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Coloring squares of planar graphs with girth six

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Abstract

Wang and Lih conjectured that for every $g \geq 5$, there exists a number $M(g)$ such that the square of a planar graph G of girth at least g and maximum degree $\Delta \geq M(g)$ is $(\Delta+1)$-colorable. The conjecture is known to be true for $g \geq 7$ but false for $g \in \{5, 6\}$. We show that the conjecture for $g = 6$ is off by just one, i.e., the square of a planar graph G of girth at least six and sufficiently large maximum degree is $(\Delta + 2)$-colorable.

1. Introduction

We study colorings of squares of planar graphs with no short cycles. The *square* G^2 of a graph G is the graph with the same vertex set in which two vertices are joined by an edge if their distance in G is at most two. The chromatic number of the square of a graph G is between $\Delta + 1$ and $\Delta^2 + 1$ where Δ is the maximum degree of G. However, it is not hard to infer from Brooks' theorem that there are only finitely many connected graphs for which the upper bound is attained. On the other hand, the chromatic number of the square of a planar graph is bounded by a function linear in the maximum degree (note that this does not

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follow directly from the 5-degeneracy of planar graphs [9]). A natural question is when the chromatic number of the square of a planar graph is the lowest possible, i.e., it is equal to $\Delta + 1$. Wang and Lih [19] conjectured that this is the case for planar graphs with sufficiently large maximum degree that have girth five or more. Borodin et al. [5] proved their conjecture for planar graphs of girth seven and more (without being actually aware of the conjecture) and showed that it is not true for planar graphs of girth five and six. In this paper, we show that the conjectured bound is off just by one for graphs with girth six, i.e., the chromatic number of the square of a planar graph with girth six and sufficiently large maximum degree is at most $\Delta + 2$.

Let us now briefly survey the rich history of coloring of the squares of planar graphs. Wegner [20] proved that the squares of cubic planar graphs are 8-colorable. He conjectured that his bound can be improved:

CONJECTURE 1.1 (WEGNER 1977): *Let G be a planar graph with maximum degree Δ. The chromatic number of G^2 is at most 7, if $\Delta = 3$, at most $\Delta + 5$, if $4 \le \Delta \le 7$, and $\lfloor \frac{3\Delta}{2} \rfloor + 1$, otherwise.*

If Conjecture 1.1 were true, the bounds would be the best possible. The reader is welcome to see Section 2.18 in [12] for more details. Though Conjecture 1.1 has been verified for several special classes of planar graphs, including outerplanar graphs [14], it remains open for all values of Δ. However, there is a series of partial results. The following upper bounds on the chromatic number of the square of a planar graph with maximum degree Δ have been established: $8\Delta - 22$ by Jonas [13], $3\Delta + 5$ by Wong [21], $3\Delta + 9$ for $\Delta \ge 8$ by Jendrol' and Skupien [10], $2\Delta + 18$ for $\Delta \ge 12$ by Madaras and Marcionová [15], $2\Delta + 25$ by van den Heuvel and McGuiness [9], $\lceil 9\Delta/5 \rceil + 2$ for $\Delta \ge 749$ by Agnarsson and Halldórsson [1, 2], and $\lceil 9\Delta/5 \rceil + 1$ for $\Delta \ge 47$ by Borodin, Broersma, Glebow and van den Heuvel [4]. The best known upper bounds are due to Molloy and Salavatipour [16, 17]: $\lceil 5\Delta/3 \rceil + 78$ for all Δ and $\lceil 5\Delta/3 \rceil + 25$ for $\Delta \ge 241$. Some of the above results were obtained by identifying so-called light structures in planar graphs—the reader is welcome to see the survey [11]. Coloring of higher powers of planar graphs was addressed by Agnarsson and Halldórsson [1, 2] who established an asymptotically tight upper bound on their chromatic numbers.

In this paper, we are interested in colorings of the squares of planar graphs with no short cycles. There are several upper bounds on the chromatic number of the squares of such planar and non-planar graphs: if the girth of a (not necessarily planar) graph G with maximum degree Δ is at least 7, then $\chi(G^2) \le O(\Delta^2/\log\Delta)$ [3]. Since the incidence graphs of finite projective planes have girth six and the chromatic number of their squares is $\Theta(\Delta^2)$, the assumption on the girth cannot be further decreased. The following bounds for planar graphs were proven by Wang and Lih [19]:

- $\chi(G^2) \le \Delta + 5$ if G is a planar graph of girth at least seven,
- $\chi(G^2) \le \Delta + 10$ if G is a planar graph of girth at least six, and

- $\chi(G^2) \leq \Delta + 16$ if G is a planar graph of girth at least five.

