PROCEEDINGS OF THE EWG-DSS LONDON-2011 WORKSHOP ON DECISION SYSTEMS
F.Dargam, B.Delibasic, J.E.Hernández, S.Liu, R.Ribeiro, P.Zaraté (editors)
London, UK, June 23-24, 2011

Ranking of non-linear qualitative decision preferences using copulas

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ABSTRACT

In this paper we address the problem of option ranking in qualitative evaluation models. Current approaches make the assumptions that when qualitative data are suitably mapped into discrete quantitative data, they form monotone or closely linear tabular value functions. Although the power of using monotone and linear functions to model decision maker's preferences is impressive, there are many cases of non-linear decision preferences that need to be modeled using non-linear functions. In this paper, we present one possibility of how to capture the discrete non-linear decision maker preferences by employing copulas. Copulas are functions that manage to capture the non-linear dependences between random variables. Mainly, they are used for aggregation of two attributes. We extend the concept to multivariate case by introducing a hierarchical copula. That way we capture the non-linear dependences among all uniformly distributed variables. We use the obtained dependence structure for copula-based median regression which results into the required option ranking.

The results show that this method may outperform the current approaches for qualitative option ranking of non-monotone decision preferences for a class of non-linear preferences. Furthermore, the mathematics behind copula functions allows extending their usage on preferences expressed with continuous attributes.

Keywords: qualitative decision models, copula, option ranking

INTRODUCTION

Solving problems described with qualitative attributes is part of our daily life. People manage to deal with small problems. However, with increased complexity of the problems, the need for computer tools to help in solving the problems increases too. The complexity of the qualitatively described problems increases when:

- the number of attributes increase,
- the set of possible values that the attributes may receive increases (in general people may consistently distinguish five levels, such as 5 grades at school, while experts may consistently distinguish up to 7 levels [1]) and
- several options belong to the same qualitative class of preferable options, therefore it is difficult to identify the best one (when one has to choose from many varieties, the decision making becomes a difficult task [2]).

In order to tackle these problems, a Qualitative-Quantitative (QQ) method [3, 4] was developed. QQ is an extension of the DEX [5] method and DEXi computer program [6] for multi-attribute decision making. DEXi has been successfully used in wide range of applications such as environmental [7], agricultural [8], and medicine and healthcare [9]. In DEX, the inference of discrete qualitative attributes is specified with a table, whose rows are interpreted as if-then rules. The QQ method maps qualitative attributes consistently into quantitative ones, which are evaluated yielding a numerical utility. QQ aggregates qualitative attributes that may be connected on one level or may build a hierarchical structure. Such a quantitative representation of the model brings the main asset of the method: it is used to distinguish among the options that belong into the same class by ordering them.

The applicability of QQ method is limited to decision problems in which the qualitative attributes can be regarded as discrete values forming monotone or closely linear functions. A monotone function is the one for which the class always increases or remains constant with the increasing values of the attributes. A function is closely linear if it can be 'sufficiently well' (usually within the [–0.5, +0.5] range) approximated by a linear function.

In order to overcome the problem of evaluation of non-monotone decision preferences, we propose to extend the QQ method by using techniques that will capture the non-linear dependencies among different attributes expressed qualitatively by the decision maker. Unlike conventional methods like correlation that summarize the linear dependence relation, we propose to use copulas [10].

Copulas are functions that can capture the non-linear dependences between random variables. To use copulas in order to perform ranking of options that belong in the same class, we go through the following steps. We start with the determination of the marginal distribution function of each variable. Next, two marginal distribution functions are used to find a copula that models the non-linear dependence between them. If we have more than two attributes at hand, we construct a hierarchical copula iteratively. In particular, in each iteration we use the constructed copula and the marginal distribution of a new variable in order to form a new copula. The final copula represents the joint distribution function of all variables. We use the final joint distribution function to express the final class of an option as a function of its attributes. This is performed by employing quantile regression, or more precisely, a median regression [11], hence obtaining the most probable regression curve. Finally, the obtained regression function is used for ordering the options in each class.

USED METHODS

Qualitative-Quantitative method

In order to understand the problem at hand we give a short overview of the QQ method. QQ consists of three stages, as schematically presented in Figure 1.

