

# Light subgraphs of graphs embedded in the plane and in the projective plane – a survey –

S. Jendrol' <sup>1)</sup> and H.-J. Voss <sup>2)†</sup>

<sup>1)</sup> Institute of Mathematics, Faculty of Science,  
P.J. Šafárik University,  
Jesenná 5, SK-040 01 Košice, Slovakia  
e-mail: stanislav.jendrol@upjs.sk

<sup>2)</sup> Department of Algebra,  
Technical University Dresden,  
Mommsenstrasse 13. D-01062 Dresden, Germany

**Abstract.** It is well known that every planar graph contains a vertex of degree at most 5. A theorem of Kotzig states that every 3-connected planar graph contains an edge whose endvertices have degree-sum at most 13. Recently, Fabrici and Jendrol' proved that every 3-connected planar graph  $G$  that contains a  $k$ -vertex path, a path on  $k$  vertices, contains also a  $k$ -vertex path  $P$  such that every vertex of  $P$  has degree at most  $5k$ . A result by Enomoto and Ota says that every 3-connected planar graph  $G$  of order at least  $k$  contains a connected subgraph  $H$  of order  $k$  such that the degree sum of vertices of  $H$  in  $G$  is at most  $8k - 1$ . Motivated by these results, a concept of light graphs has been introduced. A graph  $H$  is said to be *light* in a family  $\mathcal{G}$  of graphs if at least one member of  $\mathcal{G}$  contains a copy of  $H$  and there is an integer  $w(H, \mathcal{G})$  such that each member  $G$  of  $\mathcal{G}$  with a copy of  $H$  also has a copy  $K$  of  $H$  with degree sum  $\sum_{v \in V(K)} \deg_G(v) \leq w(H, \mathcal{G})$ .

In this paper we present a survey of results on light graphs in different families of plane and projective plane graphs and multigraphs. A similar survey dealing with the family of all graphs embedded in surfaces other than the sphere and the projective plane was prepared as well.

† Professor Voss passed away on September 7, 2003.

*2000 Mathematical Subject Classification 2000:* Primary: 05C10 Secondary: 52B10, 05C38, 57N05.

*Keywords:* vertex colouring, rainbowness, plane graphs

## Content

1. Introduction
  2. Notation and preliminaries
  3. Light edges
  4. Light subgraphs of order three
  5. Subgraphs with restricted degrees
  6. Maximum degree problems
  7. Maximum degree of light families
  8. Weight problems
  9. Light subgraphs of graphs embedded on surfaces
  10. Related topics
- References

## 1 Introduction

The study of the structure of plane graphs (i.e. planar graphs embedded in the plane without edge crossings) has its origin in the time of L. Euler. It is connected with the result (*Euler's polyhedron formula* discovered in 1750) stating that in a convex polyhedron with  $n$  vertices,  $e$  edges and  $f$  faces,  $n - e + f = 2$ . It was apparently discovered by Euler but first proved by Legendre (see, e.g. [5], [107]). The graph theory version of the formula is expressed in

**Theorem 1.1** (Euler's Polyhedron Formula). *For a plane connected graph with  $n$  vertices,  $e$  edges and  $f$  faces,*

$$n - e + f = 2. \tag{1}$$

□

Euler's formula was not systematically exploited to any extent until the late nineteenth century. Only then a renaissance of interest in metric and combinatorial properties of solids started. A renewed interest in geometry and combinatorics of convex solids culminated in the book by Steinitz and Rademacher [124] published in 1934. This book presented an extraordinary

result, known as *Steinitz' Theorem*. Unfortunately, the theorem was couched in such an archaic language that it was not appreciated for many years. Only the reformulation of the theorem by Grünbaum in 1963 released a torrent of results. This resulted in a cross-fertilization of geometry with both graph theory and combinatorics, with benefits to all three areas. To be able to formulate this, by our opinion, "*Fundamental Theorem on Polyhedral Graph Theory*" in modern terminology we need two definitions. The *graph of a polyhedron*  $P$  is the graph consisting of the vertices and edges of  $P$ . A graph  $G$  is *polyhedral* if it is isomorphic to the graph of some convex polyhedron.

**Theorem 1.2** (Steinitz' Theorem). *A graph is polyhedral if and only if it is planar and 3-connected.* □

This result is deeper than it might at first appear: proofs and excellent discussions may be found in ([43], [45], [107], [133]). What makes the theorem so remarkable is its implication that a general class of 3-dimensional structures is equivalent to a certain class of 2-dimensional ones - that is, studying convex polyhedra combinatorially does not require thinking of them in 3-dimensional space. It is sufficient to investigate their graphs. Hence any knowledge about 3-connected planar graphs indicates properties of convex polyhedra.

Probably the most important impulse to study the structure of plane graphs came from one of the most celebrated combinatorial problems - The *Four Colour Conjecture* (4CC), posed in 1852. This conjecture was proved in 1976 by Appel and Haken [4] and the result is well known as

**Theorem 1.3** (The Four Colour Theorem). *Vertices of every planar graph can be coloured with four colours in such a way that adjacent vertices are coloured with different colours.* □

When trying to solve the 4CC, Birkhoff in 1912 reviewed several ideas due to earlier writers and melded them into a systematic method of investigation. The line of enquiry he suggested lead to the solution of the problem in 1976 (see Chapter 9 in [5]). This line is a common method for proofs of many theorems concerning properties of plane graphs, see e.g. [4], [7], [9], [13], [18], [25], [30], [54], [60], [112], [116], [118], [127], and [129].

If there are plane graphs that are counterexamples to a theorem, then there must be among them a graph with the smallest number of vertices; such a graph is said to be *irreducible* (or *a minimal counterexample*) with respect to the theorem. The basic idea is to obtain more and more restrictive conditions that an irreducible graph must satisfy, in the hope that eventually we shall have enough conditions either to construct the graph explicitly, or,

alternatively, to prove that it cannot exist. These restrictive conditions are usually described in language of configurations.

A *configuration*  $H$  in a plane graph  $G$  is a connected subgraph  $H$  of  $G$  together with degrees of vertices of  $H$  in  $G$ . We call the set  $\mathcal{C}$  of configurations *unavoidable* in a given family  $\mathcal{G}$  of plane graphs if at least one member of  $\mathcal{C}$  occurs in every graph belonging to  $\mathcal{G}$ . As an example we note that the set of vertices of degree less than six is in this sense unavoidable in the family of all plane graphs. We also define a configuration  $H$  to be *reducible* (with respect to a conjecture being proved) if it cannot occur in any graph that is irreducible with respect to the conjecture.

Heawood observed in 1890 that if there is an irreducible plane graph with respect to the 4CC then it belongs to the set  $\mathcal{A}$  of plane triangulations with minimum degree 5. Hence to prove 4CC it suffices to find a finite unavoidable set  $\mathcal{U}$  of reducible configurations in  $\mathcal{A}$ . Appel and Haken [4] succeeded in finding such a set consisting of 1879 configurations (see also [56], [130]). In 1989 Appel and Haken [4] announced that proofs of the 4CC with only 1482, 1405, and 1256 configurations are possible. In a later streamlining of the proof of the 4CC Robertson, Sanders, Seymour, and Thomas [121] present an unavoidable set of 633 reducible configurations.

Already in 1904, Wernicke [128] showed that every plane triangulation of minimum degree 5 contains either two adjacent vertices of degree 5 or a vertex of degree 5 adjacent to a vertex of degree 6. In above defined terms this reads as follows: "The set  $\mathcal{A}$  of all plane triangulations with minimum degree 5 has an unavoidable set of two configurations, namely, an edge with both endpoints of degree 5, and an edge with endpoints of degrees 5 and 6."

In 1922, Franklin [42] extended Wernicke's result by proving that each plane graph from the set  $\mathcal{A}$  contains a vertex of degree 5 with two neighbours each having degree at most 6. This yields an unavoidable set of three configurations.

Attracted by the 4CC and the result of Franklin, the well-known analyst H. Lebesgue realized that it would be very helpful to identify unavoidable sets of configurations for different families of plane graphs. In his 1940 paper [100], he presented several such lists. His fundamental theorem (see Theorem 2.1 in Section 2 below) provides an unavoidable set of configurations for the family of all 3-connected plane graphs.

However, there is a limit to what can be achieved with Lebesgue's approach, and so stronger methods have been devised over the last three decades in order to solve various long-standing structural and colouring problems on plane graphs (for a survey of a portion of this area, see [18]). As a result, many unavoidable sets of configurations for different families of plane graphs have been discovered. They are scattered in many papers. It would be an

honourable achievement to collect and classify them.

We restrict ourselves to sets  $\mathcal{F}$  of connected graphs  $H$  for given families  $\mathcal{G}$  of embedded graphs that have the following property: There is a constant  $\varphi(H, \mathcal{G})$ , depending only on  $H$  and  $\mathcal{G}$ , such that if the graph  $H$  is a proper subgraph of a graph  $G$  in  $\mathcal{G}$ , then  $G$  contains a copy of  $H$  whose all vertices have degree at most  $\varphi(H, \mathcal{G})$  in  $G$ . When this holds, we call then graph  $H$  *light* in  $\mathcal{G}$ . The set  $\mathcal{F}$  is the *set of light graphs* of  $\mathcal{G}$ .

It is a well known fact easily deduced from Euler's Formula that each plane graph has a vertex of degree at most 5. Such a vertex can be interpreted as a path on one vertex. Thus, the one-vertex path is light in the family of all plane graphs. In 1955, Kotzig [97] showed that each 3-connected plane graph contains an edge (i.e. a path with two vertices) having degree sum at most 13. We say that path with two vertices is light in the family of 3-connected plane graph. The complete bipartite graphs  $K_{2,s}$  for  $s \geq 3$  show that the path with two vertices is not light in the family of all plane graphs. In 1997, Fabrici and Jendrol' proved that for each  $k$  the path with  $k$  vertices is light in the family of all 3-connected plane graphs and no plane graph other than a path is light in this family of graphs. Hence the set of light graphs of the family of all 3-connected plane graphs consists exactly of paths.

In this paper we give a survey of results on light subgraphs in several families of graphs embedded in the plane and the projective plane. Light subgraphs of graphs embedded on surfaces other than the plane or the projective plane are considered in another survey paper [88]. However, in Section 9 we briefly mention recent results together with the most important results concerning light subgraphs in graphs embedded in nonspherical surfaces.

## 2 Notation and preliminaries

All graphs considered throughout the paper have no loops or multiple edges. Multigraphs can have multiple edges and loops. An embedding of a connected planar graph (planar multigraph) into the plane  $\mathbb{M}$  is called a *plane graph* (a *plane multigraph*, respectively). If a planar (multi)graph  $G$  is embedded in  $\mathbb{M}$ , then the *faces* of  $G$  are the maximal connected regions of  $\mathbb{M} - G$ .

The *facial walk* of a face  $\alpha$  of a connected plane multigraph  $G$  is the shortest closed walk traversing all edges incident with  $\alpha$ . The *degree (or size)* of a face  $\alpha$  is the length of its facial walk. The degree of a face  $\alpha$  in  $G$  is denoted by  $\deg_G(\alpha)$  or  $\deg(\alpha)$  if  $G$  is known from context. The degree of the vertex  $v$  of a connected plane multigraph  $G$  is the number of incidences of edges with  $v$ , where loops are counted twice. Analogously, the

notation  $\deg_G(v)$  or  $\deg(v)$  is used for the degree of a vertex  $v$ . Vertices and faces of degree  $i$  are called  $i$ -vertices and  $i$ -faces (or  $i$ -gons), respectively. The numbers of  $i$ -vertices and  $i$ -faces of a connected plane multigraph  $G$  are denoted  $n_i(G)$  and  $f_i(G)$ , respectively, or  $n_i$  and  $f_i$  if  $G$  is known. We use  $\delta(G)$  to denote the minimum vertex degree of  $G$ .

We call an edge  $h$  an  $(a, b)$ -edge if the endvertices of  $h$  are an  $a$ -vertex and a  $b$ -vertex. By  $e_{a,b}(G)$  or  $e_{a,b}$  we denote the number of  $(a, b)$ -edges in a plane multigraph  $G$ .

For  $r \geq 2$ , an  $r$ -face  $\alpha$  is an  $(a_1, a_2, \dots, a_r)$ -face if vertices  $x_1, x_2, \dots, x_r$ , in order along the facial walk incident with  $\alpha$  have degrees  $a_1, a_2, \dots, a_r$ , respectively. A face of length 3, 4, or 5 is a *triangle*, a *quadrangle*, or a *pentagon*, respectively. Now we are able to state the classical theorem of Lebesgue [100] already mentioned in Section 1.

**Theorem 2.1** (Lebesgue's Theorem). *Every 3-connected plane graph contains at least one of the following faces:*

(i) *an  $(a, b, c)$ -triangle with*

*$a = 3$  and  $3 \leq b \leq 6$  and  $3 \leq c$ , or  $a = 3$  and  $b = 7$  and  $7 \leq c \leq 41$ ,*  
*or*

*$a = 3$  and  $b = 8$  and  $8 \leq c \leq 23$ , or  $a = 3$  and  $b = 9$  and  $9 \leq c \leq 17$ ,*  
*or*

*$a = 3$  and  $b = 10$  and  $10 \leq c \leq 14$ , or  $a = 3$  and  $b = 11$  and  $11 \leq c \leq 13$ ,* *or*

*$a = 4$  and  $b = 4$  and  $4 \leq c$ , or  $a = 4$  and  $b = 5$  and  $5 \leq c \leq 19$ ,* *or*

*$a = 4$  and  $b = 6$  and  $6 \leq c \leq 11$ , or  $a = 4$  and  $b = 7$  and  $7 \leq c \leq 9$ ,*  
*or*

*$a = 5$  and  $b = 5$  and  $5 \leq c \leq 9$ , or  $a = 5$  and  $b = 6$  and  $6 \leq c \leq 7$ ,* *or*

(ii) *a  $(3, b, c, d)$ -quadrangle with*

*$b = 3$  and  $c = 3$  and  $d \geq 3$ , or  $b = 3$  and  $c = 4$  and  $4 \leq d \leq 11$ ,* *or*

*$b = 4$  and  $c = 3$  and  $4 \leq d \leq 11$ , or  $b = 3$  and  $c = 5$  and  $5 \leq d \leq 7$ ,*  
*or*

*$b = 5$  and  $c = 3$  and  $5 \leq d \leq 7$ , or  $b = 4$  and  $c = 4$  and  $4 \leq d \leq 5$ ,* *or*

*$b = 4$  and  $c = 5$  and  $d = 4$ ,* *or*

(iii) *a  $(3, 3, 3, 3, d)$ -pentagon with  $3 \leq d \leq 5$ .* □

Let  $V(H)$  denote the set of vertices of a graph  $H$ . If  $H$  is a subgraph of a graph  $G$ , then the *weight*  $w_G(H)$  of  $H$  in  $G$  is the sum of the degrees in  $G$  of the vertices of  $H$ .

$$w_G(H) = \sum_{v \in V(H)} \deg_G(v).$$

Moreover, the weight  $w_G(e)$  of an  $(a, b)$ -edge  $e$  is

$$w_G(e) = a + b.$$

If  $G$  is known from context, then we simply write  $w(H) = w_G(H)$  and  $w(e) = w_G(e)$ .

A path and a cycle on  $k$  distinct vertices are referred as a  $k$ -*path* and a  $k$ -*cycle*, respectively. The *length* of a path or a cycle is the number of edges it has. A  $k$ -path with vertices  $v_1, v_2, \dots, v_k$  in order is also called an  $(a_1, a_2, \dots, a_k)$ -*path* when  $\deg(v_i) = a_i$  for  $1 \leq i \leq k$ .

A subgraph  $K_{1,3}$  of a graph  $G$  is called a  $(d; a, b, c)$ -*star* if its central vertex has degree  $d$  and its three leaves have degrees  $a, b$ , and  $c$  in  $G$ .