In addition, they conjectured the following:

Conjecture 1.2 (Wang and Lih 2003): *For any integer $g \geq 5$, there exists an integer $M(g)$ such that if G is a planar graph of girth g and maximum degree $\Delta \geq M(g)$, then $\chi(G^2) = \Delta + 1$.*

The conjecture is known to be false for $g \in \{5, 6\}$ and true for $g \geq 7$ with $M(7) = 30$ [5] and $M(9) = 16$ [6]. Our main result is that Conjecture 1.2 is also almost true for $g = 6$ (Theorem 7.1): if G is a planar graph of maximum degree $\Delta \geq 8821$ and its girth is at least six, then $\chi(G^2) \leq \Delta + 2$. Since Conjecture 1.2 does not hold for $g = 6$, the bound on the chromatic number is the best possible (also see Proposition 8.1). We are aware that the threshold on Δ can be improved, but we decided to focus solely on proving the statement for sufficiently large Δ without trying to optimize the threshold. However, one cannot expect to be able to easily obtain a significantly smaller threshold on Δ since quite a big threshold also appears in a similar result of [7] that the squares of planar graphs of girth six, sufficiently large maximum degree Δ, and with the additional assumption that each edge is incident with a vertex of degree two, are $(\Delta + 1)$-colorable.

It is natural to ask whether an analogous statement can hold for planar graphs of girth five. We conjecture this is indeed the case:

Conjecture 1.3: *There exists an integer M such that the square of every planar graph G with maximum degree $\Delta \geq M$ and girth at least 5 is $(\Delta + 2)$-colorable.*

Since in Section 8, we exhibit a construction of planar graphs G with girth six and arbitrarily large maximum degree Δ with $\chi(G^2) = \Delta + 2$, the bound given in Conjecture 1.3 would be the best possible. The only upper bound that we are aware of is $\Delta + 16$ given by Wang and Lih [19].

2. Preliminaries

In this section, we introduce notation used throughout the paper. All graphs considered in the paper are simple, i.e., without parallel edges and loops. A *d-vertex* is a vertex of degree exactly d. An *$(\leq d)$-vertex* is a vertex of degree at most d. Similarly, an *$(\geq d)$-vertex* is a vertex of degree at least d. A *k-thread* is an induced path comprised of k 2-vertices. The set of all the neighbors of a vertex v is called the *neighborhood* of v and the neighborhood enhanced by v is called the *closed neighborhood* of v.

An *ℓ-face* is a face of length ℓ (counting multiple incidences, i.e., bridges incident to the face are counted twice). If the boundary of a face f forms a connected subgraph, then the subgraph formed by the boundary (implicitly equipped with the orientation determined by the embedding) is called the *facial walk*. A face f is said to be *biconnected* if its boundary is formed by a single simple cycle. The neighbors of a vertex v on the facial walk are called *f-neighbors* of v. Note that if f is biconnected, then each vertex incident with f has exactly two f-neighbors.

Let us consider a biconnected face f, and let $v_1, \ldots, v_k$ be (≥ 3)-vertices incident to f listed in the order on the facial walk of f. The *type* of f is a k-tuple $(\ell_1, \ldots, \ell_k)$ where ℓ_i is the length of the 2-thread between v_i and v_{i+1}. In particular, if v_i and v_{i+1} are f-neighbors, then ℓ_i is zero. Two face types are considered to be the same if they can be types of the same face, i.e., they differ only by a cyclic rotation and/or a reflection.

Some of our arguments are based on elementary facts on list colorings (choosability of graphs). List colorings were introduced independently by Erdős, Rubin and Taylor [8] and Vizing [18]. A graph G is said to be ℓ-choosable if for any assignment of lists $L(v)$ of sizes ℓ to the vertices of G, there exists a proper coloring c of G such that $c(v) \in L(v)$ for every vertex v. The gap between the list chromatic number (the smallest ℓ for which the graph is ℓ-choosable) and the usual chromatic number can be arbitrary large: for every integer ℓ, there exists a bipartite graph that is not ℓ-choosable. However, the only simple fact that we need in our consideration is the following: any cycle of even length is 2-choosable. The reader can figure out details of a simple proof of this statement him/herself or can consult [8].

