Optimal parametric interpolants of circular arcs

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Abstract

The aim of this paper is a construction of quartic parametric polynomial interpolants of a circular arc, where two boundary points of a circular arc are interpolated. For every unit circular arc of an inner angle not greater than 2π we find the best interpolant, where the optimality is measured by the simplified radial error.

Keywords: geometric interpolation, circular arc, parametric polynomial, Bézier curve, optimal interpolation
2000 MSC: 65D05, 65D07, 65D17

1. Introduction

Circular arcs are basic ingredients of several graphical and control systems, so their approximation by parametric polynomials is important in Computer Aided Geometric Design (CAGD), Computer Aided Design (CAD) and Computer Aided Manufacturing (CAM). Usually we construct parametric polynomial approximant of a circular arc by interpolation of some corresponding geometric quantities. This usually include interpolation of boundary points, corresponding tangent directions, signed curvatures,... The results are so called geometric parametric polynomial interpolants ($\mathcal{G}^n$ interpolants), which can be combined to form geometrically smooth spline curves. One of the standard measures in this case is the radial distance d_r, measuring the distance of the point on the parametric polynomial to the corresponding point on the circular arc in the radial direction. Under some assumptions the metric d_r is equivalent to the Hausdorff metric ([1] and [6]). Hence to find the best interpolant of the unit circular arc $\boldsymbol{c}$ with respect to the Hausdorff metric, we have to find a polynomial $\boldsymbol{p} = (x, y)^T$ which minimizes the value $d_r(\boldsymbol{c}, \boldsymbol{p}) = \max_t \left|\|\boldsymbol{p}(t)\| - 1\right| = \max_t \left|\sqrt{x^2(t) + y^2(t)} - 1\right|$. In the very first paper in which the optimality was proved [8], Mørken observed that the distance d_r is rather cumbersome to work with, so he suggested that instead of the radial distance, we should use the simplified radial distance d_{sr} defined by $d_{sr}(\boldsymbol{c}, \boldsymbol{p}) = \max_t \left|x^2(t) + y^2(t) - 1\right|$. The involved function $x^2 + y^2 - 1$ is a polynomial, which significantly simplifies an analysis of the optimality of the best interpolant.

There are many papers where different types of geometric approximations are considered. But only a few of them are dealing with the optimality of the solution. Mørken considered the parabolic $\mathcal{G}^0$ interpolation of a circular arc [8]. Hur and Kim analyzed the cubic $\mathcal{G}^1$ and the quartic $\mathcal{G}^2$ cases [4]. In this three cases there is only one free parameter involved. Two parametric cases were considered by Vavpetič and Žagar in [9] where the optimal solutions for the cubic $\mathcal{G}^0$ and the quartic $\mathcal{G}^1$ interpolants were found. So far there are no results on the optimal solution of the quintic or higher degree interpolants. In all cases the optimality

order	2	3	4
$\mathcal{G}^0$	Mørken (1991)	Vavpetič, Žagar (2019)	this paper
$\mathcal{G}^1$	Knez, Žagar (2018)	Hur, Kim (2011)	Vavpetič, Žagar (2019)
$\mathcal{G}^2$	–	Knez, Žagar (2018)	Hur, Kim (2011)
$\mathcal{G}^3$	–	–	Knez, Žagar (2018)

Table 1: The list of results where the optimality of the solution was proved.

is measured by simplified radial distance. The only paper where the optimality of the best interpolant is proved according to the real radial distance is [10]. Let us also mention the $\mathcal{G}^n$ interpolation of order $n + 1$. For every circular arc and for every $n \in \mathbb{N}$

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there are only finitely many $\mathcal{G}^n$ interpolants of order $n+1$, and for many circular arcs there is only one such interpolant. Hence there is nothing much to optimize, the only question is the existence of the optimal interpolant, which was positively answered by Knez and Žagar [7]. In this paper we consider the only remaining case of order less than 5, i.e., the quartic $\mathcal{G}^0$ interpolation.

The paper is organised as follows. In Section 2 we review basic definitions and describe the idea of the construction of the best $\mathcal{G}^0$ interpolant of order n. In Section 3 we use our method to confirm known results about the cubic interpolations and also extend our results to circular arcs with an inner angle greater than π. The main part of the paper is Section 4, where we construct the best quartic interpolant of a circular arc and we prove its optimality according to the simplified radial distance. In Section 5 we give some concluding remarks and suggestions for possible future research.

2. Preliminaries

Let $0 \le \varphi \le \frac{\pi}{2}$ and let $\boldsymbol{c}\colon [-\varphi, \varphi] \to \mathbb{R}^2$, $\boldsymbol{c}(t) = (\cos t, \sin t)^T$, be the standard nonpolynomial parametrization of a unit circular arc. We'd like to find the best approximation of $\boldsymbol{c}$ by polynomial curve $\boldsymbol{p}\colon [-1,1] \to \mathbb{R}^2$ of degree $n \in \mathbb{N}$ for which $\boldsymbol{p}(\pm 1) = (\cos\varphi, \pm\sin\varphi)^T$. It is convenient to write $\boldsymbol{p} = (x, y)^T$, where x and y are polynomials of degree at most n. We shall choose the Bernstein-Bézier representation of $\boldsymbol{p}$, i.e.,

$$\boldsymbol{p}(t) = \sum_{j=0}^{n} B_j^n(t)\, \boldsymbol{b}_j, \tag{1}$$

where B_j^n, $j = 0, 1, \ldots, n$, are (reparameterized) Bernstein polynomials over $[-1, 1]$, given as

$$B_j^n(t) = \binom{n}{j} \left(\frac{1+t}{2}\right)^j \left(\frac{1-t}{2}\right)^{n-j},$$

and $\boldsymbol{b}_j \in \mathbb{R}^2$, $j = 0, 1, \ldots, n$, are the control points. Since we consider $\mathcal{G}^0$ interpolation, we have $\boldsymbol{b}_0 = (\cos\varphi, -\sin\varphi)^T$ and $\boldsymbol{b}_n = (\cos\varphi, \sin\varphi)^T$. The circular arc is symmetric over x axis, therefore the best interpolant possesses the same symmetry, i.e., $\boldsymbol{b}_{n-j} = r(\boldsymbol{b}_j)$ for all $j = 0, 1, \ldots, n$, where $r\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection over x axis. Therefore all possible sets of control points of desired interpolants can be described by $n-1$ parameters. The simplified signed radial error function ψ will be defined as