For a connected plane multigraph  $G$ , let  $V, E$ , and  $F$  be the vertex set, the edge set, and the face set of  $G$ , respectively. Since

$$\sum_{\alpha \in F} \deg(\alpha) = \sum_{v \in V} \deg(v) = 2|E|,$$

from (1) we can easily derive

$$\sum_{\alpha \in F} (6 - \deg(\alpha)) + 2 \sum_{v \in V} (3 - \deg(v)) = 12 \quad (2)$$

$$\sum_{v \in V} (6 - \deg(v)) + 2 \sum_{\alpha \in F} (3 - \deg(\alpha)) = 12 \quad (3)$$

$$\sum_{v \in V} (4 - \deg(v)) + \sum_{\alpha \in F} (4 - \deg(\alpha)) = 8 \quad (4)$$

Let  $\mathcal{P}(\delta, \rho)$  be the family of all 3-connected plane graphs (i.e. polyhedral graphs, see Theorem 1.2) with minimum vertex degree at least  $\delta$  and minimum face size at least  $\rho$ . Let  $\mathcal{P}(\delta, \bar{\rho})$  be the family of all graphs in  $\mathcal{P}(\delta, \rho)$  in which every face is a  $\rho$ -face. Define the family  $\mathcal{P}(\bar{\delta}, \rho)$  analogously. The family  $\mathcal{P}(3, \bar{3})$  is the family of all plane *triangulations*, and  $\mathcal{P}(3, \bar{4})$  is the family of all 3-connected plane *quadrangulations*. By  $\mathcal{P}(\delta, \rho; R)$  we mean the subfamily of  $\mathcal{P}(\delta, \rho)$  consisting of those members that satisfy the additional requirements  $R$ . It is an easy consequence of the equalities (2),(3) and (4) that  $\mathcal{P}(\delta, \rho)$  is nonempty only when  $(\delta, \rho) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$ .

Let  $\mathcal{M}(\delta, \rho)$  be the family of all connected plane multigraphs with minimum vertex degree at least  $\delta$  and minimum face size at least  $\rho$ . Note that plane multigraphs from  $\mathcal{M}(\delta, \rho)$  are allowed to have loops. A plane multigraph  $G$  from the family  $\mathcal{M}(3, 3)$  is a *normal plane map*. If a plane multigraph from  $\mathcal{M}(3, 3)$  has only 3-faces, then it is a plane *semitriangulation*. Note that semitriangulations may contain loops and multiple edges, but triangulations do not have them.

Using (1) one can easily obtain

$$|E| \leq 3|V| - 6 \tag{5}$$

for each normal plane map with  $|V| \geq 3$ .

For two graphs  $H$  and  $G$  we write  $G \cong H$  when  $H$  and  $G$  are isomorphic. For a graph  $K$  we say that  $G$  *contains a copy* of  $K$  if  $G$  has a subgraph  $H$  such that  $H \cong K$ .

### 3 Light edges

The theory of light subgraphs has its origin in two beautiful theorems of Kotzig [97]. They state that every 3-connected plane graph contains an edge of weight at most 13 in general, and at most 11 in the absence of 3-vertices, respectively. These bounds are best possible, as can be seen from the 3-connected plane graphs obtained from a graph of the icosahedron and dodecahedron, respectively, by placing a vertex inside each face and make it adjacent to all vertices of that face.

Kotzig's result was further developed in various directions. We shall discuss some of them in subsequent sections. Here we only mention that in 1972 Erdős conjectured (see [46]) that Kotzig's theorem is valid for all planar graphs with minimum vertex degree at least 3 regardless of connectivity. This conjecture was proved (but never published) by Barnette (see [46]) and independently by Borodin [7]. The theorem of Kotzig was published in 1955 in Slovak. Therefore, its original proof [97] is not readily accessible, but Grünbaum [45] in 1975 sketched a proof in English. Other proofs can be deduced from [7], [8], [10], [15], [69]. Here we present a simple proof from [68] that uses the Discharging Method, a method used in the proof of the Four Color Theorem (see [4], [121], [129]). (The idea of discharging is due to Heesch [53]). This method is a common technique for proving results on planar graphs, see e.g. [118]. We prove the theorem in a stronger form that includes Erdős' conjecture.



**Theorem 3.1** (Kotzig's Theorem). *Every normal plane map contains a  $(3, a)$ -edge with  $3 \leq a \leq 10$ , or a  $(4, b)$ -edge with  $4 \leq b \leq 7$ , or a  $(5, c)$ -edge with  $5 \leq c \leq 6$ . The bounds 10, 7, and 6 are best possible.*

*Proof.* Let  $G$  be a counterexample on a set  $V$  of  $n$  vertices that has the maximum number of edges, say  $m$ , among all counterexamples on  $n$  vertices. Let  $f$  be the number of faces of  $G$ . Call edges of the desired type *light* edges.

By the choice of  $G$ , it must be a semitriangulation. Suppose  $G$  has a  $k$ -face  $\alpha$  with  $k \geq 4$ . Because  $G$  has no light edges, each edge has an endvertex of degree at least 6, so  $\alpha$  has two vertices  $x$  and  $y$  that are not consecutive on the boundary of  $\alpha$  and both have degree at least 6.

Inserting a diagonal  $xy$  into the face  $\alpha$  yields a counterexample having the same vertex set as  $G$  but one edge more, a contradiction.

Because  $G$  is a semitriangulation, (3) may be rewritten

$$\sum_{v \in V} (\deg(v) - 6) = -12. \quad (6)$$

Consider an initial charge function  $\varphi : V \rightarrow \mathbb{Q}$  such that  $\varphi(v) = \deg(v) - 6$  for each  $v \in V$ . Therefore (3.1) is equivalent to

$$\sum_{v \in V} \varphi(v) = -12.$$

We use the following rule to transform  $\varphi$  into a new charge function  $\psi : V \rightarrow \mathbb{Q}$  by redistributing charges locally so that  $\sum_{v \in V} \varphi(v) = \sum_{v \in V} \psi(v)$ .

**Rule.** If  $uv$  is an edge of  $G$  with  $\deg(u) \geq 7$  and  $\deg(v) \leq 5$ , then the vertex  $u$  sends to  $v$  the charge  $\frac{6 - \deg(v)}{\deg(v)}$ .

Let  $\psi(x)$  denote the resulting charge at a vertex  $x$ . Since charge sent to  $v$  is deducted from  $u$ , we have

$$\sum_{x \in V} \psi(x) = -12.$$

We are going to show that  $\psi$  is a nonnegative function, which will trivially be a contradiction.

Observe that a  $k$ -vertex sends charge to at most  $\lfloor \frac{k}{2} \rfloor$  vertices. The charge sent to a neighbour of degree  $d$  is  $\frac{6-d}{d}$ , where  $d \leq 5$ . A  $k$ -vertex begins with charge  $k - 6$ . By the definition of light edges, a 7-vertex sends charge only to 5-vertices, giving charge at most  $3 \times (\frac{1}{5})$  and hence retaining positive charge. For  $k \in \{8, 9, 10\}$ , a  $k$ -vertex may give charge to neighbours of degree 4 or 5 and have given away at most  $\lfloor \frac{k}{2} \rfloor \times (\frac{1}{2})$ , retaining nonnegative charge. For

$k \geq 11$ , a  $k$ -vertex may also give charge to neighbours of degree 3, thus giving charge at most  $\lfloor \frac{k}{2} \rfloor \times 1$  and retaining nonnegative charge.

The bounds 10 and 6 are best possible, as can be seen from the graphs mentioned at the beginning of Section 3. An example showing that also 7 is best possible can be found in [9].

This finishes the proof. □

A result weaker than the Kotzig's Theorem was known already to Lebesgue in 1940. From his Theorem 2.1 it follows that every 3-connected plane graph contains an edge  $e$  with  $w(e) \leq 14$ .

Now we turn our attention to the problem of guaranteeing many light edges (i.e. edges having weight at most 13) in families of plane graphs and multigraphs.

Kotzig's Theorem (Theorem 3.1) states that  $\sum_{i+j \leq 13} e_{i,j} > 0$  for every 3-connected planar graph.

Grünbaum [46] conjectured that for every 3-connected plane graph the following is true:

$$\begin{aligned} & 20 e_{3,3} + 15 e_{3,4} + 12 e_{3,5} + 10 e_{3,6} + \frac{20}{3} e_{3,7} + 5 e_{3,8} + \frac{10}{3} e_{3,9} + 2 e_{3,10} \\ & + 12 e_{4,4} + 7 e_{4,5} + 5 e_{4,6} + 4 e_{4,7} + \frac{8}{3} e_{4,8} + \frac{2}{3} e_{4,9} \\ & + 4 e_{5,5} + 2 e_{5,6} + \frac{1}{3} e_{5,7} + 12 e_{6,6} \geq 120. \end{aligned}$$

Jucovič [94], [95] made first steps towards this conjecture by proving a weaker inequality.

For the class of normal plane maps, which includes the class of 3-connected plane graphs, Borodin [10] obtained the following result. (Recall that a normal plane map is a plane multigraph in which every vertex degree and every face size is at least 3.)

**Theorem 3.2** ([10]). *Every normal plane map satisfies*

$$\begin{aligned} & 40 e_{3,3} + 25 e_{3,4} + 16 e_{3,5} + 10 e_{3,6} + \frac{20}{3} e_{3,7} + 5 e_{3,8} + \frac{5}{2} e_{3,9} + 2 e_{3,10} \\ & + \frac{50}{3} e_{4,4} + 11 e_{4,5} + 5 e_{4,6} + \frac{5}{3} e_{4,7} + \frac{16}{3} e_{5,5} + 2 e_{5,6} \geq 120; \end{aligned}$$

Moreover, each coefficient in this inequality is best possible. □

The sharpness of coefficients in Theorem 3.2 and in those below is understood in the sense that none of the coefficients can be decreased while keeping all the other  $\alpha_{ij}$  constant without violating the correspondent relation.

In fact, the conjecture of Grünbaum is false. The following result by Fabrici and Jendrol' [36] gives the sharpest inequality of this sort for 3-connected plane graphs.

**Theorem 3.3** ([36]). *Every 3-connected plane graph satisfies*

$$20 e_{3,3} + 25 e_{3,4} + 16 e_{3,5} + 10 e_{3,6} + \frac{20}{3} e_{3,7} + 5 e_{3,8} + \frac{5}{2} e_{3,9} + 2 e_{3,10} \\ + \frac{50}{3} e_{4,4} + 11 e_{4,5} + 5 e_{4,6} + \frac{5}{3} e_{4,7} + \frac{16}{3} e_{5,5} + 2 e_{5,6} \geq 120;$$

Moreover, each coefficient is best possible.  $\square$

The inequalities for normal maps and 3-connected planar graphs differ only in the coefficient of  $e_{3,3}$ . As was later proved by Borodin, Theorem 3.3 is valid for all normal plane maps with the exception of exactly one multigraph.

In the subclass of all normal plane maps with minimum degree at least 4 we have  $e_{3,j} = 0$  for  $3 \leq j \leq 10$ . Borodin proved the corresponding sharp result for this family.

**Theorem 3.4** ([10]). *Every normal plane map of minimum degree at least 4 satisfies*

$$\frac{50}{3} e_{4,4} + 11 e_{4,5} + 5 e_{4,6} + \frac{5}{3} e_{4,7} + \frac{16}{3} e_{5,5} + 2 e_{5,6} \geq 120$$

Moreover, each of these coefficients is best possible.  $\square$

Already in 1904 in his work on the Four Color Problem, Wernicke [128] established the inequality  $e_{5,5} + e_{5,6} > 0$  for every graph  $G \in \mathcal{P}(5, 3)$ . Further contributions are due to Grünbaum [44], [46], Fisk (see [47]), Grünbaum and Shephard [47], and Borodin [11]. Borodin and Sanders [23] found the best possible light edge inequality for plane graphs of minimum degree 5.

**Theorem 3.5** ([23]). *Every normal plane map of minimum vertex degree at least 5 satisfies*

$$\frac{14}{3} e_{5,5} + 2 e_{5,6} \geq 120.$$

Moreover, the coefficients  $\frac{14}{3}$  and 2 are best possible.  $\square$

If in Theorem 3.4 we set  $e_{4,j} = 0$  for  $4 \leq j \leq 7$ , then the resulting inequality differs from that of Theorem 3.5 only in the coefficient of  $e_{5,5}$ .

We close this section with few very recent results.

A planar graph embeddable in the plane in such a way that every vertex lies on the boundary of single region is called an *outerplanar* graph. Hackmann and Kemnitz [48] recently proved

**Theorem 3.6** ([48]). *Every outerplanar graph with minimum degree at least 2 contains a (2, 2)-edge, a (2, 3)-edge, or a (2, 4, 2)-path.*  $\square$

The following theorem states that the family of plane graphs with minimum degree 5 contains a light 3-star.

**Theorem 3.7** ([72]). *Every planar graph with minimum degree 5 contains a (5; 6, 6, 6)-star or a (5; 5, b, c)-star with  $5 \leq b \leq 6$  and  $5 \leq c \leq 7$ . Moreover, the bounds 6 and 7 are best possible.*  $\square$

The next theorem is a strengthening of Theorems 3.1 and 3.7.

**Theorem 3.8** ([49]). *Every planar graph  $G$  with minimum degree at least 3 contains*

(i) *a (3, b)-edge with  $3 \leq b \leq 10$ , or*

(ii) *an (a, 4, b)-path with*

*$a = 4$  and  $4 \leq b \leq 10$ , or*

*$a = 5$  and  $5 \leq b \leq 9$ , or*

*$6 \leq a \leq 7$  and  $6 \leq b \leq 8$ , or*

(iii) *a (5; a, b, c)-star with*

*$4 \leq a \leq 5$  and  $5 \leq b \leq 6$  and*

*$5 \leq c \leq 7$ , or  $a = b = c = 6$ .*

*Moreover, for every  $S \in \{(3, 10)\text{-edge}, (4, 4, 9)\text{-path}, (5, 4, 8)\text{-path}, (6, 4, 8)\text{-path}, (7, 4, 7)\text{-path}, (5; 5, 6, 7)\text{-star}, (5; 6, 6, 6)\text{-star}\}$  there is a 3-connected plane graph  $H$  containing  $S$  and no other configuration from the above list.*  $\square$

Theorem 3.8 has an application in a problem of colouring of vertices of the square of a planar graph posed by Wegner [126] in 1977 (see also Jensen and Toft [93], p. 51).

The square  $G^2$  of a graph  $G$  is a graph with the same vertex set  $V(G)$ . Two vertices  $x$  and  $y$  are adjacent in  $G^2$  if and only if their distance in  $G$  is at most 2. For a graph  $G$  let  $\Delta(G)$  and  $\chi(G)$  denote its maximum degree and chromatic number, respectively.

**Corollary 3.9** ([49]). *Let  $G$  be a planar graph of maximum degree  $\Delta(G) \geq 11$ . Then*

$$\chi(G^2) \leq 2\Delta(G) + 19.$$

*Proof.* If there is a counterexample, then let  $H$  be one with the fewest vertices. Evidently the minimum degree of  $H$  is at least 3. By Theorem 3.8 the graph  $H$  contains an  $(a, b)$ -edge  $xy$  with  $\deg_H(x) = a$ ,  $\deg_H(y) = b$  and  $\deg_{H^2}(x) \leq 2\Delta(H) + 18$  with values  $a = 3$  and  $b \leq 10$ , or  $a = 4$  and  $b \leq 7$ , or  $a = 5$  and  $b \leq 6$ . If the edge  $uv$  is contracted into a vertex  $z$  and all multiple edges are replaced by single edges, then the resulting graph  $H^*$  has the same maximum degree because, in  $H^*$ ,  $\deg(z) \leq 11$ . By the minimality of  $H$  the graph  $H^*$  is not a counterexample. Thus the square of  $H^*$  has a colouring with  $2\Delta(H) + 19$  colours. This colouring induces a colouring with  $2\Delta(H) + 19$  colours of the graph  $H$  in which all vertices except  $x$  and  $y$  are coloured. Now assign the color of vertex  $z$  to the vertex  $y$ . As the vertex  $x$  has at most  $2\Delta(H) + 18$  neighbours in  $H^2$  it can be coloured with one of the available  $2\Delta(H) + 19$  colours, a contradiction.  $\square$

The bound of Corollary 3.9 is better than  $2\Delta + 25$ , the bound obtained recently by van den Heuvel and McGuinness [54].

