-negative.

The analysis of final charge of big vertices needs finer arguments:

Lemma 12 *If a graph G is D-minimal, then the final charge of every big vertex is non-negative.*

PROOF. Let v be a big vertex of degree d. Let $v_1, \ldots, v_d$ be the neighbors of v in a cyclic order around the vertex v and let $f_1, \ldots, f_d$ be the faces incident to v in the order such that the f_i-neighbors of v are the vertices v_i and v_{i+1}. Note that some of the faces f_i can coincide. Let $\varphi(v_i)$ be the amount of charge sent from v through a vertex v_i. Similarly, $\varphi(f_i)$ is the amount of charge sent to f_i. Note that this is a slight abuse of our notation since the faces f_i are not necessarily mutually distinct—in such case, $\varphi(f_i)$ is the amount of charge sent from v because of this particular incidence to f_i.

We show that the following holds for every $i = 1, \ldots, d$ (indices are modulo d):

$$\frac{\varphi(v_i)}{2} + \varphi(f_i) + \varphi(v_{i+1}) + \varphi(f_{i+1}) + \frac{\varphi(v_{i+2})}{2} \leq \frac{31}{16}. \tag{1}$$

Summing (1) over all $i = 1, \ldots, d$ yields the following:

$$\sum_{i=1}^{d} (2\varphi(v_i) + 2\varphi(f_i)) \leq \left(2 - \frac{1}{16}\right) d. \tag{2}$$

Recall now that the initial charge of v is $d-3$. Because v is big, its degree d is at least 96. Since the charge sent out by v is at most $d - d/32$ by (2), the final charge of v is non-negative. Therefore, in order to establish the statement of the lemma, it is enough to show that the inequality (1) holds.

Let us fix an integer i between 1 and d. We distinguish several cases according to which of the vertices v_i, v_{i+1} and v_{i+2} are of degree 2:

None of the vertices v_i, v_{i+1} and v_{i+2} is a 2-vertex. The vertex v sends through each of the vertices v_i, v_{i+1} and v_{i+2} a charge at most $5/16$ by Rules R4 and R5, i.e., $\varphi(v_i), \varphi(v_{i+1}), \varphi(v_{i+2}) \leq 5/16$. By Rule R6, both the faces f_i and f_{i+1} receive charge of $1/2$ from v, i.e., $\varphi(f_i), \varphi(f_{i+1}) \leq 1/2$. Hence, the sum (1) of charges is at most $13/8 < 31/16$.

The vertex v_{i+1} is not a 2-vertex and one of v_i and v_{i+2} is a 2-vertex. By symmetry, we can assume that v_i is a 2-vertex and v_{i+2} is a (≥ 3)-vertex. Since v_i is a 2-vertex, v sends no charge through it, i.e., $\varphi(v_i) = 0$. By Rule R6, $\varphi(f_i) = 3/4$ and $\varphi(f_{i+1}) = 1/2$. By Rules R4 and R5, the amounts of charge sent from v through v_{i+1} and v_{i+2} do not exceed $5/16$, i.e., $\varphi(v_{i+1}), \varphi(v_{i+2}) \leq 5/16$. Therefore, the sum (1) is bounded by $3/4 + 1/2 + 3/2 \cdot 5/16 < 31/16$.

The vertex v_{i+1} is not a 2-vertex and both v_i and v_{i+2} are 2-vertices. The vertex v sends a charge of $3/4$ to both the faces f_i and f_{i+1} by Rule R6, i.e., $\varphi(f_i) = \varphi(f_{i+1}) = 3/4$. No charge is sent through the vertices v_i and v_{i+2}, i.e., $\varphi(v_i) = \varphi(v_{i+2}) = 0$. The amount of charge sent through v_{i+1} is at most $5/16$ (charge can be sent through it only by Rule R4 or Rule R5), i.e., $\varphi(v_{i+1}) \le 5/16$. We conclude that the sum (1) is at most $2 \cdot 3/4 + 5/16 < 31/16$.

The vertex v_{i+1} is a 2-vertex and neither v_i nor v_{i+2} is a 2-vertex. The vertex v sends a charge of $3/4$ to both the faces f_i and f_{i+1} by Rule R6, i.e., $\varphi(f_i) = \varphi(f_{i+1}) = 3/4$. The amount of charge sent through each of v_i or v_{i+2} is at most $5/16$ (charge can be sent through it only by Rule R4 or Rule R5), i.e., $\varphi(v_i), \varphi(v_{i+2}) \le 5/16$. Since no charge is sent through v_{i+1}, i.e., $\varphi(v_{i+1}) = 0$, the sum (1) is at most $2 \cdot 3/4 + 5/16 < 31/16$.