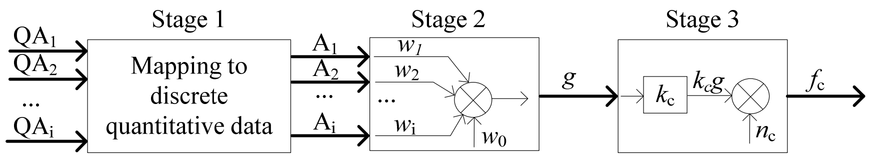

Figure 1 Schematic overview of the three stages of the QQ method

In the first stage, the qualitative attributes QA_i are mapped into discrete quantitative attributes A_i. The mapping should guarantee that the qualitative preferences of the decision maker are preserved into the quantitative space. To illustrate this mapping we consider two qualitative attributes QA_1 and QA_2 and the output class given in Table 1. They get qualitative values from the set: {*good*, *better*, *the best*}. In this set, the preferences of the decision maker are: $the\ best \succ better \succ good$, where the sign $\succ$ denotes "is more preferred than". Then, one possible mapping is into the set of ordered values {1, 2, 3}, such that the decision maker preferences are mapped to $3 > 2 > 1$. This way, the higher number corresponds to a larger preference of the decision maker. An example of this mapping is given in Table 2, which corresponds to the values in Table 1.

Table 1: Qualitatively described problem

No.	QA_1	QA_2	Class
1	good	better	good
2	good	good	good
3	the best	better	better
4	the best	good	good
5	better	better	better
6	better	the best	the best
7	the best	the best	the best

Table 2: Quantitatively described problem

No.	A_1	A_2	Class
1	1	2	1
2	1	1	1
3	3	2	2
4	3	1	1
5	2	2	2
6	2	3	3
7	3	3	3

In the second stage, QQ models the quantitative values of the decision preferences using the linear function:

$$g = \sum_i \omega_i A_i + \omega_0 \tag{1}$$

where A_i are attributes and ω_i are weights obtained by least square regression.

The final stage ensures that the qualitative and quantitative mappings are consistent with each other, that is, whenever the former yields the qualitative class *c*, the latter should yield a numerical value in the range $c \pm 0.5$. Therefore, for each class *c*, *g* is normalized into the corresponding interval $c \pm 0.5$. The set of ranking functions is given with

$$f_c = k_c g + n_c \tag{2}$$

where n_c and k_c are parameters for the normalization of *g* into the interval $c \pm 0.5$. Each of the linear functions in (2) represents a quantitative model of one class of the originally qualitatively described decision preferences. The linear functions (2) are used to determine the order of options in each class. For more details on the QQ method, the reader should refer to [3, 4].

Introduction to Copulas

The QQ method suffers from two main drawbacks: by using qualitative attributes mapped into quantitative ones like those given in Table 2, we are limited to discrete values, and QQ provides only linear functions for ordering of attributes. Linear functions usually do not manage to capture non-linearity in the preferences of the decision maker. To model non-linear preferences and to be able to use both discrete and continuous attributes, we propose to use the concept of copulas. In this paper, we explain this concept on discrete quantitative values, although it is applicable to continuous values as well.

Quantitative attributes in Table 2 may get any discrete value from the space $\Omega = \{1, 2, 3\}$. Consequently we may consider the attributes as discrete random variables and therefore suitable for the application of copulas.

The dependence between random variables is completely defined by their joint distribution function. The joint distribution function $H(x,y)$ for two random variables (r.v.) X and Y, specified on the same probability space, defines the probability of a random event in terms of both X and Y. It is given by:

$$H(x, y) = P[0 \le X \le x, 0 \le Y \le y]$$

where P is probability function. In 1946, Sklar proved that the joint distribution function of two r.v. is equal to the copula of their uniform distributions on the unit interval [0,1] [10].

Theorem 1 (Sklar's theorem [10]): Let H be a two-dimensional distribution function with marginal distribution functions $u=F(x)$ and $v=G(y)$. Then copula C exists such that for all $x, y \in \Re$:

$$H(x,y)=C(F(x),G(y)) \tag{3}$$

If $F(x)$ and $G(y)$ are continuous, then C is unique; otherwise C is uniquely determined on $Range(F) \times Range(G)$. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (3) is a joint distribution function.