The requirement of minimum degree at least 3 in the above theorems cannot be relaxed, as one can see from the graphs  $K_{1,r}$  and  $K_{2,r}$  for  $r \geq 3$ . Next theorem goes deeper to the structure of planar graphs.

**Theorem 3.10** ([2]). *Every connected planar graph of order at least 2 contains either*

- (i) *two vertices having degree sum  $\leq 4$ , or*
- (ii) *two 3-vertices at distance two, or*
- (iii) *an edge  $e$  of weight at most 11 incident with two 3-faces, or*
- (iv) *an edge  $g$  of weight at most 9 incident with a 3-face, or*
- (v) *an edge  $h$  of weight at most 7.*

*Moreover the bounds 11, 9, and 7 are tight.*  $\square$

An edge  $e$  of a 3-connected graph  $G$  is *contractible* if the graph  $G \odot e$  obtained by contracting the edge  $e$  is also 3-connected. Recently Dvořák and Škrekovski strengthened original Kotzig's Theorem by proving

**Theorem 3.11** ([28]). *Every 3-connected planar graph distinct from  $K_4$  contains a contractible  $(a, b)$ -edge with  $a + b \leq 13$ .*

For other results in this direction, see Borodin [8], [10], [11], [18], Borodin, Kostochka, and Woodall [22], Borodin and Sanders [23], Cole, Kowalik, and Škrekovski [27], Jucovič [95], Sanders [123], and Zaks [131], [132].

## 4 Light subgraphs of order three

Trying to solve the Four Color Conjecture, Franklin [42] in 1922 proved that every 3-connected plane graph  $G$  with minimum degree at least 5, contains a 3-path with weight 17, the bound being best possible. In 1993, Ando, Iwasaki, and Kaneko obtained the analogous result for all 3-connected plane graphs. Namely, they proved

**Theorem 4.1** ([3]). *Every  $G \in \mathcal{P}(3, 3)$  contains a 3-path with weight at most 21. Moreover the bound 21 is sharp.*

The following result, which strengthens Kotzig's Theorem (Theorem 3.1) was proved by Jendrol' [66].

**Theorem 4.2** ([66]). *Every  $G \in \mathcal{P}(3, 3)$  contains an  $(a, b, c)$ -path, where*

- (i)  $3 \leq a \leq 10$  and  $b = 3$  and  $3 \leq c \leq 10$ , or
- (ii)  $4 \leq a \leq 7$  and  $b = 4$  and  $4 \leq c \leq 7$ , or
- (iii)  $5 \leq a \leq 6$  and  $b = 5$  and  $5 \leq c \leq 6$ , or
- (iv)  $a = 3$  and  $3 \leq b \leq 4$  and  $4 \leq c \leq 15$ , or
- (v)  $a = 3$  and  $5 \leq b \leq 6$  and  $4 \leq c \leq 11$ , or
- (vi)  $a = 3$  and  $7 \leq b \leq 8$  and  $4 \leq c \leq 5$ , or
- (vii)  $a = 3$  and  $3 \leq b \leq 10$  and  $c = 3$ , or
- (viii)  $a = 4$  and  $b = 4$  and  $4 \leq c \leq 11$ , or
- (ix)  $a = 4$  and  $b = 5$  and  $5 \leq c \leq 7$ , or
- (x)  $a = 4$  and  $6 \leq b \leq 7$  and  $4 \leq c \leq 5$ .

Moreover, in each of the cases (iii), (iv), (vi), and (vii) the upper bounds on parameters  $a, b, c$  can be obtained simultaneously. Furthermore, there is a graph in  $\mathcal{P}(3, 3)$  having only  $(4, 7, 4)$ -paths and  $(7, 4, 7)$ -paths from the above list.  $\square$

The requirement of 3-connectedness in the above theorems is fundamental because of the following theorem. We present a proof of this theorem with a proof technique that is typical in this area.

**Theorem 4.3** ([70]). *For every connected plane graph  $H$  of order at least 3 and every integer  $m$  with  $m \geq 3$ , there exists a 2-connected plane graph  $G$  such that each copy of  $H$  in  $G$  contains a vertex  $A$  with*

$$\deg_G(A) \geq m.$$

*Proof.* Augment  $H$  to a triangulation  $T$  with vertex set  $V(H)$ . Let  $[uvw]$  be the outer 3-face of  $T$ . For  $m \geq 2$ , let  $D_m$  be the plane graph obtained from the double  $2m$ -pyramid with poles  $z_1$  and  $z_2$  by deleting every second edge of the equatorial cycle of length  $2m$ . If we insert into a 3-face  $[z_1xy]$  of  $D_m$  the triangulation  $T$  so that the vertex  $u$  coincides with  $z_1$ , the vertices  $v$  and  $w$  with  $x$  and  $y$ , respectively, then we obtain the required graph  $G$ . If  $H$  has at least three vertices, then each copy of  $H$  in  $G$  contains at least one of the vertices  $z_1$  or  $z_2$ , which have degree at least  $2m$ .  $\square$

The graph  $G$  from the proof of Theorem 4.3 contains a 3-face incident with two 3-vertices. Borodin [16] observed that if such faces are excluded the requirement of 3-connectivity can be omitted. More precisely he has proved the following statement.

**Theorem 4.4** ([16]). *Each normal plane map  $G$  having no 3-face incident with two 3-vertices has the following two properties:*

- (i)  *$G$  has either a 3-path  $P$  with  $w(P) \leq 18$  or a vertex of degree  $\leq 15$  adjacent to two 3-vertices*
- (ii)  *$G$  has either a 3-path  $P'$  with  $w(P') \leq 17$  or an edge  $e$  with  $w(e) \leq 7$ .*

$\square$

Already in 1940 Lebesgue [100] proved that each 3-connected plane graph  $G$  of minimum degree 5 contains a 3-cycle  $C$  bounding a 3-face  $\alpha$  with  $w(C) \leq 19$ . Kotzig [98] improved this result to  $w(C) \leq 18$  and conjectured in [99] that 17 is the best upper bound. Borodin confirmed this.

**Theorem 4.5** ([6]). *Every  $G \in \mathcal{P}(5, 3)$  contains a 3-face  $\alpha$  with  $w_G(\alpha) \leq 17$ . This bound is best possible.*  $\square$

This theorem has a beautiful corollary. It confirms the conjecture of Grünbaum from 1975, see [45], that every 5-connected plane graph is cyclically 11-connected. Let us sketch a proof of this statement. A graph is *cyclically  $k$ -connected* for some  $k \geq 1$  if there is no set  $S$  of fewer than  $k$  edges with property that  $G - S$  has two components each containing a cycle. Let  $G$  be a graph from  $\mathcal{P}(5, 3)$ . Consider a face  $\alpha$  of weight at most 17 of

$G$ . Choose  $S$  to be the set of edges incident with vertices of  $\alpha$  but not on the boundary of  $\alpha$ . Clearly  $|S| \leq 11$  and the graph  $G - S$  has two components each containing a cycle.

Earlier, in 1972, Plummer [115] proved that the cyclic connectivity of these graphs is at most 13.

Many papers have studied the structural properties of different classes of plane triangulations; see e.g. [98], [99], Borodin [11], [14], [17], [20], Jendrol' [69], Sanders [123], and Borodin and Sanders [23]. In 1999, Jendrol' [69] proved the following theorem, which includes earlier results by Lebesgue [100], Kotzig [98], [99], and Borodin [14].

**Theorem 4.6** ([69]). *Each plane triangulation of order at least 5 contains an  $(a, b, c)$ -triangle, where*

- (i)  $a = 3$  and  $b = 4$  and  $4 \leq c \leq 35$ , or
- (ii)  $a = 3$  and  $b = 5$  and  $5 \leq c \leq 21$ , or
- (iii)  $a = 3$  and  $b = 6$  and  $6 \leq c \leq 20$ , or
- (iv)  $a = 3$  and  $b = 7$  and  $7 \leq c \leq 16$ , or
- (v)  $a = 3$  and  $b = 8$  and  $8 \leq c \leq 14$ , or
- (vi)  $a = 3$  and  $b = 9$  and  $9 \leq c \leq 14$ , or
- (vii)  $a = 3$  and  $b = 10$  and  $10 \leq c \leq 13$ , or
- (viii)  $a = 4$  and  $b = 4$  and  $c \geq 4$ , or
- (ix)  $a = 4$  and  $b = 5$  and  $5 \leq c \leq 13$ , or
- (x)  $a = 4$  and  $b = 6$  and  $6 \leq c \leq 17$ , or
- (xi)  $a = 4$  and  $b = 7$  and  $7 \leq c \leq 8$ , or
- (xii)  $a = 5$  and  $b = 5$  and  $5 \leq c \leq 7$ , or
- (xiii)  $a = 5$  and  $b = 6$  and  $c = 6$ .

Moreover, the result is nearly sharp in the following sense: If  $c(k)$  denotes the upper bound on  $c$  in the above case (k), then  $c(\text{i}) \geq 30$ ,  $c(\text{ii}) \geq 18$ ,  $c(\text{iii}) = 20$ ,  $c(\text{iv}) \geq 7$ ,  $c(\text{v}) = 14$ ,  $c(\text{vi}) \geq 11$ ,  $c(\text{vii}) \geq 12$ ,  $c(\text{viii}) = \infty$ ,  $c(\text{ix}) \geq 10$ ,  $c(\text{x}) \geq 10$ ,  $c(\text{xi}) \geq 7$  and  $c(\text{xii}) = 7$ .  $\square$

Theorem 4.6 points out that if a plane triangulation has no  $(4, 4)$ -edge, then it contains a light triangle. Borodin [17] went further. He proved



**Theorem 4.7** ([17]). *If in a plane triangulation  $T$  there is no path consisting of  $k$  vertices of degree 4 for some  $k \geq 1$ , then*

- (i)  *$T$  contains a 3-face with weight at most  $\max\{37, 10k + 17\}$ , and*
- (ii)  *$T$  contains a  $(3, 4)$ -edge or a 3-face with weight at most  $\max\{29, 5k + 8\}$ .  
These bounds are best possible.  $\square$*

In a connection with results mentioned in this section, the following problem seems to be interesting.

**Problem 4.8.** *Find the best versions of Theorem 4.2 and Theorem 4.6.*

In each of the cases (iii), (iv), (vi), and (vii) of Theorem 4.2, the upper bounds on parameters are tight and can be obtained independently from the others. Furthermore, there is a graph  $G \in \mathcal{P}(\exists, \exists)$  having only  $(4, 7, 4)$ -paths and  $(7, 4, 7)$ -paths from the list of the Theorem.

Similarly we are interested in the best version of Theorem 4.6. As we have mentioned, in the cases (iii), (v), (viii), (xii), and (xiii) the assertions of Theorem 4.6 are best possible. For the other cases we believe the following Conjecture is true:

**Conjecture 4.9** ([68]). *If  $c(k)$  denotes the upper bound on  $c$  in the case  $(k)$  of Theorem 4.6, then  $c(\text{i}) = 30$ ,  $c(\text{ii}) = 18$ ,  $c(\text{iv}) = 14$ ,  $c(\text{vi}) = 2$ ,  $c(\text{vii}) = 12$ ,  $c(\text{ix}) = 10$  and  $c(\text{xi}) = 7$ .  $\square$*

For a plane graph  $G$  from  $\mathcal{P}(3, 3)$ , let  $f_{i,j,k}$  be the number of 3-faces that are incident with an  $i$ -vertex, a  $j$ -vertex, and a  $k$ -vertex. As proved by Lebesgue [100],  $f_{5,5,5} + \frac{2}{3}f_{5,5,6} + \frac{3}{7}f_{5,5,7} + \frac{1}{4}f_{5,5,8} + \frac{1}{9}f_{5,5,9} + \frac{1}{3}f_{5,6,6} + \frac{2}{21}f_{5,6,7} \geq 120$  holds for every graph  $G \in \mathcal{P}(5, 3)$ . This result of Lebesgue and Theorem 4.5 are strengthened in the next Theorem proved by Borodin [12] except for the optimality of the coefficient 5 of  $f_{5,5,7}$ . Its optimality was proved by Borodin and Sanders in [23].

**Theorem 4.10.** *If  $G$  is a graph from  $\mathcal{P}(5, 3)$ , then*

$$18f_{5,5,5} + 9f_{5,5,6} + 5f_{5,5,7} + 4f_{5,6,6} \geq 144.$$

*Moreover, all of these coefficients are best possible.  $\square$*

One can expect similar inequalities when restricting to other subfamilies of plane graphs.

## 5 Subgraphs with restricted degrees

In this section we prove two basic results that are typical for this topic and have served as a starting point for a theory of light subgraphs that we shall describe in further sections. Because of Theorem 4.3 we only deal with the family of 3-connected plane graphs. The first result is due to Fabrici and Jendrol'.

**Theorem 5.1** ([37]). *If  $G$  is a 3-connected plane graph having a  $k$ -path where  $k$  is a positive integer, then  $G$  contains a  $k$ -path whose all vertices have degree at most  $5k$  in  $G$ . Moreover, the bound  $5k$  is tight.*

*Proof.* (a) *The upper bound.* Suppose the theorem is not true, and let  $G$  be a counterexample on  $n$  vertices that has the most edges among all counterexamples on  $n$  vertices. Let us call a vertex *major* if its degree exceeds  $5k$ , otherwise, it is *minor*. The crucial point is that maximality of the counterexample guarantees that every major vertex is incident only with triangular faces. If a major vertex  $u$  lies on a face  $\alpha$  with size at least 4, then a chord can be inserted joining  $u$  to a vertex  $v$  not adjacent to  $u$ . The edge  $uv$  cannot lie on any path consisting of minor vertices only, so maximality guarantees the desired  $k$ -path in the original graph.

Now let  $M$  be the subgraph of  $G$  induced by the major vertices. Since  $M$  is a plane graph, it has a vertex  $x$  of degree at most 5. Since  $x$  is a major vertex in  $G$ , the claim above guarantees that the subgraph induced by  $x$  and its neighbour contains a wheel. The neighbour of  $x$  in  $G$  form a cycle  $C$  with at least  $5k + 1$  vertices and, since  $\deg_M(x) \leq 5$ , there are at most 5 major vertices on this cycle. By the pigeonhole principle, the cycle contains a  $k$ -path whose all vertices are all minor in  $G$ ; a contradiction.

(b) *The sharpness of the bound.* Now we construct a 3-connected plane graph  $G$  in which every  $k$ -path has a vertex  $u$  of degree at least  $5k$ . The construction begins with the dodecahedron. Into each of its 5-faces we insert a new vertex  $x$  and join it to all vertices incident with this face. The result is a graph all faces of which are triangles  $[xyz]$  with  $\deg(x) = 5$  and  $\deg(y) = \deg(z) = 6$ . Into every triangle of this graph we insert a subdivided 3-star consisting of a central vertex  $v$  and three paths  $p_x, p_y, p_z$  from  $v$  to  $x, y, z$ , respectively, with  $p_x$  of length  $\lceil \frac{k+2}{2} \rceil$  and  $p_y$  and  $p_z$  of length  $\lfloor \frac{k-2}{2} \rfloor$ . Now make  $x$  adjacent to all vertices of  $p_x$  and  $p_y$ , and similarly make  $y$  adjacent to all of  $p_y$  and  $p_z$ , and  $z$  adjacent to all of  $p_z$  and  $p_x$ . Observe that in the resulting graph  $G$  all the vertices of type  $x$  have degree  $5k$ , the vertices of type  $y$  and  $z$  have degrees at least  $6k - 6$  and every other vertex has degree at most 6. It is easy to see that each  $k$ -path contains at least one vertex of type  $x, y$  or  $z$ .  $\square$

It is natural to ask a more general question: Does every 3-connected plane graph  $G$  having a copy of a connected plane graph  $H$  different from a path also contain a copy of  $H$  such that its vertices have bounded degrees in  $G$ ? The answer is surprisingly negative. Fabrici and Jendrol' proved

**Theorem 5.2** ([37]). *If  $H$  is a connected plane graph other than a path, and  $m$  is an integer greater than 3, then there is a 3-connected plane graph  $T$  such that each copy of  $H$  in  $T$  has a vertex  $y$  with degree at least  $m$ .*

*Proof.* Augment  $H$  to a triangulation  $T_0$  with vertex set  $V(H)$ . Into each triangle  $[uvw]$  of  $T_0$  insert a wheel with a central vertex  $z$  and with  $m$  spokes  $zx_i$  for  $1 \leq i \leq m$ . Join the vertex  $u$  to  $x_i$  for  $1 \leq i \leq \lfloor \frac{m}{3} \rfloor$ , the vertex  $v$  to  $x_j$  for  $\lfloor \frac{m}{3} \rfloor \leq j \leq \lfloor \frac{2m}{3} \rfloor$ , and the vertex  $w$  to  $x_1$  and to  $x_t$  for  $\lfloor \frac{2m}{3} \rfloor \leq t \leq m$ . The result is the desired triangulation  $T$ . One can easily check that each vertex of  $T$  that lies in  $T_0$  has degree at least  $m$  in  $T$ , as does the center of each inserted wheel. The vertices of degrees less than  $k$  induce  $m$ -cycles in  $T$ . Therefore, as  $H$  is not a path, each copy of  $H$  in  $T$  contains at least one vertex of degree  $\geq m$ .  $\square$

Madaras improved Theorem 5.1 by proving the following result.