The vertex v_{i+1} is a 2-vertex and one of v_i and v_{i+2} is a 2-vertex. By symmetry, we can assume that v_i is a 2-vertex and v_{i+2} is a (≥ 3)-vertex. Since v_i and v_{i+1} are 2-vertices, v sends no charge through v_i or v_{i+1}, i.e., $\varphi(v_i) = \varphi(v_{i+1}) = 0$. By Rule R6, the face f_i receives a charge of at most 1 and the face f_{i+1} a charge of at most $3/4$, i.e., $\varphi(f_i) \le 1$ and $\varphi(f_{i+1}) \le 3/4$. Finally, the charge sent from v through v_{i+2} is at most $5/16$, i.e., $\varphi(v_{i+2}) \le 5/16$. We infer that the sum (1) is bounded by $1 + 3/4 + 5/32 < 31/16$.

All the vertices v_i, v_{i+1} and v_{i+2} are 2-vertices. There is no charge sent from v through any of the vertices v_i, v_{i+1} and v_{i+2}, i.e., $\varphi(v_i) = \varphi(v_{i+1}) = \varphi(v_{i+2}) = 0$. If at least one of the faces f_i and f_{i+1} is not a $(3, 2)$-face, then the total amount of charge sent to both of them by Rule R6 is at most $15/16 + 1 = 31/16$ as desired. In the rest, we consider the case when both the faces f_i and f_{i+1} are $(3, 2)$-faces. Let v' be the other big vertex incident to f_i and f_{i+1} (v' is big by Lemma 4). The vertex v_{i+1} lies in a 2-thread or a 3-thread shared by the faces f_i and f_{i+1}. If the faces f_i and f_{i+1} share a 2-thread, then the vertices v and v' are joined by two 3-threads—this is impossible by Lemma 6. On the other hand, if they share a 3-thread, then the vertices v and v' together with the two 2-threads form a 6-cycle contradicting the girth assumption.

7 Final charge of faces

In this section, we analyze the final amounts of charge of faces. First, we start with faces that are not biconnected. Recall that a maximal 2-connected subgraph of a graph is called a *block*. Blocks form a tree-like structure. The blocks that contain (at most) one vertex in common with other blocks are referred to as *end-blocks*.

Lemma 13 *Let f be a face of a D-minimal graph G. If f is not biconnected,* *then its final charge is non-negative.*

PROOF. Let P be the facial walk of f. Since f is not biconnected, P consists of two or more blocks. In particular, it contains at least one cut-vertex. Each end-block of P is a cycle by Lemma 1. In addition, observe that the end-blocks of P are cycles of length at least seven. Let C_1 and C_2 be two different end-blocks of P and w_1 and w_2 be their cut-vertices (note that w_1 may be equal to w_2), respectively.

Let k be the number of incidences of f with (≥ 3)-vertices, counting multiplicities. If $w_1 \neq w_2$, then each of w_1 and w_2 contributes at least two to k, thus w_1 and w_2 together contribute by at least 4 to k. Otherwise, the vertex $w_1 = w_2$ contributes at least two to k (it contributes two if P is comprised of two blocks).

Since the length of C_1 is at least seven, it has at least one (≥ 3)-vertex different from w_1 by Lemma 3. If C_1 contains exactly one such (≥ 3)-vertex, then it has a 3-thread (it cannot have a 4-thread by Lemma 3), and the vertex w_1 is big by Lemma 4. Similar statements hold for C_2. Therefore, there are at least two (≥ 3)-vertices incident with f that are distinct from w_1 and w_2. We conclude that $k \geq 4$. Moreover, if $w_1 \neq w_2$ or $w_1 = w_2$ is small, then $k \geq 6$. Note that in the latter case, there are at least four (≥ 3)-vertices incident with f that are distinct from $w_1 = w_2$.

If f is an ℓ-face, its initial charge is $\ell/2 - 3$. The face f sends out a charge of $(\ell - k)/2$ by Rule R1. If $k \geq 6$, then this is at most $\ell/2 - 3$ and thus the final charge of the face is non-negative.

If $k < 6$, then $w_1 = w_2$ is a big vertex (this follows from our previous discussion) and it has two incidences with f. Therefore f receives a charge of at least one unit from w_1 by Rule R6 and its final charge is $\ell/2 - 3 - (\ell - k)/2 + 1 \geq 0$.