$$\psi(t) = x^2(t) + y^2(t) - 1 = \|\boldsymbol{p}(t)\|_2^2 - 1, \quad t \in [-1, 1],$$

where $\|\cdot\|_2$ is the Euclidean norm. The function ψ also depends on φ and $n-1$ parameters which describe control points and consequently an interpolant $\boldsymbol{p}$. Our goal is for a fixed φ find an interpolant $\boldsymbol{p}$; i.e., find $n-1$ parameters for which the maximum $m_{\boldsymbol{p}} = \max\{|\psi(t)|\,; t \in [-1,1]\}$ of the corresponding simplified error function is as small as possible. Note that a maximum $m_{\boldsymbol{p}}$ can be small but the polynomial curve $\boldsymbol{p}$ is far to be a good interpolant. Namely, according to the definition of ψ a polynomial curve $\boldsymbol{p}$ with small $m_{\boldsymbol{p}}$ can be good interpolation of the circular arc $\boldsymbol{d}\colon [-\varphi, \varphi - 2\pi] \to \mathbb{R}^2$, $\boldsymbol{d}(t) = (\cos t, \sin t)^T$. Note that the mid point $\boldsymbol{p}(0)$ is always on the x-axis, so a necessary condition on $\boldsymbol{p}$ to be a good interpolant of $\boldsymbol{c}$ is that the point $\boldsymbol{p}(0)$ lies on the positive half of x-axis. And if $\boldsymbol{p}(0)$ lies on the negative half of x-axis then $\boldsymbol{p}$ can be viewed as an interpolant of the arc $\boldsymbol{d}$. Therefore an analysis of the function ψ gives us the best interpolant of the circular arc $\boldsymbol{c}$ with the inner angle $2\varphi \in (0, \pi]$ and also the best interpolant of the circular arc $\boldsymbol{d}$ with the inner angle $2(\pi - \varphi) \in [\pi, 2\pi]$.

In the case of $\mathcal{G}^0$ interpolant, the corresponding simplified error function ψ has zeros at ± 1. Eisele showed [3] that the best $\mathcal{G}^k$ interpolant of order n (if it exists) is an alternant with $2(n-k-1)+1$ extreme points, i.e., the corresponding signed simplified error function has $2(n-k-1)+1$ local extrema of the same absolute value and sequential ones have different sign (see Figure 1). In the case of $\mathcal{G}^0$ interpolants which is considered in this paper, the simplified error function of the best interpolant of order n is of the form $\chi(t) = \mu T_{2n}(\zeta_n t)$, where T_{2n} is the Chebyshev polynomial of order $2n$, $\zeta_n = \cos\frac{\pi}{2n}$ is its largest zero and $\mu \in \mathbb{R} \setminus \{0\}$ is a multiplicative constant.

The candidates for the best interpolants are those which have the corresponding simplified error function of the form χ. Hence, to find all candidates for the best interpolant of a circular arc of order n we have to solve the system of equations $\psi(u_i) = 0$, where $u_i = (\cos\frac{\pi}{2n})^{-1} \cos(\frac{2i+1}{2n}\pi)$, $i = 1, \ldots, n-1$, are zeros of the function χ on the interval $(0, 1)$. This is a nonlinear system of $n-1$ polynomial equations for $n-1$ parameters (which describe control points of an interpolant). The system is solved using Gröbner basis. Namely, we find a Gröbner basis for the ideal $\langle \psi(u_i) \mid i = 1, \ldots, n-1 \rangle$ for some monomial order, such that one element of the basis is a polynomial f in only one variable (of order n). All interpolants obtained as a solution of the above

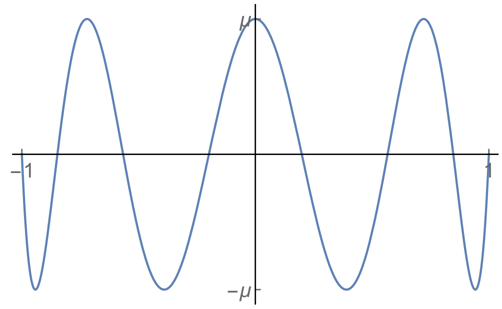

Figure 1: The graph of the error function of the best quartic $\mathcal{G}^0$ interpolant.

system of equations induce error functions of the same shape (see Figure 1), which differ only by the multiplicative constant μ. To find the best interpolant of the circular arc $\boldsymbol{c}$ (the circular arc $\boldsymbol{d}$) we have to find the zero of f, such that the mid point $\boldsymbol{p}(0)$ of the induced interpolant $\boldsymbol{p}$ lies on the positive (negative) half of x-axis, and that the absolute value of the corresponding multiplicative constant μ is the smallest possible. Note that μ is the smallest possible if and only if the absolute value of the leading coefficient (or any other coefficient) of the simplified error function is the smallest possible.

In what follows many mathematical expressions are considered as polynomial in $\cos\varphi$, so it makes sense to define new variables $c := \cos\varphi$ and $s := \sin\varphi$.

3. The cubic $\mathcal{G}^0$ case

This case was considered in [9] but only for a circular arcs with the inner angle not greater than π. In [9] we used a purely geometric argument to prove the existence of the best interpolant. Here we use an appoach with Gröbner basis which can be easier generalised to the quartic $\mathcal{G}^0$ case. The control points for an arbitrary cubic $\mathcal{G}^0$ interpolant are $\boldsymbol{b}_0 = (c, -s)^T$, $\boldsymbol{b}_1 = (\xi, -\eta)^T$, $\boldsymbol{b}_2 = (\xi, \eta)^T$ $\boldsymbol{b}_3 = (c, s)^T$, and the corresponding signed simplified error function is

$$\psi(t) = \frac{1}{16}(t^2-1)\big((3\eta-s)^2t^4 + (16s^2 - 9(\eta+s)^2 + 9(\xi-c)^2)t^2 + (16-(3\xi+c)^2)\big).$$