**Theorem 5.3** ([101]). *If  $G$  is a 3-connected plane graph containing a vertex of degree at least  $k$ , where  $k$  is a positive integer, then  $G$  contains a  $k$ -path on which  $k-1$  vertices have degree at most  $\frac{5}{2}k$  in  $G$  and the remaining vertex has degree at most  $5k$  in  $G$ .*  $\square$

## 6 Maximum degree problems

The problems mentioned in the previous sections suggest the formulation of more general problems.

**Problem 6.1.** *Let  $\mathcal{H}$  be a family of graphs and let  $H$  be a connected graph that is a proper subgraph of at least one member of  $\mathcal{H}$ . Let  $\varphi(H, \mathcal{H})$  be the smallest integer  $m$  with the property that every graph  $G$  in  $\mathcal{H}$ , that contains  $H$ , contains a copy of  $H$  whose all vertices have degree at most  $m$  in  $G$ . Determine the value of  $\varphi(H, \mathcal{H})$  for given  $H$  and  $\mathcal{H}$ .*

If such a  $\varphi(H, \mathcal{H})$  does not exist then we write  $\varphi(H, \mathcal{H}) = +\infty$ . If  $\varphi(H, \mathcal{H}) < +\infty$ , then we call the graph  $H$  *light* in the family  $\mathcal{H}$ .

Here we consider the family  $\mathcal{H} = \mathcal{P}(\delta, \rho)$  of all 3-connected plane graphs (i.e. the family of all polyhedral graphs [43]) with minimum vertex degree at

least  $\delta$  and minimum face size at least  $\rho$ , where  $\delta$  and  $\rho$  are at least 3. In the sequel, let  $P_k$  and  $C_k$  denote the  $k$ -path and the  $k$ -cycle, respectively, and let

$$\varphi(\delta, \rho; H) = \varphi(H, \mathcal{P}(\delta, \rho)) \quad \text{and} \quad \varphi(\delta, \bar{\rho}, \mathcal{H}) = \varphi(\mathcal{H}, \mathcal{P}(\delta, \bar{\rho})).$$

The result for  $P_1$  derived from Euler's formula can be rewritten as  $\varphi(3, 3; P_1) = 5$ . Kotzig [97] (see also Section 3) proved that each graph  $G \in \mathcal{P}(3, 3)$  contains an edge  $e$  with  $w(e) \leq 13$ . This implies  $\varphi(3, 3; P_2) = 10$ . Theorem 4.1 provides  $\varphi(3, 3; P_3) = 15$ . Fabrici and Jendrol' (see Section 5) generalized these results to arbitrary  $k$ -paths. Their Theorem 5.1 states that the  $k$ -path  $P_k$  is light in the family  $\mathcal{P}(3, 3)$  for every  $k$ . Next theorem summarizes results concerning the constant  $\varphi(P_k, \mathcal{G})$  for several families of 3-connected plane graphs  $\mathcal{G}$ . (Brackets indicate the papers, where the results are proved.)

**Theorem 6.2.**

- (i)  $\varphi(3, 3; P_k) = 5k$  for all  $k \geq 1$ . [36]
- (ii)  $\varphi(4, 3; P_k) = 5k - 7$  for all  $k \geq 8$ . [35]
- (iii)  $5 \lfloor \frac{k}{2} \rfloor \leq \varphi(3, 4; P_k) \leq \frac{5}{2}k$  for all  $k \geq 2$ . [50]
- (iv)  $5k - 325 \leq \varphi(5, 3; P_k) \leq 5k - 7$  for all  $k \geq 68$ . [32]
- (v)  $\frac{5}{3}k - 80 \leq \varphi(3, 5; P_k) \leq \frac{5}{3}k$  for all  $k \geq 935$ . [74]
- (vi)  $2k + 2 \leq \varphi(P_k, \mathcal{F}) \leq \epsilon || + \exists$  for all  $k \geq 2$ . [57]

(Here  $\mathcal{F}$  is the family of 4-connected planar graphs) □

Note that the precise values of  $\varphi(4, 3; P_k)$  are known also for all  $k \leq 7$ , see [35]. We can also show that the lower bound in (iv) cannot be smaller than  $5k - 125$  if  $k \geq 714$  and that the upper bound in (v) is valid for all  $k \geq 2$ .

Theorem 5.2 asserts that the paths are the only light graphs in the family  $\mathcal{P}(3, 3)$ . This motivates the following

**Problem 6.3.** For a given infinite family  $\mathcal{G}$  of plane graphs determine all connected planar graphs that are light in  $\mathcal{G}$ . □

The next theorem provides results of this type.

**Theorem 6.4.** If  $\mathcal{G} \in \{\mathcal{P}(3, 3), \mathcal{P}(3, 4), \mathcal{P}(4, 3), \mathcal{F}\}$ , where  $\mathcal{F}$  is the family of 4-connected planar graphs, then a graph  $H$  is light in the family  $\mathcal{G}$  if and only if  $H$  is  $k$ -path  $P_k$  for some  $k \geq 1$ . □

The proof of Theorem 6.4 for the family  $\mathcal{P}(3, 4)$  is by Harant, Jendrol' and Tkáč [50], for  $\mathcal{P}(4, 3)$  by Fabrici, Hexel, Jendrol' and Walther [35], and for the family  $\mathcal{F}$  by Mohar [108].

In the families  $\mathcal{P}(3, 3)$ ,  $\mathcal{P}(3, 4)$ , and  $\mathcal{P}(4, 3)$ , only the paths  $P_k$  are light. The situation changes significantly for the families  $\mathcal{P}(5, 3)$  and  $\mathcal{P}(3, 5)$ . From Theorem 6.1 it follows that each  $k$ -path for every  $k \geq 1$  is light in both families  $\mathcal{P}(5, 3)$  and  $\mathcal{P}(3, 5)$ . Recently, Hajduk and Soták [52] showed that, for every  $k \geq 6$ , the  $k$ -path  $P_k$  with a certain small graph attached to one of its ends is light in the family  $\mathcal{P}(5, 3)$  (and in  $\mathcal{P}(3, 5)$ ).

Next theorem excludes some families of graphs to be light

**Theorem 6.5** ([32], [73], [74]).

- (i) *No plane connected graph  $H$  with maximum degree at least 5 or with a block having at least 11 vertices is light in  $\mathcal{P}(5, \bar{3})$  (and, hence, in  $\mathcal{P}(5, 3)$ ).*
- (ii) *No plane connected graph  $H$  with maximum degree at least 4 or with a block having at least 19 vertices is light in  $\mathcal{P}(3, 5)$ .  $\square$*

From the classical result of Lebesgue [100] (see Theorem 2.1) it follows that the 3-cycle  $C_3$  is light in  $\mathcal{P}(5, 3)$  and the 5-cycle  $C_5$  is light in  $\mathcal{P}(3, 5)$ . Recently Jendrol' and Madaras in [72] proved that the star  $K_{1,r}$  for  $r \geq 3$  is light in  $\mathcal{P}(5, 3)$  if and only if  $r \in \{3, 4\}$ . However, the problem of determining all graphs that are light in  $\mathcal{P}(5, 3)$ , or in  $\mathcal{P}(3, 5)$  remains open. Jendrol' et al. [73] proved the following theorem (For  $\varphi(C_{10})$  see [104]).

**Theorem 6.6** ([73]). *The  $r$ -cycle  $C_r$  is light in the family  $\mathcal{P}(5, \bar{3})$  of all plane triangulations with minimum degree 5 if and only if  $3 \leq r \leq 10$ . Moreover*

$$\begin{aligned} \varphi(C_3) &= 7, & \varphi(C_4) &= 10, & \varphi(C_5) &= 10, \\ 10 \leq \varphi(C_6) &\leq 11, & 15 \leq \varphi(C_7) &\leq 17, & 15 \leq \varphi(C_8) &\leq 29, \\ 19 \leq \varphi(C_9) &\leq 41, \text{ and} & 20 \leq \varphi(C_{10}) &\leq 415, \end{aligned}$$

where  $\varphi(C_k) = \varphi(5, \bar{3}; C_k)$ .  $\square$

In [55] there is a question for which  $k \geq 11$  the cycle  $C_k$  is light in the family of all 5-connected plane triangulation. The answer is rather surprising

**Theorem 6.7** ([56]). *The  $k$ -cycle  $C_k$  is light in the family of all 5-connected plane triangulation for every  $k$  at least 3.  $\square$*

Hexel and Soták conjecture that, for every  $k \geq 3$  the  $k$ -cycle  $C_k$  is light in the family of all 5-connected planar graphs. More results concerning light graphs in subfamilies of plane graphs with high connectivity can be found in [108] and [57].

An analogue of Theorem 6.4 for the family  $\mathcal{P}(3, \bar{5})$  is

**Theorem 6.8** ([74]). *The  $r$ -cycle  $C_r$  is light in the family  $\mathcal{P}(3, \bar{5})$  if and only if  $r \in \{5, 8, 11, 14\}$ . Moreover  $\varphi(3, \bar{5}; C_5) = 5$ ,  $6 \leq \varphi(3, \bar{5}; C_8) \leq 7$ ,  $10 \leq \varphi(3, \bar{5}; C_{11}) \leq 11$  and  $10 \leq \varphi(3, \bar{5}; C_{14}) \leq 17$ .  $\square$*

By Theorem 6.8 the only cycles that are candidates for being light in  $\mathcal{P}(3, \bar{5})$  are  $C_5, C_8, C_{11}$  and  $C_{14}$ . As we mentioned above,  $C_5$  is light in this family. In [74] the cycles  $C_8$  and  $C_{11}$  are proved to be not light there. It is an open question whether  $C_{14}$  is light in  $\mathcal{P}(3, \bar{5})$ .

The 3-cycle is light in  $\mathcal{P}(5, 3)$  and not light in  $\mathcal{P}(4, 3)$ . The question arises for which subclasses other than  $\mathcal{P}(5, 3)$  the 3-cycle is light? Let  $\mathcal{P}(4, 3, \mathcal{E}_t)$  denote the family of all graphs in  $\mathcal{P}(4, 3)$  having no path consisting of  $t$  vertices all having degree 4. Borodin [17] showed that the triangle  $C_3$  is light in  $\mathcal{P}(4, \bar{3}, \mathcal{E}_t)$  for all  $t \geq 1$ . Mohar, Škrekovski, and Voss [111] showed that the cycle  $C_4$  is light in  $\mathcal{P}(4, 3, \mathcal{E}_t)$  for  $t \in \{2, 3, 4\}$ ; so  $C_4$  is light in  $\mathcal{P}(4, 3, \mathcal{E}_t)$  for  $1 \leq t \leq 4$ . Further,  $C_4$  is not light in  $\mathcal{P}(4, 3, \mathcal{E}_t)$  for all  $t \geq 23$ . For  $5 \leq t \leq 22$  this question is open. For  $r \geq 5$  and  $t \geq 3$  the cycle  $C_r$  is not light in  $\mathcal{P}(4, 3, \mathcal{E}_t)$ . For  $t = 2$  Mohar, Škrekovski, and Voss proved the following result.

**Theorem 6.9** ([111]). *The  $r$ -cycle  $C_r$  is light in the family  $\mathcal{P}(4, 3, \mathcal{E}_2)$  of all 3-connected plane graphs of minimum degree at least 4 and edge-weight at least 9 if and only if  $r \in \{3, 4, 5, 6\}$ . Moreover,*

$$\begin{aligned} \varphi(4, 3, \mathcal{E}_2; C_3) &= 12, \quad \varphi(4, 3, \mathcal{E}_2; C_4) \leq 22, \\ \varphi(4, 3, \mathcal{E}_2; C_5) &\leq 107, \quad \varphi(4, 3, \mathcal{E}_2; C_6) \leq 107. \end{aligned}$$

$\square$

From a theorem of Borodin [6] it follows:  $\varphi(5, 3; C_3) = 7$ . Soták (personal communication) proved  $\varphi(5, 3; C_4) = 11$  and  $\varphi(5, 3; C_5) = 10$ . The proof that  $\varphi(5, 3; C_6) \leq 107$  is by Mohar et al. [111]. The lightness of the 7-cycle in  $\mathcal{P}(5, 3)$  is proved by Madaras et al. [106]. These results are summarized in the next Theorem.

**Theorem 6.10.** *The  $r$ -cycle  $C_r$  is light in  $\mathcal{P}(5, 3)$  if  $r \in \{3, 4, 5, 6, 7\}$ , and is not light in  $\mathcal{P}(5, 3)$  if  $r \geq 11$ . Moreover,*

$$\begin{aligned} \varphi(5, 3; C_3) &= 7, \quad \varphi(5, 3; C_4) = 11, \quad \varphi(5, 3; C_5) = 10, \\ \varphi(5, 3; C_6) &\leq 107, \quad \varphi(5, 3; C_7) \leq 359. \end{aligned}$$

□

It is an open question whether in the class  $\mathcal{P}(5, 3)$  the cycles  $C_8, C_9, C_{10}$  are light or not.

For a plane graph  $G$  let the *edge weight* of  $G$ ,  $w(G)$ , be equal to  $\min\{w(e) : e \in E(G)\}$ ; the *dual edge weight*  $w^*(G)$  be the edge weight of the dual of the graph  $G$ . Let  $\mathcal{P}(\delta, \rho; w, w^*)$  be the family of all polyhedral maps with minimum vertex degree at least  $\delta$ , minimum face size at least  $\rho$ , edge weight at least  $w$ , and dual edge weight at least  $w^*$ . The family  $\mathcal{P}(\delta, \rho; w, w^*)$  is not empty for 35 quadruples  $(\delta, \rho; w, w^*)$ , see [40]. In fact each of these families is infinite. Ferencová and Madaras [41] have considered the question which cycles are light in the family  $\mathcal{P}(\delta, \rho; w, w^*)$ . They received several very interesting results. As an illustration we mention the following one.

**Theorem 6.11** ([41]). *The cycles  $C_3$  and  $C_{10}$  are light in the family  $\mathcal{P}(3, 3; 6, 13)$  while the cycles  $C_k$  for  $4 \leq k \leq 9$  and for  $k \geq 21$  are not light there.* □

It would be interesting to determine the sets of light cycles for families  $\mathcal{P}(\delta, \rho; w, w^*)$ .

Jendrol' and Madaras [72] showed for  $r \geq 3$  that the star  $K_{1,r}$  is light in  $\mathcal{P}(5, 3)$  if and only if  $r \in \{3, 4\}$ . Mohar, Škrekovski, and Voss [111] proved that this is also true in the class  $\mathcal{P}(4, 3, \mathcal{E}_2)$ . For  $t \geq 3$  the only star that is light in  $\mathcal{P}(4, 3, \mathcal{E}_t)$  is  $K_{1,3}$ , see [111].

## 7 Maximum degree of light families

In section 6 we have defined a light subgraph  $H$  in a family  $\mathcal{H}$  of graphs. Here we introduce the concept of a light family  $\mathcal{L}$  of graphs in a family  $\mathcal{H}$  of graphs.