The proof of our main result is based on the discharging method. For an integer $D \geq 8821$, a graph G is called *D-good* if its maximum degree is at most D and the chromatic number of G^2 is at most $D + 2$. A planar graph G of girth at least 6 and maximum degree at most D is *D-minimal* if G is not D-good but every proper subgraph of G is D-good. If G is a D-minimal graph, then G is connected. Observe that G is also 2-connected: otherwise, color the blocks of G separately and afterwards permute the colors so that the colors of the cut-vertices match and the colors of their neighbors are pairwise distinct. In particular, the minimum degree of a D-minimal graph is at least two.

A vertex is said to be *small* if its degree is at most 1763, and it is said to be *big* otherwise.

In Sections 3–7, we show that there is no D-minimal graph. We assume that there is a D-minimal graph and assign charge to its vertices and its faces. The total amount of initial charge will be negative. We then redistribute charge in two phases as determined by the rules presented in Sections 5 and 6. We eventually obtain contradiction with our assumption that there exists a D-minimal graph by showing that the total final amount of charge is non-negative.

3. Reducible configurations

Let us first describe several configurations that cannot appear in a D-minimal graph. Such a configuration is called *reducible*.

LEMMA 3.1: *The following configurations are reducible:*

1. *A small vertex u and a vertex v joined by a 2-thread.*

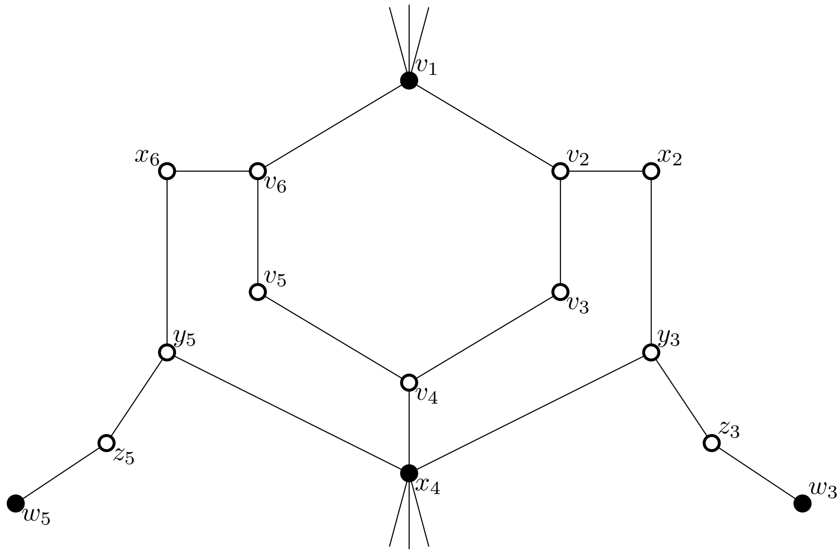

Figure 1: The reducible configuration from Lemma 3.1(5). The vertices that are not removed in the proof are represented by full circles.

2. *Vertices u and v joined by two 2-threads.*
3. *A small vertex v joined by a 1-thread to a vertex u of degree at most six, such that all the neighbors of u are small.*
4. *Two adjacent 3-vertices u and v such that all the neighbors of u and v are small and at least one of the neighbors of u is a 2-vertex.*
5. *The configuration in Figure 1, where v_2, v_4, v_6, y_3 and y_5 are 3-vertices, v_3, v_5, x_2, x_6, z_3 and z_5 are 2-vertices, and w_3 and w_5 are small vertices (there is no restriction on the degrees of v_1 and x_4).*

Proof: Let G be a D-minimal graph, in particular, $\chi(G^2) > D+2$. We deal with the configurations separately. In each of the cases, we first assume that G contains the configuration described in the statement of the lemma and we obtain contradiction by showing that G is not D-minimal.

1. Let x and y be the vertices of the 2-thread, where x is the vertex adjacent to u. Consider the graph $G' = G \setminus \{x, y\}$. Since G is D-minimal, the square of G' is $(D+2)$-colorable. Since the degree of v in G' is at most $D-1$, there are at least two colors distinct from the colors of v and its neighbors. At least one of them (call it γ) is distinct from the color of u. Assign the color γ to the vertex y. Since u is small, the degree of x in G^2 is at most $1763+3 < D$. Therefore, we can choose a color distinct from colors of u, its neighbors in G', v and y for x. We obtained a proper coloring of G^2 by $(D+2)$ colors. This contradicts the D-minimality of G.
2. Let the vertices of the 2-threads be x_1, x_2, y_1 and y_2 where x_i is adjacent