Copulas are functions that manage to formulate the multivariate distribution in such a way so that various general types of dependences including non-linear one may be captured. Copulas have the following properties:

1. For every $u, v \in [0,1]$,

   $$C(u,0) = 0 = C(0,v)$$

   and

   $$C(u,1) = u \text{ and } C(1,v) = v$$

2. For every $u_1, u_2, v_1, v_2 \in [0,1]$ such that $u_1 \le u_2$ and $v_1 \le v_2$,

   $$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0$$

The first property of copulas says that the joint probability of all variables is zero if the marginal probability of any variable is zero. Also, if the marginal probability of one variable is one, then the joint probability of all variables is the same as the probability of the second variable.

The second property says that the value of copula function is always non-negative. This comes from the fact that the second property can be interpreted as:

$$P[u_1 \le x_1 \le u_2, v_1 \le x_2 \le v_2] =$$
$$= P[0 \le x_1 \le u_2, 0 \le x_2 \le v_2] - P[0 \le x_1 \le u_1, 0 \le x_2 \le v_2] -$$
$$- P[0 \le x_1 \le u_2, 0 \le x_2 \le v_1] + P[0 \le x_1 \le u_1, 0 \le x_2 \le v_1] \ge 0$$

which is non-negative, as sketched in Figure 2.

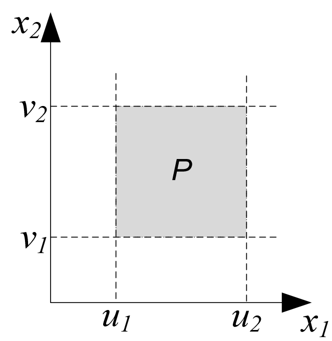

Figure 2 Sketch of the joint probability function $P[u_1 \le x_1 \le u_2, v_1 \le x_2 \le v_2]$

The properties of copulas ensure that they can be used as functions that link a multidimensional distribution to its one-dimensional marginal distributions. Therefore they are considered as a tool for modeling the dependence structure in the data.

Archimedean copulas

One important class of copulas represents the Archimedean copulas whose origin is in the probabilistic metric spaces [12]. To construct an Archimedean copula we use the following relation [8]:

$$C(u,v)=\varphi^{[-1]}(\varphi(u)+ \varphi(v)) \tag{4}$$

where $\varphi : [0,1]^2 \to [0,\infty]$ is called a generator function. If $\varphi(0) < \infty$ then the inverse function of φ is called pseudo-inverse denoted as $\varphi^{[-1]}$. Otherwise, the inverse function of φ is φ^{-1} defined in the interval $0 \le t \le \varphi(0)$. In order φ to be generator function, it must be continuous, strictly decreasing from $[0,1] \to [0,\infty]$ with $\varphi(1) = 0$.

The usage of the Archimedian copulas is mainly motivated by their nice properties:

- symmetry, i.e $C(u_1,u_2) = C(u_2,u_1), \forall u_1,u_2 \in [0,1]$,
- associativity, i.e $C(C(u_1,u_2),u_3) = C(u_1, C(u_2,u_3)), \forall u_1,u_2,u_3 \in [0,1]$
- if φ is the generator, then $k \times \varphi$ is also generator for $k > 0$.

In this paper, we focus on Clayton copula which belongs to the family of Archimedian copulas. Through the Clayton copula, we will show how the concept of copulas may be applied to the qualitative decision making. The generator function of Clayton copula is

$$\varphi(t) = \frac{t^{-\theta} - 1}{\theta} \tag{5}$$

which leads to following form of copula:

$$C_\theta(u,v) = [\max(u^{-\theta} + v^{-\theta} - 1), 0]^{-\frac{1}{\theta}} \tag{6}$$

Next step is to estimate the parameter θ of the Clayton copula.

Estimation of the parameter θ using maximum likelihood

Starting from the fact that probability density function and cumulative distribution function are connected with the derivative (integral) relation, from the distribution function of Clayton copula we get the density function:

$$c(u,v)=\frac{\partial^2 C(u,v)}{\partial u \partial v}=u^{-(\theta+1)}(-\frac{1+\theta}{\theta})(u^{-\theta}+v^{-\theta}-1)^{-\frac{1+\theta}{\theta}-1}(-\theta)v^{-(\theta+1)} \quad (7)$$

After rearranging the last equation, we get:

$$c(u,v)=(1+\theta)(uv)^{-(\theta+1)}(u^{-\theta}+v^{-\theta}-1)^{-\frac{1+2\theta}{\theta}} \quad (8)$$

