On a conjecture about Wiener index in iterated line graphs of trees

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Abstract

Let G be a graph. Denote by $L^i(G)$ its i-iterated line graph and denote by $W(G)$ its Wiener index. There is a conjecture which claims that there exists no nontrivial tree T and $i \geq 3$, such that $W(L^i(T)) = W(T)$, see [5]. We prove this conjecture for trees which are not homeomorphic to the claw $K_{1,3}$ and H, where H is a tree consisting of 6 vertices, 2 of which have degree 3.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. For any two of its vertices, say u and v, by $d(u, v)$ we denote the distance from u to v in G. The *Wiener index* of G, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of G. Wiener index was introduced by Wiener in 1947, see [15]. In the next decades, it was intensively studied by chemists, as it is related to many physical properties of organical molecules, see [9]. Graph theoretists reintroduced this parameter as the distance in 1970 and transmission in 1984, see [6] and [14], respectively. Recently, graph theoretic aspects of Wiener index are intensively studied, see e.g. [7] and [8], or surveys [3] and [4].

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By definition, if G has a unique vertex, i.e., if $G = K_1$, then $W(G) = 0$. In this case we say that the graph G is *trivial*. We set $W(G) = 0$ also when the set of vertices (and therefore also the set of edges) of G is empty.

The line graph of G, $L(G)$, has vertex set identical with the set of edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0 \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In [1], the following theorem was proved.

Theorem 1.1 [1] *If T is a tree on n vertices, then $W(L(T)) = W(T) - \binom{n}{2}$.*

Since $\binom{n}{2} > 0$ if $n \geq 2$, there is no nontrivial tree for which $W(L(T)) = W(T)$. However, there are trees T satisfying $W(L^2(T)) = W(T)$, see for example [2]. In [5] the following conjecture was posed (see also [3]).

Conjecture 1.2 [5] *Let T be a nontrivial tree and $i \geq 3$. Then $W(L^i(T)) \neq W(T)$.*

Denote by P_n a path on n vertices. If $n \geq 2$, then $W(P_n) > W(P_{n-1})$. Since $L(P_n) = P_{n-1}$ if $n \geq 2$, while $L(P_1)$ is an empty graph, it follows that $W(L^i(P_n)) < W(P_n)$ for every $i \geq 1$ provided that $n \geq 2$. Hence, Conjecture 1.2 is trivially true for paths of length at least 1.

In [11], we prove that for every graph G the function $W(L^i(G))$ is convex in variable i. The following corollary is a straightforward consequence of this fact.

Corollary 1.3 *Let T be a tree such that $W(L^3(T)) > W(T)$. Then for every $i \geq 3$ the inequality $W(L^i(T)) > W(T)$ holds.*

Let G be a graph. A *pendant path* (or a *ray* for short) R' in G is a (directed) path, the first vertex of which has degree at least 3, its last vertex has degree 1, and all of its internal vertices (if any exist) have degree 2 in G. Observe that if R' has length t, $t \geq 2$, then the edges of R' correspond to vertices of a ray R in $L(G)$ of length $t - 1$. In [11] we proved the following theorem.

Theorem 1.4 [11] *Let T be a tree distinct from a path and the claw $K_{1,3}$ such that all of its rays have length 1. Then $W(L^3(T)) > W(T)$.*

Here we extend this statement to trees with arbitrarily long rays. Denote by H a tree on 6 vertices, two of which have degree 3 and four of which have degree 1. (That is, H is the graph which "looks" like the letter H.) The main result of this paper is the following theorem.

Theorem 1.5 *Let T be a tree not homeomorphic to a path, claw $K_{1,3}$ and H. Then $W(L^3(T)) > W(T)$.*

Recall that graphs G_1 and G_2 are homeomorphic if and only if the graphs obtained from them by repeatedly removing a vertex of degree 2 (and making its two neighbours adjacent) are isomorphic. Combining Corollary 1.3 and Theorem 1.5 we obtain the following corollary, which proves Conjecture 1.2 for the trees T satisfying the assumption in Theorem 1.5.

Corollary 1.6 *Let T be a tree not homeomorphic to a path, claw $K_{1,3}$ and H. Then $W(L^i(T)) > W(T)$ for every $i \geq 3$.*

We remark that trees homeomorphic to the claw $K_{1,3}$ and the graph H are considered in forthcoming papers, see [12, 13].

For a tree T, let $D(T) = W(L^3(T)) - W(T)$. We prove $D(T) > 0$ by induction on the length of the longest ray in T. By Theorem 1.4, $D(T) > 0$ if the longest ray has length 1. Now we describe the induction step:

We suppose that $D(T) > 0$ for all trees rays of which have length at most $l+1$. We would like to extend this statement to trees with rays of length at most $l+2$. Let a' be the last vertex of a ray of length $l+1$ in T, $l \geq 0$. Since we extend only one ray in turn, namely the ray terminating at a', we assume that all rays of T have lengths at most $l+2$. Add to T one new vertex b' and the edge $a'b'$, and denote the resulting tree by T^*. Denote by a the edge of T containing a' and denote by b the edge $a'b'$. Then ab is an edge of $L(T^*)$ and the degree of b is 1 in $L(T^*)$. Moreover, a is an endvertex of a ray of length l in $L(T)$ and b is an endvertex of a ray of length $l+1$ in $L(T^*)$; see Figure 1. By the assumption, all rays of $L(T)$ have lengths at most $l+1$. Let

$$\Delta T = D(T^*) - D(T).$$

In the next section we present an exact formula for ΔT. In Section 3 we prove $\Delta T \geq 0$ and this will establish Theorem 1.5 (for more detailed explanation see the proof of Theorem 1.5).

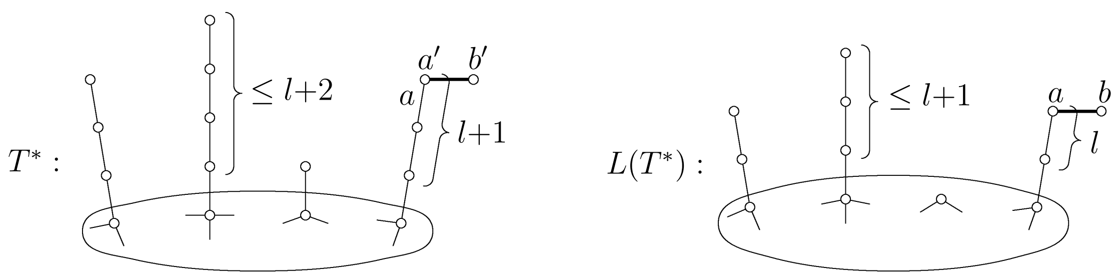

Figure 1: Description of the induction step for $l = 2$.

Now we introduce notation used throughout the paper. Since we work repeatedly with line graphs of trees, we simplify the notation and write LG rather than $L(G)$. The degree of a vertex z is denoted by d_z. If there are more graphs containing the vertex z, then d_z denotes the degree of z in LT (whatever the meaning of T at that point will be). Similarly, by $d(z,w)$ we denote the distance from z to w, and this distance is preferably considered in LT (rather than T). When no confusion is likely, any path starting at u and terminating at v will be denoted by $u-v$.

2 Preliminaries

Analogously as a vertex of $L(G)$ corresponds to an edge of G, a vertex of $L^2(G)$ corresponds to a path of length two in G. For $x \in V(L^2(G))$, we denote by $B_2(x)$ the corresponding path in G. For two subgraphs S_1 and S_2 of G, by $d(S_1,S_2)$ we denote the shortest distance in G between a vertex of S_1 and a vertex of S_2. If S_1 and S_2 share s edges, then we set $d(S_1,S_2)=-s$.

Let x and y be two vertices of $L^2(G)$, such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Then $d_{L^2(G)}(x,y)=d(B_2(x),B_2(y))+2$; see [10, 11].