**Problem 7.1.** *Let  $\mathcal{H}$  be a family of graphs, and let  $\mathcal{L}$  be a finite family of connected graphs having the property that every member of  $\mathcal{L}$  is a proper subgraph of at least one member of  $\mathcal{H}$ . Let  $\varphi(\mathcal{L}, \mathcal{H})$  be the smallest integer  $t$  with the property that every graph  $G$  in  $\mathcal{H}$  that has a subgraph  $H$  belonging to  $\mathcal{L}$ , has such a subgraph  $H$  whose vertices all have degree at most  $t$  in  $G$ . Determine the value  $\varphi(\mathcal{L}, \mathcal{H})$  for a given pair of families  $\mathcal{L}$  and  $\mathcal{H}$ .*

If such a  $\varphi(\mathcal{L}, \mathcal{H})$  does not exist, then we write  $\varphi(\mathcal{L}, \mathcal{H}) = +\infty$ . If  $\varphi(\mathcal{L}, \mathcal{H}) < +\infty$ , then we call the family  $\mathcal{L}$  *light* in the family  $\mathcal{H}$ . When  $\mathcal{L}$  is the family  $\mathcal{T}_k$  of all trees on  $k$  vertices, and  $\mathcal{H} = \mathcal{P}(\delta, \rho)$ , we write  $\tau(k, \delta, \rho)$  instead of  $\varphi(\mathcal{T}_k, \mathcal{P}(\delta, \rho))$ .

Obviously,  $\mathcal{T}_1 = \{P_1\}$ ,  $\mathcal{T}_2 = \{P_2\}$ ,  $\mathcal{T}_3 = \{P_3\}$ , and  $\{P_k\} \subset \mathcal{T}_k$  for all  $k \geq 4$ . Hence  $\tau(k, \delta, \rho) = \varphi(\delta, \rho; P_k)$  for  $1 \leq k \leq 3$ , and  $\tau(k, \delta, \rho) \leq \varphi(\delta, \rho; P_k)$ . For  $\mathcal{P}(3, 3)$  Fabrici and Jendrol' proved:

**Theorem 7.2** ([38]).

- (i)  $\tau(1, 3, 3) = 5$ ,
- (ii)  $\tau(2, 3, 3) = 10$ ,
- (iii)  $\tau(k, 3, 3) = 4k + 3$  for any  $k \geq 3$ .

The theorem can be reformulated, as follows:

**Theorem 7.3.** *Every 3-connected planar graph  $G$  of order at least  $k \geq 3$  has a connected subgraph  $K$  of order  $k$  such that the degree of every vertex of  $K$  in  $G$  is at most  $4k + 3$ .*

*The bound  $4k + 3$  is best possible.* □

If the minimum degree of graphs of  $\mathcal{H}$  is increased to 4 then a slightly smaller bound is obtained by Fabrici.

**Theorem 7.4** ([32]).  $\tau(k, 4, 3) = 4k - 1$  for any  $k \geq 4$ . □

In the paper [57] of Hexel and Walther and that of Hexel [55] the reader can find several bounds on  $\tau(k, \delta, \rho)$  for 4-connected graphs.

Here we provide a result that gives a necessary condition for a fixed family  $\mathcal{L}$  of connected plane graphs to be light in  $\mathcal{P}(3, 3)$ .

**Theorem 7.5.** *If  $\mathcal{L}$  is a finite family of connected plane graphs  $H$  such that  $\Delta(H) \geq 3$  or  $\delta(H) \geq 2$ , then  $\mathcal{L}$  is not light in  $\mathcal{P}(3, 3)$ .*

*Proof.* Let  $K$  be the disjoint union of all graphs from the family  $\mathcal{L}$ . Let  $T_0$  be a plane triangulation of  $K$ , that is the graph obtained from  $K$  by inserting necessary edges into  $K$  to obtain a plane triangulation. The rest is the same as in the proof of Theorem 5.2, using  $K$  as  $H$  and  $m$  arbitrarily large. □

**Corollary 7.6.** *If  $\mathcal{L}$  is a light family in  $\mathcal{P}(3, 3)$  then  $\mathcal{L}$  contains a  $k$ -path for some  $k$ .* □

Theorem 7.2 leads to the following problem:

**Problem 7.7.** *Find an optimal set  $\mathcal{S}_k$  of trees on  $k$  vertices such that  $\varphi(\mathcal{S}_k, \mathcal{P}(3, 3)) = 4k + 3$ .*

Applying Theorems 5.1 and 7.2 we can easily get the following.



**Theorem 7.8.** *If  $\mathcal{L} = \{P_k, K_{1,3}\}$ , then  $\varphi(\mathcal{L}, \mathcal{P}(3, 3)) = 4k + 3$ .  $\square$*

The next theorem generalizes this result. For  $i \geq 0$ , let  $S_i$  denote a *generalized 3-star* with a central vertex of degree 3, where the three paths with a common end vertex have  $i + 1$  vertices. Obviously,  $S_0 = K_1$  and  $S_1 = K_{1,3}$ .

**Theorem 7.9** ([89]). *Let  $k$  and  $i$  be integers,  $k \geq 3$  and  $1 \leq i \leq \frac{k}{2}$ . If  $\mathcal{L}_i = \{P_k, S_i\}$ , then  $\varphi(\mathcal{L}_i, \mathcal{P}(3, 3)) = \min\{5k, 4(k + i) - 1\}$ .  $\square$*

## 8 Weight problems

In generalizing Kotzig's Theorem there are several other natural directions. Two possibilities are as follows. Let  $k \geq 1$  be an integer.

**Problem 8.1.** *Find the smallest integer  $f = f(k, \delta, \rho)$  such that whenever a graph  $G \in \mathcal{P}(\delta, \rho)$  contains at least  $k$  vertices, there is a connected subgraph  $H$  of  $G$  of order  $k$  having weight*

$$w_G(H) \leq f(k, \delta, \rho).$$

**Problem 8.2.** *Find the smallest integer  $w = w(k, \delta, \rho)$  such that whenever a graph  $G \in \mathcal{P}(\delta, \rho)$  contains a  $k$ -path, there is a  $k$ -path  $P$  in  $G$  with weight*

$$w_G(P) \leq w(k, \delta, \rho).$$

The precise values of  $w(k, \delta, 3)$  and  $f(k, \rho, \rho)$  are known only for small  $k$ , e.g.  $w(1, 3, 3) = f(1, 3, 3) = 5$ ,  $w(2, 3, 3) = f(2, 3, 3) = 13$  and  $w(2, 4, 3) = f(2, 4, 3)$  by Kotzig [97],  $w(2, 5, 3) = f(2, 5, 3) = 11$  by Wernicke [128],  $w(3, 5, 3) = f(3, 5, 3) = 17$  by Franklin [42], and  $w(3, 3, 3) = f(3, 3, 3) = 21$  by Ando, Iwasaki and Kaneko [3]. For greater  $k$  only estimations are known.

The Problem 8.1 was investigated first time by Enomoto and Ota [31]. In 1999 they proved the following results.

**Theorem 8.3** ([31]). *If  $k \geq 4$ , then*

$$\begin{aligned} 8k - 5 &\leq f(k, 3, 3) \leq 8k - 1, \\ 8k - 5 &\leq f(k, 4, 3) \leq 8k - 3, \text{ and} \\ f(k, 5, 3) &\leq 7k - 2. \end{aligned}$$

*There exist infinitely many  $k$  such that  $7k - 4 \leq f(k, 5, 3)$ .  $\square$*

They expect the following to be true.

**Conjecture 8.4** ([31]). *If  $k$  is an integer,  $k \geq 4$ , then*

$$f(k, 3, 3) = 8k - 5.$$

Problem 8.2 was firstly formulated in [37]. In [37] and [38] it was proved that

$$k \log_2 k \leq w(k, 3, 3) \leq 5k^2.$$

Madaras [101] improved the upper bound showing that  $w(k, 3, 3) \leq \frac{5}{2} k(k+1)$ . Presently the best known upper bound on  $w(k, 3, 3)$  is by Fabrici, Harant, and Jendrol' [34].

**Theorem 8.5** ([34]). *Let  $k$  be an integer,  $k \geq 4$ . Then*

- (i) *every plane triangulation  $T$  that contains a  $k$ -path, contains also a  $k$ -path  $P$  with weight*

$$w_T(P) \leq k^2 + 13k, \text{ and}$$

- (ii) *every 3-connected planar graph that contains a  $k$ -path, contains also a  $k$ -path  $P$  with weight*

$$w_G(P) \leq w(k, 3, 3) \leq \frac{3}{2} k^2 + \mathcal{O}(k).$$

□

The restrictions to 4-connected plane graphs brings a different behaviour. In 2000 Mohar [108] proved a beautiful result.

**Theorem 8.6** ([108]). *Let  $G$  be a 4-connected plane graph on  $n$  vertices, and let  $k$  be an integer,  $1 \leq k \leq n$ . Then  $G$  contains a  $k$ -path  $P$  with weight*

$$w_G(P) \leq 6k - 1.$$

*The bound is sharp.*

□

In fact Theorem 8.6 is a corollary of the following results because 4-connected plane graphs are known to be hamiltonian due to a theorem of Tutte [125].

**Theorem 8.7** ([108]). *Let  $H$  be a hamiltonian planar graph on  $n$  vertices and let  $k$  be an integer,  $1 \leq k \leq n$ . Then  $H$  contains a  $k$ -path  $P$  with weight*

$$w_H(P) \leq 6k - 1.$$

*The bound is tight.*

*Proof.* For a planar hamiltonian graph  $H$  on  $n$  vertices, let  $C_n$  be a Hamilton cycle through vertices  $v_1, v_2, \dots, v_n$  in order. Let  $R_i$  be the part of  $C_n$  on vertices  $v_i, v_{i+1}, \dots, v_{i+k-1}$  (indices modulo  $n$ ). Let  $w(R_i) = \sum_{j=i}^{i+k-1} \deg_H(v_j)$  denote the weight of  $R_i$  in  $H$  (that is, the sum of degrees in  $H$  of vertices of  $R_i$ ). Then

$$\sum_{i=1}^n w(R_i) = k \sum_{v \in V(H)} \deg(v) = 2k|E(H)| \leq 2k(3n - 6).$$

The last inequality follows from the well known corollary (2.4) of Euler's formula. Hence, one of the paths, say  $R_j$ , has weight at most  $2k(3n - 6)/n$ . Since this is less than  $6k$ , we obtain  $w_H(R_j) \leq 6k - 1$ . On the other hand, there are 5-connected plane triangulations which contain precisely 12 vertices of degree 5, and all other vertices are 6-vertices. Moreover, the 5-vertices are as far away from each other as we like. This completes the proof.  $\square$

Clearly 2-connected outerplanar graphs are hamiltonian. Using the idea of Mohar, one can prove that every 2-connected outerplanar graph has a light  $k$ -path of the weight at most  $4k - 1$ . The next theorem is by Fabrici [33].

**Theorem 8.8** ([33]). *Let  $G$  be an outerplanar graph on  $n$  vertices and let  $1 \leq k \leq n$ . Then  $G$  contains a  $k$ -path  $P$  with weight*

$$w_G(P) \leq 4k - 2.$$

*The bound is tight.*  $\square$

In [38] for every  $k \geq 4$  one can find a construction of a 3-connected plane graph  $G$  in which each  $k$ -path has weight at least  $k \log k$ . Developing the ideas of Mohar's proof of Theorem 8.7, Fabrici et al. [34] showed that in 3-connected planar graphs with long cycles one can find  $k$ -paths of weight linear in  $k$  for  $k$  relatively small to the circumference.

**Theorem 8.9** ([34]). *For a 3-connected plane graph  $G$  let  $c(G)$  be the length of a longest cycle in  $G$ . If  $c(G) \geq \sigma|V(G)|$  for some constant  $\sigma > 0$  then for any  $k, 1 \leq k \leq c(G)$ ,  $G$  contains a  $k$ -path  $P$  with weight*

$$w_G(P) \leq \left(\frac{3}{\sigma} + 3\right)k.$$

$\square$

We believe that the following is true

**Conjecture 8.10.** *For every  $k \geq 4$ ,  $w(k, 3, 3) = \mathcal{O}(k \log_2 k)$ .*

For the families  $\mathcal{P}(\delta, \rho)$  with  $(\delta, \rho) \neq (3, 3)$  we know that  $w(k, 5, 3) \leq w(k, 4, 3) \leq 5k^2 - 7k$  for  $k \geq 8$  by [35],  $w(k, 3, 4) \leq \frac{5}{2}k^2$  for  $k \geq 4$  by [50] and  $w(k, 3, 5) \leq \frac{5}{3}k^2$  by [74]. Mohar [108] constructed 3-connected planar graphs proving that  $w(k, 4, 3) \geq \frac{9}{16}k \log_2 k$  and  $w(k, 5, 3) \geq \frac{3}{10}k \log_2 k$ .

Using the notion of weight one can define light graphs in an equivalent form. We start with the following.

**Problem 8.11.** *Let  $\mathcal{H}$  be family of graphs and let  $H$  be a connected graph that is a proper subgraph of at least one member of  $\mathcal{H}$ . Let  $w(H, \mathcal{H})$  be the smallest integer with the property that every graph  $G$  in  $\mathcal{H}$ , that contains  $H$ , contains a copy  $K$  of  $H$  with weight  $w_G(K) \leq w(H, \mathcal{H})$ . Determine the value  $w(H, \mathcal{H})$  for given  $H$  and  $\mathcal{H}$ .*

If such a  $w(H, \mathcal{H})$  does not exist then we write  $w(H, \mathcal{H}) = +\infty$ . It is easy to see that  $H$  is light in  $\mathcal{H}$  if and only if  $w(H, \mathcal{H}) < +\infty$ . Let  $w(\delta, \rho; H) := w(H, \mathcal{P}(\delta, \rho))$ . Problem 8.11 can be now formulated

**Problem 8.12.** *Determine the precise value of  $w(\delta, \rho; H)$  for all light graphs  $H$  in the family  $\mathcal{P}(\delta, \rho)$  for all admissible pairs  $(\delta, \rho)$ .*

The known precise results concerning this problem not mentioned in Section 4 are listed here: Theorem 2.1 of Lebesgue yield:  $w(3, 5; P_4) = 12$ , and  $w(3, 5; P_5) = 17$ . Jendrol' and Madaras [72] proved  $w(5, 3; P_4) = w(5, 3; K_{1,3}) = 23$ , and recently Borodin and Woodall [24] showed that  $w(5, \bar{3}; C_4) = 25$ , and  $w(5, \bar{3}; C_5) = 30$ , and  $w(5, 3; K_{1,4}) = 30$ . Madaras [102] proved that  $w(4, \bar{3}; P_4) \leq 31$  and  $w(5, \bar{3}; P_5) = 29$ .

Analogously we can define the value  $w(\mathcal{L}, \mathcal{H})$  for a finite family of connected graphs  $\mathcal{L}$  and a family  $\mathcal{H}$  having the properties mentioned in Section 7. Namely, every member of  $\mathcal{L}$  is a proper subgraph of at least one member of  $\mathcal{H}$ . Let  $w(\mathcal{L}, \mathcal{H})$  be the smallest integer  $t$  with the property that every graph  $G$  in  $\mathcal{H}$  that has a subgraph belonging to  $\mathcal{L}$  has such a subgraph  $H$  whose vertices have degrees in  $G$  such that sum to at most  $w(\mathcal{L}, \mathcal{H})$ .

Madaras and Škrekovski [105] investigated the conditions related to weight of a fixed subgraph of plane graphs that can enforce the existence of light graphs in families of plane graphs. For the families of plane graphs and triangulations whose edges are of weight at least  $w$  they study the necessary and sufficient conditions for lightness of certain graphs according to values of  $w$ . We like some of their results:

**Theorem 8.13** ([105]). *Let  $\mathcal{R}(w)$  be the family of all planar graphs of minimum degree at least 3 whose edges are of weight at least  $w$ .*

- (i) The 4-path  $P_4$  is light in  $\mathcal{R}(w)$  if and only if  $8 \leq w \leq 13$ .
- (ii) The  $k$ -cycle  $C_k$ ,  $k \in \{3, 4\}$ , is light in  $\mathcal{R}(w)$  if and only if  $10 \leq w \leq 13$ .
- (iii) The star  $K_{1,4}$  is light in  $\mathcal{R}(w)$  if and only if  $9 \leq w \leq 13$ . □

## 9 Light subgraphs of graphs embedded on surfaces

In this section we discuss light subgraphs of connected graphs embedded into surfaces other than the sphere.