The simplified error function of the best interpolant is of the form $\chi(t) = \mu T_6(\zeta_3 t) = c(t^2-1)(t^2-u^2)(t^2-v^2)$, where $u = \sqrt{3}-1$ and $v = 2-\sqrt{3}$. To get parameters (ξ, η) of the best interpolant, we have to solve the system of equations: $\psi(u) = \psi(v) = 0$. From $(1-v^2)^2v^2\psi(u) - (1-u^2)^2u^2\psi(v) = 0$, we get

$$\eta = \frac{1}{s}\left(\frac{2+\sqrt{3}}{8}(3\xi+c)^2 - \xi c - 3 - 2\sqrt{3}\right), \tag{2}$$

provided $s \neq 0$. To keep the symmetry we use (2) in the equality $\psi(u) - \psi(v) = 0$. We get one solution $\xi = c$ and the remaining ones satisfy the equality

$$f(\xi) := 243\xi^3 - 27c\left(11-16\sqrt{3}\right)\xi^2 - 3\left(32\left(1+2\sqrt{3}\right) - 3\left(81-32\sqrt{3}\right)c^2\right)\xi - 32\left(13+2\sqrt{3}\right)c - \left(163-112\sqrt{3}\right)c^3 = 0. \tag{3}$$

For all $c \in (0,1)$ we have $-\frac{1}{3}(4+c) < -\frac{1}{9}(8\sqrt{3}-1)c < -\frac{1}{3}c < c < \frac{1}{3}(4-c)$ and

$$\begin{aligned}
f\left(-\tfrac{1}{3}(4+c)\right) &= -64\left(7-4\sqrt{3}\right)(1+c)^3 < 0,\\
f\left(-\tfrac{1}{9}(8\sqrt{3}-1)c\right) &= \tfrac{256}{3}c(1-c^2) > 0,\\
f\left(-\tfrac{1}{3}c\right) &= -64\left(6+(7+4\sqrt{3})c^2\right)c < 0,\\
f(c) &= -256(2+\sqrt{3})c(1-c^2) < 0,\\
f\left(\tfrac{1}{3}(4-c)\right) &= 64\left(7-4\sqrt{3}\right)(1-c)^3 > 0.
\end{aligned}$$

Therefore the cubic polynomial f has zeros on the intervals $\left(-\frac{1}{3}(4+c), -\frac{1}{9}(8\sqrt{3}-1)c\right)$, $\left[-\frac{1}{9}(8\sqrt{3}-1)c, -\frac{1}{3}c\right]$ and $\left(c, \frac{1}{3}(4-c)\right)$. This is also true for $c = 0$, sice in that case f has zeros $\pm\frac{4}{9}\sqrt{2\left(1+2\sqrt{3}\right)}$ and 0.

The necessary condition on a polynomial $\boldsymbol{p} = (x, y)$ to be good interpolant of $\boldsymbol{c}$ is $0 \le x(0) = \frac{1}{4}(3\xi + c)$, hence $\xi \ge -\frac{c}{3}$. We have only two candidates $\xi = c$ and the zero of f on the interval $\left(c, \frac{1}{3}(4-c)\right)$. The constant coefficient of ψ is $-\frac{1}{16}(16-(3\xi+c)^2)$, which is negative and increasing on $\left(-\frac{c}{3}, \frac{1}{3}(4-c)\right)$. Therefore the best interpolant of $\boldsymbol{c}$ is induced by the only zero of f on the interval $\left(c, \frac{1}{3}(4-c)\right)$.

The necessary condition on a polynomial $\boldsymbol{p} = (x, y)$ to be good interpolant of $\boldsymbol{d}$ is $0 \ge x(0) = \frac{1}{4}(3\xi + c)$, hence $\xi \le -\frac{c}{3}$. For $c \ne 1$, i.e., $2(\pi - \varphi) \ne 2\pi$, we have only two candidates. The constant coefficient of ψ is negative and decreasing on $\left(-\frac{1}{3}(4+c), -\frac{c}{3}\right)$. Therefore the best interpolant of $\boldsymbol{d}$ is induced by the only zero of f on the interval $\left(-\frac{1}{3}(4+c), -\frac{c}{3}\right)$.

In the last case when we interpolate the whole unit circle we still have the necessary condition $\xi \le -\frac{c}{3} = -\frac{1}{3}$. We can not use (3), because the equation (2) is not valid for $s = 0$. But from the equality $(1-v^2)^2 v^2 \psi(u) - (1-u^2)^2 u^2 \psi(v) = 0$ we get only one desired solution $\xi = -\frac{1}{9}(8\sqrt{3}-1)$ and from $\psi(u) = 0$ we get $\eta = \frac{4}{9}\sqrt{38 + 22\sqrt{3}}$.

We have proved the following theorem.

Theorem 1. *For every $\varphi \in (0, \pi]$ there exists the unique pair of parameters (ξ, η) which induces the best polynomial interpolant of a circular arc with the inner angle 2φ. If $\varphi \in (0, \frac{\pi}{2}]$, then ξ is the only zero of the function f on the interval $\left(\cos\varphi, \frac{1}{3}(4+\cos\varphi)\right)$ and η is defined by (2). If $\varphi \in (\frac{\pi}{2}.\pi)$ then ξ is the only zero of the function f on the interval $\left(-\frac{1}{3}(4+\cos\varphi), -\frac{1}{9}(8\sqrt{3}-1)\cos\varphi\right)$ and η is defined by (2). If $\varphi = \pi$ the pair of parameters is $(\xi, \eta) = \left(-\frac{1}{9}(8\sqrt{3}-1), \frac{4}{9}\sqrt{38+22\sqrt{3}}\right)$.*

inner angle	ξ	η	radial error
2π	-1.42849	3.87726	3.25356×10^{-1}
$\frac{11\pi}{6}$	-1.50262	3.24480	2.15914×10^{-1}
$\frac{7\pi}{4}$	-1.52161	2.94230	1.71465×10^{-1}
$\frac{5\pi}{3}$	-1.52964	2.65270	1.33756×10^{-1}
$\frac{3\pi}{2}$	-1.51679	2.11990	7.68372×10^{-2}
$\frac{4\pi}{3}$	-1.47274	1.65620	4.04723×10^{-2}

inner angle	ξ	η	radial error
π	1.32800	0.94046	7.97742×10^{-3}
$\frac{2\pi}{3}$	1.16617	0.47494	7.50902×10^{-4}
$\frac{\pi}{2}$	1.09754	0.31523	1.36878×10^{-4}
$\frac{\pi}{3}$	1.04465	0.19043	1.22221×10^{-5}
$\frac{\pi}{4}$	1.02537	0.13762	2.18815×10^{-6}
$\frac{\pi}{6}$	1.01136	0.08926	1.92912×10^{-7}

Table 2: A table of the simplified radial errors of the best cubic $\mathcal{G}^0$ geometric interpolants of a circular arc with a given inner angle.