Let $u,v \in V(G)$, $u \neq v$. Denote by $\beta_i(u,v)$ the number of pairs $x,y \in V(L^2(G))$, with u being the center of $B_2(x)$ and v being the center of $B_2(y)$, such that $d(B_2(x),B_2(y))=d(u,v)-2+i$. Since $d(u,v)-2 \leq d(B_2(x),B_2(y)) \leq d(u,v)$, we see that $\beta_i(u,v)=0$ for all $i \notin \{0,1,2\}$. In [11] the following statement was proved:

Proposition 2.1 *Let G be a connected graph. Then*

$$\begin{aligned} W(L^2(G)) &= \sum_{u\neq v}\left[\binom{d_u}{2}\binom{d_v}{2}d(u,v)+\beta_1(u,v)+2\beta_2(u,v)\right] \\ &+ \sum_{u}\left[3\binom{d_u}{3}+6\binom{d_u}{4}\right], \end{aligned}$$

where the first sum runs through all unordered pairs of distinct vertices $u,v \in V(G)$ and the second one runs through all $u \in V(G)$.

We apply Proposition 2.1 to line graphs of trees. Let us recall the structure of these graphs. For any tree F, the graph LF consists of cliques in the following sense: Denote by $\mathcal{C}(LF)$ the set of maximal cliques of LF. Then every vertex of LF belongs to at most two cliques from $\mathcal{C}(LF)$; each pair of cliques from $\mathcal{C}(LF)$ intersects in at most one vertex; and the cliques of $\mathcal{C}(LF)$ have a "tree structure", i.e., there are no cliques $C_0,C_1,\dots,C_{t-1}$, $t\geq 3$, such that C_i and C_{i+1} have nonempty intersection, $0\leq i\leq t-1$, the addition being modulo t.

We start with an exact formula for ΔT. For $u \in V(LT) \setminus \{a\}$, let

$$h_{LT}(u) = \left(\binom{d_u}{2} d_a - 1\right) d(u,a) + (d_u - 1)\Big(d_u\, d_a - d_a - \frac{1}{2} d_u\Big) - 2 - \phi(u,a), \quad (1)$$

where

$$\phi(u,a) = \begin{cases} (d_a - 1)(d_u - 2) & \text{if } d(u,a) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2 *For a nontrivial tree, the following equality holds:*

$$\Delta T = \sum_u h_{LT}(u) + \frac{1}{2} d_a\,(d_a - 1)(2d_a - 1) - 3,$$

where the sum is taken over all vertices $u \in V(LT) \setminus \{a\}$.

Proof Let F be a tree and let u and v be distinct vertices of LF. Consider vertices $x, y \in V(L^2(LF))$ such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Due to the clique structure of LF, there is a unique shortest $u - v$ path in LF. Denote this path by $u = a_0, a_1, \ldots, a_t = v$. If $d(B_2(x), B_2(y)) = d(u,v) - 2$, then we must have $a_1 \in V(B_2(x))$ and $a_{t-1} \in V(B_2(y))$. There are $(d_u - 1)$ ways to choose the other endvertex of $B_2(x)$, and there are $(d_v - 1)$ ways to choose the other endvertex of $B_2(y)$. Hence, $\beta_0(u,v) = (d_u - 1)(d_v - 1)$.

Now we find $\beta_1(u,v)$. We distinguish two cases: $d(u,v) \geq 2$ and $d(u,v) = 1$.

Suppose first $d(u,v) \geq 2$. In this case u and v do not belong to a common clique from $\mathcal{C}(LF)$. If $d(B_2(x), B_2(y)) = d(u,v) - 1$, then either $a_1 \in V(B_2(x))$ or $a_{t-1} \in V(B_2(y))$, but not both. In the first case we obtain $(d_u - 1)\binom{d_v - 1}{2}$ pairs x, y and in the second case $\binom{d_u - 1}{2}(d_v - 1)$ pairs x, y. Thus

$$\beta_1(u,v) = (d_u - 1)\binom{d_v - 1}{2} + \binom{d_u - 1}{2}(d_v - 1).$$

Suppose now that $d(u,v) = 1$. In this case, u and v belong to a common clique. All pairs x, y mentioned in the previous case contribute to $\beta_1(u,v)$, but we have to add pairs x, y such that $v \notin V(B_2(x))$, $u \notin V(B_2(y))$ and $d(B_2(x), B_2(y)) = d(u,v) - 1 = 0$. For these pairs, the paths $B_2(x)$ and $B_2(y)$ share at least one of their endvertices. Denote by $\alpha_{LF}(u,v)$ the number of these extra pairs. Then

$$\beta_1(u,v) = (d_u - 1)\binom{d_v - 1}{2} + \binom{d_u - 1}{2}(d_v - 1) + \alpha_{LF}(u,v).$$

Since we do not need to evaluate $\alpha_{LF}(u,v)$ in general, we postpone this computation until later. To simplify the notation, we set $\alpha_{LF}(u,v) = 0$ for all pairs u, v such that $d(u,v) \geq 2$.

We have $\binom{d_u}{2}\binom{d_v}{2}$ pairs $x, y \in V(L^2(LF))$ such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Since

$$\begin{aligned}\binom{d_u}{2}\binom{d_v}{2} &= (d_u-1)(d_v-1) + (d_u-1)\binom{d_v-1}{2} + \binom{d_u-1}{2}(d_v-1) \\ &\quad + \binom{d_u-1}{2}\binom{d_v-1}{2},\end{aligned}$$

we obtain $\beta_2(u,v) = \binom{d_u-1}{2}\binom{d_v-1}{2} - \alpha_{LF}(u,v)$. By Proposition 2.1, it follows that

$$\begin{aligned}W(L^2(LF)) &= \sum_{u \neq v} \Bigg[\binom{d_u}{2}\binom{d_v}{2} d(u,v) + (d_u-1)\binom{d_v-1}{2} \\ &\quad + \binom{d_u-1}{2}(d_v-1) + 2\binom{d_u-1}{2}\binom{d_v-1}{2} - \alpha_{LF}(u,v)\Bigg] \\ &\quad + \sum_u \left[3\binom{d_u}{3} + 6\binom{d_u}{4}\right]. \qquad (2)\end{aligned}$$

Now we evaluate $W(L^3(T^*)) - W(L^3(T)) = W(L^2(LT^*)) - W(L^2(LT))$; see the notation following Corollary 1.6. The graph LT^* has one more vertex than LT, namely the vertex b of degree 1, and the degree of a increased by 1 to $d_a + 1$ in LT^*. Therefore, all the terms of (2) for pairs u, v which do not contain neither a nor b cancell out in $W(L^2(LT^*)) - W(L^2(LT))$. However, we need to subtract the terms for pairs u, a in LT and to add the terms for pairs u, a in LT^*, $u \in V(LT) \setminus \{a\}$. We can ignore the terms containing b in LT^*, since the degree of b is 1, and thus b cannot be a center of $B_2(y)$ for any $y \in V(L^2(LT^*))$. (Observe that all terms of (2) are 0 if one of the vertices has degree 1.) As regards the second sum in (2), we have to subtract the term corresponding to a in LT and add the terms corresponding to a and b in LT^*, the second one being 0 since the degree of b is 1 in LT^*. Denote by $\Delta\alpha(u,v)$ the difference $\alpha_{LT^*}(u,v) - \alpha_{LT}(u,v)$ and denote by ΔWL^2 the difference $W(L^2(LT^*)) - W(L^2(LT))$. By (2), it follows that