Next, we analyze biconnected faces starting with 7-faces:

Lemma 14 *The final charge of each* 7*-face* f *of a* D*-minimal graph* G *is non-negative.*

PROOF. The initial charge of the face f is $1/2$. By Lemma 3, f does not contain a 4-thread, and thus the face f is incident to at least two (≥ 3)-vertices. We distinguish five cases according to the number of (≥ 3)-vertices incident to f:

The face f is incident to two (≥ 3)-vertices. In this case, the type of f is $(3,2)$. By Lemma 4, both the (≥ 3)-vertices are big and each of them sends a charge of 1 unit to f by Rule R6. Since f sends out a charge of $5/2$ to the five incident 2-vertices, its final charge is zero.

The face f is incident to three (≥ 3)-vertices. Since f sends a charge of two units to the incident 2-vertices, it is enough to show that it receives a charge of at least $3/2$ from the incident (≥ 3)-vertices. Since G does not contain a 4-thread by Lemma 3, the type of f is $(3,1,0)$, $(2,2,0)$ or $(2,1,1)$.

If f is incident to two big vertices, then each of them sends a charge of at least $3/4$ to f by Rule R6, and the final charge of f is non-negative. In the rest, we assume that f is incident to at most one big vertex. Consequently, the type of f is $(2,2,0)$ or $(2,1,1)$ by Lemma 4 and f is incident to exactly one big vertex by Lemma 2.

Assume that the type of f is $(2,2,0)$. By our assumption, f is incident to a single big vertex and, by Lemma 2, this vertex delimits both the 2-threads of f. However, Lemma 5 yields that one of the other two (≥ 3)-vertices is also big (contrary to our assumption).

The final case to consider is that the type of f is $(2,1,1)$. Let v be the big vertex incident to f. By Lemma 2, v delimits the 2-thread. Since both f-neighbors of v are 2-vertices, v sends a charge of $15/16$ to f. Let v' be any of the other two (≥ 3)-vertices incident to f. If v' is a 3-vertex, its neighbor opposite to f is big by Lemma 7 and it sends (through v') a charge of $5/16$ to f by Rule R4. If v' is a 4-vertex, it sends a charge of $1/4$ to f, and if v' has a big neighbor opposite to f, then the big neighbor sends f an additional charge of $1/16$ by Rule R5. Finally, if v' is a small (≥ 5)-vertex, it sends a charge of $5/16$ to f by Rule R3. We conclude that if f receives a total charge of less than $3/2$, then both the (≥ 3)-vertices incident to f are 4-vertices with no big neighbors. However, this is impossible by Lemma 8.

The face f is incident to four (≥ 3)-vertices. Since f is incident to three 2-vertices, it sends out a charge of $3/2$. We show that, on the other hand, it receives a charge of at least one unit from the incident (≥ 3)-vertices. This will imply that the final charge of f is non-negative. If f is incident to two big vertices, then it receives a charge of at least $1/2$ from each of them, i.e., a charge of at least one unit in total. Hence, we can assume in the rest that f is incident to at most one big vertex. In particular, by Lemma 4, f has no 3-thread. Therefore, the type of f is one of the following: $(2,1,0,0)$, $(2,0,1,0)$ or $(1,1,1,0)$.

Assume first that f is incident to no big vertex. By Lemma 2, the type of f is $(1,1,1,0)$. Let v be any of the four (≥ 3)-vertices incident to f. Note that v has an f-neighbor that is a 2-vertex. If v is a (≥ 4)-vertex, then f receives a charge of at least $1/4$ units from v by Rules R2 and R3. If v is a 3-vertex, then its neighbor opposite to f is big by Lemma 7 and it sends a charge of $5/16$ through v to f by Rule R4. Since the choice of v was arbitrary, the amount of charge sent from (or through) each incident (≥ 3)-vertex is at least $1/4$ and f receives a charge of at least 1 unit in

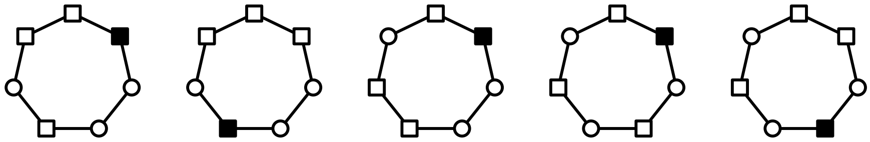

Fig. 2. All configurations (up to symmetry) of a 7-face of types $(2,1,0,0)$, $(2,0,1,0)$ and $(1,1,1,0)$ when the face is incident to a single big vertex. The big vertices are represented by full squares, the small (≥ 3)-vertices by empty squares and the 2-vertices by circles. Note that a 2-thread must be bounded by at least one big vertex by Lemma 2.

total.