Throughout this section we use terminology of [110]. However, we recall some definitions. An *orientable surface*  $\mathbb{S}_g$  of genus  $g$  is obtained from the sphere by adding  $g$  handles. Correspondingly, a *nonorientable surface*  $\mathbb{N}_q$  of genus  $q$  is obtained from the sphere by adding  $q$  crosscaps. The Euler characteristic is defined by

$$\chi(\mathbb{S}_g) = 2 - 2g \text{ and } \chi(\mathbb{N}_q) = 2 - q.$$

By a *surface*  $\mathbb{M}$  we mean either an orientable surface  $\mathbb{S}_g$  or a nonorientable surface  $\mathbb{N}_q$ . By the *genus*  $g$  (the *nonorientable genus*  $q$ ) of a graph  $G$  we mean the smallest integer  $g$  ( $q$ ) such that  $G$  has an embedding into  $\mathbb{S}_g$  ( $\mathbb{N}_q$ , respectively).

If a graph  $G$  is embedded in a surface  $\mathbb{M}$  then the connected regions of  $\mathbb{M} - G$  are called the *faces* of  $G$ . If each face is an open disc then the embedding is called a *2-cell embedding*. If each vertex has degree at least 3 and each vertex of degree  $h$  is incident with  $h$  different faces then  $G$  is called a *map* in  $\mathbb{M}$ . If, in addition,  $G$  is 3-connected and the embedding has "representativity" at least three, then  $G$  is called a *polyhedral map* in  $\mathbb{M}$ , see e.g. Robertson and Vitray [120]. Let us recall that the *representativity* (or *face width*) of a (2-cell) embedded graph  $G$  into a surface  $\mathbb{M}$  is equal to the smallest number  $k$  such that  $\mathbb{M}$  contains a noncontractible closed curve that intersects the graph  $G$  in  $k$  points. We say that  $H$  is a *subgraph* of a polyhedral map  $G$  if  $H$  is a subgraph of the underlying graph of the map  $G$ .

For a map  $G$  let  $V(G)$ ,  $E(G)$  and  $F(G)$  be the vertex set, the edge set and the face set  $G$ , respectively. For a map  $G$  on a surface  $\mathbb{M}$ , Euler's formula states

$$|V(G)| - |E(G)| + |F(G)| = \chi(\mathbb{M}).$$

In 1990, Ivančo [62] generalized the Theorem 3.1 of Kotzig in the following way:

**Theorem 9.1** ([62]). *If  $G$  is a connected graph of orientable genus  $g$  and minimum degree at least 3, then  $G$  contains an edge  $e$  of weight*

$$w(e) \leq \begin{cases} 2g + 13 & \text{for } 0 \leq g \leq 3 \\ 4g + 7 & \text{for } g \geq 3. \end{cases}$$

*Furthermore, if  $G$  does not contain 3-cycles, then*

$$w(e) \leq \begin{cases} 8 & \text{for } g = 0 \\ 4g + 5 & \text{for } g \geq 1. \end{cases}$$

*Moreover, all bounds are best possible.* □

An analogous result for graphs on non-orientable surfaces proved by Jendrol' and Tuhářsky [77] is as follows

**Theorem 9.2** ([77]). *If  $G$  is a connected graph of minimum degree at least 3 on a nonorientable surface of genus  $q \geq 1$ , then  $G$  contains an edge  $e$  with weight*

$$w(e) \leq \begin{cases} 2q + 11 & \text{for } 1 \leq q \leq 2, \\ 2q + 9 & \text{for } 3 \leq q \leq 5, \\ 2q + 7 & \text{for } q \geq 6. \end{cases}$$

*Furthermore, if  $G$  does not contain 3-cycles, then  $G$  contains on edge  $e$  with weight*

$$w(e) \leq \begin{cases} 8 & \text{for } q = 1, \\ 2q + 5 & \text{for } q \geq 2. \end{cases}$$

*Moreover, all bounds are best possible.* □

For the projective plane (the nonorientable surface of the smallest genus) using the same ideas as in the proof of Theorem 3.1, one can easily prove

**Theorem 9.3.** *Every connected projective planar graph of minimum degree at least 3 contains a  $(3, a)$ -edge with  $3 \leq a \leq 10$ , or a  $(4, b)$ -edge with  $4 \leq b \leq 7$  or a  $(5, c)$ -edge with  $5 \leq c \leq 6$ . The bounds 10, 7, and 6 are best possible.* □

The bounds in Theorem 9.1 and 9.2 can be essentially improved if embedded graphs have a "large" number of vertices. Namely, the following holds:

**Theorem 9.4** ([78]). *Let  $G$  be a normal map on surface  $\mathbb{M}$  of Euler characteristic  $\chi(\mathbb{M}) \leq 0$  and let  $n$  be the number of vertices of  $G$ . If*

(a)  $\sum(\deg_G(x) - 6) > 48|\chi(\mathbb{M})|$ , or

(b)  $n > 26|\chi(\mathbb{M})|$ ,

then  $G$  contains an  $(a, b)$ -edge such that

(i)  $a = 3$  and  $3 \leq b \leq 12$ , or

(ii)  $a = 4$  and  $4 \leq b \leq 8$ , or

(iii)  $5 \leq a \leq 6$  and  $5 \leq b \leq 6$ .

The bounds 12, 8, and 6 are best possible.  $\square$

Nothing seems to be done in the following

**Problem 9.5.** Find an analogue of Theorem 3.3 for polyhedral maps on manifolds  $\mathbb{M}$  of Euler characteristic  $\chi(\mathbb{M}) \leq 0$ .

For the projective plane, Sanders established sharp inequalities.

**Theorem 9.6** ([123]). Every normal projective planar map satisfies the following inequality:

$$\begin{aligned} 40 e_{3,3} + 25 e_{3,4} + 16 e_{3,5} + 10 e_{3,6} + \frac{20}{3} e_{3,7} + 5 e_{3,8} + \frac{5}{2} e_{3,9} + 2 e_{3,10} \\ + \frac{50}{3} e_{4,4} + 11 e_{4,5} + 5 e_{4,6} + \frac{5}{3} e_{4,7} \\ + \frac{16}{3} e_{5,5} + 2 e_{5,6} \geq 60. \end{aligned}$$

$\square$

Each of these coefficients is best possible.

In Theorem 9.6 we have the same coefficients as in the planar case (Theorem 3.2), but on the right side the value 120 has been weakened to 60.

In the subclass of all normal projective planar graphs with minimum degree at least 4 we have  $e_{3,j} = 0$  for  $3 \leq j \leq 10$ . Sanders proved that in the resulting inequality all coefficients are best possible.

**Theorem 9.7** ([123]). Every normal projective plane map of minimum degree four satisfies the inequality

$$50 e_{4,4} + 33 e_{4,5} + 15 e_{4,6} + 5 e_{4,7} + 16 e_{5,5} + 6 e_{5,6} \geq 180,$$

and each of these coefficients is best possible.  $\square$

**Theorem 9.8** ([123]). *Every projective plane map of minimum degree five satisfies the inequality*

$$16 e_{5,5} + 7 e_{5,6} \geq 210,$$

*and each of these coefficients is best possible.*  $\square$

If in Theorem 9.7 we use  $e_{4,j} = 0$  for  $4 \leq j \leq 7$  then an inequality is obtained which differs in the coefficient of  $e_{5,5}$ .

**Theorem 9.9** ([123]). *Every projective plane map of minimum degree five satisfies the inequality*

$$18 f_{5,5,5} + 9 f_{5,5,6} + 5 f_{5,5,7} + 4 f_{5,6,6} \geq 72,$$

*and each of these coefficients is best possible.*  $\square$

Euler's formula implies (with some terms left out)  $3v_3 + 2v_4 + v_5 \geq 12$  for the plane and  $3v_3 + 2v_4 + v_5 \geq 6$  for the projective plane. Most of the above inequalities differ only on the right side, where 12 appears for the normal plane maps and 6 for the normal projective plane maps. This is completely true if the minimum degrees are 3 or 4, respectively. The only inequality that does not follow precisely these lines is the light edge inequality for graphs of minimum degree five. For the plane, the coefficient of  $e_{5,5}$  went from  $8/15$  to  $7/15$ . For the projective plane, it is lowered from  $8/15$  to  $16/35$ . Each other coefficient of the inequalities in the projective plane case is equal to the corresponding coefficient in the plane case.

Using the same arguments as for the planar case it is possible to prove the following analogues of Theorem 3.1 and Theorem 7.3

**Theorem 9.10** ([38]). *Every 3-connected projective planar graph  $G$  that contains a  $k$ -path contains also such a path whose all vertices have degree at most  $5k$  in  $G$ . The bound  $5k$  is best possible.*  $\square$

**Theorem 9.11** ([83]). *Let  $k \geq 3$ . Then every 3-connected projective planar graph  $G$  of order at least  $k$  contains a connected subgraph  $H$  of order  $k$  whose vertices all have degree at most  $4k + 3$  in  $G$ .*  $\square$

We generalized these and other results on light subgraphs to surfaces  $\mathbb{M}$  with nonpositive Euler characteristics. For details on these results, see [88]. In the next subsections we give a brief survey only. We mention that all other theorems of Section 7 are also true for 3-connected projective planar graphs.

**Theorem 9.12** ([79]). *Each polyhedral map  $G$  on  $\mathbb{M}$  that contains a  $k$ -path contains also such a  $k$ -path whose all vertices have degree at most  $k \left\lfloor \frac{(5 + \sqrt{49 - 24\chi(\mathbb{M})})}{2} \right\rfloor$  in  $G$ . The equality holds for even  $k$ .*  $\square$



Let  $K_n$  and  $K_n^-$  denote the complete graph on  $n$  vertices and the graph obtained from it by deleting one edge, respectively. For odd  $k$  we can show:

**Theorem 9.13** ([82]). *For each odd  $k$  greater than  $\frac{4}{3} \left\lfloor \frac{5 + \sqrt{49 - 24\chi(\mathbb{M})}}{2} \right\rfloor + 1$ :*

- (i) *the upper bound in Theorem 9.12 is attained at infinite many orientable surfaces and at infinite nonorientable surfaces, where these surfaces are characterized by the fact that each of these surfaces has a triangular embedding of a  $K_n^-$ ;*
- (ii) *the upper bound in Theorem 9.12 is not attained at infinite many orientable surfaces and at infinite many nonorientable surfaces, where these surfaces are characterized by the fact that each of these surfaces has a triangular embedding of a  $K_n^-$  (in this case an upper bound is*

$$\left\lfloor \left(k - \frac{1}{3}\right) \frac{5 + \sqrt{49 - 24\chi(\mathbb{M})}}{2} \right\rfloor.$$

□

A polyhedral map  $G$  is called *large* if it has a large number of vertices or large positive  $k$ -charge, where the *positive charge* of  $G$  is  $\sum_{\deg_G(u) > 6k} (\deg_G(u) - 6k)$ .

**Theorem 9.14** ([80], [81]). *Every large polyhedral map on a surface  $\mathbb{M}$  of nonpositive Euler characteristic that contains a  $k$ -path contains also such a  $k$ -path whose all vertices have degree, in  $G$ , at most  $6k$  for  $k = 1$  or even  $k \geq 2$  and at most  $6k - 2$  for odd  $k \geq 3$ . Moreover, these bounds are tight.*

□

The upper bound on maximum degree of vertices of light paths of polyhedral maps on a surface  $\mathbb{M}$  depends on  $\sqrt{|\chi(\mathbb{M})|}$ . In arbitrary embedding of 3-connected graphs (multigraphs) in  $\mathbb{M}$  this degree bound is a linear function of  $|\chi(\mathbb{M})|$ .

**Theorem 9.15** ([84]). *Each 3-connected multigraph  $G$  on  $\mathbb{M}$  without loops and 2-faces that has a  $k$ -path contains also such a  $k$ -path whose all vertices have degree at most  $(6k - 2\varepsilon)(1 + |\chi(\mathbb{M})|/3)$  in  $G$ , where  $\varepsilon = 0$  if  $k \geq 2$  is even, and  $\varepsilon = 1$  if  $k \geq 3$  is odd. The bounds are best possible.*

□

**Theorem 9.16** ([85]). *For 3-connected graphs on  $\mathbb{M}$  the precise degree bound is*

$$2 + \left\lfloor (6k - 6 - 2\varepsilon) \left(1 + |\chi(\mathbb{M})|/3\right) \right\rfloor \text{ for } k \geq 4.$$

□

Fabrici, Hexel, Jendroľ, and Walther [35] proved that each 3-connected plane graph of minimum degree at least 4 that has a  $k$ -path contains also such a  $k$ -path whose all vertices have degree at most  $5k - 7$  in  $G$ . This bound is sharp for  $k \geq 8$ . For surfaces other than the sphere, we have

**Theorem 9.17** ([86]). *For  $k \geq 8$ , each large polyhedral map  $G$  on  $\mathbb{M}$  of minimum degree at least 4 that contains a  $k$ -path contains such a  $k$ -path whose vertices have degree at most  $6k - 12$  in  $G$ . This bound is sharp for even  $k$ , and it must be at least  $6k - 14$  for odd  $k$ .  $\square$*

In 3-connected plane graphs of minimum degree at least 5 only the upper bound  $5k - 7$  [35] is known. For large polyhedral maps on 2-manifolds  $\mathbb{M}$ , the degree bound is not a linear function on  $k$ .

**Theorem 9.18** ([86]). *Let  $k \geq 66$  be an integer. Each large polyhedral map  $G$  on  $\mathbb{M}$  of minimum degree at least 5 that contains a  $k$ -path contains such a  $k$ -path whose all vertices have degree at most  $6k - \log_2 k + 2$ . Moreover, the exact bound is at least  $6k - 72 \log_2 k - 132$ .  $\square$*

In families of polyhedral maps of Theorem 9.11, 9.13 and 9.16 and in embeddings of 3-connected multigraphs (Theorem 9.14), and in embeddings of 3-connected graphs (Theorem 9.15) only  $k$ -paths are light for every  $k$ .

In the families of large polyhedral maps of minimum degree at least 5 on surfaces of nonpositive genus one can prove the existence of other light graphs. We have obtained that the cycle  $C_3$  is light there (see [90]) as well as all proper spanning subgraphs  $H$  of the complete graph  $K_4$ , while  $K_4$  itself is not light (see [91]). The 5-cycle  $C_5$  and the 5-cycle with one or two diagonals are light in this class as well (see [92]). For other results see [87].

Fabrici and Jendroľ [38] proved that each 3-connected plane graph  $G$  of order at least  $k$  contains a connected subgraph of order  $k$  whose all vertices have degree at most  $4k + 3$ . We have proved that this holds also for the projective plane. For polyhedral maps on  $\mathbb{M}$ , the degree bound for connected subgraphs of order  $k$  again depends on  $\sqrt{|\chi(\mathbb{M})|}$ . (This result is not presented here). For polyhedral maps with many vertices we proved

**Theorem 9.19** ([83]). *For  $k \geq 2$ , each polyhedral map on  $\mathbb{M}$  having at least  $(8k^2 + 6k - 6)|\chi(\mathbb{M})| + 1$  vertices contains a connected subgraph of order  $k$  whose all vertices have degree at most  $4k + 4$ . This bound is best possible.  $\square$*

For polyhedral maps the bound depends on  $\sqrt{|\chi(\mathbb{M})|}$ . In arbitrary embeddings of 3-connected graphs (multigraphs) in  $\mathbb{M}$  the bound is a linear function of  $|\chi(\mathbb{M})|$ .

**Theorem 9.20** ([84]). *For  $k \geq 2$  each 3-connected multigraph  $G$  on  $\mathbb{M}$  that has no loops or 2-faces and has order at least  $k$  contains a connected subgraph of order  $k$  whose all vertices have degree at most  $\lfloor (4k + 4)(1 + \frac{|\chi(\mathbb{M})|}{3}) \rfloor$  in  $G$ . This bound is sharp.  $\square$*

**Theorem 9.21** ([85]). *For 3-connected graphs on  $\mathbb{M}$ , the precise degree bound is*

$$2 + \left\lfloor (4k - 2) \left( 1 + \frac{|\chi(\mathbb{M})|}{3} \right) \right\rfloor \text{ for } k \geq 5.$$

$\square$

**Theorem 9.22** ([84]). *For both large 3-connected multigraphs on  $\mathbb{M}$  without loops and 2-faces and for large 3-connected graphs on  $\mathbb{M}$  having at least  $k \geq 2$  vertices the precise degree bound is  $4k + 4$ .  $\square$*

We finish this section with an analogue of Theorem 8.9 recently proved by Kawarabayashi et al. [96].