$$\begin{aligned}\Delta WL^2 &= -\sum_u \Bigg[\binom{d_u}{2}\binom{d_a}{2} d(u,a) + (d_u-1)\binom{d_a-1}{2} + \\ &\quad + \binom{d_u-1}{2}(d_a-1) + 2\binom{d_u-1}{2}\binom{d_a-1}{2} - \alpha_{LT}(u,a)\Bigg] \\ &\quad + \sum_u \Bigg[\binom{d_u}{2}\binom{d_a+1}{2} d(u,a) + (d_u-1)\binom{d_a}{2} + \\ &\quad + \binom{d_u-1}{2} d_a + 2\binom{d_u-1}{2}\binom{d_a}{2} - \alpha_{LT^*}(u,a)\Bigg] \\ &\quad -3\binom{d_a}{3} - 6\binom{d_a}{4} + 3\binom{d_a+1}{3} + 6\binom{d_a+1}{4}\end{aligned}$$

$$
\begin{aligned}
&= \sum_u \Bigg[\binom{d_u}{2} d_a\, d(u,a) + (d_u-1)(d_a-1) \\
&\quad + \binom{d_u-1}{2} + 2\binom{d_u-1}{2}(d_a-1) - \Delta\alpha(u,v)\Bigg] \\
&\quad + \frac{1}{4} d_a\,(d_a-1)\Big[-2(d_a-2) - (d_a-2)(d_a-3) \\
&\quad + 2(d_a+1) + (d_a+1)(d_a-2)\Big] \\
&= \sum_u \Bigg[\binom{d_u}{2} d_a\, d(u,a) + (d_u-1)\Big(d_u\, d_a - d_a - \frac{1}{2}d_u\Big) - \Delta\alpha(u,a)\Bigg] \\
&\quad + \frac{1}{2} d_a(d_a-1)(2d_a-1). \qquad (3)
\end{aligned}
$$

Now we determine $\Delta\alpha(u,a)$. For $u \in V(LT) \setminus \{a\}$, the distance from u to a in LT is the same as in LT^*. Therefore $\Delta\alpha(u,a) = \alpha_{LT^*}(u,a) - \alpha_{LT}(u,a) = 0 - 0 = 0$ if $d(u,a) \geq 2$. If $d(u,a) = 1$, then in order to evaluate $\alpha_{LT^*}(u,a) - \alpha_{LT}(u,a)$ we need to count pairs x, y such that $b \in V(B_2(y))$. Denote by C the clique of $\mathcal{C}(LT)$ containing both a and u. The order of C is $d_a + 1$. We distinguish two cases.

- *Both endvertices of* $B_2(x)$ *are in* C: We have $\binom{d_a-1}{2}$ choices for $B_2(x)$ in this case since $a \notin V(B_2(x))$. For each of these choices there are two choices for $B_2(y)$ such that $B_2(x)$ and $B_2(y)$ share an endvertex and $b \in V(B_2(y))$. Therefore there are $2\binom{d_a-1}{2}$ pairs x, y contributing to $\Delta\alpha(u,v)$ in this case.

- *Only one endvertex of* $B_2(x)$ *is in* C: For this vertex we have $d_a - 1$ choices, since $a \notin V(B_2(x))$, and for the other endvertex of $B_2(x)$ we have $d_u - d_a$ choices. In this case, for every x there is a unique y such that $B_2(x)$ and $B_2(y)$ share an endvertex and $b \in V(B_2(y))$. Thus there are $(d_a-1)(d_u-d_a)$ pairs x, y contributing to $\Delta\alpha(u,v)$ in this case.

Hence,

$$
\Delta\alpha(u,v) = 2\binom{d_a-1}{2} + (d_a-1)(d_u-d_a) = (d_a-1)(d_u-2) = \phi(u,a). \qquad (4)
$$

Now we evaluate $W(T^*) - W(T)$. If F is a tree with n_0 vertices, then $W(LF) = W(F) - \binom{n_0}{2}$, by Theorem 1.1. Denote by n_1 the number of vertices of LF. Since $n_1 = n_0 - 1$, we have $W(F) = W(LF) + \binom{n_1+1}{2}$. Denote by n the number of vertices of LT. Then

$$
\begin{aligned}
W(T^*) - W(T) &= W(LT^*) + \binom{n+2}{2} - W(LT) - \binom{n+1}{2} \\
&= W(LT^*) - W(LT) + n + 1.
\end{aligned}
$$

In $W(LT^*) - W(LT)$, all terms for pairs u, v which do not contain b will cancell out. Therefore

$$\begin{aligned} W(T^*) - W(T) &= \sum_u d(u,b) + d(a,b) + n + 1 \\ &= \sum_u \Big(d(u,a) + 1\Big) + 1 + \sum_u 1 + 2 \\ &= \sum_u \Big(d(u,a) + 2\Big) + 3. \end{aligned} \tag{5}$$

where the sum goes once again through $n-1$ vertices $u \in V(LT) \setminus \{a\}$.

Since $\Delta T = D(T^*) - D(T) = W(L^3(T^*)) - W(T^*) - W(L^3(T)) + W(T) = \Delta WL^2 - (W(T^*) - W(T))$, combining (3), (4) and (5) we obtain the required result.

□

3 Proof of Theorem 1.5

We prove that $\Delta T \geq 0$ for every tree T which is not homeomorphic to a path, claw $K_{1,3}$ or the graph H. Let l, a', b', a, b, T^* and ΔT be as in the discussion following Corollary 1.6. As explained there, we proceed by induction on l.

First we prove that $\Delta T \geq 0$ for the case $l = 0$. In this case, a' is adjacent to a vertex of degree at least 3 in T, implying that in LT we have $d_a \geq 2$.

Let v be an endvertex of a ray (a pendant path) R in LT, i.e., $d_v = 1$. By $\overline{v}$ we denote the first vertex of R, i.e., a vertex at shortest distance to v whose degree is at least 3. Due to the clique structure of LT described after Proposition 2.1, we have:

Observation 3.1 *If u and v are distinct vertices of degree 1 in LT, then $\overline{u} \neq \overline{v}$.*

We use Obseravtion 3.1 repeatedly in the following proofs.

Lemma 3.2 *Let T be a tree different from a path in which all rays have length at most $l+2$, and let $l = 0$. Then $\Delta T \geq 0$.*

Proof We find a lower bound for $\sum_u h_{LT}(u)$. Consider four cases:

- $d_u = 1$: Then $d(u,a) > 1$, and thus $h_{LT}(u) = -d(u,a) - 2$ by (1).
- $d_u = 2$: Since $(d_a - 1)(d_u - 2) = 0$, we see that $\phi(u,a) = 0$ in this case as well. By (1) we obtain

  $$h_{LT}(u) = (d_a - 1)d(u,a) + d_a - 3 \geq d_a - 1 + d_a - 3 = 2d_a - 4 \geq 0$$

  since $d_a \geq 2$.

- $d_u \geq 3$ *and* $d(u,a) \geq 2$: By (1) it follows that

$$\begin{aligned}
h_{LT}(u) &= \left(\binom{d_u}{2} d_a - 1\right) d(u,a) + (d_u - 1)\Big(d_u\, d_a - d_a - \frac{1}{2} d_u\Big) - 2 \\
&\geq 5d(u,a) + (d_u - 1)\frac{1}{2}\Big[d_a\,(d_u - 2) + d_u\,(d_a - 1)\Big] - 2 \\
&\geq 5d(u,a) + 5 - 2 \\
&\geq d(u,a) + 11
\end{aligned}$$

as $d_u \geq 3$, $d_a \geq 2$ and $d(u,a) \geq 2$.

- $d_u \geq 3$ *and* $d(u,a) = 1$: By (1) it follows that

$$\begin{aligned}
h_{LT}(u) &= \left(\binom{d_u}{2} d_a - 1\right) d(u,a) + (d_u - 1)\Big(d_u\, d_a - d_a - \frac{1}{2} d_u\Big) - 2 \\
&\quad - (d_a - 1)(d_u - 2) \\
&\geq 5d(u,a) + {d_u}^2 d_a - \frac{1}{2}{d_u}^2 - 3d_u\, d_a + \frac{3}{2} d_u + 3d_a - \frac{3}{2} - \frac{5}{2} \\
&= 5d(u,a) + \frac{1}{2}\Big[(2d_a - 1)\Big(d_u(d_u - 3) + 3\Big) - 5\Big] \\
&\geq d(u,a) + 6
\end{aligned}$$

as $d_u \geq 3$, $d_a \geq 2$ and $d(u,a) = 1$.