We now consider the case that exactly one vertex incident to f is big. We say that a vertex x incident to f has Property S if the following conditions are satisfied:
(1) x is small,
(2) both f-neighbors of x are small, and
(3) one of the f-neighbors of x is a 2-vertex with no big f-neighbor.
It is routine to check that the following claim holds (consult Figure 2): unless the type of f is $(2,1,0,0)$ and the big vertex delimits both the 2-thread and the 1-thread of f, the face f is incident to two different (≥ 3)-vertices w_1 and w_2 that have Property S.

Under the assumption that the type of f is not $(2,1,0,0)$, we show that the face f receives a charge of at least $1/4$ from (or through) each of w_1 and w_2: if w_i is a (≥ 4)-vertex, then f receives a charge of at least $1/4$ from it. Otherwise, w_i is a 3-vertex and, by Lemma 7, its neighbor opposite to f is big. Consequently, it sends through w_i a charge of $5/16$ to f. Since f receives in addition the charge of at least $1/2$ from the big vertex, its final charge is non-negative as desired.

It remains to consider the case when the type of f is $(2,1,0,0)$ and the big vertex delimits both the 2-thread and the 1-thread of f. In this case, f receives a charge of $15/16$ from the incident big vertex by Rule R6. Moreover, there exists a vertex w that has Property S (consult Figure 2). Similarly as in the previous paragraph, the charge sent from w to f is at least $1/4$. Altogether, f receives a charge of at least 1 and the final charge is thus non-negative.

The face f is incident to five (≥ 3)-vertices. The face f sends a charge of 1 unit to the two incident 2-vertices. Thus it is enough to show that the face f receives a charge of at least $1/2$ from incident vertices. If f is incident to a big vertex, then f receives a charge of at least $1/2$ from it by Rule R6. We assume in the rest that f is only incident to small vertices. In particular, f has no 2-thread (by Lemma 2).

Let v be a 2-vertex incident to f and let v^- and v^+ be the two f-neighbors of v. Note that both v^- and v^+ are (≥ 3)-vertices. If v^- is a (≥ 4)-vertex, it sends a charge of at least $1/4$ to f. If v^- is a 3-vertex, then its neighbor opposite to f is big by Lemma 7, and it sends a charge of $5/16$ through v^-

to f. Similarly, f receives a charge of at least $1/4$ from (or through) v^+. Hence, the total charge received by f from the vertices v^- and v^+ is at least $1/2$ and the final charge of f is non-negative.

The face f is incident to six or seven (≥ 3)-vertices. Since the face f is incident to at most one 2-vertex, it sends out a charge of at most $1/2$ and its final charge is non-negative.

Next, we analyze the final charge of 8-faces.

Lemma 15 *The final charge of each biconnected 8-face f of a D-minimal graph G is non-negative.*

PROOF. First note that the initial charge of the face f is one. By Lemma 3, the face f does not contain a 4-thread. Therefore, the face f is incident to at least two (≥ 3)-vertices. We distinguish five cases based on the number of (≥ 3)-vertices incident to the face f:

The face f is incident to two (≥ 3)-vertices. Since f does not contain a 4-thread, the type of f is $(3,3)$. However, this is impossible by Lemma 6.

The face f is incident to three (≥ 3)-vertices. Since f sends a charge of $5/2$ to the incident 2-vertices, it is enough to show that it receives a charge of at least $3/2$ from the incident (≥ 3)-vertices. Since f does not contain a 4-thread, the type of f is $(3,2,0)$, $(3,1,1)$ or $(2,2,1)$.

If the type of f is $(3,2,0)$ or $(3,1,1)$, then the 3-thread is delimited by two big vertices (by Lemma 4) and f receives from each of them a charge of at least $3/4$ by Rule R6. Hence, the final charge of f is non-negative.

Assume that the type of f is $(2,2,1)$. It is enough to show that f is incident to at least two big vertices because each of them would send a charge of $3/4$ to f by Rule R6. If this is not the case, then f is incident to exactly one big vertex that is common to the two 2-threads by Lemma 2. However, by Lemma 5, at least one of the other two (≥ 3)-vertices is also big. We conclude that f is incident to at least two big vertices.

The face f is incident to four (≥ 3)-vertices. Since f is incident to four 2-vertices, f sends out a charge of two units. We claim that it also receives a charge of at least one unit from the incident vertices. This will imply that the final charge of f is non-negative. If f is incident to two big vertices, then it receives a charge of at least $1/2$ from each of them and the claim holds. We assume in the rest that f is incident to at most one big vertex. In particular, by Lemma 4, f does not have a 3-thread.