**Theorem 9.23** ([96]). *For any non-spherical surface  $\mathbb{M}$  and any positive integer  $t$  there exist positive integers  $r(\mathbb{M})$  and  $n_0 = n_0(\mathbb{M}, t)$  such that if  $G$  is a 3-connected graph with  $n \geq n_0$  vertices embedded into  $\mathbb{M}$  with representativity  $r(\mathbb{M})$ , then  $G$  has a connected subgraph  $H$  of  $t$  vertices such that*

$$w_G(H) = \sum_{v \in V(H)} \deg_G(v) \leq 8t - 1.$$

## 10 Related topics

The concept of the weight of an edge, of a face, of a path, or of a cycle respectively, as presented in this survey has served as a starting point for research in several other directions. We briefly mention some of them.

1. A graph is called 1-planar if it can be drawn in the plane so that each its edge is crossed by at most one other edge. An investigation of 1-planar graphs was introduced by Ringel [119] in 1965. He proved, among other results, that each 1-planar graph contains a vertex of degree at most 7. Fabrici and Madaras in [39] have started an investigation of the existence of light subgraphs in the family of 1-planar graphs. They have proved that each 3-connected 1-planar graph contains an edge with both endvertices of degree at most 20. They also have presented similar results concerning larger structures in 1-planar graphs with additional constrains. Hudák and Madaras [61] found out that each 1-planar graph of minimum degree 5 and girth 4 contains a 5-vertex adjacent to a vertex of degree at most 6, a 4-cycle whose

every vertex has degree at most 9 and a 4-star  $K_{1,4}$  with all vertices having degree at most 11. It would be very interesting to determine the set of all light graphs in the family of 1-planar graphs.

**2.** The idea of light edges was used by P. Erdős who formulated in 1990 at the conference in Prachatice (Czechoslovakia) the following max-min problem (see [63]): For a graph  $G = (V, E)$ , its edge weight  $w(G)$  is defined as  $\min\{w(e) | e \in E\}$ . Let  $\mathcal{G}(n, m)$  denote the family of all graphs with  $V = |n|$  vertices and  $|E| = m$  edges. Determine the value

$$W(n, m) = \max\{w(G) | G \in \mathcal{G}(n, m)\}.$$

Ivančo and Jendrol' [63] have proved several partial results. Recently Jendrol' and Schiermeyer [76] have found a complete solution to Erdős's question and characterized all graphs on  $n$  vertices and  $m$  edges attaining this minimum weight.

A graph  $G$  from  $\mathcal{G}(n, m)$  having no isolated vertices is *degree-constrained* if  $a = \frac{2m}{n} < 2\delta$ , where  $a$  is the average degree of  $G$  and  $\delta = \delta(G)$  is minimum degree of  $G$ . Bose, Smid, and Wood [26] proved that every degree-constrained graph has an edge  $uv$  with both  $\deg(u)$  and  $\deg(v)$  at most  $\lfloor d \rfloor$  where  $d = \frac{a\delta}{2\delta - a}$ . Moreover, they investigate matchings consisting of light edges in degree-constrained graphs.

**3.** The idea of a light path has been considered also in the family of all  $K_{1,r}$ -free graphs,  $r \geq 3$ , i.e. graphs without  $K_{1,r}$  as an induced subgraph. Harant et al. [51] proved among others the following rather surprising result.

**Theorem 10.1** ([51]). *If  $G$  is a  $K_{1,r}$ -free graph on  $n$  vertices, where  $r \geq 3$ , then each induced path of length at least  $2r - 1$  and each induced cycle of length at least  $2r$  in  $G$  has the weight at most  $(2r - 2)(n - \alpha_0)$ , where  $\alpha_0$  is the independence number of  $G$ . Moreover, this bound is tight.  $\square$*

Recently, Voigt presented to the conference Cycles and Colourings 2010 the following generalization of the above theorem.

**Theorem 10.2** ([117]). *Let  $G$  be a connected  $K_{1,r}$ -free graph,  $r \geq 3$ , of order  $n$  and independence number  $\alpha_0$ . If  $H$  is a  $k$ -colourable induced subgraph of  $G$ , then its weight  $w(H)$  in  $G$  is*

$$w(H) = \sum_{v \in V(H)} \deg_G(v) \leq k(r - 1)(n - \alpha_0).$$

*The bound is tight.  $\square$*

4. For a polyhedral map  $G$  the  $p$ -vector is defined in [43] to be a sequence  $p = (f_i | i \geq 3)$  where  $f_i$  denotes the number of  $i$ -gonal faces of  $G$ . Rosenfeld [122] started to investigate the problem of characterization of  $p$ -vectors of 3-connected non-regular plane graphs (i.e. non-regular polyhedral graphs) whose edges have the same, constant, weight. Jendrol' and Jucovič [71] made first steps in dealing with this problem for polyhedral maps on orientable surfaces. There is a lot of open problems in this topic, see e.g. [65], [71].

5. Similarly, Jucovič [64] suggested studying polyhedral maps with constant weight of faces. By the weight of a face  $\alpha$  we mean the sum of degrees of vertices incident with  $\alpha$ . All Platonic solids and all duals to Archimedean solids have constant face weights. Ivančo and Trenkler [64], and Horňák and Ivančo [58] determined the number of nonisomorphic 3-connected plane graphs having prescribed face weight  $w$ . There are infinitely many such graphs if and only if  $16 \leq w \leq 21$  or  $w = 23$  or  $w = 25$ . There is exactly one such graph if  $w = 9$  or  $w = 11$ ; for  $w = 14$  there are four, and for  $w = 15$  there are ten such graphs. For the remaining  $w \geq 12$  there are exactly two such graphs. For  $w = 10$  no such graph exists. Nothing is known about polyhedral maps with constant face weight on surfaces other than the sphere.

6. The idea of the weight of an edge  $e = uv$  being the degree sum of the endvertices  $u$  and  $v$  motivated Jendrol' and Ryjáček [75] to introduce the concept of *tolerance* of the edge  $e$ . The *tolerance*  $\tau(e)$  of the edge  $e = uv$  is defined to be the absolute value of the difference of degrees of the vertices  $u$  and  $v$ ,

$$\tau(e) = |\deg(u) - \deg(v)|.$$

Necessary and sufficient conditions for the existence of connected planar and 3-connected planar graphs with constant edge tolerance appeared in [75]. Several constructions of graphs with constant edge tolerance for general graphs appear in Acharya and Vartak [1]. An open problem is to find necessary and sufficient conditions for graphs with constant edge tolerance embedded into surfaces different from the sphere.

7. Motivated by "light" results, Mohar [109] considered analogical problems for infinite planar graphs. He used the discharging method to prove some new results in this direction. The general outline of the method is presented in [109]. Many applications are given there, including results on light subgraphs and the following: Planar graphs with only finitely many vertices of degree at most 5 and with subexponential growth contain arbitrarily large finite submaps of the tessellation of the plane or of some tessellation of the cylinder by equilateral triangles.

8. We feel a need to write down few remarks concerning the Lebesgue's Theorem (Theorem 2.1). It was published in 1940 but it remained unnoticed until 1967 when Ore's book [113] appeared. Ore was aware of the importance of the theorem and therefore he included it into his book together with a complete proof (using Lebesgue's theory of the Euler's contributions) and with corollaries. But only after its application in an Ore and Plummer's [114] problem on cyclic colouring of plane graphs by Plummer and Toft [116] the theorem started to attract people. (Note that the *cyclic colouring* is a colouring of vertices of plane graphs in such a way that the vertices incident with the same face receive different colours.) To prove the conjecture of Plummer and Toft [116] (see also [93]) that every 3-connected plane graph  $G$  has a cyclic colouring of its vertices with  $\Delta^* + 2$  colours (here  $\Delta^*$  means the size of the largest face in  $G$ ) there were attempts to improve some terms in the theorem.

Recall that the unavoidable set by Lebesgue's theorem (see Theorem 2.1) consists of six infinite series of faces, namely series of  $(3, b, c)$ -triangles for  $3 \leq b \leq 6$  and  $c \geq 3$ ,  $(4, 4, c)$ -triangles for  $c \geq 4$ , and  $(3, 3, 3, d)$ -quadrangles for  $d \geq 3$ , and 126 individual faces. Horňák and Jendrol' [59] reduced two infinite series of  $(3, b, c)$ -triangles for  $5 \leq b \leq 6$  and  $c \geq 5$  to finite ones. Namely they proved the existence of an unavoidable set of configurations consisting of four infinite series and 160 individual configurations.

Note that, in general, none of infinite series of  $(3, b, c)$ -triangles for  $3 \leq b \leq 4, c \geq 3$ ,  $(4, 4, c)$ -triangles for  $c \geq 4$ , and  $(3, 3, 3, d)$ -quadrangles for  $k \geq 3$  can be omitted. In [60] Horňák and Jendrol' replaced the serie of  $(4, 4, c)$ -triangles with few individual terms together with a configuration consisting of a chain of three quadrangles.

On the other side Borodin [19] succesfully reduced individual terms of the theorem to 95 by letting all six infinite series  $(3, b, c)$ -triangles for  $3 \leq b \leq 6$ ,  $(4, 4, c)$ -triangles, and  $(3, 3, 3, c)$ -quadrangles, all with  $c \geq 3$ . In his paper [19] Borodin posed the following problem: Find the best possible version(s) of Lebesgue's theorem.

For other discussions concerning the Lebesgue theorems readers are recommended to [19], [21], and [59]. (For a present situation concerning the conjecture of Plummer and Toft, see e.g. [29], [30].)

### Acknowledgements

This work was supported by the Science and Technology Assistance Agency under the contract APVV-0007-07, and by Slovak VEGA Grant 1/0428/10 is acknowledged. The authors wish to thank the anonymous referee for his careful reading of the paper and very detailed proposals that helped to improve the presentation of the paper.

## References

- [1] B. D. Acharya, M. N. Vartak *On the construction of graphs with given constant valence-difference ( $S$ ) on each of their lines*, *Wiss. Z. TH Ilmenau* 26, (1977), No. 6, 33–60.
- [2] B. A. Aksenov, O. V. Borodin, A. N. Glebov *On a structural property of plane graphs*, *Diskretn. Anal. Issled. Oper. Ser. 1*, 7 (2000), 5–16, (Russian).
- [3] K. Ando, S. Iwasaki, A. Kaneko *Every 3-connected planar graph has a connected subgraph with small degree sum*, Annual Meeting in Mathematical Society of Japan (1993), (Japanese).
- [4] K. Appel, W. Haken *Every planar map is four colorable*, *Contemp. Math.* 98, Providence, RI: American Mathematical Society (AMS). XV, 751 (1989).
- [5] N. Z. Biggs, E. K. Lloyd, R. J. Wilson *Graph Theory 1736-1936*, Clarendon Press, Oxford (1998).
- [6] O. V. Borodin *Solution of problems of Kotzig and Grünbaum concerning the isolation of cycles in planar graphs*, *Math. Notes* 46 (1989), 835–837 (English translation) *Math. Zametki* 46 (1989), 9–12, (Russian).
- [7] O. V. Borodin *On the total coloring of planar graphs*, *J. Reine Angew. Math.* 394 (1989), 180–185.
- [8] O. V. Borodin *Computing light edges in planar graphs*, in *Topics in Combinatorics and Graph Theory* (R. Bodendiek and R. Henn Eds.) *Physica-Verlag, Heidelberg* (1990), 137–144.
- [9] O. V. Borodin *Structural properties and colorings of plane graphs*, *Ann. Discrete Math.* 51 (1992), 31–37.
- [10] O. V. Borodin *Precise lower bound for the number of edges of minor weight in planar maps*, *Math. Slovaca* 42 (1992), 129–142.
- [11] O. V. Borodin *Structural properties of planar maps with minimum degree 5*, *Math. Nachr.* 158 (1992), 109–117.
- [12] O. V. Borodin *Joint extension of two theorems of Kotzig on 3-polytopes*, *Combinatorica* 13 (1993), 121–125.

- [13] O. V. Borodin *Simultaneous coloring of edges and faces of plane graphs*, Discrete Math. 128 (1994), 21–33.
- [14] O. V. Borodin *Triangles with restricted degree sum of their boundary vertices in plane graphs*, Discrete Math. 137 (1995), 45–51.
- [15] O. V. Borodin *More about the weight of edges in planar graphs*, Tatra Mt. Math. Publ. 9 (1996), 11–14.
- [16] O. V. Borodin *Minimal vertex degree sum of a 3-path in plane maps*, Discuss. Math. Graph Theory 17 (1997), 279–284.
- [17] O. V. Borodin *Triangulated 3-polytopes without faces of low weight*, Discrete Math. 186 (1998), 281–285.
- [18] O. V. Borodin *Colourings and topological representations of graphs*, Diskretn. Anal. Issled. Oper. 3 (1996), 3–27, (Russian).
- [19] O. V. Borodin *An improvement of Lebesgue’s theorem on the structure of minor faces of 3-polytopes*, Diskretn. Anal. Issled. Oper. 9, No. 3 (2002), 29–35, (Russian).
- [20] O. V. Borodin, H. Broersma, A. N. Glebov, J. van den Heuvel *The structure of plane triangulations in terms of clusters and stars*, Diskretn. Anal. Issled. Oper. Ser. 1., 8 (2001), 15–39, (Russian).
- [21] O. V. Borodin, D. V. Loparev *The height of small faces in normal plane maps*, Diskretn. Anal. Issled. Oper. 5 (1998), 6–17, (Russian).
- [22] O. V. Borodin, A. V. Kostochka, D. R. Woodall *Light edges and list total colourings of multigraphs*, J. Combin. Theory Ser. B 71 (1997), 184–204.
- [23] O. V. Borodin, D. P. Sanders *On light edges and triangles in planar graphs of minimum degree five*, Math. Nachr. 170 (1994), 19–24.
- [24] O. V. Borodin, D. R. Woodall *Short cycles of low weight in normal plane maps with minimum degree five*, Discuss. Math. Graph Theory 18 (1998), 159–164.
- [25] O. V. Borodin, D. R. Woodall *Cyclic degrees of 3-polytopes*, Graphs Combin. 15 (1999), 267–277.
- [26] P. Bose, M. Smid, D. R. Wood *Light edges in degree-constrained graph*, Discrete Math. 282 (2004), 35–41.



- [27] R. Cole, L. Kowalik, R. Škrekovski *A generalization of Kotzig's theorem and its application*, SIAM J. Discrete Math. 21 (2007), 93–106.
- [28] Z. Dvořák, R. Škrekovski *A theorem about contractible light edge*, SIAM J. Discrete Math. 20 (2006), 55–61.
- [29] H. Enomoto, M. Horňák *A general upper bound for the cyclic chromatic number of 3-connected plane graphs*, J. Graph Theory 62 (2009), 1–25.
- [30] H. Enomoto, M. Horňák, S. Jendrol' *Cyclic chromatic number of 3-connected plane graphs*, SIAM J. Discrete Math. 14 (2001), 121–137.
- [31] H. Enomoto, K. Ota *Connected subgraphs with small degree sum in 3-connected planar graphs*, J. Graph Theory 30 (1999), 191–203.
- [32] I. Fabrici *On vertex-degree restricted subgraphs in polyhedral graphs*, Discrete Math. 256 (2002), 105–114.
- [33] I. Fabrici *Light graphs in families of outerplanar graphs*, Discrete Math. 307 (2007), 866–872.
- [34] I. Fabrici, J. Harant, S. Jendrol' *Paths of low weight in planar graph*, Discuss. Math. Graph Theory 28 (2008), 121–135.
- [35] I. Fabrici, E. Hexel, S. Jendrol', H. Walther *On vertex-degree restricted paths in polyhedral graphs*, Discrete Math. 212 (2000), 61–73.
- [36] I. Fabrici, S. Jendrol' *An inequality concerning edges of minor weight in convex 3-polytopes*, Discuss. Math. Graph Theory 16 (1996), 81–87.
- [37] I. Fabrici, S. Jendrol' *Subgraphs with restricted degrees of their vertices in planar 3-connected graphs*, Graphs Combin. 13 (1997), 245–250.
- [38] I. Fabrici, S. Jendrol' *Subgraphs with restricted degrees of their vertices in planar graphs*, Discrete Math. 191 (1998), 83–90.
- [39] I. Fabrici, T. Madaras *The structure of 1-planar graphs* Discrete Math. 307 (2007), 854–865.
- [40] B. Ferencová, T. Madaras *On the structure of polyhedral graphs with prescribed edge and dual edge weight*, Acta Univ. M. Belii Math. 12 (2005), 13–18.
- [41] B. Ferencová, T. Madaras *Light graph in families of polyhedral graphs with prescribed minimum degree, face size, edge and dual edge weight*, Discrete Math. 310 (2010), 1661–1675.

