Two Extensions of Kotzig's Thm

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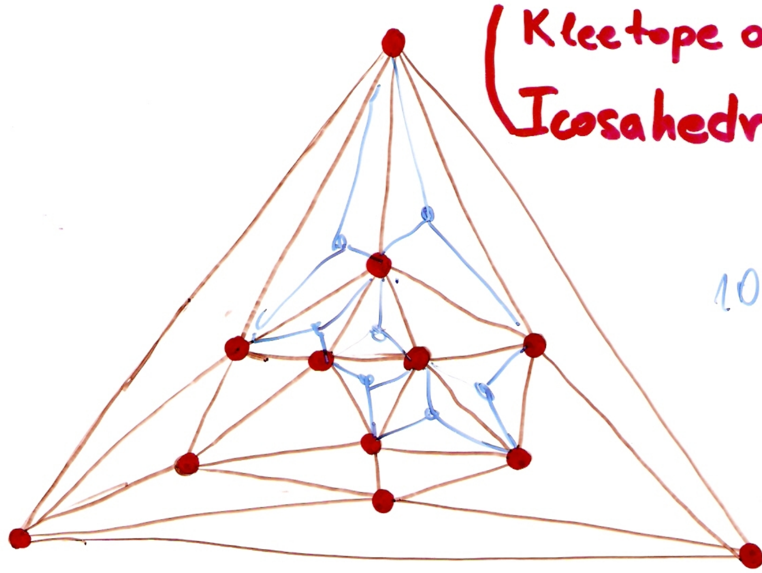
- Thm (Kotzig, 55) Every 3-connected planar graph G contains an edge of weight ≤ 13.
  (Moreover, if $\delta(G) \geq 4$ then G contains an edge of weight ≤ 11.)

- Bound 13 is sharp:

(Kleetope over Icosahedron)



$10 + 3 = 13$

- Bound 11 is sharp:
  (The DUAL OF ANY FULLEREN is "FINE".)

- An edge from Kotzig's Thm is called <u>light</u>

- **EXTENSIONS:**

1. (B. Grünbaum & G. Shephard, 82)
   Kotzig's Thm for Torus, bound ≤ 15

2. (J. Zaks, 82) Some bounds for higher surfaces

3. (Borodin, 89) 3-connectivity of G REPLACE BY minimum degree 3.

- **IMPACT:**

"LIGHT GRAPH THEORY"

- Graph H is light for class $\mathcal{H}$ if
  $\forall G \in \mathcal{H}$ which contains H, also contains copy of H with bounded weight.

- Thus, K_1, K_2 are light for 3-connected planar graphs

- (Fabrici, Jendrol 97) The paths are only light graphs for 3-connected planar graphs.

- WHY INTERESTING!

- Thm (Steinitz, 22) The skeletons of 3-polytopes are precisely 3-connected PLANAR graphs.

Kotzig's Thm Every polyhidron contains an edge of weight ≤ 13.

3-CONNECTIVITY

Thm (Tutte 61) Every 3-connected graph $\neq K_4$, contains a contractible edge.

Cor. Graph G is 3-connected iff exists $G_0, G_1, \dots, G_n$ such that:

ⓐ $G_0 = K_4$ and $G_n = G$

ⓑ $\forall i$: G_{i+1} has an edge xy with $d(x) \geq 3$, $d(y) \geq 3$ and $G_i = G_{i+1}/xy$

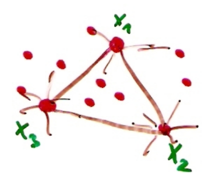
Cor. Polyhedras are precisely the graphs constructed by "vertex-splittings" of $\underbrace{\text{Tetrahedron}}_{\cong K_4}$.

Thm (Z. Dvorak, R. Š.) Every planar 3-connected graph $\neq K_4$ contains a contractible + light edge.

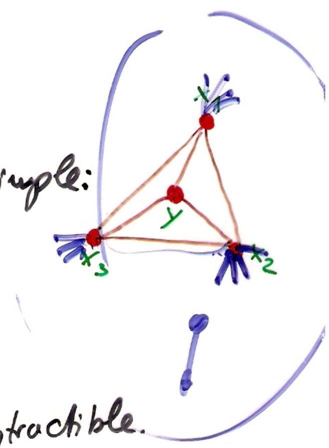
Cor. Polyhidras are precisely the graphs construted from Tetrahidron by splittings vertices of degree ≤ 11.