Assume that f contains a 2-thread. Let v and v' be the vertices delimiting the 2-thread. By Lemma 2, v or v' is big, say v. Since v is incident to a 2-vertex, it sends a charge of at least $3/4$ to f by Rule R6. If v' is a (≥ 4)-vertex, then f receives a charge of at least $1/4$ from v' and the final charge of f is non-negative. Otherwise, v' is a 3-vertex incident to a 2-thread and its f-neighbor not contained in the 2-thread is a small vertex. By Lemma 7, the neighbor of v' opposite to f is a big vertex. Hence, the face f receives a charge of $5/16$ from the big neighbor of v' and thus its final charge is non-negative.

In the rest, we assume that f has neither a 3-thread nor a 2-thread. Consequently, the type of f must be $(1,1,1,1)$. Let v_1, v_2, v_3 and v_4 be the (≥ 3)-vertices incident to f in the order as they appear on the facial walk of f. We have already established that f is incident with at most one big vertex. First assume that f is incident to a single big vertex, say v_1. Note that f receives a charge of $15/16$ from v_1 by Rule R6. If v_3 is a (≥ 4)-vertex, it sends a charge of $1/4$ to f and the final charge of f is non-negative. If v_3 is a 3-vertex, then its neighbor opposite to f is big (by Lemma 7) and sends a charge of $5/16$ to f, and thus the final charge of f is non-negative.

It remains to consider the case when the type of f is $(1,1,1,1)$ and f is not incident to a big vertex. Let us consider a vertex v_1. If v_1 is (≥ 4)-vertex, it sends a charge of at least $1/4$ to f. If v_1 is 3-vertex, then its neighbor opposite to f is big, and it sends a charge of $5/16$ to f through v_1. Similarly, we can infer that f receives a charge of at least $1/4$ from (or through) the vertices v_2, v_3 and v_4. Hence, f receives a charge of at least one unit from the incident vertices and its final charge is non-negative.

The face f is incident to five (≥ 3)-vertices. The face f sends a charge of $3/2$ units to the incident 2-vertices. Thus it is enough to show that the face f receives a charge of at least $1/2$ from incident (≥ 3)-vertices. If f is incident to a big vertex, then f receives a charge of at least $1/2$ from it and the final charge is non-negative. We assume in the rest that f is only incident to small vertices.

Let v be a 2-vertex incident to f. Since f is incident to no big vertex, both the neighbors v^- and v^+ of v are (≥ 3)-vertices by Lemma 2. If v^- is a (≥ 4)-vertex, it sends a charge of at least $1/4$ to f. And if v^- is a 3-vertex, then its neighbor opposite to f is big by Lemma 7 and it sends through v^- to f a charge of $5/16$. Similarly, f receives a charge of at least $1/4$ from (or through) v^+. Hence, f receives a charge of at least $1/2$ in total from the two neighbors of v and the final charge of f is non-negative.

The face f is incident to six or more (≥ 3)-vertices. Since the face f is incident to at most two 2-vertices, it sends out a charge of at most one unit and the final charge of f is non-negative.

Finally, we analyze the case of (≥ 9)-faces:

Lemma 16 *The final charge of each biconnected (≥ 9)-face f of a D-minimal graph is non-negative.*

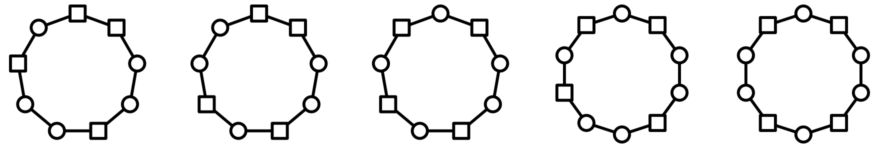

Fig. 3. Possible types of a 9-face or a 10-face with no 3-thread and at most two 2-threads. The ($\geq$ 3)-vertices are represented by squares and the 2-vertices by circles.

PROOF. Since f does not contain a 4-thread by Lemma 3, the face f is incident to at least three ($\geq$ 3)-vertices. The initial charge of f is $\ell/2 - 3$ where ℓ is the length of f. We distinguish four cases according to the number of ($\geq$ 3)-vertices incident to f:

The face f is incident to three ($\geq$ 3)-vertices. The face f sends out a charge of $(\ell - 3)/2$ to the incident 2-vertices. It is enough to show that f receives a charge of at least 3/2 from the incident vertices. If f has a 3-thread, then the 3-thread is delimited by two big vertices. Both of them send a charge of at least 3/4 to f by Rule R6. Therefore, if the total charge received by f is less than 3/2, then f has no 3-thread. Consequently, the length of f is nine and its type is $(2, 2, 2)$. By Lemma 2, at least two of the ($\geq$ 3)-vertices are big and f receives a charge of at least 3/2 from them by Rule R6 in this case.