In order to find the parameter θ in (6), we use the maximum likelihood estimation [13]. By definition, the likelihood function of a random sample of size n from density (mass) function $f(x;\theta)$ is the joint probability density (mass) function denoted by:

$$L(\theta)=L(\theta;x_1,x_2,...,x_n)=f(x_1,x_2,...,x_n;\theta) \quad (9)$$

Equation (9) provides the likelihood that the r.v. take on a particular value $x_1,x_2,...,x_n$, while θ is the unknown parameter. In order to find an estimate $\hat{\theta}$ that maximizes the likelihood function, we take the derivative of $L(\theta)$ and set it equal to zero:

$$\frac{\partial L(\theta)}{\partial \theta}=0 \quad (10)$$

The maxima of the likelihood function and the maxima of the logarithm of the likelihood function are the same. We use this fact in cases when it is easier to find the maxima of a logarithm likelihood function, such as when exponents are involved in the density function, as it is in our case. The logarithm of the density function (8) is:

$$\log c(u,v)=\log(1+\theta)-(\theta+1)\log(uv)-\frac{1+2\theta}{\theta}\log(u^{-\theta}+v^{-\theta}-1) \quad (11)$$

Knowing that the maximum likelihood is the same as the minimum negative log likelihood, we use the minimum finder routine created by Brent [14, 15] to determine θ.

Higher-dimensional copula

Although most of the research focused on copulas is limited to bivariate case, there are several proposals of how to build multi-dimensional copulas [16, 17], from which we take the fully nested Archimedian constructions (FNAC) [18]. An example of FNAC is presented in Figure 3. The basic element in the FNAC structure represents the bivariate copula. As shown in Figure 3, the two nodes u_1 and u_2 are coupled firstly forming a Clayton copula $C_1(u_1,u_2)$ with parameter θ_1. In the next step C_1 is coupled with u_3 into $C_2(u_3,C_1)$ with parameter θ_2, then C_2 is coupled with u_4 into $C_3(u_3,C_2)$ with parameter θ_3 and, finally, C_4 with parameter θ_4 is obtained by coupling C_3 and u_5 [17]:

$$C(u_1,u_2,u_3,u_4,u_5)=C_4(u_5,C_3(u_4,C_2(u_3,C_1(u_2,u_1)))) \quad (12)$$

In order to ensure that (12) represents a valid copula expression, the generator functions φ_i have to fulfil certain conditions. Following the procedures [19] for determining the conditions under which the FNAC built with Clayton copulas represents a copula itself, we obtain the following condition that must be fulfilled:

$$\theta_1 \geq ... \geq \theta_n \quad (13)$$

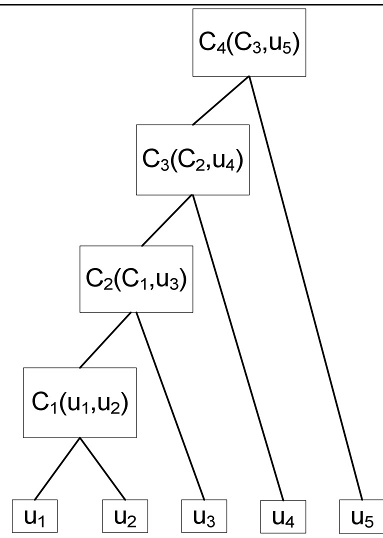

Figure 3: Fully nested Archimedian copula

Regression using copulas

Having determined the dependences that exist among all attributes, including the output one, we may proceed in finding the value of the output, given the values of the input attributes. This kind of dependence is solved with regression. For two r.v. X and Y, the regression curve that defines their mutual dependence is given with

$$y=E(Y|X=x) \tag{14}$$

In (14) y represents the mean value of Y, as the most typical value of Y, for each value of $X=x$. An alternative to the mean values of Y are the quantiles q of Y leading to quantile regression [20]. The q quantile for a r.v. is value x such that the probability

$$P[Y \le y | X=x]=q \tag{15}$$

Performing regression using copulas and converting the density to regression task is described in [11, 21, 22]. In this paper we will use the regression approach for bivariate copulas as in [11, 23]. Given the r.v. X and Y with joint distribution function $H(x,y)$ and marginal distributions $F(x)$ and $G(y)$ respectively, we have:

$$P[Y \le y \mid X = x] = P[V \le G(y) \mid U = F(x)] = \frac{\partial C(u,v)}{\partial u} = q \tag{16}$$