Proof of Thm: By contradiction, G is counterexample with min. number of vertices ≥ 5 + max. number of edges

Main Property Every non-contractible edge lies on separating triangle



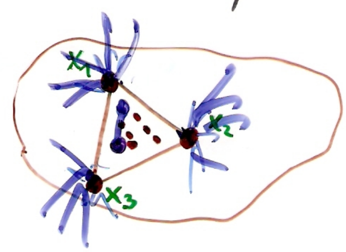
CASE 1: Every separating triangle is simple:

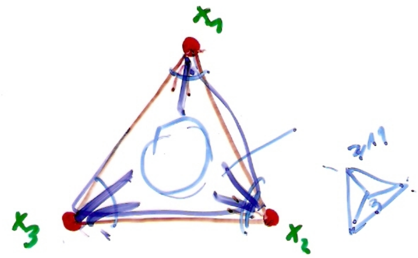


- yx_i contractibe $\Rightarrow d(x_i) \geq 11$
- no x_i is on light edge

$\Rightarrow$ Every light edge of G is contractible.

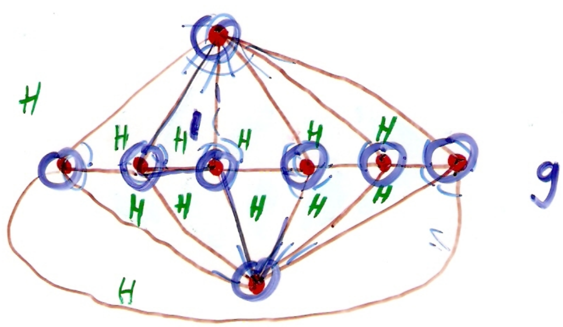
CASE 2: G has non-simple separating triangle $S = x_1 x_2 x_3$





$H = In(S)$

- Choose S that H is smallest
- We assume $d(x_1) \geq 4$, $d(x_2) \geq 4$

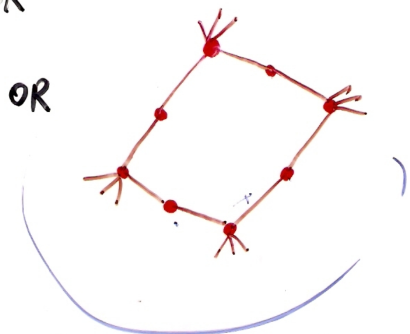


- H has light edge e non-incident with x_1, x_2, x_3
- e is contractible (as in previous case)

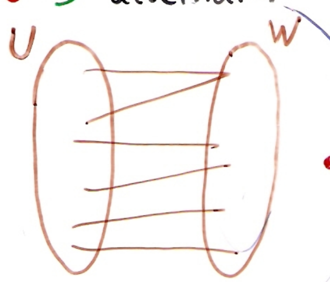
• PLANAR GRAPHS OF MIN. DEGREE 2

Thm (Borodin, Kostochka, Woodall) Every planar graph with minimum degree ≥ 2 contains

- light edge OR
- 2-alterating cycle OR



- 3-alternator



- $\forall x \in U$: $2 \leq d_F(x) = d_G(x) \leq 3$
- $\forall x \in W$: $d_F(x) \geq 3$ OR

  x has two neighbors in U, and both of them are of degree $14 - d_G(x)$

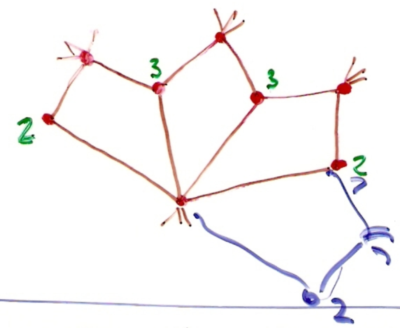
Corl. Every planar graph of max. degree $\Delta \geq 12$ is Δ-edge-list-colorable.

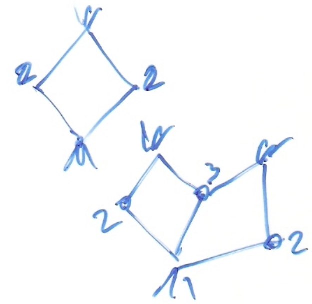
$\Delta \leq \chi' \leq \Delta + 1$

Thm (R. Cole, L. Kowalik, R. Š.) Every planar graph with minimum degree ≥ 2 contains

- light edge $=13$ OR
- k-crown, k = 1, 2, 3, 4, 5.

3-crown:





Corl ∃ linear-time algorithm for edge-list-coloring with lists of size $\max\{\Delta, 12\}$.

$O(m \cdot \Delta)$