Jaklič in [5] considered the best cubic polynomial approximant of the unite circle, i.e., an approximant where we do not assume that the boundary points are on the circle. He found the interpolant such that its error function has five local extrema on $(-1, 1)$ all having the absolute value 0.23921 but it has the value 0.27186 at the end points ± 1. According to Eisele's theorem this is not the best approximant but it is close to the best one, since the best approximant has simplified error between 0.23921 and 0.27186. We see that the best cubic $\mathcal{G}^0$ interpolant constructed above is not far from the best cubic approximant.

4. The quartic $\mathcal{G}^0$ case

This is a three-parametric problem. The control points are $\boldsymbol{b}_0 = (c, -s)^T$, $\boldsymbol{b}_1 = (\alpha, \beta)^T$, $\boldsymbol{b}_2 = (\gamma, 0)^T$, $\boldsymbol{b}_3 = (\alpha, -\beta)^T$, $\boldsymbol{b}_4 = (c, s)^T$, and the corresponding signed simplified error function is

$$\psi(t) = -1 + \frac{1}{64}\left(4\left(1-t^4\right)\alpha + 3\left(1-t^2\right)^2\gamma + \left(1+6t^2+t^4\right)c\right)^2 + \frac{1}{4}t^2\left(2\left(1-t^2\right)\beta + \left(1+t^2\right)s\right)^2.$$

By Eisele's theorem the simplified error function of the best interpolant is of the form $\chi(t) = \mu T_8(\zeta_4 t)$, with three zeros $u_1 = \sqrt{2\left(2+\sqrt{2}\right)} - 1 - \sqrt{2}$, $u_2 = \sqrt{2+\sqrt{2}} - 1$ and $u_3 = 1 + \sqrt{2} - \sqrt{2+\sqrt{2}}$ on the interval $(0, 1)$. We have to solve the system of equations $\psi(u_j) = 0$, $j = 1, 2, 3$, and find out which solution (α, β, γ) induces the best interpolation of the circular arc. In what follows it is useful to define $\sigma_1 = (1-u_1^2)+(1-u_1^2)+(1-u_1^2) \approx 1.92$, $\sigma_2 = (1-u_1^2)(1-u_2^2)+(1-u_1^2)(1-u_3^2)+(1-u_2^2)(1-u_3^2) \approx 1.11$, and $\sigma_3 = (1-u_1^2)(1-u_1^2)(1-u_1^2) \approx 0.18$.

We form the linear combination of equations of the system so that we eliminate the variable β and get

$$\begin{aligned} 0 &= 64\left(\frac{u_2^2 u_3^2 \psi(u_1)}{(u_2^2-u_1^2)(u_1^2-u_3^2)(1-u_1^2)} + \frac{u_1^2 u_3^2 \psi(u_2)}{(u_3^2-u_2^2)(u_2^2-u_1^2)(1-u_2^2)} + \frac{u_1^2 u_2^2 \psi(u_3)}{(u_1^2-u_3^2)(u_3^2-u_2^2)(1-u_3^2)}\right) \\ &= 64 + u_1^2 u_2^2 u_3^2 (4\alpha - 3\gamma - c)^2 - (4\alpha + 3\gamma + c)^2, \end{aligned} \tag{4}$$

then set $x := 4\alpha - 3\gamma - c$ and $y := 4\alpha + 3\gamma + c$. Similarly we eliminate β^2 and y^2 and get

$$0 = \sum_{j=1}^{3} \left(u_{j-1}^2 \left(1 - u_{j-1}^2\right)^2 - u_{j+1}^2 \left(1 - u_{j+1}^2\right)^2\right)\psi(u_j) - \frac{\sigma_2}{64} \left(u_1^2 - u_2^2\right) \left(u_2^2 - u_3^2\right) \left(u_3^2 - u_1^2\right) \left(64 + u_1^2 u_2^2 u_3^2 x^2 - y^2\right)$$
$$= 2 \left(u_1^2 - u_2^2\right) \left(u_2^2 - u_3^2\right) \left(u_3^2 - u_1^2\right) \left(\sigma_2 - \sigma_1 + 1\right) \left(-1 + \frac{1}{128}\sigma_3 x^2 + \frac{1}{8}(x+y)c + \beta s\right),$$

where $u_0 = u_3$ and $u_4 = u_1$, therefore

$$0 = -1 + \frac{1}{128}\sigma_3 x^2 + \frac{1}{8}(x+y)c + \beta s. \tag{5}$$

Using the equalities (4) and (5) we get

$$0 = \frac{\psi(u_1)}{1-u_1^2} + \frac{\psi(u_2)}{1-u_2^2} + \frac{\psi(u_3)}{1-u_3^2} + \frac{1}{64}\sigma_1(64 + u_1^2u_2^2u_3^2x^2 - y^2) - (\sigma_1^2 - 3\sigma_1 - 2\sigma_2 + 6)\left(-1 + \frac{1}{128}\sigma_3 x^2 + \frac{1}{8}c(x+y) + \beta s\right)$$
$$= \frac{1}{128}\left(\sigma_1^2 - 2\sigma_2 - \sigma_1\right)\left(96 - 4xy - 128\beta^2 + 16c(x-y) + 32c^2 - \left(4 - 2\sigma_1 + \sigma_3\right)x^2\right),$$

so

$$0 = 96 - 4xy - 128\beta^2 + 16c(x-y) + 32c^2 - \left(4 - 2\sigma_1 + \sigma_3\right)x^2.$$