to y_i and u for $i = 1, 2$. The square of the graph $G' = G \setminus \{x_1, x_2, y_1, y_2\}$ is $(D+2)$-colorable by the D-minimality of G. Fix a coloring of G' with $D+2$ colors. Let C_u and C_v be the sets of the colors which are assigned to no vertex in the closed neighborhood of u and v, respectively. Since the degrees of u and v in G' are at most $D-2$, both C_u and C_v have sizes at least three. Let c_u and c_v be the colors of u and v, respectively. Let $C'_u = C_u \setminus \{c_v\}$ and $C'_v = C_v \setminus \{c_u\}$. Assign the list C'_u to the vertices x_1 and x_2 and the list C'_v to the vertices y_1 and y_2. The subgraph of G^2 induced by $\{x_1, x_2, y_1, y_2\}$ is a 4-cycle. This graph is 2-choosable. Therefore, its vertices can be colored from the assigned lists. The coloring obtained by extending the coloring of G' to G in this way is a proper coloring of G^2 with $D + 2$ colors that contradicts our assumption that G is D-minimal.

3. Let x be the 2-vertex of the 1-thread. The square of the graph $G' = G \setminus \{x\}$ is $(D + 2)$-colorable. Fix such a coloring. The degree of u in G'^2 is at most $5 \cdot 1763 + 5 < D$. Therefore, we can modify the coloring by changing the color of u so that it is distinct from the color of v as well as from the colors of the neighbors if u in G'^2. The degree of x in G^2 is at most $1763 + 7 < D$. Hence, we can extend this coloring to x. This contradicts the D-minimality of G.

4. Let x be a 2-vertex adjacent to u. Let y be the vertex adjacent to x distinct from u. Let w be the neighbor of u distinct from x and v. By the D-minimality of G, the square of the graph $G' = G \setminus \{x, u\}$ is $(D+2)$-colorable. Fix such a coloring. The vertex y has degree at most $D - 1$ in G', therefore at least two colors are unused on closed neighborhood of y in G'. Choose a color for x from the unused colors so that it is distinct from the color of w. The degree of v in G'^2 is at most $2 \cdot 1763 + 2 < D$. Therefore, it is possible to change the color of v so that it is distinct from the colors of x and w. Finally choose a color for u: its degree is at most $1763 + 6 < D$ in G^2. Therefore, it is always possible. This contradicts the D-minimality of G.

5. The square of the graph $G' = G \setminus \{v_2, v_3, v_4, v_5, v_6, x_2, x_6, y_3, y_5, z_3, z_5\}$ is $(D + 2)$-colorable (the removed vertices are marked by empty circles in Figure 1). Fix a coloring of G' with $D + 2$ colors. Since the degree of x_4 in G' is at most $D - 3$, there are at least four colors which are not assigned to a vertex of the closed neighborhood of x_4 in G'. Let L_4 be the set of the unused colors. The degree of v_1 in G' is at most $D - 2$, therefore the set L_1 of colors that do not appear on closed neighborhood of v_1 has size at least three. Let c_5 be the color of w_5 and c_3 the color of w_3. Assign the list L_1 to vertices v_2 and v_6, the list L_4 to the vertex v_4, the list $L_4 \setminus \{c_5\}$ to the vertex y_5 and the list $L_4 \setminus \{c_3\}$ to the vertex y_3. All 2-vertices of the configuration are adjacent only to small vertices. Therefore, if we were able to color the subgraph G'' of G^2 induced by $\{v_2, v_4, v_6, y_3, y_5\}$ from the lists, we could choose colors for the 2-vertices of the configuration carefully

and extend the coloring to the coloring of the whole graph G^2. This would eventually contradict the D-minimality of G.

However, such a coloring of G'' always exists. Choose a color for v_4 from L_4 arbitrarily, and remove this color from the lists of the remaining four vertices. The graph $G'' \setminus \{v_4\}$ is a 4-cycle. Since it is 2-choosable, the remaining vertices of G'' can be colored from the assigned lists.

□

4. Initial charge

We now describe the amounts of initial charge of vertices. The initial charge of a d-vertex v is set to

$$\mathrm{ch}(v) = d - 3,$$

and the initial charge of an ℓ-face f to

$$\mathrm{ch}(f) = \ell/2 - 3.$$

It is easy to verify that the sum of initial charges is negative:

PROPOSITION 4.1: *If G is a connected planar graph, then the sum of all initial charges of the vertices and faces of G is -6.*

Proof: Since G is connected, Euler's formula yields that $n + f = m + 2$ where n is the number of the vertices of G, m is the number of its edges and f is the number of its faces. The sum of initial charges of the vertices of G is equal to

$$\sum_{v \in V(G)} (d(v) - 3) = 2m - 3n.$$

The sum of initial charges of the faces of G is equal to

$$\sum_{f \in F(G)} \left(\frac{\ell(f)}{2} - 3 \right) = m - 3f.$$

Therefore, the sum of initial charges of all the vertices and faces is $3m - 3n - 3f = -6$. □

Note that the amounts of initial charge were chosen such that each face of size at least 6 (consequently, each face of a D-minimal graph) has non-negative charge, the charge of 6-faces is zero and only 2-vertices have negative charge of -1 unit.