The face f is incident to four ($\geq$ 3)-vertices. The face f sends a charge of $(\ell - 4)/2$ to the incident 2-vertices. It is enough to show that f receives a charge of at least 1 from the incident vertices. If f has a 3-thread, then the 3-thread is delimited by two big vertices (by Lemma 4) and each of them sends a charge of at least 1/2 to f by Rule R6. If f has at least three 2-threads, then these threads are delimited by at least two different big vertices by Lemma 2, and f receives a charge of at least 1/2 from each of them by Rule R6. If none of the above cases holds, i.e., f has no 3-thread and at most two 2-threads, then its type must be one of the following: $(2, 2, 1, 0)$, $(2, 1, 2, 0)$, $(2, 1, 1, 1)$, $(2, 2, 1, 1)$, and $(2, 1, 2, 1)$—see Figure 3.

Assume that the type of f is one of those five types. Since f has a 2-thread, it must be incident to a big vertex v by Lemma 2. Let v', v'' and v''' be the remaining ($\geq$ 3)-vertices incident to f. The face f receives a charge of at least 1/2 from the vertex v by Rule R6. If at least one of v', v'' and v''' is big, then it sends an additional charge of at least 1/2 to f by Rule R6, and the total amount of charge received by f is at least one. Let us assume in the rest that all the vertices v', v'' and v''' are small.

Observe that in this case the type of f is $(2, 2, 1, 0)$, $(2, 1, 1, 1)$ or $(2, 2, 1, 1)$. If v' is a ($\geq$ 4)-vertex, f receives a charge of at least 1/4 from v' by Rule R2 or Rule R3. If v' is a 3-vertex, its neighbor opposite to f is big by Lemma 7 and it sends through v' to f a charge 5/16 by Rule R4. Similarly, f receives a charge of at least 1/4 from v'' and v'''. We conclude that the total charge received by f is at least one.

The face f is incident to five (≥ 3)-vertices. The face f sends a charge of $(\ell - 5)/2$ to the incident 2-vertices. It is enough to show that f receives a charge of at least 1/2 from the incident vertices. If f is incident to a big vertex, then it receives a charge of at least 1/2 by Rule R6 from this vertex. Assume in the rest that f is only incident to small vertices. In particular, the length of every 2-thread of f is one by Lemma 2. Let v be a 2-vertex incident to f and v^- and v^+ the f-neighbors of v. Note that both v^- and v^+ are (≥ 3)-vertices. If v^- is a (≥ 4)-vertex, then f receives a charge of at least 1/4 from v^- by Rule R2 or Rule R3. If v^- is a 3-vertex, then its neighbor opposite to v is big by Lemma 7 and the face f receives a charge of 5/16 from it through v. Similarly, f receives a charge of at least 1/4 from (or through) v^+. Altogether, f receives a charge of at least 1/2 as required.

The face f is incident to six or more (≥ 3)-vertices. The face f sends out a charge of at most $(\ell - 6)/2$ by Rule R1. Since the initial charge of f is $\ell/2 - 3$ and $\ell \geq 9$, the final charge is non-negative.

8 Final step

We now combine our observations from the previous sections together:

Theorem 17 *If G is a planar graph of maximum degree $\Delta \geq 190+2p$, $p \geq 1$, and the girth of G is at least seven, then G has a proper $L(p,1)$-labeling with span $2p + \Delta - 2$.*

PROOF. Consider a possible counterexample G and set $D = \Delta$. Since G is not D-good, there exists a D-minimal graph G'. Assign charge to the vertices and faces of G' as described in Section 4. Apply the rules given in Section 5 to G'. By Proposition 9, the sum of the amounts of initial charge assigned to the vertices and edges of G' is -6. On the other hand, the final amounts of charge of every vertex (Lemmas 10–12) and every face (Lemmas 13–16) are non-negative. However, this is impossible since the total amount of charge is preserved by the rules.