- [42] P. Franklin *The four color problem*, Amer. J. Math. 44 (1922), 225–236.
- [43] B. Grünbaum *Convex polytopes*, Interscience, New York (1967) or Springer, New York (2003), Second Edition.
- [44] B. Grünbaum *Acyclic colorings of planar graphs*, Israel J. Math. 14 (1973), 390–408.
- [45] B. Grünbaum *Polytopal graphs*, in Studies in Graph Theory (D. R. Fulkerson, ed.), MAA Studies in Mathematics 12 (1975), 201–224.
- [46] B. Grünbaum *New views on some old questions of combinatorial geometry*, Int. Teorie Combinatorie, Rome, (1973) 1 (1976), 451–468.
- [47] B. Grünbaum, G. C. Shephard *Analogues for tiling of Kotzig’s theorem on minimal weights of edges*, Ann. Discrete Math. 12 (1982), 129–140.
- [48] A. Hackmann, A. Kemnitz *List edge colorings of outerplanar graphs*, Ars Combin. 60 (2001), 181–185.
- [49] J. Harant, S. Jendrol’ *On the existence of specific stars in planar graphs*, Graphs Combin. 23 (2007), 529–543.
- [50] J. Harant, S. Jendrol’, M. Tkáč *On 3-connected plane graphs without triangular faces*, J. Combin. Theory Ser. B 77 (1999), 150–161.
- [51] J. Harant, S. Jendrol’, B. Randerath, Z. Ryjáček, I. Schiermeyer, M. Voigt *On weight of induced paths and cycles in claw-free graphs*, J. Graph Theory 36 (2001), 131–143.
- [52] R. Hajduk, R. Soták *On large light graphs in families of polyhedral graphs*, Discrete Math. 310 (2010), 417–422.
- [53] H. Heesch *Untersuchungen zum Vierfarbenproblem*, Hochschulsriptum 810/a/b, Bibliographisches Institut, Mannheim 1969.
- [54] J. van den Heuvel, S. McGuinness *Coloring the square of a planar graph*, J. Graph Theory 42 (2003), 110–124.
- [55] E. Hexel *On light graphs in the family of 4-connected planar graphs*, Discrete Math. 251 (2002), 103–107.
- [56] E. Hexel, R. Soták *On light cycles in the family of 5-connected triangulations*, Manuscript, June 2003.

- [57] E. Hexel, H. Walther *On vertex-degree restricted paths in 4-connected planar graphs*, Tatra Mt. Math. Publ. 18 (1999), 1–13.
- [58] M. Horňák, J. Ivančo *On the number of 3-polytopes with constant face weight*, Studia Sci Math. Hungar. 39 (2002), 1–19.
- [59] M. Horňák, S. Jendroľ *Unavoidable set of face types for planar maps*, Discuss. Math. Graph Theory 16 (1996), 123–141.
- [60] M. Horňák, S. Jendroľ *On a Conjecture by Plummer and Toft*, J. Graph Theory 30 (1999), 177–189.
- [61] D. Hudák, T. Madaras *On local structure of 1-planar graphs of minimum degree 5 and girth 4*, Discuss. Math. Graph Theory 29 (2009), 385–400.
- [62] J. Ivančo *The weight of a graph* Ann. Discrete Math. 51 (1992), 113–116.
- [63] J. Ivančo, S. Jendroľ *On extremal problems concerning weights of edges of graphs*, Colloq. Math. Soc. János Bolyai **60** Sets, Graphs and Numbers, Budapest (1991), North Holland (1992), 339–410.
- [64] J. Ivančo, M. Trenkler *3-polytopes with constant face weight*, Studia Sci. Math. Hungar. 35 (1999), 1–15.
- [65] S. Jendroľ *Convex 3-polytopes with constant edge weight*, Colloq. Math. Soc. János Bolyai **63** Intuitive geometry, Szeged (Hungary, 1991) North Holland, Budapest (1994), 173–179.
- [66] S. Jendroľ *A structural property of convex 3-polytopes*, Geom. Dedicata 68 (1997), 91–99.
- [67] S. Jendroľ *Light paths in 3-connected planar graphs (Problem 294)*, Discrete Math. 191 (1998), 248.
- [68] S. Jendroľ *A short proof of Kotzig’s theorem on minimal edge weights of convex 3-polytopes*, Proc. Internat. Scient. Conference of Math. (V. Balint Ed.) Žilina (1998) EDIS Univ. Press, Žilina (1999), 35–38.
- [69] S. Jendroľ *Triangles with restricted degrees of their boundary vertices in plane triangulations* Discrete Math. 196 (1999), 177–196.
- [70] S. Jendroľ *Paths with restricted degrees of their vertices in planar graphs* Czechoslovak Math. J. 49 (124), (1999), 481–490.

- [71] S. Jendroľ, E. Jucovič *On face-vectors of maps with constant weight of edges*, Studia Sci. Math. Hungar. 17 (1982), 159–175.
- [72] S. Jendroľ, T. Madaras *On light subgraphs in plane graphs of minimum degree five*, Discuss. Math. Graph Theory 16 (1996), 207–217.
- [73] S. Jendroľ, T. Madaras, R. Soták, Z. Tuza *On light cycles in plane triangulations*, Discrete Math. 197/198 (1999), 453–467.
- [74] S. Jendroľ, P. Owens *On light graphs in 3-connected plane graphs without triangular or quadrangular faces*, Graphs Combin. 17 (2001), 659–680.
- [75] S. Jendroľ, Z. Ryjáček *3-polytopes of constant tolerance of edges*, Comment. Math. Univ. Carolin. 22,4 (1981), 843–850.
- [76] S. Jendroľ, I. Schiermeyer *On max-min problem concerning weights of edges*, Combinatorica 21 (2001), 351–359.
- [77] S. Jendroľ, M. Tuhársky *A Kotzig type theorem for non-orientable surfaces*, Math. Slovaca 56 (2006), 245–253.
- [78] S. Jendroľ, M. Tuhársky, H.-J. Voss *A Kotzig type theorem for large maps on surfaces* Tatra Mt. Math. Publ. 27 (2003), 153–162.
- [79] S. Jendroľ, H.-J. Voss *A local property of polyhedral maps on compact 2-dimensional manifolds*, Discrete Math. 212 (2000), 111–120.
- [80] S. Jendroľ, H.-J. Voss *A local property of large polyhedral maps on compact 2-dimensional manifolds*, Graphs Combin. 15 (1999), 303–313.
- [81] S. Jendroľ, H.-J. Voss *Light paths with an odd number of vertices in large polyhedral maps*, Ann. Comb. 2 (1998), 313–324.
- [82] S. Jendroľ, H.-J. Voss *Light paths with an odd number of vertices in polyhedral maps*, Czechoslovak Math. J. 50(120) (2000), 555–564.
- [83] S. Jendroľ, H.-J. Voss *Subgraphs with restricted degrees of their vertices in large polyhedral maps on compact 2-manifolds*, European J. Combin. 20 (1999), 821–832.
- [84] S. Jendroľ, H.-J. Voss *Light subgraphs of multigraphs on compact 2-dimensional manifolds*, Discrete Math. 233 (2001), 329–351.

- [85] S. Jendrol', H.-J. Voss *Two local and one global properties of 3-connected graphs on compact 2-dimensional manifolds*, J. Combin. Theory Ser. B 93 (2005), 1–21.
- [86] S. Jendrol', H.-J. Voss *Light paths in large polyhedral maps with prescribed minimum degree*, Australas. J. Combin. 25 (2002), 79–102.
- [87] S. Jendrol', H.-J. Voss *Subgraphs with restricted degrees of their vertices in polyhedral maps on compact 2-manifolds*, Ann. Comb. 5 (2001), 211–226.
- [88] S. Jendrol', H.-J. Voss *Light subgraphs of graphs embedded in 2-dimensional manifolds of Euler characteristic  $\leq 0$  - a survey*, In "Paul Erdős and his Mathematics, I" (G. Halász, L. Lovász, M. Simonovits, T. Sos, Eds.), Springer, Budapest (Hungary) 1999, Bolyai Society Mathematicae Studies, 11. Budapest (2002), 375–411.
- [89] S. Jendrol', H.-J. Voss *Light classes of generalized stars in polyhedral maps on surfaces*, Discuss. Math. Graph Theory 24 (2004), 85–107.
- [90] S. Jendrol', H.-J. Voss *Light subgraphs of order  $\leq 3$  in large maps of minimum degree 5 on compact 2-manifold*, European J. Combin. 26 (2005), 457–471.
- [91] S. Jendrol', H.-J. Voss *Light subgraphs of order 4 in large maps of minimum degree 5 on compact 2-manifold*, Australas. J. Combin. 28 (2003), 171–199.
- [92] S. Jendrol', H.-J. Voss *The 5-cycle  $C_5$  is light in large maps of minimum degree 5 on compact 2-manifolds*, Bul. Acad. Stiinte Repub. Mold. Mat. 3(40) (2002), 106–124.
- [93] T. R. Jensen, B. Toft *Graph coloring problems*, Wiley-Interscience, New York (1995).
- [94] E. Jucovič *Strengthening of a theorem about 3-polytopes*, Geom. Dedicata 13 (1974), 233–237.
- [95] E. Jucovič *Convex polytopes*, Veda Bratislava, 1981, (Slovak).
- [96] K. Kawarabayashi, A. Nakamoto, K. Ota *Subgraphs of graphs on surfaces with high representativity*, J. Combin. Theory Ser. B 89 (2003), 207–229.

- [97] A. Kotzig *Contribution to the theory of Eulerian polyhedra*, Mat. Čas SAV (Math. Slovaca) 5 (1955), 101–113, (Slovak).
- [98] A. Kotzig *From the theory of eulerian polyhedrons*, Mat. Čas SAV (Math. Slovaca) 13 (1963), 20–34, (Russian).
- [99] A. Kotzig *Extremal polyhedral graphs*, Ann. New York Acad. Sci. 319 (1979), 565–570.
- [100] H. Lebesgue *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl. 19 (1940), 27–43.
- [101] T. Madaras *Note on weights of paths in polyhedral graphs*, Discrete Math. 203 (1999), 267–269.
- [102] T. Madaras *Note on the weight of paths in plane triangulation of minimum degree 4 and 5*, Discuss. Math. Graph Theory 20 (2000), 173–180.
- [103] T. Madaras *Two variations of Franklin's theorem*, Tatra Mt. Math. Publ. 36 (2007), 61–70.
- [104] T. Madaras, R. Soták *The 10-cycle is light in the family of all plane triangulations with minimum degree five*, Tatra Mt. Math. Publ. 18 (1999), 35–56.
- [105] T. Madaras, R. Škrekovski *Heavy paths, light stars, and big melons*, Discrete Math. 286 (2004), 115–131.
- [106] B. Madaras, R. Škrekovski, H.-J. Voss *The 7-cycle  $C_7$  is light in the family of planar graphs with minimum degree 5*, Discrete Math. 307 (2007), 1430–1435.
- [107] J. Malkevitch *Polytopal graphs*, Selected topics in Graph theory **3** (L.W. Beineke and R.J. Wilson, Eds.), Academic Press London (1988), 169–188.
- [108] B. Mohar *Light paths in 4-connected graphs in the plane and other surfaces*, J. Graph Theory 34 (2000), 170–179.
- [109] B. Mohar *Light structures in infinite planar graphs without the strong isoperimetric property*, Trans. Amer. Math. Soc. 354 (2002), 3059–3074.
- [110] B. Mohar, C. Thomassen *Graphs on Surfaces*, The Johns Hopkins University Press, Baltimore and London (2001).

- [111] B. Mohar, R. Škrekovski, H.-J. Voss *Light subgraphs in planar graphs of minimum degree 4 and edge-degree 9*, J. Graph Theory 44 (2003), 261–295.
- [112] M. Molloy, M. R. Salavatipour *A bound on the chromatic number of the square of a planar graph*, J. Combin. Theory Ser. B 94 (2005), 189–213.
- [113] O. Ore *The four color problem*, Academic Press, New York/London (1967).
- [114] O. Ore, M. D. Plummer *Cyclic coloration of plane graphs*, in Recent Progress in Combinatorics, W.T. Tutte, ed., Academic Press, New York (1969), 287–293.
- [115] M. D. Plummer *On the cyclic connectivity of planar graph*, Graph Theory and Application, Springer, Berlin (1972), 235–242.
- [116] M. D. Plummer, B. Toft *Cyclic coloration of 3-polytopes*, J. Graph Theory 11 (1987), 507–515.
- [117] A. Pruchniewski, M. Voigt *Weights of induced subgraphs in  $K_{1,r}$ -free graphs*, Manuscript, October 25, 2010.
- [118] R. Radoičić, G. Tóth *The discharging method in combinatorial geometry and the Pach-Sharir conjecture*, in: Surveys on Discrete and Computational Geometry Twenty Years Later (Eds. J. E. Pach and R. Pollack) Contemporary Mathematics 453, American Mathematical Society (2008), 319–342.
- [119] G. Ringel *Ein Sechsfarbenproblem auf der Kugel*, Abh. Math. Sem. Univ. Hamburg 29 (1965), 107–117.
- [120] N. Robertson, R. P. Vitray *Representativity of surface embeddings*, in "Paths, Flows and VLSI - Layout" (B. Korte, L. Lovász, J. Prömel, and A. Schrijver, Eds.) Springer-Verlag, Berlin/New York (1990), 293–328.
- [121] N. Robertson, D. Sanders, P. Seymour, R. Thomas *The four-color theorem*, J. Combin. Theory Ser. B 70 (1997), 2–44.
- [122] M. Rosenfeld *Polytopes of constant weight*, Israel J. Math. 21 (1975), 24–30.
- [123] D. P. Sanders *On light edges and triangles in projective planar graphs*, J. Graph Theory 21 (1996), 335–342.

- [124] E. Steinitz, H. Rademacher *Vorlesung über die Theorie der Polyeder*, Berlin (1934)
- [125] W. T. Tutte *A theorem on planar graphs*, Trans. Amer. Math. Soc. 82 (1956), 99–116.
- [126] G. Wegner *Graphs with given diameter and a coloring problem*, Technical report, University of Dortmund (1977).
- [127] W. Weifan, K.-W. Lih *Structural properties and edge choosability of planar graphs without 6-cycles*, Combinatorics, Probability and Computing 10 (2001), 267–276.
- [128] P. Wernicke *Über den kartographischen Vierfarbensatz*, Math. Ann. 58 (1904), 413–426.
- [129] R. A. Wilson *Graphs, Colourings and the Four-colour Theorem*, Oxford University Press, Oxford, 2002.
- [130] D. R. Woodall, R. J. Wilson *The Appel-Haken proof of the Four-Color Theorem*, in "Selected Topics in Graph Theory", (L.W. Beineke and R. J. Wilson, Eds.) Academic Press London-New York-San Francisco (1978), 83–101.
- [131] J. Zaks *Extending Kotzig's Theorem*, Israel J. Math. 45 (1983), 281–296.
- [132] J. Zaks *Extending two theorems of A. Kotzig*, Discrete Math. 43 (1983), 309–315.
- [133] G. M. Ziegler *Lectures on polytopes*, Grad. Texts in Math. 152, Springer-Verlag, Berlin (1995).