5. The first discharging phase

The goal of the first phase is that each 2-vertex receives 2ε units of charge and the amount of charge of other vertices and faces is not decreased too much where $\varepsilon = 1/588$.

If u is a 2-vertex, an edge $e = uv$ is *void* if either $d(v) \in \{2, 4, 5, 6\}$, or v is a 3-vertex and all its neighbors are small. Intuitively, the void edges are those through which it may be impossible to send any charge to u.

In order to simplify the analysis of final charge of big vertices, we send all charge transfered from a big vertex through the edges incident to it. Each rule that deals with big vertices specifies through which edge the charge is (considered to be) sent. The value of ε and the bound on the degree of big vertices was chosen in such a way that a big vertex is able to send $1 - \varepsilon$ units of charge through each edge incident to it, and its final charge is still non-negative.

If v is a big vertex, we call an edge uv *red* if one of the following conditions holds:

- the vertex u is a 2-vertex, $e \neq uv$ is the other edge incident to u, and e is void, or
- the vertex u is a 3-vertex, x_1 and x_2 are the neighbors of u distinct from v, both x_1 and x_2 are 2-vertices, and all the neighbors of x_1 and x_2 are small.

The edges incident to big vertices which are not red are called *green*. Intuitively, the green edges are those through which the big vertex does not need to send "too much" charge and the red ones are those through which almost one unit of charge has to be sent.

In order to simplify the description of the rules, we define the following operation: if f is a 6-face and F is the set containing f and all the 6-faces sharing an edge with f, a 6-face f is *boosted* from a vertex or face z when 3ε units of charge are transferred from z to each face of F. Note that the charge of z decreases by at most 21ε.

The discharging rules of the first phase are the following:

F1 Each (≥ 7)-face boosts all the 6-faces sharing an edge with it.

F2 If v is a big vertex, e is a green edge incident to it and f is a 6-face incident to e, then the vertex v boosts f. The charge is sent through the edge e.

F3 If v is a small vertex of degree at least 4, then it boosts all the incident 6-faces.

F4 If v is a 2-vertex and f is a face incident to v, then f sends ε units of charge to v.

Note that no charge is sent through a red edge in the first phase. We now analyze the amounts of charge after the first phase:

LEMMA 5.1: *Let G be a D-minimal graph. After the first phase of discharging, the following claims hold:*

1. at most $1/8$ *units of charge was sent through each green edge,*

2. the charge of a small vertex of degree $d \geq 4$ *has decreased by at most* $d/16$*,*

3. the charge of each 2-vertex is $2\varepsilon - 1$*, and*

4. the charge of each face is non-negative.

Proof: We prove each claim separately:

1. Charge is sent through green edges only by Rule F2. Each green edge e is incident to at most two 6-faces and thus the total amount of charge sent through e is at most $42\varepsilon \leq 1/8$.
2. Charge is sent from small vertices only by Rule F3. A d-vertex is incident to at most d 6-faces. Therefore, the total amount of sent charge is at most $21\varepsilon d \leq d/16$.
3. Each 2-vertex receives ε units of charge from both the incident faces by Rule F4. Therefore, its charge becomes $2\varepsilon - 1$.
4. Charge is sent from faces by Rules F1 and F4. A d-face f shares an edge with at most d 6-faces. Therefore, the total amount of charge sent from f by Rule F1 is at most $21\varepsilon d$. Since at most d 2-vertices are incident to f, at most εd units of charge are sent by Rule F4. In total, at most $22\varepsilon d$ units of charge are sent from f.

   The charge of a d-face with $d \geq 7$ after the first phase is at least

   $$\frac{d}{2} - 3 - 22\varepsilon d = \left(\frac{1}{2} - 22\varepsilon\right) d - 3 \geq \frac{3}{7} d - 3 \geq 0.$$

   Hence, if f is a (≥ 7)-face, its final charge is non-negative.

   It remains to consider the case when f is a 6-face. Let k be the number of 2-vertices incident to f. Observe that k does not exceed 3: otherwise f contains at least four 2-vertices and it thus contains either a 3-thread or two vertices connected by two 2-threads. Both configurations are reducible by Lemma 3.1.