Hence,

$$h_{LT}(u) \geq \begin{cases} -d(u,a) - 2 & \text{if } d_u = 1, \\ 0 & \text{if } d_u = 2, \\ d(u,a) + 6 & \text{if } d_u \geq 3. \end{cases} \tag{6}$$

Since $l = 0$, all rays of T have length at most 2, implying that all rays of LT have length at most 1. Hence, if $d_u = 1$, then $d(u, \overline{u}) = 1$ in LT. Thus

$$h_{LT}(u) + h_{LT}(\overline{u}) \geq -d(u,a) - 2 + d(\overline{u},a) + 6 = -d(\overline{u},a) - 3 + d(\overline{u},a) + 6 \geq 0.$$

Denote by V_1 the set of vertices of degree 1 in $V(LT) \setminus \{a\}$. By Observation 3.1, $\overline{u} \neq \overline{v}$ whenever $u, v \in V_1$, $u \neq v$. Therefore, by (6) it follows that

$$\sum_u h_{LT}(u) \geq \sum_{u \in V_1} \Big(h_{LT}(u) + h_{LT}(\overline{u})\Big) \geq 0.$$

Since $d_a \geq 2$, we have $\frac{1}{2} d_a\,(d_a - 1)(2d_a - 1) \geq 3$, implying that

$$\Delta T = \sum_u h_{LT}(u) + \frac{1}{2} d_a\,(d_a - 1)(2d_a - 1) - 3 \geq 0,$$

by Proposition 2.2. $\square$

Now we prove that $\Delta T \geq 0$ for all $l \geq 1$, i.e., from now on we consider $l \geq 1$. In this case $\phi(u,a) = 0$ since $d_a = 1$, which simplifies the expression for $h_{LT}(u)$ in (1). The problem is that $h_{LT}(u) < 0$ even if $d_u = 2$, suggesting that we need sharper estimations. We prove $\Delta T \geq 0$ by induction on the number of vertices of degree at least 3 in T.

Let G be a graph. A path of length at least one in G is *interior path* if its endvertices have degrees both at least 3, its interior vertices (if any) have degree 2 in G, and its edges are bridges of G. In the next lemma we show that it suffices to prove $\Delta T \geq 0$ for trees whose interior paths have lengths at most 2, i.e., we reduce the class of trees for which we need to prove $\Delta T \geq 0$.

Lemma 3.3 *Let T^s be obtained from T by subdividing one edge of an interior path of length t, $t \geq 2$, and let $l \geq 1$. Then $\Delta T^s \geq \Delta T$.*

Proof Denote by P' the interior path of T, whose edge was subdivided to obtain T^s. Since P' has length $t \geq 2$, the edges of P' form an interior path P of length $t-1 \geq 1$ in LT. Obviously, LT^s can be obtained from LT by subdividing one edge of P. Denote by e the endvertex of P, which has among the vertices of P the greatest distance from a. Let LT^s be obtained from LT by subdividing that edge of P which is incident to e. Denote the new vertex by w. Observe that for every vertex $u \in V(LT)$, the degree of u in LT is the same as its degree in LT^s.

Since the degree of a is the same in LT^s as in LT, namely 1, by Proposition 2.2 it suffices to show that $\sum_{u \in V(LT^s)\setminus\{a\}} h_{LT^s}(u) \geq \sum_{u \in V(LT)\setminus\{a\}} h_{LT}(u)$. We distinguish three cases.

- *u is a vertex of LT such that e does not lie on $u-a$ path in LT*: Then $d_{LT^s}(u,a) = d_{LT}(u,a)$, implying that $h_{LT^s}(u) = h_{LT}(u)$ and $h_{LT^s}(u) - h_{LT}(u) = 0$, see (1).
- *u is a vertex of LT, such that e lays on $u-a$ path in LT*: Then $d_{LT^s}(u,a) = d_{LT}(u,a)+1$, implying that $h_{LT^s}(u) - h_{LT}(u) = \binom{d_u}{2} - 1$ since $d_a = 1$; see (1). Thus $h_{LT^s}(u) - h_{LT}(u) = -1$ if $d_u = 1$, $h_{LT^s}(u) - h_{LT}(u) = 0$ if $d_u = 2$ and $h_{LT^s}(u) - h_{LT}(u) \geq 2$ if $d_u \geq 3$.
- $u = w$: Since the degree of w is 2 in LT^s, we see that $h_{LT^s}(w) = -2$, by (1).

Every vertex u of degree 1 in LT is an endvertex of a ray starting at vertex $\overline{u}$ of degree at least 3. By Observation 3.1, if u and v are distinct vertices of degree 1 in LT, then $\overline{u} \neq \overline{v}$. Denote by V_e the set of vertices u of LT such that $d_u = 1$ and e lays on $u-a$ path. Observe that $e \neq \overline{u}$ for any $u \in V_e$.

Denote $\Delta h = \sum_{u\in V(LT^s)\setminus\{a\}} h_{LT^s}(u) - \sum_{u\in V(LT)\setminus\{a\}} h_{LT}(u)$. By the analysis above only vertices of $V_e \cup \{w\}$ contribute negative value to Δh. Therefore

$$\begin{aligned}
\Delta h &\geq \sum_{u\in V_e} \Big[\Big(h_{LT^s}(u) - h_{LT}(u)\Big) + \Big(h_{LT^s}(\overline{u}) - h_{LT}(\overline{u})\Big)\Big] \\
&\quad + \Big(h_{LT^s}(e) - h_{LT}(e)\Big) + h_{LT^s}(w) \\
&\geq \sum_{u\in V_e}(-1+2) + 2 - 2 \geq 0.
\end{aligned}$$

Hence $\Delta T^s \geq \Delta T$. □

Let F be a tree with a ray of length $l+1$ terminating in the edge a. Denote by S_{LF} the set of first edges of rays of F. Then S_{LF} is also a set of vertices of LF. These vertices have degree at least 3, with the exception when the corresponding edge is incident to vertices of degree 1 and 3 in F. Let $u \in S_{LF}$. If there is a ray in LF starting at u, then denote by $R_{LF}(u)$ the set of vertices (other than a) of this ray; otherwise set $R_{LF}(u) = \{u\}$. Since $l \geq 1$, there is a ray in LF starting at $\overline{a}$ and $\overline{a} \neq s$. Observe also that $R_{LF}(u) \cap R_{LF}(v) = \emptyset$ whenever $u, v \in S_{LF}$, $u \neq v$.

Lemma 3.4 *Let F be a tree rays of which have length at most $l+2$, $l \geq 1$. Moreover, one ray of F has length exactly $l+1$ and this ray terminates by the edge a. Let $c \in S_{LF}$ be a vertex of a clique from $\mathcal{C}(LF)$ of order $r \geq 3$. Then*

$$\sum_{u\in R_{LF}(c)} h_{LF}(u) \geq \begin{cases} \Big(\binom{r}{2} - 3\Big)l + \binom{r-1}{2} & \textit{if } c = \overline{a} \\ \Big(\binom{r-1}{2} - 1\Big)d(c,a) + \binom{r-2}{2} - 2 & \textit{if } c \neq \overline{a} \textit{ and } |R_{LF}(c)| = 1 \\ \Big(\binom{r}{2} - 2\Big)d(c,a) - 3l + \binom{r-1}{2} - 5 & \textit{if } c \neq \overline{a} \textit{ and } |R_{LF}(c)| \geq 2. \end{cases}$$

Proof We distinguish three cases.