We multiply the last equality by s^2, use the equality $\beta^2 s^2 = (-1 + \frac{1}{128}\sigma_3 x^2 + \frac{1}{8}c(x+y))^2$ obtained from (5), then use the equality $y^2 = 64 + u_1^2u_2^2u_3^2x^2 = 64 + (1 - s_1 + s_2 - s_3)x^2$ obtained from (4), and get

$$0 = \left(-\frac{x}{32} + \frac{c}{8} + \frac{c^3}{8} - \frac{1}{512}\sigma_3 c x^2\right) y - \frac{1}{4}\left(\left(\frac{1}{64}\sigma_3 x^2 + \frac{1}{4}cx - s^2\right)^2 - \frac{1}{16}\left(1 - \sigma_2 + 2\sigma_3\right)c^2x^2 + \frac{1}{16}\left(2 - \sigma_1\right)x^2 - c(x - 8c)\right). \tag{6}$$

By combining the equalities (6) and (4), we see that we have to investigate the zeros of the function

$$f(x) := \frac{1}{16}\left(\left(\frac{1}{64}\sigma_3 x^2 + \frac{1}{4}cx - 1 + c^2\right)^2 - \frac{1}{16}\left(1 - \sigma_2 + 2\sigma_3\right)c^2x^2 + \frac{1}{16}\left(2 - \sigma_1\right)x^2 - c(x - 8c)\right)^2$$
$$- \left(64 + u_1^2u_2^2u_3^2x^2\right)\left(\frac{x}{32} - \frac{c}{8} - \frac{c^3}{8} + \frac{1}{512}\sigma_3 c x^2\right)^2. \tag{7}$$

Note that for every real zero x of the function f there are the unique (real) number y obtained from (6) and the unique (real)

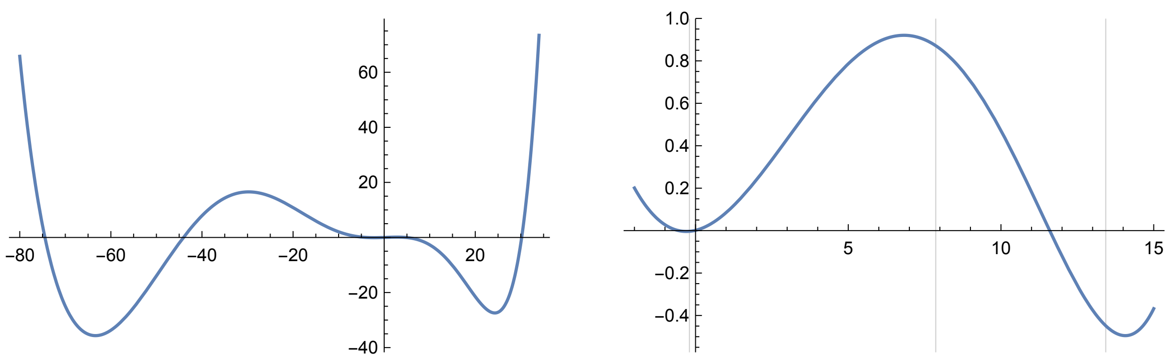

Figure 2: The graph of the function f for the angle $\varphi = \frac{1}{4}\pi$ is on the left and the graph of the function f for the angle $\varphi = \frac{1}{24}\pi$ on smaller interval around 0 where the desired zeros of f appear on the right. The vertical lines on the right graph are drawn at $x = -(1-\cos\varphi)^2$, $x = 4\cos\varphi(1+\cos^2\varphi)$, and $x = (6 - 3\sqrt{2} + \sqrt{68 - 46\sqrt{2}})(1 + \cos^4\varphi)^2$.

number β obtained from (5), such that the triple $(\alpha = \frac{1}{8}(x+y), \beta, \gamma = \frac{1}{6}(y - x - 2\cos\varphi))$ induces an interpolant with the simplified error function ψ of the form χ. The amplitude of the simplified error function ψ is the smallest possible if the leading coefficient of ψ is the smallest possible. The leading coefficient of ψ is $\frac{x^2}{64}$, hence among all zeros of f which induce a good interpolant for a given circular arc we must find the one with the smallest absolute value.

The function f is a polynomial of two variables x and c. Quite often we consider an expression $f(g(c))$, where g is a polynomial in c. Then $f \circ g$ is a polynomial of only one variable and it is easy to check if it has zero on the interval $[0, 1]$.

Remark 2. *Sometimes we will use the following argument: Let $p(x) = c_0 + c_1 x + \ldots + c_n x^n$ be a nonzero polynomial such that the sum $d_j := c_0 + \ldots + c_j$ is nonnegative for all $j = 0, \ldots, n$. Then we can write $p(x) = (1-x)(d_0 + d_1 x + \ldots + d_{n-1}x^{n-1}) + d_n x^n$, therefore $p(x) > 0$ for all $x \in (0,1)$. Similarly, if $d_j \le 0$ for all $j = 0, \ldots, n$, then $p(x) < 0$ for all $x \in (0,1)$.*

Lemma 3. *For every $c \in [0,1]$ the function f has exactly two zeros on the interval $[-(1-c)^2, 4(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}})]$, the one is on the interval $[-(1-c)^2, 0]$ and the other is on the interval $(4c(1+c^2), (6-3\sqrt{2}+\sqrt{68-46\sqrt{2}})(1+c^4)^2]$.*

Proof. For $c \in [0,1)$ we have $f\left(-(1-c)^2\right) < 0$, $f(0) > 0$, $f\left(4c(1+c^2)\right) > 0$, and $f\left((6-3\sqrt{2}+\sqrt{68-46\sqrt{2}})(1+c^4)^2\right) < 0$. If $c = 1$, f has zeros at 0 and $4(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}})$. Thus for all $c \in [0,1]$ the polynomial f has at least one zero on $[-(1-c)^2, 0]$ and at least one zero on $(4c(1+c^2), (6-3\sqrt{2}+\sqrt{68-46\sqrt{2}})(1+c^4)^2]$. To show that f has exactly two zeros on $[-(1-c)^2, 4(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}})]$ it is enough to show that f is increasing on $(-(1-c)^2, 4c^2)$, f is concave on $(4c^2, 8c)$, and f is decreasing on $(8c, 4(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}))$.