   Initial charge of f is zero and f sends out charge of $k\varepsilon$ by Rule F4. If $k = 0$, the final charge of f is non-negative. Assume that $k > 0$. It is sufficient to prove that f receives at least 3ε units of charge by Rules F1, F2 and F3. We show that f or one of the 6-faces incident to f is boosted during the first phase.

   If f shares an edge with a (≥ 7)-face, f is incident to a small vertex of degree at least 4, or f is incident to a green edge, then f itself is boosted. Therefore, we may assume that no edge incident to f is green, all the vertices incident to f are either big or have degree 2 or 3, and all the faces sharing an edge with f are 6-faces.

   Let $v_1, \ldots, v_6$ be the vertices of f in a cyclic order around the face.

Suppose first that f is incident to at least two big vertices. Assume that v_1 is a big vertex. The second big vertex of f is v_4: otherwise, the two big vertices are either f-neighbors or share an f-neighbor and at least one of the edges of f is green. If all the f-neighbors of v_1 and v_4 were 2-vertices, then v_1 and v_4 would be joined by two 2-threads, which is impossible by Lemma 3.1. Therefore at least one of the big vertices is adjacent to a 3-vertex. Assume that v_2 is a 3-vertex. But since v_4 is big, the edge v_1v_2 is green regardless of the degree of v_3. Therefore, the face f is boosted.

If f is incident to no big vertex, then no two 2-vertices of f are adjacent by Lemma 3.1(1). Assume that v_2 is a 2-vertex. Therefore, v_1 and v_3 are 3-vertices. Let x_1 and x_3 be the neighbors of v_1 and v_3 not incident to f. Since v_6 and v_4 are small, both x_1 and x_3 are big by Lemma 3.1(3). Let f' be the 6-face incident to v_2 distinct from f. Note that both x_1 and x_3 belong to the 6-face f' and share a common f'-neighbor. Hence, at least one of the edges incident to f' is green. Consequently, f' is boosted and f receives the charge of 3ε units.

It remains to consider the case when f contains exactly one big vertex, say v_1. If v_4 were a 2-vertex, we could use a similar argument as in the previous paragraph to show that the other face incident to v_4 is boosted. Therefore, we can assume that v_4 is a 3-vertex. In addition, either v_2 or v_3 is a 2-vertex, since the edge v_1v_2 is not green.

First suppose that v_2 is a 2-vertex. Hence v_3 is a 3-vertex. Let x_3 and x_4 be the neighbors of v_3 and v_4 not incident to f. If x_3 is big, then the edge v_1v_2 is green. And, if x_4 is big, then the edge x_4v_4 is green. In both the cases, f receives the required charge. If both x_3 and x_4 are small, the configuration is reducible by Lemma 3.1(4). The case that v_6 is a 2-vertex is symmetrical.

Suppose now that both v_2 and v_6 are 3-vertices and v_3 is a 2-vertex. We may assume that the neighbors of v_2 and v_6 (including v_5) distinct from v_1 are 2-vertices: otherwise, one of the edges v_1v_2 and v_1v_6 would be green. Let x_2, x_4 and x_6 be the vertices adjacent to v_2, v_4 and v_6 and not incident to f. By Lemma 3.1(3), the vertex x_4 is big. Let f_3 and f_5 be the faces incident to v_3 and v_5 and distinct from f. Let y_5 be the remaining vertex of f_5 distinct from x_6, x_4, v_4, v_5 and v_6. Let y_3 be the remaining vertex of f_3 distinct from x_2, x_4, v_2, v_3 and v_4. The degrees of both y_3 and y_5 must be 3: they cannot be two by Lemma 3.1(1) and if one of them were greater than 3, then one of the edges y_3x_4 and y_5x_4 would be green and f would receive charge because of boosting from f_3 or f_5. Let z_3 and z_5 be the neighbors of y_3 and y_5 distinct from x_6, x_4 and x_2. Both z_3 and z_5 must be 2-vertices and all their neighbors must be small, since otherwise one of edges y_3x_4 or y_5x_4 is green. However, the resulting configuration is reducible by Lemma 3.1(5). This finishes the proof of the claim.

□

6. The second phase of discharging

In this phase we redistribute the charge so that the final charge of all vertices is non-negative. The following rules are used during this phase:

S1 If v is a big vertex adjacent to a 2-vertex u, then v sends $1-\varepsilon$ units of charge to u if uv is red and it sends $3/4$ units of charge to u if uv is green. The charge is sent through the edge uv.