We use an argument applied in [24] to derive the following result for $L(p,q)$-labelings:

Corollary 18 *If G is a planar graph of maximum degree $\Delta \geq 190 + 2\lceil p/q \rceil$, $p, q \geq 1$, and girth at least seven, then G has a proper $L(p,q)$-labeling with span $2p + q\Delta - 2$.*

PROOF. Let $p' = \lceil p/q \rceil$. By Theorem 17, the graph G has a proper $L(p', 1)$-labeling c' with span $2p' + \Delta - 2$. Define a labeling c by setting $c(v) = qc'(v)$ for each vertex v. The labeling c is a proper $L(p'q, q)$-labeling. Therefore, it is also a proper $L(p, q)$-labeling of G. The span of c is at most the following:

$$q(2p' + \Delta - 2) = 2\left(p' - \frac{q-1}{q}\right)q + q\Delta - 2 \leq 2p + q\Delta - 2.$$

9 Conclusion

One may ask whether the bound proven in Theorem 17 cannot be further improved, e.g., to $2p + \Delta - 3$. However, the bound is tight for all considered pairs of Δ and p as shown in the following proposition (though the next proposition follows from results of [12], see Proposition 20, we include its short proof for the sake of completeness):

Proposition 19 *Let p and $\Delta \geq 2p$ be arbitrary integers. There exists a tree T with maximum degree Δ such that the span of an optimal $L(p, 1)$-labeling of T is $2p + \Delta - 2$.*

PROOF. It can be easily proven by induction on the order of a tree that the span of an optimal labeling of any tree with maximum degree Δ is at most $2p + \Delta - 2$. Therefore, it is enough to construct a tree with no $L(p, 1)$-labeling with span less than $2p + \Delta - 2$. Let us consider the following tree T: a vertex v_0 is adjacent to Δ vertices $v_1, \ldots, v_\Delta$ and each of the vertices $v_1, \ldots, v_\Delta$ is adjacent to $\Delta - 1$ leaves. Clearly, the maximum degree of T is Δ.

Assume that T has a proper $L(p, 1)$-labeling c of span at most $2p + \Delta - 3$. Since $\Delta \geq 2p$, the color of at least one of the vertices $v_0, \ldots, v_\Delta$ is between $p - 1$ and $p + \Delta - 2$, i.e., $c(v_i) \in \{p - 1, \ldots, p + \Delta - 2\}$ for some i. The color of each neighbor of v_i is either at most $c(v_i) - p$ or at least $c(v_i) + p$. Since there are only $\Delta - 1$ such colors, two of the neighbors of v_i have the same color and the labeling c is not proper.

One may also ask whether the condition $\Delta \geq 190 + 2p$ in Theorem 17 cannot be further weakened. The answer is positive (we strongly believe that the bound for $p = 2$ can be lowered to approximately 50) but we decided not to try to refine the discharging phase and the analysis in order to avoid adding more pages to the paper. It is also natural to consider $L(p, q)$-labelings of planar graphs with no short cycles for $q > 2$. In such case, the following result of Georges and Mauro [12, Theorems 3.2–3.5] comes to use:

Proposition 20 *Let p and q, $p \geq q$, be two positive integers. There exists a Δ_0 such that the span of an optimal $L(p,q)$-labeling of the infinite Δ-regular tree T_Δ, $\Delta \geq \Delta_0$ (Δ_0 depends on p and q), is the following:*

$$\lambda_{p,q}(T_\Delta) = \begin{cases} q\Delta + 2p - 2q & \text{if } p/q \text{ is an integer, i.e., } q|p, \\ q\Delta + p & \text{if } 1 < \frac{p}{q} \leq \frac{3}{2}, \\ q\Delta + \left\lfloor \frac{p}{q} \right\rfloor q + p - q & \text{if } 2 \leq \left\lfloor \frac{p}{q} \right\rfloor < \frac{p}{q} \leq \left\lfloor \frac{p}{q} \right\rfloor + \frac{1}{2}, \\ q\Delta + 2\left\lfloor \frac{p}{q} \right\rfloor q & \text{otherwise.} \end{cases}$$

Proposition 20 provides lower bounds on optimum spans of $L(p,q)$-labelings of planar graphs with large girth as every infinite tree T_Δ contains a finite subtree T with $\lambda_{p,q}(T) = \lambda_{p,q}(T_\Delta)$. The lower bounds can be complemented by the following (rather straightforward) upper bound which is tight if $q = 1$:

Proposition 21 *Let p and q, $p \geq q$, be two positive integers. There exists an integer Δ_0, which depends on p and q, such that every planar graph G of maximum degree Δ and of girth at least 18 has an $L(p,q)$-labeling of span at most $qD + 2p + q - 3$ where $D = \max\{\Delta_0, \Delta\}$.*

PROOF. Fix p, q and Δ and let $\Lambda = qD+2p+q-3$. We prove the proposition for $\Delta_0 = (2p-1)/q + 3$. Let G be a planar graph of the smallest order such that the maximum degree of G is at most Δ, G contains no cycle of length less than 18 and $\lambda_{p,q}(G) > \Lambda$. Clearly, G is connected. We partition the vertices of G into three classes and refer the vertices in the classes as to red, green and blue vertices: the vertices of degree one will be red, the vertices adjacent to at most two vertices that are not red will be green and the remaining vertices will be blue.