The function f is increasing on $(-(1-c)^2, 0)$: We have $f'(x) > 0$ for all $x \in (-(1-c)^2, 0)$ if and only if $g'(t) < 0$ for all $t \in (0,1)$, where $g(t) = f(-t(1-c)^2)$. We can write $g'(t) = -\left((1-c)^4 g_1(t) + \frac{1}{2^{13}}(1-c)^8\left(g_2(t)+g_3(t)+g_4(t)\right)\right)$ where

$$\begin{aligned}
g_1(t) &= \frac{1}{16}(1+c)^2\left(5+c^2\right)c + \frac{1}{128}\left(12-56c^2+28c^4+2\sigma_1\left(1-c^2\right)\left(1+5c^2+2c^4\right)+2\sigma_2\left(1-c^2\right)^2c^2+\sigma_3\left(1-c^2\right)^3\right)t \\
&\quad + \frac{3}{1024}\left(4\sigma_1\left(1-3c^2\right)+4\sigma_2\left(2-c^2+c^4\right)+\left(1-c^2\right)\left(-16+\left(-9+5c^2\right)\sigma_3\right)\right)(1-c)^2ct^2, \\
g_2(t) &= \left(\frac{5}{2}\sigma_3 - \sigma_2 c\right)^2 t^3, \\
g_3(t) &= \Big(\sigma_2\left(32s^2+\sigma_3\left(5-8cs^2\right)c\right) - \frac{1}{4}\sigma_3\left(64\left(1-5c^2+2c^4\right)+\sigma_3\left(37-24c^2+12c^4\right)\right)+8\sigma_1\left(2\sigma_2c^2-\sigma_3\left(1+c^2-2c^4\right)\right) \\
&\quad - 8\sigma_1^2 - \left(c^2+8c^4\right)\sigma_2^2\Big)t^3 - \frac{5}{8}\sigma_3\left(8\sigma_2+5c^2\sigma_3\right)(1-c)^2ct^4 - \frac{3}{64}\sigma_3^2\left(4+4c^2\sigma_1+c^2\sigma_3\right)(1-c)^4t^5 - \frac{1}{4096}\sigma_3^4(1-c)^8t^7, \\
g_4(t) &= \frac{5}{8}\sigma_3\left(4\sigma_1+4c^2\sigma_2+3\sigma_3\right)(1-c)^2ct^4 + \frac{3}{64}\sigma_3^2\left(2\sigma_1+2c^2\sigma_2+\sigma_3\right)(1-c)^4t^5 + \frac{7}{512}\sigma_3^3(1-c)^6ct^6.
\end{aligned}$$

It is easy to see that $g_2(t) > 0$ and $g_4(t) \ge 0$ for all $t \in (0,1)$ and $c \in [0,1]$. By Remark 2, also $g_1(t) \ge 0$ for all $t \in (0,1)$ and $c \in [0,1]$. The coefficients of t^4, t^5 and t^7 in the polynomial g_3 are obviously nonpositive for all $c \in [0,1]$. Hence if we replace all t^k in g_3 by t^3, we decrease the value of $g_3(t)$ and we get the expression of the form $t^3 h(c)$, where h is a polynomial. By Remark 2 we get $h(c) > 0$ for all $c \in [0,1]$, hence $g_2(t) \ge 0$ for all $t \in (0,1)$ and $c \in [0,1]$. This implies that f is increasing on the interval $(-(1-c)^2, 0)$.

The function f is increasing on $(0, 4c^2)$: Since $f'(0) \ge 0$ and $f'(4c^2) \ge 0$ for all $c \in (0,1]$ it is enough to show that f' is concave on $(0, 4c^2)$. Let us define

$$\begin{aligned}
g_1(x) &= -\frac{1}{2^{24}}\left(21\sigma_3^4x^4 + 84c\sigma_3^3\left(10+c\sigma_3\right)x^3 + 48\sigma_3^2\left(160+80\sigma_1-80\sigma_2+70\sigma_3+7\sigma_3^2\right)x^2\right)\left(4c^2-x\right), \\
g_2(x) &= -\frac{3\sigma_3^2}{2^{20}}\left(80(1+c)(2\sigma_1-\sigma_2)+\sigma_3\left(10\left(11+11c+7c^2\right)+7\left(1+c+c^2+c^3\right)\sigma_3\right)\right)(1-c)x^3, \\
g_3(x) &= -\frac{3\sigma_3(1-c)}{2^{18}}\left(80\sigma_2\left(4\left(c^2+c-1\right)-(1+c)\sigma_3\right)+80\sigma_1\left(4+(1+c)\sigma_3\right)+\sigma_3\left(70(1+c)\sigma_3-80c(3+5c)+7(1+c)\sigma_3^2\right)\right)x^2
\end{aligned}$$

It is easy to see that $g_1(x) < 0$, $g_2(x) \le 0$ and $g_3(x) \le 0$ for all $x \in (0, 4c^2)$ and $c \in (0,1]$. Hence it is enough to show that $g_0(x) = f'''(x) - g_1(x) - g_2(x) - g_3(x) \le 0$ for all $x \in (0, 4c^2)$ and $c \in (0,1]$. This is true, since for all $c \in (0,1]$ the function q_0 is a quadratic polynomial with negative leading coefficient, $g_0(0) \le 0$, and $g_0(4c^2) \le 0$.