- $c = \overline{a}$: Then $R_{LF}(c)$ has one vertex of degree r, namely c with $d(c,a) = l$, and $l-1$ vertices of degree 2. Since the degree of a is 1, by (1) we have

$$\begin{aligned}
\sum_{u\in R_{LF}(c)} h_{LF}(u) &= \left(\binom{r}{2} - 1\right)d(c,a) + (r-1)\Big(\frac{r-2}{2}\Big) - 2 + (l-1)(-2) \\
&= \left(\binom{r}{2} - 3\right)l + \binom{r-1}{2}.
\end{aligned}$$

- $c \neq \overline{a}$ *and* $|R_{LF}(c)| = 1$: Since the degree of c is $r-1$, by (1) we obtain

$$\sum_{u\in R_{LF}(c)} h_{LF}(u) = h_{LF}(c) = \left(\binom{r-1}{2} - 1\right)d(c,a) + (r-2)\Big(\frac{r-3}{2}\Big) - 2.$$

- $c \neq \overline{a}$ *and* $|R_{LF}(c)| \geq 2$: Then $R_{LF}(c)$ has one vertex of degree r, namely c, one vertex of degree 1 at distance at most $d(c,a)+l+1$ from a and at most l vertices of degree 2 since all rays of LF have length at most $l+1$. By (1) it follows that

$$\begin{aligned}\sum_{u \in R_{LF}(c)} h_{LF}(u) &\geq \left(\binom{r}{2} - 1\right) d(c,a) + (r-1)\binom{r-2}{2} - 2 \\ &\quad - (d(c,a)+l+1) - 2 + l(-2) \\ &= \left(\binom{r}{2} - 2\right) d(c,a) - 3l + \binom{r-1}{2} - 5.\end{aligned}$$

□

Before we state the lemmas necessary for the basis of induction, we give the proof of induction step. That is, we prove that if $\Delta T^h \geq 0$ for every tree T^h homeomorphic to T rays of which have lengths at most $l+2$, then $\Delta T^{gh} \geq 0$ for all trees T^{gh} homeomorphic to T^g rays of which have lengths at most $l+2$, where T^g is obtained from T by inserting a star at the end of one ray of T (of course, we cannot attach this star to a').

Let R' be a ray of T which does not terminate at a'. Remove R' from T and replace it by a path PR' of length i, $1 \leq i \leq 2$. Denote by c' the vertex of degree 1 in PR'. Now attach to c' exactly $j-1$ rays, each of length at most $l+2$, and denote the resulting graph by $T_{i,j}$, $j \geq 3$. In the next two lemmas we prove that $\Delta T_{i,j} \geq 0$.

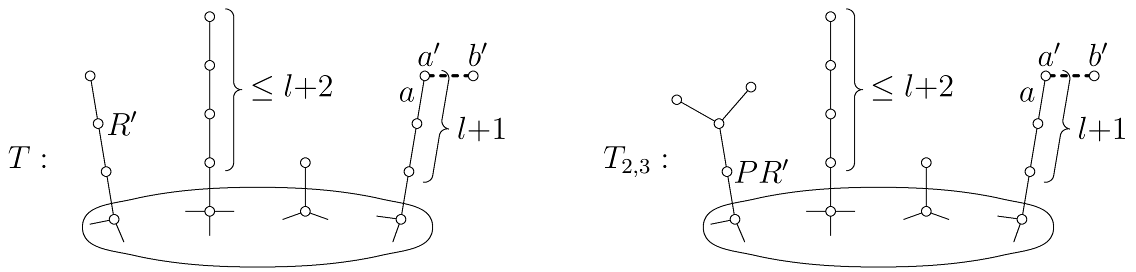

Figure 2: The trees T and $T_{2,3}$ in the case $l=2$.

Lemma 3.5 *Suppose that $\Delta T^h \geq 0$ for all trees T^h homeomorphic to T rays of which have lengths at most $l+2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l+1$. Then $\Delta T_{i,3} \geq 0$ for all $i \in \{1,2\}$.*

Proof Since $\Delta T^h \geq 0$ for all trees homeomorphic to T rays of which have length at most $l+2$, we may assume that the length of R' is exactly $l+2$, $l \geq 1$. Then the edges of R' form a ray R in LT of length $l+1$. Denote by e the first vertex of R. By (1) it follows that

$$\sum_{u \in R_{LT}(e) \setminus \{e\}} h_{LT}(u) = -2l - (d(e,a) + l + 1) - 2 = -d(e,a) - 3l - 3$$

since $R_{LT}(e)$ has l vertices of degree 2 and one vertex of degree 1 at distance $d(e,a)+l+1$ from a. We distinguish two cases.

- $i = 1$: Then PR' has length 1 and the unique edge of PR' corresponds to the vertex e in $LT_{1,3}$. In $LT_{1,3}$ the degree of e is $d_e + 3 - 2 = d_e + 1$ since e is in two cliques from $\mathcal{C}(LT_{1,3})$, one of them has order d_e and the other one has order 3. Denote by c any one of the other two vertices of this clique of order 3. Since $d(c,a) \geq l+2$, we see that $d(c,a) - 3l - 4 \geq -2l - 2$. Hence, by Lemma 3.4,

  $$\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq \begin{cases} -2 & \text{if } |R_{LT_{1,3}}(c)| = 1 \\ -2l - 2 & \text{if } |R_{LT_{1,3}}(c)| \geq 2. \end{cases}$$

  Since $-2l - 2 \leq -2$, we have $\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq -2l - 2$.

  Denote
  $$\Delta h = \sum_{u \in V(LT_{1,3}) \setminus \{a\}} h_{LT_{1,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).$$

  In Δh all terms cancel out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing e, the vertex e itself, and the vertices of $R_{LT}(e) \setminus \{e\}$. By (1) it follows that

  $$\begin{aligned} \Delta h &\geq 2(-2l-2) + \left(\binom{d_e+1}{2} - 1\right) d(e,a) + d_e\left(\frac{d_e+1}{2} - 1\right) - 2 \\ &\quad - \left(\binom{d_e}{2} - 1\right) d(e,a) - (d_e - 1)\left(\frac{d_e}{2} - 1\right) + 2 + (d(e,a) + 3l + 3) \\ &\geq (d_e+1)d(e,a) + (d_e - 1) - l - 1 \\ &\geq 4d(e,a) - l + 1 \geq 0 \end{aligned}$$

  since $d_e \geq 3$ and $d(e,a) \geq l+1$. By Proposition 2.2, $\Delta T_{1,3} - \Delta T = \Delta h \geq 0$, implying that $\Delta T_{1,3} \geq \Delta T \geq 0$.

- $i = 2$: Then PR' has length 2. One edge of PR' corresponds to e, while the other corresponds to a vertex of degree 3, say f, in $LT_{2,3}$. Observe that the degree of e is d_e in $LT_{2,3}$ and the degree of f is 3 in $LT_{2,3}$. Analogously as in the previous case, denote by c any one of the two vertices of the triangle containing

f, $c \neq f$. Since $d(c,a) = d(e,a)+2 \geq l+3$, we see that $d(c,a)-3l-4 \geq -2l-1$. Hence, by Lemma 3.4

$$\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq \begin{cases} -2 & \text{if } |R_{LT_{2,3}}(c)| = 1 \\ -2l-1 & \text{if } |R_{LT_{2,3}}(c)| \geq 2. \end{cases}$$

Since $l \geq 1$ it follows that $-2l-1 \leq -2$, implying that $\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq -2l-1$. Denote

$$\Delta h = \sum_{u \in V(LT_{2,3}) \setminus \{a\}} h_{LT_{2,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).$$

In Δh all terms cancell out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing f, the vertex f itself, and the vertices of $R_{LT}(e) \setminus \{e\}$. By (1) it follows that

$$\begin{aligned} \Delta h &\geq 2(-2l-1) + (2d(f,a)-1) + (d(e,a)+3l+3) \\ &\geq 3d(e,a) - l + 2 \geq 0 \end{aligned}$$

since $d(f,a) = d(e,a)+1$ and $d(e,a) \geq l+1$. By Proposition 2.2, $\Delta T_{2,3} - \Delta T = \Delta h \geq 0$, implying that $\Delta T_{2,3} \geq \Delta T \geq 0$.