To perform the regression task, first we find the partial derivative of the Clayton copula:

$$q = \frac{\partial C(u,v)}{\partial u} = -\frac{1}{\theta}(u^{-\theta} + v^{-\theta} - 1)^{-\frac{1+\theta}{\theta}}(-\theta u^{-(\theta+1)}) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1+\theta}{\theta}} u^{-(\theta+1)} \tag{17}$$

solving for $q \in [0,1]$ where q is the quantile, leads to v:

$$v = [1 - u^{-\theta} + (qu^{1+\theta})^{-\frac{\theta}{1+\theta}}]^{-\frac{1}{\theta}} \tag{18}$$

Using that $u=F(x)$ and $v=G(y)$ we can find different quantile regression curves for the variable Y on X:

$$y = G^{-1}[1 - (F^{-1}(x))^{-\theta} + (q(F^{-1}(x))^{1+\theta})^{-\frac{\theta}{1+\theta}}]^{-\frac{1}{\theta}} \tag{19}$$

In the example that follows, we will choose $q=1/2$ in order to perform median regression, thus finding the most probable value of the output.

ILUSTRATIVE EXAMPLE

We illustrate the methods on the example given in Table 3. The first column in Table 3 is the number of option. The next four columns are four attributes $(A_{1,...,}A_4)$. The fifth column is an output class. Next, we show the calculations for ranking of attributes obtained by using QQ, Clayton copula and Clayton copula normalized methods.

Each attribute in Table 3 may get values from the interval [1, 5], and each option may belong to one of the set of classes {1, 2, 3, 4, 5}. For the QQ calculations, we use (1), and we obtain

$$g = 0.1279 \times A_1 + 0.3152 \times A_2 - 0.2591 \times A_3 - 0.0855 \times A_4 + 2.547$$

Applying (2), we get the final output for QQ:

$$f_c = \begin{cases} 0.5917 \times g - 0.4432, \mathit{for}\ c = 1 \\ 0.5343 \times g + 0.6196, \mathit{for}\ c = 2 \\ 0.3527 \times g + 1.9395, \mathit{for}\ c = 3 \\ 0.5396 \times g + 2.3274, \mathit{for}\ c = 4 \\ 0.7715 \times g + 2.4589, \mathit{for}\ c = 5 \end{cases}$$

To obtain the Clayton copula, we first find the probability mass function of the values of attributes. Then we obtain their univariate margins $u_i = F(A_i)$, where F_i is cumulative distribution function of attribute A_i, as described in the Sklar's theorem. The cumulative distributions are used to calculate the parameters of the bivariate Clayton copulas defined in (6). The cumulative distribution functions of all attributes and the output class are aggregated into the FNAC structure, as the one in Figure 3. The obtained parameters of the bivariate copulas using the maximum likelihood approach (11) are $\theta_1 = 0.0548$, $\theta_2 = 0.0442$, $\theta_3 = 0.0257$, $\theta_4 = 0.0246$. These values fulfill the condition of equation (13) for FNAC to represent a copula. The final obtained Clayton copula with parameter θ_4 is used for defining the median regression line, or more precisely, the line that provides the ranking of the options in classes. Substituting the value of θ_4 in (19), leads to:

$$y = G^{-1}[1 - (F^{-1}(x))^{-0.0246} + (0.5 \times (F^{-1}(x))^{1.0246})^{-0.0246}]^{-40.62724}$$

where F^{-1} and G^{-1} are inverse cumulative distribution functions. From Table 3, we may notice that QQ breaches the monotonicity property on five occasions, i.e., for option pairs (10, 14), (10, 15), (14, 15), (22, 24), (23, 24). One example of these is given in Table 4, where the option 14 has the same values for the three attributes as the option 15. Regarding the attribute A3, option 15 has a higher value and, consequently, we would expect that QQ evaluated it better than the option 14. However, this is not the case with the QQ calculations. On the other hand, the Clayton copula provides correct ranking between the two options.

Using the median regression for determining the order of options, we get values in the interval [1.95, 2.64]. In order to determine the ranking of option that is consistent with the class into which the option belongs, we use (2) as described in the QQ method. By applying (2) on the results obtained with the Clayton copula, we get values that we named as a Clayton normalized copula (see the last column in Table 3). After normalization, the highest value in one class will be the same as the smallest value of the next higher class. In other words, the transition of values from one class to the next one will be smooth, in contrast with QQ, where there is usually a gap between two consecutive classes. This results from the fact that copulas aggregate the marginal distribution function of attributes. The shape of the marginal distribution function is shift invariant. This means that when a distribution function is shifted for some constant, the values of the minimum and the maximum of the distribution function, which are used for calculation of the parameters n_c and k_c in (2), will not change.