The function f is concave on $(4c^2, 8c)$: Let us define

$$\begin{aligned}
g_1(x) &= -\frac{1}{2^{25}}\Big(7\sigma_3^4x^5 + 56c\sigma_3^3\left(6+\sigma_3\right)x^4 + 64\sigma_3^2\left(60+30\sigma_1-30\sigma_2+42\sigma_3+7\sigma_3^2\right)x^3 \\
&\quad + 512c\sigma_3\left(7\left(\sigma_3+2\right)^3+80\left(\sigma_1-2\sigma_2\right)+30\left(\sigma_1-\sigma_2\right)\sigma_3-56+36\sigma_3+20\left(4\sigma_2-5\sigma_3\right)c^2\right)x^2\Big)(8c-x), \\
g_2(x) &= -\frac{1}{2^{19}}\sigma_3^2\left(60\sigma_1-30\sigma_2+\sigma_3(57+7\sigma_3)\left(1-c^2\right)x^4\right., \\
g_3(x) &= -\frac{1}{2^{13}}\Big(7\sigma_3\left(\sigma_3+2\right)^3+\left(30\sigma_1-30\sigma_2-73\right)\sigma_3^2+24\left(4-2\sigma_1+\sigma_2\right)\sigma_2+8\left(7\sigma_1-10\sigma_2-25\right)\sigma_3 \\
&\quad + \left(24\sigma_2^2+96\sigma_3-48\sigma_1\sigma_3+56\sigma_2\sigma_3-91\sigma_3^2\right)c^2\Big)\left(1-c^2\right)x^2.
\end{aligned}$$

It is easy to see that $g_1(x) < 0$, $g_2(x) < 0$, and $g_3(x) \le 0$ for all $x \in (4c^2, 8c)$ and $c \in (0,1]$. Therefore it is enough to prove that $g_0(x) = f''(x) - g_1(x) - q_2(x) - q_3(x) \le 0$. This is true, since for all $c \in (0,1]$ the function g_0 is the quadratic polynomial with negative leading coefficient, $g_0(4c^2) < 0$, and $g_0(8c) < 0$.

The function f is decreasing on $(8c, 4(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}))$: We have $f'(x) < 0$ for all $x \in (8c, 4(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}))$ if and only if $g'(t) < 0$ for all $t \in (0,1)$, where $g(t) = f(4(\sqrt{2}+\sqrt{68-46\sqrt{2}}-2c)t + 8c)$. By Remark 2, we get $g(t) < 0$ for all $t \in (0,1)$ and $c \in [0,1]$. □

Lemma 4. *Let $c \in [0,1]$.*

1. *If $x \in [-14, 0)$ the the unique solution y of the equation (6) is positive.*
2. *If $x \in \left(4c(1+c^2), \left(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}\right)(1+c^4)^2\right]$ the the unique solution y of the equation (6) is negative.*

Proof. Let us define

$$
\begin{aligned}
q_1(x) &= -\frac{x}{32} + \frac{c}{8} + \frac{c^3}{8} - \frac{1}{512}\sigma_3 c x^2, \\
q_2(x) &= \frac{1}{4}\left(\frac{\sigma_3 x^2}{64} + \frac{cx}{4} - 1 + c^2\right)^2, \\
q_3(x) &= \frac{1}{4}\left(-\frac{1}{16}\left(1-\sigma_2+2\sigma_3\right)c^2x^2 + \frac{1}{16}\left(2-\sigma_1\right)x^2 - c(x-8c)\right),
\end{aligned}
$$

then by (6) we have $q_1(x)y = q_2(x) + q_3(x)$.

Let $x \in [-14, 0)$. Because the polynomial q_1 is decreasing on $[-14, 0)$ and $q_1(0) \ge 0$, we have $q_1(x) > 0$. The polynomial q_3 is decreasing on $(-\infty, 0]$ and $q_3(0) \ge 0$, hence $q_3(x) > 0$. Because also $q_2(x) \ge 0$, we have $y > 0$.

Let $x \in \left[4c(1+c^2), \left(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}\right)(1+c^4)^2\right]$. Because the polynomial q_1 is decreasing on $[0,\infty)$ and $q_1(8c) < 0$, we have $q_1(x) < 0$. To show that $q_2(x) + q_3(x) > 0$ it is enough to show that $q_0(x) = q_2(x) + q_3(x) - \frac{1}{2^{14}}(32c\sigma_3x^3 + \sigma_3^2x^4) > 0$. For $c \ge 0.9$ the function q_0 is a quadratic polynomial with positive leading coefficient and negative discriminant, hence $q_0(x) > 0$. If $c \le 0.9$ then $q_0'(x) < 0$ and $q_0\left(\left(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}\right)(1+c^4)^2\right) > 0$, hence $q_0(x) > 0$. □

The following theorem follows from Lemma 3 and Lemma 4.

Theorem 5. *For every $\varphi \in (0,\pi]$ there exists the unique triple of parameters (α,β,γ), which induces the best polynomial interpolant of a circular arc of the inner angle 2φ, where $\alpha = \frac{1}{8}(x+y)$, $\gamma = \frac{1}{6}(y - x - 2\cos\varphi)$, x is the only zero of the function* (7) *on the interval $[-(1-\cos\varphi)^2, 0]$, if $\varphi \in (0, \frac{\pi}{2}]$, x is the only zero of the function* (7) *on the interval $\left(0, 6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}\right]$, if $\varphi \in (\frac{\pi}{2}, \pi]$, y is the solution of the equation* (6)*, and β is the solution of the equation* (5).

inner angle	α	β	γ	radial error
2π	0.72323	−2.59930	−4.00076	2.75264×10^{-2}
$\frac{11\pi}{6}$	0.25762	2.33512	−3.35153	1.45985×10^{-2}
$\frac{7\pi}{4}$	0.06000	2.19627	−3.06840	1.03522×10^{-2}
$\frac{5\pi}{3}$	−0.11542	2.05518	−2.81103	7.19767×10^{-3}
$\frac{3\pi}{2}$	−0.40437	1.77231	−2.36754	3.25472×10^{-3}
$\frac{4\pi}{3}$	−0.61961	1.49705	−2.00895	1.32486×10^{-3}

inner angle	α	β	γ	radial error
$\frac{\pi}{2}$	0.87518	0.99857	1.49995	1.42325×10^{-4}
$\frac{\pi}{3}$	0.97471	0.59188	1.20039	5.83570×10^{-6}
$\frac{\pi}{4}$	0.99193	0.42228	1.10839	5.94378×10^{-7}
$\frac{\pi}{6}$	0.99840	0.27073	1.04680	2.34778×10^{-8}
$\frac{\pi}{8}$	0.99949	0.20014	1.02605	2.36051×10^{-9}
$\frac{\pi}{12}$	0.99990	0.13203	1.01149	9.23852×10^{-11}

Table 3: A table of the radial errors of the best quartic $\mathcal{G}^0$ geometric interpolants of a circular arc with a given inner angle.