In both cases the inequality $\Delta T_{i,3} \geq 0$ holds, which completes the proof. □

Now we extend the previous lemma to trees $T_{i,j}$ with higher j.

Lemma 3.6 *Suppose that $\Delta T^h \geq 0$ for all trees T^h homeomorphic to T rays of which have lengths at most $l+2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l+1$. Then $\Delta T_{i,j} \geq 0$ for all $j \geq 4$ and $i \in \{1,2\}$.*

Proof We use the notation of the proof of Lemma 3.5. Analogously as in the proof of Lemma 3.5, assume that the length of R' is $l+2$, $l \geq 1$. Then again

$$\sum_{u \in R_{LT}(e) \setminus \{e\}} h_{LT}(u) = -d(e,a) - 3l - 3.$$

Let c be one of the $j-1$ vertices of the clique of order j obtained from the edges incident to c', other than e (in the case $i=1$) or f (in the case $i=2$). By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT_{i,j}}(c)} h_{LT_{i,j}}(u) \geq \begin{cases} \left(\binom{j-1}{2} - 1\right) d(c,a) + \binom{j-2}{2} - 2 & \text{if } |R_{LT_{i,j}}(c)| = 1 \\ \left(\binom{j}{2} - 2\right) d(c,a) - 3l + \binom{j-1}{2} - 5 & \text{if } |R_{LT_{i,j}}(c)| \geq 2. \end{cases}$$

Since $j \geq 4$ and $d(c,a) \geq l+2 \geq 3$, in any case we have $\sum_{u \in R_{LT_{i,j}}(c)} h_{LT_{i,j}}(u) \geq 0$. Now if $i = 1$, then $h_{LT_{i,j}}(e) - h_{LT}(e) \geq 0$ since the degree of e is $d_e + j - 2$ in $T_{i,j}$; see (1). On the other hand, if $i = 2$, then $h_{LT_{i,j}}(e) = h_{LT}(e)$ while $h_{LT_{i,j}}(f) \geq 0$, since the degree of f is $j \geq 4$ in $T_{i,j}$. Hence

$$\Delta h = \sum_{u \in V(LT_{i,j}) \setminus \{a\}} h_{LT_{i,j}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u) \geq (j-1) \cdot 0 + 0 + d(e,a) + 3l + 3 \geq 0.$$

By Proposition 2.2, $\Delta T_{i,j} - \Delta T = \Delta h \geq 0$, showing that $\Delta T_{i,j} \geq \Delta T \geq 0$. $\square$

Now we prove $\Delta T \geq 0$ for the basis of induction. In all graphs in this basis, a' is an endvertex of a ray of length $l+1$ and a is the edge incident with a'.

Lemma 3.7 *Let T be a tree homeomorphic to a star $K_{1,k}$, $k \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l+1$. Then $\Delta T \geq 0$.*

Proof Here $|S_{LT}| = k$ and $\cup_{u \in S_{LT}} R_{LT}(u) = V(LT) \setminus \{a\}$ where $R_{LT}(u) \cap R_{LT}(v) = \emptyset$ if $u \neq v$. Thus $\sum_u h_{LT}(u) = \sum_{c \in S_{LT}} (\sum_{u \in R_{LT}(c)} h_{LT}(u))$. We prove that $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 1$.

Choose $c \in S_{LT}$. By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq \begin{cases} \left(\binom{k}{2} - 3\right) l + \binom{k-1}{2} & \text{if } c = \overline{a} \\ \left(\binom{k-1}{2} - 1\right) d(c,a) + \binom{k-2}{2} - 2 & \text{if } c \neq \overline{a} \text{ and } |R_{LT}(c)| = 1 \\ \left(\binom{k}{2} - 2\right) d(c,a) - 3l + \binom{k-1}{2} - 5 & \text{if } c \neq \overline{a} \text{ and } |R_{LT}(c)| \geq 2. \end{cases}$$

Since $d(c,a) = l+1$ in the last two cases, $k \geq 4$ and $l \geq 1$, in all three cases we conclude $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 1$

We have $\sum_u h_{LT}(u) = \sum_{c \in S_{LT}} (\sum_{u \in R_{LT}(c)} h_{LT}(u)) \geq k \cdot 1 \geq 4$. Since $d_a = 1$, we se that $\Delta T = \sum_u h_{LT}(u) - 3$ by Proposition 2.2, implying that $\Delta T \geq 0$. $\square$

Denote by $H_{i,j}$ a tree having $i+j$ vertices, $i,j \geq 3$. One of these vertices has degree i, another one has degree j and the remaining $i+j-2$ vertices have degrees 1. Obviously, the vertices of degrees i and j must be adjacent in $H_{i,j}$ and $H = H_{3,3}$.

Lemma 3.8 *Let T be a tree homeomorphic to $H_{3,j}$, $j \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l+1$. Suppose that the interior path of $H_{3,j}$ has length at most 2 and moreover suppose that the first vertex of the ray terminating in a has degree 3. Then $\Delta T \geq 0$.*

Proof Denote $e = \overline{a}$ and let P' be the unique interior path of T. If P' has length 1, then the unique vertex of LP' (denote it by v) has degree $3 + j - 2 \geq 5$, while if P' has length 2, then one of the vertices of LP' has degree 3 and the other (denote it by v) has degree $j \geq 4$. Since by (1), $h_{LT}(u) \geq 0$ if $d_u \geq 3$ and $h_{LT}(u) \geq 5d(u,a)+1$ if $d_u \geq 4$, the vertices of LP' contribute to $\sum_{u \in V(LT)\setminus\{a\}} h_{LT}(u)$ by at least $5d(v,a) + 1 \geq 5l + 6$ since $d(v,a) \geq l + 1$.

Denote by c any one of the $j-1$ vertices of the clique of order j from $\mathcal{C}(H_{3,j})$, which is not in LP'. By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq \left\{ \begin{array}{ll} \left(\binom{j-1}{2} - 1\right) d(c,a) + \binom{j-2}{2} - 2 & \text{if } |R_{LT}(c)| = 1 \\ \left(\binom{j}{2} - 2\right) d(c,a) - 3l + \binom{j-1}{2} - 5 & \text{if } |R_{LT}(c)| \geq 2. \end{array} \right.$$

Since $j \geq 4$ and $d(c,a) \geq l + 2 \geq 3$, in any case we have $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 0$.

Now consider the rays attached to the clique of order 3 from $\mathcal{C}(H_{3,j})$. By Lemma 3.4

$$\sum_{u \in R_{LT}(e)} h_{LT}(u) = \left(\binom{3}{2} - 3\right) l + \binom{3-1}{2} = 1.$$

Denote by f that vertex of the clique of order 3 from $\mathcal{C}(H_{3,j})$ which is different from e and which is not in LP'. By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(f)} h_{LT}(u) \geq \left\{ \begin{array}{ll} -2 & \text{if } |R_{LT}(f)| = 1 \\ d(f,a) - 3l - 4 & \text{if } |R_{LT}(f)| \geq 2. \end{array} \right.$$

Since $d(f,a) = l + 1$ and $l \geq 1$, in any case we have $\sum_{u \in R_{LT}(f)} h_{LT}(u) \geq -2l - 3$.

Now summing up the inequalities above we obtain

$$\sum_u h_{LT}(u) \geq (5l + 6) + (j-1) \cdot 0 + 1 + (-2l - 3) = 3l + 4 \geq 3.$$

Since $d_a = 1$, we have $\Delta T = \sum_u h_{LT}(u) - 3$ by Proposition 2.2, showing that $\Delta T \geq 0$.