Assume first that there is a red vertex adjacent to a green vertex. Let v be that green vertex and W all red vertices adjacent to v. By the choice of G, $G \setminus W$ has an $L(p,q)$-labeling of span at most Λ. Since v is green, it is adjacent to at most two vertices that are green or blue. We consider the case that v is adjacent to two such vertices, say v_1 and v_2, and leave the other cases to be verified by the reader since our arguments readily translate to those cases. Note that the vertices v, v_1 and v_2 are the only vertices at distance at most from the vertices of W in G.

Our aim now is to find $\Delta - 2$ numbers $a_1, \ldots, a_{\Delta-2}$ such that the difference between any two numbers a_i and a_j, $i \neq j$, is at least q, the difference between any number a_i and the label of v is at least p and the difference between a_i and the label of v_1 or v_2 is at least q. The numbers are constructed inductively as follows. Set $a_1 = 0$ and $i = 1$ and apply the following three rules:

Rule 1 If the difference between a_i and the label of v is smaller than p, increase a_i by $2p-1$.

Rule 2 If the difference between a_i and the label of v_1 or v_2 is smaller than q, increase a_i by $2q-1$.

Rule 3 Suppose that neither Rule 1 nor Rule 2 applies. If $i = \Delta - 2$, stop. Otherwise, set $a_{i+1} = a_i + q$ and increase i by one.

Observe that Rule 1 can apply at most once and Rule 2 at most twice during the entire process. This yields that the value of $a_{\Delta-2}$ does not exceed $(\Delta - 3)q + 2p - 1 + 4q - 2 \leq \Lambda$. Hence, the labeling of $G \setminus W$ can be extended to G by assigning the vertices of W the labels $a_1, \ldots, a_{\Delta-2}$ to an $L(p,q)$-labeling of span Λ which contradicts our choice of G. We conclude that green vertices are adjacent to green and blue vertices only. In particular, every green vertex has degree two.

Let G' be the subgraph of G induced by all green and blue vertices. Observe that the degree of each green vertex in G' is two and the degree of each blue vertex is at least three. Hence, the minimum degree of G' is three. As the girth of G' is at least 18, G' contains a 3-thread comprised of (green) vertices v_1, v_2 and v_3. Since green vertices are adjacent to green and blue vertices in G only, the vertices v_1, v_2 and v_3 also form a 3-thread in G.

By the choice of G, the graph $G \setminus v_2$ has an $L(p,q)$-labeling of span at most Λ. We aim to extend the labeling of $G \setminus v_2$ to v_2. Let us count the number of labels that cannot be assigned to v_2. There are at most $2p-1$ labels that cannot be assigned to v_2 because of the label assigned to v_1 and there are at most $2q-1$ additional labels that cannot be assigned to v_2 because of the label assigned to the neighbor of v_1 different from v_2. Similarly, there are at most $2p+2q-2$ labels that cannot be assigned to v_2 because of the labels of v_3 and the other neighbor of v_3. In total, there at most $4p+4q-4$ labels that cannot be assigned to v_2. Since there are at least $\Lambda+1 = qD+2p+q-3+1 \geq q\Delta_0+2p+q-2 = (2p-1)+3q+2p+q-2 = 4p+4q-3$ labels, the labeling can be extended to v_2 contradicting our choice of G. The proof of the proposition is now finished.

Note that Proposition 21 can be generalized to minor-closed classes of graphs (with the bound on the girth depending on the considered class of graphs). However, we think that the assumption on the girth in the proposition is not optimal and can be weakened to seven:

Conjecture 22 *Let p and q, $p \geq q$, be two positive integers. There exists an integer Δ_0, which depends on p and q, such that every planar graph G of maximum degree Δ and of girth at least seven has an $L(p,q)$-labeling of span at most $qD + 2p + q - 3$ where $D = \max\{\Delta_0, \Delta\}$.*

The lower and upper bounds given in Propositions 20 and 21 do match for $q = 1$ but they differ for $q \neq 1$. We leave as an open problem to determine the optimal values of spans of $L(p, q)$-labelings of planar graph with large maximum degree and no short cycles for $q \neq 1$.

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