Another advantage and the reason for using the Clayton normalized copula is that it ensures consistency with the qualitative model. As stated in the problem definition, we wish to keep the quantitative evaluation consistent with the qualitative one: for each qualitative class *c*, the corresponding numerical evaluation should be in the range $c \pm 0.5$. The normalization ensures this property. For example, if we compare options 16 and 22, we see that option 16 has a higher value of A_3 than the option 22; however the decision maker ranks option 22 into a higher class than the option 16. Although Clayton calculation provides a correct ranking within the classes, it does not capture the obvious non-linearity imposed by the decision maker in the global ranking of options. This is corrected by applying the third stage of QQ, hence obtaining the Clayton normalized copula. In addition to consistency, we also obtain a better readability of the ranked results, because only a single number (result of Clayton normalised calculations) is sufficient to determine both the class and the rank of each option.

Table 3: Example of table function with four attributes and one output class, and calculations obtained with QQ, Clayton and Clayton normalized methods

No.	A_1	A_2	A_3	A_4	Class	QQ	Clayton	Clayton normalized
1	2	3	4	2	1	0.88	2.51	1.39
2	2	5	3	3	1	1.30	2.53	1.49
3	5	1	4	3	1	0.70	2.23	0.50
4	5	2	2	5	1	1.05	2.54	1.50
5	2	3	3	1	2	2.16	2.19	2.29
6	3	5	5	1	2	2.29	2.24	2.50
7	1	1	3	2	2	1.71	1.99	1.50
8	1	3	2	5	2	2.05	2.21	2.40
9	4	5	2	1	3	3.36	2.21	2.89
10	1	3	1	3	3	3.03	1.99	2.50
11	3	5	4	3	3	3.07	2.58	3.50
12	5	2	1	4	3	3.07	2.219	2.896
13	3	3	1	4	3	3.09	2.220	2.897
14	1	3	3	4	3	2.82	2.222	2.899
15	1	3	5	4	3	2.64	2.24	2.92
16	4	2	4	1	4	3.71	2.21	3.53
17	2	3	4	1	4	3.74	2.20	3.50
18	1	5	3	2	4	4.11	2.21	3.52
19	4	5	3	3	4	4.27	2.58	4.50
20	4	3	1	5	4	4.12	2.24	3.60
21	4	4	1	5	4	4.29	2.25	3.63
22	4	2	1	1	5	5.04	1.95	4.50
23	2	4	1	3	5	5.20	2.20	4.87
24	5	5	5	5	5	4.80	2.64	5.50

Table 4: Example when monotonicity is breached with QQ method

No.	A_1	A_2	A_3	A_4	Class	QQ	Clayton	Clayton normalized
14	1	3	**3**	4	3	2.82	2.222	2.899
15	1	3	**5**	4	3	2.64	2.24	2.92

DISCUSSION AND CONCLUSIONS

Results show that copulas may be used for modeling non-linear qualitative table functions. In this paper we used Clayton copula to capture the non-linear dependences in the input attributes, which is not the usual case when we apply the QQ method. One asset of copulas comes from the fact that they use the distribution functions of the attributes instead of the attribute values themselves. That makes them useful for attributes with different scale units (for example kilometers, seconds, intervals, probabilities), combined attributes (such as discrete, continuous, etc.), as long as the preferences of the values in each of the attributes are defined, or for scale-free attributes.

In order to apply copulas on multi attribute decision problems, there is a need to construct a hierarchical structure of copulas. In our paper we used FNAC that uses bivariate Clayton copula as a basic building block. The main limitation of FNAC comes from the fact that when the number of attributes increases, the probability that the final hierarchical structure also represents a copula decreases. It is difficult to estimate the number of table functions that may be solved when the number of attributes increases due to the fact that there is a combinatorial explosion in higher dimensions.

In order to extend the number of non-linear preferences that may be modeled with copulas, we should analyze how different copula types perform. Additionally, we may expand the estimation of θ by investigating different hierarchical structure of copulas thus circumventing the FNAC conditions. That is left for further research.

ACKNOWLEDGMENTS

The research of the first author was supported by Ad futura Programme of the Slovenian Human Resources and Scholarship Fund.

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