$\square$

Denote by $Y_{i,j}$, $1 \leq i, j \leq 2$, a tree having three vertices of degree 3, namely y'_1, y'_2 and y'_3. All the other vertices of $Y_{i,j}$ have degree at most 2. There are two interior paths in $Y_{i,j}$, namely $y'_1 - y'_2$ and $y'_2 - y'_3$, and their lengths are i and j, respectively. Moreover, there are five rays in $Y_{i,j}$. Two such rays start at y'_1, one starts at y'_2 and two start at y'_3. Of course, one of these rays has length exactly $l + 1$ and it terminates in a'.

Lemma 3.9 *Let T be the tree $Y_{i,j}$, $1 \leq i, j \leq 2$, in which all rays have lengths at most $l + 2$, $l \geq 1$. Then $\Delta T \geq 0$.*

Proof Denote by x_1, x_2, x_3, x_4 and x_5 the five vertices of S_{LT} corresponding to the first edges of rays starting at y'_1, y'_1, y'_2, y'_3 and y'_3, respectively. Since the degrees of y'_1, y'_2 and y'_3 are 3 in T, all $x_1, x_2, \ldots, x_5$ are vertices of cliques of order 3 in LT. Let $x_t = \overline{a}$, $1 \le t \le 5$. By Lemma 3.4

$$\sum_{u \in R_{LT}(x_t)} h_{LT}(u) = 1.$$

For all other x_r, $1 \le r \le 5$ and $r \ne t$, by Lemma 3.4 we obtain

$$\sum_{u \in R_{LT}(x_r)} h_{LT}(u) \ge \min\{-2, d(x_r, a) - 3l - 4\}.$$

Since $l \ge 1$, this minimum equals $d(x_r, a) - 3l - 4$ if $d(x_r, a) \le l+4$. If $d(x_r, a) = l+5$ then $\sum_{u \in R_{LT}(x_r)} h_{LT}(u) \ge \min\{-2, -2l+1\} \ge -2l$.

Now we consider vertices corresponding to the edges of interior paths. If such a path has length 1, then its unique edge corresponds to a vertex, say e, the degree of which is 4 in LT. By (1) it follows that

$$h_{LT}(e) = 5d(e, a) + 1.$$

On the other hand if such a path has length 2, then its edges correspond to two vetices, say e and f, both of degree 3. Suppose that e is closer to a than f. By (1) it follows that

$$h_{LT}(e) + h_{LT}(f) = 2d(e, a) - 1 + 2d(f, a) - 1 = 4d(e, a).$$

In what follows, we list contributions to $\sum_u h_{LT}(u)$ first by vertices of rays starting at $x_1, x_2, \ldots, x_5$ and then by the vertices corresponding to edges of paths $y'_1 - y'_2$ and $y'_2 - y'_3$. By symmetry, there are two cases to consider. First, suppose that $t = 1$, i.e., $\overline{a} = x_1$. We distinguish 4 subcases.

- $i = j = 1$: Then $d(x_2, a) = l+1$, $d(x_3, a) = l+2$ and $d(x_4, a) = d(x_5, a) = l+3$. Since $l \ge 1$, we see that

$$\sum_u h_{LT}(u) \ge 1+(-2l-3)+(-2l-2)+2(-2l-1)+(5l+6)+(5l+11) \ge 2l+11 \ge 3.$$

- $i = 1$ *and* $j = 2$: Analogously as above we obtain

$$\sum_u h_{LT}(u) \ge 1+(-2l-3)+(-2l-2)+2(-2l)+(5l+6)+(4l+8) \ge l+10 \ge 3.$$

- $i = 2$ *and* $j = 1$: Then

$$\sum_u h_{LT}(u) \ge 1+(-2l-3)+(-2l-1)+2(-2l)+(4l+4)+(5l+16) \ge l+17 \ge 3.$$

- $i = j = 2$: Here $d(x_4, a) = d(x_5, a) = l + 5$. Hence

$$\sum_u h_{LT}(u) \geq 1 + (-2l-3) + (-2l-1) + 2(-2l) + (4l+4) + (4l+12) \geq 13 \geq 3.$$

Now suppose that $t = 3$, i.e., $\overline{a} = x_3$. By symmetry, it suffices to consider 3 subcases.

- $i = j = 1$: Then $d(x_1, a) = d(x_2, a) = l + 2$ and also $d(x_4, a) = d(x_5, a) = l + 2$. Since $l \geq 1$, we see that

$$\sum_u h_{LT}(u) \geq 2(-2l-2) + 1 + 2(-2l-2) + (5l+6) + (5l+6) \geq 2l + 5 \geq 3.$$

- $i = 1$ *and* $j = 2$: Then the following holds

$$\sum_u h_{LT}(u) \geq 2(-2l-2) + 1 + 2(-2l-1) + (5l+6) + (4l+4) \geq l + 5 \geq 3.$$

- $i = j = 2$: Then

$$\sum_u h_{LT}(u) \geq 2(-2l-1) + 1 + 2(-2l-1) + (4l+4) + (4l+4) \geq 5 \geq 3.$$

Since $\Delta T = \sum_u h_{LT}(u) - 3$ by Proposition 2.2, we conclude that $\Delta T \geq 0$. □

Now we prove $\Delta T \geq 0$ for the last graph of the basis of induction. Denote by X_k, $k \geq 4$, the tree having two vertices of degree 3, namely y'_1 and y'_2, and one vertex of degree k, namely y'_3. All other vertices of X_k have degree at most 2. There are two interior paths in X_k, namely $y'_1 - y'_2$ and $y'_2 - y'_3$, both of length at most 2. Moreover, there are $k + 2$ rays in X_k. Two such rays start at y'_1, one starts at y'_2 and the remaining $k - 1$ start at y'_3.

Lemma 3.10 *Let T be the tree X_k, $k \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$. Suppose that the ray having length $l+1$ and terminating at a' starts at y'_1. Then $\Delta T \geq 0$.*

Proof We use the notation of the proof of Lemma 3.9. Denote by $x_1, x_2, x_3, x_4, \ldots x_{k+2}$ the $k+2$ vertices of S_{LT} corresponding to the first edges of rays starting at $y'_1, y'_1, y'_2, y'_3, \ldots, y'_3$, respectively. The vertices x_1, x_2 and x_3 are in cliques of order 3, while $x_4, \ldots, x_{k+2}$ are in the clique of order k. Assume that $\overline{a} = x_1$. As shown in the proof of Lemma 3.9, we have $\sum_{u \in R_{LT}(x_1)} h_{LT}(u) = 1$. Further, $\sum_{u \in R_{LT}(x_2)} h_{LT}(u) \geq -2l - 3$ since $d(x_2, a) = l + 1$. The vertices corresponding to edges of $y'_1 - y'_2$ path contribute to $\sum_u h_{LT}(u)$ by at least $\min\{5d(e,a)+1, 4d(e,a)\} = 4d(e,a) = 4l+4$ as $d(e,a) = l+1$. Finally, $\sum_{u\in R_{LT}(x_3)} h_{LT}(u) \geq \min\{-2, d(x_3,a) - 3l - 4\} \geq -2l-2$ as $d(x_3,a) \geq l+2$.

Since the vertices corresponding to edges of $y_2' - y_3'$ path have degree $k+1$ (in the case when the length of $y_2' - y_3'$ is 1) or 3 and k (in the case when the length of $y_2' - y_3'$ is 2), and since $h_{LT}(u) \geq 0$ if $d_u \geq 3$ by (1), the contribution of these vertices to $\sum_u h_{LT}(u)$ is nonnegative.