Numerical computations reveal that the function f has five real zeros for $\varphi \approx 0.9188$, six real zeros for $\varphi < 0.9188$, and only four real zeros for $\varphi > 0.9188$. All real zeros of f induce a quartic $\mathcal{G}^0$ interpolant which have an alternating simplified signed error function, i.e., its error function has the same shape as the error function χ of the best interpolant (only the amplitude can vary). We have proved that the largest negative zero of f induces the best interpolant of the unit circular arc with the inner angle 2φ and the smallest zero of f induces the best interpolant of the unit circular arc with the inner angle $2(\pi-\varphi)$. Most of the remaining zeros induce a non admissible interpolant, i.e., an interpolant with self intersections. The reason why interpolant has self intersections is that some of the control points b_1, b_2, b_3 of an interpolant lie left and some right from the control points b_0 and b_4. Numerical computations show that for $\varphi < 0.6772$, the second largest negative zero induces an admissible interpolant (all three control points b_1, b_2, b_3 are right of the control points b_0 and b_4) but the corresponding error function has a larger amplitude than the error function of the best interpolant (see Figure 3). We can see that the curvature of the best interpolant is much closer to the curvature of the unit circular arc than the curvature of the other interpolant. The quartic approximant of the

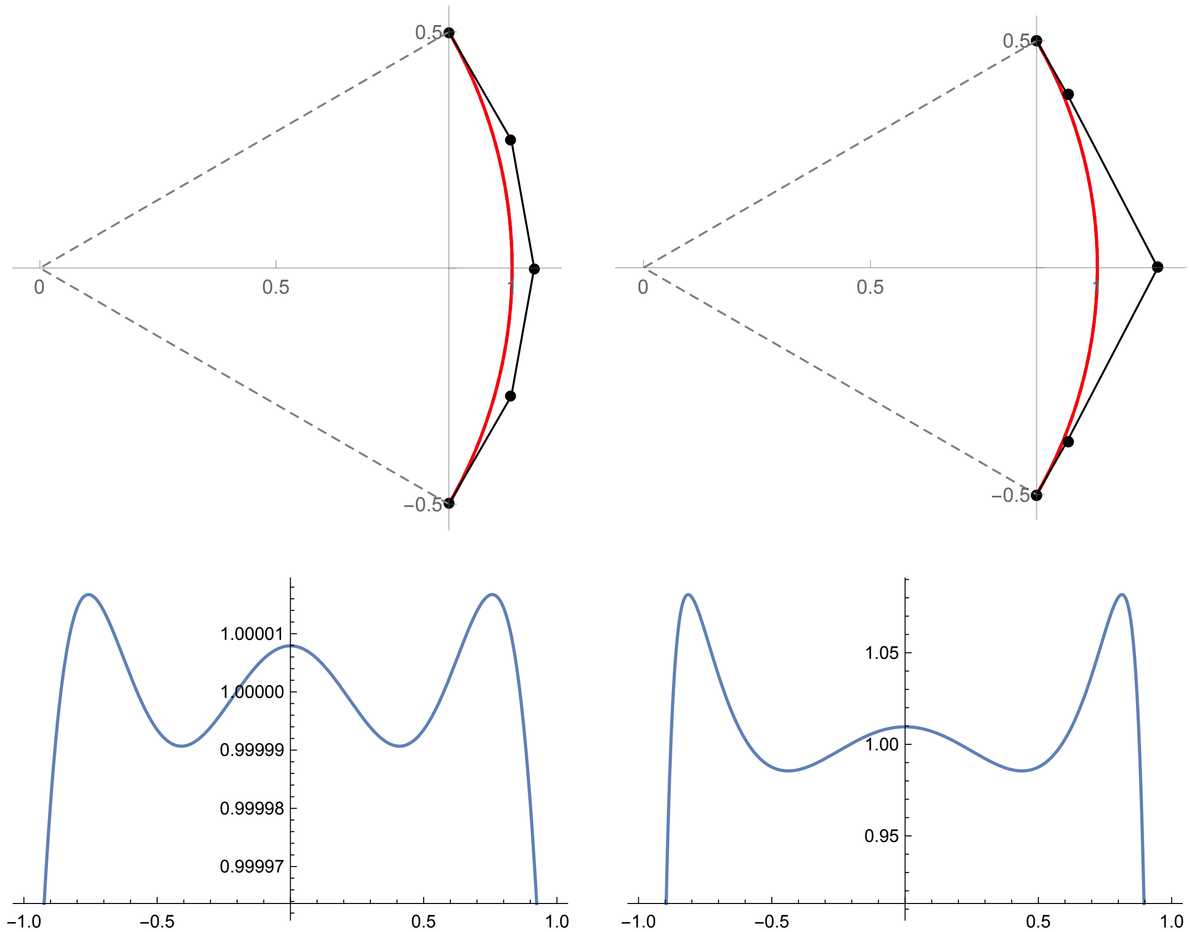

Figure 3: Two quartic $\mathcal{G}^0$ interpolants of circular arc of inner angle $\frac{\pi}{3}$ with the simplified signed error function as in the Figure 1. The left one is the best interpolant with the error 2.34778×10^{-8}, the right one has the error 4.01760×10^{-5}. In the bottom row there are graphs of the curvatures of the interpolants from the first row.

whole unit circle constructed in [5] has the error function with seven local extrema on $(-1, 1)$ all have the absolute value 0.021873 but it has the value 0.022115 at the end points ± 1. Again we can see the best quartic $\mathcal{G}^0$ interpolant constructed above is not far from the best quartic approximant of the unit circle. We can also check that the graph of the curvature of the best quartic $\mathcal{G}^0$ interpolant of the whole unit circle and the graph of the curvature of the approximant constructed in [5] are almost identical.

5. Conclusion

In this paper we presented an interpolation of a circular arc given by an inner angle $2\varphi \in (0, 2\pi]$, where both boundary points of the arc are interpolated. Our method works well in the parabolic case, where for every φ we get only one candidate for the best interpolant, and also in the cubic case, where for every φ we get only one admissible candidate. In the quartic case we get more candidates and the analysis to figure out which candidate is the best one is quite demanding. Our method could be applied for interpolation of a circular arc by higher order polynomials, but it seems that it is very hard to prove which candidate is the best one. Maybe the method can be used for some particular cases, like half circular arc or quarter circular arc.

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