S2 If v is a big vertex adjacent to a 3-vertex u and the edge uv is red, then v sends $(1-\varepsilon)/2$ units of charge to both the 2-vertices adjacent to u. The charge is sent through the edge uv.

S3 Suppose that v is a big vertex adjacent to a 3-vertex u, the edge uv is green, and x is a 2-vertex adjacent to u. If x has a big neighbor, then v sends charge of $1/4$ to x. Otherwise, v sends charge of $1/2$ to x. The charge is sent through the edge uv.

S4 If v is a big vertex adjacent to a d-vertex u, $4 \le d \le 6$, then the vertex v sends $3/4$ units of charge to u. The charge is sent through the edge uv.

S5 If v is a d-vertex, $4 \le d \le 6$, adjacent to a 2-vertex u, and if v has at least one big neighbor, then v sends $1/2$ units of charge to u.

S6 If v is a small vertex of degree $d > 6$ adjacent to a 2-vertex u, then v sends $1/2$ units of charge to u.

We now analyze the amounts of charge sent during the second phase:

LEMMA 6.1: *Let G be a D-minimal graph. The following claims hold:*

1. *at most $3/4$ units of charge was sent through each green edge during the second phase,*
2. *at most $1-\varepsilon$ units of charge was sent through each red edge during the second phase, and*
3. *the charge of each vertex is non-negative after performing the first and the second phase.*

Proof: We prove each claim separately:

1. At most one of Rules S1, S3 and S4 applies to each green edge. At most $3/4$ units of charge is sent through such an edge by any of the rules. The only case in which this is not obvious is the case of Rule S3. However, there can be at most one vertex x without a big neighbor that satisfies the assumptions of the rule: otherwise the edge uv is red.
2. At most one of Rules S1 and S2 applies to each red edge and the charge sent through such an edge is exactly $1-\varepsilon$ by any of the rules.
3. Let v be a d-vertex of G. We consider several cases regarding the degree of the vertex v:

$d = 2$**:** Let x and y be the neighbors of v. It suffices to show that v received at least $1-\varepsilon$ units of charge during the second phase because charge of v was at least 2ε after the first phase by Lemma 5.1.

Suppose first that x is big. If the edge vy is void, then the edge xv is red and v received charge of $1-\varepsilon$ from x by Rule S1. Assume that the edge vy is not void and that the edge xv is green. Consequently, v received $3/4$ units of charge by Rule S1. Additionally, since vy is not void, then either y is a 3-vertex and has a big neighbor w, or y is a (≥ 7)-vertex. In the former case, v receives $1/4$ units of charge from w by Rule S3. In the latter case, y sends $1/2$ units of charge to v by Rules S1 or S6. In both the cases, the total charge received by v is at least 1.

The final case is that both x and y are small. By Lemma 3.1(1), neither x nor y has degree 2. We show that v receives at least $(1-\varepsilon)/2$ units of charge through x. Note that by symmetry v also receives at least $(1-\varepsilon)/2$ units of charge through y, i.e., v receives $1-\varepsilon$ units of charge in total. Let d' be the degree of x. If $3 \leq d' \leq 6$, at least one neighbor of x must be big by Lemma 3.1(3). Consequently, v receives at least $(1-\varepsilon)/2$ by one of Rules S2, S3 and S5. If $d' \geq 7$, then v receives $1/2$ from x by Rule S6.

$d = 3$**:** None of the discharging rules changes the charge of a vertex of degree three. Therefore, the final charge of v is zero.

$4 \leq d \leq 6$**:** The d-vertex v sent charge of at most $d/16$ units during the first phase by Lemma 5.1(2). If v is not adjacent to a big vertex, then it does not send anything during the second phase. Otherwise, it sends at most $(d-1)/2$ units of charge by Rule S5 and receives charge of at least $3/4$ units by Rule S4. Therefore, the final charge of v is

$$d - 3 - \frac{d}{16} - \frac{d-1}{2} + \frac{3}{4} = \frac{7d}{16} - \frac{7}{4} \geq 0.$$

$d \geq 6$ **and v is small:** The vertex v sends at most $d/16$ units of charge during the first phase by Lemma 5.1(2) and at most $d/2$ units of charge during the second phase by Rule S6. Therefore, the final charge of v is at least

$$d - 3 - \frac{d}{16} - \frac{d}{2} = \frac{7d}{16} - 3 > 0.$$

v **is big:** All the charge sent out from the big vertex v was sent through some of the edges incident to it. Charge is sent through a red edge e only in the second phase and the total amount of such charge is at most $1-\varepsilon$ by the previous claim of this lemma. At most $1/8$ units of charge is sent through a green edge e in the first phase by Lemma 5.1 and at most $3/4$ units in the second phase, thus in total $7/8 < 1-\varepsilon$. Therefore, v has the final charge of at least $d-3-(1-\varepsilon)d = \varepsilon d - 3 \geq 0$ (recall that v is a d-vertex with $d > 1763$).