Finally, consider $\sum_{u\in R_{LT}(x_i)} h_{LT}(u)$ when $i \geq 4$. By Lemma 3.4 it follows that

$$\sum_{u\in R_{LT}(x_i)} h_{LT}(u) \geq \begin{cases} \left(\binom{k-1}{2} - 1\right) d(x_i,a) + \binom{k-2}{2} - 2 & \text{if } |R_{LT}(x_i)| = 1 \\ \left(\binom{k}{2} - 2\right) d(x_i,a) - 3l + \binom{k-1}{2} - 5 & \text{if } |R_{LT}(x_i)| \geq 2. \end{cases}$$

Since $d(x_i,a) \geq l+3$, $k \geq 4$ and $l \geq 1$, we obtain $\sum_{u\in R_{LT}(x_i)} h_{LT}(u) \geq \min\{7, 11\} = 7$.

Summing up these inequalities we obtain

$$\sum_u h_{LT}(u) \geq 1 + (-2l-3) + (4l+4) + (-2l-2) + 0 + (k-1)7 = 7k - 7 \geq 3.$$

Since $d_a = 1$, by Proposition 2.2 we conclude that $\Delta T = \sum_u h_{LT}(u) - 3 \geq 0$. $\square$

Now we summarize the proof of Theorem 1.5

Proof of Theorem 1.5 Let T be a tree, not homeomorphic to a path, claw $K_{1,3}$ and H. We prove that $D(T) = W(L^3(T)) - W(T) > 0$. Denote by $l+2$, $l \geq -1$, the length of a longest ray in T. If $l = -1$ then $D(T) = W(L^3(T)) - W(T) > 0$ by Theorem 1.4.

Suppose that $l \geq 0$ and suppose that the statement of the theorem is true for all trees (not homeomorphic to a path, claw $K_{1,3}$ and H) rays of which have lengths at most $l+1$. Let $R_1', R_2', \ldots, R_t'$ be rays of T having length $l+2$. Further, denote by c_i' the first vertex of R_j', denote by b_i' its last vertex and denote by a_i' the neighbour of b_i' in T, $1 \leq i \leq t$. Finally, denote by T_i a tree obtained from T by removing the vertices $b_{i+1}', b_{i+2}', \ldots, b_t'$ and edges $a_{i+1}'b_{i+1}', a_{i+2}'b_{i+2}', \ldots, a_t'b_t'$, $0 \leq i \leq t$. Then $T_t = T$ and T_0 is a tree rays of which have length at most $l+1$. By induction we have $D(T_0) = W(L^3(T_0)) - W(T_0) > 0$. Let $\Delta T_i = D(T_{i+1}) - D(T_i)$, $0 \leq i \leq t-1$.

Suppose that $l = 0$. All rays of T_i have length at most $l+2$, and the ray R_{i+1}' terminating at a_{i+1}' has length $l+1$. Moreover, T_{i+1} is obtained from T_i by adding the vertex b_{i+1}' and the edge $a_{i+1}'b_{i+1}'$. Hence $\Delta T_i \geq 0$ by Lemma 3.2, $0 \leq i \leq t-1$, where the vertex a_{i+1}' and the tree T_i play the role of a and T, respectively. Consequently

$\sum_{i=0}^{t-1} \Delta T_i \geq 0$. Since

$$0 \leq \sum_{i=0}^{t-1} \Delta T_i = D(T_t) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)],$$

we have $W(L^3(T)) - W(T) \geq W(L^3(T_0)) - W(T_0) > 0$.

Now suppose that $l \geq 1$. In T_i shorten all interior paths of length at least 3 to paths of length 2, and denote the resulting graph by T_i^-. Analogously as T_{i+1} is obtained from T_i, the tree T_{i+1}^- is obtained from T_i^- by adding the vertex b'_{i+1} and the edge $a'_{i+1}b'_{i+1}$. We prove that $\Delta T_i^- = D(T_{i+1}^-) - D(T_i^-) \geq 0$ by induction on the number of vertices of degree at least 3. Observe that T_i^-, as welll as T_i, is a tree, rays of which have length at most $l+2$ and the ray terminating at a'_{i+1} has length $l+1$, $0 \leq i \leq t-1$.

Denote by V_i^3 the set of vertices of degree at least 3 in T_i^-. We distinguish four cases.

- $|V_i^3| = 1$: Then T_i^- is homeomorphic to $K_{1,k}$. Since T is not homeomorphic to $K_{1,3}$, it follows that $k \geq 4$. By Lemma 3.7 we have $\Delta T_i^- \geq 0$.

- $|V_i^3| = 2$: If the degree of c'_{i+1} is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.8, since T is not homeomorphic to $H = H_{3,3}$. On the other hand if the degree of c'_{i+1} is $k \geq 4$, then denote by c'' the other vertex of V_i^3. Remove the rays starting at c'' from T_i^-, and denote the resulting graph by T''. Then T'' is a tree, rays of which have length at most $l+2$, and T'' is homeomorphic to $K_{1,k}$. By Lemma 3.7 we have $\Delta T'' \geq 0$. If the degree of c'' is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.5, while if the degree of c'' is at least 4 then $\Delta T_i^- \geq 0$ by Lemma 3.6.

- $|V_i^3| = 3$: Denote by T^* a graph obtained from T_i^- by removing the edges of all rays. Since T^* is a tree, it has at least two vertices of degree 1. (We remark that in this case T^* is a path.) Denote by c'' a pendant vertex in T^*, $c'' \neq c'_{i+1}$, the degree of which is the smallest possible in T_i^-. Finally, denote by T'' a tree obtained from T_i^- by removing all rays starting at c''. We distinguish two subcases.

  - *T'' is homeomorphic to H*: If the degree of c'' is 3 in T then $\Delta T_i^- \geq 0$ by Lemma 3.9. Suppose that the degree of c'' is $k \geq 4$. By the choice of c'', the vertex c'_{i+1} is a leaf of T^*. Hence T is X_k and c'_{i+1} is the vertex y'_1 in the notation of Lemma 3.10. Therefore $\Delta T_i^- \geq 0$ by Lemma 3.10.

  - *T'' is homeomorphic to $H_{i,j}$, $i \leq j$ and $j \geq 4$*: Since T'' is not homeomorphic to H, it follows that $\Delta T'' \geq 0$ by the previous case (the case $|V_i^3| = 2$). If the degree of c'' is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.5; while if the degree of c'' is at least 4, then $\Delta T_i^- \geq 0$ by Lemma 3.6.

Thus we proved $\Delta T_i^- \geq 0$ for every tree T_i^- rays of which have length at most $l+2$ and $|V_i^3| = 3$.

- $|V_i^3| \geq 4$: Analogously as in the previous case, denote by T'' a tree obtained from T_i^- by removing all rays starting at a pendant vertex c'' of T^*, $c'' \neq c'_{i+1}$. By induction we assume that $\Delta T'' \geq 0$. If the degree of c'' is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.5, while if the degree of c'' is at least 4, then $\Delta T_i^- \geq 0$ by Lemma 3.6.

Hence, in any case we have $\Delta T_i^- \geq 0$. If $T_i^- = T_i$, then it follows that also $\Delta T_i \geq 0$. Otherwise form a sequence $T_i^- = F_0, F_1, \ldots, F_r = T_i$ such that F_{j+1} is obtained from F_j by subdividing one edge of one interior path, $0 \leq j \leq r-1$. By Lemma 3.3 we have $\Delta F_{j+1} - \Delta F_j \geq 0$. Hence $\sum_{j=0}^{r-1}(\Delta F_{j+1} - \Delta F_j) \geq 0$. Since

$$0 \leq \sum_{j=0}^{r-1}(\Delta F_{j+1} - \Delta F_j) = \Delta T_i - \Delta T_i^-,$$

we see that $\Delta T_i \geq \Delta T_i^- \geq 0$.

Thus we proved that $\Delta T_i \geq 0$ for every $i \in \{0, 1, \ldots, t-1\}$. Hence $\sum_{i=0}^{t-1} \Delta T_i \geq 0$. Since

$$0 \leq \sum_{i=0}^{t-1} \Delta T_i = D(T_t) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)],$$

we conclude that $W(L^3(T)) - W(T) \geq W(L^3(T_0)) - W(T_0) > 0$. $\square$

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