□

7. Final step

We now combine our arguments from the previous sections:

THEOREM 7.1: *If G is a planar graph of maximum degree $\Delta \geq 8821$ and girth at least six, then G has a proper $L(1,1)$-labeling with span $\Delta+1$, i.e., $\chi(G^2) \leq \Delta+2$.*

Proof: If the statement of the theorem is false, then there exists a D-minimal graph. Consider such a D-minimal graph G. Assign charge to the vertices and the faces of G as described in Section 4. By Proposition 4.1, the sum of all the charges is negative. Apply the discharging rules of the two phases described in Sections 5 and 6. The final amount of charge of each face is non-negative after the first phase by Lemma 5.1 and it is preserved during the second phase, i.e., it is non-negative after the second phase. The final amount of charge of each vertex is non-negative after the second phase by Lemma 6.1. Therefore, the total final amount of charge is non-negative. We conclude that there is no D-minimal graph. □

8. The lower bound

For the sake of completeness, we also present a construction of planar graphs G with $\chi(G^2) = \Delta + 2$ and girth six. This shows that our bound is the best possible. A different construction of such graphs can be found in [5]. One of the reasons that also led us to include our construction to this paper is that our construction yields graphs with fewer vertices than that of [5].

Let G'_Δ be a graph of order $2\Delta + 2$ formed by two vertices x and y joined by $(\Delta - 1)$ 2-threads and a vertex z joined to y by a 1-thread. Let G_Δ be a graph obtained by taking $\Delta - 1$ copies of G'_Δ, identifying all the vertices z of the copies into a single vertex v, and adding a vertex u joined to v by a 1-thread and by an edge to the vertex x of each copy of G'_Δ (see Figure 2). Clearly, the girth of G_Δ is six and the maximum degree of G_Δ is Δ. The chromatic number of G_Δ is determined in the next proposition:

PROPOSITION 8.1: *The chromatic number of the square of the graph G_Δ is $\Delta+2$ for every $\Delta \geq 2$.*

Proof: It is easy to construct a coloring of G^2_Δ by $\Delta + 2$ colors. We focus on showing that it cannot be colored by $\Delta + 1$ colors.

We first show that in any proper coloring of the square of G'_Δ, the colors assigned to x and z are distinct. Suppose for contradiction that there exists a proper coloring of G'^2_Δ by the colors $0, \ldots, \Delta$ such that the colors of both x and z are the same, say 0. Since the vertex y has degree Δ, either y or one of its

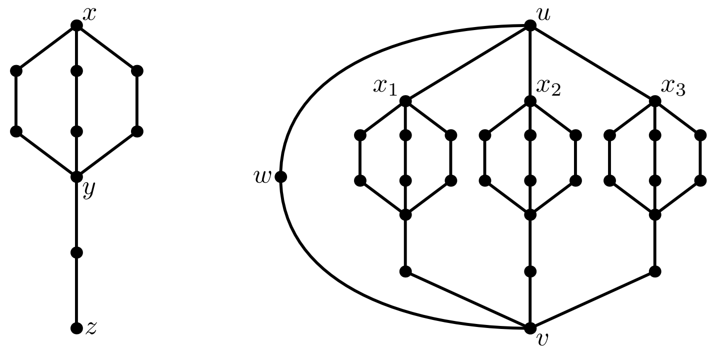

Figure 2: The graphs G_4' and G_4.

neighbors must have color 0. This is impossible because each of these vertices is at distance at most two from x or z.

Suppose now that the graph G_Δ can be colored by the colors $0, \ldots, \Delta$. Let $x_1, \ldots, x_{\Delta-1}$ be the vertices of the copies of G'_Δ adjacent to the vertex u. Let w be the vertex adjacent to u and distinct from all x_i, $1 \le i < \Delta$. We may assume that the color of v is 0. By the observation from the previous paragraph, the color of each vertex x_i is distinct from 0. The vertex u has degree Δ. Therefore, either u or one of its neighbors has color 0. This is impossible since the colors of vertices x_i are distinct from 0 and both u and w are at distance at most two from the vertex v. We conclude that there is no proper coloring of G^2_Δ with $\Delta+1$ colors. □

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